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Calculation of Stability Radii for Combinatorial Optimization Problems

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Abstract

We present algorithms to calculate the stability radius of optimal or approximate solutions of binary programming problems with a min–sum or min–max objective function. Our algorithms run in polynomial time if the optimization problem itself is polynomially solvable. We also extend our results to the tolerance approach to sensitivity analysis.

Keywords: Stability radius, Sensitivity analysis, Postoptimal analysis, Tolerance approach, Binary programming, Computational complexity

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1 Introduction

This paper is devoted to stability analysis of optimal and approximate solutions of combinatorial optimization problems of the following form:

$$(P): \ \min\{\Phi(c,x) \mid x \in X\},$$

where $c \in I\!R^n$, $x = (x_1, x_2, \dots, x_n)$ is a vector of 0/1 variables, $\Phi(c,x)$ is either of the form $\sum_{i=1}^n c_i x_i$ or $\max_{1 \le i \le n}\{c_i x_i\}$, and $X \subseteq \{0,1\}^n$ is the set of feasible solutions, which does not depend on the objective vector c.

Suppose that for a given vector c and a given $\epsilon \ge 0$, an ϵ–optimal solution $\bar{x} \in X$ is known, i.e.,

$$\Phi(c,\bar{x}) \le (1+\epsilon)\,\Phi(c,x) \quad \forall x \in X.$$

Note that we view optimality as a special case of ϵ–optimality. Also note that for $\epsilon > 0$, the concept of ϵ–optimality only makes sense if $\Phi(c,x) \ge 0$ for all $x \in X$, which is guaranteed if $c \ge 0$.

We will investigate the situation in which for one or more variables x_i, the objective coefficient may actually be different from c_i. Such components of the objective vector are referred to as *unstable*. Without loss of generality we assume that the unstable components correspond to the first w variables $x_1, x_2, \dots, x_w$. The remaining $n-w$ components of the objective vector are stable and remain equal to $c_{w+1}, c_{w+2}, \dots, c_n$.

This paper focusses on the calculation of the largest $\rho \ge 0$ for which $\bar{x}$ remains ϵ–optimal if the unstable components change simultanously, but each one not more than ρ. Hence, we are looking for ρ of maximum value such that

$$\Phi(c+\delta,\bar{x}) \le (1+\epsilon)\,\Phi(c+\delta,x) \quad \forall x \in X$$

for every $\delta \in I\!R^n$ with $||\delta||_\infty \le \rho$ and, if the objective vector is required to be non–negative, $c + \delta \ge 0$. In the literature this maximal value of ρ is called the *stability* *radius* of the ϵ–optimal solution $\bar{x}$. We refer to Sotskov, Leontev and Gordeev [5] for an extensive survey on this and related concepts. A more recent survey, which focusses on scheduling problems, is given by Sotskov, Wagelmans and Werner [6], who also present an algorithm to compute the stability radius for min–sum problems, i.e., when $\Phi(c, x) = \sum_{i=1}^{n} c_i x_i$. In general, the complexity of this algorithm is exponential, even if (P) itself is polynomially solvable.

In Ramaswamy and Chakravarti [4] and Van Hoesel and Wagelmans [7] it was shown that for $w = 1$ the existence of a polynomial algorithm for calculating the stability radius of an optimal solution implies a polynomial algorithm for problem (P). In [7] a similar implication was also proven for the case $\epsilon > 0$ when the objective function is of the min–sum type. This means that, even for $w = 1$, it is unlikely that the stability radius can be calculated in polynomial time if (P) is NP–hard. On the other hand, in [4] it was shown that if $w = 1$ and problem (P) is polynomially solvable, then the stability radius of an optimal solution can be calculated in polynomial time. It still was an open question (see [6]) whether it is possible to generalize this result to arbitrary values of w and $\epsilon > 0$.

In this paper, we will present an algorithm to compute the stability radius of an ϵ–optimal solution of min–sum problems. We also show how to compute the stability radius of optimal solutions for min–max problems, i.e., when $\Phi(c, x) = \max_{1 \leq i \leq n}\{c_i x_i\}$. Our algorithms require the solution of a polynomial number of instances of problem (P). In particular this means that, for the cases considered, we provide a positive answer to the open question mentioned before. Furthermore, we will show that it is possible to extend our results to the tolerance approach, which was proposed by Wendell [8] in the context of linear programming.

2 Calculating stability radii for min–sum problems

In this section we consider the case that $\Phi(c, x) = \sum_{i=1}^{n} c_i x_i$. To facilitate the exposition, we will first assume that the objective coefficients are unrestricted in sign.

2.1 Unrestricted objective coefficients

Suppose a problem instance with objective vector c is given and let $\bar{x}$ be an ϵ–optimal solution. We want to determine the largest $\rho \geq 0$ such that

$$\sum_{i=1}^{n}(c_i + \delta_i)\bar{x}_i \leq (1+\epsilon)\sum_{i=1}^{n}(c_i + \delta_i)x_i \tag{1}$$

for all $x \in X$ and every $\delta \in I\!R^n$ with $|\delta_i| \leq \rho$ for all $i = 1, 2, \ldots, w$ and $\delta_i = 0$ for all $i = w+1, w+2, \ldots, n$.

One can easily verify that if there exist an $x \in X$ which differs from $\bar{x}$ in at least one of the first w components, then the stability radius is finite and an upper bound is given by

$$\rho_u = \max_{1 \leq i \leq w}\{|c_i|\} + (1+\epsilon) \cdot \sum_{i=w+1}^{n} \max\{c_i, 0\} - (1+\epsilon) \cdot \sum_{i=w+1}^{n} \min\{c_i, 0\}.$$

Moreover, if all $x \in X$ have $x_i = \bar{x}_i$ for $i = 1, 2, \ldots, w$, then the stability radius is infinite. Hence, it suffices to look for the stability radius on the interval $[0, \rho_u]$. Note that for any value of ρ in this interval, the objective coefficients $c_i(1+\epsilon) - \rho d_i$, $i = 1, 2, \ldots, n$, are polynomial in c and ϵ.

Inequality (1) is equivalent to

$$\sum_{i=1}^{w} \delta_i\big(\bar{x}_i - (1+\epsilon)x_i\big) \leq \sum_{i=1}^{n} c_i\big((1+\epsilon)x_i - \bar{x}_i\big). \tag{2}$$

Let us first consider this inequality for a fixed, but unknown $x \in X$, and a fixed $\rho \geq 0$. Then the right hand side is a constant and the inequality holds if and only if it holds for those values of δ_i with $|\delta_i| \leq \rho$, $i = 1, 2 \ldots, w$, which maximize the left hand side.

Consider an $i \in \{1, 2 \ldots, w\}$ and suppose $\bar{x}_i = 0$, then the term $\delta_i\big(\bar{x}_i - (1+\epsilon)x_i\big)$ is equal to $-\delta_i(1+\epsilon)x_i$, which is maximized at $\delta_i = -\rho$, irrespective of the value of x_i. In this case, we define $d_i = 1 + \epsilon$. Hence, the maximum value is equal to $\rho d_i x_i$.

Now suppose $\bar{x}_i = 1$ for some $i \in \{1, 2 \ldots, w\}$. If $x_i = 0$ then the term $\delta_i\big(\bar{x}_i - (1+\epsilon)x_i\big)$ is equal to δ_i, which is maximized at $\delta_i = \rho$. If $x_i = 1$ then the term $\delta_i\big(\bar{x}_i - (1+\epsilon)x_i\big)$ is equal to $\delta_i(-\epsilon)$, which is maximized at $\delta_i = -\rho$. Therefore, we define $d_i = -1 + \epsilon$ in this case. The maximum value is always equal to $\rho + \rho d_i x_i$.

For convenience, we also define $d_i = 0$ for $i = w+1, w+2, \ldots, n$. Then we have derived that (2) holds if and only if

$$\rho \sum_{i=1}^{w} \bar{x}_i + \sum_{i=1}^{n} \rho d_i x_i \leq \sum_{i=1}^{n} c_i\big((1+\epsilon)x_i - \bar{x}_i\big).$$

This immediately implies the following result.

Theorem 2.1 *The stability radius is the largest $\rho \geq 0$ for which*

$$\min_{x \in X} \Big\{ \sum_{i=1}^{n} \big(c_i(1+\epsilon) - \rho d_i\big)x_i \Big\} \geq \sum_{i=1}^{n} c_i \bar{x}_i + \rho \sum_{i=1}^{w} \bar{x}_i. \tag{3}$$

The right hand side of (3) is a linear function of ρ. The left hand side is the value function of a parametric version of problem (P), where the objective coefficients are linear functions of ρ. Let us call this value function $v(\rho)$. It is well–known (see, for instance, Eisner and Severance [1] or Gusfield [2]) that $v(\rho)$ is a continuous, piecewise linear and concave function of ρ.

Lemma 2.1 *The number of linear pieces of $v(\rho)$ on $[0, \rho_u]$ is at most w^2.*

Proof. Since the slope of $v(\rho)$ is always equal to $\sum_{i=1}^{n} -d_i x_i$ for some $x \in X$, it follows from the definition of the values d_i, $i = 1, 2, \ldots, m$, that this slope takes on values in the set $\{k + m\epsilon \mid -w \leq k \leq w,\ \max\{k,1\} \leq m \leq w\}$. Moreover, because of concavity, the slope of $v(\rho)$ is non–increasing. The bound on the number of linear pieces now follows.

□

There exists a method (see [1]) to compute $v(\rho)$ in $\mathcal{O}\big(B \cdot R(|I|,\epsilon)\big)$ time, where B is the number of linear pieces, $|I|$ is the size of the problem instance with objective vector c, and $R(|I|,\epsilon)$ is the complexity of solving an instance of (P) corresponding to any value of $\rho \in [0,\rho_u]$. This complexity is a function of data which, as we have pointed out before, depends polynomially on $|I|$ and ϵ. Once $v(\rho)$ has been computed, it is trivial to find the largest value of ρ for which this function is greater than or equal to the linear function $\sum_{i=1}^n c_i\bar{x}_i + \rho\sum_{i=1}^w \bar{x}_i$. Hence, the stability radius can be calculated in $\mathcal{O}\big(w^2 \cdot R(|I|,\epsilon)\big)$ time. This has the following important implication.

Theorem 2.2 *The stability radius of an ϵ–optimal solution can be computed in polynomial time, if (P) has a min–sum objective function and if it is polynomially solvable for any objective vector.*

Proof. The only observation that we need to make is that $R(|I|,\epsilon)$ is polynomial in the size of problem instances, which are in turn polynomial in $|I|$ and ϵ.

□

2.2 Non–negative objective coefficients

Our approach can easily be extended to problems in which the objective vector is required to be non–negative. Assume, without loss of generality, that $c_1 \leq c_2 \leq \ldots \leq c_w$. Suppose that we consider only values of ρ in the interval $[c_j, c_{j+1}]$ for some $j \in \{1,2,\ldots,w-1\}$. Then, for every $i \leq j$, δ_i may not be chosen smaller than $-c_i$. Therefore, for these values of i, if $\bar{x}_i = 0$, the maximum value of the term $\delta_i\big(\bar{x}_i - (1+\epsilon)x_i\big)$ is now equal to $(1+\epsilon)c_i$. If $\bar{x}_i = 1$, then the maximum value is equal to ρ if $x_i = 0$, and equal to $c_i\epsilon$ if $x_i = 1$. Hence, in this case, the maximum is always equal to $\rho - \rho x_i + c_i\epsilon x_i$. This means that if the stability radius is an element of $[c_j, c_{j+1}]$, then it is the largest value of ρ in this interval for which

$$\min_{x\in X}\Big\{\sum_{1\leq i\leq j:\ \bar{x}_i=1}(c_i+\rho)x_i+\sum_{i=j+1}^{n}\big(c_i(1+\epsilon)-\rho d_i\big)x_i\Big\}\geq\sum_{i=1}^{n}c_i\bar{x}_i+\rho\sum_{i=1}^{w}\bar{x}_i. \quad (4)$$

As before, the value function on the left hand side of (4) is piecewise linear and concave on $[c_j, c_{j+1}]$. Note that for any fixed value of ρ, indeed a problem instance with non–negative objective coefficients results.

To find the stability radius, it is not necessary to construct the value function of every interval $[c_j, c_{j+1}]$, $j = 1, 2, \ldots, w-1$. Note that if (4) holds in the endpoints, then, because of concavity, it holds on the complete interval. Therefore, the interval which contains the stability radius can easily be found by checking only the endpoints of the intervals. This means that the correct interval (possibly $[c_w, \rho_u]$) can be found in $\mathcal{O}(w \cdot R(|I|, \epsilon))$ time. Once that interval is known, the stability radius is calculated in $\mathcal{O}\big(w^2 \cdot R(|I|, \epsilon)\big)$ time. Hence, the complexity of our approach is the same as before and the following result is obvious.

Theorem 2.3 *If (P) has a min–sum objective function with objective coefficients which are restricted to be non–negative and if (P) is polynomially solvable for any non–negative objective vector, then the stability radius of an ϵ–optimal solution can be computed in polynomial time.*

2.3 Extension to the tolerance approach

The stability radius can be viewed as a measure which focusses on absolute deviations of the unstable objective coefficients. Sometimes it may make more sense to look at relative deviations instead. For instance, suppose that the objective coeffcients are unrestricted in sign, and we would like to know the largest $\gamma \geq 0$ such that

$$\sum_{i=1}^{n} c_i(1+\delta_i)\bar{x}_i \leq (1+\epsilon)\sum_{i=1}^{n} c_i(1+\delta_i)x_i \quad (5)$$

for all $x \in X$ and every $\delta \in I\!R^n$ with $|\delta_i| \leq \gamma$ for all $i = 1, 2, \ldots, w$ and $\delta_i = 0$ for all $i = w+1, w+2, \ldots, n$. This is similar to the tolerance approach to sensitivity analysis, which was developed by Wendell [8] for linear programming. Therefore, we will refer to the largest value of γ satisfying (5) as the *tolerance radius.*

To caculate the tolerance radius, we can essentially follow the same approach as in Subsection 2.1. It boils down to finding the largest γ such that

$$\min_{x \in X} \Big\{ \sum_{i=1}^{n} \big(c_i(1+\epsilon) - \gamma d_i |c_i|\big) x_i \Big\} \geq \sum_{i=1}^{n} c_i \bar{x}_i + \sum_{1 \leq i \leq w:\ \bar{x}_i = 1} \gamma |c_i|. \tag{6}$$

However, it is not possible to bound the number of linear pieces of the value function on the left hand side in a similar way as in Lemma 2.1. Therefore, we need to calculate the largest intersection point of the value function with the right hand side of (6), without constructing the complete value function. This is possible by using a technique due to Gusfield [2], which is based on a method by Megiddo [3] for solving minimum ratio combinatorial optimization problems. The only requirement is that (P) can be solved by an algorithm with the property that if the input data consists of linear functions of a single parameter, the algorithm performs only operations which preserve the linear dependence of the data on the parameter (in ours case: γ). Gusfield calls such algorithms *suitable.* Note that most combinatorial algorithms are of this type. Given a suitable algorithm with complexity $\mathcal{O}\big(R(|I|, \epsilon)\big)$, Gusfield's technique will determine the tolerance radius in $\mathcal{O}\big(R(|I|, \epsilon)^2\big)$ time. This implies the following result.

Theorem 2.4 *The tolerance radius of an ϵ–optimal solution can be computed in polynomial time if (P) has a min–sum objective function and if it is solvable, for any objective vector, by a suitable polynomial algorithm.*

It is left to the reader to verify that the above results can be extended to the case of non–negative objective coefficients. To end this section, we note that Wendell's tolerance approach is actually more general, since it also allows the components of δ to be weighted by a vector different from c. Our approach can also be generalized in this way.

3 Calculating stability radii for min–max problems

In this section we consider the case that $\Phi(c,x) = \max_{1\leq i\leq n}\{c_i x_i\}$, i.e., (P) is a min–max (bottleneck) problem. The case in which the objective vector is required to be non–negative requires no particular care, so it will not be treated separately.

Suppose that a problem instance with objective vector c is given and that an *optimal* solution $\bar{x}$ is available. We will derive an explicit expression for the stability radius of $\bar{x}$ and will also show that it may be calculated by solving at most polynomially many instances of (P) of about the same size as the given instance.

3.1 All components unstable

In order to simplify the discussion, we first analyze the case that $w = n$, i.e., all components of the objective vector are unstable. Define $J_1 = \{j \mid \bar{x}_j = 1\}$. For each $j \in J_1$, we let x^j denote an optimal solution solution for the modified problem instance in which x_j is required to be 0, and we let b^j denote the corresponding objective value. We define $\rho_j = (b^j - c_j)/2$. If $x_j = 1$ for all $x \in X$, b^j and ρ_j are ∞.

Theorem 3.1 *If $w = n$, then the stability radius is equal to* $\min_{j\in J_1}\{\rho_j\}$.

Proof. We will first show that the stability radius is at least $\min_{j\in J_1}\{\rho_j\}$. Consider a vector δ with $|\delta_i| \leq \min_{j\in J_1}\{\rho_j\}$ for all $i = 1, 2, \ldots, w$. Note that any solution $x \in X$ with $x_j = 1$ for all $j \in J_1$ has always a value greater than or equal to $\bar{x}$. Therefore it suffices to consider only solutions which have $x_j = 0$ for some $j \in J_1$. For such a solution, let $k \in J_1$ be such that $x_k = 0$ and $c_k \geq c_j$ for all $j \in J_1$ with $x_j = 0$. Note that $x_j = 1$ for all $j \in J_1$ with $c_j > c_k$. This implies

$$\Phi(c+\delta, x) \geq \max\{c_i + \delta_i \mid i \in J_1,\ c_i > c_k\}. \tag{7}$$

Furthermore, it follows from $\Phi(c,x) \geq b^k$ that

$$\Phi(c+\delta,x) \geq b^k - \min_{j\in J_1}\{\rho_j\} \geq b^k - \rho_k \geq (b^k + c_k)/2. \tag{8}$$

We also have

$$c_i + \delta_i \leq c_k + \min_{j\in J_1}\{\rho_j\} \leq c_k + \rho_k \leq (b^k + c_k)/2 \quad \forall i \in J_1 \text{ with } c_i \leq c_k. \tag{9}$$

Using the lower bounds on $\Phi(c+\delta,x)$ defined in (7) and (8), as well as (9), we obtain

$$\begin{aligned} \Phi(c+\delta,x) &\geq \max\{(b^k+c_k)/2,\ \max\{c_i+\delta_i \mid i \in J_1,\ c_i > c_k\} \\ &\geq \max\{c_i + \delta_i \mid i \in J_1\} = \Phi(c+\delta,\bar{x}). \end{aligned}$$

This establishes the inequality.

We will next show that for any ρ strictly greater than $\min_{j\in J_1}\{\rho_j\}$, there exists a vector δ such that $|\delta_i| \leq \rho$ for each i, while $\Phi(c+\delta,x) < \Phi(c+\delta,\bar{x})$ for some $x \in X$. To be more specific, suppose that $\rho > \rho_k$ for some k with $\bar{x}_k = 1$. Consider the vector δ where $\delta_i = -\rho_k$ if $x_i^k = 1$ and $c_i \geq (b^k + c_k)/2$, $\delta_k = \rho$ and $\delta_i = 0$ otherwise. Note that $\Phi(c,x^k) = b^k \geq \Phi(c,\bar{x}) \geq c_k$, which implies $\Phi(c,x^k) \geq (b^k+c_k)/2$. Therefore, $\Phi(c+\delta,x^k) = b^k - \rho_k = (b^k+c_k)/2 < c_k + \rho \leq \Phi(c+\delta,\bar{x})$. This establishes that the stability radius is at most $\min_{j\in J_1}\{\rho_j\}$ and completes the proof.

□

To compute b^j, we just need to solve the instance of (P) with objective vector $\tilde{c}$, where $\tilde{c}_j$ is equal to a value M, which is strictly greater than the largest of $c_1, c_2, \ldots, c_n$, and $\tilde{c}_i = c_i$ for $i \neq j$. If the optimal objective value of this problem instance turns out to be M, then $x_j = 1$ for each feasible solution and $b^j = \infty$. Otherwise, the optimal objective value is exactly b^j. It therefore follows from Theorem 3.1 that when all components are unstable, the stability radius can be calculated by solving $\sum_{i=1}^n \bar{x}_i$ instances of (P).

3.2 Stable and unstable components

Let us now permit w to be any arbitrary integer less than or equal to n. Assume without loss of generality that $c_{w+1} \leq c_{w+2} \leq \ldots \leq c_n$. We will compute the stability radius as the minimum of certain values $\hat{\rho}_j$, $j \in J_1$, which we will define below.

For $j \in J_1$, $j \leq w$, and $m \geq w$ we let $f^{(j,m)}$ denote the optimal value of the problem instance with objective vector c and the additional restrictions $x_j = 0$ and $x_i = 0$ for all $i > m$. If this instance does not have a feasible solution, then $f^{(j,m)}$ is set to ∞. Let $x^{(j,m)}$ be any optimal solution of this problem instance and let $d^{(j,m)}$ denote the value of the largest c_i, $i > w$ for which $x_i^{(j,m)} = 1$. We define $d^{(j,m)}$ to be $-\infty$ if $x_i^{(j,m)} = 0$ for all $i > w$. For $j \in J_1$, $j \leq w$, we now define $\hat{\rho}_j = \min_{m \geq w} \max\{(f^{(j,m)} - c_j)/2,\ d^{(j,m)} - c_j\}$.

To compute $f^{(j,m)}$, $j \in J_1$, $j \leq w$, $m \geq w$, we solve the instance of (P) with objective vector $\tilde{c}$, where $\tilde{c}_i = M$ for $i = j$ and all $i > m$, and $\tilde{c}_i = c_i$ for all other components. If the optimal objective value of this problem instance turns out to be M, then $f^{(j,m)} = \infty$. Otherwise, the optimal objective value is exactly $f^{(j,m)}$ and we obtain a solution $x^{(j,m)}$ and the corresponding value $d^{(j,m)}$. (To compute $\hat{\rho}_j$ it actually suffices to calculate $f^{(j,m)}$ in order of decreasing m until a value of m is reached for which $(f^{(j,m)} + c_j)/2, \geq d^{(j,m)}$, because $f^{(j,m)}$ is non–increasing in m.)

For $j \in J_1$, $j > w$, we let g^j denote the optimal value of the problem instance with objective vector c and the additional restrictions $x_j = 0$ and $x_i = 0$ for all $i > w$ with $c_i \geq c_j$; $g^j = \infty$ if this problem instance does not have a feasible solution. For $j \in J_1$, $j > w$, we now define $\hat{\rho}_j = g^j - c_j$. The calculation of g^j is obvious.

Lemma 3.1 *Suppose that $x \in X$ is a solution with $x_j = 0$ for some $j \in J_1$, $j \leq w$. Let l be the largest index such that $l > w$ with $x_l = 1$; define $c_l = -\infty$ if no such index exists. Then $c_j + \hat{\rho}_j \leq \max\{(\Phi(c,x) + c_j)/2,\ c_l\}$.*

Proof. Suppose $x_i = 0$ for all $i > w$, then $\Phi(c,x) \geq f^{(j,w)}$. Since $d^{(j,w)} = -\infty$, it follows that $c_j + \hat{\rho}_j \leq \max\{(f^{(j,w)} + c_j)/2,\ d^{(j,w)}\} \leq (\Phi(c,x) + c_j)/2$.

If $x_i = 1$ for some $i > w$, then $\Phi(c,x) \geq f^{(j,l)}$ and $c_l \geq d^{(j,l)}$. Therefore, $c_j + \hat{\rho}_j \leq$

$\max\{(f^{(j,l)} + c_j)/2,\ d^{(j,l)}\} \le \max\{\Phi(c,x) + c_j)/2,\ c_l\}$.

□

Theorem 3.2 *The stability radius is equal to* $\min_{j \in J_1}\{\hat{\rho}_j\}$.

Proof. We will first show that the stability radius is at least equal to $\min_{j\in J_1}\{\hat{\rho}_j\}$. Consider a vector δ such that $\delta_i = 0$ for all $i > w$, and $|\delta_i| \le \min_{j\in J_1}\{\hat{\rho}_j\}$ for all $i \le w$. We have

$$\Phi(c+\delta,\bar{x}) \ = \max\{ \begin{array}{l} \max\{c_i + \delta_i \mid i \in J_1,\ i \le w,\ c_i \le c_k\}, \\ \max\{c_i + \delta_i \mid i \in J_1,\ i \le w,\ c_i > c_k\}, \\ \max\{c_i \mid i \in J_1,\ i > w,\ x_i = 1\}, \\ \max\{c_i \mid i \in J_1,\ i > w,\ x_i = 0\} \qquad \} \end{array} \tag{10}$$

For any solution $x \in X$, we will show that the four expressions on the right hand side of (10) are all lower bounds on $\Phi(c+\delta,x)$.

Let $k = \mathrm{argmax}\{c_i \mid i \in J_1,\ i \le w,\ x_i = 0\}$. The first expression, $\max\{c_i + \delta_i \mid i \in J_1,\ i \le w,\ c_i \le c_k\}$, is less than or equal to $c_k + \min_{j\in J_1}\{\hat{\rho}_j\} \le c_k + \rho_k$. Because of Lemma 3.1 this is at most $\max\{(\Phi(c,x) + c_k)/2,\ c_l\}$, where c_l is defined as in the lemma. Clearly, $c_l \le \Phi(c+\delta,x)$. Furthermore, if $(\Phi(c,x)+c_k)/2 > c_l$, then $\Phi(c+\delta,x) \ge \Phi(c,x) - \hat{\rho}_k = (\Phi(c,x) + c_k)/2$.

To see that the second expression is a lower bound, it suffices to observe that if $i \in J_1$ and $c_i > c_k$, then $x_i = 1$. The third expression is an obvious lower bound.

Define $r = \mathrm{argmax}\{c_i \mid i \in J_1,\ i > w,\ x_i = 0\}$, then the fourth expression is equal to c_r. To show that this is a lower bound on $\Phi(c+\delta,x)$, we first note that this is certainly true if it is not greater than c_l. Now suppose that $c_r > c_l$, i.e., $x_i = 0$ for all $i > w$ with $c_i \ge c_r$. Then $\Phi(c,x) \ge g^r$, and we have $\Phi(c+\delta,x) \ge \Phi(c,x) - \hat{\rho}_r \ge g^r - \hat{\rho}_r = c_r$.

This establishes the desired inequality.

If we are given any $\hat{\rho}$ strictly greater than $\min_{j\in J_1}\{\hat{\rho}_j\}$, then we can find a vector δ such that $|\delta_i| \le \hat{\rho}$ for each $i \le w$, $\delta_i = 0$ for each $i > w$, while $\Phi(c+\delta,x) < \Phi(c+\delta,\bar{x})$

for some x. The argument is quite similar to that used in the proof of Theorem 3.1 and therefore we omit details. This completes the proof.

$\square$

The main result of this section is summarized as follows.

Theorem 3.3 *The stability radius of an optimal solution can be computed in polynomial time, if (P) has a min–max objective function and if it is polynomially solvable for any objective vector.*

3.3 Extensions

A straightforward extension to the tolerance approach is possible for min–max problems as well. In the case in which $w = n$, i.e., all components are unstable, we define $\gamma_j = (b^j - c_j)/(b^j + c_j)$ and the tolerance radius is equal to $\min_{j \in J_1}\{\gamma_j\}$. The proof is quite similar to that of Theorem 3.1 and is therefore omitted. The more general case, in which w is an arbitrary integer between 1 and n may be dealt with quite similarly.

It appears that the stability radius of an ϵ–optimal solution to a min–max problem may be determined by techniques which are conceptually similar, but more intricate than the ones presented in this section. We have therefore refrained from carrying out a full investigation of this topic.

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