APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 18, no. 2 (2017), 241-253
doi:10.4995/agt.2017.5185


# An equivalence of results in $C^*$-algebra valued $b$-metric and $b$-metric spaces

Nguyen Van Dung [a,b], Vo Thi Le Hang [c,d] and Diana Dolicanin-Djekic [e]

[a] Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[b] Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam (nguyenvandung2@tdt.edu.vn)
[c] Journal of Science, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam
[d] Faculty of Mathematics and Information Technology Teacher Education, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam (vtlhang@dthu.edu.vn)
[e] Faculty of Technical Sciences, University of Pristina-Kosovska Mitrovica, Serbia (diana.dolicanin@pr.ac.rs)

*Communicated by J. Galindo*

Abstract

*In this paper, we first construct a b-metric, which is a special type of $C^*$-algebra-valued b-metrics, from a given $C^*$-algebra-valued b-metric and prove some equivalences between them. Then we show that not only fixed point results but also topological properties in $C^*$-algebra-valued b-metric spaces may be deduced from certain results in b-metric spaces. In particular, every $C^*$-algebra-valued b-metric space is metrizable.*

2010 MSC: *Primary 47H10; 54H25; Secondary 54D99; 54E99.*

Keywords: *metric; b-metric; $C^*$-algebra-valued metric; $C^*$-algebra-valued b-metric; fixed point.*

## 1. Introduction and preliminaries

In metric fixed point theory, many generalised metric spaces leading to fixed point theorems have been introduced, see [18] for example. One of them, that

*Received 28 March 2016 – Accepted 15 April 2017*

attracted many authors, is the notion of the $b$-metric [9], [10] or, equivalently, of the metric-type [17]. However, many of these generalised metric spaces are topologically equivalent to a metric space or to a known generalised metric space and many fixed point results in such spaces may be deduced from certain fixed point results in metric spaces or in known generalised metric spaces [5]. In particular, a $b$-metric space is also metrizable, see Theorem 1.11 below and more details in [11].

Based on the notion and properties of $C^*$-algebras, Ma *et al.* [21] introduced the notion of $C^*$-algebra-valued metric spaces and gave some fixed point theorems for self-maps with contractive or expansive conditions in such spaces. As applications, existence and uniqueness results for a type of integral equations and operator equations were given. Similar to the relation between a metric and a $b$-metric, Ma and Jiang [20] then generalised the notion of a $C^*$-algebra-valued metric to a $C^*$-algebra-valued $b$-metric. The classes of $C^*$-algebra-valued metric spaces and $C^*$-algebra-valued $b$-metric spaces were then studied by Batul and Kamran [7], Shehwar and Kamran [26], Klin-eama and Kaskasem [19], Qiaoling *et al.* [24], Shehwar *et al.* [25], Tianqing [27].

Very recently, Alsulami *et al.* [4] pointed out that the $C^*$-algebra-valued metric does not bring about a real extension in metric fixed point theory and showed that fixed point results in the $C^*$-algebra-valued metric can be derived from the desired Banach mapping principle and its famous consecutive theorems.

Motivated by the work of Alsulami *et al.* [4] we construct a $b$-metric from a given $C^*$-algebra-valued $b$-metric and prove some equivalences between them. Then we show that not only fixed point results but also topological properties in $C^*$-algebra-valued $b$-metric spaces may be deduced from certain results in $b$-metric spaces. A direct consequence is that every $C^*$-algebra-valued $b$-metric space is metrizable.

We next recall definitions and properties which will be useful in what follows.

**Definition 1.1** ([13], [22])**.** Let $\mathbb{A}$ be an algebra and $* : \mathbb{A} \longrightarrow \mathbb{A}$ be a map that maps $a \in \mathbb{A}$ to $a^* \in \mathbb{A}$.

(1) $\mathbb{A}$ is called a *$*$-algebra* if $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in \mathbb{A}$.
(2) $\mathbb{A}$ is called a *unital $*$-algebra* if there exists an identity element $1_{\mathbb{A}}$ of $\mathbb{A}$, that is, $\|1_{\mathbb{A}}\| = 1$ and $1_{\mathbb{A}}a = a1_{\mathbb{A}} = a$ for all $a \in \mathbb{A}$.
(3) $\mathbb{A}$ is called a *Banach $*$-algebra* if $\mathbb{A}$ is a complete normed unital $*$-algebra such that $\|ab\| \leq \|a\|\|b\|$ and $\|a\| = \|a^*\|$ for all $a, b \in \mathbb{A}$.
(4) $\mathbb{A}$ is called a *$C^*$-algebra* if $A$ is a Banach $*$-algebra and $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$.
(5) An element $a \in \mathbb{A}$ is called *positive*, written as $0 \preceq a$ or $a \succeq 0$, if $a^* = a$ and $\sigma(a) \subset [0, \infty)$ where
$$\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}} - a \text{ is not invertible } \}$$
is the spectrum of $a$.
(6) For all $a, b \in \mathbb{A}$, $a$ is called *less than* $b$ or $b$ is called *greater than* $a$, written as $a \preceq b$ or $b \succeq a$, if $b - a \succeq 0$.

 

The following lemma plays an important role in next proofs.

**Lemma 1.2** ([22], Theorem 2.2.5.(3))**.** *Let $\mathbb{A}$ be a $C^*$-algebra. If $a, b \in \mathbb{A}$ and $0 \preceq a \preceq b$ then $\|a\| \leq \|b\|$.*

The following notion of a $C^*$-algebra-valued metric is an analogue of a metric where a $C^*$-algebra $\mathbb{A}$ plays the role of the field of real numbers $\mathbb{R}$.

**Definition 1.3** ([21], Definition 2.1)**.** Let $X$ be a non-empty set, $\mathbb{A}$ be a $C^*$-algebra and $d : X \times X \longrightarrow \mathbb{A}$ be a map such that for all $x, y, z \in X$,

(1) $0 \preceq d(x, y)$; and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \preceq d(x, y) + d(y, z)$.

Then $d$ is called a *$C^*$-algebra-valued metric* on $X$ and $(X, \mathbb{A}, d)$ is called a *$C^*$-algebra-valued metric space.*

The convergence, Cauchy sequence and completeness in $C^*$-algebra-valued metric spaces are defined with the same manner as that in metric spaces as follows.

**Definition 1.4** ([21], Definition 2.2)**.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued metric space and $\{x_n\}$ be a sequence in $X$.

(1) The sequence $\{x_n\}$ is called *convergent* to $x \in X$, written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|d(x_n, x)\| = 0$.
(2) The sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} \|d(x_n, x_m)\| = 0$ .
(3) The space $(X, \mathbb{A}, d)$ is called *compete* if each Cauchy sequence is a convergent sequence.

The following notion of a $C^*$-algebra-valued $b$-metric is a generalisation of a $C^*$-algebra-valued metric. Note that the relation between a $C^*$-algebra-valued $b$-metric and a $C^*$-algebra-valued metric is very similar to that between a $b$-metric and a metric.

**Definition 1.5** ([10])**.** Let $X$ be a non-empty set and $\rho : X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$ and some $K \geq 1$,

(1) $\rho(x, y) = 0$ if and only if $x = y$.
(2) $\rho(x, y) = \rho(y, x)$.
(3) $\rho(x, z) \leq K [\rho(x, y) + \rho(y, z)]$.

Then $\rho$ is called a *$b$-metric* on $X$ and $(X, \rho)$ is called a *$b$-metric space* with the coefficient $K$.

**Definition 1.6** ([17], Definition 7)**.** Let $(X, \rho)$ be a $b$-metric space and $\{x_n\}$ be a sequence in $X$.

(1) The sequence $\{x_n\}$ is called *convergent* to $x$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \rho(x_n, x) = 0$.
(2) The sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.

(3) The space $(X, \rho)$ is called *complete* if each Cauchy sequence is a convergent sequence.

**Definition 1.7** ([20], Definition 2.1)**.** Let $X$ be a non-empty set, $\mathbb{A}$ be a $C^*$-algebra and $d : X \times X \longrightarrow \mathbb{A}$ be a map such that for all $x, y, z \in X$ and some $a \in \mathbb{A}$ with $a \succeq 1_{\mathbb{A}}$,

(1) $0 \preceq d(x, y)$; and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \preceq a[d(x, y) + d(y, z)]$.

Then $d$ is called a *$C^*$-algebra-valued $b$-metric* on $X$ and $(X, \mathbb{A}, d)$ is called a *$C^*$-algebra-valued $b$-metric space* with the coefficient $a$.

If the coefficient $a = 1_{\mathbb{A}}$ then every $C^*$-algebra-valued $b$-metric space is a $C^*$-algebra-valued metric space. The convergence, Cauchy sequence and completeness in $C^*$-algebra-valued $b$-metric spaces are defined in the same manner of that in $C^*$-algebra-valued metric spaces as follows.

**Definition 1.8** ([20], Definition 2.2)**.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued $b$-metric space and $\{x_n\}$ be a sequence in $X$.

(1) The sequence $\{x_n\}$ is called *convergent* to $x \in X$, written $\lim\limits_{n\to\infty} x_n = x$, if $\lim\limits_{n\to\infty} \|d(x_n, x)\| = 0$.
(2) The sequence $\{x_n\}$ is called *Cauchy* if $\lim\limits_{n\to\infty} \|d(x_n, x_m)\| = 0$.
(3) The space $(X, \mathbb{A}, d)$ is called *compete* if each Cauchy sequence is a convergent sequence.

**Definition 1.9** ([19], Definition 3.6)**.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued $b$-metric space. A subset $S$ of $X$ is called *bounded* if there exists $\overline{x} \in X$ and $M \geq 0$ such that $\|d(x, \overline{x})\| \leq M$ for all $x \in S$.

**Definition 1.10** ([16])**.** Let $S, T : X \longrightarrow X$ be two maps. Then $S$ and $T$ are called *weakly compatible* if $STx = TSx$ provided $Sx = Tx$.

The following result is the key in metrization of a $b$-metric space.

**Theorem 1.11** ([3], Theorem I)**.** *Let $(X, \rho)$ be a $b$-metric space with the coefficient $K$. Then there exists $0 < \beta \leq 1$, depending only on $K$, such that*

$$m(x, y) = \inf \Big\{ \sum_{i=1}^{n} \rho^\beta(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \Big\} \tag{1.1}$$

*is a metric on $X$ satisfying $\frac{1}{2}\rho^\beta \leq m \leq \rho^\beta$. In particular, if $\rho$ is a metric then $m = \rho$.*

A $b$-metric is not continuous in general, for example see [6, Example 3.9.(3)]. To overcome the non-continuity of a $b$-metric in proving fixed point results in $b$-metric spaces, the following result was used in the literature.

**Lemma 1.12** ([2], Lemma 2.1)**.** *Let $(X, \rho)$ be a $b$-metric with the coefficient $K$. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then*

$$\frac{1}{K^2}\rho(x,y) \leq \liminf_{n\to\infty} \rho(x_n, y_n) \leq \limsup_{n\to\infty} \rho(x_n, y_n) \leq K^2\rho(x,y).$$

*In particular, if $x = y$ then $\lim_{n\to\infty} \rho(x_n, y_n) = 0$. Moreover, for any $z \in X$,*

$$\frac{1}{K}\rho(x,z) \leq \liminf_{n\to\infty} \rho(x_n, z) \leq \limsup_{n\to\infty} \rho(x_n, z) \leq K\rho(x,y).$$

## 2. Main results

First we construct a $b$-metric from a given $C^*$-algebra-valued $b$-metric and state some equivalences between them. This result shows that each $C^*$-algebra-valued $b$-metric space is topologically equivalent to a $b$-metric space. Note that every $b$-metric space is metrizable by Theorem 1.11. So, every $C^*$-algebra-valued $b$-metric space is also metrizable.

**Theorem 2.1.** *Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued $b$-metric space with the coefficient $a$ and $\rho(x,y) = \|d(x,y)\|$ for all $x, y \in X$. Then*

(1) *$\rho$ is a $b$-metric on $X$ with the coefficient $\|a\|$.*
(2) *A sequence $\{x_n\}$ is convergent to $x$ in the $C^*$-algebra-valued $b$-metric space $(X, \mathbb{A}, d)$ if and only if it is convergent to $x$ in the $b$-metric space $(X, \rho)$.*
(3) *A sequence $\{x_n\}$ is Cauchy in the $C^*$-algebra-valued $b$-metric space $(X, \mathbb{A}, d)$ if and only if it is Cauchy in the $b$-metric space $(X, \rho)$.*
(4) *The $C^*$-algebra-valued $b$-metric space $(X, \mathbb{A}, d)$ is complete if and only if the $b$-metric space $(X, \rho)$ is complete.*
(5) *A subset $S$ is bounded in the $C^*$-algebra-valued $b$-metric space $(X, \mathbb{A}, d)$ if and only if it is bounded in the $b$-metric space $(X, \rho)$.*

*Proof.* (1). For all $x, y \in X$ we have

$\rho(x,y) = \|d(x,y)\| \geq 0$;

$\rho(x,y) = 0$ if and only if $\|d(x,y)\| = 0$, if and only if $d(x,y) = 0$, that is, $x = y$;

$\rho(x,y) = \|d(x,y)\| = \|d(y,x)\| = \rho(y,x)$.

For all $x, y, z \in X$, since $0 \preceq d(x,z) \preceq a[d(x,z) + d(z,y)]$, by Lemma 1.2 we have

$$0 \leq \|d(x,z)\| \leq \|a[d(x,z) + d(z,y)]\| \leq \|a\|[\|d(x,z)\| + \|d(z,y)\|].$$

Then $\rho(x,z) \leq \|a\|[\rho(x,y) + \rho(y,z)]$. So $\rho$ is a $b$-metric on $X$ with the coefficient $\|a\|$.

(2) and (3). They are obvious from Definition 1.8 and the formulation of $\rho$.

(4). It is a direct consequence of (2) and (3).

(5). We find that $S$ is bounded in the $C^*$-algebra-valued $b$-metric space $(X, \mathbb{A}, d)$ if and only if there exists $\overline{x} \in X$ and $M \geq 0$ such that $\|d(x, \overline{x})\| \leq M$ for all $x \in S$. It is equivalent to $\rho(x, \overline{x}) \leq M$ for all $x \in S$, that is, $S$ is bounded in the $b$-metric space $(X, \rho)$. □

If the coefficient $a = 1_{\mathbb{A}}$ then the $C^*$-algebra-valued $b$-metric space is obviously a $C^*$-algebra-valued metric space. So from Theorem 2.1 we get the following result that was the key in the proof of [4, Theorem 2.1].

**Corollary 2.2.** *Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued metric space and $\rho(x, y) = \|d(x, y)\|$ for all $x, y \in X$. Then*

*(1) $\rho$ is a metric on $X$.*

*(2) A sequence $\{x_n\}$ is convergent to $x$ in the $C^*$-algebra-valued metric space $(X, \mathbb{A}, d)$ if and only if it is convergent to $x$ in the metric space $(X, \rho)$.*

*(3) A sequence $\{x_n\}$ is Cauchy in the $C^*$-algebra-valued metric space $(X, \mathbb{A}, d)$ if and only if it is Cauchy in the metric space $(X, \rho)$.*

*(4) The $C^*$-algebra-valued metric space $(X, \mathbb{A}, d)$ is complete if and only if the metric space $(X, \rho)$ is complete.*

By using Corollary 2.2 Alsulami *et al.* claimed that the main results of [21], [7] and [26] can be derived from the existing corresponding fixed point theorems in the setting of the standard metric space in the literature.

In [24] Qiaoling *et al.* established coincidence fixed point and common fixed point theorems for two maps in complete $C^*$-algebra-valued metric spaces. In the next we present short proofs to show that those results can be deduced from certain coincidence fixed point and common fixed point results in metric spaces.

**Corollary 2.3** ([24], Theorem 2.1)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space and $T, S : X \longrightarrow X$ be two maps such that*

$$d(Tx, Sy) \preceq b^* d(x, y) b \tag{2.1}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}$ with $\|b\| < 1$. Then $T$ and $S$ have a unique common fixed point in $X$.*

*Proof.* Let $\rho$ be the metric defined as in Corollary 2.2. Then $(X, \rho)$ is a complete metric space. By (2.1) we have $0 \preceq d(Tx, Sy) \preceq b^* d(x, y) b$. Then by Lemma 1.2 we get

$$\rho(Tx, Sy) = \|d(Tx, Sy)\| \leq \|b^* d(x, y) b\| \leq \|b^*\| \|d(x, y)\| \|b\| = \|b\|^2 \rho(x, y).$$

Note that $\|b\|^2 < 1$ since $\|b\| < 1$. By using [14, Theorem 3.8] with $A = T$, $B = S$ and $\phi(t) = \|b\|^2 t$ for all $t \geq 0$, we get that $T$ and $S$ have a unique common fixed point on $X$. $\square$

**Corollary 2.4** ([24], Theorem 2.2)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space and $T, S : X \longrightarrow X$ be two maps such that*

$$d(Tx, Ty) \preceq b^* d(Sx, Sy) b \tag{2.2}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}$ with $\|b\| < 1$. If $T(X) \subset S(X)$ and $S(X)$ is complete then $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$.*

*Proof.* Let $\rho$ be the metric defined as in Corollary 2.2. Then $(X, \rho)$ is a complete metric space. By (2.2) we have $0 \preceq d(Tx, Ty) \preceq b^* d(Sx, Sy)b$. Then by Lemma 1.2 we get

$$\begin{aligned}\rho(Tx, Sy) &= \|d(Tx, Ty)\| \leq \|b^* d(Sx, Sy)b\| \\ &\leq \|b^*\|\|d(Sx, Sy)\|\|b\| = \|b\|^2 \rho(Sx, Sy).\end{aligned}$$

Note that $\|b\|^2 < 1$ since $\|b\| < 1$. By using [1, Theorems 2.1-2.2], we find that $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$. □

Using Corollary 2.2 to prove two following results we need not use even the assumption $b \in \mathbb{A}'_+$, where $\mathbb{A}'_+ = \{b \in \mathbb{A} : b \succeq 0, bc = cb \text{ for all } c \in \mathbb{A}\}$. So the assumption $b \in \mathbb{A}'_+$ may be replaced by $b \in \mathbb{A}$.

**Corollary 2.5** ([24], Theorem 2.3)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space and $T, S : X \longrightarrow X$ be two maps such that*

$$d(Tx, Ty) \preceq b[d(Tx, Sx) + d(Ty, Sy)] \tag{2.3}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}'_+$ with $\|b\| < \frac{1}{2}$. If $T(X) \subset S(X)$ and $S(X)$ is complete then $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$.*

*Proof.* Let $\rho$ be the metric defined as in Corollary 2.2. Then $(X, \rho)$ is a complete metric space. By (2.3) we have $0 \preceq d(Tx, Ty) \preceq b[d(Tx, Sx)+d(Ty, Sy)]$. Then by Lemma 1.2 we get

$$\begin{aligned}\rho(Tx, Sy) &= \|d(Tx, Ty)\| \leq \|b[d(Tx, Sx) + d(Ty, Sy)]\| \\ &\leq 2\|b\| \max\{\rho(Tx, Sx), \rho(Ty, Sy)\}.\end{aligned}$$

Since $\|b\| < \frac{1}{2}$, we get $2\|b\| < 1$. By using [1, Theorems 2.1-2.2], we find that $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$. □

**Corollary 2.6** ([24], Theorem 2.4)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space and $T, S : X \longrightarrow X$ be two maps such that*

$$d(Tx, Ty) \preceq b[d(Tx, Sy) + d(Sx, Ty)] \tag{2.4}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}'_+$ with $\|b\| < \frac{1}{2}$. If $T(X) \subset S(X)$ and $S(X)$ is complete then $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$.*

*Proof.* Let $\rho$ be the metric defined as in Corollary 2.2. Then $(X, \rho)$ is a complete metric space. By (2.4) we have $0 \preceq d(Tx, Ty) \preceq b[d(Tx, Sy)+d(Sx, Ty)]$. Then by Lemma 1.2 we get

$$\begin{aligned}\rho(Tx, Sy) &= \|d(Tx, Ty)\| \leq \|b[d(Tx, Sy) + d(Sx, Ty)]\| \\ &\leq 2\|b\| \frac{\rho(Tx, Sy) + \rho(Sx, Ty)}{2}.\end{aligned}$$

Since $\|b\| < \frac{1}{2}$, we get $2\|b\| < 1$. By using [1, Theorems 2.1-2.2], we find that $T$ and $S$ have a unique point of coincidence in $X$. Furthermore, if $T$ and $S$ are weakly compatible then $T$ and $S$ have a unique common fixed point in $X$. □

By using Theorem 2.1 we show that many results in $C^*$-algebra-valued $b$-metric spaces can be deduced from certain results in $b$-metric spaces. Note that the proof of [20, Theorem 2.1] is correct only for the case of $\|B\|^2 < \|A\|$, see the equality in lines -8, -7, -6 on page 4 of [20]. We next reprove that result as follows.

**Corollary 2.7** ([20], Theorem 2.1)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued b-metric space with the coefficient $a$ and $T : X \longrightarrow X$ be a map such that*

$$d(Tx, Ty) \preceq b^* d(x, y) b \tag{2.5}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}$ with $\|b\| < 1$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $\rho$ be the $b$-metric defined as in Theorem 2.1. Then $(X, \rho)$ is a complete $b$-metric space with the coefficient $\|a\|$. By (2.5) we have

$$0 \preceq d(Tx, Ty) \preceq b^* d(x, y) b.$$

Then by Lemma 1.2 we get

$$\rho(Tx, Ty) = \|d(Tx, Ty)\| \leq \|b^* d(x, y) b\| \leq \|b^*\| \|d(x, y)\| \|b\| = \|b\|^2 \rho(x, y).$$

Note that $\|b\|^2 < 1$ since $\|b\| < 1$. So $T$ is a contraction on the complete $b$-metric space $(X, \rho)$. By the Banach contraction principle in $b$-metric spaces [12, Theorem 2.1], $T$ has a unique fixed point in $X$. □

In the proof of [20, Theorem 2.2], Ma and Jiang claimed in lines -3 and -2 on page 6 that

$$\|d(Tx, x)\| \leq \|(1_{\mathbb{A}} - a^2 b)^{-1} a^2 b\| \|d(x, x_n)\| + \|(1_{\mathbb{A}} - a^2 b)^{-1}(ab + a)\| \|d(x_{n+1}, x)\|.$$

However, this fact is not correct since the operator $1_{\mathbb{A}} - a^2 b$ may not be invertible from the assumption $\|ba\| < \frac{1}{2}$. In the next we revise [20, Theorem 2.2] by replacing the assumption $\|ba\| < \frac{1}{2}$ by $\|b\| \|a\|^2 < \frac{1}{2}$. Furthermore, we also replace the assumption $b \in \mathbb{A}'_+$ by the weaker one $b \in \mathbb{A}$.

**Corollary 2.8** (Revision of Theorem 2.2 in [20])**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued b-metric space with the coefficient $a$ and $T : X \longrightarrow X$ be a map such that*

$$d(Tx, Ty) \preceq b[d(Tx, y) + d(Ty, x)] \tag{2.6}$$

*for all $x, y \in X$ and some $b \in \mathbb{A}$ with $\|b\| \|a\|^2 < \frac{1}{2}$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $\rho$ be the $b$-metric defined as in Theorem 2.1. Then $(X, \rho)$ is a complete $b$-metric space with the coefficient $\|a\|$. By (2.6) we have

$$0 \preceq d(Tx, Ty) \preceq b[d(Tx, y) + d(Ty, x)].$$

Then by Lemma 1.2 we get

$$\rho(Tx, Ty) = \|d(Tx, Ty)\| \leq \|b[d(Tx, y) + d(Ty, x)]\| \leq \|b\|[\rho(Tx, y) + \rho(Ty, x)].$$

Note that $\|b\| < \frac{1}{2\|a\|^2}$. Using [15, Corollary 3.9] with $f = T$, $K = \|a\|$ and $a_4 = a_5 = \|b\|$, we see that $T$ has a unique fixed point in $X$. □

In the proof of [20, Theorem 2.3], Ma and Jiang claimed in line -5 on page 8 that

$$d(Tx, x) \leq (1_{\mathbb{A}} - ab)^{-1}abd(Tx_n, Tx_{n-1}) + (1_{\mathbb{A}} - ab)^{-1}ad(Tx_n, x).$$

However, this fact is not correct since the member $1_{\mathbb{A}} - ab$ may not be invertible from the assumption $\|b\| < \frac{1}{2}$. In the next we revise [20, Theorem 2.3] by replacing the assumption $\|b\| < \frac{1}{2}$ by $\|b\|\|a\| < \frac{1}{2}$. Furthermore, we also replace the assumption $b \in \mathbb{A}'_+$ by the weaker one $b \in \mathbb{A}$.

**Corollary 2.9** (Revision of Theorem 2.3 in [20])**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued b-metric space with the coefficient $a$ and $T : X \longrightarrow X$ be a map such that*

$$d(Tx, Ty) \preceq b[d(Tx, x) + d(Ty, y)]$$

*for all $x, y \in X$ and some $b \in \mathbb{A}$ with $\|b\|\|a\| < \frac{1}{2}$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Similar to the proof of Corollary 2.8 where $\|b\| < \frac{1}{2\|a\|}$ and using [15, Corollary 3.9] with $f = T$, $K = \|a\|$ and $a_2 = a_3 = \|b\|$. □

In [19] Klin-eama and Kaskasem studied fundamental properties of $C^*$-algebra-valued $b$-metric spaces and gave some fixed point theorems for cyclic maps. By using Theorem 2.1 we also show that fundamental properties of $C^*$-algebra-valued $b$-metric spaces presented in [19] may be deduced from certain properties of $b$-metric spaces.

**Theorem 2.10** (Fundamental properties)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued b-metric space with the coefficient $a$.*

(1) [19, Theorem 3.5] *If $\{x_n\}$ is a convergent sequence then $\{x_n\}$ is a Cauchy sequence.*
(2) [19, Theorem 3.7]
  (a) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$.
  (b) *If $\{x_n\}$ is convergent then it is bounded and its limit is unique.*
  (c) *If $\{x_n\}$ is a Cauchy sequence then it is bounded.*
(3) [19, Theorem 3.8] *If $\lim_{n\to\infty} x_n = x$ then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and $\lim_{k\to\infty} x_{n_k} = x$.*
(4) [19, Theorem 3.9] *If $\{x_n\}$ is a Cauchy sequence then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a Cauchy sequence.*

(5) [19, Theorem 3.10] *If* $\{x_n\}$ *is a Cauchy sequence and has a convergent subsequence then it is convergent.*

(6) [19, Theorem 3.12] *If* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$ *then*

$$\frac{1}{\|a\|^2}\|d(x,y)\| \leq \liminf_{n\to\infty}\|d(x_n,y_n)\| \leq \limsup_{n\to\infty}\|d(x_n,y_n)\| \leq \|a\|^2\|d(x,y)\|.$$

*In particular, if* $x = y$ *then* $\lim_{n\to\infty}\|d(x_n,y_n)\| = 0$. *Moreover, for any* $z \in X$,

$$\frac{1}{\|a\|}\|d(x,z)\| \leq \liminf_{n\to\infty}\|d(x_n,z)\| \leq \limsup_{n\to\infty}\|d(x_n,z)\| \leq \|a\|\|d(x,y)\|.$$

*Proof.* Let $\rho$ be the $b$-metric defined as in Theorem 2.1. Then $(X,\rho)$ is a $b$-metric space with the coefficient $\|a\|$. By using Lemma 1.12 for the $b$-metric space $(X,\rho)$ with the coefficient $\|a\|$ we get (6).

By using Theorem 1.11 we see that every $b$-metric space $(X,\rho)$ is metrizable by the metric $m$. Moreover, we find that convergence, Cauchy sequence, completeness and boundedness between the metric space $(X,m)$ and the $b$-metric space $(X,\rho)$ are equivalent since $\frac{1}{2}\rho^\beta \leq m \leq \rho^\beta$. By Theorem 2.1 we also find that convergence, Cauchy sequence, completeness and boundedness between the $b$-metric space $(X,\rho)$ and the $C^*$-algebra-valued $b$-metric space $(X,\mathbb{A},d)$ are equivalent. Note that all statements (1), (2), (3), (4) and (5) hold in the metric space $(X,m)$. So they also hold in the $C^*$-algebra-valued $b$-metric space $(X,\mathbb{A},d)$. □

Klin-eama and Kaskasem claimed that for a $C^*$-algebra-valued $b$-metric space $(X,\mathbb{A},d)$ with the coefficient $a$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} d(x_n,y_n) = a^2 d(x,y)$, see [19, Theorem 3.11]. This claim is not correct for a $b$-metric space with a continuous $b$-metric and the coefficient $K > 1$. The confusion in the proof of [19, Theorem 3.11] is applying [19, Theorem 2.11] to the inequality

$$d(x_n,y_n) - a^2 d(x,y) \preceq a d(x_n,x) + a^2 d(y,y_n)$$

where $d(x_n,y_n) - a^2 d(x,y)$ is not positive in general. In fact, the property of $\lim_{n\to\infty} d(x_n,y_n)$ is stated in Theorem 2.10.(6).

Finally we show that fixed point results for cyclic maps in [19] may be deduced from certain fixed point results for cyclic maps in the the setting of $b$-metric spaces. We need not use even the assumption $b \in \mathbb{A}'_+$ in proofs of Corollary 2.12 and Corollary 2.13 and it may be replaced by $b \in \mathbb{A}$.

**Corollary 2.11** ([19], Theorem 4.1)**.** *Let* $(X,\mathbb{A},d)$ *be a complete* $C^*$*-algebra-valued* $b$*-metric space with the coefficient* $a$*,* $A$ *and* $B$ *be non-empty closed subsets of* $X$ *and* $T : A\cup B \longrightarrow A\cup B$ *be a cyclic map, that is* $TA \subset B$ *and* $TB \subset A$*, such that*

$$d(Tx,Ty) \preceq b^* d(x,y) b$$

*for all $x \in A$, $y \in B$ and some $b \in \mathbb{A}$ with $\|b\| < \frac{1}{\|a\|}$. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* Let $\rho$ be the $b$-metric defined as in Corollary 2.2. Then $(X, \rho)$ is a complete $b$-metric space. By (2.11) we have $0 \preceq d(Tx, Ty) \preceq b^* d(x, y) b$ for all $x \in A$ and $y \in B$. Then by Lemma 1.2 we get, for all $x \in A$ and $y \in B$,

$$\rho(Tx, Ty) = \|d(Tx, Ty)\| \leq \|b^* d(x, y) b\| \leq \|b^*\| \|d(x, y)\| \|b\| = \|b\|^2 \rho(x, y).$$

Note that $\|b\|^2 < 1$ since $\|b\| < 1$. By using [23, Corallary 3.4] with $\varphi(t) = \|b\|^2 t$ for all $t \geq 0$ we get that $T$ has unique fixed point in $A \cap B$. □

**Corollary 2.12** ([19], Theorem 4.5)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space, $A$ and $B$ be non-empty closed subsets of $X$ and $T : A \cup B \longrightarrow A \cup B$ be a cyclic map such that*

$$d(Tx, Ty) \preceq b[d(Tx, x) + d(Ty, y)]$$

*for all $x, y \in X$ and some $b \in \mathbb{A}'_+$ with $\|b\| < \frac{1}{2\|a\|}$. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* Similar to the argument in the proof of Corollary 2.11 and using [23, Corallary 3.8] with $\varphi(t) = 2\|b\| t$ for all $t \geq 0$. □

**Corollary 2.13** ([19], Theorem 4.7)**.** *Let $(X, \mathbb{A}, d)$ be a complete $C^*$-algebra-valued metric space, $A$ and $B$ be non-empty closed subsets of $X$ and $T : A \cup B \longrightarrow A \cup B$ be a cyclic map such that*

$$d(Tx, Ty) \preceq b[d(Tx, y) + d(Ty, x)]$$

*for all $x, y \in X$ and some $b \in \mathbb{A}'_+$ with $\|b\| < \frac{1}{2\|a\|^2}$. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* Similar to the argument in the proof of Corollary 2.12 and using [23, Corallary 3.6] with $\varphi(t) = 2\|b\| t$ for all $t \geq 0$. □

Though many results in $C^*$-algebra valued metric spaces and $C^*$-algebra valued $b$-metric spaces are consequences of certain results in metric spaces or in $b$-metric spaces, we must say that we do not know whether the Caristi's fixed point theorem in $C^*$-algebra valued metric spaces [25, Theorem 3.5] may be deduced from Caristi's fixed point theorem in metric spaces [8, Theorem 1] or not. The difficulty is that from the inequality (3.6) on page 587 of [25] we may not get $\rho(x, Tx) \leq \|\phi(x)\| - \|\phi(Tx)\|$ for all $x \in X$. So the following question is still open.

**Question 2.14.** *Is it possible to derive Caristi's fixed point theorem in $C^*$-algebra valued metric spaces* [25, Theorem 3.5] *from Caristi's fixed point theorem in metric spaces* [8, Theorem 1]*?*

Acknowledgements. *The authors wish to express their thanks to anonymous reviewers and the Editor for several helpful comments concerning the paper.*

## References


[1] M. Abbas, G. V. R. Babu and G. N. Alemayehu, On common fixed points of weakly compatible mappings satisfying 'generalized condition $(B)$', Filomat 25, no. 2 (2011), 9–19.

[2] A. Aghajani, M. Abbas and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered $b$-metric spaces, Math. Slovaca 64, no. 4 (2014), 941–960.

[3] H. Aimar, B. Iaffei and L. Nitti, On the Macías-Segovia metrization of quasi-metric spaces, Rev. Un. Mat. Argentina 41 (1998), 67–75.

[4] H. H. Alsulami, R. P. Agarwal, E. Karapinar and F. Khojasteh, A short note on $C^*$-valued contraction mappings, J. Inequal. Appl. 2016:50 (2016), 1–3.

[5] T. V. An, N. V. Dung, Z. Kadelburg and S. Radenović, Various generalizations of metric spaces and fixed point theorems, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 109 (2015), 175–198.

[6] T. V. An, L. Q. Tuyen and N. V. Dung, Stone-type theorem on $b$-metric spaces and applications, Topology Appl. 185/186 (2015), 50–64.

[7] S. Batul and T. Kamran, $C^*$-valued contractive type mappings, Fixed Point Theory Appl. 2015:142 (2015), 1–9.

[8] J. Caristi and W. A. Kirk, Geometric fixed point theory and inwardness conditions, In: The geometry of metric and linear spaces, vol. 490, pp. 74–83, Springer Berlin Heidelberg, 1975.

[9] S. Czerwik, Contraction mappings in $b$-metric spaces, Acta Math. Univ. Ostrav. 1, no. 1 (1993), 5–11.

[10] S. Czerwik, Nonlinear set-valued contraction mappings in $b$-metric spaces, Atti Sem. Math. Fis. Univ. Modena 46 (1998), 263–276.

[11] N. V. Dung, T. V. An, and V. T. L. Hang, Remarks on Frink's metrization technique and applications, Fixed Point Theory, to appear.

[12] N. V. Dung and V. T. L. Hang, On relaxations of contraction constants and Caristi's theorem in $b$-metricspaces, J. Fixed Point Theory Appl. 18, no. 2 (2016), 267–284.

[13] I. Gelfand and M. Naimark, On the embedding of normed rings into the ring of operators in Hilbert space, Mat. Sb. 12 (1943), 197–213.

[14] J. Jachymski, Common fixed point theorems for some families of maps, Indian J. Pure Appl. Math. 25, no. 9 (1994), 925–937.

[15] M. Jovanović, Z. Kadelburg and S. Radenović, Common fixed point results in metric-type spaces, Fixed Point Theory Appl. 2010 (2010), 1–15.

[16] G. Jungck, Common fixed points for noncontinuous nonself mappings on a nonmetric space, Far East J. Math. Sci. 4, no. 2 (1996), 199–212.

[17] M. A. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal. 73, no. 9 (2010), 3123–3129.

[18] W. Kirk and N. Shahzad, Fixed point theory in distance spaces, Springer, Cham, 2014.

[19] C. Klin-eam and P. Kaskasem, Fixed point theorems for cyclic contractions in $C^*$-algebra-valued $b$-metric spaces, J. Funct. Spaces 2016, Art. ID 7827040, 16 pp.

[20] Z. Ma and L. Jiang, $C^*$-algebra-valued $b$-metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2015, 2015:222, 12 pp.

[21] Z. Ma, L. Jiang and H. Sun, $C^*$-algebra-valued metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2014, 2014:206, 11 pp.

[22] G. J. Murphy, $C^*$-algebras and operator theory, Academic Press, Inc., 1990.

[23] H. K. Nashine and Z. Kadelburg, Cyclic generalized $\varphi$-contractions in $b$-metric spaces and an application to integral equations, Filomat 28, no. 10 (2014), 2047–2057.

[24] X. Qiaoling, J. Lining and M. Zhenhua, Common fixed point theorems in $C^*$-algebra-valued metric spaces, arXiv:1506.05545v2 (2015).

[25] D. E. Shehwar, S. Batul, T. Kamran and A. Ghiura, Caristi's fixed point theorem on $C^*$-algebra valued metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 584–588.

[26] D. E. Shehwar and T. Kamran, $C^*$-valued $G$-contractions and fixed points, J. Inequal. Appl. 2015, 2015:304, 8 pp.
[27] C. Tianqing, Some coupled fixed point theorems in $C^*$-algebra-valued metric spaces, arXiv:1601.07168v1 (2016).