

Research Article

Study of Stability of Rotational Motion of Spacecraft with Canonical Variables

**William Reis Silva,¹ Maria Cecília F. P. S. Zanardi,¹
Regina Elaine Santos Cabette,^{2,3}
and Jorge Kennety Silva Formiga^{3,4}**

¹ *Group of Orbital Dynamics and Planetology, São Paulo State University (UNESP), Guaratinguetá 12516-410, SP, Brazil*

² *Department of Mathematics, São Paulo Salesian University (UNISAL), Lorena 12600-100, SP, Brazil*

³ *Space Mechanic and Control Division, National Institute for Space Research (INPE), São José dos Campos 12227-010, SP, Brazil*

⁴ *Department of Aircraft Maintenance and Aeronautical Manufacturing, Faculty of Technology (FATEC), São José dos Campos 12247-004, SP, Brazil*

Correspondence should be addressed to William Reis Silva, reis.william@gmail.com

Received 16 July 2011; Revised 10 October 2011; Accepted 23 October 2011

Academic Editor: Tadashi Yokoyama

Copyright © 2012 William Reis Silva et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This work aims to analyze the stability of the rotational motion of artificial satellites in circular orbit with the influence of gravity gradient torque, using the Andoyer variables. The used method in this paper to analyze stability is the Kovalev-Savchenko theorem. This method requires the reduction of the Hamiltonian in its normal form up to fourth order by means of canonical transformations around equilibrium points. The coefficients of the normal Hamiltonian are indispensable in the study of nonlinear stability of its equilibrium points according to the three established conditions in the theorem. Some physical and orbital data of real satellites were used in the numerical simulations. In comparison with previous work, the results show a greater number of equilibrium points and an optimization in the algorithm to determine the normal form and stability analysis. The results of this paper can directly contribute in maintaining the attitude of artificial satellites.

1. Introduction

Stability analysis of the rotational motion of a satellite taking into account the influence of external torques is very important in maintaining the attitude to ensure the success of a space mission.

Recently, some studies on the subject have been developed, and they motivated the development of this paper. In [1, 2] is presented a study on the stability of the rotational motion of artificial satellites in an elliptic orbit, considering disturbance due to gravity

gradient torque, using canonical formulations and the Andoyer canonical variables. These studies use the procedure presented in [3] to determine a normal form of the Hamiltonian up to 4th order.

In [4] is developed a numerical-analytical method for normalization of Hamiltonian systems with 2 and 3 degrees of freedom. The normal form is obtained using the Lie-Hori method [5]. The stability analysis of the system is made by the Kovalev-Savchenko theorem [6]. The most important role of this work is the results obtained analytically for the generating function of 3rd order, necessary for determining the coefficients of the normal Hamiltonian of 4th order.

Thus, the objective of this work is to optimize the stability analysis developed in [2] to determine the equilibrium points, the normal form for dynamical systems with two degrees of freedom, and the applications of the Kovalev-Savchenko theorem [6]. It will be done by applying the expressions obtained in [4] for the coefficients of the normal 4th-order Hamiltonian.

In this paper equilibrium points and/or regions of stability are established when parcels associated with gravity gradient torque acting on the satellite are included in the equations of rotational motion.

The Andoyer variables are used to describe the rotational motion of the satellite in order to facilitate the application of methods of stability of Hamiltonian systems. The Andoyer canonical variables [7] are represented by generalized moments (L_1, L_2, L_3) and by generalized coordinates (ℓ_1, ℓ_2, ℓ_3) that are outlined in Figure 1. The angular variables ℓ_1, ℓ_2, ℓ_3 are angles related to the satellite system Oxyz (with axes parallel to the spacecraft's principal axes of inertia) and equatorial system OXYZ (with axes parallel to the axis of the Earth's equatorial system). Variables metrics L_1, L_2, L_3 are defined as follows: L_2 is the magnitude of the angular momentum of rotation \vec{L}_2 , L_1 is the projection of \vec{L}_2 on the z-axis of principal axis system of inertia ($L_1 = L_2 \cos J_2$, where J_2 is the angle between the z-satellite axis and \vec{L}_2), and L_3 is the projection of \vec{L}_2 on the Z-equatorial axis ($L_3 = L_2 \cos I_2$, where I_2 is the angle between Z-equatorial axis and \vec{L}_2).

The nonlinear stability of equilibrium points of the rotational motion is analyzed here by the Kovalev-Savchenko theorem [6], which requires the normalized Hamiltonian up to terms of fourth order around the equilibrium points.

The equilibrium points are found from the equations of motion. With the application of the Kovalev-Savchenko theorem, it is possible to verify if they remain stable under the influence of terms of higher order of the normal Hamiltonian.

In this paper numerical simulations were made for two hypothetical groups of artificial satellites, considering them in a circular orbit and with symmetric shape in relation to their physical and geometric characteristics. The satellites are classified as medium and small sized; they have orbital data and physical characteristics similar to real satellites.

2. Equations of Motion

The Andoyer variables, defined above, are used to characterize the rotational motion of a satellite around its center of mass [7], and the Delaunay variables describe the translational motion of the center of mass of the satellite around the Earth [8].

The Delaunay variables (L, G, H, l, g, h) are defined as [8] $L = M\sqrt{\mu a}$, $G = L\sqrt{1 - e^2}$, $H = G \cos I$, l is the mean anomaly, g is the argument of perigee, h is the longitude

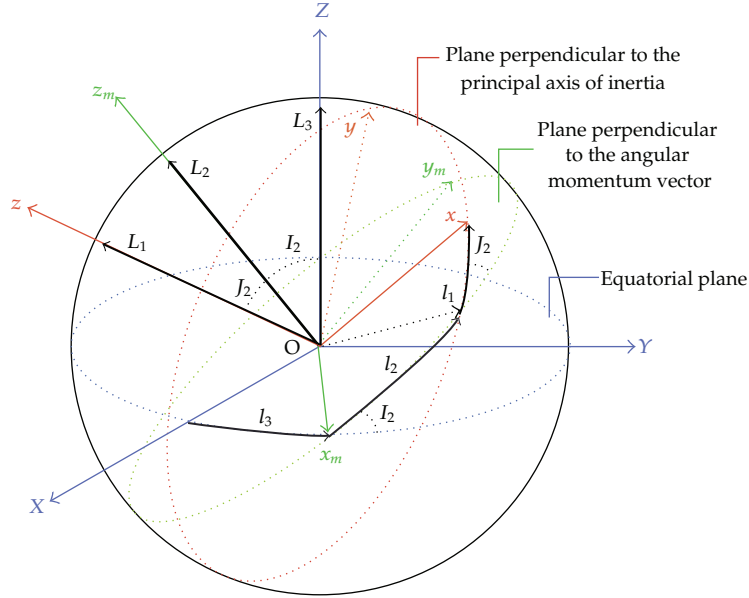


Figure 1: The Andoyer canonical variables.

of the ascending node, M is the mass of the satellite, μ is the Earth gravitational parameter, I is the inclination of the orbit, a is the semimajor axis, and e is the orbital eccentricity.

Here it is assumed that the satellites are in a circular orbit, which differs from the study presented in [2] which considers satellites in eccentric orbits. This consideration was adopted to simplify the Hamiltonian of the problem, which is extensive [1], and to facilitate the stability analysis of the equilibrium points.

Thus, assuming that the satellites have well-defined circular orbit, the main focus of the work is only aimed at stability analysis of the rotational movement of the satellites in study.

In this case the Hamiltonian of the problem is expressed in terms of the Andoyer and Delaunay variables [9] as follows:

$$F(L_1, L_2, L_3, \ell_2, \ell_3, L, G, H, l, g, h) = F_0(L, L_1, L_2) + F_1(L_1, L_2, L_3, \ell_2, \ell_3, L, G, H, l, g, h), \quad (2.1)$$

where F_0 is the unperturbed Hamiltonian and F_1 is the term of the Hamiltonian associated with the disturbance due to the gravity gradient torque, both are described respectively by [10]

$$F_0 = -\frac{\mu^2 M^3}{2L^2} + \frac{1}{2} \left(\left(\frac{1}{C} - \frac{1}{2A} - \frac{1}{2B} \right) L_1^2 + \frac{1}{2} \left(\frac{1}{A} + \frac{1}{B} \right) L_2^2 \right) + \frac{1}{4} \left(\frac{1}{B} - \frac{1}{A} \right) (L_2^2 - L_1^2) \cos 2\ell_1, \\ F_1 = \frac{\mu^4 M^6}{L^6} \left[\frac{2C - A - B}{2} \mathcal{H}_1(\ell_m, L_n) + \frac{A - B}{4} \mathcal{H}_2(\ell_m, L_n) \right], \quad (2.2)$$

where $m = 2, 3$ and $n = 1, 2, 3$; A, B , and C are the principal moments of inertia of the satellite on x -axis, y -axis, and z -axis, respectively; \mathcal{L}_1 and \mathcal{L}_2 are functions of the variables (ℓ_m, L_n) , where ℓ_2 and ℓ_3 appear in the arguments of cosines. The complete analytical expression is presented in [1] for eccentric orbits. In this paper F_1 will be simplified considering the satellite in a circular orbit; its complete analytical expression is given in the Appendix.

Thus, the equations of motion associated with the Hamiltonian (2.1) are given by

$$\frac{d\ell_i}{dt} = \frac{\partial F}{\partial L_i}, \quad \frac{dL_i}{dt} = -\frac{\partial F}{\partial \ell_i} \quad (i = 1, 2, 3). \quad (2.3)$$

These equations are used to find the possible equilibrium points of the rotational movement.

In this study, it is also considered that the satellite has two of its principal moments of inertia equal, $B = A$. With this relationship, the variable ℓ_1 will not be present in the Hamiltonian, reducing the dynamic system to two degrees of freedom, a necessary condition for applying the stability theorem chosen for analysis of equilibrium points.

3. Determining the Normal Form of the Hamiltonian of the Rotational Motion

3.1. Normal Form of the Hamiltonian of 2nd Order and Linear Stability Analysis

Because the variable ℓ_1 is a cyclic coordinate, the Hamiltonian (2.1) is written as $F(L_2, L_3, \ell_2, \ell_3)$ and the equations of motion (2.3) can be written in vector form as [1]

$$\dot{\mathbf{w}} = JF_{\mathbf{w}}, \quad (3.1)$$

where \mathbf{w} is the state vector and $F_{\mathbf{w}}$ is the matrix of partial derivatives with respect to their respective variables given by

$$\mathbf{w} = \begin{pmatrix} \ell_2 \\ L_2 \\ \ell_3 \\ L_3 \end{pmatrix}, \quad F_{\mathbf{w}} = \begin{pmatrix} \frac{\partial F}{\partial \ell_2} \\ \frac{\partial F}{\partial L_2} \\ \frac{\partial F}{\partial \ell_3} \\ \frac{\partial F}{\partial L_3} \end{pmatrix} \quad (3.2)$$

and J a symplectic matrix given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (3.3)$$

Now, it is necessary to linearize the system around the equilibrium point. However, it is convenient to make a translation of the coordinates in the Hamiltonian (2.1) so that the origin coincides with the equilibrium point under study; it means that

$$\begin{aligned} L_1 &= L_{1e} , \\ \ell_2 &= \ell_{2e} + q_1, \\ L_2 &= L_{2e} + p_1, \\ \ell_3 &= \ell_{3e} + q_2, \\ L_3 &= L_{3e} + p_2, \end{aligned} \quad (3.4)$$

where $L_{1e}, \ell_{2e}, L_{2e}, \ell_{3e}$, and L_{3e} are the coordinates of equilibrium points.

With this translation we can expand the Hamiltonian in Taylor series around the new origin that is nothing more than expand the Hamiltonian in the neighborhoods of the equilibrium point when $q_1 = p_1 = q_2 = p_2 = 0$. Then,

$$F(\ell_m, L_n) = \sum_{k=2}^{\infty} F_k(\ell_m, L_n) = F_2(\ell_m, L_n) + F_3(\ell_m, L_n) + F_4(\ell_m, L_n) + \dots, \quad (3.5)$$

where $m = 2, 3, n = 1, 2, 3$, and F_k is the Hamiltonian expanded up to terms of order k , with $k = 2, 3, 4, \dots$

The Hessian P of the problem is calculated from the Hamiltonian expanded up to 2nd order around the equilibrium point:

$$P = \begin{pmatrix} \frac{\partial^2 F}{\partial q_1^2} & \frac{\partial^2 F}{\partial q_1 \partial p_1} & \frac{\partial^2 F}{\partial q_1 \partial q_2} & \frac{\partial^2 F}{\partial q_1 \partial p_2} \\ \frac{\partial^2 F}{\partial p_1 \partial q_1} & \frac{\partial^2 F}{\partial p_1^2} & \frac{\partial^2 F}{\partial p_1 \partial q_2} & \frac{\partial^2 F}{\partial p_1 \partial p_2} \\ \frac{\partial^2 F}{\partial q_2 \partial q_1} & \frac{\partial^2 F}{\partial q_2 \partial p_1} & \frac{\partial^2 F}{\partial q_2^2} & \frac{\partial^2 F}{\partial q_2 \partial p_2} \\ \frac{\partial^2 F}{\partial p_2 \partial q_1} & \frac{\partial^2 F}{\partial p_2 \partial p_1} & \frac{\partial^2 F}{\partial p_2 \partial q_2} & \frac{\partial^2 F}{\partial p_2^2} \end{pmatrix}. \quad (3.6)$$

Thus, the system of linearized equations can be written as

$$\dot{\mathcal{W}} = JP\mathcal{W}, \quad (3.7)$$

where \mathcal{W} is the state vector in the new variables

$$\mathcal{W} = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}. \quad (3.8)$$

So, to find the eigenvalues of the reduced linear system, we should diagonalize the matrix JP present in (3.7) by means of the determinant $\det(\lambda I - JP) = 0$, where I is an identity matrix and λ are the eigenvalues of the matrix JP . If these eigenvalues are pure imaginary $\lambda_j = \pm i\omega_j$, the normal form of 2nd-order Hamiltonian can be expressed as [2]

$$H_2^*(q_j, p_j) = \sum_{j=1}^2 \frac{\omega_j}{2} (q_j^2 + p_j^2). \quad (3.9)$$

3.2. Extension of the Normal Form of the Hamiltonian to Higher Orders

Extending the process of normalization of the Hamiltonian to higher orders (H_3, H_4, \dots), the Hamiltonian normal in terms of variables q_i and p_i can be expressed by

$$H(q_j, p_j) = \sum_{j=1}^2 \frac{\omega_j}{2} (q_j^2 + p_j^2) + H_3^*(q_j, p_j) + H_4^*(q_j, p_j) + \dots. \quad (3.10)$$

To obtain the Hamiltonian normal H_3^*, H_4^*, \dots , it is necessary to use the method of Lie-Hori. According to [11], it is desirable that the normal Hamiltonian is written in complex variables before using this method.

This transformation of variables is given by

$$q_j = \frac{1}{\sqrt{2}}(x_j + iy_j), \quad p_j = \frac{1}{\sqrt{2}}(ix_j + y_j) \quad (3.11)$$

with inverse given by

$$x_j = \frac{1}{\sqrt{2}}(q_j - ip_j), \quad y_j = \frac{1}{\sqrt{2}}(-iq_j + p_j). \quad (3.12)$$

Thus, the normal Hamiltonian of 2nd order is given by

$$H_2^*(x_j, y_j) = \sum_{j=1}^2 i\omega_j x_j y_j \quad (3.13)$$

and normal Hamiltonian until the 4th order in complex variables is represented by

$$H^*(x_j, y_j) = \sum_{j=1}^2 i\omega_j x_j y_j + H_3^*(x_j, y_j) + H_4^*(x_j, y_j) + \dots, \quad (3.14)$$

where $H_3^*(x, y)$ and $H_4^*(x, y)$ can be expressed briefly as

$$H_3^*(x, y) = \sum_{|\alpha|+|\beta|=3} h_{3,\alpha,\beta} x^\alpha y^\beta, \quad (3.15)$$

$$H_4^*(x, y) = \sum_{|\alpha|+|\beta|=4} h_{4,\alpha,\beta} x^\alpha y^\beta. \quad (3.16)$$

In (3.15) and (3.16), it is always assumed that $\alpha, \beta > 0$. For (3.15), $|\alpha| + |\beta| = 3$ means that the sum of the exponents of the new variables must obey the order of the Hamiltonian H_3^* , and, for (3.16), $|\alpha| + |\beta| = 4$ means that the sum of the exponents of the new variables must obey the order of the Hamiltonian H_4^* . These equations are presented entirely in [1, 4].

Thus, we have the Hamiltonian written in complex variables. It is important to note that H_2 is already normalized, and now we can apply the method of Lie-Hori to find the normal form to higher orders H_3^*, H_4^* .

3.2.1. Series Method of Lie-Hori

The normal form is obtained from the expansion of the Hamiltonian in terms of the Lie series given by [12, 13]

$$H^{\text{new}} = H^* + \{H^*, G\} + \frac{1}{2!} \{\{H^*, G\}, G\} + \frac{1}{3!} \{\{\{H^*, G\}, G\}, G\} + \dots, \quad (3.17)$$

where H^{new} is normal Hamiltonian obtained by canonical transformations around the equilibrium point, $G = G(x_i, y_i)$ is the generating function developed as a power series, where each degree of G_n is a sum of homogeneous polynomials of degree n , $H^* = H^*(x, y)$ is the original Hamiltonian, and $\{H^*, G\}$ is the Poisson brackets of order $r + s - 2$, where r and s represent the order of the polynomials H^* and G , respectively.

The Poisson bracket is defined as

$$\{H^*, G\} = \frac{\partial H^*}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial H^*}{\partial y} \frac{\partial G}{\partial x}. \quad (3.18)$$

The new Hamiltonian H^{new} after the new canonical transformation in the neighborhoods of identity can be expressed as [2]

$$H^{\text{new}} = H_2^{\text{new}} + H_3^{\text{new}} + H_4^{\text{new}} + \dots. \quad (3.19)$$

So the new Hamiltonian, ordered by degree up to 4th order, can be expressed following the next form [11, 13]:

$$\text{2nd order: } H_2^{\text{new}} = H_2^*, \quad (3.20)$$

$$\text{3rd order: } H_3^{\text{new}} = H_3^* + \{H_2^*, G_3\}, \quad (3.21)$$

$$\text{4th order: } H_4^{\text{new}} = H_4^* + \{H_3^*, G_3\} + \frac{1}{2!} \{ \{H_2^*, G_3\}, G_3 \} + \{H_2^*, G_4\}. \quad (3.22)$$

As H_2^{new} is already given in its normal form, the next step is to calculate the generating function which leads to the normal form up to 4th-order terms. This procedure allows to obtain the minimum number of nonresonant monomials defined in [12]; thus, the generating function G_n is chosen to eliminate the complex variables in the total Hamiltonian (3.5) in which the monomials x_j and y_j have different exponents of the desired degree, leaving only the monomials that carry on the resonance of the intrinsic Hamiltonian systems. The following is a procedure to obtain the normal Hamiltonian to 4th order [1, 4].

(I) It is known that Hamiltonian systems are naturally resonant in even orders, 2nd, 4th, and so forth [10, 12], so to find the normal Hamiltonian until the 4th order, the 3rd disturbance H_3^{new} must first be removed from H^{new} , it means

$$H_3^{\text{new}} = 0. \quad (3.23)$$

Assuming H_3^* is briefly defined in (3.15) to eliminate terms of 3rd order, we consider the generating function $G_3(x, y)$ expressed in abbreviated form as

$$G_3(x, y) = \sum_{|\alpha|+|\beta|=3} g_{3,\alpha,\beta} x^\alpha y^\beta. \quad (3.24)$$

Then it is necessary to find $G_3(x, y)$ of the third-order homological equation (3.21), where

$$\{H_2^*, G_3\} = -H_3^*. \quad (3.25)$$

After some algebraic manipulations, we find $G_3(x, y)$ given by [4]

$$G_3(x, y) = \sum_{|\alpha|+|\beta|=3} -\frac{h_{3,\alpha,\beta}}{\langle \beta - \alpha, W \rangle} x^\alpha y^\beta. \quad (3.26)$$

Note that this equation is well defined if $\langle \beta - \alpha, W \rangle \neq 0$, where $\langle \beta - \alpha, W \rangle$ represents an inner product between $\beta - \alpha = (\beta_1 - \alpha_1, \beta_2 - \alpha_2, \dots, \beta_n - \alpha_n)$ and $W = (\lambda_1, \lambda_2, \dots, \lambda_n) = (i\omega_1, i\omega_2, \dots, i\omega_n)$ which are pure imaginary eigenvalues obtained from the quadratic part of normal Hamiltonian. The complete analytical representation for $\langle \beta - \alpha, W \rangle$ was presented in [4].

For the resonance condition it is not accepted that $\langle \beta - \alpha, W \rangle \neq 0$, which is always true for monomials of third-order Hamiltonian when considering only the natural resonance of the Hamiltonian.

(II) This process is repeated for the polynomial of degree 4 finding $G_4(x, y)$ that will eliminate some of monomials H_4^* in (3.16). The remaining monomials in H_4^{new} are called resonant monomials [3]. Then (3.22) can be reduced as [1]

$$\{G_4, H_2^*\} + H_4^{\text{new}} = H_4^* + \{H_3^*, G_3\} + \frac{1}{2!} \{\{H_2^*, G_3\}, G_3\}, \quad (3.27)$$

$$\{G_4, H_2^*\} + H_4^{\text{new}} = F_4. \quad (3.28)$$

Equation (3.28) is called homological equation of 4th order, and it can be rewritten as

$$\begin{aligned} \{G_4, H_2^*\} &= F_4 - H_4^{\text{new}}, \\ \{G_4, H_2^*\} &= R_4. \end{aligned} \quad (3.29)$$

Assume that $R_4(x, y)$ and $G_4(x, y)$ will be expressed in short form as

$$R_4(x, y) = \sum_{|\alpha|+|\beta|=4} r_{4,\alpha,\beta} x^\alpha y^\beta, \quad (3.30)$$

$$G_4(x, y) = \sum_{|\alpha|+|\beta|=4} g_{4,\alpha,\beta} x^\alpha y^\beta. \quad (3.31)$$

After some algebraic manipulations, we find $G_4(x, y)$ given by [4]

$$G_4(x, y) = \sum_{|\alpha|+|\beta|=4} \frac{r_{4,\alpha,\beta}}{\langle \alpha - \beta, W \rangle} x^\alpha y^\beta, \quad (3.32)$$

where $\langle \alpha - \beta, W \rangle \neq 0$ represents an inner product as mentioned above.

Thus, using (3.34) in (3.29) it is possible to find H_4^{new} after some algebraic manipulations.

This procedure can be repeated for any order that is required in order to find a normal form of the new Hamiltonian H^{new} . In this paper it is considered the 4th order of the H^{new} .

3.3. Normal Form of the Hamiltonian of 4th Order and Nonlinear Stability Analysis

As we get the normal form of 2nd-order Hamiltonian (3.9), we can apply the method described previously to find the Hamiltonian normal until the 4th order using the Hamiltonian in complex variables (3.14) represented by [1, 4]

$$H^*(x_j, y_j) = i\omega_1 x_1 y_1 + i\omega_2 x_2 y_2 + H_3^*(x_1, y_1, x_2, y_2) + H_4^*(x_1, y_1, x_2, y_2) + \dots, \quad (3.33)$$

where $H_3^*(x_1, y_1, x_2, y_2)$ and $H_4^*(x_1, y_1, x_2, y_2)$ are obtained from expressions

$$\begin{aligned} H_3^*(x_1, y_1, x_2, y_2) &= \sum_{|\alpha|+|\beta|=3} h_{3,\alpha_1,\beta_1,\alpha_2,\beta_2} x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}, \\ H_4^*(x_1, y_1, x_2, y_2) &= \sum_{|\alpha|+|\beta|=4} h_{4,\alpha_1,\beta_1,\alpha_2,\beta_2} x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}. \end{aligned} \quad (3.34)$$

Using the method of Lie-Hori presented before, it is possible to determine the generating function $G_3(x_1, y_1, x_2, y_2)$ in order to satisfy the homological equation (3.25), where $G_3(x_1, y_1, x_2, y_2)$ was determined analytically in [4]. This generating function is responsible for the elimination of the 3rd-order terms of the Hamiltonian (3.33).

The generating function $G_3(x_1, y_1, x_2, y_2)$ is used to find the terms of 4th order, F_4 , and perform the separation described in (3.28), obtaining H_4 and $G_4(x_1, y_1, x_2, y_2)$. The coefficients of the normal form H_4^{new} can be calculated using the terms obtained in H_4 and the eigenvalues of H_2 .

Thus, the normal Hamiltonian in complex variables separated by degree can be expressed as follows[4]:

$$\text{2nd order: } H_2^{\text{new}} = i\omega_1 x_1 y_1 + i\omega_2 x_2 y_2, \quad (3.35)$$

$$\text{3rd order: } H_3^{\text{new}} = 0, \quad (3.36)$$

$$\text{4th order: } H_4^{\text{new}} = \delta_{11}(x_1 y_1)^2 + \delta_{12}(x_1 y_1 x_2 y_2) + \delta_{22}(x_2 y_2)^2, \quad (3.37)$$

where $\omega_j (j = 1, 2)$ is the imaginary part of eigenvalues associated with the matrix defined by the product of a matrix symplectic of order 4 with the Hessian of the Hamiltonian expanded in Taylor series up to 2nd order around the equilibrium point; δ_{ij} are real coefficients obtained from the combination of $h_{k,\alpha,\beta} x^\alpha y^\beta$ ($k = 3, 4$).

However, the Kovalev-Savchenko theorem requires that the normal Hamiltonian is in real variables, and with the applications of the transformation of variables (3.12) the normal form Hamiltonian can be expressed as follows:

$$\text{2nd order: } H_2^{\text{new}} = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \frac{\omega_2}{2}(q_2^2 + p_2^2), \quad (3.38)$$

$$\text{3rd order: } H_3^{\text{new}} = 0, \quad (3.39)$$

$$\begin{aligned} \text{4th order: } H_4^{\text{new}} &= \delta_{11}(q_1^4 + p_1^4 + 2q_1^2 p_1^2) + \delta_{12}(q_1^2 q_2^2 + q_1^2 p_2^2 + p_1^2 q_2^2 + p_1^2 p_2^2) \\ &+ \delta_{22}(q_2^4 + p_2^4 + 2q_2^2 p_2^2), \end{aligned} \quad (3.40)$$

where δ_{ij} are the coefficients of the Hamiltonian normal 4th order which are expressed analytically in [4].

4. Kovalev-Savchenko Theorem

To use the stability theorem, the normal Hamiltonian of the problem is necessary as it was discussed above.

Considering the normal Hamiltonian \mathcal{H}^o , an analytics function of coordinates (q_ν) and generalized moments (p_ν) to a fixed point P is expressed by

$$\mathcal{H}^o = \sum_{\nu=1}^2 \frac{\omega_\nu^o}{2} \mathcal{R}_\nu + \sum_{\nu,v=1}^2 \frac{\delta_{\nu,v}^o}{4} \mathcal{R}_\nu \mathcal{R}_v + O_5, \quad (4.1)$$

where O_5 represents higher-order terms; ω_ν^o is the imaginary part of eigenvalues associated with the matrix defined by the product of a 4th-order matrix symplectic with the Hessian of the Hamiltonian expanded in Taylor series up to 2nd order around the equilibrium point; $\delta_{\nu,v}^o$ depend on the eigenvalues ω_ν^o and the coefficients of the Hamiltonian expanded in Taylor series of 3rd and 4th order around the equilibrium point and they are presented analytically in Formiga [4], and

$$\mathcal{R}_m = q_m^2 + p_m^2 \quad \text{with } m = 1, 2. \quad (4.2)$$

The stability analysis is performed here by the Kovalev-Savchenko theorem [6] which ensures that the motion is Lyapunov stable if the following conditions are satisfied.

- (i) The eigenvalues of the reduced linear system are pure imaginary $\pm i\omega_1^o$ and $\pm i\omega_2^o$.
- (ii) The condition

$$k_1\omega_1^o + k_2\omega_2^o \neq 0 \quad (4.3)$$

is valid for all k_1 and k_2 integers satisfying the inequality

$$|k_1| + |k_2| \leq 4. \quad (4.4)$$

- (iii) The determinant D^o must satisfy the inequality

$$D^o = -\left(\delta_{11}^o \omega_2^{o2} - 2\delta_{12}^o \omega_1^o \omega_2^o + \delta_{22}^o \omega_1^{o2}\right) \neq 0, \quad (4.5)$$

where $\delta_{\nu,v}^o$ are the coefficients of the normal 4th-order Hamiltonian.

This theorem says that a Hamiltonian reduced in its normal form up to 4th order, in the absence of the resonance condition of the eigenvalues associated and if the condition (4.5) is satisfied, it is guaranteed the existence of tori invariant in a neighborhood small enough of equilibrium position [14–16].

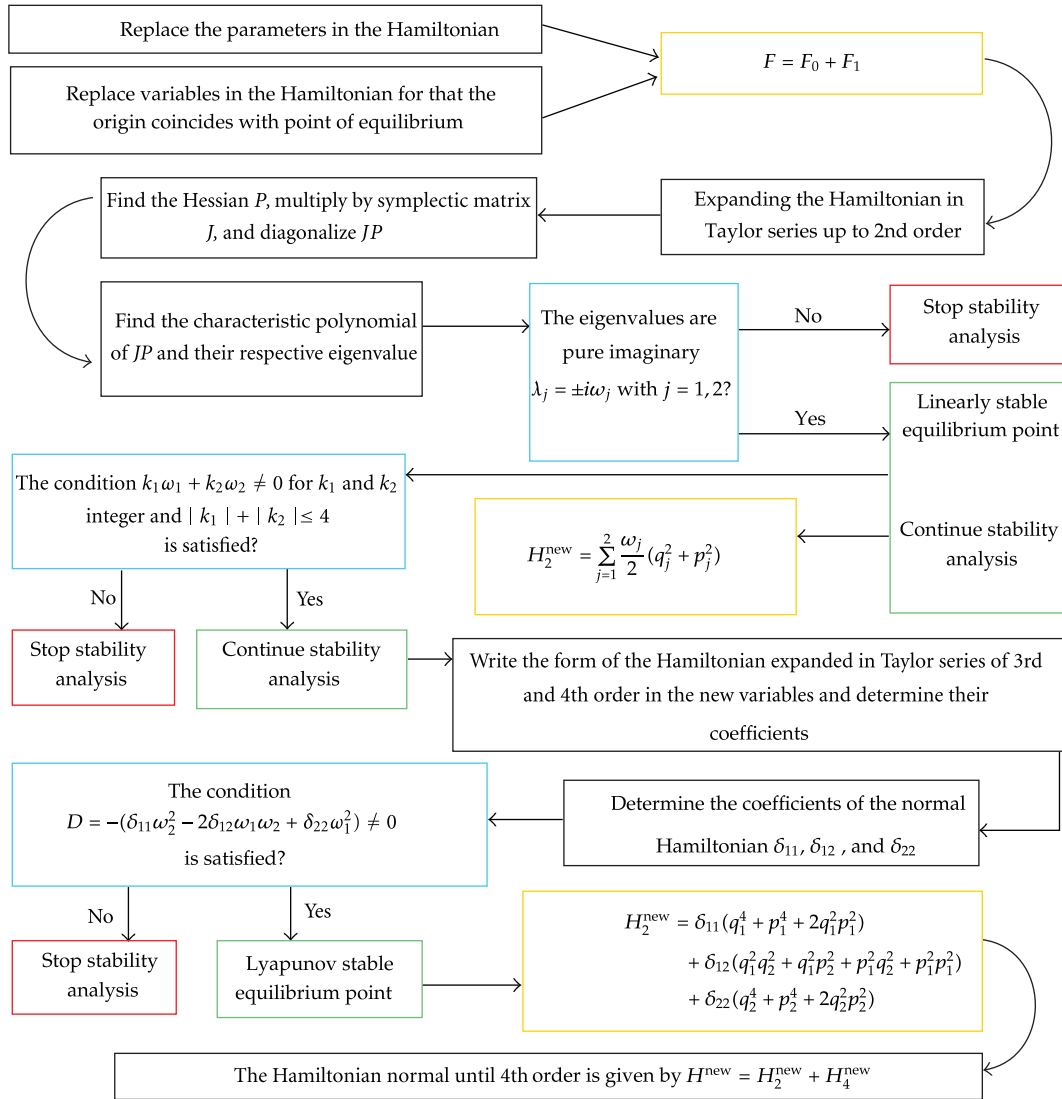


Figure 2: Flowchart representative of the stability analysis of equilibrium points.

5. Computational Algorithm for the Normal Form of the Hamiltonian of Rotational Motion and Analysis of Stability

In order to synthesize the process to analyze the stability of equilibrium points, the logical sequence of the algorithm is now presented.

5.1. Stability Analysis of Equilibrium Points

See Figure 2.

Table 1: Quantitative summary of the classification of equilibrium points.

Satellite	Stable points		Unstable points	
MP	2	Failed 1st condition	Failed 2nd condition	Failed 3rd condition
		58	0	0
PP-1	7	Failed 1st condition	Failed 2nd condition	Failed 3rd condition
		43	0	0
PP-2	10	Failed 1st condition	Failed 2nd condition	Failed 3rd condition
		40	0	0

6. Numerical Simulations

As already mentioned, we consider two types of satellites: medium sized (MP), which has similar orbital characteristics with the American satellite PEGASUS [17], and small sized (PP), which has similar orbital characteristics of the Brazilian data collection satellite SCD-1 and SCD-2 [18]. All the numerical simulations were developed using the software MATHEMATICA.

Table 1 shows a quantitative summary of the equilibrium points found and their stability, according to the criteria of Kovalev-Savchenko theorem [6]. It can be observed that, when the first condition is not satisfied, the eigenvalues associated with the matrix JP are real or are not pure imaginary. This equilibrium point is not linearly stable.

There were found totals of 60 equilibrium points for the MP satellite, 50 equilibrium points to the PP-1 satellite and 50 equilibrium points for the PP-2 satellite. Tables 2, 3 and 4, show two equilibrium points found in the simulations, one being Lyapunov stable and the other unstable, to satellites MP, PP1 and PP2, respectively.

For the Lyapunov stable equilibrium points of Tables 2, 3, and 4 respectively, Table 5 shows the values of the angles I_2 , J_2 , the rotation speed ω , and the rotation period T . These values characterize the nonexistence of singularities in these points; it means that the angles I_2 and J_2 are not null or close to zero. By Figure 1, when these angles I_2 and J_2 are null or close to zero, the Andoyer variables l_1 , l_2 , and l_3 are indeterminate, because it is difficult to determine the intersection between the involved planes in the definitions of these variables. This analysis was performed for all equilibrium points found in the simulations. Table 6 shows the same conditions for the unstable points of Tables 2, 3, and 4.

7. Conclusion

In this paper we presented a semianalytical stability of the rotational motion of artificial satellites, considering the influence of gravity gradient torque for symmetric satellite in a circular orbit. Applications in numerical simulations, performed with the software MATHEMATICA, were made to two types of satellites: medium (MP) and small (PP).

Initially the points of equilibrium were determined using the physical, orbital, and attitude characteristics of each satellite. Then the algorithm for stability analysis was applied and it was obtained 2 stable equilibrium points for the satellite MP, 7 stable points for the satellite PP-1, and 10 stable points for the satellite PP-2.

Table 2: Satellite MP: $A = B = 3,9499 * 10^{-1} \text{ kg} \cdot (\text{km})^2$, $C = 1,0307 * 10^{-1} \text{ kg} \cdot (\text{km})^2$.

Equilibrium points	Lyapunov stable	Unstable
$L_1(\text{kg} \cdot \text{km}^2/\text{s})$	$2,50725 * 10^{-4}$	$-1,34202 * 10^{-11}$
$L_2(\text{kg} \cdot \text{km}^2/\text{s})$	$3,85388 * 10^{-4}$	$-4,19645 * 10^{-10}$
$L_3(\text{kg} \cdot \text{km}^2/\text{s})$	$-2,28562 * 10^{-4}$	$-3,53991 * 10^{-10}$
$l_2(\text{rad})$	$-1,6 * 10^{-7}$	0,0934286
$l_3(\text{rad})$	0,5235	0,41354

Table 3: Satellite PP-1: $A = B = 9.20 * 10^{-6} \text{ kg} \cdot (\text{km})^2$, $C = 13 * 10^{-6} \text{ kg} \cdot (\text{km})^2$.

Equilibrium points	Lyapunov stable	Unstable
$L_1(\text{kg} \cdot \text{km}^2/\text{s})$	$2,69365 * 10^{-7}$	$7,13725 * 10^{-11}$
$L_2(\text{kg} \cdot \text{km}^2/\text{s})$	$2,07921 * 10^{-5}$	$2,14416 * 10^{-10}$
$L_3(\text{kg} \cdot \text{km}^2/\text{s})$	$1,05633 * 10^{-5}$	$1,07843 * 10^{-10}$
$l_2(\text{rad})$	0,066560	-0,090962
$l_3(\text{rad})$	0,4	0,07

Table 4: PP-2: $A = B = 28,7 * 10^{-4} \text{ kg} \cdot (\text{km})^2$, $C = 36,9 * 10^{-4} \text{ kg} \cdot (\text{km})^2$.

Equilibrium points	Lyapunov stable	Unstable
$L_1(\text{kg} \cdot \text{km}^2/\text{s})$	$6,25509 * 10^{-5}$	$2,48081 * 10^{-11}$
$L_2(\text{kg} \cdot \text{km}^2/\text{s})$	$7,24797 * 10^{-4}$	$2,77796 * 10^{-11}$
$L_3(\text{kg} \cdot \text{km}^2/\text{s})$	$2,17564 * 10^{-4}$	$-2,54792 * 10^{-11}$
$l_2(\text{rad})$	0,098581	-1,68042
$l_3(\text{rad})$	1,42527	1,05253

For the satellite MP it was gotten only two equilibrium points because this satellite has similar characteristics to the satellite PEGASUS, which is tumbling [17]. For satellites PP-1 and PP-2 were obtained many other equilibrium points, but most of them were discarded, because they lead to the Andoyer variables, a condition of uniqueness (it means that the angles I_2 and J_2 are null or close to zero, and the Andoyer variables l_1 , l_2 , and l_3 are indeterminate).

It can be observed that a larger number of stable equilibrium points were determined in comparison with the results of [1, 2], which show the stability analysis of the rotational motion with the gravity gradient torque, but with the satellite in an elliptic orbit.

An optimization was done in the algorithm of determining the normal form of the Hamiltonian and the stability analysis, using expressions obtained by [4] for the coefficients of the normal 4th-order Hamiltonian. The introduction of these expressions enabled a more effective stability analysis for equilibrium points in comparison with the results of [1, 2]. It is possible to say that the numerical simulations have become less laborious allowing analysis of data in more numbers .

This paper presents results that can directly contribute in maintaining the attitude of artificial satellites. Once the regions of stability are known for the rotational motion, a smaller number of maneuvers to maintain the desired attitude can be accomplished. In this case, a fuel economy can be generated to the satellite with propulsion system, increasing the satellite's lifetime.

Table 5: Analysis of possible singularities of the Anloyer variables for satellites MP, PP1 of and PP2.

Lyapunov stable equilibrium point of Table 2, 3 and 4, respectively.					
Satellite	I_2 (rad)	J_2 (rad)	ω (rad/s)	T (s)	Situation
MP	2,20566	0,862451	0,00373909	1680,4	Accepted
PP-1	1,03788	1,55784	1,59939	3,92848	Accepted
PP-2	1,26592	1,48439	0,196422	31,9882	Accepted

Table 6: Analysis of possible singularities of the Anloyer variables for satellites MP, PP1 and PP2.

Unstable equilibrium point of Table 2, 3, and 4, respectively.					
Satellite	I_2 (rad)	J_2 (rad)	ω (rad/s)	T (s)	Situation
MP	0,56693	1,53881	$-4,071 * 10^{-9}$	$-1,543 * 10^9$	Accepted
PP-1	1,04377	1,23145	$1,64936 * 10^{-5}$	380948,0	Accepted
PP-2	2,73176	0,46676	$7,52836 * 10^{-9}$	$8,346 * 10^8$	Accepted

Appendix

The disturbance Hamiltonian F_1 , due to the gravity gradient torque, for a satellite in a circular orbit with two of its principal moments of inertia equal, $B = A$, is given by [19]

$$\begin{aligned}
 F_1 = & \frac{\mu^4 M^7}{L^6} \left\{ \frac{C - A}{M} \left\{ \left(\left(\frac{3}{2} \left(\frac{L_1}{L_2} \right)^2 - \frac{1}{2} \right) \right. \right. \right. \\
 & \times \left(-\frac{1}{2} + \frac{3}{8} \left(1 + \left(\frac{H}{G} \right)^2 + \left(\frac{L_3}{L_2} \right)^2 - 3 \left(\frac{H}{G} \right)^2 \left(\frac{L_3}{L_2} \right)^2 \right) \right. \\
 & - \frac{3}{8} \sin[2I] \sin[2I_2] \cos[h - \ell_3] \\
 & \left. \left. \left. - \frac{3}{8} \sin^2[I] \sin^2[I_2] \cos[2h - 2\ell_3] \right) \right. \right. \\
 & - \frac{3}{16} \left(1 - 3 \left(\frac{H}{G} \right)^2 \right) \sin[2I_2] \sin[2J_2] \cos[\ell_2] \\
 & + \frac{3}{16} \left(1 - 3 \left(\frac{H}{G} \right)^2 \right) \sin^2[I_2] \sin^2[J_2] \cos[2\ell_2] \\
 & + \frac{3}{16} \left(1 - \frac{L_3}{L_2} \right) \left(1 + 2 \frac{L_3}{L_2} \right) \sin[2I] \sin[2J_2] \cos[h - \ell_3 - \ell_2] \\
 & + \frac{3}{16} \left(1 + \frac{L_3}{L_2} \right) \left(1 - 2 \frac{L_3}{L_2} \right) \sin[2I] \sin[2J_2] \cos[h - \ell_3 + \ell_2] \\
 & + \frac{3}{16} \left(1 - \frac{L_3}{L_2} \right) \sin^2[I] \sin[I_2] \sin[2J_2] \cos[2h - 2\ell_3 + \ell_2] \\
 & \left. \left. \left. - \frac{3}{16} \left(1 - \frac{L_3}{L_2} \right) \sin^2[I] \sin[I_2] \sin[2J_2] \right] \cos[2h - \ell_3 + 2\ell_2] \right. \right. \\
 & \left. \left. \right. \right\}
 \end{aligned}$$

$$\begin{aligned}
& -\frac{3}{16} \left(1 - \frac{L_3}{L_2}\right) \sin[2I] \sin[I_2] \sin^2[J_2] \cos[h - \ell_3 + 2\ell_2] \\
& + \frac{3}{16} \left(1 + \frac{L_3}{L_2}\right) \sin[2I] \sin[I_2] \sin^2[J_2] \cos[h - \ell_3 - 2\ell_2] \\
& - \frac{3}{32} \left(1 - \frac{L_3}{L_2}\right)^2 \sin^2[I] \sin^2[J_2] \cos[2h - 2\ell_3 + 2\ell_2] \\
& - \frac{3}{32} \left(1 + \frac{L_3}{L_2}\right)^2 \sin^2[I] \sin^2[J_2] \cos[2h - 2\ell_3 - 2\ell_2] \Big) \\
& + \left(\frac{3}{2} \left(\frac{L_1}{L_2}\right)^2 - \frac{1}{2}\right) \left\{ \frac{3}{8} \sin^2[I] \left(1 - 3 \left(\frac{L_3}{L_2}\right)^2\right) \right. \\
& \quad \times \cos[2l + 2g] \\
& \quad - \frac{3}{8} \left(1 - \frac{H}{G}\right) \sin[I] \sin[I_2] \\
& \quad \times \cos[2l + 2g - (h - \ell_3)] \\
& \quad - \frac{3}{16} \left(1 + \frac{H}{G}\right)^2 \sin^2[I_2] \\
& \quad \times \cos[2l + 2g + (2h - 2\ell_3)] \\
& \quad \left. - \frac{3}{16} \left(1 - \frac{H}{G}\right)^2 \sin^2[I_2] \cos[2l + 2g - (2h - 2\ell_3)] \right\} \\
& + \frac{9}{32} \sin^2[I] \sin[2I_2] \sin[2J_2] \cos[2l + 2g + \ell_2] \\
& + \frac{9}{32} \sin^2[I] \sin[2I_2] \sin[2J_2] \cos[2l + 2g - \ell_2] \\
& - \frac{3}{16} \left(1 + \frac{H}{G}\right) \left(1 - \frac{L_3}{L_2}\right) \left(1 + 2 \frac{L_3}{L_2}\right) \sin[I] \sin[2J_2] \\
& \quad \times \cos[2l + 2g + (h - \ell_3) + \ell_2] \\
& - \frac{3}{16} \left(1 + \frac{H}{G}\right) \left(1 + \frac{L_3}{L_2}\right) \left(1 - 2 \frac{L_3}{L_2}\right) \sin[I] \sin[2J_2] \\
& \quad \times \cos[2l + 2g + (h - \ell_3) - \ell_2] \\
& + \frac{3}{16} \left(1 - \frac{H}{G}\right) \left(1 - \frac{L_3}{L_2}\right) \left(1 + 2 \frac{L_3}{L_2}\right) \sin[I] \sin[2J_2] \\
& \quad \times \cos[2l + 2g - (h - \ell_3) - \ell_2] \\
& + \frac{3}{16} \left(1 - \frac{H}{G}\right) \left(1 + \frac{L_3}{L_2}\right) \left(1 - 2 \frac{L_3}{L_2}\right) \sin[I] \sin[2J_2] \\
& \quad \times \cos[2l + 2g - (h - \ell_3) + \ell_2]
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{32} \left(1 + \frac{H}{G}\right)^2 \left(1 - \frac{L_3}{L_2}\right) \sin[I_2] \sin[2J_2] \\
& \times \cos[2l + 2g + (2h - 2\ell_3) - \ell_2] \\
& - \frac{3}{32} \left(1 + \frac{H}{G}\right)^2 \left(1 + \frac{L_3}{L_2}\right) \sin[I_2] \sin[2J_2] \\
& \times \cos[2l + 2g + (2h - 2\ell_3) + \ell_2] \\
& + \frac{3}{32} \left(1 - \frac{H}{G}\right)^2 \left(1 - \frac{L_3}{L_2}\right) \sin[I_2] \sin[2J_2] \\
& \times \cos[2l + 2g - (2h - 2\ell_3) + \ell_2] \\
& - \frac{3}{32} \left(1 - \frac{H}{G}\right)^2 \left(1 + \frac{L_3}{L_2}\right) \sin[I_2] \sin[2J_2] \\
& \times \cos[2l + 2g - (2h - 2\ell_3) - \ell_2] \\
& - \frac{9}{32} \sin^2[I] \sin^2[I_2] \\
& \times \sin^2[J_2] \cos[2l + 2g + 2\ell_2] \\
& - \frac{9}{32} \sin^2[I] \sin^2[I_2] \sin^2[J_2] \cos[2l + 2g - 2\ell_2] \\
& + \frac{3}{16} \left(1 + \frac{H}{G}\right) \left(1 - \frac{L_3}{L_2}\right) \sin[I] \sin[I_2] \sin^2[J_2] \\
& \times \cos[2l + 2g + (h - \ell_3) + 2\ell_2] \\
& - \frac{3}{16} \left(1 + \frac{H}{G}\right) \left(1 + \frac{L_3}{L_2}\right) \sin[I] \sin[I_2] \sin^2[J_2] \\
& \times \cos[2l + 2g + (h - \ell_3) - 2\ell_2] \\
& - \frac{3}{16} \left(1 - \frac{H}{G}\right) \left(1 - \frac{L_3}{L_2}\right) \sin[I] \sin[I_2] \sin^2[J_2] \\
& \times \cos[2l + 2g - (h - \ell_3) - 2\ell_2] \\
& + \frac{3}{16} \left(1 - \frac{H}{G}\right) \left(1 + \frac{L_3}{L_2}\right) \sin[I] \sin[I_2] \sin^2[J_2] \\
& \times \cos[2l + 2g - (h - \ell_3) + 2\ell_2] \\
& - \frac{3}{64} \left(1 - \frac{H}{G}\right)^2 \left(1 - \frac{L_3}{L_2}\right)^2 \sin^2[J_2] \\
& \times \cos[2l + 2g + (2h - 2\ell_3) + 2\ell_2] \\
& + \frac{3}{64} \left(1 - \frac{H}{G}\right)^2 \left(1 + \frac{L_3}{L_2}\right)^2 \sin^2[J_2]
\end{aligned}$$

$$\begin{aligned}
& \times \cos[2l + 2g + (2h - 2\ell_3) - 2\ell_2] \\
& + \frac{3}{64} \left(1 + \frac{H}{G}\right)^2 \left(1 - \frac{L_3}{L_2}\right)^2 \sin^2[J_2] \\
& \times \cos[2l + 2g - (2h - 2\ell_3) - 2\ell_2] \\
& + \frac{3}{64} \left(1 + \frac{H}{G}\right)^2 \left(1 + \frac{L_3}{L_2}\right)^2 \sin^2[J_2] \\
& \times \cos[2l + 2g - (2h - 2\ell_3) + 2\ell_2] \Big\} \Big\},
\end{aligned} \tag{A.1}$$

where: $A = B$ and C are the principal moments of inertia on x -axis, y -axis and z -axis respectively, M is the mass of the satellite, μ is the gravitational constant, I is the inclination of the orbit, I_2 is the inclination of the plane of angular momentum with the plane of the equator, and J_2 is the inclination of the principal plane with the plane of the angular momentum.

In (A.1) the generalized moments (L_1 , L_2 , L_3) are implicit in some terms, using the definitions of generalized moments present in the introduction and trigonometric properties, they can be explicit replacing.

$$\begin{aligned}
\sin[I_2] &= \sqrt{1 - \frac{L_3^2}{L_2^2}}, \\
\sin^2[I_2] &= 1 - \frac{L_3^2}{L_2^2}, \\
\sin[2I_2] &= \frac{2L_3}{L_2} \sqrt{1 - \frac{L_3^2}{L_2^2}}, \\
\sin[J_2] &= \sqrt{1 - \frac{L_1^2}{L_2^2}}, \\
\sin^2[J_2] &= 1 - \frac{L_1^2}{L_2^2}, \\
\sin[2J_2] &= \frac{2L_1}{L_2} \sqrt{1 - \frac{L_1^2}{L_2^2}}.
\end{aligned} \tag{A.2}$$

Acknowledgment

This present work was supported by CAPES.

References

- [1] R. E. S. Cabette, *Estabilidade do movimento rotacional de satélites artificiais*, Dissertation, National Institute for Space Research, São Paulo, Brazil, 2006.

- [2] R. V. de Moraes, R. E. S. Cabette, M. C. Zanardi, T. J. Stuchi, and J. K. Formiga, "Attitude stability of artificial satellites subject to gravity gradient torque," *Celestial Mechanics & Dynamical Astronomy*, vol. 104, no. 4, pp. 337–353, 2009.
- [3] A. L. F. Machuy, *Cálculo efetivo da forma normal parcial para o problema de Hill*, Dissertação de mestrado—instituto de matemática, UFRJ—Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, 2001.
- [4] J. K. Formiga, *Formas normais no estudo da estabilidade para L4 no problema fotogravitacional*, Dissertation, National Institute for Space Research, São José dos Campos, Brazil, 2009.
- [5] G. Hori, "Theory of general perturbations for non-canonical system," *Astronomical Society of Japan*, vol. 23, no. 4, pp. 567–587, 1971.
- [6] A. M. Kovalov and A. Ia. Savchenko, "Stability of uniform rotations of a rigid body about a principal axis," *Prikladnaia Matematika i Mekhanika*, vol. 39, no. 4, pp. 650–660, 1975.
- [7] H. Kinoshita, "First order perturbation of the two finite body problem," *Publications of the Astronomical Society of Japan*, vol. 24, pp. 423–439, 1972.
- [8] C. D. Murray and S. F. Dermott, *Solar system dynamics*, Cambridge University Press, Cambridge, 1999.
- [9] M. C. Zanardi, *Movimento rotacional e translacional acoplado de satélites artificiais*, Dissertação de mestrado, Instituto de Tecnologia Aeronáutica, São José dos Campos, São Paulo, Brazil, 1983.
- [10] M. C. Zanardi, "Study of the terms of coupling between rotational and translational motions," *Celestial Mechanics*, vol. 39, no. 2, pp. 147–158, 1986.
- [11] T. J. Stuchi, "KAM tori in the center manifold of the 3-D Hill problem," in *Advanced in Space*, O. C. Winter and A. F. B. Prado, Eds., vol. 2, pp. 112–127, INPE, São Paulo, Brazil, 2002.
- [12] G. Hori, "Theory of general perturbations with unspecified canonical variables," *Publications of the Astronomical Society of Japan*, vol. 18, pp. 287–299, 1966.
- [13] S. Ferraz-Mello, *Canonical Perturbation Theories-Degenerate Systems and Resonance*, Springer, New York, NY, USA, 2007.
- [14] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, vol. 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*, Springer, New York, NY, USA, 1983.
- [15] V. I. Arnold, *Equações Diferenciais Ordinárias*, Mir, Moscow, Russia, 1985.
- [16] V. I. Arnold, "Métodos matemáticos da mecânica clássica," *Editora Mir Moscou*, p. 479, 1987.
- [17] J. U. Crenshaw and P. M. Fitzpatrick, "Gravity effects on the rotational motion of a uniaxial artificial satellite," *AIAAJ*, vol. 6, article 2140, 1968.
- [18] H. K. Kuga, V. Orlando, and R. V. F. Lopes, "Flight dynamics operations during leap for the INPE's second environmental data collecting satellite SCD 2," *Journal of the Brazilian Society of Mechanical Sciences*, vol. 21, pp. 339–344, 1999.
- [19] W. R. Silva, *Estudo da estabilidade do movimento rotacional de satélites artificiais com variáveis canônicas*, Qualificação de mestrado, UNESP—Universidade Estadual Paulista, São Paulo, Brazil, 2011.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

