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Research Article

Characterization of Holomorphic Bisectional Curvature of *GCR*-Lightlike Submanifolds

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We obtain the expressions for sectional curvature, holomorphic sectional curvature, and holomorphic bisectional curvature of a *GCR*-lightlike submanifold of an indefinite Kaehler manifold. We discuss the boundedness of holomorphic sectional curvature of *GCR*-lightlike submanifolds of an indefinite complex space form. We establish a condition for a *GCR*-lightlike submanifold of an indefinite complex space form to be null holomorphically flat. We also obtain some characterization theorems for holomorphic sectional and holomorphic bisectional curvature.

1. Introduction

The study of *CR* submanifolds of Kaehler manifolds was initiated by Bejancu [1], as a generalization of totally real and complex submanifolds and further developed by [2–7]. The *CR* structures on real hypresurfaces of complex manifolds have interesting applications to relativity. Penrose [8] discovered a correspondence, called Penrose correspondence, between points of a Minkowski space and projective lines of a certain real hypersurfaces in a complex projective space, which is an interesting means of passing from the geometry of a Minkowski space to the geometry of a *CR* manifold. Duggal [9, 10] studied the geometry of *CR* submanifolds with Lorentzian metric and obtained their interaction with relativity. The theory of lightlike submanifolds has interaction with some results on Killing horizon, electromagnetic, and radition fields and asymptotically flat spacetimes (for detail see chapters 7, 8, and 9 of [11]). Thus due to the significant applications of CR structures in relativity and growing importance of lightlike submanifolds in mathematical physics and relativity, Duggal and Bejancu [11] introduced the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds which have direct relation with physically important asymptotically flat space time which further lead to Twistor theory of Penrose and Heaven theory of Newman. Moreover, they concluded that, contrary to the CR-non degenerate submanifolds, CR-lightlike submanifolds do not include invariant (complex) and totally real lightlike submanifolds. Therefore, Duggal and Sahin [12] introduced SCR-lightlike submanifolds of indefinite Kaehler manifold which contain complex and totally real subcases but there was no inclusion relation between CR and SCR cases. Later on, Duggal and Sahin [13] introduced GCR-lightlike submanifolds of indefinite Kaehler manifolds, which behaves as an umbrella of invariant (complex), screen real and CR-lightlike submanifolds and also studied the existence (or nonexistence) of this new class in an indefinite space form. R. Kumar et al. [14] studied geodesic GCR-lightlike submanifolds of indefinite Kaehler manifolds and obtained some characterization theorems for a GCR-lightlike submanifold to be a GCR-lightlike product.

Since sectional curvature offers a lot of information concerning the intrinsic geometry of Riemannian manifolds, therefore in this paper, we obtain the expressions for sectional curvature, holomorphic sectional curvature, and holomorphic bisectional curvature of a GCR-lightlike submanifold of an indefinite Kaehler manifold. In [15], Kulkarni showed that the boundedness of the sectional curvature on a semi-Riemannian manifold implies the constancy of the sectional curvature. In [16], Bonome et al. showed that the boundedness of the holomorphic sectional curvature on indefinite almost Hermitian manifolds leads to the space of pointwise constant holomorphic sectional curvature. Therefore in Section 4, we discuss the boundedness of holomorphic sectional curvature of GCR-lightlike submanifolds of an indefinite complex space form. In Section 5, we established a condition for a GCR-lightlike submanifold of an indefinite complex space form to be null holomorphically flat. We also obtain some characterization theorems on holomorphic sectional and holomorphic bisectional curvature.

2. Lightlike Submanifolds

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$, (M, g) an m-dimensional submanifold of $\overline{M}$ and g the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\overline{M}$ (see [11]). For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\operatorname{Rad} T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $\operatorname{Rad} TM : x \in M \to \operatorname{Rad} T_xM$ defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\overline{M}$ is called an r-lightlike submanifold and $\operatorname{Rad} TM$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $\operatorname{Rad}(TM)$ in TM, that is

$$TM = \operatorname{Rad} TM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $\operatorname{Rad} TM$ in $TM^{\perp}$. Let $\operatorname{tr}(TM)$ and $\operatorname{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $\operatorname{Rad} TM$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$\operatorname{tr}(TM) = \operatorname{ltr}(TM) \perp S\left(TM^{\perp}\right), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus \operatorname{tr}(TM) = (\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)) \perp S(TM) \perp S\left(TM^{\perp}\right). \tag{2.3}$$

For a quasi-orthonormal fields of frames on TM, we have the following.

Theorem 2.1 (see [11]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a complementary vector bundle $\operatorname{ltr}(TM)$ of $\operatorname{Rad} TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\operatorname{ltr}(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighborhood of M such that*

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0, \quad \text{for any } i, j \in \{1, 2, \ldots, r\}, \tag{2.4}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\operatorname{Rad}(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$, then, according to the decomposition (2.3), the Gauss and Weingarten formulas are given by the following:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \overline{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\operatorname{tr}(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\operatorname{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, and A_U is a linear a operator on M and known as shape operator.

According to (2.2) considering the projection morphisms L and S of $\operatorname{tr}(TM)$ on $\operatorname{ltr}(TM)$ and $S(TM^{\perp})$, respectively, then Gauss and Weingarten formulas become

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \qquad \overline{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(\operatorname{ltr}(TM))$ valued and $\Gamma(S(TM^{\perp}))$ valued, respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \qquad \overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X \in \Gamma(TM), N \in \Gamma(\operatorname{ltr}(TM))$, and $W \in \Gamma(S(TM^{\perp}))$. Using (2.6) and (2.7), we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}\left(Y, D^l(X, W)\right) = g(A_W X, Y), \tag{2.8}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(A_W X, N), \tag{2.9}$$

for any $X \in \Gamma(TM), W \in \Gamma(S(TM^{\perp}))$, and $N, N' \in \Gamma(\operatorname{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$, then, using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as follows:

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \qquad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\operatorname{Rad} TM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\operatorname{Rad} TM)$, respectively. Using (2.6) and (2.10), we obtain

$$\overline{g}\left(h^l(X, PY), \xi\right) = g\left(A^*_\xi X, PY\right), \qquad \overline{g}(h^*(X, PY), N) = \overline{g}(A_N X, PY), \tag{2.11}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(\operatorname{Rad}(TM))$ and $N \in \Gamma(\operatorname{ltr}(TM))$.

Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively, then by straightforward calculations ([11]), we have

$$\begin{aligned}\overline{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)} Y - A_{h^l(Y,Z)} X + A_{h^s(X,Z)} Y \\ & - A_{h^s(Y,Z)} X + \left(\nabla_X h^l\right)(Y, Z) - \left(\nabla_Y h^l\right)(X, Z) \\ & + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ & - (\nabla_Y h^s)(X, Z) + D^s\left(X, h^l(Y, Z)\right) - D^s\left(Y, h^l(X, Z)\right),\end{aligned} \tag{2.12}$$

where

$$(\nabla_X h^s)(Y, Z) = \nabla^s_X h^s(Y, Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z) \tag{2.13}$$

$$\left(\nabla_X h^l\right)(Y, Z) = \nabla^l_X h^l(Y, Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z). \tag{2.14}$$

Then Codazzi equation is given, respectively, by the following:

$$\begin{aligned}\left(\overline{R}(X, Y)Z\right)^\perp = {} & \left(\nabla_X h^l\right)(Y, Z) - \left(\nabla_Y h^l\right)(X, Z) \\ & + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ & - (\nabla_Y h^s)(X, Z) + D^s\left(X, h^l(Y, Z)\right) - D^s\left(Y, h^l(X, Z)\right).\end{aligned} \tag{2.15}$$

Barros and Romero [17] defined indefinite Kaehler manifolds as follows.

Definition 2.2. Let $(\overline{M}, J, \overline{g})$ be an indefinite almost Hermitian manifold and $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, with respect to an indefinite metric $\overline{g}$. Then $\overline{M}$ is called an indefinite Kaehler manifold if J is parallel, with respect to $\overline{\nabla}$, that is

$$J^2 = -I, \quad \left(\overline{\nabla}_X J\right) Y = 0, \quad \overline{g}(JX, JY) = \overline{g}(X, Y) \quad \forall X, Y \in \Gamma\left(T\overline{M}\right). \tag{2.16}$$

3. Generalized Cauchy-Riemann Lightlike Submanifolds

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\overline{M}, \overline{g}, J)$, then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied.

(A) There exist two subbundles D_1 and D_2 of $\text{Rad}(TM)$ such that

$$\text{Rad}(TM) = D_1 \oplus D_2, \qquad J(D_1) = D_1, \qquad J(D_2) \subset S(TM). \tag{3.1}$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{JD_2 \oplus D'\} \perp D_0, \qquad J(D_0) = D_0, \qquad J(D') = L_1 \perp L_2, \tag{3.2}$$

where D_0 is a nondegenerate distribution on M, L_1 and L_2 are vector bundle of $\text{ltr}(TM)$ and $S(TM)^{\perp}$, respectively.

Then the tangent bundle TM of M is decomposed as $TM = D \perp D'$, where $D = \text{Rad}(TM) \oplus D_0 \oplus JD_2$. M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$, and $L_2 \neq \{0\}$. Let Q, P_1 and P_2 be the projections on D, $J(L_1) = M_1$, and $J(L_2) = M_2$, respectively. Then, for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X, \tag{3.3}$$

applying J to (3.3), we obtain

$$JX = TX + wP_1X + wP_2X, \tag{3.4}$$

and we can write (3.4) as follwos:

$$JX = TX + wX, \tag{3.5}$$

where TX and wX are the tangential and transversal components of JX, respectively. Similarly,

$$JV = BV + CV, \tag{3.6}$$

for any $V \in \Gamma(\text{tr}(TM))$, where BV and CV are the sections of TM and $\text{tr}(TM)$, respectively. Applying J to (3.5) and (3.6), we get

$$T^2 = -I - Bw, \qquad C^2 = -I - wB. \tag{3.7}$$

Differentiating (3.4) and using (2.6), (2.7), and (3.6), we have

$$D^s(X, wP_1Y) = -\nabla^s_X wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y), \tag{3.8}$$

$$D^l(X, wP_2Y) = -\nabla^l_X wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y). \tag{3.9}$$

Using Kaehlerian property of $\overline{\nabla}$ with (2.7), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\overline{M}$. Then one has*

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y), \qquad (\nabla^t_X w)Y = Ch(X, Y) - h(X, TY), \tag{3.10}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \qquad (\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y. \tag{3.11}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\overline{M}$. Then one has*

$$(\nabla_X B)V = A_{CV}X - TA_VX, \qquad (\nabla^t_X C)V = -wA_VX - h(X, BV), \tag{3.12}$$

where $X \in \Gamma(TM)$, $V \in \Gamma(\mathrm{tr}(TM))$, and

$$(\nabla_X B)V = \nabla_X BV - B\nabla^t_X V, \qquad (\nabla^t_X C)V = \nabla^t_X CV - C\nabla^t_X V. \tag{3.13}$$

4. Holomorphic Sectional Curvature of a GCR-Lightlike Submanifold

Let $\overline{M}$ be a complex space form of constant holomorphic curvature c. Then the curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by the following:

$$\begin{aligned}\overline{R}(X, Y)Z = \frac{c}{4}\{&\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(JY, Z)JX \\ &- \overline{g}(JX, Z)JY + 2\overline{g}(X, JY)JZ\},\end{aligned} \tag{4.1}$$

for X, Y, Z vector fields on $\overline{M}$. Using (4.1) and (2.12), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) = {} & \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) \\ & -g(JX,Z)g(JY,W) + 2g(X,JY)g(JZ,W)\} - g\left(A_{h^l(X,Z)}Y,W\right) \\ & + g\left(A_{h^l(Y,Z)}X,W\right) - g\left(A_{h^s(X,Z)}Y,W\right) + g\left(A_{h^s(Y,Z)}X,W\right) \\ & - \overline{g}\left(\left(\nabla_X h^l\right)(Y,Z),W\right) + \overline{g}\left(\left(\nabla_Y h^l\right)(X,Z),W\right) - \overline{g}\left(D^l(X,h^s(Y,Z)),W\right) \\ & + \overline{g}\left(D^l(Y,h^s(X,Z)),W\right). \end{aligned} \tag{4.2}$$

Using (2.8) in (4.2), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) = {} & \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) \\ & -g(JX,Z)g(JY,W) + 2g(X,JY)g(JZ,W)\} - g\left(A_{h^l(X,Z)}Y,W\right) \\ & + g\left(A_{h^l(Y,Z)}X,W\right) - \overline{g}(h^s(Y,W),h^s(X,Z)) + \overline{g}(h^s(X,W),h^s(Y,Z)) \\ & - \overline{g}\left(\left(\nabla_X h^l\right)(Y,Z),W\right) + \overline{g}\left(\left(\nabla_Y h^l\right)(X,Z),W\right). \end{aligned} \tag{4.3}$$

Then the sectional curvature $K_M(X,Y) = g(R(X,Y)Y,X)$ of M determined by orthonormal vectors X and Y of $\Gamma(D_0 \oplus M_2)$ and given by the following:

$$\begin{aligned} K_M(X,Y) = {} & \frac{c}{4}\left\{1 + 3g(X,JY)^2\right\} - g\left(A_{h^l(X,Y)}Y,X\right) + g\left(A_{h^l(Y,Y)}X,X\right) \\ & - \overline{g}(h^s(Y,X),h^s(X,Y)) + \overline{g}(h^s(X,X),h^s(Y,Y)). \end{aligned} \tag{4.4}$$

Corollary 4.1. *Let M be a GCR-lightlike submanifold of an indefinite Complex space form $\overline{M}(c)$. Then sectional curvature of M is given by $K_M(X,Y) = (c/4)\{1 + 3g(X,JY)^2\}$, if*

(i) *M_2 defines a totally geodesic foliation in $\overline{M}$,*

(ii) *D_0 defines a totally geodesic foliation in $\overline{M}$,*

(iii) *M is totally geodesic in $\overline{M}$.*

Definition 4.2. The holomorphic sectional curvature $H(X) = g(R(X,JX)JX,X)$ of M determined by a unit vector $X \in \Gamma(D_0)$ is the sectional curvature of a plane section $\{X, JX\}$.

Then using (2.11) and (4.4), for a unit vector field $X \in \Gamma(D_0)$, we get

$$\begin{aligned} H(X) = {} & c - \overline{g}\left(h^l(X,JX),h^*(JX,X)\right) + \overline{g}\left(h^l(JX,JX),h^*(X,X)\right) \\ & - \overline{g}(h^s(JX,X),h^s(X,JX)) + \overline{g}(h^s(X,X),h^s(JX,JX)). \end{aligned} \tag{4.5}$$

From (3.8), for any $X, Y \in \Gamma(D_0)$, we have

$$h^s(X, JY) = wP_2\nabla_X Y + Ch^s(X, Y), \tag{4.6}$$

and further using (3.7) and (4.6), we have

$$h^s(JX, JY) = wP_2\nabla_{JX} Y - wBh^s(X, Y) - h^s(X, Y). \tag{4.7}$$

Hence using (4.6) and (4.7) in (4.5), we obtain the expression for holomorphic sectional curvature as follows:

$$\begin{aligned} H(X) = c &- \overline{g}\left(h^l(X, JX), h^*(JX, X)\right) + \overline{g}\left(h^l(JX, JX), h^*(X, X)\right) - \|wP_2\nabla_X X\|^2 \\ &- \|Ch^s(X, X)\|^2 + \overline{g}\left(h^s(X, X), wP_2\nabla_{JX} X\right) + \|Bh^s(X, X)\|^2 - \|h^s(X, X)\|^2. \end{aligned} \tag{4.8}$$

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$. If M is totally geodesic in $\overline{M}(c)$, then $H(X) \le c$, for any unit vector field $X \in \Gamma(D_0)$.*

Proof. Using the hypothesis in (4.8), we get $H(X) = c - \|wP_2\nabla_X X\|^2$. Hence the result follows. □

Theorem 4.4 (see [13]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, then the distribution D is integrable if and only if $h(X, JY) = h(Y, JX)$, for any $X, Y \in \Gamma(D)$.*

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$, and D_0 is integrable, then $H(X) \le c$ for any unit vector field $X \in \Gamma(D_0)$.*

Proof. Since D_0 is integrable therefore using Theorem 4.4, we have $h(JX, JY) = -h(X, Y)$, for any unit vector field $X \in \Gamma(D_0)$. Therefore, from (4.5), we obtain

$$\begin{aligned} H(X) = c &- \overline{g}\left(h^l(X, JX), h^*(JX, X)\right) - \overline{g}\left(h^l(X, X), h^*(X, X)\right) \\ &- \|h^s(X, JX)\|^2 - \|h^s(X, X)\|^2 \le c. \end{aligned} \tag{4.9}$$

□

Theorem 4.6. *A GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ is D_0-totally geodesic if and only if*

(i) *D_0 is integrable,*

(ii) *$H(X) = c$, for any unit vector field $X \in \Gamma(D_0)$.*

Proof. Proof follows from (4.9). □

Theorem 4.7 (see [13]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if $Bh(X,Y)=0$, for any $X,Y\in\Gamma(D)$.*

Theorem 4.8. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$, and D_0 defines a totally geodesic foliation in M, then $H(X)\le c$, for any unit vector field $X\in\Gamma(D_0)$.*

Proof. Since D_0 defines a totally geodesic foliation in M, therefore by definition $\nabla_X X\in\Gamma(D_0)$, this implies that $h^*(X,X)=0$. Also by using Theorem 4.7, we have $Bh(X,X)=0$, for any $X\in\Gamma(D_0)$; hence, (4.8) becomes $H(X)=c-2\|Ch^s(X,X)\|^2$ and the result follows. □

Definition 4.9. The horizontal distribution D is called parallel with respect to the induced connection ∇ on M if $\nabla_X Y\in D$ for any $X\in\Gamma(TM)$ and $Y\in\Gamma(D)$.

Theorem 4.10. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ and D_0 is parallel, with respect to ∇, then $H(X)\le c$, for any unit vector field $X\in\Gamma(D_0)$.*

Proof. Since D_0 is parallel, with respect to the induced connection ∇ on M, therefore $\nabla^*_X Y\in\Gamma(D_0)$ and $h^*(X,Y)=0$, for any $X\in\Gamma(TM)$ and $Y\in\Gamma(D_0)$. Hence, from (4.8), we obtain $H(X)=c+\|Bh^s(X,X)\|^2-\|h^s(X,X)\|^2-\|Ch^s(X,X)\|^2$, and then by using (3.6), we get $H(X)=c-2\|Ch^s(X,X)\|^2$. Hence the result is complete. □

Lemma 4.11. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If the distribution D_0 defines a totally geodesic foliation in $\overline{M}$, then M is D_0-geodesic.*

Proof. By the definition of GCR-lightlike submanifold, M is D_0-geodesic if $\overline{g}(h^l(X,Y),\xi)=\overline{g}(h^s(X,Y),W)=0$, for any $X,Y\in\Gamma(D_0)$, $\xi\in\Gamma(\mathrm{Rad}(TM))$, and $W\in\Gamma(S(TM^\perp))$. Since D_0 defines a totally geodesic foliation in $\overline{M}$, therefore $\overline{g}(h^l(X,Y),\xi)=\overline{g}(\overline{\nabla}_X Y,\xi)=0$ and $\overline{g}(h^s(X,Y),W)=\overline{g}(\overline{\nabla}_X Y,W)=0$. Hence, the assertion follows. □

Theorem 4.12. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$. If D_0 defines a totally geodesic foliation in $\overline{M}$, then $H(X)=c$, for any unit vector field $X\in\Gamma(D_0)$.*

Proof. The result follows directly using Lemma 4.11 and (4.8). □

5. Null Holomorphically Flat GCR-Lightlike Submanifold

Let $x\in\overline{M}$ and U be a null vector of $T_x\overline{M}$. A plane π of $T_x\overline{M}$ is called a null plane directed by U if it contains U, $\overline{g}_x(U,V)=0$, for any $V\in\pi$, and there exists $V_0\in\pi$ such that $\overline{g}_x(V_0,V_0)\neq 0$. Following Beem-Ehrlich [18], the null sectional curvature of π, with respect to U and $\overline{\nabla}$, as a real number, is defined as follows:

$$\overline{K}_U(\pi)=\frac{\overline{g}_x\left(\overline{R}(V,U)U,V\right)}{\overline{g}_x(V,V)},\tag{5.1}$$

where V is an arbitrary non null vector in π. Clearly $\overline{K}_U(\pi)$ is independent of V but depends in a quadratic fashion on U.

Consider $u \in M$ and a null plane π of T_uM directed by $\xi_u \in \mathrm{Rad}(TM)$, then the null sectional curvature of π, with respect to ξ_u and ∇, as a real number is defined as

$$K_{\xi_u}(\pi) = \frac{g_u(R(V_u,\xi_u)\xi_u, V_u)}{g_u(V_u,V_u)}, \tag{5.2}$$

where V_u is an arbitrary non-null vector in π.

Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ then using (4.3), the null sectional curvature of π, with respect to ξ, is given by the following:

$$\begin{aligned} K_\xi(\pi) = \frac{1}{g(V,V)}\{ & g(A_{h^l(\xi,\xi)}V,V) - g(A_{h^l(V,\xi)}\xi,V) + \overline{g}(h^s(V,V),h^s(\xi,\xi)) \\ & - \overline{g}(h^s(\xi,V),h^s(V,\xi)) - \overline{g}\left(\left(\nabla_V h^l\right)(\xi,\xi),V\right) + \overline{g}\left(\left(\nabla_\xi h^l\right)(V,\xi),V\right)\}. \end{aligned} \tag{5.3}$$

Then, using (2.11), we obtain

$$\begin{aligned} K_\xi(\pi) = \frac{1}{g(V,V)}\Big\{ & g\left(h^*(V,V),h^l(\xi,\xi)\right) - g\left(h^*(\xi,V),h^l(V,\xi)\right) + \overline{g}(h^s(V,V),h^s(\xi,\xi)) \\ & - \overline{g}(h^s(\xi,V),h^s(V,\xi)) - \overline{g}\left(\left(\nabla_V h^l\right)(\xi,\xi),V\right) + \overline{g}\left(\left(\nabla_\xi h^l\right)(V,\xi),V\right)\Big\}. \end{aligned} \tag{5.4}$$

We know that a plane π is called holomorphic if it remains invariant under the action of the almost complex structure $\overline{J}$, that is, if $\pi = \{Z, \overline{J}Z\}$. The sectional curvature associated with the holomorphic plane is called the holomorphic sectional curvature, denoted by $\overline{H}(\pi)$ and given by $\overline{H}(\pi) = \overline{R}(Z,\overline{J}Z,Z,\overline{J}Z)/\overline{g}(Z,Z)^2$. The holomorphic plane $\pi = \{Z,\overline{J}Z\}$ is called null or degenerate if and only if Z is a null vector. A manifold $(\overline{M},\overline{g},\overline{J})$ is called null holomorphically flat if the curvature tensor $\overline{R}$ satisfies (see [19]).

$$\overline{R}\left(Z,\overline{J}Z,Z,\overline{J}Z\right) = 0, \tag{5.5}$$

for all null vectors Z. Put $\overline{g}(\overline{R}(X,Y)Z,W) = \overline{R}(X,Y,Z,W)$, then, from (5.4), we obtain

$$\begin{aligned} R\left(\xi,\overline{J}\xi,\xi,\overline{J}\xi\right) = {} & g\left(h^*(\xi,\xi),h^l\left(\overline{J}\xi,\overline{J}\xi\right)\right) - g\left(h^*\left(\overline{J}\xi,\xi\right),h^l\left(\xi,\overline{J}\xi\right)\right) \\ & + \overline{g}\left(h^s(\xi,\xi),h^s\left(\overline{J}\xi,\overline{J}\xi\right)\right) - \overline{g}\left(h^s\left(\overline{J}\xi,\xi\right),h^s\left(\xi,\overline{J}\xi\right)\right) \\ & - \overline{g}\left(\left(\nabla_\xi h^l\right)\left(\overline{J}\xi,\overline{J}\xi\right),\xi\right) + \overline{g}\left(\left(\nabla_{\overline{J}\xi} h^l\right)\left(\xi,\overline{J}\xi\right),\xi\right). \end{aligned} \tag{5.6}$$

Definition 5.1 (see [20]). A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be a totally umbilical in $\overline{M}$ if there is a smooth transversal vector field

$H \in \Gamma(\text{tr}(TM))$ on M, called the transversal curvature vector field of M, such that, for $X, Y \in \Gamma(TM)$,

$$h(X,Y) = H\overline{g}(X,Y). \tag{5.7}$$

Using (2.6), it is clear that M is a totally umbilical if and only if on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(\text{ltr}(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X,Y) = H^l g(X,Y), \qquad h^s(X,Y) = H^s g(X,Y), \qquad D^l(X,W) = 0, \tag{5.8}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$. A lightlike submanifold is said to be totally geodesic if $h(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$. Therefore, in other words, a lightlike submanifold is totally geodesic if $H^l = 0$ and $H^s = 0$.

Let M be a totally umbilical lightlike submanifold, then, using above definition, we have $h(\overline{J}\xi, \overline{J}\xi) = Hg(\overline{J}\xi, \overline{J}\xi) = Hg(\xi,\xi) = 0$ and $h(\xi, \overline{J}\xi) = Hg(\xi, \overline{J}\xi) = 0$, for any $\xi \in \Gamma(\text{Rad}(TM))$. Thus, from (5.6), we have the following theorem.

Theorem 5.2. *Let M be a GCR-lightlike submanifold of an indefinite complex space form $\overline{M}(c)$. If M is totally umbilical lightlike submanifold in $\overline{M}(c)$, then M is null holomorphically flat.*

Moreover, from (5.6), it is clear that the expression of $R(\xi, \overline{J}\xi, \xi, \overline{J}\xi)$ is expressed in terms of screen second fundamental forms of M, thus GCR-lightlike submanifold M of an indefinite complex space form $\overline{M}(c)$ is null holomorphically flat if M is totally geodesic.

6. Holomorphic Bisectional Curvature of a GCR-Lightlike Submanifold

Definition 6.1. The holomorphic bisectional for the pair of unit vector fields $\{X, Y\}$ on $\overline{M}$ is given by $\overline{H}(X,Y) = \overline{g}(\overline{R}(X, JX)JY, Y)$.

Theorem 6.2. *Let M be a mixed totally geodesic GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with D_0 parallel distribution. Then $\overline{H}(X,Z) = 0$, for any unit vector fields $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$.*

Proof. Let $X, Y \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$ then, by using that hypothesis that the distribution D_0 is a parallel, with respect to ∇ on M, we have $g(T\nabla_X Z, Y) = -\overline{g}(\overline{\nabla}_X Z, TY) = g(Z, \nabla_X TY) = 0$. Hence, the non degeneracy of the distribution D_0 implies that

$$\nabla_X Z \in \Gamma(D'), \tag{6.1}$$

for any $Z \in \Gamma(M_2)$. Now replacing Y by JX, respectively, in (2.15) and then taking inner product with JZ, for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$, we get

$$\begin{aligned}\overline{H}(X,Z) = {}& \overline{g}(h^s(\nabla_X JX, Z), JZ) - \overline{g}(\nabla_X h^s(JX,Z), JZ) + \overline{g}(h^s(JX,\nabla_X Z), JZ) \\ & + \overline{g}(\nabla_{JX} h^s(X,Z), JZ) - \overline{g}(h^s(\nabla_{JX}X, Z), JZ) - \overline{g}(h^s(X,\nabla_{JX}Z), JZ) \\ & - \overline{g}\Big(D^s\Big(X, h^l(JX,Z)\Big), JZ\Big) + \overline{g}\Big(D^s\Big(JX, h^l(X,Z)\Big), JZ\Big).\end{aligned} \tag{6.2}$$

Hence by using that M is mixed totally geodesic with (6.1), the assertion follows. □

Theorem 6.3. *In order that an indefinite complex space form $\overline{M}(c)$ may admit a mixed totally geodesic GCR-lightlike submanifold M with parallel horizontal distribution D_0, it is necessary that $c = 0$.*

Proof. Let $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$ be unit vector fields, then (4.1) implies that $\overline{H}(X,Z) = -(c/2)g(X,X)g(Z,Z)$, then the non degeneracy of the distributions D_0 and M_2 with the Theorem 6.2, we obtain $c = 0$. Hence, the result follows. □

Lemma 6.4. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then we have the following:*

(i) *if D' defines a totally geodesic foliation in $\overline{M}$ then $\overline{g}(h^s(D', D_0), JD') = 0$,*

(ii) *if D_0 is a parallel distribution, with respect to ∇, then $h(Z, JX) = Ch(Z,X)$ for any $X \in \Gamma(D_0)$, $Z \in \Gamma(M_2)$.*

Proof. (i) Let D' define a totally geodesic foliation in $\overline{M}$ this implies that $\overline{\nabla}_X Y = \nabla_X Y \in \Gamma(D')$ and $h(X,Y) = 0$, for any $X, Y \in \Gamma(D')$. Therefore by using (3.10), we obtain $A_{wY}X = -Bh(X,Y) = 0$. Let $Z \in \Gamma(D_0)$, then by using (2.8), we get $0 = g(A_{wY}X, Z) = \overline{g}(h^s(X,Z), wY)$. Thus we have $\overline{g}(h^s(D', D_0), JD') = 0$.

(ii) Let D_0 is a parallel distribution with respect to the induced connection ∇, therefore $\nabla_X Y \in \Gamma(D_0)$, for any $Y \in \Gamma(D_0), X \in \Gamma(TM)$. Since $\overline{M}$ is Kaehler manifold, therefore for $Z \in \Gamma(M_2)$ and $X \in \Gamma(D_0)$, we have $\overline{\nabla}_Z JX = J\overline{\nabla}_Z X$. This implies that $\nabla_Z JX + h(Z, JX) = J\nabla_Z X + Bh(Z,X) + Ch(Z,X)$, then by equating transversal components on both sides, we get the result. □

Theorem 6.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If D_0 is parallel with respect to the induced connection ∇, and M_2 defines a totally geodesic foliation in $\overline{M}$, then*

$$\begin{aligned}\overline{H}(X,Z) = {}& \overline{g}(h(JX,Z), J\nabla_X Z) - \overline{g}(h(X,Z), J\nabla_{JX} Z) + g(A_{JZ}JX, \nabla_X Z) \\ & - g(A_{JZ}X, \nabla_{JX} Z) + 2\|Ch^s(X,Z)\|^2,\end{aligned} \tag{6.3}$$

for any unit vector fields $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$.

Proof. Let $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$, then the equation of Codazzi (2.15) becomes

$$\begin{aligned}\overline{g}\Big(\overline{R}(X, JX)Z, JZ\Big) &= \overline{g}((\nabla_X h^s)(JX, Z), JZ) - \overline{g}((\nabla_{JX} h^s)(X, Z), JZ) \\ &\quad + \overline{g}\Big(D^s\Big(X, h^l(JX, Z)\Big), JZ\Big) - \overline{g}\Big(D^s\Big(JX, h^l(X, Z)\Big), JZ\Big).\end{aligned} \tag{6.4}$$

By using (2.13) and (6.1) with the Lemma 6.4 (i), we obtain

$$\begin{aligned}\overline{H}(X, Z) = \overline{g}\Big(\nabla^s_{JX} h^s(X, Z), JZ\Big) &- \overline{g}(\nabla^s_X h^s(JX, Z), JZ) \\ &+ g(A_{JZ} JX, \nabla_X Z) - g(A_{JZ} X, \nabla_{JX} Z) \\ &- \overline{g}\Big(D^s\Big(X, h^l(JX, Z)\Big), JZ\Big) + \overline{g}\Big(D^s\Big(JX, h^l(X, Z)\Big), JZ\Big).\end{aligned} \tag{6.5}$$

Now by using (2.7) with the Lemma 6.4 (ii), we have

$$\begin{aligned}\overline{g}(\nabla^s_X h^s(JX, Z), JZ) &= \overline{g}\Big(\overline{\nabla}_X h^s(JX, Z), JZ\Big) \\ &= -\overline{g}(h^s(JX, Z), J\nabla_X Z) - \|Ch^s(X, Z)\|^2,\end{aligned} \tag{6.6}$$

and similarly

$$\overline{g}\Big(\nabla^s_{JX} h^s(X, Z), JZ\Big) = -\overline{g}(h^s(X, Z), J\nabla_{JX} Z) + \|Ch^s(X, Z)\|^2. \tag{6.7}$$

By using (2.7), we have

$$\overline{g}\Big(D^s\Big(X, h^l(JX, Z)\Big), JZ\Big) = -\overline{g}\Big(h^l(JX, Z), J\nabla_X Z\Big), \tag{6.8}$$

$$\overline{g}\Big(D^s\Big(JX, h^l(X, Z)\Big), JZ\Big) = -\overline{g}\Big(h^l(X, Z), J\nabla_{JX} Z\Big). \tag{6.9}$$

Hence by using (6.6)–(6.9) in (6.5), the result follows. □

Definition 6.6 (see [13]). A GCR-lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is called a GCR-lightlike product if both the distributions D and D' define totally geodesic foliations in M.

Theorem 6.7 (see [14]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, M is a GCR-lightlike product if and only if $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$.*

Theorem 6.8. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(TM)$, then*

$$\overline{H}(X, Z) = 2\overline{g}(h^s(JX, Z), Jh^s(X, Z)), \tag{6.10}$$

for any unit vector fields $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$.

Proof. Let $(\nabla_X T)Y = 0$, then (3.10) implies that

$$A_{wY}X + Bh(X, Y) = 0, \tag{6.11}$$

for any $X, Y \in \Gamma(TM)$. Let $Y \in \Gamma(D), X \in \Gamma(TM)$, then (6.11) gives

$$Bh(X, Y) = 0. \tag{6.12}$$

Let $Z \in \Gamma(D')$ and $X \in \Gamma(D)$, then using (6.11) and (6.12), we obtain

$$A_{wZ}X = 0. \tag{6.13}$$

Particularly choosing $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$ in (2.15), we get

$$\begin{aligned}\overline{g}\left(\overline{R}(X, JX)Z, JZ\right) = {} & \overline{g}((\nabla_X h^s)(JX, Z), JZ) - \overline{g}((\nabla_{JX} h^s)(X, Z), JZ) \\ & + \overline{g}\left(D^s\left(X, h^l(JX, Z)\right), JZ\right) - \overline{g}\left(D^s\left(JX, h^l(X, Z)\right), JZ\right).\end{aligned} \tag{6.14}$$

Since $(\nabla_X T)Y = 0$ therefore, by using Theorem 6.7, the distributions D and D' define totally geodesic foliations in M. Then D' defines totally geodesic foliation in M implies that for any $Z_1, Z_2 \in \Gamma(D')$, we have $\nabla_{Z_1} Z_2 \in \Gamma(D')$. Therefore, using (3.10) and (3.11), we get $A_{wZ_2}Z_1 + Bh(Z_1, Z_2) = 0$. By taking inner product with $X \in \Gamma(D_0)$ and using (2.8), we get

$$\overline{g}(h^s(Z_1, X), wZ_2) = 0. \tag{6.15}$$

Also, (3.11) implies that $T\nabla_X Z = 0$, that is, $\nabla_X Z \in \Gamma(D')$. Therefore using (2.8), (2.13), (6.13), and (6.15) in (6.14), we obtain

$$\begin{aligned}\overline{g}\left(\overline{R}(X, JX)Z, JZ\right) = {} & \overline{g}(\nabla^s_X h^s(JX, Z), JZ) - \overline{g}\left(\nabla^s_{JX} h^s(X, Z), JZ\right) \\ & + \overline{g}\left(D^s\left(X, h^l(JX, Z)\right), JZ\right) - \overline{g}\left(D^s\left(JX, h^l(X, Z)\right), JZ\right).\end{aligned} \tag{6.16}$$

Now using (2.6), (2.7), (2.8), and (6.15), we have

$$\overline{g}(\nabla^s_X h^s(JX, Z), JZ) = \overline{g}(Jh^s(JX, Z), h^s(X, Z)). \tag{6.17}$$

Similarly,

$$\overline{g}\left(\nabla^s_{JX} h^s(X,Z), JZ\right) = -\overline{g}(Jh^s(JX,Z), h^s(X,Z)). \tag{6.18}$$

Also using (2.7), we have

$$\overline{g}\left(D^s\left(X, h^l(JX,Z)\right), JZ\right) = 0, \qquad \overline{g}\left(D^s\left(JX, h^l(X,Z)\right), JZ\right) = 0. \tag{6.19}$$

Hence, using (6.17)–(6.19) in (6.16), the result follows. □

Lemma 6.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold such that $(\nabla_X B)Y = 0$. Then $h^s(JX,Z) \in \Gamma(L_2^{\perp})$ for any $X \in \Gamma(D_0)$, $Z \in \Gamma(M_2)$.*

Proof. Since $(\nabla_X B)Y = 0$, therefore, from (3.12) we have $TA_W X = 0$, for any $X \in \Gamma(TM)$ and $W \in \Gamma(L_2)$, this implies that

$$A_W X \in \Gamma(D'), \tag{6.20}$$

for any $W \in \Gamma(L_2)$ and $X \in \Gamma(TM)$. Since $h^s(JX,Z) \in \Gamma(S(TM^{\perp}))$, therefore, to prove that $h^s(JX,Z) \in \Gamma(L_2^{\perp})$, it is sufficient to prove that $\overline{g}(h^s(JX,Z),W) = 0$, for any $W \in \Gamma(L_2)$. Let $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$ such that $W = wZ$, we have $g(\nabla_U Z, X) = \overline{g}(\overline{\nabla}_U Z, X) = \overline{g}(\overline{\nabla}_U JZ, JX) = -g(A_{JZ}U, JX) = g(JA_{JZ}U, X)$, then using (2.8) we obtain

$$\overline{g}(h^s(JX,Z),W) = -g(JA_W Z, X) = -g(\nabla_Z Z, X) = -g(T\nabla_Z Z, TX). \tag{6.21}$$

Since, from (3.10), we have $T\nabla_Z Z = -A_{wZ}Z - Bh(Z,Z)$, then using (6.20) in (6.21), the result follows. □

Theorem 6.10. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold such that $(\nabla_X B)Y = 0$, then*

$$\overline{H}(X,Z) = 2\overline{g}(Jh^s(X,Z), h^s(JX,Z)) - g(A_{JZ}\nabla_{JX}X, Z) + g(A_{JZ}\nabla_X JX, Z), \tag{6.22}$$

for any unit vector fields $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$.

Proof. Let $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$, then, from (2.15) and (2.13), we obtain

$$\begin{aligned} \overline{g}\left(\overline{R}(X,JX)Z, JZ\right) = {} & \overline{g}(\nabla^s_X h^s(JX,Z), JZ) - \overline{g}(h^s(\nabla_X JX, Z), JZ) \\ & - \overline{g}(h^s(JX, \nabla_X Z), JZ) - \overline{g}\left(\nabla^s_{JX} h^s(X,Z), JZ\right) \\ & + \overline{g}(h^s(\nabla_{JX}X, Z), JZ) + \overline{g}(h^s(X, \nabla_{JX}Z), JZ) \\ & + \overline{g}\left(D^s\left(X, h^l(JX,Z)\right), JZ\right) - \overline{g}\left(D^s\left(JX, h^l(X,Z)\right), JZ\right), \end{aligned} \tag{6.23}$$

using (6.20) in (2.9), we obtain

$$\overline{g}\Big(D^s\Big(X, h^l(JX, Z)\Big), JZ\Big) = 0, \qquad \overline{g}\Big(D^s\Big(JX, h^l(X, Z)\Big), JZ\Big) = 0. \tag{6.24}$$

Now consider

$$\overline{g}(\nabla^s_X h^s(JX, Z), JZ) = -\overline{g}\Big(h^s(JX, Z), J\overline{\nabla}_X Z\Big) = \overline{g}(h^s(X, Z), Jh^s(JX, Z)), \tag{6.25}$$

and similarly

$$\overline{g}\Big(\nabla^s_{JX} h^s(X, Z), JZ\Big) = -\overline{g}(h^s(X, Z), Jh^s(JX, Z)). \tag{6.26}$$

Also using (2.8), we have

$$\begin{aligned} \overline{g}(h^s(\nabla_X JX, Z), JZ) &= g(A_{JZ}\nabla_X JX, Z), \\ \overline{g}(h^s(\nabla_{JX} X, Z), JZ) &= g(A_{JZ}\nabla_{JX} X, Z). \end{aligned} \tag{6.27}$$

Using (6.20) in (2.8), we have

$$\overline{g}(h^s(JX, \nabla_X Z), JZ) = 0, \qquad \overline{g}(h^s(X, \nabla_{JX} Z), JZ) = 0. \tag{6.28}$$

Thus using (6.24)–(6.28) in (6.23), the result follows. □

Theorem 6.11. *Let M be a mixed foliate GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, and $S(TM)$ is parallel distribution, with respect to the induced connection ∇, then*

$$\overline{H}(X, Z) = 2\overline{g}(A_{JZ}JX, JA_{JZ}X), \tag{6.29}$$

for any unit vector fields $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$.

Proof. Since M is mixed foliate therefore for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$, Codazzi equation (2.15) and (2.13), imply that

$$\begin{aligned} \overline{H}(X, Z) = {} & \overline{g}(h^s(\nabla_X JX, Z), JZ) + \overline{g}(h^s(JX, \nabla_X Z), JZ) \\ & - \overline{g}(h^s(\nabla_{JX} X, Z), JZ) - \overline{g}(h^s(X, \nabla_{JX} Z), JZ). \end{aligned} \tag{6.30}$$

Since using (2.8) and the hypothesis, we have $g(A_{JZ}X, JW) = 0$ and $g(A_{JZ}X, J\xi) = \overline{g}(h^s(X, J\xi), JZ) = \overline{g}(\overline{\nabla}_X J\xi, JZ) = -g(\xi, \nabla_X Z) = 0$. Therefore, by the definition of a *GCR*-lightlike submanifold, we have

$$A_{JZ}X \in \Gamma(D). \tag{6.31}$$

Thus, using (2.7), (2.8), and (6.31) with the hypothesis, we obtain

$$\begin{aligned}\overline{g}(h^s(JX,\nabla_X Z),JZ) &= \overline{g}\Big(A_{JZ}JX,\overline{\nabla}_X Z\Big) = \overline{g}\Big(JA_{JZ}JX,J\overline{\nabla}_X Z\Big)\\ &= \overline{g}(A_{JZ}JX,JA_{JZ}X) - \overline{g}\Big(A_{JZ}JX,JD^l(X,JZ)\Big)\\ &= \overline{g}(A_{JZ}JX,JA_{JZ}X).\end{aligned} \tag{6.32}$$

Similarly, we obtain

$$\begin{aligned}\overline{g}(h^s(X,\nabla_{JX}Z),JZ) &= \overline{g}(A_{JZ}X,JA_{JZ}JX),\\ \overline{g}(h^s(\nabla_{JX}X,Z),JZ) &= g(A_{JZ}Z,\nabla_{JX}X) - \overline{g}\Big(\nabla_{JX}X,D^l(JZ,Z)\Big),\\ \overline{g}(h^s(\nabla_X JX,Z),JZ) &= g(A_{JZ}Z,\nabla_X JX) - \overline{g}\Big(\nabla_X JX,D^l(JZ,Z)\Big).\end{aligned} \tag{6.33}$$

Thus, (6.30) becomes

$$\begin{aligned}\overline{H}(X,Z) = 2\overline{g}(A_{JZ}JX,JA_{JZ}X) - g(A_{JZ}Z,\nabla_{JX}X) + \overline{g}\Big(\nabla_{JX}X,D^l(JZ,Z)\Big)\\ + g(A_{JZ}Z,\nabla_X JX) - \overline{g}\Big(\nabla_X JX,D^l(JZ,Z)\Big).\end{aligned} \tag{6.34}$$

Since the distribution D is integrable, therefore $\nabla_{JX}X - \nabla_X JX = [X,JX] = X' \in \Gamma(D)$, then (6.34) becomes $\overline{H}(X,Z) = 2\overline{g}(A_{JZ}JX,JA_{JZ}X) - g(A_{JZ}Z,X') + \overline{g}(X',D^l(JZ,Z))$. Hence, using the hypothesis and (2.8), the result follows. □

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