TI 2001-046/4
Tinbergen Institute Discussion Paper

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ON THE DECAY OF INFINITE PRODUCTS OF TRIGONOMETRIC POLYNOMIALS

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March 9, 2001

TINBERGEN INSTITUTE REPORT EI-2001-10

Abstract[2]

We consider infinite products of the form $f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi)$, where $\{m_k\}$ is an arbitrary sequence of trigonometric polynomials of degree at most n with uniformly bounded norms such that $m_k(0) = 1$ for all k. We show that $f(\xi)$ can decrease at infinity not faster than $O(\xi^{-n})$ and present conditions under which this maximal decay attains. This result proves the impossibility of the construction of infinitely differentiable nonstationary wavelets with compact support and restricts the smoothness of nonstationary wavelets by the length of their support. Also this generalizes well-known similar results obtained for stable sequences of polynomials (when all m_k coincide). In several examples we show that by weakening the boundedness conditions one can achieve an exponential decay.

Key words. trigonometric polynomial, infinite product, wavelets, roots.

AMS subject classification. 26C10, 39B32, 42A05, 42A38

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[2] The author is very grateful to Professor Igor Novikov at Voronezh State University for useful discussions and constant attention to this work..

I. Introduction

The following notation will be used: $\mathbb{N}, \mathbb{R}, \mathbb{C}$ are the sets of natural, real and complex numbers respectively, $\mathcal{P}_n$ is the space of algebraic polynomials over $\mathbb{C}$ of degree at most n, $\mathcal{L}_1$ is the space of summable functions on $\mathbb{R}$, S' is the space of distributions of slow growth (the dual for the Schwartz space S); for given functions $g_0(x)$ and $g_1(x)$ we say that $g_1 = o(g_0)$ as $x \to \infty$ if $g_1/g_0 \to 0$ as $x \to \infty$; also $g_1 = O(g_0)$ as $x \to \infty$ if there exists a constant $C > 0$ such that $|g_1| \le C|g_0|$ for all sufficiently large x. We will work with trigonometric polynomials of the form $m(\xi) = \sum_{s=0}^{n} c_s e^{-is\xi}$, i.e., trigonometric polynomials without positive powers. We denote by $\tilde{m}$ the corresponding algebraic polynomial $\tilde{m}(z) = \sum_{s=0}^{n} c_s z^s, z \in \mathbb{C}$, so $m(\xi) = \tilde{m}(e^{-i\xi})$. As usual we denote $\|m\| = \sup\limits_{\xi \in \mathbb{R}} |m(\xi)|$, also for the associated algebraic polynomial $\tilde{m}$ we denote by $\|\tilde{m}\|$ its norm on the unit circle: $\|\tilde{m}\| = \sup\limits_{z \in \mathbb{C}, |z|=1} |\tilde{m}(z)|$. Thus $\|m\| = \|\tilde{m}\|$. We assume everywhere that the leading coefficient c_n is nonzero, so $\deg m = \deg \tilde{m} = n$. Further, for a given function $g(\xi)$ its *rate of decreasing* $\mathbf{d}(g)$ is the largest integer l such that $g(\xi) = o(\xi^{-l})$ as $|\xi| \to \infty$; if g decreases faster than polynomially, then $\mathbf{d}(g) = +\infty$, if such integers l do not exist, then $\mathbf{d}(g) = -\infty$

For an arbitrary sequence of trigonometric polynomials $\{m_k\}_{k \in \mathbb{N}}$ such that $m_k(0) = 1, k \in \mathbb{N}$ we consider the following infinite product:

$$f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi). \tag{1}$$

Such products are used in the study of wavelets and subdivision schemes in approximation theory and curve design (we will discuss it below), they also applied in some problems of probability theory ([Der], [DDL], [P2]). Products of this kind arise naturally in the study of fractal curves (for instance, DeRham curves, see [CDM], [M]), Bernoulli convolutions ([E], [PS]) and combinatorial number theory ([Re]).

Under some appropriate conditions (for example, if the norms and the degrees of these polynomials are uniformly bounded) product (1) converges uniformly on any compact set and hence represents an analytic function. How fast can this function decrease on infinity? In this paper we analyze the rate of decreasing of the function f in terms of the sequence $\{m_k\}$.

In a special case, when all the polynomials m_k are the same, this problem is studied in great detail (see References). It is easy to see that the inverse Fourier transform of the function $f_m(\xi) = \prod_{k=1}^{\infty} m(2^{-k}\xi)$ exists at least in the sense of distributions and satisfies the following *refinement equation*:

$$\check{f}_m(x) = 2 \sum_{s=0}^{n} c_s \check{f}_m(2x - s), \tag{2}$$

where $c_0, \dots, c_n$ are coefficients of the polynomial m. This equation plays an exceptional role in the study of wavelets (see [C] for more references), and also has found a lot of applications in approximation theory and curve design (see [CDM],[DGL], [DyL], [Re] and references therein). It is not difficult to show that equation (2) either has no $\mathcal{L}_1$-solutions at all or has the only solution $\check{f}_m$ (if this function belongs to $\mathcal{L}_1$). If the function f_m has a positive rate of decreasing, then $\check{f}_m$ is indeed in $\mathcal{L}_1$ and, moreover, is $\mathbf{d}(f_m)-1$ time differentiable. So the smoothness of solutions of refinement equations can be estimated by the decay of the corresponding polynomial products f_m. This idea was put into good use in [D], [CD], [RS] and many other works on this subject. It is well known that the rate of decreasing of the function $f_m(\xi)$ cannot exceed $n-1$, where $n = \deg m$, and this maximal decay attains only for the polynomial $m(\xi) = \left(\frac{e^{-i\xi}+1}{2}\right)^n$, which will be denoted in the sequel by $w_{n-1}(\xi)$. The corresponding product $f_w(\xi) = \prod_{k=1}^{\infty} w_{n-1}(2^{-k}\xi)$ is the Fourier transform of the cardinal B-spline of order $n-1$: $B_{n-1}(x) = \chi_{[0,1]}(x) * \cdots * \chi_{[0,1]}(x)$ ($n-1$ convolutions altogether, $\chi_{[0,1]}(x)$ is the characteristic function of the segment $[0,1]$). Indeed:

$$f_w(\xi) = \prod_{k=1}^{\infty} \frac{e^{-i2^{-k}\xi}+1}{2} = \frac{[i(e^{-i\xi}-1)]^n}{\xi^n} = \hat{B}_{n-1}(\xi).$$

Thus for any polynomial m of degree n we have $\mathbf{d}(f_m) \le n-1$ and the corresponding equality takes place only for the polynomial w_{n-1}, for which moreover $f_w(\xi) = O(\xi^{-n})$. This is the *maximality property* of the cardinal B-spline (see [CDM],[DL1] for the corresponding proofs, see also [Sc],[DL2] for more properties of B-splines). This implies, in particular, that the smoothness of wavelets supported on the segment of length n cannot exceed $n-1$.

A natural question arises what can be said about the decreasing of function (1) in the case when the polynomials $m_k, k \in \mathbb{N}$ do not have to coincide? Whether it is possible to obtain a faster decay by choosing a special sequence of polynomials of degree at most n ? This problem appeared in connection with the study of nonstationary wavelets introduced in [BN], [BVR]. These wavelets are also defined by product (1), but in this case the polynomials $\{m_k\}$ are a priori different. The problem was first formulated by I.Novikov in 1999: can product (1), with the only requirements that the degrees of m_k do not exceed n and their norms are bounded uniformly, have an infinite rate of decreasing? The positive answer would lead to a construction of infinitely-smooth nonstationary wavelets with compact support (see [N]). In Theorem 1 we show that the answer is negative. The rate of decreasing cannot be infinite, moreover it still cannot exceed $n-1$. Thus the maximality property of B-splines extends now onto a much wider class of polynomial products: on functions of type (1) for all possible sets of polynomials $\{m_k\}$ (with bounded degrees and norms). Furthermore, if such a function has the maximal rate of decreasing $n-1$, then the corresponding sequence m_k converges to w_{n-1}, which means

$\|m_k - w_{n-1}\| \to 0$ as $k \to \infty$. In several examples we show that none of the assumptions of Theorem 1 can be weakened. In particular (Example 2), without the condition of uniform boundedness by norm the function f may have an exponential decay. Further, in Theorem 2, we establish a criterion on the sequence $\{m_k\}$, which ensures that the corresponding product f has the fastest decay, i.e., decreases as $O(\xi^{-n})$.

II. The main result.

Theorem 1. *Let us have a family of trigonometric polynomials $\{m_k\}_{k\in\mathbb{N}}$ and numbers $n \in \mathbb{N}, M \geq 1$ such that for all $k \in \mathbb{N}$ we have $m_k(0) = 1$, $\deg m_k \leq n$, $\|m_k(\cdot)\| \leq M$. Then the rate of decreasing of the function $f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi)$ does not exceed $n-1$. Moreover, if the rate of decreasing is equal to $n-1$, then $m_k \to w_{n-1}$ as $k \to \infty$.*

Proof. Suppose that the rate of decreasing is at least $n-1$. This means that

$$\xi^{n-1} f(\xi) \to 0 \quad \text{as } |\xi| \to \infty. \tag{3}$$

Let $\sigma = \frac{1}{2nM}$. For any $\xi \in [-\sigma, \sigma]$, $j \geq 0$ and $N \in \mathbb{N}$ or $N = +\infty$ we have

$$\left| \prod_{k=1}^{N} m_{j+k}(2^{-k}\xi) \right| > \frac{1}{2}. \tag{4}$$

Indeed, by Bernstein inequality we have $\|m'_{j+k}\| \leq nM$, therefore $\left|m_{j+k}(2^{-k}\xi)\right| \geq |m_{j+k}(0)| - 2^{-k}|\xi| \cdot \|m'_{j+k}\| \geq 1 - nM2^{-k}|\xi| \geq 1 - nM2^{-k}\sigma = 1 - 2^{-k-1}$. It now follows that $\left|\prod_{k=1}^{N} m_{j+k}(2^{-k}\xi)\right| \geq \prod_{k=1}^{N}\left(1 - 2^{-k-1}\right) > \frac{1}{2}$. Let now p be the smallest integer such that $2^p \geq 8\pi nM$. For arbitrary $\delta \in (0, \sigma)$ and $l \in \mathbb{N}$ we have

$$f(2\pi \cdot 2^l + \delta) = \prod_{k=1}^{l} m_k\left(2\pi \cdot 2^{l-k} + 2^{-k}\delta\right) \prod_{k=1}^{p} m_{l+k}\left(2\pi \cdot 2^{-k} + 2^{-l-k}\delta\right) \prod_{k=1}^{\infty} m_{l+p+k}\left(2\pi \cdot 2^{-p-k} + 2^{-l-p-k}\delta\right).$$

Applying (4) for $\xi = \delta, j = 0$ and $N = l$ we obtain

$$\left|\prod_{k=1}^{l} m_k\left(2\pi \cdot 2^{l-k} + 2^{-k}\delta\right)\right| = \left|\prod_{k=1}^{l} m_k\left(2^{-k}\delta\right)\right| > \frac{1}{2};$$

also apply (4) for $\xi = 2\pi \cdot 2^{-p} + 2^{-l-p}\delta$, $j = l + p$ and $N = +\infty$ and get

$$\left|\prod_{k=1}^{\infty} m_{l+p+k}\left(2\pi \cdot 2^{-p-k} + 2^{-l-p-k}\delta\right)\right| > \frac{1}{2}.$$

Therefore

$$\left|f(2\pi \cdot 2^l + \delta)\right| > \frac{1}{4}\left|m_{l+1}\left(\pi + 2^{-l-1}\delta\right) m_{l+2}\left(\frac{\pi}{2} + 2^{-l-2}\delta\right) \cdots m_{l+p}\left(2^{1-p}\pi + 2^{-l-p}\delta\right)\right|. \tag{5}$$

By our assumption $2^{l(n-1)}f(2\pi\cdot 2^l+\delta)\to 0$ as $l\to\infty$, consequently

$$2^{l(n-1)}m_{l+1}\Bigl(\pi+2^{-l-1}\delta\Bigr)m_{l+2}\Bigl(\frac{\pi}{2}+2^{-l-2}\delta\Bigr)\cdots m_{l+p}\Bigl(2^{1-p}\pi+2^{-l-p}\delta\Bigr)\to 0 \quad \text{as} \quad l\to\infty. \tag{6}$$

This holds for all $\delta\in(0,\sigma)$. Take now a positive R and surround each of the points $\Bigl\{e^{-2^{1-k}\pi i}\Bigr\}_{k=1}^{p}$ and $\Bigl\{e^{-3\cdot 2^{1-k}\pi i}\Bigr\}_{k=1}^{p}$ with a closed ball of radius R. Denote these balls by $\Bigl\{\beta_k\Bigr\}_{k=1}^{p}$ and $\Bigl\{\gamma_k\Bigr\}_{k=1}^{p}$ respectively. So we obtain a family of $2p$ equal balls, where the two balls β_1 and γ_1 coincide. Suppose R is small enough, so that all the balls are disjoint.

Take some $l\geq 0$ and consider the chain of the corresponding algebraic polynomials $\tilde m_{l+1},\dots,\tilde m_{l+p}$. Let β_k contains s_k roots of the polynomial $\tilde m_{l+k}$ (all roots are counted with multiplicity). If $s_k>0$, then we denote these roots by $z_1^{(k)},\dots,z_{s_k}^{(k)}$ and all other roots of m_{l+k} by $z_{s_k+1}^{(k)},\dots,z_{n_k}^{(k)}$, where $n_k=\deg\tilde m_{l+k}$. We have $\tilde m_{l+k}(z)=A_k\tilde\psi_k(z)\tilde q_k(z)$, where $\tilde\psi_k(z)=\prod\limits_{j=1}^{s_k}(z-z_j^{(k)})$, $\tilde q_k(z)=\prod\limits_{\nu=s_k+1}^{n_k}\Bigl(\frac{z-b_k}{z_\nu^{(k)}-b_k}-1\Bigr)$, $b_k=e^{-2^{1-k}\pi i}$ is the center of the ball β_k, and A_k is a constant (if $s_k=0$ we put $\tilde\psi_k=1$, if $s_k=n_k$, then $\tilde q_k=1$). Certainly all $s_k, A_k, \tilde\psi_k, \tilde q_k$ depend not only on k but also on l. We do not write this for the sake of simplicity. For any z from the unit circle we have $|z+z_j|\leq 1+R$ for $j=1,\dots,s_k$ and $\Bigl|\frac{z-b_k}{z_\nu^{(k)}-b_k}-1\Bigr|<\frac{2}{R}+1$ for $\nu=s_k+1,\dots,n_k$. Therefore $\|\tilde\psi_k\|\leq(1+R)^{s_k}$ and $\|\tilde q_k\|\leq\bigl(\frac{2}{R}+1\bigr)^{n_k-s_k}$. Now applying $\|\tilde m_{l+k}\|\geq|\tilde m_{l+k}(1)|=1$ we obtain

$$|A_k|\geq\frac{\|\tilde m_{l+k}\|}{\|\tilde\psi_k\|\cdot\|\tilde q_k\|}\geq\frac{1}{\bigl((1+R)^{s_k}\bigl(\frac{2}{R}-1\bigr)^{n_k-s_k}\bigr)}\geq\frac{1}{\bigl((1+R)^{n}\bigl(\frac{2}{R}+1\bigr)^{n}\bigr)}=\Bigl(\frac{2}{R}+R+3\Bigr)^{-n}. \tag{7}$$

Finally denote by $\bar\beta_k$ the closed ball of radius $R/2$ with the center b_k. For any $z\in\bar\beta_k$ we have $\Bigl|\frac{z-b_k}{z_\nu^{(k)}-b_k}\Bigr|<\frac{1}{2}$, and hence $|\tilde q_k(z)|>2^{-n}$. If we combine this with (7) we get

$$|\tilde m_{l+k}(z)|>2^{-n}\Bigl(\frac{2}{R}+R+3\Bigr)^{-n}|\tilde\psi_k(z)| \quad \text{for every} \quad z\in\bar\beta_k.$$

This yields that for all sufficiently large l, more precisely, for l such that $2^{-l}\sigma\leq R$, we have

$$\Bigl|\prod_{k=1}^{p}m_{l+k}\left(2^{1-k}\pi+2^{-l-k}\delta\right)\Bigr|>\Bigl[2^{-n}\bigl(\frac{2}{R}+R+3\bigr)^{-n}\Bigr]^{p}\Bigl|\prod_{k=1}^{p}\psi_k\left(2^{1-k}\pi+2^{-l-k}\delta\right)\Bigr|,$$

where ψ_k is the trigonometric polynomial associated to $\tilde\psi$, i.e., $\psi(\xi)=\tilde\psi(e^{-i\xi})$. Substituting this in (6) we get

$$2^{l(n-1)}\prod_{k=1}^{p}\psi_k\left(2^{1-k}\pi+2^{-l-k}\delta\right)\to 0 \quad \text{as } l\to\infty. \tag{8}$$

Now we come to the conclusive step of the proof. Surround each root of the polynomial $\prod_{k=1}^{p}\tilde\psi_k(z)$ with a ball of radius $r=\frac{\sigma}{2^{l+p}\pi S}$, where $S=\sum_{k=1}^{p}s_k$. Each of these balls intersects the unit circle with an arch of length smaller than πr. Hence the total length of the set

$$\Delta_1 = \Big\{\delta \in (0,\sigma), \quad \exists j \le s_1 \ \ |e^{-i(\pi+\delta 2^{-l-1})} - z_j^{(1)}| < r\Big\}$$

is smaller than $2^{l+1}\pi r s_0$. Similarly we show that for every $k = 2, \dots, p$ the total length of the set

$$\Delta_k = \Big\{\delta \in (0,\sigma), \quad \exists j \le s_k \ \ |e^{-i(\pi 2^{-k}+\delta 2^{-l-k})} - z_j^{(k)}| < r\Big\}$$

is smaller than $2^{l+k}\pi r s_k$. Therefore the total length of all the sets $\Delta_1, \dots, \Delta_p$ is smaller than $2^l\pi r\sum_{k=1}^p 2^k s_k \le 2^l\pi r 2^p\sum_{k=1}^p s_k = 2^{l+p}\pi r S = \sigma$. Thus there exists a point $\delta_0 \in (0,\sigma)$ that belongs to none of these sets. For this point we have $\left|\prod_{k=1}^p \psi_k\left(2^{1-k}\pi + 2^{1-l-k}\delta_0\right)\right| \ge r^S$. Since S does not exceed the number of all roots of the polynomials $\{m_{l+k}\}_{k=1}^p$, and hence $S \le np$, it follows that

$$r^S = \Big(\frac{\sigma}{2^{l+p}\pi S}\Big)^S = \Big(2^{l+p+1}Mn\pi S\Big)^{-S} \ge 2^{-S(l+p+1)}\Big(Mn\pi np\Big)^{-np} \ge 2^{-Sl}\Big(2^{p+1}Mn^2\pi p\Big)^{-np}.$$

Thus $\left|\prod_{k=1}^p \psi_k(2^{1-k}\pi + \delta_0 2^{-l-k})\right| \ge 2^{-Sl}\Big(2^{p+1}Mn^2\pi p\Big)^{-np}$. Combining this with (8) we obtain $2^{l(n-1)}\cdot 2^{-lS} \to 0$ as $l \to +\infty$, consequently $S \ge n$.

Let us recall that we took the total number of roots of the polynomial $\tilde m_{l+k}$ in the ball β_k and computed the sum S of these numbers for all k from 1 to p. We have shown that for all $l \ge 0$ this sum is at least n. In the same way we prove that the analogous sum computed for the balls $\{\gamma_k\}_{k=1}^p$ is also bigger than or equal to n. Now join these two results: consider the total number of roots of the polynomial $\tilde m_{l+k}$ in the balls β_k and γ_k and take the sum S_l of these numbers for all k from 1 to p. We have proven that for every $l \ge 0$ there are only two possibilities:

1) $S_l = n$, in this case all the corresponding roots lie the ball β_0;

2) $S_l \ge n+1$.

Let us show that for all sufficiently large l case 1) takes place. Take an integer N and estimate the total number of roots of the polynomials $\tilde m_{l+1}, \dots, \tilde m_{l+p+N}$. Obviously this number does not exceed $n(p+N)$. On the other hand the total number of roots is at least $\sum_{j=0}^N S_j$, since each root is counted only once in this sum. Suppose that case 2) takes place for ℓ chains $\{\tilde m_{l+j+k}\}_{k=1}^p$; then $\sum_{j=0}^N S_l \ge n(N+1)+\ell$ and therefore $n(p+N) \ge n(N+1)+\ell$. Thus $\ell \le n(p-1)$. Whence, beginning with some large l_0 we have case 1) only. This implies that for each $k \ge l_0$ the polynomial $\tilde m_k$ has exactly n roots in the ball β_0. Under the assumptions of our theorem this means that $\deg \tilde m_k = n$ and all its n roots belong to β_0. Thus, for any $R > 0$ there exists $l_0(R)$ such that for every $k \ge l_0(R)$ the polynomial $\tilde m_k$ has degree n and has all its roots in the ball $|z+1| < R$.

So all n roots of $\tilde{m}_k$ converge to -1 as $k \to \infty$. Since, moreover, $\tilde{m}_k(1) = 1$, we see that $\tilde{m}_k \to (\frac{1+z}{2})^n$ and correspondingly $m_k \to w_{n-1}$ as $k \to \infty$.

If we now suppose that the rate of decreasing of the function $f(\xi)$ is at least n, then by the same argument we obtain $m_k \to w_n$ as $k \to \infty$. Therefore $\deg m_k \geq n+1$ whenever k is large enough. The contradiction concludes the proof.

□

III. Remarks and examples.

Remark 1. Actually we have proved a stronger version of Theorem 1: if the rate of decreasing of the function f *from one side* is at lest $n-1$, i.e. $f(\xi) = o(\xi^{1-n})$ as $\xi \to +\infty$ or as $\xi \to -\infty$, then $m_k \to w_{n-1}$.

Remark 2. If the powers of polynomials m_k are not bounded uniformly, then in general the rate of decreasing of the function f may be infinite. There are families of polynomials, whose degrees grow arbitrarily slow, but nevertheless the function f decreases faster than polynomially. More precisely,

for every nondecreasing sequence of positive integers $\{a_k\}_{k\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} a_k = +\infty$ *there exists a sequence of trigonometric polynomials* $\{m_k\}_{k\in\mathbb{N}}$ *such that for all* k $\|m_k\| \leq 1, m_k(0) = 1, \deg m_k \leq a_k$ *and for any* $d \geq 0$ $f(\xi) = o(\xi^{-d})$ *as* $|\xi| \to \infty$.

To prove this take the sequence $d_k = \min\{a_k, k\}$ and the polynomials $m_k = w_{d_k - 1}$. It is seen easily that the product $f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi)$ converges uniformly on every compact subset of $\mathbb{R}$, hence the function f is well defined. Since $|m_k(\xi)| \leq 1$ for all k and ξ, it follows that

$$|f(\xi)| \leq \left|\prod_{k=r}^{\infty} m_k(2^{-k}\xi)\right| = \left|\prod_{k=r}^{\infty}\left(\frac{1+e^{-2^{-k}i\xi}}{2}\right)^{d_k}\right| \leq \left|\prod_{k=r}^{\infty}\left(\frac{1+e^{-2^{-k}i\xi}}{2}\right)\right|^{d_r} = \left|\widehat{B_{d_r-1}}(2^{1-r}\xi)\right| \leq 2^{rd_r}|\xi|^{-d_r}.$$

Since $d_r \to +\infty$ as $r \to \infty$, we see that the function f decreases faster than any power of $\frac{1}{\xi}$. Another example can be found in [N].

Remark 3. It follows from Theorem 1 that for any sequence of trigonometric polynomials of power n with uniformly bounded norms the function f cannot decrease faster than $O(\xi^{-n})$. This maximal decay attains for the identical sequence $m_k = w_{n-1}, k \in \mathbb{N}$, but not for this one only. For example, for any $a > 0$ the sequence $m_k(\xi) = \left(\frac{e^{-i\xi}+a^{(1/2)^k}}{1+a^{(1/2)^k}}\right)^n$ possesses the same property. In Theorem 2 we will present a criterion for a sequence of polynomials to provide the maximal decay.

Remark 4. The natural question arises if the condition of uniform boundedness by norm can be omitted in the statement of Theorem 1? Can we replace it by the assumption that the product $f(\xi) = \prod_{k=1}^{\infty} m_k(2^{-k}\xi)$

converges uniformly on any compact set? Surprisingly enough, the answer is negative. At first sight this may seem strange: for an unbounded family of polynomials this product should be still larger than for a bounded one. Nevertheless, in Example 2 we present a sequence of polynomials $\{m_k\}$ such that $\deg m_k = 2, m_k(0) = 1$ for all $k \ge 1$, and the function $f(\xi)$ has an exponential decay as $|\xi| \to \infty$.

Remark 5. For algebraic polynomials an analog of Theorem 1 is also true, but in this case it is trivial. It can be shown that for any sequence of algebraic polynomials $\{p_k\}$ whose degrees and norms (on the unit circle) are bounded uniformly and with the property $p_k(0) = 1,\ k \in \mathbb{N}$ the corresponding product $f(x) = \prod_{k=1}^{\infty} p_k(2^{-k}x)$ cannot converge to zero as $x \to \infty$ at all. However, without the condition of uniform boundedness by norm this analog of Theorem 1 does not hold (see the example below).

Example 1. *For any even integer $d \ge 2$ and for any $\varepsilon > 0$ there exists a sequence of algebraic polynomials $\{p_k\}$ such that for all $k \ge 1$ $\deg p_k \le d,\ p_k(0) = 1,$ the product $f(x) = \prod_{k=1}^{\infty} p_k(2^{-k}x)$ converges uniformly on any compact set, and $|f(x)| \le Ce^{-|x|^{d-\varepsilon}},\ x \in \mathbb{R}$.*

For arbitrary $\alpha \in (1,2)$ consider the family of polynomials $Q_k(x) = 1 - \frac{2^k}{k^\alpha}x,\ k \in \mathbb{N}$. Clearly, the product $f_Q(x) = \prod_{k=1}^{\infty} Q_k(2^{-k}x) = \prod_{k=1}^{\infty}\left(1 - \frac{x}{k^\alpha}\right)$ converges uniformly on any compact set. Take now $x = y^\alpha$ for some $y > 2$ and denote by I_y the set of four integers $[y]-1, [y], [y]+1, [y]+2$, where $[y]$ is the largest integer that does not exceed y. Denote also $J_y = \mathbb{N} \setminus I_y$. We have $f_Q(x) = \prod\limits_{k \in J_y} Q_k(2^{-k}x) \prod\limits_{k \in I_y} Q_k(2^{-k}x)$ and $Q_k(2^{-k}x) \ne 0$ for all $k \in J_y$. Furthermore,

$$\ln\left|\prod_{k \in J_y} Q_k(2^{-k}x)\right| = \sum_{k \in J_y} \ln\left|1 - \left(\frac{y}{k}\right)^\alpha\right| = y \sum_{k \in J_y} \frac{1}{y} \ln\left|1 - \left(\frac{k}{y}\right)^{-\alpha}\right|.$$

Observe that $\lim\limits_{y \to +\infty} \sum\limits_{k \in J_y} \frac{1}{y} \ln\left|1 - \left(\frac{k}{y}\right)^{-\alpha}\right| = \int\limits_0^{+\infty} \ln|1 - t^{-\alpha}|dt$ (an accurate proof of this limit passage is left to the reader). If we now denote $b(\alpha) = \int\limits_0^{+\infty} \ln|1 - t^{-\alpha}|dt$, we obtain

$$\left|f_Q(y^\alpha)\right| = e^{b(\alpha)y + \omega(y)} \prod_{k \in I_y} \left|Q_k(2^{-k}y^\alpha)\right|,$$

where $\omega(y) = o(y)$ as $y \to +\infty$. It is shown easily that for large y the product $\prod\limits_{k \in I_y} \left|Q_k(2^{-k}y^\alpha)\right| = \prod\limits_{k \in I_y} \left|1 - (\frac{k}{y})^{-\alpha}\right|$ is smaller than one, whence

$$|f_Q(y^\alpha)| \le e^{b(\alpha)y + \omega(y)}. \tag{9}$$

Note that the function $b(\alpha)$ strictly increases on $(1,2]$ and $b(2) = 0$, therefore $b(\alpha)$ is negative for all $\alpha \in (1,2)$. Thus the function f_Q has an exponential decay as $y \to +\infty$.

Consider now the family of polynomials $\{p_k(x) = 1 - \frac{2^k}{k^\alpha} x^d\}_{k\in\mathbb{N}}$ for an even d and for $\alpha \in (1,2)$ chosen so that $\frac{d}{\alpha} > d - \varepsilon$ ($\varepsilon > 0$ is given). Using (9) we get

$$|f(x)| = |f_Q(x^d)| \leq e^{b(\alpha)|x|^{d/\alpha} + \omega(|x|^{d/\alpha})}$$

whenever $|x|$ is large enough. Since $b(\alpha)$ is negative, it follows that $|f(x)| \leq Ce^{-|x|^{d-\varepsilon}}$ for some constant C.

Example 2. *There exists a family of trigonometric polynomials $\{m_k\}$ such that $\deg m_k = 2$, $m_k(0) = 1$ for all $k \geq 1$ and the corresponding function $f(\xi)$ has an exponential decay as $|\xi| \to \infty$.*

Let us show that for any $\beta \in (\frac{1}{2}, 1)$ the family

$$\left\{ m_k(\xi) = e^{-i\xi} \frac{\cos\xi - \cos\frac{k^\beta}{2^k}}{1 - \cos\frac{k^\beta}{2^k}} \right\}_{k\in\mathbb{N}}$$

is the one we are looking for. Since $|m_k(\xi)| = |m_k(-\xi)|$, and therefore $|f(\xi)| = |f(-\xi)|$, we can consider the case $\xi \geq 0$ only. Further, since $\cos t = 1 - \frac{t^2}{2} + h(t)$, where $0 \leq h(t) \leq \frac{t^4}{24}$, we decompose so each cosine and obtain after simplification:

$$|m_k(2^{-k}\xi)| = \left| \frac{1 - \frac{\xi^2}{k^{2\beta}} - \frac{2^{2k+1}}{k^{2\beta}}\left(h\left(\frac{k^\beta}{2^k}\right) - h\left(\frac{\xi}{2^k}\right)\right)}{1 - \frac{2^{2k+1}}{k^{2\beta}} h\left(\frac{k^\beta}{2^k}\right)} \right|. \tag{10}$$

Now we split the product $f(\xi)$ into three parts:

$$f(\xi) = \prod_{k\leq 6\log_2\xi} m_k\left(2^{-k}\xi\right) \prod_{k\in I(y),\ k>6\log_2\xi} m_k\left(2^{-k}\xi\right) \prod_{k\in J(y),\ k>6\log_2\xi} m_k\left(2^{-k}\xi\right), \tag{11}$$

where $y = \xi^{1/\beta}$ and $I_y,\ J_y$ are the sets from Example 1, and estimate these parts separately.

a) $\underline{k \leq 6\log_2\xi}$. Since $\|m_k\| = 2/\left(1 - \cos\frac{k^\beta}{2^k}\right) = 2/\left(\frac{1}{2}\frac{k^{2\beta}}{2^{2k}} - h(k^\beta/2^k)\right) \leq 2/\left(\frac{1}{2}\frac{k^{2\beta}}{2^{2k}} - \frac{1}{24}\frac{k^{4\beta}}{2^{4k}}\right)$, it follows that

$$\prod_{k\leq 6\log_2\xi} \left|m_k\left(2^{-k}\xi\right)\right| \leq \prod_{k\leq 6\log_2\xi} \frac{1}{\frac{k^{2\beta}}{2^{2k+2}}\left(1 - \frac{1}{12}\frac{k^{2\beta}}{2^{2k}}\right)} \leq C_0 \prod_{k\leq 6\log_2\xi} \frac{2^{2k+2}}{k^{2\beta}} \leq C_0 \prod_{k\leq 6\log_2\xi} 2^{2k+2} \leq C_0 2^{36\log_2^2\xi + 18\log_2\xi},$$

where $C_0 = \left(\prod_{k\in\mathbb{N}}\left(1 - \frac{1}{12}\frac{k^2}{2^{2k}}\right)\right)^{-1}$ is an absolute constant.

b) $\underline{k \in I(y),\ k > 6\log_2\xi}$. The reader will have no difficulty in showing that for large ξ $\prod_{k\in I(y)}\left|m_k\left(2^{-k}\xi\right)\right| \leq 1$ (for large ξ each term of this product is smaller than 1).

c) $\underline{k \in J(y), k > 6\log_2\xi}$ (Denominators). Now we use formula (10) and estimate the product of the denominators. Since $\frac{2^{2k+1}}{k^{2\beta}} h\left(\frac{k^\beta}{2^k}\right) \leq \frac{2^{2k+1}}{k^{2\beta}}\frac{1}{24}\frac{k^{4\beta}}{2^{4k}} = \frac{1}{12}\frac{k^{2\beta}}{2^{2k}}$, we see that the product $\prod\limits_{k\in\mathbb{N}}\left(1 - \frac{2^{2k+1}}{k^{2\beta}} h\left(\frac{k^\beta}{2^k}\right)\right)$ converges to some positive constant C_1. It is clear that $\prod\limits_{k\in J(y), k>6\log_2\xi}\left(1 - \frac{2^{2k+1}}{k^{2\beta}} h\left(\frac{k^\beta}{2^k}\right)\right) \geq C_1$

(Numerators). We have

$$\prod_{k\in J(y),\ k>6\log_2\xi}\left|1-\frac{\xi^2}{k^{2\beta}}-\frac{2^{2k+1}}{k^{2\beta}}\left(h\Big(\frac{k^\beta}{2^k}\Big)-h\Big(\frac{\xi}{2^k}\Big)\right)\right|\le\prod_{k\in J(y),\ k>6\log_2\xi}\left|1-\frac{\xi^2}{k^{2\beta}}\right|\left|1-\frac{\frac{2^{2k+1}}{k^{2\beta}}\left[h\left(\frac{k^\beta}{2^k}\right)-h\left(\frac{\xi}{2^k}\right)\right]}{1-\frac{\xi^2}{k^{2\beta}}}\right|.$$

If ξ is large enough, then for all $k\in J(y)$ $\left|1-\frac{\xi^2}{k^{2\beta}}\right|=\left|1-\left(\frac{y}{k}\right)^{2\beta}\right|\ge\left|1-\left(\frac{y}{y+2}\right)^{2\beta}\right|\ge\frac{2}{y+2}\ge\frac{1}{y}=\xi^{-1/\beta}\ge\xi^{-2}$. Thus

$$\prod_{k\in J(y),\ k>6\log_2\xi}\left|1-\frac{\frac{2^{2k}}{k^{2\beta}}\left[h\left(\frac{k^\beta}{2^k}\right)-h\left(\frac{\xi}{2^k}\right)\right]}{1-\frac{\xi^2}{k^{2\beta}}}\right|\le\prod_{k\in J(y),\ k>6\log_2\xi}\left(1+\frac{\frac{2^{2k}}{k^{2\beta}}\frac{1}{24}\left(\frac{k^{4\beta}}{2^{4k}}+\frac{\xi^4}{2^{4k}}\right)}{\xi^{-2}}\right)\le$$

$$\prod_{k\in J(y),\ k>6\log_2\xi}\left(1+\left(\frac{\xi^6}{2^k}\right)^{1/3}\frac{k^{2\beta}}{24\cdot2^{\frac{5}{3}k}}+\frac{\xi^6}{2^k}\frac{1}{24\cdot2^kk^{2\beta}}\right)\le\prod_{k\in J(y),\ k>6\log_2\xi}\left(1+\frac{k^{2\beta}}{24\cdot2^{\frac{5}{3}k}}+\frac{1}{24\cdot2^kk^{2\beta}}\right)\le C_2.$$

Using the result of Example 1 for $d=2$ and $\alpha=2\beta$ we see that $\prod_{k\in J(y)}\left|1-\frac{\xi^2}{k^{2\beta}}\right|\le e^{b(2\beta)\xi^{1/\beta}+\omega(\xi^{1/\beta})}$. Besides, it is clear that for sufficiently large ξ $\prod_{k\le6\log_2\xi}\left|1-\frac{\xi^2}{k^{2\beta}}\right|\ge\left(\frac{\xi^2}{36\log_2^2\xi}-1\right)^{6\log_2\xi}$. Combining these inequalities we obtain $\left|\prod_{k\in J(y),\ k>6\log_2\xi}m_k\left(2^{-k}\xi\right)\right|\le\frac{C_2}{C_1}e^{b(2\beta)\xi^{1/\beta}+\omega(\xi^{1/\beta})}\left(\frac{\xi^2}{36\log_2^2\xi}-1\right)^{-6\log_2\xi}$

Now we substitute **a), b)** and **c)** into (11) and get

$$|f(\xi)|\le C_02^{36\log_2^2\xi+18\log_2\xi}\cdot\frac{C_2}{C_1}e^{b(2\beta)\xi^{1/\beta}+\omega(\xi^{1/\beta})}\left(\frac{\xi^2}{36\log_2^2\xi}-1\right)^{-6\log_2\xi}\le e^{b(2\beta)\xi^{1/\beta}+\omega_1(\xi^{1/\beta})},$$

where $\omega_1(y)=o(y)$ as $y\to+\infty$. Now for any $\varepsilon>0$ one can choose $\beta\in(1/2,1)$ so that $f(\xi)=O\left(e^{-|\xi|^{2-\varepsilon}}\right)$.

IV. A criterion of the fastest decay.

Now we formulate the condition under which the function f has the fastest possible decay (Remark 3). We say that a system of trigonometric polynomials $\{m_k\}_{k\in\mathbb{N}}$ satisfies condition $(*)$ if there exists a positive constant C_0 such that for all sufficiently large k $\deg m_k=n$ and all n roots of the polynomial $\tilde m_k$ lie in the ball $|z+1|\le C_02^{-k}$. This condition turns out to be equivalent to the maximal decay.

Theorem 2. *Under the assumptions of Theorem 1 $f(\xi)=O(\xi^{-n})$ if and only if the system $\{m_k\}$ satisfies condition $(*)$.*

Remark 6. Thus the product $f(\xi)=\prod m_k(2^{-k}\xi)$ has the fastest decay if the roots of polynomials $\tilde m_k$ converge to -1 sufficiently fast. It is easy to see that condition $(*)$ combined with the assumption $m_k(0)=1, k\in\mathbb{N}$ implies that $\|m_k-w_{n-1}\|=O(2^{-k})$. In general the converse is not true: the maximal distance between roots of $\tilde m_k$ and the point -1 can decrease slower than $\|m_k-w_{n-1}\|$.

Proof of Theorem 2. *(Necessity)* As in the proof of Theorem 1 take arbitrary $\delta \in (0,\sigma)$ and consider the value $f(2\pi \cdot 2^l + \delta)$. By the assumption $f(2\pi \cdot 2^l + \delta) = O(2^{-ln})$ as $l \to \infty$. Applying (5) we obtain

$$\left| m_{l+1}\Big(\pi + 2^{-l-1}\delta\Big) m_{l+2}\Big(\frac{\pi}{2} + 2^{-l-2}\delta\Big) \cdots m_{l+p}\Big(2^{1-p}\pi + 2^{-l-p}\delta\Big)\right| = O(2^{-ln}) \quad \text{as} \quad l \to \infty. \tag{12}$$

By Theorem 1 $m_j \to w_{n-1}$ as $j \to \infty$, whence $\deg m_j = n$ for large j and furthermore, for the kth multiplier ($k = 2, \dots, p$) in (12) we have $\lim\limits_{l\to\infty} m_{l+k}\Big(2^{1-k}\pi + 2^{-l-k}\delta\Big) = w_{n-1}(2^{1-k}\pi) \neq 0$. Therefore (12) implies now that $\left|m_{l+1}\Big(\pi + 2^{-l-1}\delta\Big)\right| = O(2^{-ln})$ as $l \to \infty$. Let $\{z_s\}_{s=1}^n$ be roots of the polynomial $\tilde m_{l+1}$, so $\tilde m_{l+1}(z) = a_{l+1}\prod_{s=1}^n (z - z_s)$, where by Theorem 1 $\lim_{l\to\infty} a_{l+1} = 2^{-n}$ (2^{-n} is the leading coefficient of w_{n-1}). Let also $\rho = \max_s |z_s + 1|$; without loss of generality we assume $|z_n + 1| = \rho$. In the same way as in the proof of Theorem 1, surrounding each point $z_1, \dots, z_{n-1}$ with a ball of radius $r = \frac{\sigma}{2^{l+1}\pi(n-1)}$, we show that there exists $\delta_0 \in (0,\sigma)$ such that $\prod\limits_{s=1}^{n-1} \left|e^{-i(\pi + 2^{-l-1}\delta_0)} - z_s\right| \geq r^{n-1}$. Therefore $\left|m_{l+1}\Big(\pi + 2^{-l-1}\delta_0\Big)\right| \geq \left|a_{l+1}(\rho - r)r^{n-1}\right|$, and hence $a_{l+1}(\rho - r)2^{-(l+1)(n-1)} = O(2^{-ln})$. This yields that $\rho = O(2^{-l})$ as $l \to \infty$.

(Sufficiency) We prove the statement for positive ξ (the case of negative ξ is considered in the same way). We can restrict ourselves to large values of ξ, so we assume $\xi \geq 4\pi$. Furthermore, it suffices to realize the proof for the case $m = 1$ only, in other words we shall prove the following: if a sequence $\{z_k\}$ of complex numbers satisfies $(*)$, i.e., converges to -1 so that $|z_k + 1| \leq C_0 2^{-k}$, then

$$|f_1(\xi)| = \left| \prod_{k=1}^{\infty} \frac{e^{-i2^{-k}\xi} - z_k}{2} \right| = O\Big(\frac{1}{\xi}\Big) \quad \text{as} \quad \xi \to +\infty. \tag{13}$$

Indeed, if we decompose each polynomial $m_k(\xi) = a_k \cdot 2^n \left(\frac{e^{-i\xi} - z_{1k}}{2}\right) \cdots \left(\frac{e^{-i\xi} - z_{nk}}{2}\right)$ and note that under the assumptions of Theorem 2 $|a_k \cdot 2^n - 1| = O(2^{-k})$ as $k \to \infty$ (Remark 6), we see that the product $\prod\limits_{k\in\mathbb{N}} (a_k \cdot 2^n)$ converges to some constant. Now applying (13) to each sequence of polynomials $\left\{\left(\frac{e^{-i\xi} - z_{sk}}{2}\right)\right\}_{k\in\mathbb{N}}$, $s = 1, \dots, n$ we establish the theorem. It remains us prove (13).

Denote $\xi = 2\pi x$ and represent x in its binary extension: $x = d_1 \dots d_l.d_{l+1} \dots$. Consider the sequence of digits before the point: $d_1 \dots d_l$. This sequence begins with a series of consecutive ones $d_1 \dots d_{k_1}$, then a series of consecutive zeros $d_{k_1+1} \dots d_{k_2}$ follows, and so on. Suppose that the last digit d_l is zero (the opposite case is considered similarly), so the sequence finishes with a series of consecutive zeros $d_{k_s+1} \dots d_l$. Finally put $D = \{k_1, \dots, k_s\}$ to be the set of the last numbers of the series. Now denote $h_k(\xi) = \frac{e^{-i2^{-k}\xi} - z_k}{2}$ and decompose product (13) into three parts:

$$f_1(2\pi x) = \left(\prod_{k=1}^{l-k_s} h_k\right) \cdot h_{l-k_s+1} \prod_{k=l-k_s+2}^{\infty} h_k.$$

We estimate these parts separately.

a) Since $|h_k| \le \frac{1+|z_k|}{2} \le \frac{2+|z_k-1|}{2} \le 1 + C_0 2^{-k-1} \le e^{C_0 2^{-k-1}}$, we see that $\left|\prod_{k=1}^{l-k_s} h_k(2\pi x)\right| \le e^{C_0/2}$.

b) $h_{l-k_s+1}(2\pi x) = \frac{e^{-2\pi i\bar{x}} - z_{l-k_s+1}}{2}$, where $\bar{x} = \{2^{k_s-l-1}x\} = 2^{k_s-l-1}x - [2^{k_s-l-1}x] = 0.10\dots0d_{l+1}\dots$ ($l-k_s$ zeros after the one). Thus $|2\pi\bar{x} - \pi| \le \pi 2^{k_s-l}$, hence

$$\left|\frac{e^{-2\pi i\bar{x}} - z_{l-k_s+1}}{2}\right| \le \frac{|e^{-2\pi i\bar{x}} - e^{-i\pi}| + |e^{-i\pi} - z_{l-k_s+1}|}{2} \le \frac{\pi 2^{k_s-l} + C_0 2^{k_s-l-1}}{2} \le (2\pi + C_0)2^{k_s-l-2}.$$

c) We have

$$\prod_{k=l-k_s+2}^{\infty} |h_k| = \left|\prod_{k=l-k_s+2}^{\infty} \frac{e^{-2\pi i 2^{-k}x}+1}{2}\right| \cdot \left|\prod_{k=l-k_s+2}^{\infty}\left(1 - \frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right)\right| \le$$
$$\left|\prod_{k=l-k_s+2}^{\infty} w_0(2^{-k}\cdot 2\pi x)\right| \cdot \prod_{k=l-k_s+2}^{\infty}\left(1 + \left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right|\right) \le \frac{2}{2^{k_s-l-1}\cdot 2\pi x}\prod_{k=l-k_s+2}^{\infty}\left(1 + \left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right|\right)$$
$$\le \frac{2}{2^{k_s-l-1}\cdot 2\pi x}\exp\left(\sum_{k\ge l-k_s+2}\left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right|\right).$$

To estimate the terms in the last sum consider two possible subcases:

1) For the number $\bar{x} = \{2^{-k}x\}$ the two first digits after the binary point coincide. This is the case if k is not of the form $k = l - k_j + 1$ for some $k_j \in D$, in other words $l - k + 1 \notin D$. It follows that $|\bar{x} - \frac{1}{2}| \ge \frac{1}{4}$, therefore $\left|e^{-2\pi i\bar{x}} + 1\right| > 1$ and so $\left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right| \le |z_k + 1| \le C_0 2^{-k}$. Thus

$$\sum_{\substack{k\ge l-k_s+2\\ l-k+1\notin D}}\left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right| \le \sum_{\substack{k\ge l-k_s+2\\ l-k+1\notin D}} C_0 2^{-k} \le \sum_{k\in\mathbb{N}} C_0 2^{-k} \le C_0 \tag{14}$$

2) For the number $\bar{x}$ the two first digits after the binary point are distinct, i.e., k is of the form $k = l - k_j + 1$ for some $k_j \in D$. In this case $|\bar{x} - \frac{1}{2}| \ge 2^{k_j - k_{j+1} - 1}$, hence $\left|e^{-2\pi i\bar{x}} + 1\right| \ge 2^{k_j - k_{j+1} + 1}$. This implies

$$\sum_{\substack{k\ge l-k_s+2\\ l-k+1\in D}}\left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right| \le \sum_{\substack{k=l-k_j+1\\ k_j\in D}} \frac{C_0 2^{-k}}{2^{k_j-k_{j+1}+1}} = \sum_{j=1}^{s}\frac{C_0 2^{k_j-l-1}}{2^{k_j-k_{j+1}+1}} = \sum_{j=1}^{s} C_0 2^{k_{j+1}-l-2} \le \sum_{k=1}^{l} C_0 2^{k-l-2} < C_0/2.$$

Combining this with (14) we obtain $\sum_{k\ge l-k_s+2}\left|\frac{z_k+1}{e^{-2\pi i 2^{-k}x}+1}\right| \le \frac{3}{2}C_0$. Thus in the case **c)** we have

$$\prod_{k\ge l-k_s+2} |h_k| \le \frac{1}{2^{k_s-l-2}\cdot 2\pi x} e^{\frac{3}{2}C_0}.$$

Now **a)**, **b)** and **c)** altogether give

$$|f_1(\xi)| = |f_1(2\pi x)| < e^{C_0/2}\cdot(2\pi + C_0)2^{k_s-l-2}\cdot\frac{1}{2\pi\cdot 2^{k_s-l-2}x}e^{\frac{3}{2}C_0} \le \frac{e^{2C_0}(2\pi + C_0)}{2\pi x} = \frac{e^{2C_0}(2\pi + C_0)}{\xi},$$

this completes the proof.

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