Axiomatic Characterization of the Interval Function of a Graph

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Abstract

A fundamental notion in metric graph theory is that of the interval function $I : V \times V \to 2^V - \{\emptyset\}$ of a (finite) connected graph $G = (V, E)$, where $I(u, v) = \{ w \mid d(u, w) + d(w, v) = d(u, v) \}$ is the interval between u and v. An obvious question is whether I can be characterized in a nice way amongst all functions $F : V \times V \to 2^V - \{\emptyset\}$. This was done in [13, 14, 16] by axioms in terms of properties of the functions F. The authors of the present paper, in the conviction that characterizing the interval function belongs to the central questions of metric graph theory, return here to this result again. In this characterization the set of axioms consists of five simple, and obviously necessary, axioms, already presented in [9], plus two more complicated axioms. The question arises whether the last two axioms are really necessary in the form given or whether simpler axioms would do the trick. This question turns out to be non-trivial. The aim of this paper is to show that these two supplementary axioms are optimal in the following sense. The functions satisfying only the five simple axioms are studied extensively. Then the obstructions are pinpointed why such functions may not be the interval function of some connected graph. It turns out that these obstructions occur precisely when either one of the supplementary axioms is not satisfied. It is also shown that each of these supplementary axioms is independent of the other six axioms. The presented way of proving the characterizing theorem (Theorem 3 here) allows us to find two new separate "intermediate" results (Theorems 1 and 2). In addition some new characterizations of modular and median graphs are presented. As shown in the last section the results of this paper could provide a new perspective on finite connected graphs.

Keywords: organizing function, geometric function, interval function, geodetic betweenness.

MCS: 05C12, 05C38, 05C75, 05C99.

1 Introduction

A *graphic metric space* (V, d) is a finite metric space that is derived from a finite connected graph $G = (V, E)$, where V is the vertex set of G and d is the distance function of G. In [6] Kay and Chartrand characterized the finite metric spaces that are graphic: (V, d) has to have two simple properties, viz. integrality of distances and if $d(u, v) > 1$ then there exists a w distinct from u and v with $d(u, w)+d(w, v) = d(u, v)$. In this sense, finite connected graphs and such finite metric spaces are just two manifestations of the same discrete structure. But there is a striking difference when these two manifestations are seen as a point of view on the structure. In a graph not only the distance between two vertices u and v is relevant but also the set of *all* shortest paths (geodesics) between u and v. This prominent distinction makes *metric graph theory* an area of its own interest. Hence, within this area, one of the fundamental notions is that of the *interval function* $I : V \times V \to 2^V - \{\emptyset\}$ of a connected graph $G = (V, E)$. Here $I(u, v)$ is the *interval* between u and v, that is, the set of vertices that are on u, v-geodesics. For a first extensive study of the interval function see [9].

An obvious question is, given some function $F : V \times V \to 2^V - \{\emptyset\}$, what properties should F have to make it the interval function of some connected graph with vertex set V. In Proposition 1.1.2 of [9] some simple properties of the interval function were listed. These were phrased in terms of the function only and without any reference to graphs. They are given in the next section as the five so-called classical axioms. It is obvious that, for F to be an interval function, it should satisfy these five axioms. Below various simple examples are given of such functions that are *not* the interval function of a graph. So this Proposition posed the challenge to find additional axioms that 'characterize' the interval function. In [13, 14, 16] such characterizations are given. Two additional axioms were needed, which were more complicated than the five classical ones. These two axioms seemed to be "heavy duty axioms". So this posed a new challenge: are there other, much simpler axioms that would do the trick. This challenge turned out to be far from trivial. The aim of this paper is to find the 'optimal', or, if one prefers, 'minimal' axioms that are needed besides the five classical axioms to characterize the interval function. The main result of this paper is that the two additional axioms in [13, 14, 16] are precisely the 'minimal' ones. But along the way, our approach here provides us with some new results, and most of all, new insight in the problem. Moreover, we apply the results on characterizations of the interval function of special classes of graphs, e.g. modular and median graphs, which are new.

So far, we have presented the metric point of view. But the area of such functions $F : V \times V \to 2^V - \{\emptyset\}$ has more perspectives. The set $F(u, v)$ might signify the possible ways to get from u to v. For instance, instead of shortest u, v-paths one might use induced u, v-paths, or other types of paths, see e.g. [3]. Or $F(u, v)$ might signify the way how to get from one logical statement u to another logical statement v. This is just to name a few examples of the possible use of such functions. Because of this broader perspective, we have chosen a terminology that emphasizes that $F(u, v)$ is a set that *leads us from u to v*, either metrically, or logically, or otherwise. Therefore we have chosen the term *organizing function* for F.

Another point of view is to say that x is *between* u and v if x lies in $I(u, v)$. This can be formulated in terms of a ternary algebra $T \subseteq V \times V \times V$. That x lies between u and v is then algebraically denoted as $(u, x, v) \in T$. Now algebraic axioms are needed. Our results can be phrased in these terms. The area of ternary algebras is again a well-developed area, but we will touch it only in passing. The idea of x being between u and v has been studied also in the guise of the notion of *betweenness*. Various types of betweenness have been proposed defined by slightly different sets of betweenness axioms, see e.g. [20, 9, 4, 19, 7]. Of these the geodetic betweenness may present an alternative approach to the results below.

In Section 2 we introduce organizing functions, and present the five 'classical' axioms, the simple properties of the interval function from [9] that have been the starting point for much related research. We collect some basic ideas and lemmata from earlier papers of the second author that we need here. In Section 3 we focus on organizing functions satisfying all the five classical axioms, which we call *geometric functions*. We prove 'as much as possible' using the five classical axioms only. Thus we try to find the obstructions why a geometric function might not be the interval function of a graph. This requires quite some efforts, but finally we can pinpoint these obstructions very precisely. The culmination of these efforts are the two new Theorems 1 and 2, which are the main results of this paper. These theorems are in a way 'intermediate' results on the way to the characterizing theorems in Section 4. In Section 4 we deduce from Theorems 1 and 2 two immediate Corollaries, which provide us with two necessary, supplementary axioms that will overcome the obstructions. Thus we get the characterizations of the interval function in Theorems 3 and 4 that were already obtained in [13, 16] and in [14], respectively. Moreover, we discuss the modular and median case. In the concluding section we make some observations about the implications of our results.

2 Organizing functions

Throughout this paper V is a finite, nonempty set. Note that the assumption that V is finite is essential for most of the results and proofs in this paper.

A mapping $F : V \times V \to 2^V - \{\emptyset\}$ into the power set of V is called an *organizing function* on V. Since our focus in this paper is on graphs, we call the elements of V *vertices*. Two vertices u and v in V are *adjacent in* F if

$$u \neq v \text{ and } F(u, v) = \{u, v\} = F(v, u).$$

Let F be an organizing function on V. The *underlying graph* G_F of F has V as its vertex set, and distinct vertices u and v are adjacent in G_F if and only if they are adjacent in F. By abuse of language we will sometimes say that F is an organizing function on G_F.

Let G be a connected graph with V as its vertex set, and let I denote the interval function of G. Recall that I is defined by

$$I(u, v) = \{\ w \mid d(u, w) + d(w, v) = d(u, v)\ \},$$

see [9] for an extensive study of the interval function. Obviously, I is an organizing function on V. Moreover, the underlying graph of I is G. By Proposition 1.1.2 in [9], if F is the interval function of G, then F satisfies the five simple axioms $(c1)$, ... , $(c5)$ given below. So if an organizing function is going to be the interval function of a connected graph, then it should at least satisfy axioms $(c1)$ up to $(c5)$. We will call these essential axioms the *classical axioms*. Here u, v, x, y are variables in V.

- $(c1)$ $u, v \in F(u, v)$ for all u, v,
- $(c2)$ $F(v, u) = F(u, v)$ for all u, v,
- $(c3)$ if $x \in F(u, v)$ and $y \in F(u, x)$, then $y \in F(u, v)$ for all u, v, x, y,
- $(c4)$ if $x \in F(u, v)$, then $F(u, x) \cap F(x, v) = \{x\}$ for all u, v, x,
- $(c5)$ if $x \in F(u, v)$ and $y \in F(u, x)$, then $x \in F(y, v)$ for all u, v, x, y.

Axiom $(c3)$ has a slightly different form than in [9], where it was formulated as "if $x \in F(u, v)$ then $F(u, x) \subseteq F(u, v)$". We have chosen the above form because now all five axioms can be formulated in a language of first-order logic, which we need in Section 5.

Note that axioms $(c1)$ and $(c4)$ imply the axiom

$(c4)'$ $F(u,u) = \{u\}$ for all $u \in V$.

Under the assumption of axioms $(c1)$, $(c2)$ and $(c5)$ axioms $(c4)'$ and $(c4)$ are equivalent, as is shown by the following lemma.

Lemma 1 *Let F be an organizing function on a finite set V satisfying axioms $(c1)$, $(c2)$, $(c4)'$, and $(c5)$. Then F satisfies $(c4)$.*

Proof. Choose any x in $F(u,v)$. By $(c1)$, we have $x \in F(u,x) \cap F(x,v)$. Take any $y \in F(u,x) \cap F(x,v)$. Since y lies in $F(x,v)$, it follows from $(c2)$ that y lies in $F(v,x)$. Moreover, x lies in $F(v,u)$. Now $(c5)$ implies that x lies in $F(y,u)$, which is $F(u,y)$ by $(c2)$. Recall that y lies in $F(u,x)$. Now applying $(c5)$ with $y = v$, we deduce that x lies in $F(y,y)$. Hence, by $(c4)'$, we conclude that $x = y$, so that $(c4)$ holds. □□□

An organizing function satisfying axioms $(c1)$ and $(c2)$ is called an *interval operator* in [21]; if it also satisfies $(c4)'$, then it is called a *transit function* in [**?**]. An interval operator satisfying axioms $(c3)$, $(c4)'$, and $(c5)$ as well is called a *geometric interval operator* in [22], see also [21]. Following this usage, we will call an organizing function satisfying the above five classical axioms a *geometric function*, see the next section. In [4] Hedlíková studied *ternary spaces.* These are ternary algebras satisfying certain algebraic axioms. This structure is equivalent to Verheul's geometric interval operator and our geometric function. The approach of ternary algebras allows a completely algebraic treatment of the results below. We postpone this to the last section. Finally, a geometric function that is the interval function of a graph is called a *graphic interval operator* in [21].

As stated above, the interval function of a connected graph satisfies the five classical axioms. The 'converse' is not true as the following example shows. Recall that the *wheel* W_n is the graph consisting of a cycle C of length $n \geq 4$ and an additional vertex a, called the *axis*, adjacent to all vertices of the cycle. Let the n-cycle be $v_1 \to v_2 \to \ldots \to v_n \to v_1$. We will write $v_0 = v_n$ and $v_{n+1} = v_1$. The *spokes* of the wheel are the edges incident with the axis a. We call two spokes of the type av_i, av_{i+1} *consecutive spokes.* A *broken wheel* is obtained from a wheel by deleting some non-consecutive spokes. By abuse of language, a wheel is also a broken wheel. A special instance of a broken wheel is the *cogwheel* M_k: it is obtained from the wheel W_{2k} by deleting k non-consecutive spokes (so that there are no consecutive spokes left). Note that the cogwheels are the only bipartite broken wheels. Now let B be a broken wheel with axis a and cycle C. Let $V = V(C) \cup \{a\}$. Let I be the interval function of B and I_C that of C. Then we define the organizing function F on V as follows:

$F(u,v) = I_C(u,v)$ for any two vertices u, v on the cycle,

$F(a,v) = F(v,a) = I(a,v)$ for the axis a and any vertex v.

Trivially, $F \neq I$. Note that, for any two vertices u, v on C the classical axioms reduce to statements on the interval function of the cycle C. If we take u to be the axis a, then the classical axioms reduce to statements of the interval function of a K_2 or a 4-cycle, which are easily verified. So F satisfies the five classical axioms.

Let F be an organizing function on V, and let $u_0, \ldots, u_m, v$ be a sequence of vertices in V, where $m \geq 1$. We say that F *leads us from* u_0 *to* v *along* $u_1, \ldots, u_m$ if

$$u_k \in F(u_{k-1}, v) \text{ for all } k = 1, \ldots, m.$$

We denote this by $u_0 \ldots u_m[F]v$. It is clear that, if $u_0 \ldots u_m[F]v$, then $u_i \ldots u_j[F]v$, for any $0 \leq i < j \leq m$. Moreover, if F satisfies axiom $(c1)$, then $u_0 \ldots u_m v[F]v$. First we collect some basic facts from [17] without proofs. The proofs of Lemmas A and B are straightforward anyway.

Lemma A (Proposition 3 in [17]) *Let F be an organizing function on V satisfying $(c2)$, $(c3)$, and $(c5)$. If $u_0, \ldots, u_{m-1}[F]u_m$, then $u_j \in F(u_i, u_k)$, for $0 \leq i < j < k \leq m$.*

Lemma B (Corollary 4 in [17]) *Let F be an organizing function on V satisfying axioms $(c2)$, $(c3)$, and $(c5)$, and let $u_0, u_1, \ldots u_{n-1}, u_n \in V$. If $u_0 u_1 \ldots u_{n-1}[F]u_n$, then $u_n u_{n-1} \ldots u_1[F]u_0$.*

Note that if F is an organizing function satisfying $(c2)$, $(c3)$ and $(C5)$, then, if u and v are adjacent in F, the only sequences that can lead us from u to v are those that consist of u's and v's only. Moreover, if F satisfies $(c4)'$, then the only sequences that lead us from u to v are of the type $u, u, \ldots, u, v, v, \ldots, v$. Our main concern will be sequences of a special type. Similarly as in [17], we define a u_0, u_m*-process* in F to be a sequence

$$\pi = (u_0, \ldots, u_m),\ m \geq 1 \text{ with } u_0, \ldots, u_m \in V$$

such that

$$u_{i-1} \text{ and } u_i \text{ are adjacent in } F \text{ for } i = 1, \ldots, m \tag{1}$$

and

$$\text{if } m \geq 2, \text{ then } u_0 \ldots u_{m-1}[F]u_m.$$

The *length* of the process π is m. Note that, for $0 \leq i < j \leq m$, the sequence $u_i, u_{i+1}, \ldots, u_j$ is a u_i, u_j-process as well. If u is adjacent to u_0 in F and $u_0 \in F(u, u_m)$, then $(u, u_0, \ldots, u_m)$ is a u, u_m-process in F if and only if π is a u_0, u_m-process in F. The proof of the following lemma is easy.

Lemma C (Proposition 6 and Corollary 7 in [17]) *Let F be an organizing function on V, and let F satisfy axioms $(c1)$, $(c2)$, $(c3)$, and $(c4)$. Then there exists a u, v-process in F for all u, v in V. Therefore, the underlying graph of F is connected.*

Note that finiteness of V is essential in proving Lemma C.

Lemma 2 *Let F be an organizing function on V satisfying axioms $(c2)$, $(c3)$, and $(c5)$, let G be the underlying graph of F, let $u_0, \ldots, u_m \in V$, with $m \geq 1$, and let $(u_0, \ldots, u_m)$ be a u_0, u_m-process in F. Then $(u_m, \ldots, u_0)$ is a u_m, u_0-process in F. If F satisfies axioms $(c1)$ and $(c4)$ as well, then $(u_0, \ldots, u_m)$ is a path in G.*

Proof. Obviously, $u_0 \ldots u_{m-1}[F]u_m$. By Lemma B, we have $u_m \ldots u_1[F]u_0$ and therefore $(u_m, \ldots, u_0)$ is also a process in F.

Assume that F satisfies axioms $(c1)$ and $(c4)$ as well, so that F satisfies axiom $(c4)'$. Suppose, to the contrary, that $(u_0, \ldots u_m)$ is not a path in G. Then there exist i and j, $0 \leq i < j \leq m$, such that $u_j = u_i$. Then $(u_i, u_{i+1}, \ldots, u_j)$ is a u_i, u_j-process in F. Hence $u_{i+1} \in F(u_i, u_j) = F(u_i, u_i)$. Then, by $(c4)'$, we get $u_{i+1} = u_i$, which contradicts the fact that in a process consecutive vertices should be adjacent in F, and hence distinct. This completes the proof. □□□

Lemma 3 *Let F satisfy axioms $(c2)$, $\ldots$,$(c5)$, and let $u_0, \ldots, u_m, u_{m+1} \in V$, with $m \geq 1$. Let $(u_0, \ldots, u_m)$ be a process in F, and let u_m and u_{m+1} be adjacent in F. Then $(u_0, \ldots, u_m, u_{m+1})$ is a process in F if and only if $u_m \in F(u_0, u_{m+1})$.*

Proof. First note that if $(u_0, \ldots, u_m, u_{m+1})$ is a process in F, then, by Lemma A, we have $u_m \in F(u_0, u_{m+1})$.

Conversely, by $(c3)$, we have $F(u_0, u_m) \subseteq F(u_0, u_{m+1})$. Take any i with $0 \leq i < m$. Since $\pi = (u_0, \ \ldots \ , u_m)$ is a process, it follows from Lemma A that $u_i \in F(u_0, u_m)$. So, by $(c5)$, we have $u_m \in F(u_i, u_{m+1})$. Hence, by $(c3)$, we have $F(u_i, u_m) \subseteq F(u_i, u_{m+1})$. Since π is a process, we have $u_{i+1} \in F(u_i, u_m)$. So we conclude that $u_{i+1} \in F(u_i, u_{m+1})$. The condition on the adjacencies in $(u_0, \ \ldots \ , u_m, u_{m+1})$ is trivially satisfied, so $(u_0, \ \ldots \ , u_m, u_{m+1})$ indeed is a process. □□□

3 Geometric Functions and the Interval Function of their Underlying Graph

Recall that V is a finite nonempty set. A *geometric function* on V is an organizing function F on V satisfying all the five classical axioms $(c1)$, $\ldots$,$(c5)$.

Let F be a geometric function on V. By Lemma C, the underlying graph of F is connected. It is easy to see that, if F is the interval function of some connected graph G, then G is the underlying graph of F. It was proved in [13] and [16] that F is the interval function of the underlying graph of F if and only if F satisfies two axioms equivalent to axioms $(s1)$ and $(s2)$ presented in the next section. (Note that it was assumed in [13] that G is connected, whereas in this paper connectedness follows from the chosen axioms). This characterization of the interval function was extended in [14] (note that a simpler but stronger modification of axiom $(s1)$ was used in [14]). In the present paper we will give a new proof of the (extended) characterization: axioms $(s1)$ and $(s2)$ will be used *only* at the very end of the proof. This is an essential feature of the approach chosen in this paper, as is explained below. Note that a characterizing theorem for the interval function of an infinite connected graph can be found in [15].

In the rest of this section we assume that a geometric function F on the set V is given. We denote by $\mathcal{P}$ the set of all processes in F and by G the underlying graph of F. Recall that, by Lemma C, the graph G is connected. Moreover, we denote by d, I, and $\mathcal{G}$ the distance function of G, the interval function of G, and the set of all geodesics in G, respectively.

Recall that if $(u_0, \ldots, u_m)$ is a process in F, with $m \geq 1$ and $u_0, \ldots, u_m \in V$, and if $0 \leq i < j \leq m$, then both $(u_i, u_{i+1}, \ldots, u_j)$ and $(u_j, u_{j-1}, \ldots, u_i)$ are processes in F.

Lemma 4 *Let $u, w \in V$. If $F(u, w) - I(u, w) \neq \emptyset$, then there exists a u, w-process ϕ in F such that the length of ϕ is greater than $d(u, w)$.*

Proof. Assume that there exists some $v \in F(u, w) - I(u, w)$. Since v is not in $I(u, w)$, we have $u \neq v \neq w$. By virtue of Lemma C and axiom $(c2)$, there exist j and m with $0 < j < m$, and $y_0, \ldots, y_m \in V$ such that $y_0 = u$, $y_j = v$, $y_m = w$, and

$$(y_j, \ldots, y_1, y_0) \text{ and } (y_j, \ldots, y_{m-1}, y_m)$$

are processes in F. This implies that $y_j \ldots y_1[F]y_0$ and $y_j \ldots y_{m-1}[F]y_m$, so that consecutive vertices are adjacent. Recall that $y_j \in F(y_0, y_m)$. By axiom $(c2)$, we have $y_j \in F(y_m, y_0)$. Hence $y_m y_j \ldots y_1[F]y_0$. By Lemma B, we have $y_0 y_1 \ldots y_j[F]y_m$. Since $y_j \ldots y_{m-1}[F]y_m$, we get $y_0 \ldots y_j \ldots y_{m-1}[F]y_m$. Then $(y_0, \ldots, y_j, \ldots, y_m)$ is an u, w-process in F, say a process ϕ of length m. Obviously, $m \geq d(u, w)$. If $m = d(u, w)$, then ϕ is a geodesic in G and therefore $v \in I(u, w)$, which is a contradiction. Thus $m > d(u, w)$, which completes the proof. □□□

So far, using only the five classical axioms, we have established that processes are paths. But to obtain an axiomatic characterization of the interval function of a connected graph this is not sufficient, as the example of the broken wheels in Section 2 shows. So we need more axioms. The question is then: how do we find the axioms that serve this purpose? In search of such axioms we proceed as follows. We use freely the classical axioms and try to get as close as possible to our goal of characterizing the interval function by organizing functions. Then we hope to find 'minimal' axioms that will do the trick to complete the proof. An essential step is that we would like to prove that processes in the organizing function F and geodesics in the underlying graph G coincide. The strategy to follow is, of course, induction on the lengths of the geodesics, that is, the distance between vertices in G. An important tool in this induction step is the following Lemma. To formulate the induction hypothesis and the induction step more smoothly, we first introduce the following notation.

Let $n \geq 1$. We will write $\mathbf{S}_{<n}(F, =, I)$ if and only if the following statement holds:

$$F(r, s) = I(r, s) \text{ for all } r, s \in V \text{ such that } d(r, s) < n.$$

It is easy to see that if $n \leq 2$, then $\mathbf{S}_{<n}(F, =, I)$.

Lemma 5 *Let $n \geq 2$, and let $\mathbf{S}_{<n}(F, =, I)$. Consider $u_0, \ldots, u_k \in V$, where $k \geq 1$. Assume that $d(u_0, u_k) < n$.*

$$\textit{If } (u_k, \ldots, u_0) \in \mathcal{P} \cup \mathcal{G}, \textit{ then } (u_k, \ldots, u_0) \in \mathcal{P} \cap \mathcal{G}.$$

Proof. The lemma can be easily proved by induction on k. The case when $k = 1$ is obvious. Let $k \geq 2$. Assume that $(u_k, \ldots, u_0) \in \mathcal{P} \cup \mathcal{G}$. Then $(u_{k-1}, \ldots, u_0) \in \mathcal{P} \cup \mathcal{G}$ and thus, by the induction hypothesis, $(u_{k-1}, \ldots, u_0) \in \mathcal{P} \cap \mathcal{G}$. Moreover, we get $u_{k-1} \in F(u_k, u_0)$ or $u_{k-1} \in I(u_k, u_0)$. Since $d(u_0, u_k) < n$, it follows from $\mathbf{S}_{<n}(F, =, I)$ that $u_{k-1} \in F(u_k, u_0)$ and $u_{k-1} \in I(u_k, u_0)$. This implies that $(u_k, \ldots, u_0) \in \mathcal{P} \cap \mathcal{G}$. □□□

In the next two lemmata and two theorems we search for the obstructions that might prevent us from establishing the induction step, when we can only use the five classical axioms. This will provide us with the insight what extra axioms we actually need to obtain a full axiomatic characterization of the interval function using organizing functions. After all this preliminary work has been done the actual characterization in the next section is then relatively easy to obtain. To find these obstructions we develop some further notation to facilitate the proofs. Basically, we consider the situation that we have a geodesic and a process between two vertices y_0, y_m, where the length of the process is at least the length of the geodesic. We view the process as going from y_0 to y_m first up then down in a circular arc and the geodesic from y_m to $y_{m+n} = y_0$ first down then up in an inverted circular arc. The underlying idea is that we choose y_0 and y_m as our initial positions (the first vertex of the process and the geodesic, respectively), and then move clockwise simultaneously along the two circular arcs to the two positions u_1 and u_{m+1}, and so forth. We continue until we arrive at some vertices y_k and y_{m+k} that together with their successors y_{k+1} and y_{m+k+1} provide us with an obstruction that is 'minimal'. What this minimality condition exactly comprises will be made clear below. The notation we use is as follows.

Assume that there exist $y_0, y_1, \ldots, y_m, y_{m+1}, \ldots, y_{m+n} \in V$, where $m \geq n \geq 2$ such that $y_{m+n} = y_0$,

$$(y_0, y_1, \ldots, y_m) \in \mathcal{P}, \text{ and } (y_m, y_{m+1}, \ldots, y_{m+n}) \in \mathcal{G}.$$

Put

$$y_{m+n+1} = y_1, \ldots, y_{m+2n} = y_n. \tag{2}$$

It follows from $\mathbf{S}_{<2}(F, =, I)$ that y_{h-1} and y_h are adjacent in F for each h, $1 \leq h \leq m + n$. Define

$$\begin{array}{l} \phi_i = (y_i, y_{i+1}, \ldots, y_{i+m}),\ \phi_i^- = (y_{i+1}, \ldots, y_{i+m}), \\ \psi_i = (y_{i+m}, y_{i+m+1}, \ldots, y_{i+m+n}), \text{ and } \psi_i^- = (y_{i+m+1}, \ldots, y_{i+m+n}) \\ \text{for each } i \text{ with } 0 \leq i \leq n. \end{array} \tag{3}$$

Note that ψ_i contains a path between y_{i+m} and y_{i+m+n}, which is of length at most n, so $d(y_{i+m}, y_{i+m+n}) \leq n$, for each i with $0 \leq i \leq n$.

We will be searching for some fixed integer k, $0 \leq k < n$, that will play a special role. For this k we define

$$x = y_k, \bar{x} = y_{k+1}, z = y_{k+m} \text{ and } \bar{z} = y_{k+m+1}. \tag{4}$$

Note that $y_k = y_{k+m+n}$, so by the above observation we have $d(x, z) \leq n$.

Let F_1 and F_2 be geometric functions on V, and let $n \geq 0$. Assume that both the underlying graph of F_1 and the underlying graph of F_2 is G. We write $\mathbf{S}_n(F_1, \subseteq, F_2)$ if and only if

$$F_1(r, s) \subseteq F_2(r, s) \text{ for all } r, s \in V \text{ such that } d(r, s) = n.$$

Moreover, by $\neg\mathbf{S}_n(F_1, \subseteq, F_2)$ we denote the negation of $\mathbf{S}_n(F_1, \subseteq, F_2)$.

In the following lemma and theorem we search for the obstruction that causes $\neg\mathbf{S}_n(F, \subseteq, I)$.

Lemma 6 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $\mathbf{S}_{<n}(F, =, I)$ and $\neg(\mathbf{S}_n(F, \subseteq, I))$, for some $n \geq 2$, then there exist $x, \bar{x}, z, \bar{z} \in V$ such that*

$$x \text{ and } \bar{x} \text{ are adjacent in } F, \text{ and } z \text{ and } \bar{z} \text{ are adjacent in } F, \tag{5}$$

$$d(x, z) = n \text{ and } d(x, \bar{z}) = n - 1, \tag{6}$$

$$d(\bar{x}, z) \geq n, \tag{7}$$

$$\bar{x} \in F(x, z), \tag{8}$$

and

$$z \notin F(\bar{x}, \bar{z}). \tag{9}$$

Proof. Since $\neg(\mathbf{S}_n(F, \subseteq, I))$ holds, there exist $u, w \in V$ such that $d(u, w) = n$ and $F(u, w) - I(u, w) \neq \emptyset$. By virtue of Lemma 4, there exist $y_0, y_1, \ldots, y_m \in V$, such that $y_0 = u$, $y_m = w$,

$$(y_0, y_1, \ldots, y_m) \in \mathcal{P} \text{ and } m > n.$$

Put $\phi^* = (y_m, \ldots, y_1, y_0)$. Since $d(u, w) = n$, there exist $y_{m+1}, \ldots, y_{m+n} \in V$ such that $y_{m+n} = y_0$ and

$$(y_m, y_{m+1}, \ldots, y_{m+n}) \in \mathcal{G}.$$

We use conventions (2) and (3). Note that $\phi_0 \in \mathcal{P}$. By Lemma 2, we have $\phi^* \in \mathcal{P}$. Assume that $\phi_n \in \mathcal{P}$, so that $y_{n+1} \in F(y_n, y_{m+n}) = F(y_n, y_0)$. Since $m > n$ and $\phi^* \in \mathcal{P}$, we also have $y_n \in F(y_{n+1}, y_0)$. By $(c2)$ we have $y_{n+1} \in F(y_0, y_n)$ and $y_{n+1} \in F(y_0, y_{n+1})$. It follows from $(c5)$ with $u = y_0$, $v = y = y_n$ and $x = y_{n+1}$

that $y_n \in F(y_n, y_n)$. Hence, by $(c4)'$ we have $y_{n+1} = y_n$, which is impossible. This contradiction tells us that $\phi_n \notin \mathcal{P}$.

We conclude that there exists a k with $0 \leq k < n$ such that $\phi_k \in \mathcal{P}$ and $\phi_{k+1} \notin \mathcal{P}$. Let x, $\bar{x}$, z, and $\bar{z}$ be defined as in (4). Clearly, we have (5).

As observed above, we have $d(x, z) \leq n$. Suppose that $d(x, z) < n$. Since $\phi_k \in \mathcal{P}$, Lemma 5 would imply that $\phi_k \in \mathcal{G}$ and therefore $m < n$, a contradiction. Hence $d(x, z) = n$ and therefore $\psi_k \in \mathcal{G}$. Moreover, we have $d(x, \bar{z}) = n - 1$, which settles (6).

Suppose that $d(\bar{x}, z) \leq n - 1$. Since $\phi_k^- \in \mathcal{P}$, Lemma 5 would imply that $\phi_k^- \in \mathcal{G}$ and $m - 1 \leq n - 1$, which is a contradiction. Thus we have (7).

Since $\phi_k \in \mathcal{P}$, we have $\bar{x} \in F(x, z)$, so that (8) holds. Since $\phi_k^- \in \mathcal{P}$ and $\phi_{k+1} \notin \mathcal{P}$, Lemma 3 implies that $z \notin F(\bar{x}, \bar{z})$, which settles (9). □□□

Theorem 1 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $\mathbf{S}_{<n}(F, =, I)$, $\mathbf{S}_n(I, \subseteq, F)$, and $\neg(\mathbf{S}_n(F, \subseteq, I))$, for some $n \geq 2$, then there exist $x, \bar{x}, z, \bar{z} \in V$ such that* (5)*,* (6)*,* (7)*,* (8)*,* (9)*,*

$$\bar{z} \in F(x, z) \tag{10}$$

and

$$x \in F(\bar{x}, \bar{z}). \tag{11}$$

Proof. Lemma 6 implies that there exist $x, \bar{x}, z, \bar{z} \in V$ such that (5), (6), (7), (8) and (9).

It follows from (6) that $\bar{z} \in I(x, z)$. By $\mathbf{S}_n(I, \subseteq, F)$, we have $\bar{z} \in F(x, z)$. Thus we have (10).

Since $d(x, \bar{z}) = n - 1$, (5) implies that $d(\bar{x}, \bar{z}) \leq n$. Suppose that $d(\bar{x}, \bar{z}) \leq n - 1$. By (7), we would have $d(\bar{x}, z) \geq n$. Thus we would get $d(\bar{x}, z) = n$ and $d(\bar{x}, \bar{z}) = n-1$. Hence we would have $\bar{z} \in I(\bar{x}, z)$. By $\mathbf{S}_n(I, \subseteq, F)$, we would get $\bar{z} \in F(\bar{x}, z)$. By (8), we would have $\bar{x} \in F(x, z)$. Axioms $(c2)$ and $(c5)$ would then imply that $\bar{x} \in F(x, \bar{z})$. By (6), we would have $d(x, \bar{z}) = n - 1$. As follows from $\mathbf{S}_{<n}(F, =, I)$, we would have $\bar{x} \in I(x, \bar{z})$ and therefore $d(\bar{x}, \bar{z}) = n - 2$, which is a contradiction. Thus we conclude that $d(\bar{x}, \bar{z}) = n$. Since $d(x, \bar{z}) = n - 1$, we get $x \in I(\bar{x}, \bar{z})$. Hence, by $\mathbf{S}_n(I, \subseteq, F)$, we have $x \in F(\bar{x}, \bar{z})$, by which we have settled (11). □□□

In the following lemma and theorem we search for the obstruction that causes $\neg\mathbf{S}_n(I, \subseteq, F)$.

Lemma 7 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $\mathbf{S}_{<n}(F, =, I)$, and $\neg(\mathbf{S}_n(I, \subseteq, F))$, for some $n \geq 2$, then there exist $x, \bar{x}, z, \bar{z} \in V$ such that* (5)*,* (6)*,* (8)*,*

$$\bar{z} \notin F(x, z), \tag{12}$$

$$z \notin F(\bar{x}, \bar{z}) \text{ or } x \in F(\bar{x}, \bar{z}), \tag{13}$$

$$\bar{x} \notin F(x, \bar{z}), \tag{14}$$

and

$$\textit{if } x \in F(\bar{x}, \bar{z}) \textit{ then } d(\bar{x}, \bar{z}) = n. \tag{15}$$

Proof. Since $\neg(\mathbf{S}_n(I, \subseteq, F))$ holds, there exist $u, w \in V$ such that $d(u, w) = n$ and $I(u, w) - F(u, w) \neq \emptyset$. As follows from Lemma C, there exist $y_0, y_1, \ldots, y_m \in V$, $m \geq n$, such that $y_0 = u$, $y_m = w$, and

$$(y_0, y_1, \ldots, y_m) \in \mathcal{P}.$$

Since $I(u, w) - F(u, w) \neq \emptyset$, there exist $y_{m+1}, \ldots, y_{m+n} \in V$ such that $y_{m+n} = y_0$ and

$$(y_m, y_{m+1}, \ldots, y_{m+n}) \in \mathcal{G} - \mathcal{P}.$$

We use conventions (2) and (3). Note that $\phi_0 \in \mathcal{P}$ and $\psi_0 \notin \mathcal{P}$. Suppose that $\phi_n \in \mathcal{P}$. Then, if we compare ψ_0 and ϕ_n, we see that we would also have $\psi_0 \in \mathcal{P}$, which yields a contradiction. Hence we have $\phi_n \notin \mathcal{P}$. This implies that there exists a k, $0 \leq k < n$, such that $\phi_k \in \mathcal{P}$, $\psi_k \notin \mathcal{P}$, and

$$\phi_{k+1} \notin \mathcal{P} \text{ or } \psi_{k+1} \in \mathcal{P}. \tag{16}$$

Let x, $\bar{x}$, z, and $\bar{z}$ be defined as in (4). Obviously, we have (5). Recall that $d(x, z) \leq n$. Suppose that $d(x, z) < n$. Then, similarly as in the proof of Lemma 6, we would get $m < n$, which yields a contradiction. Hence $d(x, z) = n$, and therefore $\psi_k \in \mathcal{G}$. This implies that $d(x, \bar{z}) = n - 1$, so that we have (6).

Recall that $\phi_k \in \mathcal{P}$. Hence we have $\phi_k^- \in \mathcal{P}$ and $\bar{x} \in F(x, z)$, so that we have (8).

Obviously, $\psi_k^- \in \mathcal{G}$. Since $d(x, \bar{z}) = n - 1$, Lemma 5 implies that $\psi_k^- \in \mathcal{P}$. From $\psi_k \notin \mathcal{P}$ it follows that $\bar{z} \notin F(x, z)$, so that we have (12).

Since $\phi_k^-, \psi_k^- \in \mathcal{P}$, combining (16) with Lemma 3, we get $z \notin F(\bar{x}, \bar{z})$ or $x \in F(\bar{x}, \bar{z})$. This settles (13).

Suppose that $\bar{x} \in F(x, \bar{z})$. Recall that, by (6), we would have $d(x, \bar{z}) = n - 1$ and $d(x, z) = n$. Then it follows from $\mathbf{S}_{<n}(F, =, I)$ that $\bar{x} \in I(x, \bar{z})$ and therefore $d(\bar{x}, \bar{z}) = n - 2$. This would imply that $d(\bar{x}, z) = n - 1$, whence $\bar{z} \in I(\bar{x}, z)$. Then, by $\mathbf{S}_{<n}(F, =, I)$, we would have $\bar{z} \in F(\bar{x}, z)$. Recall that, by (8), this would imply that $\bar{x} \in F(x, z)$. Axioms $(c2)$ and $(c3)$ then imply that $\bar{z} \in F(x, z)$, which is a contradiction with (12). Hence $\bar{x} \notin F(x, \bar{z})$, which settles (14).

Suppose that $x \in F(\bar{x}, \bar{z})$. Since $\psi_k^- \in \mathcal{P}$, Lemma 3 implies that $\psi_{k+1} \in \mathcal{P}$. Obviously, the length of ψ_{k+1} is n. So $d(\bar{x}, \bar{z}) \leq n$. Assume that $d(\bar{x}, \bar{z}) < n$. Then, by Lemma 5, we would have $\psi_{k+1} \in \mathcal{G}$. Hence the length of ψ_{k+1} would be $d(\bar{x}, \bar{z}) < n$, which is impossible. Therefore $d(\bar{x}, \bar{z}) = n$, and we have (15), by which the proof is complete. □□□

Theorem 2 *Let F be a geometric function on a nonempty finite set V with underlying graph G, and let I be the interval function of G. If $\mathbf{S}_{<n}(I,=,F)$, and $\neg(\mathbf{S}_n(F,=,I))$, for some $n \geq 2$, then*

$$\begin{array}{l}\text{there exist } x,\bar{x},z,\bar{z}\in V \text{ such that (5)}, d(x,z)=n,\\ \bar{x}\in F(x,z),\ x\in F(\bar{x},\bar{z}),\ \bar{z}\in F(x,z), \text{ and } z\notin F(\bar{x},\bar{z}),\end{array} \tag{17}$$

or

$$\begin{array}{l}\text{there exist } x,\bar{x},z,\bar{z}\in V \text{ such that (5)}, d(x,z)=n,\\ \bar{x}\in F(x,z),\ z\notin F(\bar{x},\bar{z}),\ \bar{z}\notin F(x,z), \text{ and } \bar{x}\notin F(x,\bar{z}).\end{array} \tag{18}$$

Proof. Obviously, we have

$$\mathbf{S}_n(I,\subseteq,F) \text{ and } \neg(\mathbf{S}_n(F,\subseteq,I))$$

or

$$\neg(\mathbf{S}_n(I,\subseteq,F)).$$

It follows from Theorem 1 and Lemma 7 that we have (17) or that

$$\begin{array}{l}\text{there exist } x,\bar{x},z,\bar{z}\in V \text{ such that (5), (15)}, d(x,z)=n,\ \bar{x}\in F(x,z),\\ \bar{z}\notin F(x,z),\ \bar{x}\notin F(x,\bar{z}), \text{ and moreover } z\notin F(\bar{x},\bar{z}) \text{ or } x\in F(\bar{x},\bar{z}).\end{array} \tag{19}$$

If (17) holds, then we are done. So assume that (17) does not hold. Then we have (19). Suppose that $z \in F(\bar{x},\bar{z})$. Then (19) would imply that $x \in F(\bar{x},\bar{z})$. By (15) we would have $d(\bar{x},\bar{z}) = n$. Since (17) does not hold, we would have

$$\begin{array}{l}\text{if } r \text{ and } \bar{r} \text{ as well as } s \text{ and } \bar{s} \text{ are adjacent in } F,\ d(r,s)=n,\\ r\in F(\bar{r},\bar{s}), \text{ and } \bar{r},\bar{s}\in F(r,s), \text{ then } s\in F(\bar{r},\bar{s})\end{array} \tag{20}$$

for all $r,\bar{r},s,\bar{s}$ in V.

If we put $r=\bar{x}$, $\bar{r}=x$, $s=\bar{z}$, and $\bar{s}=z$, then (20) implies that $\bar{z}\in F(x,z)$, which creates a conflict with (19). Hence we have $z\notin F(\bar{x},\bar{z})$. Thus we get (18), which completes the proof. □□□

Now we are able to get a clear understanding of the obstructions that prevent an organizing function F to be the interval function of a graph when we assume that F satisfies the five classical axioms only. We formulate these obstructions in the next section, where we use these to formulate additional axioms for geometric functions.

4 Characterizing the Interval Function

Again recall that V is a finite set. In this section we assume that a geometric function F on V is given. We denote by G the underlying graph of F. By Lemma C, the graph G is connected. We denote by d and I the distance function of G and the interval function of G respectively. Clearly, F is the interval function of a connected graph if and only if $F = I$.

For two organizing functions F_1 and F_2 we write $\mathbf{S}(F_1, \subseteq, F_2)$ if and only if $F_1(r, s) \subseteq F_2(r, s)$ for all $r, s \in V$. So, in a characterization of the interval function using the organizing function F we would like to have both $\mathbf{S}(I, \subseteq, F)$ and $\mathbf{S}(F, \subseteq, I)$. The induction we use will be in the guise of a minimal counterexample. This allows us to reformulate Theorems 1, and 2 in a much simpler form in the next two Corollaries: we can eliminate the parameter n.

Corollary 1 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $\mathbf{S}(I, \subseteq, F)$, and $F \neq I$, then*

$$\begin{aligned}&\textit{there exist } x, \bar{x}, z, \bar{z} \in V \textit{ such that } (5),\ \bar{x} \in F(x, z),\\ &x \in F(\bar{x}, \bar{z}),\ \bar{z} \in F(x, z), \textit{ and } z \notin F(\bar{x}, \bar{z}).\end{aligned} \tag{21}$$

Proof. Obviously, there exists some $n \geq 2$ such that

$$\mathbf{S}_{<n}(I, =, F),\ \mathbf{S}_n(I, \subseteq, F) \text{ and } \neg(\mathbf{S}_n(F, \subseteq, I)).$$

By Theorem 1, there exist $x, \bar{x}, z, \bar{z} \in V$ such that (5), (8), (9), (10), and (11). Thus we get (21), which completes the proof. □□□

Corollary 2 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $F \neq I$, then* (21) *holds or*

$$\begin{aligned}&\textit{there exist } x, \bar{x}, z, \bar{z} \in V \textit{ such that } (5),\ \bar{x} \in F(x, z),\\ &z \notin F(\bar{x}, \bar{z}),\ \bar{z} \notin F(x, z), \textit{ and } \bar{x} \notin F(x, \bar{z}).\end{aligned} \tag{22}$$

Proof. Obviously, there exists $n \geq 2$ such that

$$\mathbf{S}_{<n}(I, =, F), \text{ and } \neg\mathbf{S}_n(I, =, F).$$

Thus the result follows immediately from Theorem 2. □□□

It is not difficult to show that if $F = I$, then neither (21) nor (22) holds. This means that if $F = I$, then F satisfies the following *supplementary axioms* ($s1$) and ($s2$). Here $u, \bar{u}, v, \bar{v}$ are variables in V.

> ($s1$) if u and $\bar{u}$ are adjacent in F, v and $\bar{v}$ are adjacent in F, $u \in F(\bar{u}, \bar{v})$, and $\bar{u}, \bar{v} \in F(u, v)$, then $v \in F(\bar{u}, \bar{v})$ for all $u, \bar{u}, v, \bar{v}$.
>
> ($s2$) if u and $\bar{u}$ are adjacent in F, v and $\bar{v}$ are adjacent in F, $\bar{u} \in F(u, v)$, $\bar{v} \notin F(u, v)$, and $v \notin F(\bar{u}, \bar{v})$, then $\bar{u} \in F(u, \bar{v})$ for all $u, \bar{u}, v, \bar{v}$.

Note that the classical axioms can be described using simple Venn-type diagrams. This is not so easily done for the supplementary axioms. The difference in character between the classical axioms and the supplementary axioms is that in the supplementary ones there is a pairing of the four variables into two pairs, each of which consists of vertices that are adjacent in F. This makes these axioms less 'obvious' than the classical axioms.

It is straightforward to verify that the interval function of a graph satisfies (s1) and (s2), cf. [13]. Consider the example given in Section 2 of the geometric function F on a broken wheel B with axis a and cycle C of length at least 4. We consider the two axioms with respect to the broken wheels.

First let av_i be a missing spoke, so that $a \to v_{i-1} \to v_i \to v_{i+1} \to a$ is an induced 4-cycle D in B. Now take $v = a$, take $\bar{u}$ and $\bar{v}$ to be the neighbors of v in D and take u to be the fourth vertex in D. Then F does not satisfy $(s1)$ on these four vertices. Hence F does not satisfy axiom $(s1)$ on any broken wheel missing a spoke. It is straightforward to check that F on the wheel does satisfy axiom $(s1)$.

Second, let av_{j-1}, av_j, av_{j+1} be three consecutive spokes in B. Now take $\bar{v} = a$, $u = v_{j-1}$, $\bar{u} = v_j$, and $v = v_{j+1}$. Then F does not satisfy $(s2)$ on these four vertices. So any broken wheel that is not a cogwheel does *not* satisfy axiom $(s2)$. In particular, F on the wheel does *not* satisfy axiom $(s2)$. Consider the cogwheel M_k with $k \geq 4$. Now take $u = v_1, \bar{u} = v_2, v = v_5, \bar{v} = a$. Then these four vertices violate axiom $(s2)$. So the supplementary axioms are really necessary.

After Theorem 4 we present an example of a geometric function satisfying $(s2)$ but not $(s1)$. This shows that the two supplementary axioms are independent.

In view of the supplementary axioms we can reformulate Corollary 2 as follows: Let F be a geometric function with underlying graph G, and let I be the interval function of G. If $F \neq I$, then F does not satisfy $(s1)$ or $(s2)$ (or both). Thus we have new proofs for the following characterizations of the interval function of a connected graph from [13, 14, 16].

Theorem 3 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. The following statements are equivalent:*

(*a*) *$F = I$,*

(*b*) *F satisfies axioms $(s1)$ and $(s2)$.*

Due to new the approach that we took in this paper, we now understand why we have no simpler supplementary axioms: if we assume that F satisfies the five classical axioms, then precisely both the supplementary axioms in their full 'complexity' are needed to overcome any possible obstruction to the induction step in proving that F is indeed the interval function of a graph.

Using only Corollary 1 we get, as a bonus, the following characterization of the interval function, cf. [14].

Theorem 4 *Let F be a geometric function with underlying graph G, and let I be the interval function of G. The following statements are equivalent:*

(*a*) $F = I$,

(*c*) $\mathbf{S}(I, \subseteq, F)$ *and F satisfies axiom* $(s1)$,

In a recent paper of the second author [18] Theorems 3 and 4 are derived from a characterization of the set of geodesics in a connected graph, but this derivation is not trivial.

In view of Theorem 4, one might wonder whether a similar theorem would hold involving axiom $(s2)$ together with either $\mathbf{S}(I, \subseteq, F)$ or $\mathbf{S}(F, \subseteq, I)$. This is not the case, as our next examples show. Let C be a cycle, let $V = V(C)$, and let I be the interval function of C. Now we define an organizing F on V, where we distinguish between the even and the odd case.

First let C be the $2m$-cycle $v_1 \to v_2 \to \ldots \to v_{2m} \to v_1$ with $m \geq 3$. We define

$F(v_1, v_{m+1}) = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$,

$F(v_i, v_j) = I(v_i, v_j)$ for all i, j such that $1 \leq i \leq j \leq 2m$ and $(i \neq 1$ or $j \neq m+1)$.

For this example we have $\mathbf{S}(F, \subseteq, I)$ and $\neg\mathbf{S}(F, =, I)$. It is straightforward to check that F is geometric and satisfies $(s2)$. Take $\bar{u} = v_1$, $u = v_2$, $\bar{v} = v_{m+1}$, and $v = v_{m+2}$. Then F does not satisfy $(s1)$ for these vertices.

Next let C be the odd $2m+1$-cycle $v_1 \to v_2 \to \ldots \to v_{2m+1} \to v_1$ with $m \geq 3$. We define

$F(v_1, v_{m+2}) = V$,

$F(v_i, v_j) = I(v_i, v_j)$ for all i, j such that $1 \leq i \leq j \leq 2m$ and $(i \neq 1$ or $j \neq m+2)$.

For this example we have $\mathbf{S}(I, \subseteq, F)$ and $\neg\mathbf{S}(I, =, F)$. It is straightforward to check that F is geometric and satisfies $(s2)$. Take $u = v_1$, $\bar{u} = v_2$, $v = v_{m+2}$, and $\bar{v} = v_{m+3}$. Then F does not satisfy $(s1)$ for these vertices.

In the characterization of the interval function of an arbitrary connected graph we need the two rather complicated supplementary axioms, as we have shown above. For special classes of graphs we need additional axioms on the organizing function to achieve the same goal. But in the following instances it turns out that we can get rid of the supplementary axioms so that only a set of relatively simple axioms remain. Let G be a connected graph with interval function I. Then G is a *modular graph* if

$$I(u, v) \cap I(v, w) \cap I(w, v) \neq \emptyset$$

for any three vertices u, v, w of G. These graphs were introduced in [5], see also [2]. If we require

$$|I(u, v) \cap I(v, w) \cap I(w, v)| = 1$$

for any three vertices u, v, w of G, then G is a *median graph*. These graphs were introduced independently in [1, 12, 11], see also [8]. We consider two more axioms for organizing functions, here u, v, w are variables in V.

(mo) $F(u, v) \cap F(v, w) \cap F(w, v) \neq \emptyset$,

(me) $|F(u, v) \cap F(v, w) \cap F(w, v)| = 1$.

Lemma 8 *Let F be a geometric function on a finite nonempty set V. If F satisfies (mo), then F satisfies $(s1)$ and $(s2)$.*

Proof. Verification of $(s1)$. Assume, to the contrary, that there exist vertices $u, \bar{u}, v, \bar{v}$ such that u and $\bar{u}$ are adjacent in F, and v and $\bar{v}$ are adjacent in F, and moreover $\bar{u} \in F(u, v)$, $u, v \in F(\bar{u}, \bar{v})$, and $\bar{v} \notin F(u, v)$.

Obviously we have

$$F(u, v) \cap F(u, \bar{v}) \cap F(v, \bar{v}) \subseteq F(v, \bar{v}) = \{v, \bar{v}\}.$$

Since $\bar{v} \notin F(u, v)$, it follows from (mo) that

$$F(u, v) \cap F(u, \bar{v}) \cap F(v, \bar{v}) = \{v\}.$$

This implies that $v \in F(u, \bar{v})$. Since $u \in F(\bar{u}, \bar{v})$, it follows from $(c2)$ and $(c5)$ that $u \in F(\bar{u}, v)$. Since $\bar{u} \in F(u, v)$, it follows from $(c2)$ and $(c5)$ that $\bar{u} \in F(u, u) = \{u\}$, which is impossible. Hence F satisfies $(s1)$.

Verification of $(s2)$. Consider arbitrary vertices $u, \bar{u}, v, \bar{v}$ in V such that u and $\bar{u}$ are adjacent in F and v and $\bar{v}$ are adjacent in F, and $\bar{u} \in F(u, v)$. Clearly, we have

$$F(\bar{u}, v) \cap F(v, \bar{v}) \cap F(\bar{u}, \bar{v}) \subseteq F(v, \bar{v}) = \{v, \bar{v}\}.$$

Assume that $v \notin F(\bar{u}, \bar{v})$. Then (mo) implies that

$$F(\bar{u}, v) \cap F(v, \bar{v}) \cap F(\bar{u}, \bar{v}) = \{\bar{v}\}.$$

Hence $\bar{v} \in F(\bar{u}, v)$. Since $\bar{u} \in F(u, v)$, it follows from $(c2)$ and $(c5)$ that $\bar{u} \in F(u, \bar{v})$. Thus we have $v \in F(\bar{u}, \bar{v})$ or $\bar{u} \in F(u, \bar{v})$. This implies that F satisfies $(s2)$. □□□

The following proposition is a simple corollary of Lemma 8.

Proposition 1 *Let F be a geometric function on a finite nonempty set V with underlying graph G. Then*

(i) F satisfies (mo) if and only if G is a modular graph and F is the interval function of G,

(ii) F satisfies (me) if and only if G is a median graph and F is the interval function of G.

Clearly this proposition can be reformulated as a characterization of modular and median graphs.

5 A new view of connected graphs

The approach we have chosen in this paper provides us with a new view on the notion of connectedness. Again let V be a nonempty finite set. Let $\mathcal{G}_V$ be the family of all connected graphs with vertex set V, and let $\mathcal{F}_V$ be the family of all geometric functions on V satisfying both supplementary axioms ($s1$) and ($s2$). As follows from Lemma C, the underlying graph of any geometric function in $\mathcal{F}_V$ belongs to $\mathcal{G}_V$. Recall that we need finiteness in the proof of Lemma C. For $F \in \mathcal{F}_V$ we define $\alpha(F)$ to be the underlying graph of F, that is, the graph with F as its interval function. Then the following proposition is a consequence of Theorem 3.

Proposition 2 *For a finite set V, $\alpha : \mathcal{F}_V \to \mathcal{G}_V$ is a bijection.*

The essence of this proposition is that we can translate connectedness of a finite graph into axioms on an organizing function. Note that each of the axioms ($c1$), ... ,($c5$), ($s1$) and ($s2$) could be formulated in a language of first-order logic. This suggests a new perspective on the notion of a finite connected graph, which might promise a new approach to the study of finite connected graphs.

As already observed above, an alternative view on organizing functions is that of ternary algebras or relations. As usual, by a ternary relation on V we mean a subset T of $V \times V \times V$. We say that T is a ternary relation *onto* V if for every $u \in V$ there exist $v, w \in V$ such that $(u, v, w) \in T$ or $(v, u, w) \in T$ or $(v, w, u) \in T$.

Let T be a ternary relation onto V. If $u, v \in V$, then we say that u and v are *adjacent* in T if $u \neq v$ and the following condition holds for all $w \in V$:

$$\text{if } (u, w, v), (v, w, u) \in T, \text{ then } w = u \text{ or } w = v.$$

The *underlying graph* G_T of T is the graph with vertex set V such that u and v are adjacent in G_T if and only if they are adjacent in T for all $u, v \in V$.

Let G be a connected graph with the vertex set V, and let d and I denote the distance function of G and the interval function of G respectively. The *geodetic betweenness* of G is the ternary relation T exactly on V defined as follows:

$$(u, w, v) \in T \text{ if and only if } d(u, w) + d(w, v) = d(u, v) \text{ for all } u, v, w \in V$$

or, equivalently, defined as follows:

$$(u, w, v) \in T \text{ if and only if } w \in I(u, v) \text{ for all } u, v, w \in V.$$

It is easy to see that, if T is the geodetic betweenness of a connected graph G, then G is the underlying graph of T.

Let T be a geodetic betweenness of a connected graph whose vertex set is V. The five classical axioms and the two supplementary axioms can be easily translated into axioms for a geodetic betweenness. Moreover, it is easy to see that T satisfies these axioms, that we can formulate a proposition analogous to Proposition 2, and that these axioms can be formulated in a language of first-order logic. This is again a different perspective on finite connected graphs.

References

[1] S.P. Avann, Metric ternary distributive semi-lattices, *Proc. Amer. Math. Soc.* **12** (1961), 407–414.

[2] H.J. Bandelt and H.M. Mulder, Pseudo-modular graphs, *Discrete Math.* **62** (1986), 245 – 260.

[3] M. Changat, S. Klavžar and H.M. Mulder, The all-paths transit function of a graph, *Czechoslovak Math. J.* **51** (126) (2001), 439 – 447.

[4] J. Hedlíková, Ternary spaces, media, and Chebyshev sets, *Czechoslovak Math. J.* **33** (108) (1983), 373 – 389.

[5] E. Howorka, Betweenness in graphs, *Abstracts Amer. Math. Soc.* **2** (1982), *783-06-5.

[6] D.C. Kay and G. Chartrand, A characterization of certain ptolemaic graphs, *Canadian J. Math.* **17** (1965), 342 – 346.

[7] M.A. Morgana and H.M. Mulder, The induced path convexity, betweenness, and svelte graphs, *Discrete Math.* **254** (2002), 349 – 379.

[8] H.M. Mulder, The structure of median graphs, *Discrete Math.* **24** (1978), 197 – 204.

[9] H.M. Mulder, *The interval function of a graph*, Math. Centre Tracts 132, *Math. Centre*, Amsterdam, Netherlands 1980.

[10] H.M. Mulder, Transit functions on graphs (and posets), in: M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar (Eds.), *Convexity in Discrete Structures. RMS Lecture Notes Series 5.* 2008, pp. 117–130.

[11] H.M. Mulder and A. Schrijver, Median graphs and Helly hypergraphs, *Discrete Math.* **25** (1979), 41 – 50.

[12] L. Nebeský, Median graphs, *Comment. Math. Univ. Carolinae* **12** (1971), 317 – 325.

[13] L. Nebeský, A characterization of the interval function of a connected graph, *Czechoslovak Math. J.* **44** (119), (1994) 173 – 178.

[14] L. Nebeský, Characterizing the interval function of a connected graph, *Math. Bohemica* **123** (1998), 137 – 144.

[15] L. Nebeský, A characterization of the interval function of a (finite or infinite) connected graph, *Czechoslovak Math. J.* **51** (126) (2001), 635 – 642.

[16] L. Nebeský, The interval function of a connected graph and a characterization of geodetic graph, *Math. Bohemica* **126** (2001), 247 – 254.

[17] L. Nebeský, Intervals and steps in a connected graph, *Discrete Math.* **286** (2004), 151 – 156.

[18] L. Nebeský, The interval function of a connected graph and road systems, *Discrete Math.* **307** (2007), 2067 – 2073.

[19] E. Sampathkumar, B-systems, in: S. Arumugam, B.D. Acharya and E. Sampathkumar eds., *Graph Theory and its Applications, Proceedings of the National Workshop*, Manonmaniam Sundaranar University, Tirunelveli, India 1996.

[20] M. Sholander, Trees, lattices, order, and betweenness, *Proc. Amer. Math. Soc.* **3** (1952), 369 – 381.

[21] M. van de Vel, *Theory of convex structures*, *North Holland*, Amsterdam, Netherlands 1993.

[22] E.R. Verheul, *Multimedians in Metric and Normed Spaces*, PhD thesis, 1991, also published as CWI Tract 91, *CWI*, Amsterdam, Netherlands 1993.