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Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants

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Abstract. Let $\{F_n : n \geqslant 1\}$ be a normalized sequence of random variables in some fixed Wiener chaos associated with a general Gaussian field, and assume that $E[F_n^4] \to E[N^4] = 3$, where N is a standard Gaussian random variable. Our main result is the following general bound: there exist two finite constants $c, C > 0$ such that, for n sufficiently large, $c \times \max(|E[F_n^3]|, E[F_n^4] - 3) \leqslant d(F_n, N) \leqslant C \times \max(|E[F_n^3]|, E[F_n^4] - 3)$, where $d(F_n, N) = \sup |E[h(F_n)] - E[h(N)]|$, and h runs over the class of all real functions with a second derivative bounded by 1. This shows that the deterministic sequence $\max(|E[F_n^3]|, E[F_n^4] - 3)$, $n \geqslant 1$, completely characterizes the rate of convergence (with respect to smooth distances) in CLTs involving chaotic random variables. These results are used to determine optimal rates of convergence in the Breuer-Major central limit theorem, with specific emphasis on fractional Gaussian noise.

1. Introduction and main results

Let $X = \{X(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process (defined on an adequate space $(\Omega, \mathcal{F}, P)$) over some real separable Hilbert space $\mathfrak{H}$, fix an integer $q \geqslant 2$, and let $\{F_n : n \geqslant 1\}$ be a sequence of random variables belonging to the qth

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Wiener chaos of X (see Section 2.1 for precise definitions). Assume that $E[F_n^2]=1$ for every n. In recent years, many efforts have been devoted to the characterization of those chaotic sequences $\{F_n\}$ verifying a Central Limit Theorem (CLT), that is, such that F_n converges in distribution to $N \sim \mathscr{N}(0,1)$ (as $n\to\infty$), where $\mathscr{N}(0,1)$ denotes a centered Gaussian law with unit variance. An exhaustive solution to this problem was first given by Nualart and Peccati in Nualart and Peccati (2005), in the form of the following "fourth moment theorem".

Theorem 1.1 (Fourth Moment Theorem – see Nualart and Peccati (2005))**.** *Fix an integer $q\geqslant 2$, and consider a sequence of random variables $\{F_n : n\geqslant 1\}$ belonging to the qth Wiener chaos of X and such that $E[F_n^2]=1$ for all $n\geqslant 1$. Then, as $n\to\infty$, F_n converges in distribution to $N\sim\mathscr{N}(0,1)$ if and only if $E[F_n^4]\to E[N^4]=3$.*

Note that Theorem 1.1 represents a drastic simplification of the usual *method of moments and cumulants*, as described e.g. in Peccati and Taqqu (2011). Combining the so-called *Stein's method* for normal approximations (see Chen et al. (2011); I. Nourdin (2012), as well as Section 3.1 below) with Malliavin calculus (see Malliavin (1997); Nualart (2006), as well as Section 2.2), one can also prove the forthcoming Theorem 1.2, providing explicit upper bounds in the *total variation distance*. We recall that the total variation distance $d_{TV}(F,G)$ between the laws of two real-valued random variables F,G is defined as

$$d_{TV}(F,G)=\sup_{A\in\mathscr{B}(\mathbb{R})}\left|P[F\in A]-P[G\in A]\right|,$$

where the supremum runs over the class of all Borel sets $A\subset\mathbb{R}$. Given a smooth functional of the isonormal process X, we shall also write DF to indicate the Malliavin derivative of F (thus DF is a $\mathfrak{H}$-valued random element – see again Section 2.2 for details).

Theorem 1.2 (Fourth Moment Bounds – see Nourdin and Peccati (2009b); Nourdin et al. (2010))**.** *Fix $q\geqslant 2$, let F be an element of the qth Wiener chaos of X with unit variance, and let $N\sim\mathscr{N}(0,1)$. The following bounds are in order:*

$$d_{TV}(F,N)\leqslant 2\sqrt{E\left[\left(1-\frac{1}{q}\|DF\|^2_{\mathfrak{H}}\right)^2\right]}\leqslant 2\sqrt{\frac{q-1}{3q}\left(E[F^4]-3\right)}. \qquad (1.1)$$

Remark 1.3. (1) The two inequalities in (1.1) were discovered, respectively, in Nourdin and Peccati (2009b) and Nourdin et al. (2010). Using the properties of the Malliavin derivative DF (see Section 2.2 below), one sees immediately that

$$E\left[\left(1-\frac{1}{q}\|DF\|^2_{\mathfrak{H}}\right)^2\right]=\mathrm{Var}\left(\frac{1}{q}\|DF\|^2_{\mathfrak{H}}\right).$$

(2) One can prove the following refinement of the second inequality in (1.1) (see Nourdin and Peccati (2010b, Lemma 3.5)): for every random variable F belonging to the qth Wiener chaos of X and with unit variance

$$\mathrm{Var}\left(\frac{1}{q}\|DF\|^2_{\mathfrak{H}}\right)\leqslant\frac{q-1}{3q}\left(E[F^4]-3\right)\leqslant(q-1)\mathrm{Var}\left(\frac{1}{q}\|DF\|^2_{\mathfrak{H}}\right). \qquad (1.2)$$

(3) Theorem 1.2 implies that, not only the condition $E[F_n^4] \to 3$ is necessary and sufficient for convergence to Gaussian, as stated in Theorem 1.1, but also that the sequences

$$\beta(n) := \sqrt{E[F_n^4] - 3} \quad \text{and} \quad \gamma(n) := \sqrt{\operatorname{Var}\left(\frac{1}{q}\|DF_n\|^2_{\mathfrak{H}}\right)}, \quad n \geqslant 1, \tag{1.3}$$

bound from above (up to a constant) the speed of convergence of the law of F_n to that of N in the topology induced by d_{TV}.

(4) If one replaces the total variation distance with the Kolmogorov distance or with the Wasserstein distance (see e.g. Nourdin and Peccati (2009b); I. Nourdin (2012) for definitions), then the bounds (1.1) hold without the multiplicative factor 2 before the square roots.

(5) When $E[F] = 0$ and $E[F^2] = 1$, the quantity $E[F^4] - 3$ coincides with the *fourth cumulant* of F, see Definition 3.3. One can also prove that, if F belongs to a fixed Wiener chaos and has unit variance, then $E[F^4] > 3$ (see Nualart and Peccati (2005)).

(6) Throughout the paper, in order to simplify the notation, we only consider sequences of random variables having unit variance. The extension to arbitrary sequences whose variances converge to a constant can be deduced by a straightforward adaptation of our arguments.

A natural problem is now the following.

Problem 1.4. *Assume that $\{F_n\}$ is a unit variance sequence belonging to the qth Wiener chaos of the isonormal Gaussian process X. Suppose that F_n converges in distribution to $N \sim \mathscr{N}(0,1)$ and fix a distance $d_0(\cdot,\cdot)$ between the laws of real-valued random variables. Can one find an explicit optimal rate of convergence associated with the distance d_0?*

The notion of optimality adopted throughout the paper is contained in the next definition.

Definition 1.5. Assume that, as $n \to \infty$, F_n converges in distribution to N, and fix a generic distance $d_0(\cdot,\cdot)$ between the laws of real-valued random variables. A deterministic sequence $\{\varphi(n) : n \geqslant 1\}$ such that $\varphi(n) \downarrow 0$ is said to provide an *optimal rate of convergence* with respect to d_0 if there exist constants $0 < c < C < \infty$ (not depending on n) such that, for n large enough,

$$c \leqslant \frac{d_0(F_n, N)}{\varphi(n)} \leqslant C. \tag{1.4}$$

The problem of finding optimal rates is indeed quite subtle. A partial solution to Problem 1.4 is given by Nourdin and Peccati in Nourdin and Peccati (2009a). In this reference, a set of sufficient conditions are derived, ensuring that the sequences $\beta(n), \gamma(n)$ in (1.3) yield optimal rates for the distance $d_0 = d_{TV}$. In particular, these conditions involve the joint convergence of the two-dimensional vectors

$$\left(F_n, \frac{1 - q^{-1}\|DF_n\|^2_{\mathfrak{H}}}{\gamma(n)}\right), \quad n \geqslant 1. \tag{1.5}$$

The following statement constitutes one of the main finding of Nourdin and Peccati (2009a) (note that the reference Nourdin and Peccati (2009a) only deals with the Kolmogorov distance but, as far as lower bounds are concerned, it is no more difficult to work directly with d_{TV}).

Theorem 1.6 (See Nourdin and Peccati (2009a))**.** *Let $\{F_n\}$ be a unit variance sequence belonging to the qth Wiener chaos of X, and suppose that, as $n \to \infty$, F_n converges in distribution to $N \sim \mathscr{N}(0,1)$. Assume moreover that the sequence of two-dimensional random vectors in (1.5) converges in distribution to a Gaussian vector (N_1, N_2) such that $E[N_1^2] = E[N_2^2] = 1$, and $E[N_1 N_2] =: \rho \in (-1, 1)$. Then, for every $z \in \mathbb{R}$:*

$$\gamma(n)^{-1}[P(F_n \leqslant z) - P(N \leqslant z)] \to \frac{\rho}{3}(z^2 - 1)\frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{as } n \to \infty. \tag{1.6}$$

In particular, if $\rho \neq 0$ the sequences $\beta(n), \gamma(n)$ defined in (1.3) provide optimal rates of convergence with respect to the total variation distance d_{TV}, in the sense of Definition 1.5.

As shown in Nourdin and Peccati (2009a, Theorem 3.1) the conditions stated in Theorem 1.6 can be generalized to arbitrary sequences of smooth random variables (not necessarily belonging to a fixed Wiener chaos). Moreover, the content of Theorem 1.6 can be restated in terms of contractions (see Nourdin and Peccati (2009a, Theorem 3.6)) or, for elements of the second Wiener chaos of X, in terms of cumulants (see Nourdin and Peccati (2009a, Proposition 3.8)).

One should note that the techniques developed in Nourdin and Peccati (2009a) also imply analogous results for the Kolmogorov and the Wasserstein distances, that we shall denote respectively by d_{Kol} and d_W. However, although quite flexible and far-reaching, the findings of Nourdin and Peccati (2009a) *do not allow* to deduce a complete solution (that is, a solution valid for arbitrary sequences $\{F_n\}$ in a fixed Wiener chaos) of Problem 1.4 for either one of the distances d_{TV}, d_{Kol} and d_W. For instance, the results of Nourdin and Peccati (2009a) provide optimal rates in the Breuer-Major CLT only when the involved subordinating function has an *even* Hermite rank, whereas the general case remained an open problem till now – see Nourdin and Peccati (2009a, Section 6).

The aim of this paper is to provide an exhaustive solution to Problem 1.4 in the case of a suitable smooth distance between laws of real-valued random variables. The distance we are interested in is denoted by $d(\cdot,\cdot)$, and involves test functions that are twice differentiable. The formal definition of d is given below.

Definition 1.7. Given two random variables F, G with finite second moments we write $d(F,G)$ in order to indicate the quantity

$$d(F,G) = \sup_{h \in \mathcal{U}} \big| E[h(F)] - E[h(G)] \big|,$$

where $\mathcal{U}$ stands for the set of functions $h : \mathbb{R} \to \mathbb{R}$ which are $\mathcal{C}^2$ (that is, twice differentiable and with continuous derivatives) and such that $\|h''\|_\infty \leqslant 1$.

Observe that $d(\cdot,\cdot)$ defines an actual distance on the class of the distributions of random variables having a finite second moment. Also, the topology induced by d on this class is stronger than the topology of the convergence in distribution, that is: if $d(F_n, G) \to 0$, then F_n converges in distribution to G.

The forthcoming Theorem 1.9 and Theorem 1.11 contain the principal upper and lower bounds proved in this work: once merged, they show that the sequence

$$\max\left\{|E[F_n^3]|, E[F_n^4] - 3\right\}, \quad n \geqslant 1, \tag{1.7}$$

always provides optimal rates for the distance d, whenever $\{F_n\}$ lives inside a fixed Wiener chaos. As anticipated, this yields an exhaustive solution to Problem 1.4

in the case $d_0 = d$. One should also note that the speed of convergence to zero of the quantity (1.7) can be given by either one of the two sequences $\{|E[F_n^3]|\}$ and $\{E[F_n^4] - 3\}$; see indeed Corollary 6.8 for explicit examples of both situations.

Remark 1.8. Let $\{F_n : n \geqslant 1\}$ be a sequence of random variables living inside a finite sums of Wiener chaoses. Assume that $E[F_n^2] \to 1$ and that F_n converges in distribution to $N \sim \mathscr{N}(0,1)$. Then, the hypercontractivity property of the Wiener chaos (see e.g. Janson (1997, Chapter V)) imply that $E[F_n^k] \to E[N^k]$ for every integer $k \geqslant 3$. In particular, one has necessarily that $E[F_n^3] \to 0$.

Theorem 1.9 (Upper bounds)**.** *Let $N \sim \mathscr{N}(0,1)$ be a standard Gaussian random variable. Then, there exists $C > 0$ such that, for all integer $q \geqslant 2$ and all element F of the qth Wiener chaos with unit variance,*

$$d(F,N) \leqslant C \max\left\{|E[F^3]|, E[F^4] - 3\right\}. \tag{1.8}$$

Remark 1.10. (1) In the statement of Theorem 1.9, we may assume without loss of generality that N is stochastically independent of F. Then, by suitably using integration by parts and then Cauchy-Schwarz (see e.g. Nourdin et al. (2011, Theorem 3.2)), one can show that, for every $h \in \mathcal{U}$ (see Definition 1.7),

$$\begin{aligned} \left|E[h(F)] - E[h(N)]\right| &= \frac{1}{2}\left|\int_0^1 E\left[h''(\sqrt{1-t}F + \sqrt{t}N)\left(1 - \frac{1}{q}\|DF\|_{\mathfrak{H}}^2\right)\right] dt\right| \\ &\leqslant \frac{1}{2}\sqrt{\operatorname{Var}\left(\frac{1}{q}\|DF\|_{\mathfrak{H}}^2\right)}. \end{aligned} \tag{1.9}$$

Since (1.2) is in order, one sees that the inequality (1.9) does not allow to obtain a better bound than

$$d(F,N) \leqslant C\sqrt{E[F^4] - 3},$$

which is not sharp in general, compare indeed with (1.8). One should observe that the rate $\sqrt{E[F_n^4] - 3}$ may happen to be optimal in some instances, precisely when $E[F_n^3]$ and $\sqrt{E[F_n^4] - 3}$ have the same order. In the already quoted paper Nourdin and Peccati (2009a) one can find several explicit examples where this phenomenon takes place.

(2) Let F be an element of the qth Wiener chaos of some isonormal Gaussian process, and assume that F has variance 1. It is shown in Nourdin and Peccati (2010b, Proposition 3.14) that there exists a constant C, depending only on q, such that $|E[F^3]| \leqslant C\sqrt{E[F^4] - 3}$. Using this fact, one can therefore obtain yet another proof of Theorem 1.1 based on the upper bound (1.8).

Theorem 1.11 (Lower bounds)**.** *Fix an integer $q \geqslant 2$ and consider a sequence of random variables $\{F_n : n \geqslant 1\}$ belonging to the qth Wiener chaos of some isonormal Gaussian process and such that $E[F_n^2] = 1$. Assume that, as $n \to \infty$, F_n converges in distribution to $N \sim \mathscr{N}(0,1)$. Then there exists $c > 0$ (depending on the sequence $\{F_n\}$, but not on n) such that*

$$d(F_n, N) \geqslant c \times \max\left\{|E[F_n^3]|, E[F_n^4] - 3\right\}, \quad n \geqslant 1. \tag{1.10}$$

Our proofs revolve around several new estimates (detailed in Section 4), that are in turn based on the analytic characterization of cumulants given in Nourdin and Peccati (2010a). Also, a fundamental role is played by the Edgeworth-type expansions introduced by Barbour in Barbour (1986).

1.1. *Plan.* The paper is organized as follows. Section 2 is devoted to some preliminary results of Gaussian analysis and Malliavin calculus. Section 3 deals with Stein's method, cumulants and Edgeworth-type expansions. Section 4 contains the main technical estimates of the paper. Section 5 focuses on the proofs of our main findings, whereas in Section 6 one can find several applications to the computation of optimal rates in the Breuer-Major CLT.

2. Elements of Gaussian analysis and Malliavin calculus

This section contains the essential elements of Gaussian analysis and Malliavin calculus that are used in this paper. See the classical references Malliavin (1997); Nualart (2006) for further details.

2.1. *Isonormal processes and multiple integrals.* Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geqslant 1$, we write $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ to indicate, respectively, the qth tensor power and the qth symmetric tensor power of $\mathfrak{H}$; we also set by convention $\mathfrak{H}^{\otimes 0} = \mathfrak{H}^{\odot 0} = \mathbb{R}$. When $\mathfrak{H} = L^2(A, \mathcal{A}, \mu) =: L^2(\mu)$, where μ is a σ-finite and non-atomic measure on the measurable space $(A, \mathcal{A})$, then $\mathfrak{H}^{\otimes q} = L^2(A^q, \mathcal{A}^q, \mu^q) =: L^2(\mu^q)$, and $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^q, \mu^q) := L^2_s(\mu^q)$, where $L^2_s(\mu^q)$ stands for the subspace of $L^2(\mu^q)$ composed of those functions that are μ^q-almost everywhere symmetric. We denote by $X = \{X(h) : h \in \mathfrak{H}\}$ an *isonormal Gaussian process* over $\mathfrak{H}$. This means that X is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, P)$, with a covariance structure given by the relation $E[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$. We also assume that $\mathcal{F} = \sigma(X)$, that is, $\mathcal{F}$ is generated by X.

For every $q \geqslant 1$, the symbol $\mathcal{H}_q$ stands for the qth *Wiener chaos* of X, defined as the closed linear subspace of $L^2(\Omega, \mathcal{F}, P) =: L^2(\Omega)$ generated by the family $\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial given by

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\big(e^{-\frac{x^2}{2}}\big). \tag{2.1}$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $q \geqslant 1$, the mapping $I_q(h^{\otimes q}) = H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and the qth Wiener chaos $\mathcal{H}_q$. For $q = 0$, we write $I_0(c) = c$, $c \in \mathbb{R}$. A crucial fact is that, when $\mathfrak{H} = L^2(\mu)$, for every $f \in \mathfrak{H}^{\odot q} = L^2_s(\mu^q)$ the random variable $I_q(f)$ coincides with the q-fold multiple Wiener-Itô stochastic integral of f with respect to the centered Gaussian measure (with control μ) canonically generated by X (see Nualart (2006, Section 1.1.2)).

It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. It follows that any square-integrable random variable $F \in L^2(\Omega)$ admits the following *Wiener-Itô chaotic expansion*

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.2}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. For every $q \geqslant 0$, we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.2), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

Let $\{e_k,\, k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.3}$$

Notice that the definition of $f \otimes_r g$ does not depend on the particular choice of $\{e_k,\, k \geqslant 1\}$, and that $f \otimes_r g$ is not necessarily symmetric; we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. When $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ and $r = 1, ..., p \wedge q$, the contraction $f \otimes_r g$ is the element of $L^2(\mu^{p+q-2r})$ given by

$$\begin{aligned} & f \otimes_r g(x_1, ..., x_{p+q-2r}) \qquad (2.4) \\ &= \int_{A^r} f(x_1, ..., x_{p-r}, a_1, ..., a_r) g(x_{p-r+1}, ..., x_{p+q-2r}, a_1, ..., a_r) d\mu(a_1)...d\mu(a_r). \end{aligned}$$

It can also be shown that the following *multiplication formula* holds: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{2.5}$$

2.2. *Malliavin operators.* We now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X. Let $\mathcal{S}$ be the set of all cylindrical random variables of the form

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right), \tag{2.6}$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function such that its partial derivatives have polynomial growth, and $\phi_i \in \mathfrak{H}$, $i = 1, \ldots, n$. The *Malliavin derivative* of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\left(X(\phi_1), \ldots, X(\phi_n)\right) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the mth derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geqslant 2$. For $m \geqslant 1$ and $p \geqslant 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p \;=\; E\left[|F|^p\right] + \sum_{i=1}^{m} E\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

We often use the notation $\mathbb{D}^\infty := \bigcap_{m \geqslant 1} \bigcap_{p \geqslant 1} \mathbb{D}^{m,p}$.

Remark 2.1. Any random variable Y that is a finite linear combination of multiple Wiener-Itô integrals is an element of $\mathbb{D}^\infty$. Moreover, if $Y \neq 0$, then the law of Y admits a density with respect to the Lebesgue measure – see Shigekawa (1980).

The Malliavin derivative D obeys the following *chain rule*. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\,\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i. \tag{2.7}$$

Remark 2.2. By approximation, it is easily checked that equation (2.7) continues to hold in the following two cases: (i) $F_i \in \mathbb{D}^\infty$ and φ has continuous partial derivatives with at most polynomial growth, and (ii) $F_i \in \mathbb{D}^{1,2}$ has an absolutely continuous distribution and φ is Lipschitz continuous.

Note also that a random variable F as in (2.2) is in $\mathbb{D}^{1,2}$ if and only if $\sum_{q=1}^\infty q\|J_qF\|^2_{L^2(\Omega)} < \infty$ and, in this case, $E\left[\|DF\|^2_{\mathfrak{H}}\right] = \sum_{q=1}^\infty q\|J_qF\|^2_{L^2(\Omega)}$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.2) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_xF = \sum_{q=1}^{\infty} qI_{q-1}\left(f_q(\cdot, x)\right), \quad x \in A. \tag{2.8}$$

We denote by δ the adjoint of the operator D, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, noted $\mathrm{Dom}\,\delta$, if and only if it verifies $|E\langle DF, u\rangle_{\mathfrak{H}}| \leqslant c_u\,\|F\|_{L^2(\Omega)}$ for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (customarily called *integration by parts formula*)

$$E[F\delta(u)] = E[\langle DF, u\rangle_{\mathfrak{H}}], \tag{2.9}$$

which holds for every $F \in \mathbb{D}^{1,2}$.

The operator L, defined as $L = \sum_{q=0}^\infty -qJ_q$, is the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. The domain of L is

$$\mathrm{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2\,\|J_qF\|^2_{L^2(\Omega)} < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators D, δ and L. A random variable F belongs to $\mathbb{D}^{2,2}$ if and only if $F \in \mathrm{Dom}\,(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$) and, in this case,

$$\delta DF = -LF. \tag{2.10}$$

For any $F \in L^2(\Omega)$, we define $L^{-1}F = \sum_{q=1}^\infty -\frac{1}{q}J_q(F)$. The operator L^{-1} is called the *pseudo-inverse* of L. Indeed, for any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \mathrm{Dom}L = \mathbb{D}^{2,2}$, and

$$LL^{-1}F = F - E(F). \tag{2.11}$$

The following result, whose content appears in a slightly different form in Üstünel (1995, p. 62), is used throughout the paper.

Lemma 2.3. *Suppose that $H \in \mathbb{D}^{1,2}$ and $G \in L^2(\Omega)$. Then, $L^{-1}G \in \mathbb{D}^{2,2}$ and*

$$E[HG] = E[H]E[G] + E[\langle DH, -DL^{-1}G\rangle_{\mathfrak{H}}]. \tag{2.12}$$

Proof. By (2.10) and (2.11),

$$E[HG] - E[H]E[G] = E[H(G - E[G])] = E[H \times LL^{-1}G] = E[H\delta(-DL^{-1}G)],$$

and the result is obtained by using the integration by parts formula (2.9). □

3. Stein's equations and cumulants

In order to prove our main results, we shall combine the integration by parts formula of Malliavin calculus, both with a standard version of the Stein's method for normal approximations (see Chen et al. (2011) for an exhaustive presentation of this technique) and with a fine analysis of the cumulants associated with random variables living in a fixed chaos. One of our main tools is an Edgeworth-type expansion (inspired by Barbour's paper Barbour (1986)) for smooth transformations of Malliavin differentiable random variables. These fundamental topics are presented in the three subsections to follow.

3.1. *Stein's equations and associated bounds.* Let $N \sim \mathscr{N}(0,1)$ be a standard Gaussian random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Remark 3.1. In the literature about Stein's method and normal approximation, it is customary at this stage to assume that h is merely Borel measurable. However, this leads to some technical issues that are not necessary here. See e.g. I. Nourdin (2012, Chapter 3).

We associate with h the following *Stein's equation*:

$$h(x) - E[h(N)] = f'(x) - xf(x), \quad x \in \mathbb{R}. \tag{3.1}$$

It is easily checked that, if $E|h(N)| < \infty$, then the function

$$f_h(x) = e^{x^2/2} \int_{-\infty}^{x} (h(y) - E[h(N)]) e^{-y^2/2} dy, \quad x \in \mathbb{R}, \tag{3.2}$$

is the unique solution of (3.1) verifying the additional asymptotic condition

$$\lim_{x \to \pm\infty} f_h(x) e^{-x^2/2} = 0.$$

In this paper, we will actually deal with Stein's equations associated with functions h that are differentiable up to a certain order. The following statement (proved by Daly in Daly (2008)) is an important tool for our analysis. Throughout the following, given a smooth function $g : \mathbb{R} \to \mathbb{R}$, we shall denote by $g^{(k)}$, $k = 1, 2, ...$, the kth derivative of g; we sometimes write $g' = g^{(1)}$, $g'' = g^{(2)}$, and so on.

Proposition 3.2. *Let the previous notation prevail, fix an integer $k \geqslant 0$, and assume that the function h is $(k+1)$-times differentiable and such that $h^{(k)}$ is absolutely continuous. Then, f_h is $(k+2)$-times differentiable, and one has the estimate*

$$\|f_h^{(k+2)}\|_\infty \leqslant 2\|h^{(k+1)}\|_\infty. \tag{3.3}$$

Moreover, the continuity of $h^{(k+1)}$ implies the continuity of $f_h^{(k+2)}$.

Proof. The first part, i.e., inequality (3.3), is exactly Theorem 1.1 of Daly (2008), whereas the transfer of continuity is easily checked by induction and by using (3.1). □

3.2. *Cumulants.* We now formally define the cumulants associated with a random variable.

Definition 3.3 (Cumulants)**.** Let F be a real-valued random variable such that $E|F|^m < \infty$ for some integer $m \geqslant 1$, and define $\phi_F(t) = E[e^{itF}]$, $t \in \mathbb{R}$, to be the characteristic function of F. Then, for $j = 1, ..., m$, the jth *cumulant* of F, denoted by $\kappa_j(F)$, is given by

$$\kappa_j(F) = (-i)^j \frac{d^j}{dt^j} \log \phi_F(t)|_{t=0}. \tag{3.4}$$

Remark 3.4. The first four cumulants are the following: $\kappa_1(F) = E[F]$, $\kappa_2(F) = E[F^2] - E[F]^2 = \mathrm{Var}(F)$, $\kappa_3(F) = E[F^3] - 3E[F^2]E[F] + 2E[F]^3$, and

$$\kappa_4(F) = E[F^4] - 3E[F]E[F^3] - 3E[F^2]^2 + 12E[F]^2E[F^2] - 6E[F]^4.$$

In particular, when $E[F] = 0$ one sees that $\kappa_3(F) = E[F^3]$ and $\kappa_4(F) = E[F^4] - 3E[F^2]^2$.

The reader is referred to Peccati and Taqqu (2011, Chapter 3) for a self-contained presentation of the properties of cumulants and for several combinatorial characterizations. The following relation shows that moments can be recursively defined in terms of cumulants (and vice-versa): fix $m = 1, 2...$, and assume that $E|F|^{m+1} < \infty$, then

$$E[F^{m+1}] = \sum_{s=0}^{m} \binom{m}{s} \kappa_{s+1}(F) E[F^{m-s}]. \tag{3.5}$$

Remark 3.5. It is not difficult to deduce formula (3.5) from the classical *Leonov and Shyraev relations* between moments and cumulants – see Leonov and Sirjaev (1959) for the original reference, as well as Peccati and Taqqu (2011, Proposition 3.2.1). A direct proof of (3.5), based on characteristic functions, can be found e.g. in Smith (1995, Section 2), Nourdin and Peccati (2010a, Proposition 2.2) or Peccati and Taqqu (2011, Corollary 3.2.2).

We now want to characterize cumulants in terms of Malliavin operators. To do so, we need the following recursive definition (taken from Nourdin and Peccati (2010a)).

Definition 3.6. Let $F \in \mathbb{D}^\infty$. The sequence of random variables $\{\Gamma_j(F) : j \geqslant 0\} \subset \mathbb{D}^\infty$ is recursively defined as follows. Set $\Gamma_0(F) = F$ and, for every $j \geqslant 1$,

$$\Gamma_j(F) = \langle DF, -DL^{-1}\Gamma_{j-1}(F)\rangle_{\mathfrak{H}}.$$

Note that each $\Gamma_j(F)$ is a well-defined element of $\mathbb{D}^\infty$, since F is assumed to be in $\mathbb{D}^\infty$ – see Nourdin and Peccati (2010a, Lemma 4.2(3))

For instance, one has that $\Gamma_1(F) = \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$. The following statement provides two explicit relations ((3.6) and (3.7)) connecting the random variables $\Gamma_j(F)$ to the cumulants of F. Equation (3.6) has been proved in Nourdin and Peccati (2010a, Theorem 4.3), whereas (3.7) is new.

Proposition 3.7. *Let $F \in \mathbb{D}^\infty$. Then F has finite moments of every order, and the following relation holds for every $s \geqslant 0$:*

$$\kappa_{s+1}(F) = s!E[\Gamma_s(F)]. \tag{3.6}$$

If moreover $E(F) = 0$ then, for every $s \geqslant 1$,

$$\kappa_{s+2}(F) = \frac{1}{2}(s+1)!\, E\left[F^2\left(\Gamma_{s-1}(F) - \frac{\kappa_s(F)}{(s-1)!}\right)\right]. \tag{3.7}$$

Proof. In view of Nourdin and Peccati (2010a, Theorem 4.3), we have only to prove (3.7). Applying Lemma 2.3 in the special case $H = F^2$ and $G = \Gamma_{s-1}(F)$, and using the relation $DF^2 = 2FDF$, one deduces that

$$E[F^2\Gamma_{s-1}(F)] = E[F^2]E[\Gamma_{s-1}(F)] + 2E[F\Gamma_s(F)].$$

Now apply Lemma 2.3 in the case $H = F$ and $G = \Gamma_s(F)$: exploiting the fact that F is centered together with (3.6), we infer that

$$E[F\Gamma_s(F)] = E[F(\Gamma_s(F) - E[\Gamma_s(F)])] = E[\Gamma_{s+1}(F)] = \frac{\kappa_{s+2}(F)}{(s+1)!}. \tag{3.8}$$

Since (3.6) implies that $(s-1)!E[\Gamma_{s-1}(F)] = \kappa_s(F)$, the conclusion follows. □

Remark 3.8. (1) Relation (3.6) continues to hold under weaker assumptions on the regularity of F. See again Nourdin and Peccati (2010a, Theorem 4.3).

(2) Relation (3.6) generalizes the following well-known fact: if $F \in \mathbb{D}^{1,2}$, then $\Gamma_1(F) = \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$ is in $L^1(\Omega)$ and

$$\mathrm{Var}(F) = E[\Gamma_1(F)]. \tag{3.9}$$

The following statement provides an explicit expression for $\Gamma_s(F)$, $s \geqslant 1$, when F has the form of a multiple integral.

Proposition 3.9 (see Nourdin and Peccati (2010a), formula (5.25))**.** *Let $q \geqslant 2$, and assume that $F = I_q(f)$ with $f \in \mathfrak{H}^{\odot q}$. Then, for any $s \geqslant 1$, we have*

$$\Gamma_s(F) = \sum_{r_1=1}^{q} \cdots \sum_{r_s=1}^{[sq-2r_1-\ldots-2r_{s-1}]\wedge q} c_q(r_1,\ldots,r_s)\mathbf{1}_{\{r_1<q\}}\cdots \mathbf{1}_{\{r_1+\ldots+r_{s-1}<\frac{sq}{2}\}}\times I_{(s+1)q-2r_1-\ldots-2r_s}\big((\ldots(f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\ldots f)\widetilde{\otimes}_{r_s}f\big), \tag{3.10}$$

where the constants $c_q(r_1,\ldots,r_{s-2})$ are recursively defined as follows:

$$c_q(r) = q(r-1)!\binom{q-1}{r-1}^2,$$

and, for $a \geqslant 2$,

$$c_q(r_1,\ldots,r_a) = q(r_a-1)!\binom{aq-2r_1-\ldots-2r_{a-1}-1}{r_a-1}\binom{q-1}{r_a-1}c_q(r_1,\ldots,r_{a-1}).$$

Remark 3.10. By combining (3.6) with (3.10), we immediately get a representation of cumulants that is alternative to the one based on 'diagram formulae'. See Nourdin and Peccati (2010a, Theorem 5.1) for details on this point.

3.3. *Assessing Edgeworth-type expansions.* The following Edgeworth-type expansion also plays a crucial role in the following.

Proposition 3.11. *Let F be an element of $\mathbb{D}^\infty$. Then, for every $M \geqslant 1$ and every function $f : \mathbb{R} \to \mathbb{R}$ that is M times continuously differentiable with derivatives having at most polynomial growth, we have*

$$E[Ff(F)] = \sum_{s=0}^{M-1} \frac{\kappa_{s+1}(F)}{s!} E[f^{(s)}(F)] + E[\Gamma_M(F) f^{(M)}(F)]. \tag{3.11}$$

Proof. Using twice Lemma 2.3, first in the case $H = F$ and $G = f(F)$ and then in the case $F = \Gamma_1(F)$ and $G = f'(F)$, we deduce that

$$\begin{aligned} E[Ff(F)] &= E[F]E[f(F)] + E[f'(F)\Gamma_1(F)] \\ &= E[F]E[f(F)] + E[f'(F)]E[\Gamma_1(F)] + E[f''(F)\Gamma_2(F)], \end{aligned}$$

where we have used the chain rule (2.7) as well as Remark 2.2. Therefore, (3.11) holds for $M = 1, 2$ (see also (3.9)). The case of a general M follows immediately from an induction argument and by using (3.6). □

The following statements contain two important consequences of (3.11). They will be used in order to prove our main findings.

Corollary 3.12. *Let $N \sim \mathscr{N}(0,1)$ and fix $F \in \mathbb{D}^\infty$ such that $E[F] = 0$, $E[F^2] = \kappa_2(F) = 1$. For $M \geqslant 2$, let $h : \mathbb{R} \to \mathbb{R}$ be $(M-1)$ times continuously differentiable with bounded derivatives, and define f_h according to (3.2). Then,*

$$\left| E[h(N)] - E[h(F)] - \sum_{s=2}^{M-1} \frac{\kappa_{s+1}(F)}{s!} E[f_h^{(s)}(F)] \right| \leqslant 2\|h^{(M-1)}\|_\infty E|\Gamma_M(F)|. \tag{3.12}$$

Proof. From Proposition 3.2, we deduce that the function f_h is M-times continuously differentiable and that, for $k = 2, ..., M$, $\|f_h^{(k)}\|_\infty \leqslant 2\|h^{(k-1)}\|_\infty$. Using a Taylor expansion, we deduce that f_h' has at most polynomial growth. It follows that (3.11) can be applied to the function f_h, and the conclusion is obtained from the relation $E[h(N)] - E[h(F)] = E[Ff_h(F)] - E[f_h'(F)]$. □

Corollary 3.13. *Let $N \sim \mathscr{N}(0,1)$ and fix $F \in \mathbb{D}^\infty$ such that $E[F] = 0$ and $E[F^2] = \kappa_2(F) = 1$. Let $h : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and such that $\|h''\|_\infty \leqslant 1$, and define f_h according to (3.2). Then,*

$$\big|E[h(F)] - E[h(N)]\big| \leqslant K|E[F^3]| + 2E|\Gamma_3(F)|; \tag{3.13}$$

where $K := 1 + E[|F|]$.

Proof. We first observe that $E[h(F)] - E[h(N)] = E[\tilde{h}(F)] - E[\tilde{h}(N)]$, where $\tilde{h}(x) = h(x) - h(0) - h'(0)x$, so that we can assume without loss of generality that $h(0) = h'(0) = 0$. Thus, because $\|h''\|_\infty \leqslant 1$, we get that $|h(x)| \leqslant \frac{x^2}{2}$ and $|h'(x)| \leqslant |x|$ for all $x \in \mathbb{R}$, while $|E[h(N)]| \leqslant \frac{1}{2}$. It follows from (3.2) that

$$|f_h(0)| \leqslant \int_0^\infty \left(\frac{y^2}{2} + \frac{1}{2}\right) e^{-y^2/2} dy = \frac{\sqrt{2\pi}}{2} \leqslant 2.$$

Next, Proposition 3.2 shows that f_h is thrice continuously differentiable with $\|f_h'''\|_\infty \leqslant 2\,\|h''\|_\infty \leqslant 2$. On the other hand, for all $x \in \mathbb{R}$,

$$\begin{aligned} f_h'(x) &= xf_h(x) + h(x) - E[h(N)], \\ f_h''(x) &= f_h(x) + xf_h'(x) + h'(x). \end{aligned}$$

Consequently, $f_h''(0) = f_h(0)$ and

$$|f_h''(x)| \leqslant |f_h(0)| + |f_h''(x) - f_h''(0)| \leqslant 2 + \|f_h'''\|_\infty |x| \leqslant 2 + 2|x|. \tag{3.14}$$

We deduce that $|E[f_h''(F)]| \leqslant 2K$. Applying (3.11) to f_h in the case $M = 3$ yields therefore

$$E[Ff_h(F)] = E[f_h'(F)] + \frac{1}{2}E[f_h''(F)]E[F^3] + E[f_h'''(F)\Gamma_3(F)], \tag{3.15}$$

implying in turn that

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \frac{1}{2}|E[f_h''(F)]||E[F^3]| + |f_h'''|_\infty E|\Gamma_3(F)|,$$

from which the desired conclusion follows. □

Remark 3.14. (1) The idea of bounding quantities of the type

$$\left| E[Ff(F)] - \sum_{s=0}^{M-1} \frac{\kappa_{s+1}(F)}{s!} E[f^{(s)}(F)] \right|,$$

in order to estimate the distance between F and $N \sim \mathscr{N}(0,1)$, dates back to Barbour's seminal paper Barbour (1986). Due to the fact that F is a smooth functional of a Gaussian field, observe that the expression of the 'rest' $E[\Gamma_M(F)f^{(M)}(F)]$ appearing in (3.11) is remarkably simpler than the ones computed in Barbour (1986).

(2) For a fixed M, the expansion (3.11) may hold under weaker assumptions on F and f. For instance, if $F \in \mathbb{D}^{1,2}$ has an absolutely continuous law, then, for every Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$E[Ff(F)] = E[F]E[f'(F)] + E[f'(F)\Gamma_1(F)]. \tag{3.16}$$

Equation (3.16) is the starting point of the analysis developed in Nourdin and Peccati (2009b).

4. Some technical estimates

This section contains several estimates that are needed in the proof of Theorem 1.9 and Theorem 1.11.

4.1. *Inequalities for kernels.* For every integer $M \geqslant 1$, we write $[M] = \{1, ..., M\}$. Fix a set Z, as well as a vector $\mathbf{z} = (z_1, ..., z_M) \in Z^M$ and a nonempty set $b \subseteq [M]$: we denote by $\mathbf{z}_b$ the element of $Z^{|b|}$ (where $|b|$ is the cardinality of b) obtained by deleting from $\mathbf{z}$ the entries whose index is not contained in b. For instance, if $M = 5$ and $b = \{1, 3, 5\}$, then $\mathbf{z}_b = (z_1, z_3, z_5)$. Now consider the following setting:

(α) $(Z, \mathcal{Z})$ is a measurable space, and μ is a measure on it;

(β) $B, q \geqslant 2$ are integers, and $b_1, ..., b_q$ are nonempty subsets of $[B]$ such that $\cup_i b_i = [B]$, and each $k \in [B]$ appears in exactly two of the b_i's (this implies in particular that $\sum_i |b_i| = 2B$, and also that, if $q = 2$, then necessarily $b_1 = b_2 = [B]$);

(γ) $F_1, ..., F_q$ are functions such that $F_i \in L^2(Z^{|b_i|}, \mathcal{Z}^{|b_i|}, \mu^{|b_i|}) = L^2(\mu^{|b_i|})$ for every $i = 1, ..., q$ (in particular, each F_i is a function of $|b_i|$ variables).

The following generalization of the Cauchy-Schwarz inequality is crucial in this paper.

Lemma 4.1 (Generalized Cauchy-Schwarz Inequality)**.** *Under assumptions* (α)-(β)-(γ)*, the following inequality holds:*

$$\int_{Z^B} \prod_{i=1}^{q} |F_i(\mathbf{z}_{b_i})| \mu(dz_1)\cdots\mu(dz_B) \leqslant \prod_{i=1}^{q} \|F_i\|_{L^2(\mu^{|b_i|})}. \tag{4.1}$$

Proof. The case $q = 2$ is just the Cauchy-Schwarz inequality, and the general result is obtained by recursion on q. The argument goes as follows: call A the left-hand side of (4.1), and assume that the desired estimate is true for $q-1$. Applying the Cauchy-Schwarz inequality, we deduce that (with obvious notation)

$$A \leqslant \|F_1\|_{L^2(\mu^{|b_1|})} \times \left(\int_{Z^{|b_1|}} \Phi(\mathbf{z}_{b_1})^2 \mu^{|b_1|}(d\mathbf{z}_{b_1}) \right)^{1/2},$$

where the quantity $\Phi(\mathbf{z}_{b_1})$ is obtained by integrating the product $\prod_{i=2}^{q} |F_i(\mathbf{z}_{b_i})|$ over those variables z_j such that $j \notin b_1$. More explicitly, writing J for the class of those $i \in \{2, ..., q\}$ such that $b_i \subseteq b_1$,

$$\Phi(\mathbf{z}_{b_1}) = \prod_{i\in J} |F_i(\mathbf{z}_{b_i})| \times \int_{Z^{|B|-|b_1|}} \prod_{i\in J^c} |F_i(\mathbf{z}_{b_i\cap b_1},\, \mathbf{z}_{b_i\cap b_1^c})| \mu^{|B|-|b_1|}(d\mathbf{z}_{b_1^c}), \tag{4.2}$$

where b_1^c and J^c indicate, respectively, the complement of b_1 (in $[B]$) and the complement of J (in $\{2, ..., q\}$), and $\int \prod_{j\in\emptyset} = 1$ by convention. By construction, one has that the sets b_i such that $i \in J$ are disjoint, and also that $b_i \cap b_j = \emptyset$, for every $i \in J$ and $j \in J^c$. If $J^c = \emptyset$, there is nothing to prove. If $J^c \neq \emptyset$, one has to observe that the blocks $b_i' = b_i \cap b_1^c$, $i \in J^c$, verify assumption (β) with respect to the set $[B]\backslash b_1$ (that is, the class $\{b_i' : i \in J^c\}$ is composed of nonempty subsets of $[B]\backslash b_1$ such that $\cup_i b_i' = [B]\backslash b_1$, and each $k \in [B]\backslash b_1$ appears in exactly two of the b_i''s). Since $|J^c| \leqslant q-1$, the recurrence assumption can be applied to the integral on the right-hand side of (4.2), thus yielding the desired conclusion. □

Let $\mathfrak{H}$ be a real separable Hilbert space. The next estimates also play a pivotal role in our arguments.

Lemma 4.2. *Let* $p, q \geqslant 1$ *be two integers,* $r \in \{0, \ldots, p\wedge q\}$*, and* $u \in \mathfrak{H}^{\odot p}$*,* $v \in \mathfrak{H}^{\odot q}$ *Then*

$$\|u \,\widetilde{\otimes}_r\, v\|_{\mathfrak{H}^{\otimes(p+q-2r)}} \leqslant \|u \otimes_r v\|_{\mathfrak{H}^{\otimes(p+q-2r)}} \tag{4.3}$$

$$\|u \otimes_r v\|_{\mathfrak{H}^{\otimes(p+q-2r)}} \leqslant \|u\|_{\mathfrak{H}^{\otimes p}} \sqrt{\|v \otimes_{q-r} v\|_{\mathfrak{H}^{\otimes(2r)}}} \leqslant \|u\|_{\mathfrak{H}^{\otimes p}} \|v\|_{\mathfrak{H}^{\otimes q}}. \tag{4.4}$$

Moreover, if $q!\|v\|^2_{\mathfrak{H}^{\otimes q}} = 1$ *(that is, if* $E[I_q(v)^2] = 1$*), then*

$$\max_{1\leqslant r\leqslant q-1} \|v \otimes_r v\|^2_{\mathfrak{H}^{\otimes 2q-2r}} \leqslant \frac{\kappa_4(I_q(v))}{q!^2 q^2}. \tag{4.5}$$

Proof. The proof of (4.3) is evident, by using the very definition of a symmetrized function. To show the first inequality in (4.4), apply first Fubini to get that $\|u \otimes_r v\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} = \langle u \otimes_{p-r} u, v \otimes_{q-r} v\rangle_{\mathfrak{H}^{\otimes(2r)}}$, and then Cauchy-Schwarz to get the desired conclusion. The second inequality in (4.4) is an immediate consequence of Cauchy-Schwarz. Finally, the proof of (4.5) is obtained by using Nualart and Peccati (2005, first equality on p. 183). □

4.2. *Inequalities for cumulants and related quantities.* The following proposition contains all the estimates that are necessary for proving the main results in the paper. For every random variable Y such that $E|Y|^m < \infty$ ($m \geqslant 1$), we denote by $\kappa_m(Y)$ the mth cumulant of Y – see Definition 3.3. Given a vector $\mathbf{z} = (z_1, ..., z_d)$ and a permutation σ of $[d]$, we write $\sigma(\mathbf{z}) = (z_{\sigma(1)}, ..., z_{\sigma(d)})$. Given a function $F(z_1, ..., z_d)$ of d variables and a permutation σ of $[d]$, we write

$$(F)_\sigma(\mathbf{z}) = F(\sigma(\mathbf{z})) = F(z_{\sigma(1)}, ..., z_{\sigma(d)}).$$

Also, for vectors $\mathbf{z} = (z_1, ..., z_j)$ and $\mathbf{y} = (y_1, ..., y_k)$, we shall write $\mathbf{z} \vee \mathbf{y}$ for the vector of dimension $j + k$ obtained by juxtaposing $\mathbf{z}$ and $\mathbf{y}$, that is, $\mathbf{z} \vee \mathbf{y} = (z_1, ..., z_j, y_1, ..., y_k)$. Finally, in the subsequent proofs we will identify vectors of dimension zero with the empty set: if $\mathbf{z}$ has dimension zero, then integration with respect to $\mathbf{z}$ is removed by convention.

Proposition 4.3. *We use the notation introduced in Definitions 3.3 and 3.6. For each integer $q \geqslant 2$ there exists positive constants $c_2(q), c_3(q), c_4(q)$ (only depending on q) such that, for all $F = I_q(f)$ with $f \in \mathfrak{H}^{\odot q}$ and $E[F^2] = 1$, we have*

$$E\left[\left|\Gamma_2(F) - \frac{1}{2}\kappa_3(F)\right|\right] \leqslant c_2(q) \times \kappa_4(F)^{\frac{3}{4}}, \tag{4.6}$$

$$E[|\Gamma_3(F)|] \leqslant c_3(q) \times \kappa_4(F), \tag{4.7}$$

$$E[|\Gamma_4(F)|] \leqslant c_4(q) \times \kappa_4(F)^{\frac{5}{4}}. \tag{4.8}$$

Proof. By (3.6), we have $s!E(\Gamma_s) = \kappa_{s+1}(F)$ for every $s \geqslant 1$. Moreover, when $F = I_q(f)$ is as in the statement, recall the following explicit representation (3.10):

$$\Gamma_s(F) = \sum_{r_1=1}^{q} \cdots \sum_{r_s=1}^{[sq-2r_1-\ldots-2r_{s-1}]\wedge q} c_q(r_1, \ldots, r_s) \mathbf{1}_{\{r_1<q\}} \ldots \mathbf{1}_{\{r_1+\ldots+r_{s-1}<\frac{sq}{2}\}} \times I_{(s+1)q-2r_1-\ldots-2r_s}\big((\ldots(f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\ldots f)\widetilde{\otimes}_{r_s}f\big). \tag{4.9}$$

Without loss of generality, throughout the proof we shall assume that $\mathfrak{H} = L^2(Z, \mathcal{Z}, \mu)$, where Z is a Polish space, $\mathcal{Z}$ is the associated Borel σ-field, and μ is a σ-finite measure.

Proof of (4.6). According to (3.6), one has that $E[\Gamma_2(F)] = \frac{1}{2}\kappa_3(F)$, so that the random variable $E[\Gamma_2(F)] - \frac{1}{2}\kappa_3(F)$ is obtained by restricting the sum in (4.9) (in the case $s = 2$) to the terms such that $r_1 + r_2 < \frac{3q}{2}$. By virtue of (4.5), the inequality (4.6) will follow once it is shown that, for any choice of integers r_1, r_2 verifying such a constraint,

$$\|((f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\|_{\mathfrak{H}^{\otimes(3q-2r_1-2r_2)}} \leqslant \max_{1\leqslant r\leqslant q-1} \|f \otimes_r f\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes 2q-2r}}. \tag{4.10}$$

Let us first assume that $r_2 < q$. Then r_1 and $q - r_2$ both belong to $\{1, \ldots, q-1\}$. Thus, using the two inequalities (4.3) and (4.4), we infer that

$$\begin{aligned}&\|((f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\|_{\mathfrak{H}^{\otimes(3q-2r_1-2r_2)}}\\ \leqslant\ &\sqrt{\|f \otimes_{q-r_2} f\|_{\mathfrak{H}^{\otimes(2r_2)}}}\,\|f \otimes_{r_1} f\|_{\mathfrak{H}^{\otimes(2q-2r_1)}} \leqslant \max_{1\leqslant r\leqslant q-1}\|f \otimes_r f\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}}.\end{aligned}$$

Let us now consider the case when $r_2 = q$ and $r_1 < \frac{q}{2}$. The expression

$$(f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_q f = \langle (f\widetilde{\otimes}_{r_1}f), f\rangle_{\mathfrak{H}^{\otimes q}}$$

defines a function of $q-2r_1$ variables. Taking into account the symmetry of f and the symmetrization of contractions, such a function can be written as a convex linear combination of functions of the type

$$F(\mathbf{t}) = \int f(\mathbf{x}_1, \mathbf{t}_1, \mathbf{w}) f(\mathbf{x}_2, \mathbf{t}_2, \mathbf{w}) f(\mathbf{x}_1, \mathbf{x}_2) d\mu^{q+r_1}(\mathbf{w}, \mathbf{x}_1, \mathbf{x}_2),$$

where $\mathbf{w}$ has length r_1, and $\mathbf{t}_1 \vee \mathbf{t}_2 = \sigma(\mathbf{t})$ for some permutation σ and with $\mathbf{t} = (t_1, \ldots, t_{q-2r_1})$. Without loss of generality we can assume that $\mathbf{t}_1$ has positive length (recall that $r_1 < q/2$ so that $q - 2r_1 > 0$). We denote by s_j the length of the vector $\mathbf{x}_j$. We then have $1 \leqslant s_1 < q - r_1$ and $r_1 < s_2 \leqslant q-1$. Exchanging the order of integrations, we can write

$$F(\mathbf{t}) = \int f(\mathbf{x}_1, \mathbf{t}_1, \mathbf{w}) \left(f \otimes_{s_2} f\right)(\mathbf{x}_1, \mathbf{t}_2, \mathbf{w}) d\mu^{r_1+s_1}(\mathbf{w}, \mathbf{x}_1).$$

Squaring F and integrating, one sees that

$$\|F\|^2_{\mathfrak{H}^{\otimes(q-2r_1)}} = \int \prod_{i=1}^{3} (f \otimes_{\tau_i} f)_{\sigma_i}(\mathbf{z}_{b_i}) d\mu^B(\mathbf{z}_{b_1}, \mathbf{z}_{b_2}, \mathbf{z}_{b_3}),$$

with two of the τ_i's equal to s_2 and one to $q - r_1 - s_1$, where $B = q + 2s_1$, σ_i, $i = 1, 2, 3, 4$, is a permutation of $[2q - 2\tau_i]$, and the sets b_1, b_2, b_3 verify property (β), as defined at the beginning of the present section. It follows from Lemma 4.1 that

$$\|F\|_{\mathfrak{H}^{\otimes(q-2r_1)}} \leqslant \max_{1\leqslant r\leqslant q-1} \|f \otimes_r f\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes 2q-2r}},$$

from which we deduce (4.10).

Proof of (4.7). Our aim is to prove that for any choice of (r_1, r_2, r_3) appearing in the sum (4.9) in the case $s = 3$ one has the inequality

$$\|((f \widetilde{\otimes}_{r_1} f) \widetilde{\otimes}_{r_2} f) \widetilde{\otimes}_{r_3} f\|_{\mathfrak{H}^{\otimes(4q-2r_1-2r_2-2r_3)}} \leqslant \max_{1\leqslant r\leqslant q-1} \|f \otimes_r f\|^2_{\mathfrak{H}^{\otimes 2q-2r}}. \tag{4.11}$$

Remark that $((f \widetilde{\otimes}_{r_1} f) \widetilde{\otimes}_{r_2} f)$ has already been considered when looking at $\Gamma_2(F) - \frac{1}{2}\kappa_3(F)$, because of the assumption that $r_1 + r_2 < \frac{3q}{2}$.

So, using the previous estimates and (4.4), we conclude directly for $r_3 < q$. It remains to consider the case when $r_3 = q$.

As before, taking into account the symmetry of f and the symmetrization of contractions, it is sufficient to consider functions of $2(q - r_1 - r_2)$ variables of the type

$$\begin{aligned} F(\mathbf{t}) &= F(t_1, ..., t_{2(q-r_1-r_2)}) \\ &= \int_{Z^{q+r_1+r_2}} f(\mathbf{x}_1, \mathbf{a}_1, \mathbf{t}_1, \mathbf{w}) f(\mathbf{x}_2, \mathbf{a}_2, \mathbf{t}_2, \mathbf{w}) \times \\ &\qquad f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{t}_3, \mathbf{x}_3) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mu^{q+r_1+r_2}(d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3, d\mathbf{w}, d\mathbf{a}_1, d\mathbf{a}_2), \end{aligned}$$

where $\mathbf{w}$ has length r_1, $\mathbf{a}_1 \vee \mathbf{a}_2$ has length r_2 (with either $\mathbf{a}_1$ or $\mathbf{a}_2$ possibly equal to the empty set), and $\mathbf{t}_1 \vee \mathbf{t}_2 \vee \mathbf{t}_3 = \sigma(\mathbf{t})$ for some permutation σ. Squaring F and integrating, we claim that there exist integers $s_1, s_2, s_3, s_4 \in \{1, ..., q-1\}$ such that

$$\|F\|^2_{\mathfrak{H}^{2(q-r_1-r_2)}} = \int_{Z^B} \prod_{i=1}^{4} (f \otimes_{s_i} f)_{\sigma_i}(\mathbf{z}_{b_i}) \mu(dz_1) \cdots \mu(dz_B),$$

where $B = 4q - 2(s_1 + s_2 + s_3 + s_4)$, σ_i, $i = 1, 2, 3, 4$, is a permutation of $[2q - 2s_i]$, and the sets b_1, b_2, b_3, b_4 verify property (β), as defined at the beginning of the present section. We have to consider separately two cases.
(a): the length of $\mathbf{x}_3$ is not 0: we can then take $s_1 = s_2 = r_1$ and $s_3 = s_4$ equal to the length of $\mathbf{x}_3$.
(b): the length of $\mathbf{x}_3$ is 0. Then either $\mathbf{a}_1$ or $\mathbf{a}_2$ is not empty. Assuming that $\mathbf{a}_1$ is not empty, we can take for $s_1 = s_2$ the length of $\mathbf{a}_1$ and for $s_3 = s_4$ the length of $\mathbf{x}_2$, which is not 0.

As before, it follows from Lemma 4.1 that

$$\|F\|_{\mathfrak{H}^{2(q-r_1-r_2)}} \leqslant \max_{1\leqslant r\leqslant q-1} \|f \otimes_r f\|^2_{\mathfrak{H}^{\otimes 2q-2r}},$$

from which we deduce (4.11).
Proof of (4.8). Our aim is to prove that, for any choice of (r_1, r_2, r_3, r_4) which is present in the sum (4.9) (in the case $s = 4$), we have

$$\|(((f\widetilde{\otimes}_{r_1} f)\widetilde{\otimes}_{r_2} f)\widetilde{\otimes}_{r_3} f)\widetilde{\otimes}_{r_4} f\|_{\mathfrak{H}^{\otimes(5q-2r_1-2r_2-2r_3-2r_4)}} \leqslant \max_{1\leqslant r\leqslant q-1} \|f \otimes_r f\|^{\frac{5}{2}}_{\mathfrak{H}^{\otimes 2q-2r}}. \tag{4.12}$$

To do so, using the previous estimate (4.7) and (4.4) we conclude directly for $r_4 < q$. Hence, once again it remains to consider the case when $r_4 = q$.

As before, taking into account the symmetry of f and the symmetrization of contractions, one has that the function $(((f\widetilde{\otimes}_{r_1} f)\widetilde{\otimes}_{r_2} f)\widetilde{\otimes}_{r_3} f)\widetilde{\otimes}_{r_4} f$ is a linear combination (with coefficients not depending on f) of functions in $3q - 2r_1 - 2r_2 - 2r_3$ variables having the form

$$\begin{aligned} F(\mathbf{t}) = {} & F(t_1, ..., t_{3q-2r_1-2r_2-2r_3}) \\ = {} & \int_{Z^{q+r_1+r_2+r_3}} f(\mathbf{x}_1, \mathbf{a}_1, \mathbf{b}_1, \mathbf{t}_1, \mathbf{w}) f(\mathbf{x}_2, \mathbf{a}_2, \mathbf{b}_2, \mathbf{t}_2, \mathbf{w}) f(\mathbf{x}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_3, \mathbf{t}_3) \times \\ & f(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{t}_4, \mathbf{x}_4) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \mu^{q+r_1+r_2+r_3}(d\mathbf{x}, d\mathbf{w}, d\mathbf{a}, d\mathbf{b}), \end{aligned}$$

where $\mathbf{w}$ has length r_1, $\mathbf{a} = \mathbf{a}_1 \vee \mathbf{a}_2$ has length r_2 (with either $\mathbf{a}_1$ or $\mathbf{a}_2$ possibly equal to the empty set), $\mathbf{b} = \mathbf{b}_1 \vee \mathbf{b}_2 \vee \mathbf{b}_3$ has length r_3 (with some of the $\mathbf{b}_i$'s possibly equal to the empty set), $\mathbf{x} = \mathbf{x}_1 \vee \mathbf{x}_2 \vee \mathbf{x}_3 \vee \mathbf{x}_4$ and $\mathbf{t}_1 \vee \mathbf{t}_2 \vee \mathbf{t}_3 \vee \mathbf{t}_4 = \sigma(\mathbf{t})$ for some permutation σ. Squaring F and integrating, we claim that there exist integers $s_1, s_2, s_3, s_4, s_5 \in \{1, ..., q-1\}$ such that

$$\|F\|^2_{\mathfrak{H}^{3q-2r_1-2r_2-2r_3}} = \int_{Z^B} \prod_{i=1}^{5} (f \otimes_{s_i} f)_{\sigma_i}(\mathbf{z}_{b_i}) \mu(dz_1) \cdots \mu(dz_B),$$

where $B = 5q - 2(s_1 + s_2 + s_3 + s_4 + s_5)$, σ_i, $i = 1, 2, 3, 4, 5$, is a permutation of $[2q - 2s_i]$, and the sets b_1, b_2, b_3, b_4, b_5 verify property (β), as defined at the beginning of this section. We have to consider separately different cases.
(a): the length of $\mathbf{x}_4$ is not 0. We can then consider separately the three first factors, for which the same expressions as in the proof of (4.6) are available, and the two last ones, which give rise to $s_4 = s_5$ equal to the length of $\mathbf{x}_4$.
(b): the length of $\mathbf{x}_4$ is 0 and the length of $\mathbf{t}_4$ is not 0. Then we consider separately the four factors which are distinct from the fourth one and proceed as in the proof of (4.7) for them, while the fourth one gives rise to $f \otimes_\tau f$, with τ equal to the length of $\mathbf{t}_4$.

(c): the lengths of $\mathbf{x}_4$ and $\mathbf{t}_4$ are 0, but the length of $\mathbf{x}_3$ is not 0. We then separate the five factors into two groups, one with $f(\mathbf{x_3}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_3, \mathbf{t}_3)$ and $f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, the other one with the three other factors. The first group gives rise to factors $f \otimes_\tau f$, with τ equal to the length of $\mathbf{x}_3$, while the second group can be treated as in the proof of (4.6).
(d): the lengths of $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{t}_4$ are 0, but the length of $\mathbf{t}_3$ is not 0. We then consider separately the factor $f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_3, \mathbf{t}_3)$, which gives rise to a factor $f \otimes_\tau f$, with τ equal to the length of $\mathbf{t}_3$. The four other factors can be treated as in the proof of (4.7).
(e): the lengths of $\mathbf{x}_3$, $\mathbf{x}_4$, $\mathbf{t}_3$ and $\mathbf{t}_4$ are 0. Remark that $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{b}_3$ are non empty and, without loss of generality we can assume that $\mathbf{a}_2$ is non empty. As before, we can conclude by separating the five factors into two groups: for the first one we take the first factor and $f(\mathbf{x}_1, \mathbf{x}_2)$ whereas, for the second one, we choose the three remaining factors.

The desired conclusion (that is, (4.12)) follows once again from Lemma 4.1. □

5. Proof of the main results

5.1. *Proof of Theorem 1.9.* The assumption $E[F^2] = 1$ implies that $K := 1 + E[|F|] \leqslant 2$. The proof follows then immediately from (3.13) and (4.7).

5.2. *Proof of Theorem 1.11.* Since $E[F_n] = 0$ and $E[F_n^2] = 1$, one has that $\kappa_4(F_n) = E[F_n^4] - 3 > 0$. Moreover, because $F_n \overset{\text{Law}}{\to} N \sim \mathscr{N}(0,1)$ by assumption and due again to the hypercontractivity of chaotic random variables, we have that $\kappa_4(F_n) = E[F_n^4] - E[N^4] \to 0$ as $n \to \infty$. In the forthcoming proof we will need the following lemma.

Lemma 5.1. *There exists $g, h \in \mathcal{U} \cap \mathcal{C}^\infty$ with bounded derivatives of all orders (except possibly the first one) such that $E[f_g''(N)] \neq 0$, $E[f_g'''(N)] = 0$, $E[f_h''(N)] = 0$ and $E[f_h'''(N)] \neq 0$.*

Proof. Let H_p, $p \geqslant 1$, denote the sequence of Hermite polynomials. Using the well-known formula

$$\phi(x) = \sum_{p=0}^{\infty} \frac{1}{p!} E[\phi^{(p)}(N)] H_p(x), \quad x \in \mathbb{R} \text{ a.e.}, \quad N \sim \mathcal{N}(0,1),$$

valid for $\phi \in \mathcal{C}^\infty$ whose derivatives are all square integrable, it is readily checked that, for almost all $x \in \mathbb{R}$,

$$\frac{\sqrt{e}}{1+\sqrt{e}} \cos x = \sum_{q=0}^{\infty} \frac{(-1)^q}{(2q)!(1+\sqrt{e})} H_{2q}(x) \quad \text{and} \quad \sin x = \sum_{q=0}^{\infty} \frac{(-1)^q}{(2q+1)!\sqrt{e}} H_{2q+1}(x). \tag{5.1}$$

On the other hand, by applying several integration by parts, we can write, for $h \in \mathcal{U}$,

$$\begin{aligned} E[f_h''(N)] &= \int_{-\infty}^{+\infty} f_h''(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{-\infty}^{+\infty} f_h(x)(x^2-1) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\ &= \int_{-\infty}^{+\infty} dx\, H_2(x) \int_{-\infty}^{x} dy \big(h(y) - E[h(N)]\big) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \\ &= -\frac{1}{3} \int_{-\infty}^{+\infty} H_3(x)\big(h(x) - E[h(N)]\big) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = -\frac{1}{3} E[h(N)H_3(N)]. \end{aligned} \tag{5.2}$$

Similarly, we can prove that, for all $h \in \mathcal{U}$,

$$E[f_h'''(N)] = -\frac{1}{4} E[h(N)H_4(N)]. \tag{5.3}$$

Now, let us consider $g(x) = \sin x$. Using (5.1) and then (5.2)-(5.3), we get that $E[f_g''(N)] = \frac{1}{3\sqrt{e}} \neq 0$ and $E[f_g'''(N)] = 0$. Moreover, g belongs to $\mathcal{U}$ because $|g''(x)| = |\sin x| \leqslant 1$, and has bounded derivatives. Similarly, consider $h(x) = \frac{1}{1+\sqrt{e}}\big(\sqrt{e}\cos x - 1 + \frac{1}{2}H_2(x)\big)$. Using once again (5.1) and then (5.2)-(5.3), we get this time that $E[f_h''(N)] = 0$ whereas $E[f_h'''(N)] = -\frac{1}{4+4\sqrt{e}} \neq 0$, also with $|h''(x)| = \left|\frac{1}{1+\sqrt{e}}\big(1 - \sqrt{e}\cos x\big)\right| \leqslant 1$ so that $h \in \mathcal{U}$. □

Proof of Theorem 1.11. Recall that $\kappa_3(F_n) = E[F_n^3]$ and $\kappa_4(F_n) = E[F_n^4] - 3$, and let $g, h \in \mathcal{U}$ be as in the statement of Lemma 5.1. From Corollary 3.12 and Proposition 4.3, we deduce that

$$\begin{aligned} &\left| E[g(N)] - E[g(F_n)] - \frac{1}{2} E[f_g''(N)]\kappa_3(F_n) \right| \\ &\qquad \leqslant \frac{1}{2}\big|\kappa_3(F_n)\big| \, \big|E[f_g''(F_n)] - E[f_g''(N)]\big| + \\ &\qquad\quad \frac{1}{6}\big|E[f_g'''(F_n)]\big| \kappa_4(F_n) + 2c_4\|g'''\|_\infty \kappa_4(F_n)^{5/4}. \end{aligned}$$

Set

$$c = \frac{1}{3} \min\left\{ \frac{1}{2}|E[f_g''(N)]|, \frac{1}{6}|E[f_h'''(N)]| \right\}.$$

As $n \to \infty$, we have $E[f_g''(F_n)] - E[f_g''(N)] \to 0$, $E[f_g'''(F_n)] \to E[f_g'''(N)] = 0$, and $\kappa_4(F_n) \to 0$. Therefore, for n large enough we have that

$$d(F_n, N) \geqslant \big|E[g(N)] - E[g(F_n)]\big| \geqslant 3c|\kappa_3(F_n)| - \frac{c}{2}\max\{|\kappa_3(F_n)|, \kappa_4(F_n)\}.$$

Similarly, we have

$$\begin{aligned} &\left| E[h(N)] - E[h(F_n)] - \frac{1}{6} E[f_h'''(N)]\kappa_4(F_n) \right| \\ &\qquad \leqslant \frac{1}{2}\big|\kappa_3(F_n)\big| \, |E[f_h''(F_n)]| + \\ &\qquad\quad \frac{1}{6}|E[f_h'''(F_n)] - E[f_h'''(N)]| \kappa_4(F_n) + 2c_4\|h'''\|_\infty \kappa_4(F_n)^{5/4}, \end{aligned}$$

from which we deduce, again for n large enough, that

$$d(F_n, N) \geqslant \big|E[h(N)] - E[h(F_n)]\big| \geqslant 3c\,\kappa_4(F_n) - \frac{c}{2}\max\{|\kappa_3(F_n)|, \kappa_4(F_n)\}.$$

Finally taking the mean of the two previous upper bounds for $d(F_n, N)$ yields

$$\begin{aligned} d(F_n, N) &\geqslant \frac{3c}{2}\big(|\kappa_3(F_n)| + \kappa_4(F_n)\big) - \frac{c}{2}\max\{|\kappa_3(F_n)|, \kappa_4(F_n)\} \\ &\geqslant c\max\{|\kappa_3(F_n)|, \kappa_4(F_n)\}. \end{aligned}$$

The proof is concluded. □

6. Application: estimates in the Breuer-Major CLT

In this final section, we determine optimal rates of convergence associated with the well-known *Breuer-Major CLT* for Gaussian-subordinated random sequences – see Breuer and Major (1983) for the original paper, or Nourdin et al. (2011) for a more modern reference. In order to be able to directly apply our previous results, we focus on sequences that can be represented as partial sums of Hermite polynomials.

6.1. *General framework.* Consider a centered stationary Gaussian sequence $(X_k)_{k\in\mathbb{Z}}$ with unit variance and covariance function given by $E[X_kX_l] = \rho(k-l)$, $k, l \in \mathbb{Z}$. Fix an integer $q \geqslant 2$, and set

$$V_n = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_q(X_k), \quad n \geqslant 1.$$

Here, H_q stands for the qth Hermite polynomial defined by (2.1). Let also $N \sim \mathscr{N}(0,1)$, and define $v_n := E[V_n^2]$, $n \geqslant 1$. Finally, set

$$F_n = \frac{V_n}{\sqrt{v_n}} = \frac{1}{\sqrt{n\, v_n}}\sum_{k=0}^{n-1} H_q(X_k).$$

Without loss of generality, we may assume that $X_k = X(h_k)$, where $X = \{X(h) : h \in \mathfrak{H}\}$ is some isonormal Gaussian process and $\langle h_k, h_l\rangle_{\mathfrak{H}} = \rho(k-l)$ for every $k, l \in \mathbb{Z}$. We then have

$$F_n = I_q(f_n), \quad \text{with } f_n = \frac{1}{\sqrt{nv_n}}\sum_{k=0}^{n-1} h_k^{\otimes q}.$$

For $p \geqslant 1$, we introduce the Banach space $\ell^p(\mathbb{Z})$ of p-summable sequences equipped with the norm $\|u\|_{\ell^p} = \big(\sum_{k\in\mathbb{Z}}|u(k)|^p\big)^{1/p}$. In what follows, we shall assume that ρ belongs to $\ell^q(\mathbb{Z})$. Under this assumption, the celebrated Breuer-Major CLT (see Breuer and Major (1983), as well I. Nourdin (2012, Chapter 7)) asserts that

$$F_n \overset{\text{Law}}{\to} \mathscr{N}(0,1) \quad \text{as } n \to \infty.$$

Remark 6.1. One has that

$$v_n \to q!\sum_{k\in\mathbb{Z}} \rho(k)^q > 0 \quad \text{as } n \to \infty.$$

It follows that, in the subsequent discussion, the role of the sequence v_n, $n \geqslant 1$, will be immaterial as far as rates of convergence are concerned.

6.2. *Explicit formulas for the third and fourth cumulants.* Let us compute, in terms of ρ, explicit expressions for the third and fourth cumulants of F_n. According to Proposition 3.9 (and using the notation introduced in Section 3), one has

$$\Gamma_1(F_n) = q\sum_{r=1}^{q}(r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}\left(f_n\widetilde{\otimes}_r f_n\right). \tag{6.1}$$

Since $\kappa_3(F_n) = 2E[F_n\Gamma_1(F_n)]$ by (3.8), we deduce the following expression of $\kappa_3(F_n)$ in terms of the sequence f_n, $n \geqslant 1$:

$$\kappa_3(F_n) = \begin{cases} 2qq!(q/2-1)!\binom{q-1}{q/2-1}^2\langle f_n, f_n\widetilde{\otimes}_{q/2}f_n\rangle_{\mathfrak{H}^{\otimes q}} & \text{for even } q \\ 0 & \text{for odd } q \end{cases}. \tag{6.2}$$

Hence, when q is even (observe that $\langle f_n, f_n\widetilde{\otimes}_{q/2}f_n\rangle_{\mathfrak{H}^{\otimes q}} = \langle f_n, f_n \otimes_{q/2} f_n\rangle_{\mathfrak{H}^{\otimes q}}$ because f_n is symmetric), we have

$$\kappa_3(F_n) = \frac{d_3(q)}{v_n^{3/2}\, n\sqrt{n}}\sum_{j,k,l=0}^{n-1}\rho(k-l)^{\frac{q}{2}}\rho(k-j)^{\frac{q}{2}}\rho(l-j)^{\frac{q}{2}}, \tag{6.3}$$

with $d_3(q) = 2qq!(q/2-1)!\binom{q-1}{q/2-1}^2$. It will be often useful to transform the previous expression as follows: we have

$$\begin{aligned}\kappa_3(F_n) &= \frac{d_3(q)}{v_n^{3/2}\, n\sqrt{n}}\sum_{j=0}^{n-1}\sum_{k,l=-j}^{n-1-j}\rho(k-l)^{q/2}\rho(k)^{q/2}\rho(l)^{q/2} \\ &= \frac{d_3(q)}{v_n^{3/2}\sqrt{n}}\sum_{k,l\in\mathbb{Z}}\eta_n(k,l)\rho(k-l)^{q/2}\rho(k)^{q/2}\rho(l)^{q/2}, \end{aligned} \tag{6.4}$$

where

$$\eta_n(k,l) = \left(1 - \frac{\max(k,l)_+}{n} + \frac{\min(k,l)_-}{n}\right)\mathbf{1}_{\{|k|<n,|l|<n\}}. \tag{6.5}$$

Remarks 6.2. (1) When q is even, one has $\kappa_3(F_n) > 0$ for all n; indeed,

$$\sum_{j,k,l=0}^{n-1}\rho(k-l)^{\frac{q}{2}}\rho(k-j)^{\frac{q}{2}}\rho(l-j)^{\frac{q}{2}} = \frac{1}{(q/2)!}E\left[\sum_{j=0}^{n-1}\left(\sum_{k=0}^{n-1}H_{q/2}(X_k)\rho(k-j)^{q/2}\right)^2\right].$$

(2) When $q = 2$, one can even prove that $\Gamma_2(F_n) > 0$ for all n (recall that $\kappa_3(F_n) = 2\,E[\Gamma_2(F_n)]$).

Now, let have a look at the fourth cumulant. Recall from Nualart and Peccati (2005) that

$$\kappa_4(F_n) = \sum_{r=1}^{q-1}q!^2\binom{q}{r}^2\left\{\|f_n\otimes_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} + \binom{2q-2r}{q-r}\|f_n\widetilde{\otimes}_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}}\right\}. \tag{6.6}$$

For the non-symmetrized contractions of (6.6), the link with ρ is easily obtained; indeed, for any $r = 1, \ldots, q-1$, we have:

$$\|f_n\otimes_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} = \frac{1}{v_n^2\, n^2}\sum_{i,j,k,l=0}^{n-1}\rho(k-l)^r\rho(i-j)^r\rho(k-i)^{q-r}\rho(l-j)^{q-r}. \tag{6.7}$$

On the other hand, we will actually face no problem due to symmetrized contractions. Indeed, we may forget them when deriving lower bounds, whereas we can use the inequality

$$\|f_n \widetilde{\otimes}_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} \leqslant \|f_n \otimes_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} \tag{6.8}$$

when dealing with upper bounds.

6.3. *Estimates for the third and fourth cumulants.* We start with the following result about the asymptotic behavior of the third cumulant of F_n. Of course, by virtue of (6.2), only the case where q is even must be considered.

Proposition 6.3. *Assume that $q \geqslant 2$ is even. Then,*

$$\kappa_3(F_n) \leqslant \frac{d_3(q)}{v_n^{3/2}\sqrt{n}} \left(\sum_{|k|<n} |\rho(k)|^{3q/4} \right)^2 .$$

Moreover, if $\rho \in \ell^{\frac{3q}{4}}(\mathbb{Z})$, then

$$\sqrt{n}\,\kappa_3(F_n) \to \frac{d_3(q)}{q!^{3/2}} \sqrt{2\pi} \frac{\int_{\mathbb{T}} g_{q/2}(t)^3 dt}{\left(\int_{\mathbb{T}} g_{q/2}(t)^2 dt\right)^{3/2}} \quad \text{as } n \to \infty, \tag{6.9}$$

where $g_{q/2}(t) := \sum_{l\in\mathbb{Z}} \rho(k)^{q/2} e^{ikt}$ is almost everywhere positive on the torus $\mathbb{T} = \mathbb{R}\backslash(2\pi\mathbb{Z})$.

Proof. Recall the identity (6.4) and, for any $k \in \mathbb{Z}$ and any $n \geqslant 1$, set $\rho_n(k) = \rho(k)\mathbf{1}_{\{|k|<n\}}$ and $|\rho_n|(k) = |\rho_n(k)|$. Since $0 \leqslant \eta_n(k,l) \leqslant 1$ for all $n \geqslant 1$ and all $k,l \in \mathbb{Z}$, we deduce that

$$\kappa_3(F_n) \leqslant \frac{d_3(q)}{v_n^{3/2}\sqrt{n}} \sum_{l\in\mathbb{Z}} \big(|\rho_n|^{q/2} * |\rho_n|^{q/2}\big)(l)\big(|\rho_n|^{q/2}\big)(l).$$

At this stage, let us recall the (well-known) Young inequality: if $s,p,p' \in [1,\infty]$ are such that $\frac{1}{p} + \frac{1}{p'} = 1 + \frac{1}{s}$, then

$$\|u * v\|_{\ell^s} \leqslant \|u\|_{\ell^p} \|v\|_{\ell^{p'}}. \tag{6.10}$$

Hence, using first Hölder inequality and then Young inequality yield

$$\sum_{l\in\mathbb{Z}} \big(|\rho_n|^{q/2} * |\rho_n|^{q/2}\big)(l)\big(|\rho_n|^{q/2}\big)(l) \leqslant \||\rho_n|^{q/2} * |\rho_n|^{q/2}\|_{\ell^3} \|\rho_n^{q/2}\|_{\ell^{\frac{3}{2}}} \leqslant \|\rho_n^{q/2}\|^3_{\ell^{\frac{3}{2}}},$$

which proves the first statement of the proposition.

Let us now further assume that $\rho \in \ell^{\frac{3q}{4}}(\mathbb{Z})$. It implies that $\rho \in \ell^q(\mathbb{Z})$ or, equivalently, that $\rho^{q/2}$ belongs to $\ell^2(\mathbb{Z})$. Thus, the function $g_{q/2}$ is well defined in $L^2(\mathbb{T})$, as being the Fourier series with coefficients $\rho^{q/2}$. In particular, we deduce from Bessel-Parseval equality that

$$v_n \to q! \sum_{k\in\mathbb{Z}} \rho(k)^q = \frac{q!}{2\pi} \int_{\mathbb{T}} g_{q/2}(t)^2 dt \quad \text{as } n \to \infty.$$

We also have that $\rho^{q/2} * \rho^{q/2}$ belongs to $\ell^2(\mathbb{Z})$, that is, that $\sum_{l\in\mathbb{Z}} \rho^{q/2} * \rho^{q/2}(l)\rho(l)^{q/2}$ is an absolutely convergent series whose value is given by $\frac{1}{2\pi}\int_{\mathbb{T}} g_{q/2}(t)^3 dt$ (Bessel-Parseval equality). Then, using (6.3)-(6.4) and dominated convergence, we get (6.9). Finally, $\rho^{q/2}$ being a covariance sequence as well (those of the stationary sequence $\frac{1}{\sqrt{(q/2)!}}H_{q/2}(X_k))$, its spectral density $g_{q/2}$ is positive almost everywhere. (See e.g. Doob (1953).) □

The situation for the fourth cumulant turns out to be not so easy, except when $q = 2$.

Proposition 6.4. *There exists $C > 0$ such that, for all $n \geqslant 1$,*

$$\kappa_4(F_n) \leqslant \begin{cases} \frac{C}{v_n^2 n}\left(\sum_{|k|<n} |\rho(k)|^{2q/3}\right)^3 & \text{if } q \leqslant 3 \\ \frac{C}{v_n^2 n}\left(\sum_{k=-n+1}^{n-1} |\rho(k)|^{q-1}\right)^2 \left(\sum_{k=-n+1}^{n-1} |\rho(k)|^2\right) & \text{if } q \geqslant 3 \end{cases}. \tag{6.11}$$

If $q = 2$ and $\rho \in \ell^{4/3}(\mathbb{Z})$, then

$$n\,\kappa_4(F_n) \to 24\pi \frac{\int_{\mathbb{T}} g_1(t)^4 dt}{\left(\int_{\mathbb{T}} g_1(t)^2 dt\right)^2} \quad \text{as } n \to \infty, \tag{6.12}$$

where $g_1(t) := \sum_{l \in \mathbb{Z}} \rho(k)e^{ikt}$ is almost everywhere positive on the torus $\mathbb{T}$. If $q \geqslant 3$ and $\rho \in \ell^2(\mathbb{Z})$, then $\liminf_{n\to\infty} n\kappa_4(F_n) > 0$.

Proof. Thanks to (6.6), (6.7) and (6.8), we 'only' have to estimate, for any $1 \leqslant r \leqslant q-1$,

$$I(r) = \frac{1}{n}\sum_{i,j,k,l=0}^{n-1} |\rho(k-l)|^r |\rho(i-j)|^r |\rho(k-i)|^{q-r} |\rho(l-j)|^{q-r}.$$

We immediately see that

$$I(r) \leqslant \frac{1}{n}\sum_{k,j=0}^{n-1} \left(|\rho_n|^r * |\rho_n|^{q-r}\right)^2 (k-j) \leqslant \sum_{j=-n}^{n} \left(|\rho_n|^r * |\rho_n|^{q-r}\right)^2 (j).$$

Let us first assume that $q = 2$, so that $r = 1$ necessarily. In this case,

$$I(1) \leqslant \||\rho_n| * |\rho_n|\|_{\ell^2}^2 \leqslant \|\rho_n\|_{\ell^{\frac{4}{3}}}^4, \tag{6.13}$$

where we have used Young inequality (6.10) to get the last inequality. This proves (6.11) when $q = 2$. Assume now that $q \geqslant 3$. By symmetry, we may and will assume that $r \leqslant q/2$. Young inequality (6.10) yields that

$$I(r) \leqslant \||\rho_n^r| * |\rho_n|^{q-r}\|_{\ell^2}^2 \leqslant \|\rho_n^r\|_{\ell^2}^2 \|\rho_n^{q-r}\|_{\ell^1}^2. \tag{6.14}$$

This shows the desired bound when $r = 1$. For the other values of r (if any), we can make use of the log-convexity of the ℓ^p norms. More precisely, for α, β such that $2r = 2(1-\alpha) + (q-1)\alpha$ and $q - r = 2(1-\beta) + (q-1)\beta$, recall that

$$\sum_{j\in\mathbb{Z}} |\rho_n(j)|^{2r} \leqslant \|\rho_n\|_{\ell^2}^{2(1-\alpha)} \|\rho_n\|_{\ell^{q-1}}^{(q-1)\alpha} \quad \text{and}$$

$$\sum_{j\in\mathbb{Z}} |\rho_n(j)|^{q-r} \leqslant \|\rho_n\|_{\ell^2}^{2(1-\beta)} \|\rho_n\|_{\ell^{q-1}}^{(q-1)\beta}.$$

We then conclude that (6.11) holds true for any $q \geqslant 3$ as well. To finish the proof of Proposition 6.4, let us compute the limit of $n\|f_n \otimes_1 f_n\|_{\mathfrak{H}^{\otimes 2(q-1)}}^2$ under the additional assumption that $\rho \in \ell^{4/3}(\mathbb{Z})$ if $q = 2$ and $\rho \in \ell^2(\mathbb{Z})$ if $q \geqslant 3$. We can write

$$
\begin{aligned}
n\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2(q-1)}} &= \frac{1}{v_n^2\, n} \sum_{i,j,k,l=0}^{n-1} \rho(k-l)^{q-1}\rho(i-j)^{q-1}\rho(k-i)\rho(l-j) \\
&= \frac{1}{v_n^2} \sum_{j,k,l\in\mathbb{Z}} \eta_n(j,k,l)\rho(l-k)\rho(j)\rho(k)^{q-1}\rho(l-j)^{q-1} \qquad (6.15)
\end{aligned}
$$

where $\eta_n(j,k,l) = \left(1 - \frac{\max(j,k,l)_+}{n} + \frac{\min(j,k,l)_-}{n}\right) \mathbf{1}_{\{|j|<n,|k|<n,|l|<n\}}$ is bounded by 1 and tends to 1 as $n \to \infty$. We know from (6.13) if $q = 2$ and from (6.14) if $q \geqslant 3$ that the series

$$
\sum_{j,k,l\in\mathbb{Z}} \rho(l-k)\rho(j)\rho(k)^{q-1}\rho(l-j)^{q-1}
$$

is absolutely convergent under our assumption on ρ. By dominated convergence, we get that

$$
n\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2(q-1)}} \to \sum_{l\in\mathbb{Z}} \left(\rho * \rho^{q-1}\right)^2(l) = \|\rho * \rho^{q-1}\|^2_{\ell^2} \quad \text{as } n \to \infty,
$$

implying in turn that $\liminf_{n\to\infty} n\kappa_4(F_n) > 0$ as expected. Finally, when $q = 2$ we have $v_n \to \frac{1}{\pi}\int_{\mathbb{T}} g_1(t)^2 dt$, so that (6.12) holds true thanks to (6.6).

□

6.4. *Breuer-Major CLT.* The following result is the so-called Breuer-Major theorem (it is called this way in honor of the seminal paper Breuer and Major (1983)). For sake of completeness, we provide here a modern proof, that relies on the bounds (6.11) for $\kappa_4(F_n)$ and Theorem 1.1.

Theorem 6.5 (Breuer-Major)**.** *Assume $\rho \in \ell^q(\mathbb{Z})$. Then F_n converges to $\mathscr{N}(0,1)$ in total variation as $n \to \infty$.*

Proof. By virtue of Theorem 1.1, it is (surprisingly) enough to prove that $\kappa_4(F_n)$ tends to 0 as $n \to \infty$. Let us first assume that $q = 2$. Then, using Hölder inequality for $k > M$, we have, for any $n > M$,

$$
\frac{1}{n}\left(\sum_{k=-n+1}^{n-1} |\rho(k)|^{\frac{4}{3}}\right)^3 \leqslant \frac{4}{n}\left(\sum_{|k|\leqslant M} |\rho(k)|^{\frac{4}{3}}\right)^3 + 4\left(\sum_{M<|k|<n} |\rho(k)|^2\right)^2.
$$

We can then conclude by a standard argument (choosing M large enough). For $q \geqslant 3$, we proceed analogously, after having noticed that

$$
\begin{aligned}
\frac{1}{n}\left(\sum_{k=-n+1}^{n-1} |\rho(k)|^{q-1}\right)^2 \left(\sum_{k=-n+1}^{n-1} |\rho(k)|^2\right) &= \left(\frac{1}{n^{1/q}} \sum_{k=-n+1}^{n-1} |\rho(k)|^{q-1}\right)^2 \\
&\quad \left(\frac{1}{n^{1-2/q}} \sum_{k=-n+1}^{n-1} |\rho(k)|^2\right),
\end{aligned}
$$

with

$$
\frac{1}{n^{1/q}} \sum_{k=-n+1}^{n-1} |\rho(k)|^{q-1} \leqslant 2^{1/q}\left(\sum_{|k|<n} |\rho(k)|^q\right)^{1-1/q} \leqslant 2^{1/q}\|\rho\|^{q-1}_{\ell^q},
$$

and, for any $M < n$,

$$\frac{1}{n^{1-2/q}} \sum_{k=-n+1}^{n-1} |\rho(k)|^2 \leqslant \frac{1}{n^{1-2/q}} \sum_{|k|\leqslant M} |\rho(k)|^2 + 2^{1-2/q} \left(\sum_{M<|k|<n} |\rho(k)|^q \right)^{2/q}.$$

Hence, by choosing M large enough, we get that $\kappa_4(F_n)$ tends to 0, and the proof is concluded. □

6.5. *The discrete-time fractional Brownian motion.* Let B_H be a fractional Brownian motion with Hurst index $H \in (0,1)$. We recall that $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ is a centered Gaussian process with continuous paths such that

$$E[B_H(t)B_H(s)] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad s,t \in \mathbb{R}.$$

The process B_H is self-similar with stationary increments, and we refer the reader to Nualart Nualart (2006) and Samorodnitsky and Taqqu Samorodnitsky and Taqqu (1994) for its main properties. In this section, we offer fine estimates for $\kappa_3(F_n)$ and $\kappa_4(F_n)$ respectively, when the sequence $(X_k)_{k\in\mathbb{Z}}$ corresponds to increments of B^H, that is,

$$X_k = B_H(k+1) - B_H(k), \quad k \in \mathbb{Z}. \tag{6.16}$$

The X_k's are usually called 'fractional Gaussian noise' in the literature, and are centered stationary Gaussian random variables with covariance

$$\rho(k) = E[X_r X_{r+k}] = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right), \quad r,k \in \mathbb{Z}.$$

The covariance behaves asymptotically as

$$\rho(k) \sim H(2H-1)|k|^{2H-2} \quad \text{as } |k| \to \infty. \tag{6.17}$$

In particular, when $H > \frac{1}{2}$ we observe that, for $|k|$ large enough,

$$\rho(k) \geqslant H(H-\frac{1}{2})(1+|k|)^{2H-2}. \tag{6.18}$$

In the following results, we let the notation introduced in Section 6.1 prevail, and we assume that the sequence $(X_k)_{k\in\mathbb{Z}}$ is given by (6.16). We also use the following convention for non-negative sequences (u_n) and (v_n) (possibly depending on q and/or H): we write $v_n \asymp u_n$ to indicate that $0 < \liminf_{n\to\infty} v_n/u_n \leqslant \limsup_{n\to\infty} v_n/u_n < \infty$.

Proposition 6.6. *Assume that $q \geqslant 2$ is even. We have:*

$$\kappa_3(F_n) \asymp \begin{cases} n^{-\frac{1}{2}} & \text{if } \quad 0 < H < 1 - \frac{2}{3q} \\ n^{-\frac{1}{2}} \log^2 n & \text{if } \quad H = 1 - \frac{2}{3q} \\ n^{\frac{3}{2}-3q+3qH} & \text{if } \quad 1 - \frac{2}{3q} < H < 1 - \frac{1}{2q} \end{cases}.$$

Proof. When $H < 1 - \frac{2}{3q}$, we have that $\rho \in \ell^{3q/4}(\mathbb{Z})$, so that

$$0 < \liminf_{n\to\infty} \sqrt{n}\,\kappa_3(F_n) = \limsup_{n\to\infty} \sqrt{n}\,\kappa_3(F_n) < \infty$$

by Proposition 6.3. For the other values of H, consider first the limsup. We have that

$$\sum_{|k|<n} |\rho(k)|^{3q/4} \leqslant C \begin{cases} \log n & \text{if} \quad H = 1 - \frac{2}{3q} \\ n^{1-\frac{3q}{2}+\frac{3qH}{2}} & \text{if} \quad H > 1 - \frac{2}{3q} \end{cases},$$

from which we deduce the finiteness of the limsup by Proposition 6.3 together with the fact that $v_n \to q! \sum_{k\in\mathbb{Z}} \rho(k)^q \in (0,\infty)$ when $H < 1 - \frac{1}{2q}$.

Let us now focus on the liminf. Since $H > 1 - \frac{2}{3q} > \frac{1}{2}$, the lower bound (6.18) holds for $|k|$ large enough. In fact, by considering a decomposition of the type $\rho = \rho_+ + \tau$ with τ having compact support, we can do as if the inequality (6.18) were valid for the small values of $|k|$ as well. Indeed, it suffices to use the fact that $\kappa_3(F_n)$ comes from a trilinear form (see (6.4)) as well as the inequality $|x|^{q/2}|y|^{q/2} \leqslant \frac{1}{2}\big(|x|^q + |y|^q\big)$, so to deduce that, when there is at least one term τ, the asymptotic order of the corresponding contribution in $\kappa_3(F_n)$ is $O(n^{-\frac{1}{2}})$. Recall the definition (6.5) of $\eta_n(k,l)$. With the extra assumption that (6.18) holds true for all k, we get that

$$\sum_{k,l\in\mathbb{Z}} \eta_n(k,l)\rho(k-l)^{q/2}\rho(k)^{q/2}\rho(l)^{q/2}$$

is bounded by below by a constant time

$$\sum_{|l|\leqslant n/4}\ \sum_{|l|\leqslant|k|\leqslant 2|l|} (1+|k-l|)^{(H-1)q}(1+|k|)^{(H-1)q}(1+|l|)^{(H-1)q}.$$

Note that, for $k,l \in \mathbb{Z}$ with $|l| \leqslant |k| \leqslant 2|l|$, we have $1+|k-l| \leqslant 2(1+|k|)$ as well as $1+|k-l| \leqslant 3(1+|l|)$, so that

$$(1+|k-l|)^{(H-1)q} \geqslant 6^{(H-1)q/2}(1+|k|)^{(H-1)q/2}(1+|l|)^{(H-1)q/2},$$

which, by summing first over k and then over l, concludes the proof for the liminf. □

By reasoning similarly, we obtain an estimate for $\kappa_4(F_n)$. (In the following statement, the limsup is partially known from Biermé et al. (2011).)

Proposition 6.7. *For $q \in \{2,3\}$, we have*

$$\kappa_4(F_n) \asymp \begin{cases} n^{-1} & \text{if} \quad 0 < H < 1 - \frac{3}{4q} \\ n^{-1}\log^3 n & \text{if} \quad H = 1 - \frac{3}{4q} \\ n^{4qH-4q+2} & \text{if} \quad 1 - \frac{3}{4q} < H < 1 - \frac{1}{2q} \end{cases}. \tag{6.19}$$

whereas, for $q > 3$,

$$\kappa_4(F_n) \asymp \begin{cases} n^{-1} & \text{if} \quad 0 < H < \frac{3}{4} \\ n^{-1}\log(n) & \text{if} \quad H = \frac{3}{4} \\ n^{4H-4} & \text{if} \quad \frac{3}{4} < H < 1 - \frac{1}{2q-2} \\ n^{4H-4}\log^2 n & \text{if} \quad H = 1 - \frac{1}{2q-2} \\ n^{4qH-4q+2} & \text{if} \quad 1 - \frac{1}{2q-2} < H < 1 - \frac{1}{2q} \end{cases}. \tag{6.20}$$

Proof. The proof of the limsup is straightforward by using Proposition 6.4. To get the liminf result, we first consider $\rho = \rho_+ + \tau$ as in the proof of Proposition 6.6, and verify that the contribution of all the terms containing at least one τ are of lower order. Here again, we can therefore do as if the inequality (6.18) were valid for all k. This allows us to bound (6.15) by below by following the same line of reasoning as in the proof of Proposition 6.6. We finally conclude thanks to (6.6). Details are left to the reader. □

By comparing the asymptotic behaviors of $\kappa_3(F_n)$ and $\kappa_4(F_n)$, we observe the following non-expected fact.

Corollary 6.8. *When $q \geqslant 6$, there exists a non-trivial range of values of H for which $\kappa_4(F_n)$ tends less rapidly to 0 than $\kappa_3(F_n)$.*

Proof. The two functions that give the behavior of $\kappa_3(F_n)$ and $\kappa_4(F_n)$ are piecewise linear and concave. It is therefore sufficient to consider the value $H = 1 - \frac{2}{3q}$. For this value, one has $\kappa_4(F_n) \asymp n^{4H-4} \gg n^{-1/2}$ when $\frac{1}{q-1} < \frac{4}{3q} < \frac{1}{4}$. □

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