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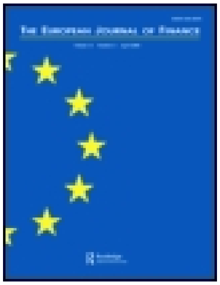

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Time varying costs of capital and the expected present value of future cash flows

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Time varying costs of capital and the expected present value of future cash flows

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The use of an inter-temporally constant discount rate or cost of capital is a strong assumption in many *ex ante* models of finance and in applied procedures such as capital budgeting. We investigate how robust this assumption is by analysing the implications of allowing the cost of capital to vary stochastically over time. We use the Feynman–Kac functional to demonstrate how there will, in general, be systematic differences between present values computed on the assumption that the currently prevailing cost of capital will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital that apply up until the point in time at which cash flows are to be received. Our analysis is based on three interpretations of the Feynman–Kac functional. The first assumes that the cost of capital evolves in terms of a state variable characterised by an Uhlenbeck and Ornstein ("On the Theory of the Brownian Motion." *Physical Review* 36(5): 823–841) process. The second and third interpretations of the Feynman–Kac functional are based on the continuous time branching process. The first of these assumes that the state variable tends to drift upwards over time, whilst the second assumes that there is no drift in the state variable. Our analysis shows that for all three stochastic processes there are significant differences between present values computed under the assumption that the currently prevailing cost of capital will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital that apply up until the point in time at which cash flows are to be received. Comparisons are also made with the environmental economics literature where similar problems have been addressed by invoking a 'gamma discounting' methodology.

Keywords: cost of capital; Feynman–Kac functional; gamma discounting; present value; wiener process

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1. Introduction

A widely used convention in the asset valuation and capital investment literature takes the discount rate applied to future cash flows to be a constant, independent of the time at which the cash flows are to be received.[1] In particular, present value calculations are invariably based on the (often implicit) assumption that the currently prevailing cost of capital will last indefinitely into the future. This is a strong assumption that merits further analysis, as failure to allow for the risks associated with changes in the cost of capital could potentially lead to serious errors in the determination of asset intrinsic (or fundamental) values.[2] In this paper, we demonstrate the dangers associated with invoking the constant discount rate assumption by using the Feynman–Kac functional to model the evolution of the expected present values implied by a time varying cost of capital. Our analysis shows that there will be systematic differences of varying degrees (depending on the nature of the underlying stochastic process) between present values computed under the assumption that

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the current discount rate will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital that apply up until the point in time at which cash flows are to be received.

We commence our analysis in the next section with a formal statement of the Feynman–Kac functional. We then employ three examples in order to demonstrate how the Feynman–Kac functional can be used to determine the expected present value of a cash flow stream when the cost of capital fluctuates stochastically in time. Probably the most familiar of these examples is based on the Uhlenbeck and Ornstein (1930) process that underscores the Cox, Ingersoll, and Ross (1985) model of the term structure of interest rates. However, two other examples based on the continuous time branching process of Feller (1951) and others are also employed. We demonstrate that for all three processes, there will generally be significant differences between present values computed under the assumption that the currently prevailing cost of capital will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital which apply up until the point in time at which cash flows are to be received. A final section contains our summary conclusions and recommendations for further work in the area.

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2. The Feynman–Kac functional

Suppose a "state variable", $x(t)$, evolves in terms of the stochastic differential equation

$$\mathrm{d}x(t) = \mu(x)\mathrm{d}t + \sigma(x)\mathrm{d}z(t), \tag{1}$$

where $\mu(x)$ is the instantaneous mean drift (per unit time) in the state variable, t is time and $\sigma(x)$ is an instantaneous "intensity (or risk) factor" defined on the "white noise" process $\mathrm{d}z(t)$.[3] Now let

$$F(x,t) = E\left[\exp\left\{-\int_0^t r(x(s))\mathrm{d}s\right\}\right], \tag{2}$$

where $r(x)$ is a strictly non-negative function over its entire domain and $E(\cdot)$ is the expectation operator. It then follows that $F(x,t)$ will be the unique solution of the following partial differential equation:

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2} + \mu(x)\frac{\partial F}{\partial x} - r(x)F(x,t) = \frac{\partial F}{\partial t}, \tag{3}$$

under the initial condition $F(x,0) = 1$.[4] The important point to make here is that if one defines $r(x)$ to be the cost of capital (on an annualised basis) and takes it to be completely characterised in terms of the state variable, x (Cox, Ingersoll, and Ross 1985), then it follows that $F(x,t)$ will be the expected present value of a unit of currency (dollar, euro, pound, etc.) to be received t years into the future.

One can demonstrate the application of the Feynman–Kac functional in determining the expected present value of future cash flows by supposing that the state variable evolves in terms of the following continuous time branching process (Feller 1951):

$$\mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma\sqrt{x(t)}\cdot\mathrm{d}z(t), \tag{4}$$

where $\mu > 0$ is a parameter, $\mu(x(t)) = \mu x(t)$ is the expected instantaneous upward drift in the state variable and $\sigma(x(t)) = \sigma\sqrt{x(t)}$ is an intensity factor defined on the white noise process $\mathrm{d}z(t)$. Note in particular how the above process conforms to the commonly held belief that the variance associated with increments in an economic variable must become larger as the variable grows in magnitude (Cox, Ingersoll, and Ross 1985).

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A significant difficulty with the implementation of the branching process given here, however, is that a closed form solution is not available for Equation (4). Fortunately, Cox and Miller (1965, 236) have determined the moment generating function for the conditional probability density associated with the branching process and this shows that the conditional expectation of the state variable will be

$$E[x(t)] = x(0)\mathrm{e}^{\mu t}, \tag{5}$$

where $x(0)$ is the current (i.e. time zero) value of the state variable and, as previously, $E(\cdot)$ is the expectation operator. Likewise, the conditional variance of the state variable turns out to be

$$\mathrm{Var}[x(t)] = \frac{\sigma^2 x(0)}{\mu}\mathrm{e}^{\mu t}(\mathrm{e}^{\mu t} - 1), \tag{6}$$

where $\mathrm{Var}(\cdot)$ is the variance operator. These results show that one can expect the state variable to grow at a rate of μ per unit time although the uncertainty associated with the evolution of the state variable increases exponentially the further one looks into the future. Now, here one can follow Karlin and Taylor (1981, 393) and Cox, Ingersoll, and Ross (1985, 390) in letting the instantaneous cost of capital be defined in terms of the state variable itself; that is, $r(x(t)) = x(t)$.[5] Moreover, the function describing the expected present value of a unit of currency to be received t years into the future – that is, $F(x, t)$ – will satisfy the following interpretation of the Feynman–Kac functional:

$$\frac{1}{2}\sigma^2 x \frac{\partial^2 F}{\partial x^2} + \mu x \frac{\partial F}{\partial x} - xF(x, t) = \frac{\partial F}{\partial t}, \tag{7}$$

with the initial condition being $F(x, 0) = 1$. In the Appendix, we demonstrate that the unique solution to this initial value problem is

$$F(x, t) = \mathrm{e}^{-xf(t)} = \exp\left[-x\left\{\frac{\mu+\gamma}{\sigma^2} - \frac{2\gamma}{\sigma^2}\frac{\mu+\gamma}{(\mu+\gamma) + (\gamma-\mu)\mathrm{e}^{\gamma t}}\right\}\right], \tag{8}$$

where $\gamma = \sqrt{\mu^2 + 2\sigma^2}$. Thus, if one takes $r(x) = x$ to be the cost of the firm's capital as characterised in terms of the state variable x, then $F(x, t)$ represents the expected present value of a unit of currency to be received t years into the future.[6]

Now suppose the state variable currently (i.e. at time $t = 0$) has a value of $x(0) = 0.10 = r(0)$ in which case the opening cost of capital is $r(0) = 10\%$ on an annualised basis. It then follows that $\mathrm{e}^{-0.1t}$ will be the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital. One can then compare this with the expected present value of a unit of currency to be received t years into the future, $F(x, t) = F(0.10, t)$ as obtained from the Feynman–Kac functional. Figure 1 plots the relationship between $F(0.10, t)$ and $\mathrm{e}^{-0.1t}$ when the drift parameter assumes a value of $\mu = 0.01$ and the intensity parameter amounts to $\sigma = 0.0375, \sigma = 0.0625, \sigma = 0.0875$ and $\sigma = 0.1125$, respectively. These graphs show that when the intensity factor, $\sigma(x)$ assumes relatively small values there will not be too much difference between the expected present value, $F(x, t)$, of a unit of currency to be received t years into the future and the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital. However, there will, in general, be significant differences between the two present value measures as the intensity factor grows in magnitude.[7] This in turn will mean that determining the expected present value of the cash flows associated with a given asset or capital project on the assumption that the cost of capital is an

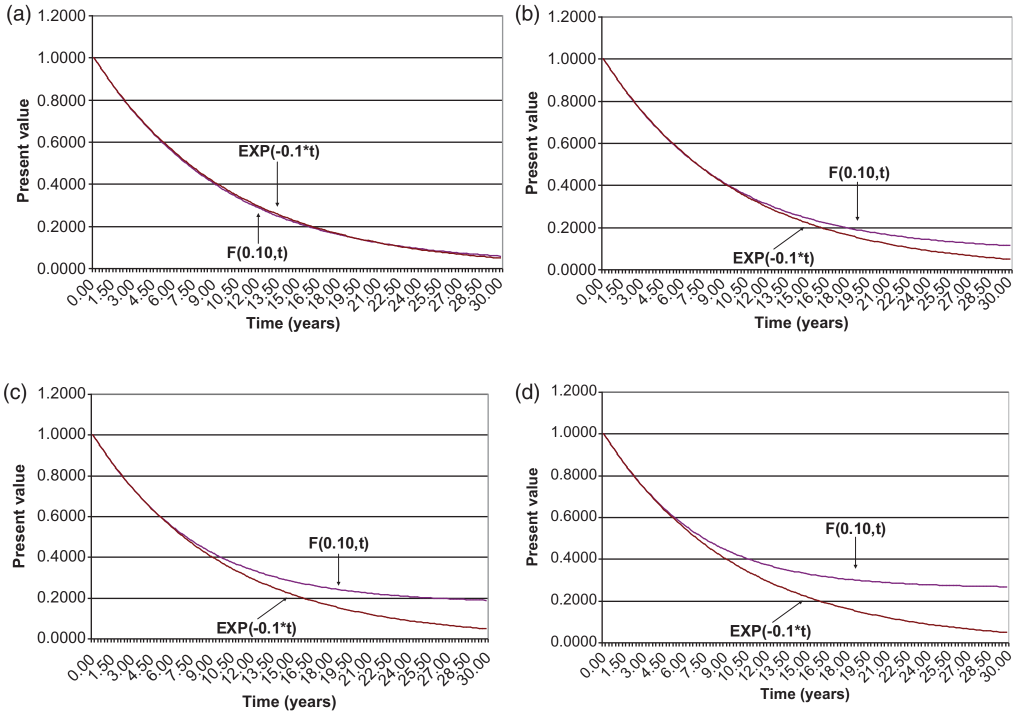

Figure 1. Branching process: present value of a unit of currency received t years into the future when discounted (i) at 10% per annum and (ii) at the expected cost of capital, $F(.10, t)$, prevailing on that date when the drift parameter is $\mu = 0.01$ and the intensity parameter is (a) $\sigma = 0.0375$, (b) $\sigma = 0.0625$, (c) $\sigma = 0.0875$ and (d) $\sigma = 0.1125$.

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inter-temporal constant independent of time will almost inevitably lead to biased estimates of the intrinsic or fundamental value of the asset or capital project under consideration.[8]

It might be argued that the disparity in the two present value measures documented here has been caused by the upward drift, μ, in the state variable as captured by Equation (4). Given this, we now assume that the state variable, x, evolves in terms of a continuous time branching process without drift. This will mean that $\mu(x(t)) = 0$ or that the state variable is characterised by the following differential equation:

$$\mathrm{d}x(t) = \sigma\sqrt{x(t)} \cdot \mathrm{d}z(t), \tag{9}$$

where, as previously, $\sigma(x(t)) = \sigma\sqrt{x(t)}$ is an intensity factor defined on the white noise process, $\mathrm{d}z(t)$. Again, there is no closed-form solution for the above differential equation and so relatively little is known about the distributional properties of state variables that evolve in terms of this differential equation. However, setting $\mu = 0$ in Equation (5) shows that the conditional expectation of the state variable will be $E[x(t)] = x(0)$. Furthermore, one can let $\mu \to 0$ in the expression for the conditional variance as summarised in Equation (6) in which case we have

$$\underset{\mu\to 0}{\text{Limit}}\ \mathrm{Var}[x(t)] = \underset{\mu\to 0}{\text{Limit}}\ \frac{\sigma^2 x(0)}{\mu}\mathrm{e}^{\mu t}(\mathrm{e}^{\mu t} - 1) = \underset{\mu\to 0}{\text{Limit}}\ \sigma^2 x(0) t (2\mathrm{e}^{2\mu t} - \mathrm{e}^{\mu t}) \tag{10}$$

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by virtue of L'Hôpital's Rule (Courant and John 1965, 464–467). It then follows that the conditional variance of a state variable which evolves in terms of a branching process without drift will be:

$$\mathrm{Var}[x(t)] = x(0)\sigma^2 t. \tag{11}$$

Note how these latter two results show that in expectations, all future values of the state variable, $x(t)$, will be equal to the current value of the state variable, $x(0)$ – although the uncertainty associated with the state variable increases the further one looks into the future. Finally, one can set $\mu = 0$ and $\gamma = \sqrt{\mu^2 + 2\sigma^2} = \sqrt{2} \cdot \sigma$ in Equation (8) in which case the expected present value of a unit of currency to be received t years into the future will be:

$$F(x,t) = \exp\left[-x\left\{\frac{\sqrt{2}}{\sigma} - \frac{2\sqrt{2}}{\sigma}\frac{1}{(1+\mathrm{e}^{\sqrt{2}\sigma t})}\right\}\right]$$

or equivalently

$$F(x,t) = \exp\left[-x\left\{\frac{(\sqrt{2}/\sigma)(\mathrm{e}^{\sqrt{2}\sigma t}-1)}{(\mathrm{e}^{\sqrt{2}\sigma t}+1)}\right\}\right].$$

Here one can factor out the term $\exp((\sqrt{2}/2)\sigma t) = \exp(\sigma t/\sqrt{2})$ from the above equation in which case it follows:

$$\begin{aligned} F(x,t) &= \exp\left[-x\left\{\frac{(\sqrt{2}/\sigma)(e^{\sqrt{2}\sigma t}-1)}{(e^{\sqrt{2}\sigma t}+1)}\right\}\right] \\ &= \exp\left[-x\left\{\frac{(\sqrt{2}/\sigma)(\exp(\sigma t/\sqrt{2}) - \exp(-\sigma t/\sqrt{2}))}{(\exp(\sigma t/\sqrt{2}) + \exp(-\sigma t/\sqrt{2}))}\right\}\right] \end{aligned} \tag{12}$$

or equivalently

$$F(x,t) = \exp\left[-\frac{\sqrt{2}\cdot x}{\sigma}\tanh\left(\frac{\sigma t}{\sqrt{2}}\right)\right]. \tag{13}$$

Now suppose, as with previous examples, the state variable currently (i.e. at time $t = 0$) has a value of $x(0) = 0.10 = r(0)$ or that the cost of equity is currently $r(0) = 10\%$ on an annualised basis. It then follows that $\mathrm{e}^{-0.1t}$ will be the present value of a unit of currency received t years into the future when discounted at the current (time $t = 0$) cost of equity capital. One can then compare this with the expected present value of a unit of currency to be received t years into the future, $F(x,t) = F(0.10,t)$ as summarised by Equation (13). Figure 2 plots the relationship between $F(0.10,t)$ and $\mathrm{e}^{-0.1t}$ when the intensity parameter assumes values of $\sigma = 0.0375$, $\sigma = 0.0625$, $\sigma = 0.0875$ and $\sigma = 0.1125$. These graphs again show that when the intensity parameter, σ, assumes relatively small values there will not be too much of a difference between the expected present value, $F(x,t)$, of a unit of currency to be received t years into the future and the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital. However, there will, in general, be significant differences between the two present value measures as the intensity parameter grows in magnitude. This in turn will mean that determining the present value of cash flows associated with a given capital project on the assumption that the currently prevailing cost of capital will last indefinitely into the future will again most likely lead to significant errors in the present value calculations.

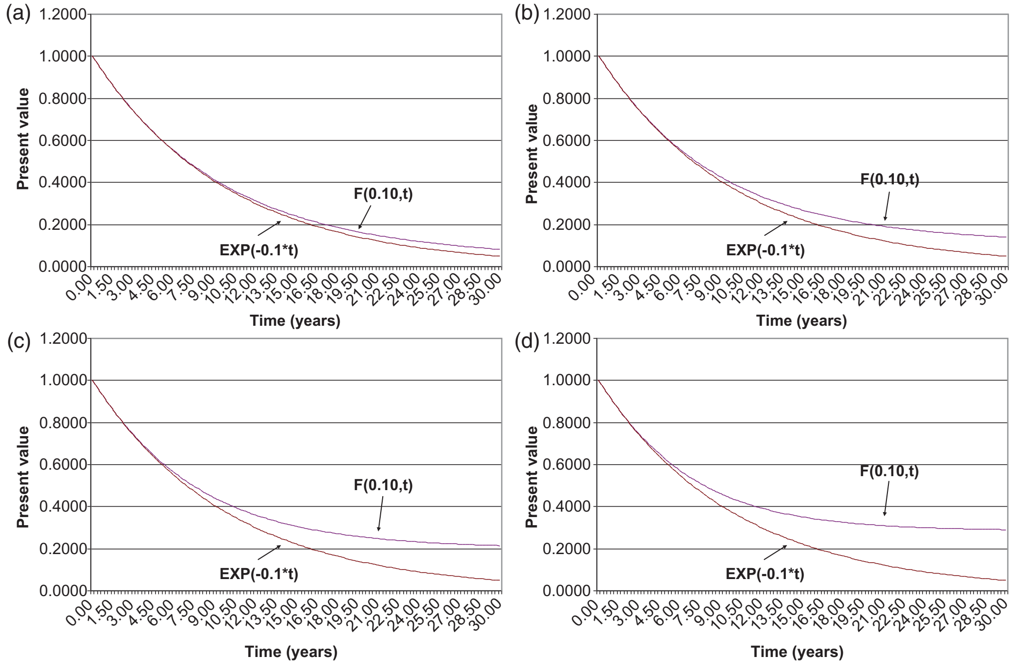

Figure 2. Continuous time branching process without drift: present value of a unit of currency received t years into the future when discounted (i) at 10% per annum and (ii) at the expected cost of capital, $F(.10, t)$, prevailing on that date when the intensity parameter is (a) $\sigma = 0.0375$, (b) $\sigma = 0.0625$, (c) $\sigma = 0.0875$ and (d) $\sigma = 0.1125$.

Now here we would emphasise that there is a steadily expanding strand of literature which argues that the discount rates used in the evaluation of capital projects ought to elastically fluctuate around a constant long run mean or perhaps, to even decline with time (Weitzman 1998, 2001, 2010; Newell and Pizer 2003; Groom et al. 2007; Gollier 2013). Weitzman (1998, 202; 2001, 261), for example, proposes a form of 'gamma discounting' under which 'people generally discount the future at declining rates of interest'. Given this, we now develop an interpretation of the Feynman–Kac functional which leads to discount rates that are based on the gamma probability density employed by Weitzman (2001, 2010). We begin by supposing that the state variable, $x(t)$, evolves in terms of an Uhlenbeck and Ornstein (1930) process; namely[9]

$$dx(t) = -\theta x(t)dt + \sigma dz(t), \tag{14}$$

where $\theta > 0$ is a speed of adjustment (or mean reversion) coefficient, $\mu(x) = -\theta x$ is the instantaneous drift in the state variable and $\sigma(x) = \sigma$ is a (constant) intensity parameter defined on the white noise process $dz(t)$. This process characterises the state variable, $x(t)$, as an elastic random walk in the sense that it has a greater tendency to revert towards its long-run mean of zero the further it is removed from it. This is evident from the solution of the Uhlenbeck and Ornstein (1930, 827) differential equation (14) which takes the form

$$x(t) = x(0)e^{-\theta t} + \int_0^t e^{-\theta(t-s)}dz(s). \tag{15}$$

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Note how this shows that the state variable has a mean of $E[x(t)] = x(0)\mathrm{e}^{-\theta t}$ whilst its variance will be $\mathrm{Var}[x(t)] = (\sigma^2/2\theta)(1 - \mathrm{e}^{-2\theta t})$ (Uhlenbeck and Ornstein 1930, 827–828). Since $\theta > 0$ this in turn will imply:

$$\underset{t\to\infty}{\text{Limit}}\ E[x(t)] = 0 \tag{16}$$

or that the state variable has a long-run mean of zero.

Now, suppose one follows Karlin and Taylor (1981, 393) and Cox, Ingersoll, and Ross (1985, 390) in letting the instantaneous cost of capital be defined in terms of the square of the state variable, $x(t)$, or

$$r(x) = x^2. \tag{17}$$

One can then substitute Equation (15) into Equation (17) and take expectations to thereby show that the expected cost of capital at some future point in time, t, will be:

$$E[r(t)] = r(0)\mathrm{e}^{-2\theta t} + \frac{\sigma^2}{2\theta}(1 - \mathrm{e}^{-2\theta t}). \tag{18}$$

Note how this result shows that the expected cost of capital will asymptotically converge towards a long run inter-temporally constant mean of[10]

$$\underset{t\to\infty}{\text{Limit}}\ E[r(t)] = \frac{\sigma^2}{2\theta}. \tag{19}$$

This in turn implies that the expected cost of capital at some future point in time is a weighted average of the current cost of capital, $r(0)$, and the long-run mean cost of capital, $\sigma^2/2\theta$ where the weights are given by $\mathrm{e}^{-2\theta t}$ and $(1 - \mathrm{e}^{-2\theta t})$ respectively.[11] Moreover, one can determine the function describing the expected present value of a unit of currency to be received t years into the future – i.e. $F(x,t)$ – by appropriate substitution into the Feynman–Kac functional; namely

$$\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - \theta x\frac{\partial F}{\partial x} - x^2 F(x,t) = \frac{\partial F}{\partial t}, \tag{20}$$

with the initial condition being $F(x,0) = 1$. Here one can use procedures similar to those illustrated in the Appendix to demonstrate that the unique solution to this initial value problem is:

$$\begin{aligned} F(x,t) &= \exp\left[\frac{(\theta-\gamma)t}{2}\right]\cdot\sqrt{\frac{\theta(\theta+\gamma)+2\sigma^2}{\theta(\theta+\gamma)+\sigma^2(1+\mathrm{e}^{-2\gamma t})}} \\ &\times \exp\left\{\left[\frac{-2}{(\theta+\gamma)} + \frac{2\gamma\mathrm{e}^{-2\gamma t}}{\theta(\theta+\gamma)+\sigma^2(1+\mathrm{e}^{-2\gamma t})}\right]\frac{x^2}{2}\right\}, \end{aligned} \tag{21}$$

where $\gamma = \sqrt{\theta^2 + 2\sigma^2}$. Thus, if one takes $r(x) = x^2$ to be the cost of capital as characterised in terms of the state variable, x, then $F(x,t)$ represents the expected present value of a unit of currency to be received t years into the future.

Now suppose the state variable currently (that is, at time $t = 0$) has a value of $x(0) = 0.316628$ in which case the opening cost of capital amounts to $r(x(0)) = r(0.316228) = 0.316228^2 = 0.10$ or 10% on an annualised basis. It then follows that $\mathrm{e}^{-0.1t}$ will be the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital. One can then compare this with the expected present value of a unit of currency to be received t years into the future, $F(x,t) = F(0.316228, t)$, as obtained from the Feynman–Kac

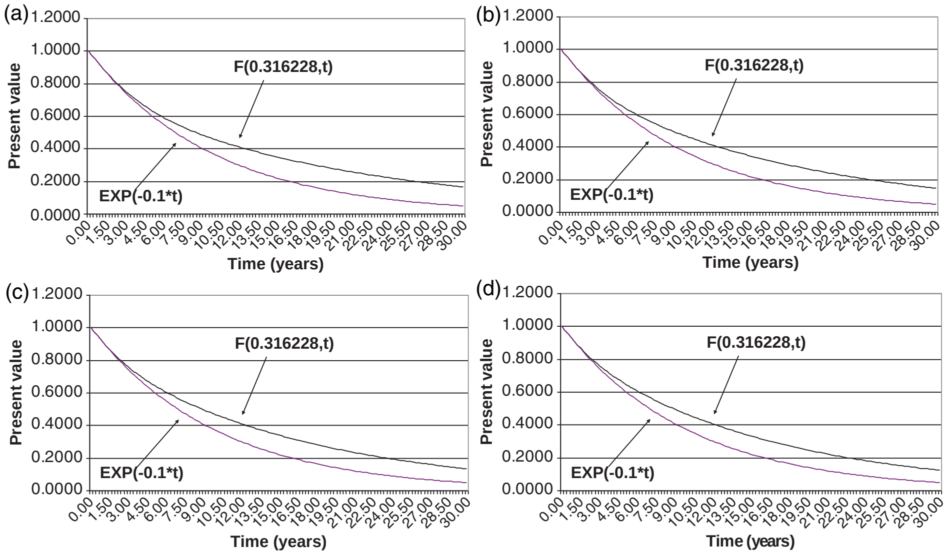

Figure 3. Uhlenbeck and Ornstein process: present value of a unit of currency received t years into the future when discounted (i) at 10% per annum and (ii) at the expected cost of capital, $F(.316228, t)$, prevailing on that date when (a) the speed of adjustment coefficient is $\theta = 0.05$ and the (square of the) intensity parameter is $\sigma^2 = 0.01$, (b) the speed of adjustment coefficient is $\theta = 0.075$ and the (square of the) intensity parameter is $\sigma^2 = 0.015$, (c) the speed of adjustment coefficient is $\theta = 0.10$ and the (square of the) intensity parameter is $\sigma^2 = 0.02$ and (d) the speed of adjustment coefficient is $\theta = 0.125$ and the (square of the) intensity parameter is $\sigma^2 = 0.025$.

functional. Figure 3 plots the relationship between $F(0.316228, t)$ and $e^{-0.1t}$ for values of $(\theta, \sigma^2) = (0.05, 0.01)$, $(\theta, \sigma^2) = (0.075, 0.015)$, $(\theta, \sigma^2) = (0.10, 0.02)$ and $(\theta, \sigma^2) = (0.125, 0.025)$. Note how for each example the long-run mean cost of capital amounts to:

$$\frac{\sigma^2}{2\theta} = \frac{0.01}{2 \times 0.05} = \frac{0.015}{2 \times 0.075} = \frac{0.02}{2 \times 0.10} = \frac{0.025}{2 \times 0.125} = 0.10$$

or 10% and is equal to the current – i.e. time zero – cost of capital, $r(x(0)) = r(0.316228) = 0.316228^2 = 0.10$. However, despite this, the graphs summarised in Figure 3 show that there will, in general, be systematic differences between the expected present value, $F(x, t)$, of a unit of currency to be received t years into the future and the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital.[12] This in turn will mean that determining the present value of the expected cash flows associated with a given asset or capital project on the assumption that the cost of capital is an inter-temporal constant independent of time will almost inevitably lead to biased estimates of the intrinsic (or fundamental) values of the given asset or capital project. This is consistent with our analysis based on the branching process considered earlier for which there are also significant differences between expected present values, $F(x, t)$, as determined from the Feynman–Kac functional and present values determined using the currently prevailing cost of capital, $r(0)$.

There are several explanations that one might offer as to why the inter-temporally constant discount rate assumption provides unreliable estimates of present values for the stochastic processes we have examined here. The first of these can be demonstrated by considering a discount rate that evolves in terms of the Uhlenbeck and Ornstein (1930) process given earlier. Here Equation (18)

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shows that if a stochastic shock engenders a situation where the current discount rate, $r(0)$, differs substantially from its long-run mean, $\sigma^2/2\theta$ then it can take an inordinately large period of time for future discount rates to adjust back to $\sigma^2/2\theta$ – depending on the magnitude of the speed of the adjustment coefficient, θ. This in turn will mean that employing $r(0)$ as an inter-temporally constant discount rate will lead to significantly different estimates of present values to those determined under the Feynman–Kac functional.

A second and more compelling reason, however, arises out of Jensen's inequality which says that under mild regularity conditions, the expectation of a convex function will be greater than the convex function of an expectation (Feller 1971, 153–154). In the present context, this will mean:

$$F(x,t) = E\left[\exp\left\{-\int_0^t r(x(s))\mathrm{d}s\right\}\right] \geq \exp\left\{-\int_0^t E[r(x(s))]\mathrm{d}s\right\} \tag{22}$$

or that the expected present value of the future cash flows will generally exceed present values computed by discounting cash flows at their expected costs of capital at the time the cash flows are to be received. Thus, if one discounts future cash flows at the long run expected discount rate, $\sigma^2/2\theta$ arising from the Uhlenbeck and Ornstein (1930) process then Jensen's inequality tells us that this will underestimate expected present value – as is demonstrated by the examples summarised in Figure 3(a)–3(d).

A more detailed understanding of how Jensen's inequality implies that the expected present value of the future cash flows will generally exceed present values computed by discounting cash flows at their expected costs of capital at the time the cash flows are to be received follows from the fact that there is a non-trivial probability that the state variable, x, can both fall to zero and then depending on the stochastic process, remain there permanently. Thus, for a continuous time branching process – i.e. Equation (4) – it may be shown that the probability of the state variable being permanently absorbed at zero before some future time, t, is given by (Cox and Miller 1965, 236):

$$\mathrm{Prob}[x(t) = 0] = \exp\left[\frac{-2x(0)}{\sigma^2} \cdot \frac{\mu}{(1 - \mathrm{e}^{-\mu t})}\right], \tag{23}$$

where, as previously, $x(0)$ is the initial (time zero) value of the state variable, μ is the expected instantaneous proportionate upwards drift in the state variable and σ is the intensity parameter associated with the white noise process $\mathrm{d}z(t)$. Table 1 summarises the probability of absorption at $x = 0$ before time $t = 1$ out to time $t = 300$ years for the data on which Figure 1 is based; namely, that the state variable has an initial value of $x(0) = 0.1 = r(0)$, a proportionate upwards drift of $\mu = 0.01$ and an intensity parameter which assumes values of $\sigma = 0.0375, \sigma = 0.0625, \sigma = 0.0875$ and $\sigma = 0.1125$, respectively.[13] Note how this table shows that for low values of the intensity parameter, σ, there is only a small probability that the state variable – and by implication the cost of capital – will fall permanently to zero over the first 10–15 years or so. However, beyond this period, the probability of absorption grows exponentially. Furthermore, all cash flows are discounted at a rate of zero beyond the point at which the state variable falls to zero. In such circumstances, applying the initial (time zero) discount rate, $r(0) = x(0)$, to all future cash flows will understate the capital project's present value. Whilst this may not pose a serious problem in the evaluation of 'immediate and near future' capital investment projects (Wietzman 2001, 261), it can present a major problem in the cost–benefit analysis of long-dated environmental projects and activities where costs and benefits can 'be spread out over hundreds of years' (Weitzman 1998, 201; Weitzman 2001, 2010; Newell and Pizer 2003; Groom et al. 2007; Gollier 2013).[14]

Table 1. Continuous time branching process: probability of absorption of state variable at $x = 0$ before time t when the state variable has an initial (time zero) value of $x(0) = 0.10$, a proportionate upwards drift of $\mu = 0.01$ and an intensity parameter of $\sigma = 0.0375$, $\sigma = 0.0625$, $\sigma = 0.0875$, and $\sigma = 0.1125$, respectively.

Time before absorption at $x = 0$	Probability $\sigma = 0.0375$	Probability $\sigma = 0.0625$	Probability $\sigma = 0.0875$	Probability $\sigma = 0.1125$
1	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0003
3	.0000	.0000	.0001	.0048
4	.0000	.0000	.0013	.0178
5	.0000	.0000	.0047	.0392
6	.0000	.0002	.0113	.0663
7	.0000	.0005	.0210	.0966
8	.0000	.0013	.0335	.1280
9	.0000	.0026	.0481	.1594
10	.0000	.0046	.0642	.1900
11	.0000	.0073	.0814	.2194
12	.0000	.0108	.0993	.2472
13	.0000	.0150	.1173	.2735
14	.0000	.0199	.1354	.2983
15	.0000	.0253	.1533	.3216
16	.0001	.0313	.1709	.3434
17	.0001	.0378	.1881	.3639
18	.0002	.0447	.2048	.3832
19	.0003	.0519	.2210	.4012
20	.0004	.0593	.2367	.4182
25	.0016	.0988	.3070	.4895
30	.0041	.1387	.3650	.5435
35	.0081	.1766	.4129	.5856
40	.0134	.2116	.4528	.6192
50	.0269	.2722	.5148	.6692
60	.0428	.3215	.5605	.7045
70	.0593	.3617	.5952	.7306
80	.0756	.3946	.6223	.7505
90	.0910	.4220	.6439	.7662
100	.1054	.4449	.6615	.7788
125	.1362	.4879	.6934	.8013
150	.1603	.5173	.7144	.8159
175	.1788	.5381	.7289	.8259
200	.1930	.5531	.7393	.8330
250	.2124	.5725	.7523	.8418
300	.2239	.5834	.7596	.8468

Tables 2 and 3 provide additional information about the impact the drift parameter, μ, can have on the probability that the state variable, x, will eventually be absorbed into the zero state. When taken in conjunction with Table 1, these tables show the probability that the state variable will eventually be absorbed into the zero state declines as the drift parameter grows in magnitude. Thus, when the state variable has an initial value of $x(0) = 0.1 = r(0)$, a proportionate upwards drift of $\mu = 0.01$ and an intensity parameter of $\sigma = 0.1125$ then Table 1 shows that the probability of the state variable being absorbed at $x = 0$ before $t = 25$ years amounts to 0.4895. However, when the drift parameter assumes the larger value of $\mu = 0.05$ Table 2 shows that the probability of the state variable being absorbed at $x = 0$ before $t = 25$ years declines significantly to 0.0281.

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Table 2. Continuous time branching process: probability of absorption of state variable at $x = 0$ before time t when the state variable has an initial (time zero) value of $x(0) = 0.10$, a proportionate upwards drift of $\mu = 0.05$ and an intensity parameter of $\sigma = 0.0375$, $\sigma = 0.0625$, $\sigma = 0.0875$ and $\sigma = 0.1125$, respectively.

Time before absorption at $x = 0$	Probability $\sigma = 0.0375$	Probability $\sigma = 0.0625$	Probability $\sigma = 0.0875$	Probability $\sigma = 0.1125$
1	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0000
3	.0000	.0000	.0000	.0000
4	.0000	.0000	.0000	.0000
5	.0000	.0000	.0000	.0000
6	.0000	.0000	.0000	.0000
7	.0000	.0000	.0000	.0000
8	.0000	.0000	.0000	.0000
9	.0000	.0000	.0000	.0001
10	.0000	.0000	.0000	.0002
11	.0000	.0000	.0000	.0005
12	.0000	.0000	.0000	.0009
13	.0000	.0000	.0000	.0015
14	.0000	.0000	.0000	.0024
15	.0000	.0000	.0001	.0034
16	.0000	.0000	.0001	.0048
17	.0000	.0000	.0002	.0064
18	.0000	.0000	.0004	.0083
19	.0000	.0000	.0005	.0104
20	.0000	.0000	.0007	.0128
25	.0000	.0000	.0027	.0281
30	.0000	.0001	.0065	.0474
35	.0000	.0002	.0120	.0689
40	.0000	.0004	.0190	.0910
50	.0000	.0015	.0362	.1342
60	.0000	.0034	.0553	.1736
70	.0000	.0062	.0747	.2081
80	.0000	.0096	.0933	.2382
90	.0000	.0134	.1107	.2641
100	.0000	.0174	.1267	.2865
125	.0000	.0277	.1603	.3304
150	.0001	.0371	.1861	.3617
175	.0002	.0451	.2058	.3843
200	.0003	.0518	.2208	.4010
250	.0004	.0615	.2410	.4228
300	.0006	.0676	.2530	.4354

Finally, Table 3 shows that for a drift parameter of $\mu = 0.10$, the probability of the state variable being absorbed at $x = 0$ before $t = 25$ before years is a mere 0.00008. The reader will be able to confirm from the other figures summarised in Tables 1, 2 and 3 that if one holds the initial (time zero) value of the state variable, $x(0)$, and the variance parameter, σ, both constant then the probability of the state variable being absorbed at $x = 0$ before any given time, t, gradually declines as the drift parameter, μ, increases in magnitude.

We conclude this section by noting how it is common practice to add a premium to the current (that is, time zero) discount rate in order to account for the risks associated with stochastic variations in the discount rate in subsequent periods (Fama 1977; Bodurtha and Mark 1991;

Table 3. Continuous time branching process: probability of absorption of state variable at $x = 0$ before time t when the state variable has an initial (time zero) value of $x(0) = 0.10$, a proportionate upwards drift of $\mu = 0.10$ and an intensity parameter of $\sigma = 0.0375$, $\sigma = 0.0625$, $\sigma = 0.0875$ and $\sigma = 0.1125$, respectively.

Time before absorption at $x = 0$	Probability $\sigma = 0.0375$	Probability $\sigma = 0.0625$	Probability $\sigma = 0.0875$	Probability $\sigma = 0.1125$
1	.0000	.0000	.0000	.0000
2	.0000	.0000	.0000	.0000
3	.0000	.0000	.0000	.0000
4	.0000	.0000	.0000	.0000
5	.0000	.0000	.0000	.0000
6	.0000	.0000	.0000	.0000
7	.0000	.0000	.0000	.0000
8	.0000	.0000	.0000	.0000
9	.0000	.0000	.0000	.0000
10	.0000	.0000	.0000	.0000
11	.0000	.0000	.0000	.0000
12	.0000	.0000	.0000	.0000
13	.0000	.0000	.0000	.0000
14	.0000	.0000	.0000	.0000
15	.0000	.0000	.0000	.0000
16	.0000	.0000	.0000	.0000
17	.0000	.0000	.0000	.0000
18	.0000	.0000	.0000	.0001
19	.0000	.0000	.0000	.0001
20	.0000	.0000	.0000	.0002
25	.0000	.0000	.0000	.0008
30	.0000	.0000	.0000	.0022
35	.0000	.0000	.0001	.0047
40	.0000	.0000	.0004	.0083
50	.0000	.0000	.0013	.0180
60	.0000	.0000	.0031	.0301
70	.0000	.0000	.0056	.0433
80	.0000	.0001	.0087	.0567
90	.0000	.0002	.0123	.0697
100	.0000	.0003	.0160	.0821
125	.0000	.0008	.0257	.1092
150	.0000	.0014	.0346	.1308
175	.0000	.0020	.0424	.1477
200	.0000	.0027	.0487	.1608
250	.0000	.0038	.0581	.1788
300	.0000	.0046	.0640	.1896

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Halliwell 2011). However, it is clear from Figures 1, 2 and 3 that the forms of the stochastic discount rates are more complicated than would be the case if generated by the addition of a simple term premium to the current discount rate. One can demonstrate this by recalling from previous analysis that $i \equiv r(x(0))$ is the current (that is, time zero) discount rate whilst $F(x,t)$ is the expected stochastic discount factor at some future time, t, as determined from the Feynman–Kac functional. It then follows that the term premium, p, added to the current discount rate, i, at some future point in time will be implicitly defined by the following equation:

$$e^{-(i+p)t} = F(x,t). \tag{24}$$

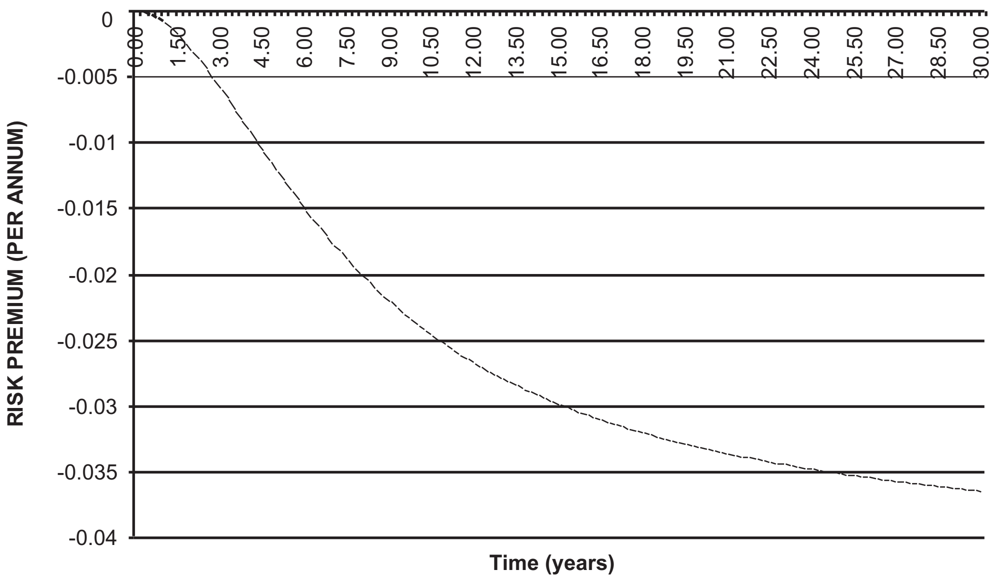

Figure 4. Uhlenbeck and Ornstein process. Term premium, *p*, on an asset that pays out a unit of currency at time t when the discount rate is $i = 10\%$ the speed of adjustment coefficient is $\theta = 0.075$ and the (square of the) intensity parameter is $\sigma^2 = 0.015$.

Moreover, one can solve this equation for the term premium in which case we have:

$$p = -\left\{\frac{\log[F(x,t)]}{t} + i\right\}. \tag{25}$$

In Figure 4, we plot the term premium against time, t, when the speed of adjustment coefficient associated with the Uhlenbeck and Ornstein (1930) process is $\theta = 0.075$ and the (square of the) intensity parameter is $\sigma^2 = 0.015$; that is, for the parameter values on which Figure 3(b) is based. Observe how the term premium gradually increases in absolute magnitude as one looks further and further into the future. This confirms our assertion that the addition of a simple term premium to the currently prevailing cost of capital cannot address the problems which arise in the determination of expected present values from a stochastically varying cost of capital. This result is particularly important, given that the Uhlenbeck and Ornstein (1930) process has been widely applied by empirical researchers in the modelling of the term structure of interest rates (Gibbons and Ramaswamy 1993).

3. Summary conclusions

We examine the integrity of the widely invoked practice of applying an inter-temporally constant discount rate to the future cash flows of a given asset or capital project in order to determine its fundamental (or intrinsic) value. In particular, we employ the Feynman–Kac functional to determine the expected present value of a unit income stream when the cost of capital evolves stochastically in time. One can then compare the profile of expected present values under the assumption that the cost of capital varies stochastically in time with the present value profile determined under the assumption that the currently prevailing cost of capital will last indefinitely

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into the future. Our analysis of this issue is based on three interpretations of the Feynman–Kac functional. The most familiar of these is based on the Uhlenbeck and Ornstein (1930) process which underscores the Cox, Ingersoll, and Ross (1985) model of the term structure of interest rates. The two other interpretations are based on the continuous time branching process of Feller (1951) and others. Our analysis shows that for all three processes, there will generally be significant differences between the expected present values determined from the Feynman–Kac functional and present values computed under the assumption that the currently prevailing cost of capital will last indefinitely into the future. The conclusion we reach from our analysis is that there is an urgent need to reassess the assumption of an inter-temporally constant discount rate on which most valuation models and capital budgeting procedures are based. Here, the emerging strand of literature which focuses on the problem of determining discount rates for long-dated environmental projects and activities is potentially of considerable importance (Weitzman 1998, 2001, 2010; Gollier 2013).

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Notes

1. See Ohlson (1995), Ashton, Cooke, and Tippett (2003) and Easton (2009) for examples of valuation models based on this convention and for references to the numerous other papers that also invoke this convention.
2. This issue is of relevance in a much broader set of circumstances than is examined here – particularly in relation to the evaluation of environmental projects. Weitzman (1998, 201–202), for example, notes that '. . . today, we are being asked to analyse environmental projects or activities whose effects will be spread out over hundreds of years. Prominent examples include: global climate change, radioactive waste disposal, loss of biodiversity, thinning of stratospheric ozone, groundwater pollution, minerals depletion, and many others . . . Few are the economists who have not sensed in their heart of hearts that something is amiss about treating a distant future event [in these areas] as just another term to be discounted away at the same constant exponential rate gotten from extrapolating past rates of return to capital. . . To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong, somewhere'. The observation 'that something is wrong, somewhere' has spawned an extensive cost–benefit literature under which long-dated environmental projects are evaluated using an inter-temporally declining discount rate (Weitzman 1998, 2001, 2010; Newell and Pizer 2003; Groom et al. 2007; Gollier 2013).
3. A Wiener process, $z(t)$, is a continuous time stochastic process that is normally distributed with a mean of zero and variance of t (Hoel, Port, and Stone 1987, 122–124). The derivative of a Wiener process is called a white noise process. Hoel, Port, and Stone (1987, 142) note that the white noise process "is not a stochastic process in the usual sense. Rather $\mathrm{d}z(t) = z'(t)\mathrm{d}t$ is a 'functional" that . . . can be used to define certain stochastic differential equations".
4. The proof of this proposition proceeds by breaking the present value expression, $F(x, t)$, into an integral involving the "current period" plus an integral involving all "later periods"; that is, by iteration of the "single period" present value expression. One can then apply a simple Taylor series expansion to the "later periods" component of the present value expression and substitute equation (1) for the increments in the state variable implied by the Taylor series expansion. Stating the analysis on a "per unit" time basis and letting the "current period" shrink to zero (i.e. the taking of limits) then retrieves the partial differential equation which defines the Feynman–Kac functional. See Karlin and Taylor (1981, 222–225) for further details.
5. There are a variety of ways in which one might justify this procedure. One could, for example, follow Cox, Ingersoll, and Ross (1985, 390) in defining $x(t)$ to be a "technological uncertainty" variable which when appropriately transformed has instantaneous increments that are perfectly correlated with instantaneous increments in the cost of capital for the given asset. Alternatively, one could base the evolution of $x(t)$ (and by implication the cost of capital itself) on the discrete time binomial filtration procedures formulated in Cox, Ross, and Rubinstein (1979) which in turn is an adaptation of the earlier work of Chandrasekhar (1943) and Kac (1947). If one is modelling capital projects in a

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commercial context, then one might base the state variable on the rate of consumptive time preference (Fisher 1930) or the gradual increase in uncertainty associated with long-dated cash flow streams. If one is modelling environmental projects and activities, then one might base the state variable on the volume of greenhouse gases or the volume of free radicals in the stratospheric ozone, both of which will tend to increase with time (Weitzman 1998, 2001, 2010; Gollier 2013).

6. This interpretation of the Feynman–Kac functional has the important property that:

$$\underset{t\to\infty}{\text{Limit}}\, F(x,t) = \exp\left[\frac{-x(\mu+\gamma)}{\sigma^2}\right].$$

Note how this means that the discount factor associated with long-dated cash flows does not fall away to zero – something that is of considerable importance in the evaluation of environmental projects and activities (Weitzman 2001, 261–262; Gollier 2013).

7. We compared present values computed under the assumption that the current discount rate will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital that apply at the time the cash flows are to be received for a wide range of alternative values of the state variable, x, and the parameters μ and σ. In virtually every instance there were systematic differences between the expected present value, $F(x,t)$, of a unit of currency to be received t years into the future and the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital.
8. One can differentiate through Equation (8) and use the results summarised in the Appendix to show that the instantaneous discount rate at time t will be:

$$\frac{-1}{F(x,t)}\frac{\partial F(x,t)}{\partial t} = xf'(t) = x\left(1+\mu f(t) - \frac{1}{2}\sigma^2[f(t)]^2\right).$$

Substituting $x = 0.10$, $\mu = 0.01$ and $\sigma = 0.0625$ into the above equation yields the instantaneous discount rates which lie behind Figure 1(b). These calculations show that the instantaneous discount rate is slightly above 10% over the first five years of the capital project's existence but then gradually declines to a little over 9% after 10 years, to $7\frac{1}{2}$% after 15 years, to a little less than 6% after 20 years and to a little over 4% after 25 years. Instantaneous discount rates for the other graphs comprising Figure 1 display similar time series properties.

9. The Uhlenbeck and Ornstein (1930) process is one of the most widely cited and applied stochastic processes in financial economics. For some examples of and further references to its applications, see Gibson and Schwartz (1990), Barndorff-Nielsen and Shephard (2001) and Hong and Satchell (2012).
10. Alternatively one can apply Itô's Lemma to $r(x) = x^2$ in which case we have $\text{d}r = (\text{d}r/\text{d}x)\text{d}x + 1/2 \cdot (\text{d}^2r)/\text{d}x^2(\text{d}x)^2 = 2x\text{d}x + (\text{d}x)^2$. Substituting Equation (14) into this expression and using the fact that $(\text{d}x)^2 = \sigma^2\text{d}t$ will then show that $\text{d}r = 2x(-\theta x\text{d}t + \sigma\text{d}z) + \sigma^2\text{d}t$ or equivalently $\text{d}r = 2\theta(\sigma^2/2\theta - r)\text{d}t + 2\sigma\sqrt{r}\cdot\text{d}z$. This latter result shows that the cost of capital, r, evolves as an elastic random walk with a long run mean of $\sigma^2/2\theta$ and a speed of adjustment coefficient equal to 2θ. Moreover, one can use the stochastic differential equation determined here in conjunction with the Fokker–Planck equation to show that in the steady state (i.e. as $t \to \infty$) the cost of capital, r, will possess the gamma probability density with parameters $1/2$ and σ^2/θ (Cox and Miller 1965, 213–215). This is of particular significance, given the widely cited model of Weitzman (2001, 261) which is based on the concept of 'gamma discounting'.
11. Note that if the current discount rate, $r(0)$, exceeds its long run mean, $\sigma^2/2\theta$ then in expectations the discount rate will decline over time. The term structure of discount rates will then be compatible with the 'gamma discounting' model of Weitzman (2001, 262–270).
12. We again compared present values computed under the assumption that the current discount rate will last indefinitely into the future and present values determined by discounting cash flows at the expected costs of capital that apply at the time the cash flows are to be received for a wide range of alternative values of the state variable, x, and the parameters θ and σ^2. Again, in virtually every instance, there were systematic differences between the expected present value, $F(x,t)$, of a unit of currency to be received t years into the future and the present value of a unit of currency to be received t years into the future when discounted at the current (time $t = 0$) cost of capital.
13. One can take limits across Equation (23) and thereby show that the probability the state variable will ultimately be absorbed into the zero state will be:

$$\underset{t\to\infty}{\text{Limit}}\ \text{Prob}[x(t)=0] = \underset{t\to\infty}{\text{Limit}}\ \exp\left[\frac{-2x(0)}{\sigma^2}\frac{\mu}{(1-\text{e}^{-\mu t})}\right] = \exp\left[\frac{-2\mu x(0)}{\sigma^2}\right] < 1.$$

This in turn will mean that there is a non-trivial probability that the discount rate will remain permanently in a non-zero state. This contrasts with the branching process without drift for which the probability of absorption into the zero state before time t is given by:

$$\text{Prob}[x(t) = 0] = \exp\left[\frac{-2x(0)}{\sigma^2 t}\right].$$

This latter result can be obtained by letting $\mu \to 0$ in Equation (23) based on the application of L'Hôpital's Rule (Courant and John 1965, 464–467). One can then take limits across this expression:

$$\underset{t\to\infty}{\text{Limit}}\ \text{Prob}[x(t) = 0] = \underset{t\to\infty}{\text{Limit}}\ \exp\left[\frac{-2x(0)}{\sigma^2 t}\right] = 1.$$

This shows that when the state variable is defined in terms of a branching process without drift, the instantaneous discount rate will eventually fall into the zero state almost surely. Finally, zero is a natural reflecting barrier of the discount rate process when the state variable is defined in terms of an Uhlenbeck and Ornstein (1930) process. This means that when the discount rate falls to zero all uncertainty momentarily vanishes and the discount rate instantaneously returns to a positive (i.e. non-zero) state (Cox, Ingersoll, and Ross 1985, 391–392).

14. It is instructive to compare the probability for the branching process that the state variable will ultimately be absorbed into the zero state with the probability that the state variable will ultimately be absorbed into the zero state for the well-known geometric Brownian motion. Unfortunately, an expression for the probability of ultimate extinction for a state variable that evolves in terms of the geometric Brownian motion is not available. Nevertheless, some useful observations can still be made. A state variable, x(t), that evolves in terms of the geometric Brownian motion will satisfy the stochastic differential equation:

$$\mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}z(t),$$

where $\mu > 0$ is the expected instantaneous proportionate upwards drift in the state variable and σ is the intensity parameter associated with the white noise process $\mathrm{d}z(t)$. One can use this differential equation in conjunction with results summarised in Karlin and Taylor (1981, 192–193) to show that the probability the state variable will fall from its initial (i.e. time zero) value of x(0) to a lower limit of $a < x(0)$ before it rises to a higher limit of $b > x(0)$ will be:

$$\frac{b^{(1-2\mu/\sigma^2)} - x^{(1-2\mu/\sigma^2)}}{b^{(1-2\mu/\sigma^2)} - a^{(1-2\mu/\sigma^2)}},$$

where $2\mu/\sigma^2 > 1$. Now, suppose $b \to \infty$. Then the probability that the state variable will decline to $x(t) = a$ at some future point in time will be:

$$\underset{t\to\infty}{\text{Limit}}\ \text{Prob}[x(t) = a] = \left(\frac{a}{x(0)}\right)^{(2\mu/\sigma^2-1)}.$$

One can then set $a > 0$ to an arbitrarily small figure and thereby obtain an approximation to the probability that the state variable will ultimately be absorbed into the zero state. The important point here is that state variables defined in terms of the branching process and the geometric Brownian motion will both have a non-trivial probability of absorption into the zero state.

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Appendix Expected present values when the state variable evolves in terms of a branching process

Suppose one lets:

$$F(x,t) = \mathrm{e}^{-xf(t)},$$

where $f(t)$ is a continuously differentiable functions of time. It then follows that $\partial F/\partial x = -f(t)\mathrm{e}^{-xf(t)}$, $\partial^2 F/\partial x^2 = [f(t)]^2\mathrm{e}^{-xf(t)}$ and $\partial F/\partial t = -xf'(t)\mathrm{e}^{-xf(t)}$. Substitution will then show:

$$\frac{1}{2}\sigma^2 x\frac{\partial^2 F}{\partial x^2} + \mu x\frac{\partial F}{\partial x} - xF(x,t) = \frac{\partial F}{\partial t}$$

or equivalently:

$$\frac{1}{2}\sigma^2 x[f(t)]^2\mathrm{e}^{-xf(t)} - \mu x f(t)\mathrm{e}^{-xf(t)} - x\mathrm{e}^{-xf(t)} = -xf'(t)\mathrm{e}^{-xf(t)}.$$

Dividing the above expression by $x\mathrm{e}^{-xf(t)}$ leads to the Riccati equation (Boyce and Diprima 2005, 132)

$$f('t) = 1 + \mu f(t) - \frac{1}{2}\sigma^2[f(t)]^2.$$

Now, suppose one makes the following substitution in the above equation

$$f(t) = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2}}{\sigma^2} + \frac{1}{\upsilon(t)},$$

where $\upsilon(t)$ is a continuously differentiable function of time. It then follows that $f'(t) = -\upsilon'(t)/[\upsilon(t)]^2$ in which case substitution shows that the Riccati equation becomes:

$$\upsilon'(t) - \gamma\upsilon(t) = \frac{1}{2}\sigma^2,$$

where $\gamma = \sqrt{\mu^2 + 2\sigma^2}$. Multiplying the above equation by $\mathrm{e}^{-\gamma t}$ will then show:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\gamma t}\upsilon(t)] = \frac{1}{2}\sigma^2\mathrm{e}^{-\gamma t}$$

or equivalently:

$$\upsilon(t) = -\frac{\sigma^2}{2\gamma} + k\mathrm{e}^{\gamma t},$$

where k is a constant of integration. It then follows:

$$f(t) = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2}}{\sigma^2} + \frac{1}{\upsilon(t)} = \frac{\mu + \gamma}{\sigma^2} + \frac{1}{-\sigma^2/2\gamma + k\mathrm{e}^{\gamma t}}.$$

Now, here it will be recalled that the solution to the Feynman–Kac functional must satisfy the initial condition $F(x, 0) = e^{-xf(0)} = 1$. This in turn will mean that $f(0) = 0$. It thus follows that:

$$f(0) = \frac{\mu + \gamma}{\sigma^2} + \frac{1}{-\sigma^2/2\gamma + k} = 0,$$

in which case we have:

$$k = \frac{\sigma^2}{2\gamma}\frac{\mu - \gamma}{\mu + \gamma}.$$

It then follows that:

$$f(t) = \frac{\mu + \gamma}{\sigma^2} - \frac{2\gamma}{\sigma^2}\frac{\mu + \gamma}{(\mu + \gamma) + (\gamma - \mu)\mathrm{e}^{\gamma t}}.$$

Finally, this will mean that when the state variable evolves in terms of a branching process, the unique solution of the Feynman–Kac functional will be:

$$F(x, t) = \exp\left[-x\left\{\frac{\mu + \gamma}{\sigma^2} - \frac{2\gamma}{\sigma^2}\frac{\mu + \gamma}{(\mu + \gamma) + (\gamma - \mu)\mathrm{e}^{\gamma t}}\right\}\right].$$

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