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The structure of $f(R)$-brane model

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Abstract Recently, a family of interesting analytical brane solutions were found in $f(R)$ gravity with $f(R) = R + \alpha R^2$ in Bazeia et al. (Phys Lett B 729:127 2014). In these solutions, the inner brane structure can be turned on by tuning the value of the parameter α. In this paper, we investigate how the parameter α affects the localization and the quasilocalization of the tensorial gravitons around these solutions. It is found that, in a range of α, despite the brane having an inner structure, there is no graviton resonance. However, in some other regions of the parameter space, although the brane has no internal structure, the effective potential for the graviton Kaluza–Klein (KK) modes has a singular structure, and there exist a series of graviton resonant modes. The contribution of the massive graviton KK modes to Newton's law of gravity is discussed briefly.

1 Introduction

The braneworld scenarios [1–5], have attracted more and more attention. They present new insights and solutions for many issues such as the hierarchy problem, the cosmological problem, the nature of dark matter and dark energy, and so on. In some braneworld scenarios, all the fields in the standard model are assumed to be trapped on a submanifold (called brane) in a higher-dimensional spacetime (called bulk), and only gravity transmits in the whole bulk. So a natural and interesting question is how to reproduce the effective four-dimensional Newtonian gravity from a bulk gravity.

In models with compact extra dimension, such as the ADD braneworld model [2,3], the effective gravity on the brane is transmitted by the massless graviton Kaluza–Klein (KK)

mode (also called the graviton zero mode). Since the graviton zero mode is separated from the first KK excitation by a mass gap, gravity is effectively four-dimensional at low energy. In models with infinitely large extra dimensions, like the one proposed by Randall and Sundrum (the RS2 model) [5], although the graviton spectrum is gapless now, the effective gravity on the brane is shown to be the Newtonian gravity plus a subdominant correction, so the low energy effective gravity is also four-dimensional [6–9]. This is because the wave functions of all the massive graviton KK modes are suppressed on the brane, so the massive modes only contribute a small correction to the Newtonian gravity at large distance.

Another class of interesting models with infinite extra dimensions is the so called thick branes. In these modes, the original singular thin brane in the RS2 model is replaced by some smooth thick domain walls generated by one or a few background scalars [10–12]. One of the interesting features of the thick domain wall brane is that the massive graviton modes feel an effective volcano-like potential, which might support some resonances. This feature was first noticed in Ref. [10].

Such resonant modes of the graviton can be interesting both phenomenologically and theoretically. Phenomenologically, the wave function of a graviton resonance peaks at the location of the brane and behaves as a plain wave in the infinity of the extra dimension. As compared to the RS2 model, the massive modes of a thick brane might contribute a different correction to the Newtonian gravity. As for the theoretical aspect, the metastable massive graviton in thick brane models provides an alternative for massive gravity theory [13,14]. Unfortunately, the early proposal for a thick brane [10–12] failed in finding a massive graviton resonance.

To construct a model that supports a graviton resonance, one has to tune the shape of the effective potential for the graviton. For typical thick brane models, where the gravity is taken as in general relativity, the only possible way is to

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tune the shape of the warp factor. Some successful models can be found in Refs. [15–18]. There is another way, however, to tune the effective potential for the graviton if the gravity is described by a more general theory. For example, in $f(R)$ gravity (see [19–21] for comprehensive reviews on $f(R)$ gravity and its applications in cosmology), the effective potential is determined by both the warp factor and the form of $f(R)$ (see Ref. [22] for details). So, in principle, graviton resonances can be turned on by tuning gravity.

However, the construction of a thick $f(R)$-brane model is not easy in practice (see Refs. [23–33] for work on $f(R)$-branes). First of all, the dynamical equation in $f(R)$-gravity is of fourth order. So, traditional methods that help us to find analytical brane solutions in second order systems, such as the superpotential method (also known as the first order formalism [34,35]) do not work in a general $f(R)$-brane model. Usually one can only solve the system either by using numerical methods [27], or by imposing strict constraints on the model, for example, by assuming the scalar curvature to be constant [24].

The linearization of a $f(R)$-brane model is also a challenging task. Naively, the linear perturbations around an arbitrary $f(R)$-brane solution should also satisfy some fourth order differential equations. However, in Ref. [22], the authors found that the tensor perturbation equation can be finally rewritten as a second order Schrödinger-like equation. The results of Ref. [22] enable us to analyze the graviton mass spectrum of any thick $f(R)$-branes solutions. For example, in Ref. [28], the authors constructed an analytical thick brane solution in a model with $f(R) = R + \alpha R^2$. The solution is stable against the tensor perturbation. The localization of zero modes of both graviton and fermion is also shown to be possible. Unfortunately, no graviton resonance was found in the model of Ref. [28]. Recently, a more general analytical thick $f(R)$-brane solution was reported in Ref. [33]. The solution of Ref. [33] is a generalization of the one in [28]. The model in Ref. [33] has an interesting feature: an inner brane structure appears for a particular range of the parameter α. Usually, the appearance of an inner brane structure is accompanied by graviton resonances [15–18]. So it is interesting to see if it is possible to find graviton resonances in the model of [33], and what the relation is between the inner brane structure and the graviton resonance. These two questions constitute the motivation for the present work.

In Sect. 2, we briefly review the model and corresponding solutions in Refs. [28,33]. Then, in Sect. 3, we study the localization of the graviton zero mode and the condition for graviton resonances in the model of [33]. The correction from the massive graviton KK modes at small distance is discussed in order to compare with the constraints of breaking the Newton inverse square law from experiments given in Ref. [36]. The conclusion and discussions will be given in Sect. 4.

2 Review of the $f(R)$-brane model and solutions

We start with the five-dimensional action of $f(R)$ gravity minimally coupled with a canonical scalar field,

$$S = \int d^4x dy \sqrt{-g} \left(\frac{1}{2\kappa_5^2} f(R) - \frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) \right), \tag{1}$$

where $f(R)$ is a function of the scalar curvature R, and $\kappa_5^2 = 2M_*^3$ with M_* the fundamental five-dimensional Planck mass. In the following, we set $M_* = 1$. The signature of the metric is taken as $(-, +, +, +, +)$, and the bulk coordinates are denoted by capital Latin indices, $M, N, \cdots = 0, 1, 2, 3, 4$, and the brane coordinates are denoted Greek indices, $\mu, \nu, \cdots = 0, 1, 2, 3$.

We are interested in the static Minkowski brane embedded in a five-dimensional spacetime, so the line element is assumed as

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{2}$$

where $e^{2A(y)}$ is the warp factor, $y = x^4$ stands for the extra dimension, and $\eta_{\mu\nu}$ is the induced metric on the brane.

For static brane solutions with the setup (2), the background scalar field ϕ is only a function of the extra dimension, i.e., $\phi = \phi(y)$. Therefore, the equations of motion for the $f(R)$-brane system are

$$\phi'' + 4A'\phi' = V_\phi, \tag{3}$$

$$f + 2f_R(4A'^2 + A'') - 6f_R'A' - 2f_R'' = \kappa_5^2(\phi'^2 + 2V), \tag{4}$$

$$-8f_R(A'' + A'^2) + 8f_R'A' - f = \kappa_5^2(\phi'^2 - 2V), \tag{5}$$

where the prime denotes the derivative with respect to y, $f_R \equiv df(R)/dR$, and $V_\phi \equiv dV(\phi)/d\phi$.

In Ref. [28], a toy model with

$$f(R) = R + \alpha R^2 \tag{6}$$

and the ϕ^4 potential

$$V(\phi) = \lambda(\phi^2 - v^2)^2 + \Lambda_5 \tag{7}$$

was considered, where $\lambda > 0$ is the self-coupling constant of the scalar field, and v is the vacuum expectation value of the scalar field. An analytical solution was found in Ref. [28]:

$$e^{A(y)} = \text{sech}(ky), \tag{8}$$

$$\phi(y) = v \tanh(ky), \tag{9}$$

where the parameters are given by

$$k = \sqrt{\frac{3}{232\alpha}}, \quad \lambda = \frac{3}{784} \frac{\kappa_5^2}{\alpha}, \tag{10}$$

$$v = 7\sqrt{\frac{3}{29\kappa_5^2}}, \quad \Lambda_5 = -\frac{477}{6728} \frac{1}{\alpha\kappa_5^2}. \tag{11}$$

The scalar field satisfies $\phi(0) = 0$ and $\phi(\pm\infty) = \pm v$, and the potential reaches the minimum (the vacuum) at $\phi = \pm v$. The energy density $\rho = T_{00} = e^{2A}(\frac{1}{2}\phi'^2 + V(\phi))$ peaks at the location of the brane, $y = 0$, and it tends to vanish at the boundary of the extra dimension. The brane is embedded in an anti-de Sitter spacetime.

The tensor perturbation of the above brane solution has been analyzed in Ref. [28]. It was shown that the solution is stable against the tensor perturbation and the gravity zero mode is localized on the brane. Furthermore, it was found that although there are fermion resonant KK modes on the brane, there are no graviton resonant modes [28].

Recently, a general solution was constructed in Ref. [33] with the same $f(R)$ as given in Eq. (6). The warp factor is assumed as the general form of (8) with two positive parameters B and k:

$$e^{A(y)} = \text{sech}^B(ky). \tag{12}$$

The corresponding Ricci tensor at the boundary of the extra dimension is given by

$$R_{MN}(y \to \pm\infty) \to -4B^2k^2 g_{MN} \equiv \Lambda_{\text{eff}} g_{MN}, \tag{13}$$

from which we can see that the background spacetime is asymptotical AdS with the effective cosmological constant $\Lambda_{\text{eff}} = -4B^2k^2$. The curvature is

$$R(y) = -4Bk^2[5B - 5(B+2)\text{sech}^2(ky)]. \tag{14}$$

Then from Eq. (4) we can give the derivative of the scalar field:

$$\phi'^2 = Bk^2\text{sech}^2(ky)\left\{\frac{3}{2} - 4\alpha k^2\left[5B^2 + 16B + 8\right.\right.$$
$$\left.\left. - (5B^2 + 32B + 12)\text{sech}^2(ky)\right]\right\}. \tag{15}$$

Note that the α here corresponds to $-\alpha$ in Ref. [33]. Thus, $\phi'^2 \geq 0$ implies

$$\alpha_1 \equiv -\frac{3}{32(1+4B)k^2} \leq \alpha \leq \frac{3}{8(8+16B+5B^2)k^2} \equiv \alpha_2. \tag{16}$$

The energy-momentum tensor density is

$$\rho = e^{2A(y)}\left(\frac{1}{2}\partial^M\phi\partial_M\phi + V(\phi)\right)$$
$$= Bk^2\text{sech}^{2B}(ky)\left\{-3B + 3\left(B + \frac{1}{2}\right)\text{sech}^2(ky)\right.$$
$$+ 4\alpha k^2\left[5B^3 - (10B^3 + 37B^2 + 32B + 8)\text{sech}^2(ky)\right.$$
$$\left.\left. + (5B^3 + 37B^2 + 44B + 12)\text{sech}^4(ky)\right]\right\}. \tag{17}$$
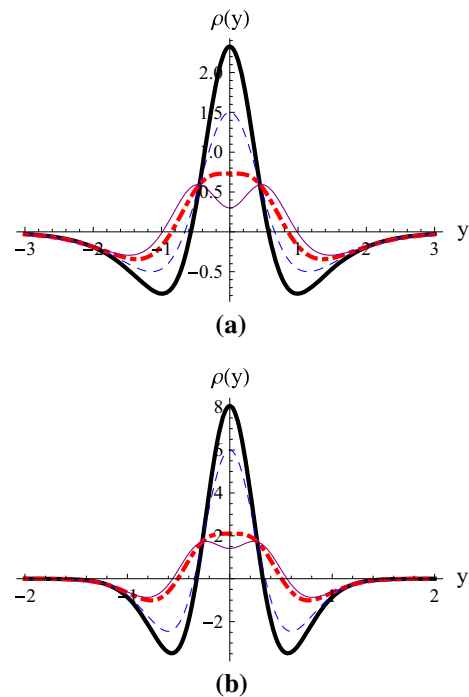

Fig. 1 The energy density $\rho(y)$ of the brane system. The parameters are set to $B = 1$ (**a**) and $B = 4$ (**b**), $k = 1$, and $\alpha = \alpha_1$ (*solid thin purple line*), $\alpha = \alpha_s$ (*dotted–dashing red line*), $\alpha = 0$ (*dashed blue line*), and α_2 (*thick black line*).

Solving $\frac{d^2\rho}{dy^2}\big|_{y=0} = 0$ results in

$$\alpha = \alpha_s \equiv -\frac{3 + 9B}{8k^2(16 + 60B + 49B^2)}, \tag{18}$$

where $\alpha_1 < \alpha_s < \alpha_2$ for finite B. So $y = 0$ is an inflection point of ρ when $\alpha = \alpha_s$, and the brane will have an internal structure when $\alpha \leq \alpha_s$. The energy density $\rho(y)$ of the brane system is shown in Fig. 1.

For an arbitrary B and a fixed α, the analytical solutions for the scalar field and scalar potential were found in Ref. [33]: when $\alpha = \alpha_1$, the solution is

$$\phi(y) = v_1[1 - \text{sech}(ky)]\text{sign}(y), \tag{19}$$
$$V(\phi) = c_2(|\phi| - v_1)^2\left[(|\phi| - v_1)^2 - c_1\right] - c_0; \tag{20}$$

when $\alpha = 0$, we have

$$\phi(y) = \sqrt{6B}\arctan\left[\tanh\left(\frac{ky}{2}\right)\right], \tag{21}$$
$$V(\phi) = -3B^2k^2 + 3Bk^2\left(B + \frac{1}{4}\right)\cos^2\left(\sqrt{\frac{2}{3B}}\phi\right); \tag{22}$$

when $\alpha = \alpha_2$, the result is just the one found in Ref. [28]:

$$\phi(y) = v_2\tanh(ky), \tag{23}$$
$$V(\phi) = \lambda_1(\phi^2 - v_2^2)^2 - \lambda_0. \tag{24}$$

Here c_i, v_i, and λ_i are positive parameters determined by B and k. We only list the expressions of v_i:

$$v_1 = \sqrt{\frac{3B(6+B)(2+5B)}{8+32B}}, \tag{25}$$

$$v_2 = \sqrt{\frac{3B(6+B)(2+5B)}{16+2B(16+5B)}}. \tag{26}$$

Now, it is clear that, when $B = 1$ and $\alpha = \alpha_2 = \frac{3}{232k^2}$, the exact solution described by Eqs. (12), (23), (24), and (26) is just the one given in Eqs. (6)–(11), for which the brane has no internal structure.

3 Localization and resonant KK modes of the tensor fluctuation

In this section, we investigate the stability of the solution against tensor fluctuations of the metric as well as the localization of gravity on the brane. Especially, we will find that the effective potential for the KK modes of the tensor fluctuation has a rich structure and it will support some resonant KK modes when B is large enough and $\alpha > 0$.

We start with the transverse-traceless (TT) tensor perturbations of the background spacetime:

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{27}$$

where $h_{\mu\nu} = h_{\mu\nu}(x^\rho, y)$ depends on all the spacetime coordinates and satisfies the TT condition

$$\eta^{\mu\nu}h_{\mu\nu} = \partial_\mu h_\nu^\mu = 0. \tag{28}$$

It can be shown that the TT tensor perturbations $h_{\mu\nu}$ is decoupled from the scalar and vector perturbations of the metric as well as the perturbation of the scalar field $\delta\phi = \tilde{\phi}(x^\rho, y)$. The perturbed Einstein equations for the TT tensor perturbations are given by [22]

$$\Box^{(5)}h_{\mu\nu} = \frac{f'_R}{f_R}\partial_y h_{\mu\nu}, \tag{29}$$

or, equivalently,

$$\left(a^{-2}\Box^{(4)}h_{\mu\nu} + 4\frac{a'}{a}h'_{\mu\nu} + h''_{\mu\nu}\right)f_R + h'_{\mu\nu}f'_R = 0, \tag{30}$$

where $a(y) = e^{A(y)}$. By making the coordinate transformation $dz = a^{-1}dy$, Eq. (30) becomes

$$\left[\partial_z^2 + \left(3\frac{\partial_z a}{a} + \frac{\partial_z f_R}{f_R}\right)\partial_z + \Box^{(4)}\right]h_{\mu\nu} = 0. \tag{31}$$

Then, by making the decomposition

$$h_{\mu\nu}(x^\rho, z) = (a^{-3/2}f_R^{-1/2})\epsilon_{\mu\nu}(x^\rho)\psi(z), \tag{32}$$

where $\epsilon_{\mu\nu}(x^\rho)$ satisfies the TT condition $\eta^{\mu\nu}\epsilon_{\mu\nu} = 0 = \partial_\mu\epsilon_\nu^\mu$, we can get from (31) the equation for the KK modes

$\psi(z)$ of the tensor perturbations, which is a Schrödinger-like equation [22]:

$$[-\partial_z^2 + W(z)]\psi(z) = m^2\psi(z), \tag{33}$$

where the effective potential $W(z)$ is

$$W(z) = \frac{3}{4}\frac{(\partial_z a)^2}{a^2} + \frac{3}{2}\frac{\partial_z^2 a}{a} + \frac{3}{2}\frac{\partial_z a \partial_z f_R}{a f_R} - \frac{1}{4}\frac{(\partial_z f_R)^2}{f_R^2} + \frac{1}{2}\frac{\partial_z^2 f_R}{f_R}. \tag{34}$$

Equation (33) can be factorized as

$$\mathcal{K}\mathcal{K}^\dagger\psi(z) = m^2\psi(z), \tag{35}$$

with

$$\mathcal{K} = +\partial_z + \left(\frac{3}{2}\frac{\partial_z a}{a} + \frac{1}{2}\frac{\partial_z f_R}{f_R}\right), \tag{36}$$

$$\mathcal{K}^\dagger = -\partial_z + \left(\frac{3}{2}\frac{\partial_z a}{a} + \frac{1}{2}\frac{\partial_z f_R}{f_R}\right), \tag{37}$$

which indicates that there is no graviton mode with $m^2 < 0$. For Eq. (33), the solution of the zero mode with $m = 0$ is

$$\psi^{(0)}(z) = N_0 a^{3/2}(z)f_R^{1/2}(z). \tag{38}$$

It is easy to show that $\psi^{(0)}(z)$ is normalizable, i.e.,

$$\int_{-\infty}^{\infty}|\psi^{(0)}(z)|^2 dz < \infty, \tag{39}$$

which implies that the zero mode is localized on the brane. Here we note that if $f_R(z) = 1 + 2\alpha R(z) = 0$ has a solution $z = \pm z_0$, then the effective potentials W would have singularities at $z = \pm z_0$ and the corresponding zero mode would vanish at that point. We note here that by making the KK reduction for the zero mode we will get the following relation between the effective four-dimensional Planck mass M_{Pl} and the fundamental five-dimensional Planck mass M_*:

$$M_{Pl}^2 \sim M_*^3/k, \tag{40}$$

which results in the effective four-dimensional general relativity. Hence, it is natural to set M_{Pl}, M_*, and k as the same scale (as in the RS-2 model),

$$M_{Pl} \sim M_* \sim k, \tag{41}$$

so that there is no hierarchy between them.

There is no analytical expression for the potential W with respect to the coordinate z. However, we can express it in the coordinate y as

$$W = \frac{3}{4}\left(\frac{a(y)a'(y)}{a(y)}\right)^2 + \frac{3}{2}\frac{a(y)a'^2(y) + a(y)^2 a''(y)}{a(y)}$$
$$+ \frac{3\alpha a^2(y)a'(y)R'(y)}{a(y)(1+2\alpha R(y))} - \frac{\alpha^2 a^2(y)R'^2(y)}{(1+2\alpha R(y))^2}$$

$$+ \frac{\alpha \left[a(y)a'(y)R'(y) + a^2(y)R''(y) \right]}{1 + 2\alpha R(y)}$$

$$= \frac{k^2}{4} \mathrm{sech}^{2B}(ky) \Bigg(15B^2 - (2 + 3B)(4 + 5B)\mathrm{sech}^2(ky)$$

$$- \frac{128B(2 + 5B)(1 + 16Bk^2\alpha)k^2\alpha}{(1 + 8B(4 + 5B)k^2\alpha + (1 - 40B^2k^2\alpha)\cosh(2ky))^2}$$

$$+ \frac{16(1 + 2B)(1 + 16Bk^2\alpha)}{1 + 8B(4 + 5B)k^2\alpha + (1 - 40B^2k^2\alpha)\cosh(2ky)} \Bigg). \tag{42}$$

From now on, we define dimensionless variables $\bar{\alpha} \equiv \alpha k^2$, $\bar{y} \equiv ky$, $\bar{z} \equiv kz$, $\bar{W} \equiv W/k^2$, and $\bar{m} \equiv m/k$. Then $\bar{z}$ is a function of just $\bar{y}$ and Eq. (33) becomes

$$\left[-\partial_{\bar{z}}^2 + \bar{W}(\bar{z}) \right] \psi(\bar{z}) = \bar{m}^2 \psi(\bar{z}), \tag{43}$$

where the effective potential is

$$\bar{W}(\bar{z}(\bar{y})) = \frac{1}{4} \mathrm{sech}^{2B} \bar{y} \Bigg[15B^2 - (2 + 3B)(4 + 5B)\mathrm{sech}^2 \bar{y}$$

$$- \frac{128B(2 + 5B)(1 + 16B\bar{\alpha})\bar{\alpha}}{\left[1 + 8B(4 + 5B)\bar{\alpha} + (1 - 40B^2\bar{\alpha})\cosh(2\bar{y}) \right]^2}$$

$$+ \frac{16(1 + 2B)(1 + 16B\bar{\alpha})}{1 + 8B(4 + 5B)\bar{\alpha} + (1 - 40B^2\bar{\alpha})\cosh(2\bar{y})} \Bigg]. \tag{44}$$

Now the parameter k does not appear, which implies that we only need to consider different values of the dimensionless varieties B and $\bar{\alpha}$ to search for resonant modes. For convenience, we remove the bars on all variables, which is equivalent to letting $k = 1$. So we have

$$\alpha_1 = -\frac{3}{32(1 + 4B)}, \tag{45}$$

$$\alpha_s = -\frac{3 + 9B}{8(16 + 60B + 49B^2)}, \tag{46}$$

$$\alpha_2 = \frac{3}{8(8 + 16B + 5B^2)}. \tag{47}$$

Note that $f_R = 1 + 2\alpha R(y)$ appears in the denominator of the expression of $W(z)$, (34), or $W(z(y))$, (42). For general relativity, we have $f_R = 1$ and so the effective potential $W(z(y))$ is always regular for a smooth solution of the thick brane. However, for the $f(R)$ gravity with $f(R) = R + \alpha R^2$, f_R may be vanishing and the effective potential will be divergent, i.e.,

$$1 + 2\alpha R(y) = 1 - 8\alpha B[-2 + (2 + 5B)\tanh^2(y)] = 0 \tag{48}$$

will lead to the singularities of the effective potential W. This will result in two δ-like potential wells which are related to the appearance of the resonant KK modes of the tensor perturbations.
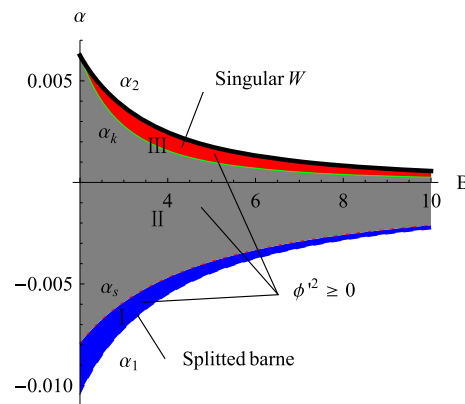

Fig. 2 The structure of the parameter space of (α, B) for the $f(R)$-brane model. Areas I ($\alpha_1 < \alpha < \alpha_s$ for a fixed B), II ($\alpha_s < \alpha < \alpha_k$ for a fixed B), and III ($\alpha_k < \alpha < \alpha_2$ for a fixed B) correspond to $\phi'^2 \geq 0$. Area I corresponds to the split branes. Area III corresponds to the singular W

Now we analyze the parameter space of (B, α) where Eq. (48) has solutions. Since $1 + 2\alpha R(y)$ is an even function of y, we only consider the region $y \in (0, +\infty)$. We first discuss the case of $\alpha > 0$. It is clear that $1 + 2\alpha R(y)$ decreases with y and $1 + 2\alpha R(0) = 1 + 16\alpha B > 0$. So the sufficient and necessary condition that Eq. (48) has a solution is $1 + 2\alpha R(y \to +\infty) = 1 - 40\alpha B^2 < 0$. Therefore, the relation between α and B is

$$\alpha > \frac{1}{40B^2} > 0. \tag{49}$$

On the other hand, according to the constraint on α (16), we should have $\frac{1}{40B^2} < \alpha_2$, which results in $B > 2$. Next, we discuss the case of $\alpha < 0$, for which $1 + 2\alpha R(y)$ increases with y and $1 + 2\alpha R(y \to +\infty) = 1 - 40\alpha B^2 > 0$. So the sufficient and necessary condition that Eq. (48) has a solution is $1 + 2\alpha R(0) = 1 + 16\alpha B \leq 0$. Therefore, the relation between α and B is

$$\alpha \leq -\frac{1}{16B} < 0. \tag{50}$$

Furthermore, the constraint on α (16) requires $-\frac{1}{16B} \geq \alpha_1$, namely $B \leq -\frac{2}{5}$, which contradicts with $B > 0$. So Eq. (48) has no solution for negative α. In conclusion, only when the constraint conditions

$$B > 2 \quad \text{and} \quad \alpha_k \equiv \frac{1}{40B^2} < \alpha \leq \alpha_2 \tag{51}$$

are satisfied, does Eq. (48) have a solution. The relations among $\alpha_1, \alpha_s, \alpha_k, \alpha_2$ are shown in Fig. 2. Because of the monotony of it, f_R (the derivative of $f(R)$ with respect of R) is negative in some range of y as long as Eq. (48) has a solution. This will lead to the existence of ghosts.

In order to judge whether there are resonant modes, we consider the partner equation of the Schrödinger-like equation (35): $\mathcal{K}^\dagger \mathcal{K} \psi(z) = m^2 \psi(z)$, for which the corresponding
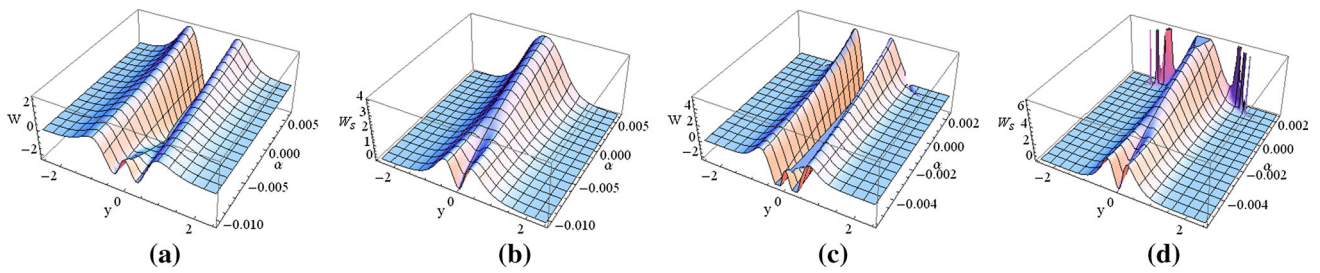
Fig. 3 The effective potential W and the dual one W_s as functions of y and α ($\alpha_1 \leq \alpha \leq \alpha_2$). The parameters are set to $B = 2$ (**a**, **b**), and $B = 4$ (**c**, **d**)

potential is given by

$$
W_s(z) = \frac{15}{4} \frac{(\partial_z a)^2}{a^2} - \frac{3}{2} \frac{\partial_z^2 a}{a} + \frac{3}{2} \frac{\partial_z a \partial_z f_R}{a f_R}
$$
$$
+ \frac{3}{4} \frac{(\partial_z f_R)^2}{f_R^2} - \frac{1}{2} \frac{\partial_z^2 f_R}{f_R}. \tag{52}
$$

Similar to $W(z)$, there is no analytical expression for $W_s(z)$. In the y coordinate, we have

$$
W_s(z(y)) = \frac{1}{4} \text{sech}^{2B} y \left[3B^2 - (2B + 3B^2) \text{sech}^2 y \right.
$$
$$
+ \frac{384B(2 + 5B)(1 + 16B\alpha)\alpha}{[1 + 8B(4 + 5B)\alpha + (1 - 40B^2\alpha) \cosh(2y)]^2}
$$
$$
\left. + \frac{16B - 128B(2 + 3B)\alpha}{1 + 8B(4 + 5B)\alpha + (1 - 40B^2\alpha) \cosh(2y)} \right]. \tag{53}
$$

The structure of W and W_s is shown in Fig. 3.

Because $\frac{dz}{dy} = a(y)$ is positive, the shape of $W_s(z)$ is the same as that of $W_s(z(y))$. Thus, in order to estimate whether there are resonant modes we just need to consider the shape of $W_s(z(y))$. The appearance of the "quasiwell" of the dual potential may lead to resonances of the graviton KK modes. Noticing

$$
W_s(0) = -W(0) > 0, \tag{54}
$$
$$
0 < W_s|_{z \to \pm\infty} \to 0, \tag{55}
$$
$$
0 < W|_{z \to \pm\infty} \to 0, \tag{56}
$$

it is convenient to judge whether there are resonant modes for Eq. (33) from W_s. Solving $\partial_y^2 W|_{y=0} = 0$ and $\partial_y^2 W_s|_{y=0} = 0$, respectively, results in

$$
\alpha = \frac{-8 - 36B - 46B^2 + (2 + 5B)\sqrt{(16 + 52B + 67B^2)}}{16B(2 + 3B)(4 + 7B)} \equiv \alpha',
$$
$$
\alpha = \frac{8 + 24B - 2B^2 - (2 + 5B)\sqrt{(16 + 16B + 13B^2)}}{16B^2(38 + 107B)} \equiv \alpha''.
$$

This implies that both the effective potential and the corresponding dual one have an interval structure in the range of $\alpha_1 < \alpha < \alpha'$ and the range of $\alpha_1 < \alpha < \alpha''$, respectively

(see Fig. 3). Furthermore, it can be seen from Fig. 3 that there is also another interesting quasiwell with singularity for large B and α satisfying the constraint conditions (51).

To get the numerical solution of Eq. (33), we impose the following conditions:

$$
\psi_{\text{even}}(0) = 1, \quad \partial_z \psi_{\text{even}}(0) = 0; \tag{57}
$$
$$
\psi_{\text{odd}}(0) = 0, \quad \partial_z \psi_{\text{odd}}(0) = 1. \tag{58}
$$

Here ψ_{even} and ψ_{odd} denote the even and odd parity modes of $\psi(z)$, respectively. The solution under this imposition does not affect the relative probability defined below.

The function $|\psi(z)|^2$ can be interpreted as the probability of finding the massive KK modes at the position z along the extra dimension [37]. In order to find the resonant modes, we definite the relative probability [37],

$$
P(m^2) = \frac{\int_{-z_b}^{z_b} |\psi(z)|^2 dz}{\int_{-z_{\text{max}}}^{z_{\text{max}}} |\psi(z)|^2 dz}, \tag{59}
$$

where $2z_b$ is approximately the width of the brane, and z_{max} is taken as $z_{\text{max}} = 10z_b$. It is clear that large relative probabilities $P(m^2)$ of finding massive KK modes within a narrow range $-z_b < z < z_b$ around the brane location indicate the existence of resonant modes.

From Fig. 3, we can see that there is a quasiwell when $\alpha \to \alpha_1$, which is consistent with our analysis. So, it seems that we may find resonant modes for small α. When $\alpha \to \alpha_2$ and $B \geq 4$, which satisfies the condition (51), the effective potential has two δ-like potential wells and there are found resonant modes.

Firstly, we discuss the case of $B \leq 2$. When α approaches α_1, the dual potential $W_s(z)$ has a quasiwell in the middle, so resonant modes of Eq. (33) may appear. We use the numerical method to search for resonant modes for $B \leq 2$, but no resonant mode is found. The reason is that the quasiwell is not deep or wide enough. The potential $W(z)$ for $B = 2$ is shown in Fig. 4a, b. The relations between the relative probability P and the mass square m^2 for $B = 2$ are shown in Fig. 5.

Secondly, we discuss the case of $B > 2$. Similar to the case of $B \leq 2$, there is a quasiwell in the middle of the
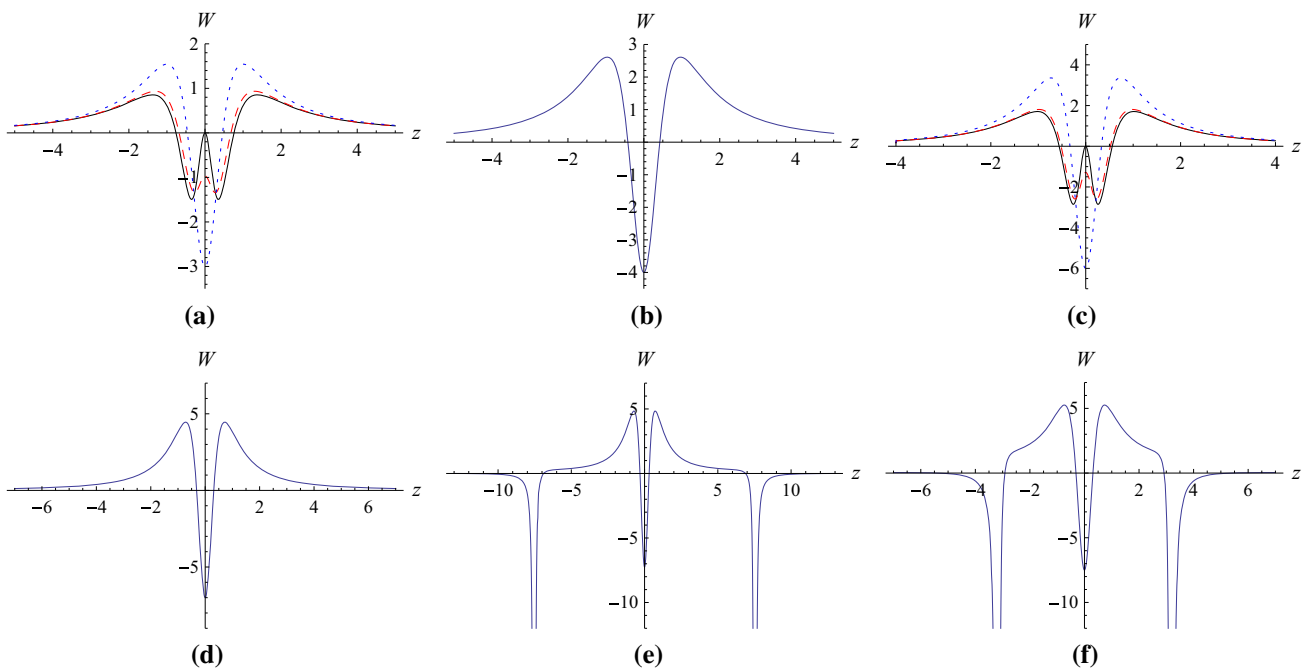
Fig. 4 The shapes of the effective potential $W(z)$. The parameters are set to $B = 2$ (**a**, **b**) and $B = 4$ (**c**, **d**), $\alpha = \alpha_1$ (*black line*), $\alpha = \alpha_s$ (*dashed red line*), and $\alpha = 0$ (*dotted blue line*) (**a**, **c**)
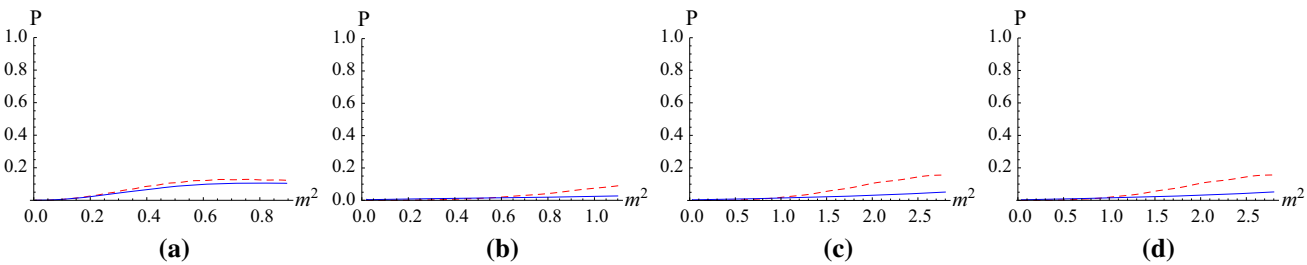

Fig. 5 The relative probability $P(m^2)$ of the KK modes for Eq. (33) with $B = 2$. The *dashed red* and *solid blue lines* are for odd and even parity KK modes, respectively

dual potential $W_s(z)$, but no resonant modes are found when $\alpha_1 \leq \alpha \leq \alpha_k$ (see Figs. 4c, d, and 6a–d). When $\alpha_k < \alpha \leq \alpha_2$, the potential $W(z)$, which has Z_2 symmetry and is shown in Fig. 4e, f, will have two δ-like potential wells. A series of resonant modes appears (see Fig. 6e, f) and their wave functions are shown in Fig. 7 for $B = 4$.

The last point of interest for this section is to calculate the contribution of the massive graviton KK modes including the resonances to Newton's law. The correction to Newton's law between two mass points m_1 and m_2 localized at $z = 0$, with a distance r from the KK modes, is given by [6,8]

$$U(r) \sim G_N \frac{m_1 m_2}{r} \left[1 + \int_0^\infty \frac{dm}{k} e^{-mr} \psi_m^2(0) \right], \quad (60)$$

where $\psi_m(z)$ is normalized such that $\psi_m(z)|_{z \to \infty} \simeq \cos(z)$, and the effective four-dimensional Newton's constant G_N is given by

$$G_N \sim M_{Pl}^{-2}. \quad (61)$$

Next, we first calculate the expression of $\psi_m(0)$. As examples, we just discuss two representative cases: $B = 4$, $\alpha = 0$, and $B = 4$, $\alpha = \alpha_2$. The corresponding numerical results are shown in Fig. 8. As we can see from Fig. 8a, b, the trend of $\psi_m(0)$ with m changing is analogous to the relative probability $P(m^2)$ (see Fig. 6c, f). Secondly, in order to do the integration in Eq. (60) we need to obtain analytical expression of $\psi_m(0)$ by using some approximate methods. One of the simplest methods is to simulate the original function with a few simple linear functions. For example, for the case of $B = 4$, $\alpha = 0$, the original function can be divided into two parts (see the blue line in Fig. 8a):

$$\begin{aligned} \psi_m(0) &= \frac{m}{4k}, \quad 0 \leq m \leq 4k, \\ \psi_m(0) &= 1, \quad m > 4k. \end{aligned} \quad (62)$$
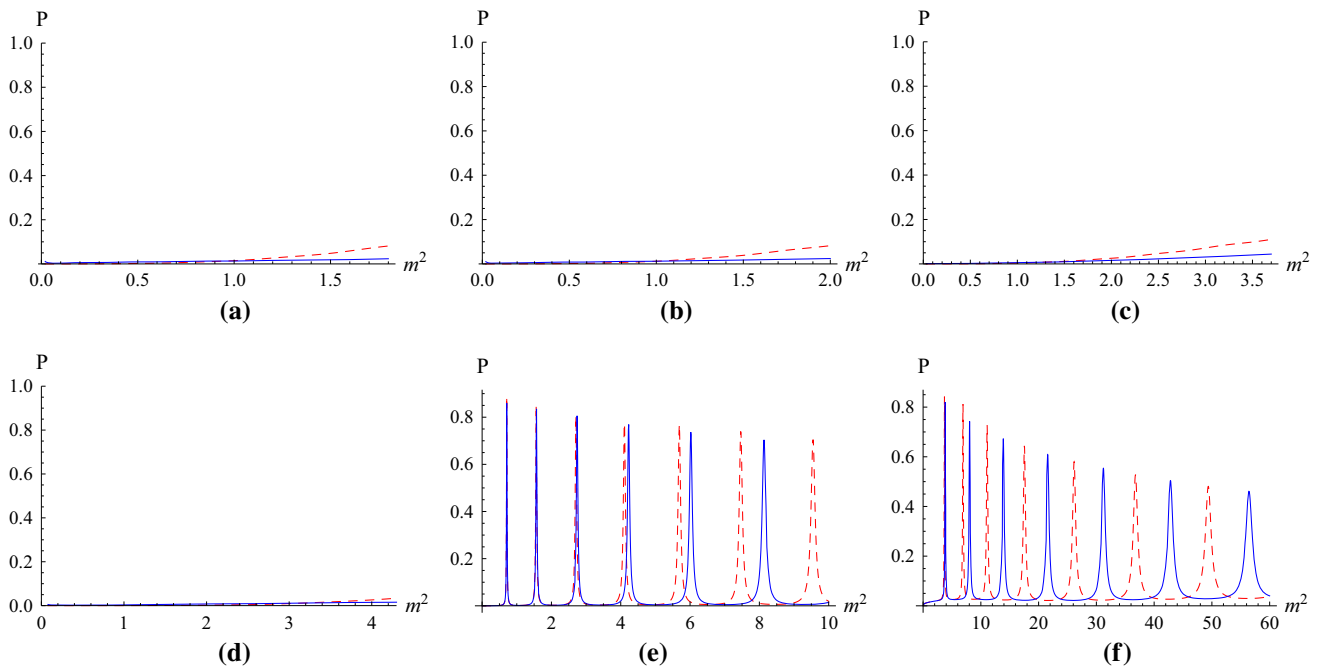
Fig. 6 The relative probability $P(m^2)$ of the KK modes for Eq. (33) with $B = 4$ and $k = 1$. The *dashed red* and *solid blue lines* are for odd and even parity KK modes, respectively
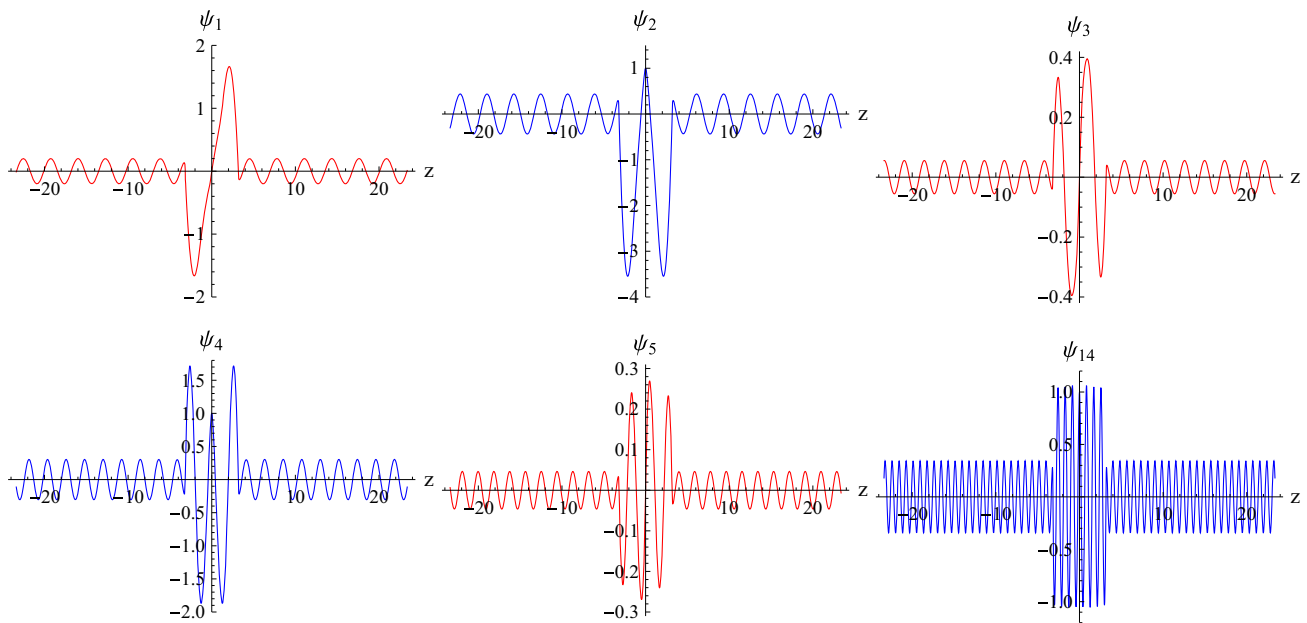

Fig. 7 The resonant wave function $\psi_n(z)$ for Eq. (33) with $B = 4$, $k = 1$, and $\alpha = \alpha_2$. The *red* and *solid blue lines* are for odd and even parity KK modes, respectively. The coordinate axis ϕ_n denotes the wave function for the nth resonant mode

Note that $\psi_m(0)$ is dimensionless. Substituting this result into Eq. (60), we get the approximate expression for the gravity potential $U(r)$:

$$U(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{e^{-4kr}(-4kr + e^{4kr} - 1)}{8(kr)^3} \right). \quad (63)$$

We expand $U(r)$ in terms of kr in two extreme situations:

$$U(r \ll 1/k) \sim \frac{G_N m_1 m_2}{r} \left(1 + \frac{1}{kr} - \frac{8}{3} + 4kr + \mathcal{O}[(kr)^2] \right),$$

$$U(r \gg 1/k) \sim \frac{G_N m_1 m_2}{r} \left(1 + \frac{1}{8(kr)^3} + \mathcal{O}\left[\frac{1}{(kr)^4}\right] \right). \quad (64)$$

Fig. 8 $\psi_m(0)$ for resonant wave function $\psi(z)$ with $B = 4$, $\alpha = 0$, and $B = 4$, $\alpha = \alpha_2$. The *red lines* are the corresponding numerical result and the *blue line* is an approximate linear fitting
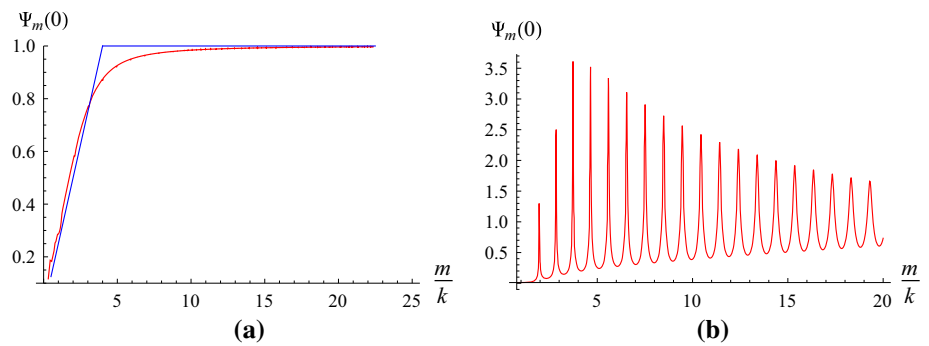
For the case without resonance, the correction to Newton's law has the following obvious characteristic. The main correction occurs at a short distance r of $r < 1/k \sim 1.6 \times 10^{-33}$ cm and it is proportional to $\frac{1}{r^2}$. So, in the case of the current accuracy of the experiment [36], such a correction is unobservable and undetectable in this brane model. At a large distance ($r \gg 1/k$), the correction can be neglected. For the case of $B = 4$, $\alpha = \alpha_2$ with multiple resonances, the analysis and method are similar. Because the value of $\psi_m(0)$ inevitably tends 1 when the parameter m approaches infinity, the leading correction to Newton's law at a large distance $r \gg 1/k$ is also proportional to $\frac{1}{r^4}$. At a short distance $r < 1/k$, the leading term is $\frac{G_N m_1 m_2}{kr^2}$. The effect of the resonant modes occurs at a few Planck lengths. It is very difficult to give the expression of this correction.

4 Conclusion

In this paper, we investigated localization and resonant KK modes of the tensor fluctuation for the $f(R)$-brane model with $f(R) = R + \alpha R^2$. This investigation was based on the interesting analytical brane solution found in Ref. [33], where the warp factor was given by $e^{A(y)} = \text{sech}^B(ky)$ and the parameter α is constrained as $\alpha_1 \leq \alpha \leq \alpha_2$ (see Eq. (16)).

It was found that, when $\alpha_1 \leq \alpha \leq \alpha_s(< 0)$, the brane has an internal structure, but the effective potential $W(z)$ for the KK modes of the tensor fluctuation has only a simple structure (volcano-like potential with a single or double well) and no resonances were found. The reason is that the quasiwell of the effective potential is not deep enough. This is different from the brane models in general relativity, for which there are usually resonant modes when the brane has an internal structure [15–17].

However, the effective potential may have a rich structure with singularities and it will support a series of resonant KK modes when B is large enough ($B > 2$) and $\alpha > 0$, although the brane has no inner structure anymore. The reason is that the effective potential for the $f(R)$-brane model is decided by both the warp factor and the function $f(R)$. It was found

that, when $B > 2$ and $\frac{1}{40B^2k^2} < \alpha \leq \alpha_2$, the effective potential has two δ-like potential wells and there are a series of resonant modes on the brane.

The existence of resonant modes contributes a correction to the Newtonian graviton potential at short distance of the Planck scale. But f_R is negative in some range so long as the effective potential has singularities because of the monotony of f_R. However, this anomaly will result in the appearance of ghosts. So we hope to construct a $f(R)$-brane model with positive f_R and graviton resonances in the future.

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