RESEARCH

Open Access

CrossMark

Qualitative analysis of eigenvalues and eigenfunctions of one boundary value-transmission problem

Kadriye Aydemir[1*] and Oktay S Mukhtarov[2,3]

*Correspondence:
kadriyeaydemr@gmail.com
[1]Faculty of Education, Giresun
University, Giresun, 28100, Turkey
Full list of author information is
available at the end of the article

Abstract

The aim of this study is to investigate various qualitative properties of eigenvalues and corresponding eigenfunctions of one Sturm-Liouville problem with an interior singular point. We introduce a new Hilbert space and integral operator in it such a way that the problem under consideration can be interpreted as a spectral problem of this operator. By using our own approaches we investigate such properties as uniform convergence of the eigenfunction expansions, the Parseval equality, the Rayleigh-Ritz formula, the minimax principle, and the monotonicity of eigenvalues for the considered boundary value-transmission problem (BVTP).

Keywords: Sturm-Liouville problems; boundary-transmission conditions; eigenvalues; Fourier series of eigenfunctions; minimax principle

1 Introduction

Sturm-Liouville eigenvalue problems appear frequently in solving several classes of partial differential equations, particularly in solving the heat equation or a wave equation by separation of variables. Other examples of Sturm-Liouville boundary value problems are Hermite equations, Airy equations, Legendre equations *etc.* Also, many physical processes, such as the vibration of strings, the interaction of atomic particles, electrodynamics of complex medium, aerodynamics, polymer rheology or the earth's free oscillations, yield Sturm-Liouville eigenvalue problems (see, for example, [1–6] and references therein).

In different areas of applied mathematics and physics many problems arise in the form of boundary value problems involving transmission conditions at the interior singular points. Such problems are called boundary value-transmission problems (BVTPs). For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see [7] and references therein). Another completely different field is that of 'hydraulic fracturing' (see [8]) used in order to increase the flow of oil from a reservoir into a producing oil well. Some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (*i.e.* a plate composed by materials with different characteristics piled in the thickness; see [9]). Some aspects of spectral problems for differential equations having singularities with classical boundary conditions at the endpoints were studied among others in [10–23] and references therein.

In this paper we shall investigate some qualitative properties of the eigenvalues and the corresponding eigenfunctions of one boundary value problem which consists of the Sturm-Liouville equation,

$$\tau(u) := -u'' + q(x)u = \lambda u(x), \quad x \in (-\pi, 0) \cup (0, \pi), \tag{1}$$

together with end-point conditions given by

$$\ell_1 u := \cos \alpha u(-\pi) + \sin \alpha u'(-\pi) = 0, \tag{2}$$

$$\ell_2 u := \cos \beta u(\pi) + \sin \beta u'(\pi) = 0, \tag{3}$$

and with transmission conditions at the interior singular point $x = 0$ given by

$$t_1 u := \gamma_1 u(0^-) - \delta_1 u(0^+) = 0, \tag{4}$$

$$t_2 u := \gamma_2 u'(0^-) - \delta_2 u'(0^+) = 0, \tag{5}$$

where $q(x)$ is a real-valued function; δ_i, γ_i ($i = 1, 2$) are real numbers; $\alpha, \beta \in [0, \pi)$; λ is a complex spectral parameter. Throughout we shall assume that $q(x)$ is continuous in $\Omega_1 := [-\pi, 0)$ and $\Omega_2 := (0, \pi]$ with finite one-hand limits $q(0^\pm)$; $\gamma_1 \gamma_2 > 0$, and $\delta_1 \delta_2 > 0$.

It is the aim of this study to investigate such important spectral properties as the eigenfunction expansion, Parseval's equality, the Rayleigh-Ritz formula (minimization principle), the minimax principle, and monotonicity of the eigenvalues for the Sturm-Liouville problem (1)-(5). The 'Rayleigh quotient' is the basis of an important approximation method that is used in solid state physics as well as in quantum mechanics. In the latter, it is used in the estimation of energy eigenvalues of nonsolvable quantum systems.

Often in physical problems, the sign of the eigenvalue λ is quite important. For example, the equation $\frac{dh}{dt} + \lambda h = 0$ occurs in certain heat flow problems. Here, positive λ corresponds to exponential decay in time, while negative λ corresponds to exponential growth. In the vibration problems $\frac{d^2 h}{dt^2} + \lambda h = 0$ only positive λ corresponds to the 'usual' expected oscillations.

The Rayleigh quotient cannot be used to explicitly determine the eigenvalue since the eigenfunction is unknown. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation. Particularly, it can be quite useful in estimating the eigenvalues.

2 Preliminary results about eigenvalues and eigenfunctions

In the direct sum of the Lebesgue spaces $H := L_2(\Omega_1) \oplus L_2(\Omega_2)$ we shall define a new inner product in terms of the coefficients of the considered transmission conditions as follows:

$$\langle f, g \rangle_H := \gamma_1 \gamma_2 \int_{-\pi}^{0^-} f(x)\overline{g(x)} \, dx + \delta_1 \delta_2 \int_{0^+}^{\pi} f(x)\overline{g(x)} \, dx. \tag{6}$$

Remark 2.1 It is easy to see that the space H is also a Hilbert space with respect to the modified inner product (6).

Lemma 2.2 *Let u and υ be eigenfunctions of BVTP* (1)-(5) *corresponding to distinct eigenvalues λ and μ, respectively. If* $\lambda \neq \overline{\mu}$ *then u and υ are orthogonal in the Hilbert space H, i.e.*

$$\gamma_1 \gamma_2 \int_{-\pi}^{0^-} u\overline{\upsilon}\, dx + \delta_1 \delta_2 \int_{0^+}^{\pi} u\overline{\upsilon}\, dx = 0.$$

Proof Since $\tau(u) = \lambda u$ and $\tau(\upsilon) = \mu \upsilon$,

$$(\lambda - \overline{\mu})\langle u, \upsilon \rangle_H = \langle \lambda u, \upsilon \rangle_H - \langle u, \mu \upsilon \rangle_H$$

$$= \langle \tau(u), \upsilon \rangle_H - \langle u, \tau(\upsilon) \rangle_H. \tag{7}$$

By using the Lagrange identity we have

$$\langle \tau(u), \upsilon \rangle_H - \langle u, \tau(\upsilon) \rangle_H = \gamma_1 \gamma_2 W(u, \upsilon; x)|_{-\pi}^{0^-} + \delta_1 \delta_2 W(u, \upsilon; x)|_{0^+}^{\pi}, \tag{8}$$

where $W(u, \upsilon; x)$ denotes the Wronskians of u and υ. The boundary conditions (2) and (3) implies $W(u, \upsilon; -\pi) = W(u, \upsilon; \pi) = 0$. Further the transmission conditions (4) and (5) imply

$$\gamma_1 \gamma_2 W(u, \upsilon; 0^-) = \delta_1 \delta_2 . W(u, \upsilon; 0^+). \tag{9}$$

By using these equations we get $(\lambda - \overline{\mu})\langle u, \upsilon \rangle_H$. Thus, $\lambda \neq \overline{\mu}$ implies $\langle u, \upsilon \rangle_H = 0$, which completes the proof. □

Theorem 2.3 *All eigenvalues of the BVTP* (1)-(5) *are real.*

Proof Let $(\lambda_0, u_0(x))$ be any eigen-pair of the problem (1)-(5). Taking the complex-conjugate of the BVTP (1)-(5) we see that the pair $(\overline{\lambda}_0, \overline{u_0(x)})$ is also an eigen-pair of this problem. From the boundary-transmission conditions (2)-(5) it follows easily that

$$W(u_0, \overline{u}_0; -\pi) = W(u_0, \overline{u}_0; \pi) = 0 \tag{10}$$

and

$$\gamma_1 \gamma_2 W(u_0, \overline{u}_0; 0^-) - \delta_1 \delta_2 W(u_0, \overline{u}_0; 0^+) = 0. \tag{11}$$

Putting these equalities in the equality (7) we have $(\lambda - \overline{\lambda}_0)\|u_0\|^2 = 0$. This implies that $\lambda_0 - \overline{\lambda}_0 = 0$, *i.e.* λ_0 is real. □

Remark 2.4 Let λ_0 be an eigenvalue of (1)-(5) with corresponding eigenfunction $u_0(x) = \upsilon_0(x) + i\omega_0(x)$, where $\upsilon_0(x)$ and $\omega_0(x)$ are real-valued. Then both $\upsilon_0(x)$ and $\omega_0(x)$ are also eigenfunctions corresponding to the same eigenvalue λ_0. Indeed, putting $u = u_0 = \upsilon_0 + i\omega_0$ and $\lambda = \lambda_0$ in (1)-(5) and in view of λ_0 being real, we have

$$\tau(\upsilon_0) + i\tau(\omega_0) = (\lambda \upsilon_0) + i(\lambda \omega_0),$$

$$\ell_i(\upsilon_0) + i\ell_i(\omega_0) = 0 \quad \text{and} \quad t_i(\upsilon_0) + it_i(\omega_0) = 0, \quad i = 1, 2,$$

from which it follows that both $\upsilon_0(x)$ and $\omega_0(x)$ are eigenfunctions corresponding to the same eigenvalue λ_0.

Theorem 2.5 *There exists only one independent eigenfunction corresponding to each eigenvalue of the BVTP* (1)-(5), *i.e. each of eigenvalues of this problem is geometrically simple.*

Proof By way of contradiction suppose that there exist two linearly independent eigenfunctions $u_0(x)$ and $\upsilon_0(x)$ corresponding to the same eigenvalue λ_0. The boundary conditions (1)-(3) imply that $W(u_0, \upsilon_0; \pi) = 0$ and consequently $W(u_0, \upsilon_0; x) = 0$ for all $x \in [-\pi, 0)$. Since $u_0(x)$ and $\upsilon_0(x)$ satisfy equation (1), $u_0(x)$ and $\upsilon_0(x)$ are linearly dependent on Ω_1 by the well-known theorem of ordinary differential equation theory, *i.e.* there exists a constant $c_1 \neq 0$ such that $u_0(x) = c_1 \upsilon_0(x)$ for all $x \in \Omega_1$. Similarly, from the second boundary condition it follows that there exists a constant $c_2 \neq 0$ such that $u_0(x) = c_2 \upsilon_0(x)$ for all $x \in \Omega_2$. Hence

$$u_0(x) = \begin{cases} c_1 \upsilon_0(x) & \text{for } x \in \Omega_1, \\ c_2 \upsilon_0(x) & \text{for } x \in \Omega_2. \end{cases} \tag{12}$$

Substituting the transmission conditions (4)-(5) we have

$$(c_1 - c_2)\gamma_1 \upsilon_0\left(0^-\right) = (c_1 - c_2)\delta_1 \upsilon_0\left(0^+\right) = 0$$

and

$$(c_1 - c_2)\gamma_2 \upsilon_0'\left(0^-\right) = (c_1 - c_2)\delta_2 \upsilon_0'\left(0^+\right) = 0.$$

From these equalities we get $c_1 - c_2 = 0$. Consequently $u_0(x)$ and $\upsilon_0(x)$ are linearly dependent on the whole $\Omega = \Omega_1 \cup \Omega_2$. Hence we have obtained a contradiction, which completes the proof. □

Remark 2.6 By virtue of Theorem 2.5 the eigenfunctions of a BVTP (1)-(5) can be chosen to be real-valued. Indeed, let λ_0 be an eigenvalue with the eigenfunction $u_0(x) = \upsilon_0(x) + i\omega_0(x)$. By Remark 2.4 both $\upsilon_0(x)$ and $\omega_0(x)$ are also eigenfunctions corresponding to the same eigenvalue. By Theorem 2.5 there is a complex number $C_0 \neq 0$ such that $\omega_0(x) = C_0 \upsilon_0(x)$. Hence $u_0(x) = \upsilon_0(x) + i\omega_0(x) = (1 + iC_0)\upsilon_0(x)$, *i.e.* here is only one real-valued eigenfunction, except for a constant factor, corresponding to each eigenvalue. In view of this fact, from now on we can assume that all eigenfunctions of the BVTP (1)-(3) are real-valued.

Now from Lemma 2.2, Theorem 2.3, and Remark 2.6 we have the next corollary.

Corollary 2.7 *Let u_1 and u_2 be eigenfunctions of BVTP* (1)-(5) *corresponding to distinct eigenvalues λ_1 and λ_2. Then u_1 and u_2 are orthogonal in the sense of the following equality*:

$$\gamma_1\gamma_2 \int_{-\pi}^{0^-} u_1(x)u_2(x)\,dx + \delta_1\delta_2 \int_{0^+}^{\pi} u_1(x)u_2(x)\,dx = 0. \tag{13}$$

3 Reduction of (1)-(5) to the integral equation with the Green kernel

Let $u_1(x, \lambda)$ be the solution of equation (1) on the left interval Ω_1 (the so-called left-hand solution) satisfying $u_1(-\pi) = \sin\alpha$, $u'_1(-\pi) = -\cos\alpha$. Next we proceed from $u_1(x, \lambda)$ to define the right-hand solution $u_2(x, \lambda)$ of equation (1) on the right-hand interval Ω_2 by the initial conditions

$$u(0) = \frac{\delta_1}{\gamma_1} u_1(0^-, \lambda), \qquad u'(0) = \frac{\delta_2}{\gamma_2} u'_1(0^-, \lambda).$$

Now, let $v_2(x, \lambda)$ be the solution of equation (1) on the right-hand interval Ω_2 satisfying the initial conditions $v_1(-\pi) = \sin\beta$, $v'_1(-\pi, \lambda) = -\cos\beta$. Similarly we proceed from $v_2(x, \lambda)$ to define the left-hand solution $v_1(x, \lambda)$ of equation (1) on the left-hand interval Ω_2 by the initial conditions

$$v(0) = \frac{\delta_1}{\gamma_1} v_2(0^+, \lambda), \qquad v'_2(0) = \frac{\delta_2}{\gamma_2} v_2(0^+, \lambda).$$

The existence of the solutions u_i and v_i ($i = 1, 2$) is obvious. Moreover, by using totally similar arguments as in [24] we can prove that each of these solutions is an entire function of the parameter $\lambda \in \mathbb{C}$ for each fixed x. Since the Wronskian $W[u_i(x, \lambda), v_i(x, \lambda)]$ is independent of the variable $x \in \Omega_i$ ($i = 1, 2$), we can denote $\omega_i(\lambda) := W[v_i(\cdot, \lambda), v_i(\cdot, \lambda)]$ ($i = 1, 2$). Using the transmission conditions (4)-(5) it is easy to see that $\gamma_1\gamma_2\omega_1(\lambda) = \delta_1\delta_2\omega_2(\lambda)$. Both sides of this equality we shall denote by $\omega(\lambda)$. Now consider the following nonhomogeneous BVTP:

$$\tau(u) - \lambda u = f, \qquad \ell_i(u) = t_i(u) = 0, \quad i = 1, 2. \tag{14}$$

Let us define a Banach space $\oplus C^k(\Omega)$ as

$$\oplus C^k(\Omega) := \left\{ f = \begin{cases} f_{(1)}(x) & \text{for } x \in \Omega_1, \\ f_{(2)}(x) & \text{for } x \in \Omega_2 \end{cases} : f_{(1)}(x) \in C^k[-\pi, 0], f_{(2)}(x) \in C^k[0, \pi] \right\}$$

($k = 0, 1, 2, \ldots$) with the norm $\|f\|_{\oplus C^k(\Omega)} := \max\{\|f_{(1)}\|_{C^k[-\pi, 0]}, \|f_{(2)}\|_{C^k[0, \pi]}\}$. Below instead of $\oplus C^0(\Omega)$ we shall write $\oplus C(\Omega)$.

Theorem 3.1 *Let $f \in \oplus C(\Omega)$. Then for λ not an eigenvalue, the nonhomogeneous BVTP* (14) *has a unique solution u_f for which the following formula holds*:

$$u_f(x, \lambda) = \frac{1}{\omega(\lambda)} \begin{cases} \gamma_1\gamma_2\{v_1(x, \lambda) \int_{-\pi}^{x} u_1(\xi, \lambda) f(\xi)\, d\xi + u_1(x, \lambda) \int_{x}^{0^-} v_1(\xi, \lambda) f(\xi)\, d\xi\} \\ \quad + \delta_1\delta_2 u_1(x, \lambda) \int_{0^+}^{\pi} v_2(\xi, \lambda) f(\xi)\, d\xi \quad \text{for } x \in [-\pi, 0), \\ \delta_1\delta_2\{v_2(x, \lambda) \int_{0^+}^{x} u_2(\xi, \lambda) f(\xi)\, d\xi + u_2(x, \lambda) \int_{x}^{\pi} v_2(\xi, \lambda) f(\xi)\, d\xi\} \\ \quad + \gamma_1\gamma_2 v_2(x, \lambda) \int_{-\pi}^{0^-} u_1(\xi, \lambda) f(\xi)\, d\xi \quad \text{for } x \in (0, \pi]. \end{cases} \tag{15}$$

Proof By differentiating equation (15) twice we can easily see that $\tau(u) = \lambda u + f$, $\ell_i(u_f) = t_i(u_f) = 0$ ($i = 1, 2$) so the function u_f given by (15) is the solution of the problem. We shall prove the uniqueness by way of contradiction. Suppose that there are two different solutions u_0 and v_0 of the system (14) corresponding to the same λ_0, which is not an eigenvalue. Denoting $\omega_0 := u_0 - v_0$ we get $\tau(\omega_0) = \lambda_0\omega_0$, $\ell_i(\omega_0) = t_i(\omega_0) = 0$ for $i = 1, 2$, *i.e.*

λ_0 is an eigenvalue with the corresponding eigenfunction ω_0. So we get a contradiction, which completes the proof. $\qquad\square$

Let us introduce to the consideration the function $G(x, \xi, \lambda)$ given by

$$
G(x, \xi; \lambda) = \frac{1}{\omega(\lambda)}
\begin{cases}
u_1(x, \lambda)v_1(\xi, \lambda) & \text{for } -\pi \leq \xi \leq x < 0, \\
u_1(\xi, \lambda)v_1(x, \lambda) & \text{for } -\pi \leq x \leq \xi < 0, \\
u_2(\xi, \lambda)v_1(x, \lambda) & \text{for } -\pi \leq x < 0 < \xi \leq \pi, \\
u_2(x, \lambda)v_1(\xi, \lambda) & \text{for } -\pi \leq \xi < 0 < x \leq \pi, \\
u_2(x, \lambda)v_2(\xi, \lambda) & \text{for } 0 < \xi \leq x \leq \pi, \\
u_2(\xi, \lambda)v_2(x, \lambda) & \text{for } 0 < x \leq \xi \leq \pi.
\end{cases}
\tag{16}
$$

Then equation (15) can be written in the following form:

$$
u_f(x, \lambda) = \gamma_1\gamma_2 \int_{-\pi}^{0^-} G(x, \xi; \lambda)f(\xi)\, d\xi + \delta_1\delta_2 \int_{0^+}^{\pi} G(x, \xi; \lambda)f(\xi)\, d\xi,
\tag{17}
$$

i.e. $u_f(x, \lambda) = \langle G(x, \cdot, \lambda), f(\cdot) \rangle_H$. Consequently the function $G(x, \xi, \lambda)$ given by (16) is the Green's function for the considered BVTP. Now suppose that $\lambda = 0$ is not an eigenvalue and let $f \in \oplus C(\Omega)$ be an arbitrary function. Denoting $G(x, \xi) = G(x, \xi; 0)$ we have

$$
\tau(u) = f, \qquad \ell_i(u) = t_i(u) = 0, \quad i = 1, 2,
\tag{18}
$$

has an unique solution $u = u(x)$ given by

$$
u(x) = \gamma_1\gamma_2 \int_{-\pi}^{0^-} G(x, \xi)f(\xi)\, d\xi + \delta_1\delta_2 \int_{0^+}^{\pi} G(x, \xi)f(\xi)\, d\xi.
\tag{19}
$$

Putting $f = \lambda u$ in equation (19) we have the following integral equation with Green's kernel:

$$
u(x) = \lambda \left(\gamma_1\gamma_2 \int_{-\pi}^{0^-} G(x, \xi)u(\xi)\, d\xi + \delta_1\delta_2 \int_{0^+}^{\pi} G(x, \xi)u(\xi)\, d\xi \right).
\tag{20}
$$

4 Uniform and mean-square convergence of the eigenfunction expansions

Let us define the integral operator $\mathfrak{F}$ by

$$
(\mathfrak{F}u)(x) = \gamma_1\gamma_2 \int_{-\pi}^{0^-} G(x, \xi)u(\xi)\, d\xi + \delta_1\delta_2 \int_{0^+}^{\pi} G(x, \xi)u(\xi)\, d\xi.
\tag{21}
$$

Then the BVTP (1)-(5) converts to the spectral problem for the integral operator $\mathfrak{F}$ given by

$$
(I - \lambda\mathfrak{F})u = 0,
$$

where I is the identity operator. Since the kernel $G(x, \xi)$ of the integral operator $\mathfrak{F}$ is symmetric and continuous we can apply the well-known extremal principle (see, for example, [25]). Let $\{\lambda_n\}$ be a sequence of eigenvalues of the integral operator $\mathfrak{F}$ determined by the extremal principles and $\{\phi_n(x)\}$ be the corresponding sequence of orthonormal eigenfunctions.

Lemma 4.1 *Let $g \in \oplus C(\Omega)$. Then*

$$\lim_{m \to \infty} \left(\gamma_1 \gamma_2 \int_{-\pi}^{0^-} \left(\mathfrak{F}g - \sum_{i=1}^{m} c_i(\mathfrak{F}g)\phi_i \right)^2 dx \right.$$

$$\left. + \delta_1 \delta_2 \int_{0^+}^{\pi} \left(\mathfrak{F}g - \sum_{i=1}^{m} c_i(\mathfrak{F}g)\phi_i \right)^2 dx \right) = 0, \tag{22}$$

where $c_i(\mathfrak{F}g) = \langle \mathfrak{F}g, \phi_i \rangle_H$ denote the Fourier coefficients of $\mathfrak{F}g$ with respect to the orthonormal set (ϕ_i).

Proof Denote $g_m(x) = g(x) - \sum_{i=1}^{m} \langle g, \phi_i \rangle_H \phi_i$. Since $\{\phi_n\}$ is the orthonormal system in H, $\langle g_m, \phi_i \rangle_H = 0$ for $i = 1, \ldots, m$. From the fact that the eigenvalues λ_n are determined by the extremal principle with the corresponding sequence of orthonormal eigenfunctions $\{\phi_n\}$ we have $\|\mathfrak{F}g_m\|_H \le |\lambda_{m+1}| \|g_m\|_H$. Since $\lambda_{m+1} \to 0$, $\|\mathfrak{F}g_m\|_H \to 0$. Then we have

$$\mathfrak{F}g = \mathfrak{F}g_m + \sum_{i=1}^{m} \langle g, \phi_i \rangle_H \mathfrak{F}\phi_i = \mathfrak{F}g_m + \sum_{i=1}^{m} \lambda_i \langle g, \phi_i \rangle_H \phi_i$$

$$= \mathfrak{F}g_m + \sum_{i=1}^{m} \langle g, \mathfrak{F}\phi_i \rangle_H \phi_i = \mathfrak{F}g_m + \sum_{i=1}^{m} \langle \mathfrak{F}g, \phi_i \rangle_H \phi_i \tag{23}$$

for arbitrary $m = 1, 2, \ldots$. Letting $m \to \infty$ we get

$$\mathfrak{F}g = \sum_{i=1}^{\infty} \langle \mathfrak{F}g, \phi_i \rangle_H \phi_i, \tag{24}$$

where the convergence is in the Hilbert space H, *i.e.* the equality (22) holds. □

Corollary 4.2 *If $g \in \oplus C(\Omega)$ then the Parseval equality*

$$\|\mathfrak{F}g\|_H^2 = \sum_{i=1}^{\infty} c_i^2(\mathfrak{F}g)$$

holds.

Corollary 4.3 *The set of orthonormal eigenfunction of the integral operator $\mathfrak{F}$ is complete in the range of the integral operator $\mathfrak{F}$ given by*

$$R(\mathfrak{F}) = \left\{ h \in \oplus C(\Omega) | \text{there exists } g \in \oplus C(\Omega) \text{ such that } h = \mathfrak{F}g \right\}.$$

Theorem 4.4 *Let the hypotheses and notation of Lemma 4.1 hold. Then, for any $h \in R(\mathfrak{F})$,*

$$h = \sum_{i=1}^{\infty} \left(\gamma_1 \gamma_2 \int_{-\pi}^{0^-} h\phi_i \, dx + \delta_1 \delta_2 \int_{0^+}^{\pi} h\phi_i \, dx \right) \phi_i(x),$$

where the series converges with respect to the norm $\oplus C(\Omega)$, i.e. uniformly on $\Omega = \Omega_1 \cup \Omega_2$.

Proof Let $h = \mathfrak{F}g$. Then for any n, p we have

$$\sum_{i=n}^{n+p} \lambda_i \langle g, \phi_i \rangle_H \phi_i = \mathfrak{F}\left[\sum_{i=n}^{n+p} \langle g, \phi_i \rangle_H \phi_i \right]. \tag{25}$$

In view of the fact that the integral operator $\mathfrak{F}$ is a bounded linear operator in the Banach space $\oplus C(\Omega)$ we get from (25)

$$\left| \sum_{i=n}^{n+p} \lambda_i \langle g, \phi_i \rangle_H \phi_i \right| \le C \left[\sum_{i=n}^{n+p} |\langle g, \phi_i \rangle_H|^2 \right]^{1/2} \tag{26}$$

for some constant C independent of n. By Bessel's inequality, the right-hand side of this inequality tends to zero as $n \to \infty$ uniformly. Thus the series

$$\sum_{i=1}^{\infty} \langle \mathfrak{F}g, \phi_i \rangle_H \phi_i(x) \tag{27}$$

converges in the Banach space $\oplus C(\Omega)$. Let $\tilde{h}(x)$ be the sum of the last series. Consequently $\tilde{h} \in \oplus C(\Omega)$ and

$$\tilde{h}(x) = \sum_{i=1}^{\infty} \langle \mathfrak{F}g, \phi_i \rangle_H \phi_i(x). \tag{28}$$

From (23) and (28) it follows that $\|\mathfrak{F}g - \tilde{h}\|_H = 0$, *i.e.* $h(x) = \tilde{h}(x)$ almost everywhere. Since h is also continuous in Ω we have $h(x) = \tilde{h}(x)$ for all $x \in \Omega$. Thus

$$h = \sum_{i=1}^{\infty} \langle h, \phi_i \rangle_H \phi_i, \tag{29}$$

where the series converges with respect to the norm of $\oplus C(\Omega)$, *i.e.* uniformly on Ω. $\quad\square$

Theorem 4.5 *The set of all nonzero eigenvalues of the integral operator $\mathfrak{F}$ coincide with the set of the eigenvalues (λ_n) which are obtained from the extremal principle.*

Proof By way of contradiction, suppose there is a nonzero eigenvalue λ^* distinct from all eigenvalues (λ_n). Let u^* be the eigenfunction corresponding to the eigenvalue λ^*. Then from Theorem 4.4 we get

$$\lambda^* u^* = \mathfrak{F}u^* = \sum_{i=1}^{\infty} \langle \mathfrak{F}u^*, \phi_i \rangle_H \phi_i = \lambda^* \sum_{i=1}^{\infty} \langle u^*, \phi_i \rangle_H \phi_i = 0 \tag{30}$$

since $\langle u^*, \phi_i \rangle_H = 0$ for all $i = 1, 2, \ldots$ by Theorem 2.7. Thus we get a contradiction. $\quad\square$

Theorem 4.6 *Let $f \in \oplus C^2(\Omega)$ and satisfy the boundary-transmission conditions (2)-(5). Then the Fourier series of f with respect to $\{\phi_i\}$ converges uniformly on $\Omega_1 \cup \Omega_2$, i.e.*

$$\lim_{n \to \infty} \left\{ \sup_{x \in \Omega} \left| f(x) - \sum_{i=1}^{n} \left(\gamma_1 \gamma_2 \int_{-\pi}^{0^-} f\phi_i \, dx + \delta_1 \delta_2 \int_{0^+}^{\pi} f\phi_i \, dx \right)^2 \phi_i(x) \right| \right\} = 0.$$

Proof Let $f \in \oplus C^2(\Omega)$ satisfy the boundary-transmission conditions (2)-(5) and denote $g = \tau(f)$. Then $g \in \oplus C(\Omega)$. By virtue of (17) and (21) we have $f = \mathfrak{F}g$. From Lemma 4.1,

$$f = \mathfrak{F}g = \sum_{i=1}^{\infty} \langle \mathfrak{F}g, \phi_i \rangle_H \phi_i = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle_H \phi_i, \tag{31}$$

where the series is convergent in the Banach space $\oplus C(\Omega)$. □

Theorem 4.7 *The set of eigenfunctions $\{\phi_i(x)\}$ is a complete orthonormal set in the Hilbert space H.*

Proof Denote by $\oplus C_0^k(\Omega)$ the set of all functions $f \in C^k(\Omega)$ which vanishes at some neighborhoods of the points $x = -\pi$, $x = 0$, and $x = \pi$. Let $f \in H$ and $\epsilon > 0$ be given. Then there exists a function $g \in \oplus C_0^2(\Omega)$ such that $\|f - g\|_{\mathcal{H}} < \frac{\epsilon}{3}$ since the set $\oplus C_0^2(\Omega)$ is dense in the Hilbert space H, *i.e.* $\overline{\oplus C_0^2(\Omega)} = H$ (see, for example, [26]). It is clear that

$$\left\| f - \sum_{i=1}^{m} \langle f, \phi_i \rangle_H \phi_i \right\|_H \le \|f - g\|_H + \left\| g - \sum_{i=1}^{m} \langle g, \phi_i \rangle_H \phi_i \right\|_H$$
$$+ \left\| \sum_{i=1}^{m} \langle (g - f), \phi_i \rangle_H \phi_i \right\|_H \tag{32}$$

for arbitrary m. By Bessel's inequality we have

$$\left\| \sum_{i=1}^{m} \langle (g - f), \phi_i \rangle_H \phi_i \right\|_H^2 = \sum_{i=1}^{m} \left| \langle (g - f), \phi_i \rangle_H \right|^2 \le \|f - g\|_H^2 < \left(\frac{\epsilon}{3} \right)^2$$

and, by Theorem 4.6, there exists an integer $n_0 = n_0(\epsilon)$ such that, for $m > n_0$,

$$\left\| g - \sum_{i=1}^{m} \langle g, \phi_i \rangle_H \phi_i \right\|_H < \frac{\epsilon}{3}. \tag{33}$$

Finally, from (32) and (33) we get

$$\left\| f - \sum_{i=1}^{m} \langle f, \phi_i \rangle_H \phi_i \right\|_H < \epsilon$$

for $m > n_0$. The proof is complete. □

Now we are ready to prove the next important result.

Theorem 4.8 *The set of eigenfunctions $(\phi_i(x))$ of the problem* (1)-(5) *form an orthonormal basis in the Hilbert space H and for any $f \in H$ the Parseval equality*

$$\|f\|_H^2 = \sum_{i=1}^{\infty} \left(\gamma_1 \gamma_2 \int_{-\pi}^{0^-} f \phi_i \, dx + \delta_1 \delta_2 \int_{0^+}^{\pi} f \phi_i \, dx \right)^2$$

holds.

Proof Without loss of generality we shall assume that $\lambda = 0$ is not an eigenvalue. Otherwise, we can select a real $\lambda_0 \neq 0$ such that the problem $\tau u = \lambda_0 u$, $\ell_i(u) = t_i(u) = 0$, $i = 1, 2$ has no nontrivial solutions. Then denoting $\tilde{\lambda} = \lambda - \lambda_0$ and $\tilde{q}(x) = q(x) - \lambda_0$ we see that the problem

$$-u'' + \tilde{q}(x)u = \tilde{\lambda}u, \qquad \ell_i(u) = t_i(u) = 0, \quad i = 1, 2, \tag{34}$$

has the same properties for the eigenfunctions and eigenvalues as the considered problem (1)-(5). Namely, the pair $(\tilde{\lambda}, u(x))$ is the eigen-pair of the problem (34) if and only if the pair $(\lambda, u(x))$ is an eigen-pair of (1)-(5). Clearly, $\tilde{\lambda} = 0$ is not an eigenvalue of the problem (34). Hence, without loss of generality we can assume that $\lambda = 0$ is not an eigenvalue of the considered BVTP (1)-(5). Moreover, if $\lambda \neq 0$, then the pair $(\lambda, u(x))$ is the eigen-pair of the BVTP (1)-(5) if and only if the pair $(\frac{1}{\lambda}, u(x))$ is the eigen-pair of the integral operator $\mathfrak{F}$. Consequently the set ϕ_i form an orthonormal set of eigenfunctions either for $\mathfrak{F}$ and (1)-(5). Moreover, this set is complete by Theorem 4.7. It is well known that any complete orthonormal set in a Hilbert space forms an orthonormal basis. Consequently, every function $f \in H$ may be expanded in a Fourier series with respect to the orthonormal set of eigenfunctions (ϕ_i), *i.e.* the equality

$$f = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle_H \phi_i$$

holds, where the series converges with respect to the norm of the Hilbert space H. Further, the Parseval equality follows immediately from the last equality. □

5 The Rayleigh-Ritz principle for the BVTP (1)-(5)

In the last sections of this study we will investigate some extremal properties of the eigenvalues and corresponding eigenfunctions of the considered BVTP (1)-(5) by using some variational methods.

Lemma 5.1 *Let $q(x) \geq 0$ for all $x \in \Omega$. Then all eigenvalues of the problem* (1)-(5) *are non-negative.*

Proof Let $(\lambda, u(x))$ be any eigen-pair of the problem (1)-(5). Multiplying (1) by $u(x)$ and integrating by parts from $x = -\pi$ to $x = 0$, and from $x = 0$ to $x = \pi$, we have

$$\gamma_1 \gamma_2 \int_{-\pi}^{0^-} u\left[u'' - qu + \lambda u\right] dx + \delta_1 \delta_2 \int_{0^+}^{\pi} u\left[u'' - qu + \lambda u\right] dx$$

$$= \gamma_1 \gamma_2 \int_{-\pi}^{0^-} \left[-u'^2 - qu^2 + \lambda u^2\right] dx + \delta_1 \delta_2 \int_{0^+}^{\pi} \left[-u'^2 - qu^2 + \lambda u^2\right] dx$$

$$+ \gamma_1 \gamma_2 uu'|_{-\pi}^{0^-} + \delta_1 \delta_2 uu'|_{0^+}^{\pi} = 0. \tag{35}$$

By using the equalities (4)-(5), we get $\gamma_1 \gamma_2 uu'|_{-\pi}^{0^-} + \delta_1 \delta_2 uu'|_{0^+}^{\pi} = 0$. Hence

$$\gamma_1 \gamma_2 \int_{-\pi}^{0^-} \left[-u'^2 - qu^2 + \lambda u^2\right] dx + \delta_1 \delta_2 \sigma \int_{0^+}^{\pi} \left[-u'^2 - qu^2 + \lambda u^2\right] dx = 0. \tag{36}$$

Consequently

$$\lambda = \frac{\gamma_1 \gamma_2 \int_{-\pi}^{0^-} [u'^2 + qu^2]\,dx + \delta_1 \delta_2 \int_{0^+}^{\pi} [u'^2 + qu^2]\,dx}{\gamma_1 \gamma_2 \int_{-\pi}^{0^-} u^2\,dx + \delta_1 \delta_2 \int_{0^+}^{\pi} u^2\,dx} \geq 0 \tag{37}$$

since $q \geq 0$ on $\Omega_1 \cup \Omega_2$ by assumption. $\qquad\square$

Theorem 5.2 *Let $q(x) \geq 0$ for all $x \in \Omega$. Then, all the eigenvalues of the problem (1)-(5) are positive if any one of the following conditions holds*:

(1) $q \not\equiv 0$, *i.e. there exists at least one $x_0 \in \Omega$ such that $q(x_0) > 0$*;

(2) $\cos^2 \alpha + \cos^2 \beta \neq 0$.

Proof Let λ_1 be the first eigenvalue with the corresponding eigenfunction $u_1(x)$. Show that $\lambda_1 > 0$.

(1) Since $q(x)$ is continuous in Ω there are $\delta > 0$ and $q_0 > 0$ such that $q(x) \geq q_0$ for all $x \in [x_0 - \delta, x_0 + \delta] \subset \Omega$. Then from (37) it follows immediately that $\lambda_1 > 0$.

(2) Suppose that it possible that $\lambda_1 = 0$. Then from (37), $u_1'(x) = 0$ for all $x \in \Omega$, *i.e.* $u_1(x)$ is a constant function in each of Ω_1 and Ω_2. Putting in (2) and (3) we have $\cos \alpha u_1(-\pi) = \cos \beta u_1(\pi) = 0$. Consequently at least one of $u_1(-\pi)$ and $u_1(\pi)$ is equal to zero and therefore u_1 is identically zero Ω_1 or Ω_2. Then by applying the transmission conditions (3) and (4) we see that u_1 is identically zero on the whole $\Omega = \Omega_1 \cup \Omega_2$. We have a contradiction, which completes the proof. $\qquad\square$

Theorem 5.3 *Suppose that any one of the following conditions holds*:

(1) $q \not\equiv 0$ *and* $q(x) \geq 0$;

(2) $q(x) \geq 0$ *and* $\cos^2 \alpha + \cos^2 \beta \neq 0$.

Let $\lambda_1 < \lambda_2 < \cdots$ be the sequence of eigenvalues with the corresponding normalized eigenfunctions $\phi_1(x), \phi_2(x), \ldots$ and let

$$S_n = \big\{ u(x) | u \in \oplus C^2(\Omega); u \not\equiv 0; \ell_i(u) = t_i(u) = 0 \text{ for } i = 1, 2;$$
$$\langle u, \phi_k \rangle_H = 0 \text{ for } k = 1, 2, \ldots, n-1 \big\}, \quad n = 1, 2, \ldots$$

(naturally by S_1 we mean $S_1 = \{u(x)| u \in \oplus C^2(\Omega); u \not\equiv 0; \ell_i(u) = t_i(u) = 0 \text{ for } i = 1, 2\}$). Then for all $n = 1, 2, \ldots$ we have

$$\lambda_n = \min\big\{ I(u) | u \in S_n \big\},$$

where the functional $I(u)$ is given by

$$I(u) = \frac{1}{\|u\|_H^2} \left\{ \gamma_1 \gamma_2 \int_{-\pi}^{0^-} [u'^2 + qu^2]\,dx + \delta_1 \delta_2 \int_{0^+}^{\pi} [u'^2 + qu^2]\,dx \right\}. \tag{38}$$

Moreover, the minimizing function is ϕ_n, i.e. $\lambda_n = I(\phi_n)$.

Proof Let $\varphi(\cdot) \in \oplus C^2(\Omega)$ with $\ell_i \varphi = t_i \varphi = 0$, $i = 1, 2$. Then by Theorem 4.6 we have

$$\varphi(x) = \sum_{n=1}^{\infty} \langle \varphi(\cdot), \phi_n(\cdot) \rangle_H \phi_n(x),$$

where the convergence is uniform on Ω. Then, by integration by parts, we get

$$\langle \tau(\varphi), \phi_n \rangle_H = \langle \varphi, \tau(\phi_n) \rangle_H = \lambda_n \langle \phi_n(\cdot), \varphi(\cdot) \rangle_H. \tag{39}$$

Since $\{\phi_n(x)\}$ is a complete orthonormal set, by Parseval's equation

$$\langle \varphi, \varphi \rangle_H = \sum_{n=1}^{\infty} c_n^2(\varphi), \tag{40}$$

where $c_n(\varphi) = \langle \phi_n(\cdot), \varphi(\cdot) \rangle_H$. By using (35) we get

$$\gamma_1 \gamma_2 \int_{-\pi}^{0} \left[\varphi'^2 + q\varphi^2 \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[\varphi'^2 + q\varphi^2 \right] dx$$
$$= \langle \varphi, \tau(\varphi) \rangle_H = -\left\langle \left(\sum_{n=1}^{\infty} c_n \phi_n \right), \tau(\varphi) \right\rangle_H = -\sum_{n=1}^{\infty} c_n \langle \phi_n, \tau(\varphi) \rangle_H$$
$$= \sum_{n=1}^{\infty} c_n \lambda_n \langle \varphi, \phi_n \rangle_H = \sum_{n=1}^{\infty} \lambda_n c_n^2 \geq \lambda_1 \|\varphi\|_H^2. \tag{41}$$

Consequently

$$\lambda_1 \leq \frac{1}{\|\varphi\|_H^2} \left\{ \gamma_1 \gamma_2 \int_{-\pi}^{0} \left[\varphi'^2 + q\varphi^2 \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[\varphi'^2 + q\varphi^2 \right] dx \right\}. \tag{42}$$

Putting $\varphi = \phi_1$ in equation (41) we have $c_n(\varphi) = \langle \phi_1, \phi_n \rangle = 0$ for $n = 2, 3, \ldots$ and

$$\lambda_1 = \frac{1}{\|\phi_1\|_H^2} \left\{ \gamma_1 \gamma_2 \int_{-\pi}^{0^-} \left[\phi_1'^2 + q\phi_1^2 \right] dx + \delta_1 \delta_2 \int_{0^+}^{\pi} \left[\phi_1'^2 + q\phi_1^2 \right] dx \right\}. \tag{43}$$

From (42)-(43) it follows immediately that

$$\lambda_1 = \min\{ I(\varphi) | \varphi \in S_1 \}$$

and the minimizing function is $\varphi = \phi_1(x)$, *i.e.* $\lambda_1 = I(\phi_1)$. Next, let $\varphi(x) \in \oplus C^2(\Omega)$ with $\ell_i \varphi = t_i \varphi = 0$, $i = 1, 2$, and $\langle \varphi, \phi_n \rangle = 0$ for $n = 1, \ldots, k$. Then

$$\gamma_1 \gamma_2 \int_{-\pi}^{0^-} \left[\varphi'^2 + q\varphi^2 \right] dx + \delta_1 \delta_2 \int_{0^+}^{\pi} \left[\varphi'^2 + q\varphi^2 \right] dx$$
$$= \sum_{n=k+1}^{\infty} \lambda_n c_n(\varphi)^2 \geq \lambda_{k+1} \sum_{n=k+1}^{\infty} c_n(\varphi)^2 = \lambda_{k+1} \|\varphi\|_H^2. \tag{44}$$

Hence, by the same arguments as before, we have

$$\lambda_{k+1} = \min\{ I(\varphi) | \varphi \in S_{k+1} \}$$

for $k = 1, 2, \ldots$ and the minimizing function is ϕ_{k+1}, *i.e.* $\lambda_{k+1} = I(\phi_{k+1})$. $\qquad \square$

Remark 5.4 By applying the minimization principle directly, it is not possible to determine explicitly the eigenvalues and corresponding eigenfunctions, since we do not know how to minimize over all 'admissible' functions. Nevertheless, using the Rayleigh functional (38) with appropriate test functions one can obtain useful approximations for the eigenvalues.

6 The minimax property of eigenvalues

According to the minimization principle which is given by the preceding Theorem 5.3 we can find the nth eigen-pair (λ_n, ϕ_n) only after the previous eigenfunctions $\phi_1(x), \phi_2(x), \ldots, \phi_{n-1}(x)$ are known. But in many applications it is important to have a characterization of any eigen-pair (λ_k, ϕ_k) that makes no reference to other eigen-pairs. By applying the following theorem we can determine the nth eigen-pair (λ_n, ϕ_n) without using the preceding eigenfunctions $\phi_1(x), \phi_2(x), \ldots, \phi_{n-1}(x)$.

Theorem 6.1 *Let $u_1(x), u_2(x), \ldots, u_{n-1}(x) \in \oplus C(\Omega)$ be arbitrary functions. Denote*

$$D_{n-1}(u_1, u_2, \ldots, u_{n-1}) = \big\{ u \in \oplus C^1(\Omega) | \ell_j(u) = t_j(u) = 0, j = 1, 2;$$

$$\langle u, u_i \rangle = 0 \text{ for } i = 1, 2, \ldots, n-1; \big\}, \quad n = 1, 2, \ldots$$

(naturally by $D_0(u_1, u_2, \ldots, u_{n-1})$ we mean the linear manifold

$$\big\{ u \in \oplus C^1(\Omega) | \ell_j(u) = t_j(u) = 0, j = 1, 2 \big\}).$$

Then the nth eigenvalue of the BVTP (1)-(5) is

$$\lambda_n = \max \big\{ \min \big\{ I(u) | u \in D_{n-1}(u_1, u_2, \ldots, u_{n-1}) \big\} | \phi_i \in \oplus C(\Omega)$$

$$\text{for } i = 1, 2, \ldots, n-1 \big\}. \tag{45}$$

Proof Let $0 < \lambda_1 < \lambda_2 < \cdots$ be the sequence of eigenvalues determined by the extremal principles and $\phi_1(x), \phi_2(x), \ldots$ be the corresponding sequence of orthonormal eigenfunctions and let $u_1, u_2, \ldots, u_{n-1} \in \oplus C(\Omega)$ be arbitrary functions. Define

$$F_n(u_1, u_2, \ldots, u_{n-1}) = \inf \big\{ I(u) | u \in D_{n-1}(u_1, u_2, \ldots, u_{n-1}) \big\}, \quad n = 1, 2, \ldots.$$

Now let $\psi_1(x), \psi_2(x), \ldots, \psi_{n-1}(x) \in \oplus C^1(\Omega)$ be any given functions, such that $\ell_i(\psi_j) = t_i(\psi_j) = 0$ $(i = 1, 2; j = 1, 2, \ldots, n-1)$. Denoting $a_{ij} = \langle \phi_i, \psi_i \rangle_H$ for $i, j = 1, 2, \ldots, n-1$, consider a system of $n-1$ homogeneous linear equations in n unknowns $z_1, z_2, \ldots, z_n$ given by

$$\sum_{i=1}^{n} a_{ij} z_i = 0, \quad j = 1, 2, \ldots, n-1. \tag{46}$$

Obviously this system of $n-1$ homogeneous linear equations in n unknowns has a nontrivial solution. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be any nontrivial solution of system (46). Define a function $\upsilon_n(x)$ by

$$\upsilon_n(x) = \alpha_1 \phi_1(x) + \cdots + \alpha_n \phi_n(x). \tag{47}$$

It is easy to see that

$$\langle \upsilon_n, \psi_j \rangle_H = 0 \quad \text{for } j = 1, 2, \ldots, n - 1. \tag{48}$$

Consequently $\upsilon_n \in D_{n-1}(\psi_1, \psi_2, \ldots, \psi_{n-1})$ and $\|\upsilon\|_H^2 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$. Then we have

$$\gamma_1 \gamma_2 \int_{-\pi}^{0} \left[\upsilon_n'^2 + q \upsilon_n^2 \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[\upsilon_n'^2 + q \upsilon_n^2 \right] dx$$

$$= \sum_{i,j=1}^{n} \alpha_i \alpha_j \left(\gamma_1 \gamma_2 \int_{-\pi}^{0} \left[\phi_i' \phi_j' + q \phi_i \phi_j \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[\phi_i' \phi_j' + q \phi_i \phi_j \right] dx \right).$$

Integrating by parts we get

$$\gamma_1 \gamma_2 \int_{-\pi}^{0} \left[\phi_i' \phi_j' + q \phi_i \phi_j \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[\phi_i' \phi_j' + q \phi_i \phi_j \right] dx$$

$$= \phi_i' \phi_j \big|_{-\pi}^{0^-} + \phi_i' \phi_j \big|_{0^+}^{\pi} + \langle \phi_j, \tau(\phi_i) \rangle_H = \lambda_i \langle \phi_j, u \phi_i \rangle_H = \lambda_i \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then we have

$$I(\upsilon_n) = \frac{1}{\|\upsilon_n\|_H^2} \sum_{i=1}^{n} \lambda_i \alpha_i^2 \leq \lambda_n.$$

Consequently $F_n(u_1, u_2, \ldots, u_{n-1}) \leq I(\upsilon_n) \leq \lambda_n$ for all $u_1, u_2, \ldots, u_{n-1} \in \oplus C(\Omega)$. Furthermore, by virtue of the preceding theorem

$$F_n(\phi_1, \phi_2, \ldots, \phi_{n-1}) = \lambda_n.$$

Hence

$$\lambda_n = \max \left\{ F(u_1, u_2, \ldots, u_{n-1}) | \phi_j \in \oplus C(\Omega), j = 1, 2, \ldots, n - 1 \right\},$$

which completes the proof. □

Remark 6.2 In many problems of mathematical physics, the smallest eigenvalue (the so-called principal eigenvalue) plays an important role. For example, the principal eigenvalue of the simple boundary value problem

$$-u'' = \lambda \rho(x) u, \quad x \in [-\pi, \pi], \qquad u(-\pi) = u(\pi) = 0,$$

where $\rho(x) > 0$ is the given function, can be interpreted as the square of the lowest frequency of vibration of a rod of nonuniform cross section given by $\rho(x)$. Therefore it is significant to determine explicitly the principal eigenvalue, or at least a 'good' estimation of it. Note that useful approximation values for the principal eigenvalue can be drawn from the minimax property (45) by using certain principles of the theory of the calculus of variations. In particular we can find an upper bound for the lowest eigenvalue λ_1. In fact let

$\oplus C(\Omega)$ be any nontrivial function satisfying the boundary-transmission conditions (2)-(5) called the trial function. Then by virtue of the minimax principle we have the inequality

$$\lambda_1 \leq \frac{1}{\|v\|_H^2} \left\{ \gamma_1 \gamma_2 \int_{-\pi}^{0} \left[v'^2 + qv^2 \right] dx + \delta_1 \delta_2 \int_{0}^{\pi} \left[v'^2 + qv^2 \right] dx \right\},$$

which gives us an upper bound for the principal eigenvalue. By taking the trial function $v(x)$ as close as possible to the corresponding eigenfunction we can expect to get the 'good' estimation for principal eigenvalue λ_1. In many special cases the useful trial function can be found by applying some principles of variational analysis.

7 Dependence of eigenvalues on the potential

The minimax principle of the eigenvalues, *i.e.* equation (45) for the eigenvalues makes it possible to study the dependence of the eigenvalues on the coefficients of the differential equation. In this section we shall establish the monotonicity of the eigenvalues with respect to the potential $q(x)$ for fixed boundary-transmission conditions.

Theorem 7.1 *Let $\lambda_n(q)$ be the nth eigenvalue of the BVTP* (1)-(5). *Then $\lambda_n(q)$ is a monotonically increasing function with respect to the variable $q = q(x)$, i.e. if $q_1(x) \leq q_2(x)$ for all $x \in \Omega$ then $\lambda_n(q_1) \leq \lambda_n(q_2)$.*

Proof Define

$$I_i(u) = \frac{1}{\|u\|_H^2} \left\{ \gamma_1 \gamma_2 \int_{-\pi}^{0-} \left[u'^2 + q_i u^2 \right] dx + \delta_1 \delta_2 \int_{0+}^{\pi} \left[u'^2 + q_i u^2 \right] dx \right\}, \quad i = 1, 2.$$

Let the notation of the preceding theorem hold and let $u_1(x), u_2(x), \ldots, u_{n-1}(x) \in \oplus C(\Omega)$ be any arbitrary functions.

Since $0 \leq q_1(x) \leq q_2(x)$ for all $x \in \Omega$ it is obvious that $I_1(u) \leq I_2(u)$ for all $u \in D_{n-1}(u_1, u_2, \ldots, u_{n-1})$. Then by virtue of Theorem 6.1, we find the required inequality $\lambda_n(q_1) \leq \lambda_n(q_2)$. The proof is complete. □

Author details
[1]Faculty of Education, Giresun University, Giresun, 28100, Turkey. [2]Department of Mathematics, Faculty of Arts and Science, Gaziosmanpaşa University, Tokat, 60250, Turkey. [3]Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan.

References
1. Gesztesy, F, Macedo, C, Streit, L: An exactly solvable periodic Schrödinger operator. J. Phys. A, Math. Gen. **18**, 503-507 (1985)
2. Kong, Q, Wu, H, Zettl, A: Geometric Aspects of Sturm-Liouville Problems. Preprint
3. Petrovsky, IG: Lectures on Partial Differential Equations, 1st edn. Interscience, New York (1954) (translated from Russian by A Shenitzer)
4. Pryce, JD: Numerical Solution of Sturm-Liouville Problems. Oxford University Press, London (1993)

5. Rotenberg, M: Studies of the quadratic Zeeman effect. I. Application of the sturmian functions. Adv. At. Mol. Phys. **6** (1970)
6. Sherstyuk, AI: Problems of Theoretical Physics, vol. 3. Leningrad. Gos. Univ., Leningrad (1988)
7. Huy, HP, Sanchez-Palencia, E: Phénomènes des transmission à travers des couches minces de conductivité élevée. J. Math. Anal. Appl. **47**, 284-309 (1974)
8. Cannon, JR, Meyer, GH: On a diffusion in a fractured medium. SIAM J. Appl. Math. **3**, 434-448 (1971)
9. Titeux, I, Yakubov, Y: Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients Math. Models Methods Appl. Sci. **7**(7), 1035-1050 (1997)
10. Allahverdiev, BP, Bairamov, E, Ugurlu, E: Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions. J. Math. Anal. Appl. **401**(1), 388-396 (2013)
11. Allahverdiev, BP, Ugurlu, E: On dilation, scattering and spectral theory for two-interval singular differential operators. Bull. Math. Soc. Sci. Math. Roum. **58(106)**(4), 383-392 (2015)
12. Aydemir, K: Boundary value problems with eigenvalue depending boundary and transmission conditions. Bound. Value Probl. **2014**, 131 (2014)
13. Aydemir, K, Mukhtarov, OS: Second-order differential operators with interior singularity. Adv. Differ. Equ. **2015**, 26 (2015). doi:10.1186/s13662-015-0360-7
14. Bairamov, E, Sertbaş, M, Ismailov, ZI: Self-adjoint extensions of singular third-order differential operator and applications. AIP Conf. Proc. **1611**(1), 177-182 (2014)
15. Hıra, F, Altınışık, N: Sampling theorems for Sturm-Liouville problem with moving discontinuity points. Bound. Value Probl. **2015**, 237 (2015)
16. Kadakal, M, Mukhtarov, OS: Sturm-Liouville problems with discontinuities at two points. Comput. Math. Appl. **54**, 1367-1379 (2007)
17. Kandemir, M: Irregular boundary value problems for elliptic differential-operator equations with discontinuous coefficients and transmission conditions. Kuwait J. Sci. Eng. **39**(1A), 71-97 (2010)
18. Kandemir, M, Mukhtarov, OS, Yakubov, S: Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter. Mediterr. J. Math. **6**, 317-338 (2009)
19. Mukhtarov, OS, Olğar, H, Aydemir, K: Resolvent operator and spectrum of new type boundary value problems. Filomat **29**(7), 1671-1680 (2015)
20. Mukhtarov, OS, Kandemir, M: Asymptotic behavior of eigenvalues for the discontinuous boundary-value problem with functional-transmission conditions. Acta Math. Sci. Ser. B **22**(3), 335-345 (2002)
21. Ugurlu, E, Allahverdiev, BP: On selfadjoint dilation of the dissipative extension of a direct sum differential operator. Banach J. Math. Anal. **7**(2), 194-207 (2013)
22. Aliyev, ZS, Kerimov, NB: Spectral properties of the differential operators of the fourth-order with eigenvalue parameter dependent boundary condition. Int. J. Math. Math. Sci. **2012**, Article ID 456517 (2012)
23. Mamedov, KR, Cetinkaya, FA: Inverse problem for a class of Sturm-Liouville operator with spectral parameter in boundary condition. Bound. Value Probl. **2013**, 183 (2013)
24. Akdoğan, Z, Demirci, M, Mukhtarov, OS: Green function of discontinuous boundary-value problem with transmission conditions. Math. Methods Appl. Sci. **30**, 1719-1738 (2007)
25. Courant, R, Hilbert, D: Methods of Mathematical Physics, vol. 1. Interscience, New York (1953)
26. Indritz, J: Methods in Analysis. Macmillan Co., New York (1963)