Compact Topological Spaces Inspired By Combinatorial Constructions

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Abstract

Due to Mrówka [24], polyadic spaces are compact Hausdorff spaces that are continuous images of some power of the one point compactification $\alpha\lambda$ of a discrete space λ. It turns out that many results about polyadic spaces hold for a more general class spaces, as we shall show in this thesis. For a sequence $\overline{\lambda} = \langle \lambda_i : \ i \in I \rangle$ of cardinals, a compact Hausdorff space X is $\overline{\lambda}$-multiadic if it is a continuous image of $\prod\limits_{i \in I} \alpha\lambda_i$. It is easy to observe that a $\overline{\lambda}$-multiadic space is λ-polyadic, but whether the converse is true is a motivation of this dissertation.

To distinguish the polyadic spaces and multiadic spaces, we consider $(\alpha\lambda)^I$ and $\prod\limits_{i \in I} \alpha\lambda_i$. We investigate two cases regarding λ: if it is a successor or a limit cardinal. For an inaccessible cardinal λ we clarify by an example that the polyadic space $(\alpha\lambda)^\lambda$ is not an image of $\prod\limits_{i<\lambda} \alpha\lambda_i$. Beside this result we find a model of set theory using Prikry-like forcing to get an analogous result when λ is singular. Although the individual polyadic and multiadic spaces differ, we show that the class of polyadic spaces is the same as multiadic class!

Moreover, this dissertation is concerned with the combinatorics of multiadic compacta that can be used to give some of their topological structure. We give a Ramsey-like property for the class of multiadic compacta called Q_λ where λ is a regular cardinal. For Boolean spaces this property is equivalent to the following: every uncountable collection of clopen sets contains an uncountable subcollection which is either linked or disjoint. We give generalizations of the Standard Sierpiński graph and use them to show that the property of being κ-multiadic is not inherited by regular closed sets for arbitrarily large κ.

Contents

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Chapter 1

Introduction

1.1 The classes of dyadic and polyadic spaces

According to a definition due to Alexandroff [1] in 1936, a compact Hausdorff topological space is called dyadic if it is a continuous image of a Cantor cube D^τ, where τ is some infinite cardinal number. As usual, D^τ denotes the product of τ copies of a discrete space $D = \{0, 1\}$. This notion of dyadicity was a natural generalization of his amazing result in 1926 that asserted that every compact metric space is a continuous image of the Cantor set D^ω, which became as standard that appears in many books of real analysis and topology, e.g see [10]. Roughly speaking, the class of dyadic compacta is the smallest class of compacta containing all metric compacta and which is closed with respect to the Tychonoff product and continuous mappings.

Research that dealt with the concept of dyadicity widened to cover the study of its topological structure and its generalizations. It showed that the class of dyadic compacta behaves very nicely with respect to the topological cardinal invariants. In particular, it is a one parameter class; if X is a dyadic space of weight τ, and ϕ is one of the topological cardinal functions then $\phi(X)$ depends only on τ.

Šanin proved [9] that, if X is an infinite dyadic space that is an image of 2^I, then the smallest possible cardinality for the exponent I is the weight of X. Another observation concerning the significance of $w(X)$ for an infinite dyadic space was due to Esenin-Volpin (see [11]) who showed that $w(X)$ is the least upper bound of the characters of the points of X, i.e $w(x) = \chi(X)$. From this follows that a dyadic compactum satisfying the first axiom of countability is metrizable. Regarding the density, it is shown by Peterson [21] that a dyadic space having a dense subset of cardinality κ must have weight no greater than 2^κ.

Moreover, in 1941, Marczewski [10] showed, in solving a problem raised by Alexandroff of whether every compact space is dyadic, that any Cantor cube D^τ satisfies the countable chain condition(i.e every system of disjoint open sets in it is of at most countable). Hence for every dyadic space X, $c(X) = \aleph_0$. Since this condition is preserved by continuous maps, he remarked that for an infinite cardinal κ, if $\alpha\kappa = \kappa \cup \{\infty\}$ is the one point compactification of the discrete space κ, then $c(\alpha\kappa) = \kappa$ so $\alpha\kappa$ is not dyadic for uncountable κ. This gives a simplest example of a non-dyadic space.

The wonderful result of Marczewski regarding the cellularity was the genesis behind a new class of spaces introduced by Mrówka [24] in 1970 that concerned spaces of uncountable cellularity. Mrówka generalized the notion of dyadicity to the κ-polyadic class which is the class of all compact Hausdorff spaces that are continuous images of some power of the one point compactification $\alpha\kappa$ of discrete space κ. Here for cardinals $\kappa,\ \lambda$, $(\alpha\kappa)^\lambda$ is the product of λ copies of $\alpha\kappa$, endowed with the product topology. He asserted that a polyadic space of weight κ is dyadic iff $2 \leq \kappa \leq \aleph_0$ [1]. At the end of his paper he raised the question whether any compact space X is a κ-polyadic for a suitable cardinal κ. The class $\mathscr{PC}$ of polyadic compacta was further studied by many topologists such as Marty [23], Gerlits [12], [13], and Bell [4].

Marty [23] answered affirmatively a question of Mrówka [24] as to whether there exists a first countable compact space that is not polyadic. He showed that a separable compactum can be polyadic iff it is metrizable. The Marty studies of the topological structure of the $\mathscr{PC}$ class was followed by Gerlits [12] who proved that the character and the weight of a polyadic compactum coincide.

In 1978 Gerlits [13] identified the class $\mathscr{PC}$ to be the smallest class that contains the one point space $D(1)$ and that is closed with respect to Hausdorff continuous images and topological products of compact spaces and such that for any system $\{R_i : \ i \in I\}$ of polyadic spaces there exists a polyadic space which is a compactification of the topological sum $\sum_{i \in I} R_i$. In that paper he also investigated the relationship of the usual topological cardinal functions for that class, he ended up with the result that asserts that the class $\mathscr{PC}$ is a two-parameter class. In particular, the values of any cardinal invariant of polyadic space can be computed from its cellularity and its tightness.

[1] define a metric d on $\alpha\aleph_0^\omega$ such that $d(f,g) = \sum_{n\in\omega} \frac{1}{2^n} \, d_i'(f_i, g_i)$ where $\forall i, \ d_i'$ is a metric defined on $\alpha\aleph_0$ as:

$$d_i'(f_i, g_i) = \begin{cases} 1 & \text{if } f_i \neq g_i \\ 0 & \text{otherwise.} \end{cases}$$

Regarding the topological structure of closed sets of type G_δ, he showed that a compact G_δ subset of a polyadic compactum is polyadic. This result was an analogue of a result given by Efimov [8] for dyadic spaces.

In this thesis we generalize the class of polyadic spaces to one that consists of all compact Hausdorff spaces that are continuous images of the product $\prod_{i\in I}\alpha\kappa_i$ of the one-point compactification of a discrete spaces κ_i for $i \in I$ and for any sequence $\overline{\kappa} = \langle \kappa_i : \ i \in I\rangle$ of cardinals. We call the class of these spaces the multiadic compacta, $\mathscr{MC}$. As a first natural step for this thesis we should distinguish the classes of polyadic and multiadic spaces. It easy to show that κ-multiadicity of any compact space implies κ-polyadicity but whether the converse is true is the key investigation of this research. Specifically, we are interested to know whether the κ-polyadic space, $(\alpha\kappa)^I$ is an image of $\prod_{i<I}\alpha\kappa_i$.

To answer this query we split it into two cases, when κ is a limit cardinal or when κ is a successor cardinal. When κ is a weakly inaccessible cardinal, we study some of the topological cardinal functions for the classes of polyadic and multiadic spaces to show that for a sequence $\langle \kappa_i : \ i < \kappa\rangle$ where $\kappa_i < \kappa$ for all $i < \kappa$, there exists no maps from $\prod_{i<\kappa}\alpha\kappa_i$ onto $(\alpha\kappa)^\kappa$. For the singular case we use Prikry-like forcing to show that there is no such map.

During the research we demonstrate that many theorems which were originally proved for dyadic and polyadic spaces remain true for the class of multiadic spaces, although they are slightly different according to their cardinal invariants. We also give a result about a measure of multiadicity that states: Suppose λ is a cardinal and let $\langle \lambda_i : \ i < i^*\rangle$ and $\langle \kappa_j : \ j < j^*\rangle$ be increasing sequences with the limit λ such that $|i^*| \geq |j^*|$. Then $\prod_{j<j^*} \alpha\kappa_j$ is a continuous image of $\prod_{i<i^*} \alpha\lambda_i$.

1.2 Relevance To Banach space Theory

A related class of spaces are the Uniform Eberlein Compacta that were introduced by Benyamini and Starbird in 1976 [5]. An Eberlein Compact is a space homeomorphic to a weakly compact subspace of a Banach space and a Uniform Eberlein Compact is a space homeomorphic to a weakly compact subspace of a Hilbert space. In 1977, Benyamini, Rudin and Wage in [6] showed that for an infinite cardinal κ, a Uniform Eberlein Compact space of weight at most κ are precisely the images of closed subspaces of $\sigma_1(\kappa)^\omega$. Here $\sigma_1(\kappa)^\omega$ is the ω^{th} power of the compact subspace of $\{0,1\}^\kappa$ which consists of the characteristic functions of subsets of κ of cardinality at most one. At the end of their paper, they raised a question of whether any Uniform Eberlein Compact space of a given weight κ is a continuous image of the universal space $\sigma_1(\kappa)^\omega$.

In order to answer the above question of the existence of a universal Uniform Eberlein Compact space of a given weight κ, Bell introduced in [4] a property called a property Q_λ where λ is a regular cardinal. For a Boolean space X (space that has a clopen base) this property is equivalent to the following: Every collection $\mathcal{O}$ of clopen sets in X of size λ contains a subcollection $\mathcal{O}'$ of size λ which is either linked or disjoint. By linked we mean that the intersection of any two disjoint element in $\mathcal{O}'$ is nonempty. He showed that all polyadic spaces fulfill the property Q_λ. Then he used the standard Sierpiński graph on ω_1 to construct a counterexample to the Benyamini, Rudin and Wage question showing that $\alpha\kappa^{\omega}$ is not a universal preimage for Uniform Eberlein Compact spaces of weight at most κ. He concluded from his example that the property of being polyadic is not a regular closed hereditary property. In particular, there exists a closed set of the polyadic space $(\alpha\omega_1)^2$ that does not satisfy property Q_{ω_1}. As a consequence of Bell's example the property of being multiadic is not a regular closed hereditary property although this analogue is refuted in the case of dyadic spaces.

Beside these results, Bell also applied the standard Sierpiński graph to prove that the property S on ω_1 that deals with a single family of open sets of a topological space X is equivalent to the property of Knaster that states: every uncountable collection of its open sets contains an uncountable linked subcollection. He provided an example of a polyadic space that does not have property S.

Some questions related to our class can be posed here: Does any multiadic space satisfy property Q_λ? Is it still true that for any regular cardinal $\lambda > \aleph_0$ the property S_λ is equivalent to K_λ? Does there exist another model where the properties K_λ and S_λ are not equivalent? This dissertation answers these questions. We study a generalization of the Sierpiński graph to get the equivalence of the properties K_λ and S_λ and to give an example of a multiadic space that has property Q_λ that does not satisfy property K_λ hence not property S_λ.

In 2007 there was a revival in this area by Aviles [2] who showed that[1] for any set Γ the unit ball of $l_p(\Gamma)$ in its weak topology is an example of a Uniform Eberlein space that is a continuous image of the full $\sigma_1(\Gamma)^{\omega}$. To justify this

[1]We shall define $l_p(\Gamma)$ and $B(\Gamma)$ in section 1.4.3.

result, Aviles proved that the unit ball of $l_p(\Gamma)$ is homeomorphic in its weak topology to a closed subset $B(\Gamma)$ of the Tychonoff cube $[-1, 1]^\Gamma$. Then Aviles proved that $B(\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\omega$ by exhibiting a mapping as a composition of many continuous functions. In the case where Γ is of size ω_1, Aviles used the standard Sierpiński graph to provide an example of two equivalent norms in a nonseparable $l_p(\Gamma)$, whose closed unit balls are not homeomorphic in the weak topology and do not satisfy property (Q_{ω_1}). In particular, he showed the existence of equivalent norms in the nonseparable $l_p(\Gamma)$ whose closed unit balls are not homeomorphic in the weak topology. This is refuted with the separable spaces, since the balls of all separable reflexive Banach spaces are weakly homeomorphic [3]. In this thesis we used the generalized standard Sierpiński graphs to give analogous results at regular cardinals larger than ω_1.

1.3 The structure of this Thesis

We have organised this dissertation into 5 chapters. Let us now briefly describe the contents of these chapters:

We begin in chapter 2 by giving the notion of multiadic spaces and some of their basic properties. We prove that every multiadic space is polyadic. In this chapter we also prove a result on a measure of multiadicity, Theorem 2.1.12, that shows to what extent the choice of the sequence of cardinals which are used to show that a certain space is multiadic is important. This chapter also concerns closed sets of multiadic spaces, in particular closed sets of type G_δ. We conclude that every space X which can embedded as a closed G_δ of a multiadic space is itself multiadic. Moreover, in the end of this chapter we study the property of being AD compact and we show that multiadic spaces X belong to the class of AD compacta. This gives us more properties of the associated class.

In chapter 3 we define a new cardinal invariant called $\sigma-$character and compute some cardinals functions of multiadic spaces. By using some of these cardinal invariants, in particularly the point character and the point $\sigma-$character of polyadic and multiadic spaces, we attempt to find differences between these spaces. We also prove that for some cardinal λ a polyadic space X that is an image of $(\alpha\lambda)^\lambda$ is not an image of the product of the $\alpha\lambda_i$'s where $\langle\lambda_i : i \in \lambda\rangle$ is a sequence of cardinals with limit λ. At the end of this chapter we divide the class of λ-polyadic, for any λ, into 2 disjoint subclasses. First, those spaces which are $\langle\lambda_i : i \in I\rangle$-multiadic for a sequence of cardinals $\lambda_i < \lambda$, $\langle\lambda_i : i \in I\rangle$, with limit λ, while for the other one there exists no such sequence!

Chapter 4 is concerned with the combinatorics of multiadic spaces. With arguments analogous to Bell in [4] we give some Ramsey properties of multiadic spaces. We show that the property Q_λ is satisfied by all multiadic spaces by showing it is an imaging property (it is transferred from a space to all of its images) and that the product $\prod_{i\in I}\alpha\kappa_i$ has property R_λ for appropriate value of λ. We also recall an argument of Mrówka [24] showing that all polyadic spaces satisfy property W_1 (a space X that has the property: the closure of the union of arbitrarily many G_δ sets of X coincides with its sequential closure). He used this property to conclude that if X is a compact ordered space that is not first countable then neither X nor $H(X)$ are polyadic, so it follows that they are not multiadic either.

Moreover in this chapter we give generalizations of the standard Sierpiński graph and we use them to show that under GCH, for any regular cardinal and any topological space X the two properties S_λ and K_λ are equivalent. By invoking these generalized graphs and property Q_λ, we give an example that shows the property of being an image of $\prod_{i \in I} \alpha\kappa_i$ is not preserved by regular closed sets.

In the final chapter, our motivation is to search whether for a singular cardinal λ there exists an explicit continuous map from $\prod_{i<I} \alpha\lambda_i \twoheadrightarrow (\alpha\lambda)^\lambda$ for any sequence $\langle \lambda_i : i \in I \rangle$ that is cofinal in λ. We use Prikry-like extensions to tackle with this problem and give a negative answer. As we deal with Prikry forcing, it is convenient to present its definition and some of the basic properties. This can be seen in section 5.2. We finish chapter 5 by showing further results in this area and pose an open question relating to our work.

1.4 Notation and Preliminaries

In this section we warm up by outlining some background material needed for this dissertation. Since we are working with topological spaces and their cardinal invariants it will be helpful to set two parts in this section for them. Also we present some definitions regarding Banach spaces as we provided in the introduction some applications of polyadic spaces that attacked a problem in Banach spaces.

1.4.1 General notions and topological spaces

Throughout we assume all spaces are Hausdorff, spaces that have the property that distinct points have disjoint neighborhoods.

The following set theoretic notion is adopted: We denote by ω_α, the α^{th} infinite order type of a well ordered set. The α^{th} infinite cardinal will be denoted by $\aleph_\alpha$. Often we interchange ω_α and $\aleph_\alpha$. ω is the smallest infinite ordinal and cardinal, ω_1 is the smallest uncountable ordinal and the cardinal κ^+ is the smallest cardinal after κ. A cardinal κ is successor cardinal if $\kappa = \lambda^+$ for some λ. A cardinal which is not successor is a limit cardinal. This means if $\lambda < \kappa$ then $\lambda^+ < \kappa$. A subset B of an ordered set A is said to be cofinal if it satisfies the following condition: For every $a \in A$, there exists some $b \in B$ such that $a \leq b$. The cofinality of κ, denoted $cf(\kappa)$, is the smallest cardinal λ such that κ has a cofinal subset of cardinality λ. A regular cardinal is a cardinal number that is equal to its own cofinality, $cf(\kappa) = \kappa$. An infinite cardinal which is not regular is called singular cardinal. Note that a singular cardinal is always a limit cardinal. A regular limit cardinal is called weakly inaccessible cardinal.

Given two sets X and Y we let Y^X denote the collection of all Y valued functions with domain X. So if $f \in Y^X$ then f is a function from X into Y. The power set of a set X is denoted by $\mathcal{P}(X)$. Given a set X we will write $[X]^\lambda$ to mean the collection of all subset of X of cardinality λ. The collection of all subset of X of cardinality less than λ will be denoted by $[X]^{<\lambda}$.

Definition 1.4.1. (Product Spaces) *Given a sequence of non empty sets* $\langle X_i : i \in I \rangle$*, the Cartesian product* $\prod_{i \in I} X_i$ *of the* $X_i^{\prime}s$ *is the set of functions* f *defined in* I *with values for* $i \in I$ *in* X_i*. We always equip* X *with the*

Tychonoff topology, where each X_i is equipped with some topology. The open sets in the product topology X are unions (finite or infinite) of sets of the form $\prod_{i \in I} U_i$, where each U_i is open in X_i and $U_i \neq X_i$ only finitely many times. The coordinate projections $\pi_i : X \to X_i$ are defined by $\pi_i(f) = f(i)$ for each $i \in I$, and are continuous open mappings on X_i's.

Definition 1.4.2. (Boolean Spaces) *A compact Hausdorff topological space X is called* Boolean *if it is totally disconnected. This means any two distinct points are separated by a clopen (closed and open) set*

$$(\forall x \neq y \in X \ \ \exists \text{ a clopen set } U \text{ such that } x \in U \text{ and } y \in X \backslash U).$$

Definition 1.4.3. (One-Point Compactification) *For any locally compact topological space X,* the (Alexandroff) one-point compactification *of X is obtained by adding one extra point ∞ and defining the topology on $X \cup \{\infty\}$ to consist of the open sets of X together with the sets of the form $U \cup \{\infty\}$, where U is an open subset of X and $X \backslash U$ is compact. With this topology, $X \cup \{\infty\}$, is always compact. We denote this compactification by αX.*

Remark 1.4.4. *We will show in the proof of Lemma 2.1.5 that if X is Hausdorff then so is αX.*

Definition 1.4.5. (Polyadic Space [24])*: For any cardinals κ, τ, a Hausdorff space X is* polyadic *if it is a continuous image of some power of the one-point compactification of a discrete space, denoted by $\alpha\kappa^\tau$.*

Remark 1.4.6. *$\alpha\kappa$ is a 0-dimensional space* (contains a base of clopen sets).

Definition 1.4.7. (Countable Chain Condition (ccc)) *A topological space (X, τ) satisfies ccc if every family of pairwise disjoint open subsets of X is at most countable.*

Definition 1.4.8. (First Countable) *Let X be a topological space and let $x \in X$. X is said to be* first countable at x *if x has a countable neighborhood base (local base). Notice that this means that there is a sequence $(B_n)_{n\in\omega}$ of open sets such that whenever U is an open set containing x, there is $n \in \omega$ such that $x \in B_n \subseteq U$. The space X is said to be* first countable *if for every $x \in X$, X is first countable at x.*

Definition 1.4.9. (Metrizable Space) *A topological space (X, τ) is said to be* metrizable *if there is a metric d such that the topology induced by d is τ.*

Definition 1.4.10. (Zero Set) *A subset H of X is called* a zero set *provided there exists a continuous $f : X \to [0, 1]$ such that $H = f^{-1}(0)$.*

Definition 1.4.11. *A G_δ set is a countable intersection of open sets.*

Remark 1.4.12. *A subset A of a compact Hausdorff space X, is a closed G_δ iff there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$.*

Definition 1.4.13. (Regular Closed) *We say that a subset A of a topological space is* regular closed *if A is the closure of an open set (i.e. $A = \overline{Int(A)}$).*

1.4.2 Cardinal functions and inequalities

Cardinal functions (or cardinal invariants) are functions on topological spaces that return cardinal numbers. They are widely used in topology as a tool for describing various topological properties. Throughout this dissertation, we are dealing with several cardinal invariants. It will be helpful to highlight the standard terminology and notation of such cardinal invariants. An obvious cardinal function is a function which assigns to a set A its cardinality, denoted by $|A|$. The most frequently used cardinal functions here are, $w(X)$, $\chi(x, X)$,

$\chi(X)$, $d(X)$ and $c(X)$ that denote the topological weight, the character at a point $x \in X$, the character of X, the density and the cellularity of a topological space X, respectively. We use [18] and [10] as references.

Definition 1.4.14. *The smallest possible cardinality of a base is called* the weight of the topological space X *and it is denoted by* $w(X)$.

Remark 1.4.15. *X is a compact metrizable space iff $w(X) = \omega$.*

Definition 1.4.16. The character at a point x in a space X*, is defined to be*

$$\chi(x, X) = \min\{|B_x| : B_x \textit{ is a local base for } x \in X\},$$

The character of the space *is defined to be*

$$\begin{aligned}\chi(X) &= \sup\{\chi(x, X) : x \in X\} \\ &= \min\{\kappa : \textit{ every point in } X \textit{ has a neighborhood base of size } \leq \kappa\}.\end{aligned}$$

Remark 1.4.17. *If X is first countable then $\chi(X) = \omega$.*

Definition 1.4.18.

$$d(X) = min\{|S| : \ S \subseteq X \textit{ and } \overline{S} = X, \}$$

is called the density *of X.*

Definition 1.4.19. *A collection $\mathcal{C}$ of open sets of the topological space X is called* a cellular family *if the members of $\mathcal{C}$ are pairwise disjoint.* The cellularity of X *is*

$$c(X) = \sup\{|\mathcal{C}| : \ \mathcal{C} \textit{ cellular in } X\}\}$$

Proposition 1.4.20. *1. Engelking [10]: If for each $i \in I$ the $w(X_i) \leq \mu \geq \aleph_0$ and $|I| \leq \mu$, then $w(\prod_{i \in I} X_i) \leq \mu$.*

2. *Engelking [10]: If for each $i \in I$ the $\chi(X_i) \leq \mu \geq \aleph_0$ and $|I| \leq \mu$, then $\chi(\prod_{i \in I} X_i) \leq \mu$.*

3. *Hewitt-Marczewski-Pondiczery [22, 16, 26]: If $d(X_i) \leq \mu \geq \aleph_0$ for all $i \in I$ and $|I| \leq 2^\mu$, then $d(\prod_{i \in I} X_i) \leq \mu$ (i.e for example, the product of at most continuum many separable spaces is separable).*

1.4.3 Banach spaces

Definition 1.4.21. *Let X be a topological vector space and let X^* be its dual space that consists of all continuous linear functionals from X into the base field $\mathbb{R}$ or $\mathbb{C}$. This is a normed space, with $\|\phi\| = \sup\{|\phi(x)| : \|x\| \leq 1\}$ where $\phi \in X^*$.* The weak topology on *X is the weakest topology (the topology with the fewest open sets) such that all elements of X^* remain continuous. Explicitly,* a subbase for the weak topology *is the collection of sets of the form $\phi^{-1}(U)$ where $\phi \in X^*$ and U is an open subset of the base field $\mathbb{R}$ or $\mathbb{C}$. In other words, a subset of X is open in the weak topology if and only if it can be written as a union of (possibly infinitely many) sets, each of which is an intersection of finitely many sets of the form $\phi^{-1}(U)$.*

Definition 1.4.22. *Let X be a normed space and $X^{**} = (X^*)^*$ denote the second dual space of X. There is a natural continuous linear transformation $J : X \to X^{**}$ defined by*

$$J(x)(\phi) = \phi(x) \text{ for every } x \in X \text{ and } \phi \in X^*.$$

That is, J maps x to the functional on X^* given by evaluation at x. As a consequence of the Hahn-Banach theorem, see e.g. [10] , J is norm-preserving *(i.e., $\|J(x)\| = \|x\|$) and hence injective. The space X is called* reflexive *if J is bijective.*

Definition 1.4.23. ($l_p(\Gamma)$ Spaces) *Let Γ be a set of reals and p a real number where $1 < p < \infty$. $l_p(\Gamma)$ is the family of all sequences $\langle x_\gamma : \gamma \in \Gamma \rangle$ where each $x_\gamma \in \mathbb{R}$ and $\sum_{\gamma \in \Gamma} |x_\gamma|^p < \infty$ (which means that for any countable set $A = \{\gamma : x_\gamma \neq 0\}$, $\exists n \in \mathbb{N}$ such that $\sum_{\gamma \in A} |x_\gamma|^p \leq n$) . The real-valued operation $\|\cdot\|_p$ defined by $\|x\|_p = (\sum_{\gamma \in \Gamma} |x_\gamma|^p)^{1/p}$ defines a norm on l_p. In fact, $l_p(\Gamma)$ is a complete metric space with respect to this norm, and therefore it is a Banach space.*

Remark 1.4.24. *The Banach space $l_p(\Gamma)$ is an example of a reflexive space.*

Aviles [2] in his paper analysed the unit ball of $l_p(\Gamma)$ as follows: From the reflexivity of $l_p(\Gamma)$ and the fact that the closed unit ball of a reflexive Banach space is compact in the weak topology [7], we get that the closed unit ball $B_{l_p(\Gamma)}$ of $l_p(\Gamma)$ is compact in the weak topology. In fact, $B_{l_p(\Gamma)}$ is homeomorphic to the following closed subset of the Tychonoff cube $[-1,1]^\Gamma$:

$$B(\Gamma) = \left\{ x \in [-1,1]^\Gamma : \sum_{\gamma \in \Gamma} |x_\gamma| \leq 1 \right\}.$$

Precisely, the function $h : B_{l_p(\Gamma)} \to B(\Gamma)$ given by $h(x)_\gamma = sign(x)_\gamma \cdot |x_\gamma|^p$ is continuous in both directions, and bijective, therefore it is a homeomorphism.

Hence by definition of an Eberlein compact, a class of the Eberlein compacts are the spaces homeomorphic to closed subsets of some $B(\Gamma)$. If $p = 2$ we get Uniform Eberlein compacts. The space $\sigma_k(\Gamma)$, the compact subset of $\{0,1\}^\Gamma$ which consists of the functions with at most k nonzero coordinates (k

a positive integer) is an example of a Uniform Eberlein compact. Namely, let C_k be the closed subset of $B(\Gamma)$ consisting of functions that have at most k non-zero coordinates. Then we can define a homeomorphism function ϕ from the closed subset C_k of $B(\Gamma)$ to $\sigma_k(\Gamma)$ by replacing each non-zero coordinate of each sequence x of C_k by one. That is

$$\phi(x)_\gamma = \begin{cases} 1 & \text{if } x_\gamma \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 2

Multiadic Spaces

In this chapter we construct a new class of spaces which are called multiadic spaces. We will study some basic properties of these spaces that are in common with dyadic and polyadic spaces. Beside these results we will prove a theorem inspired by a result of Shapirovskii which will then allow us to approximate the measure of multiadicity of various spaces. Also we investigate closed subsets of multiadic spaces and we show that the property of being multiadic is inherited by closed G_δ subsets of multiadic spaces. Finally we show that the property of being AD-compact is satisfied by multiadic spaces.

2.1 Notion of Multiadicity

Let us first give two generalizations of the notion of a polyadic space and discuss the difference between them.

Definition 2.1.1. *For any sequence $\overline{\kappa} = \langle \kappa_i : \ i \in I \rangle$ of cardinals, a Haus-*

dorff space X is multiadic *if it is a continuous image of the product* $\prod_{i\in I}\alpha\kappa_i$ *of the one-point compactification of discrete spaces κ_i for $i \in I$. We say X is $\overline{\kappa}$-multiadic.*

It is easy to see that for a sequence $\overline{\kappa} = \langle \kappa_i : \ i \in I\rangle$ if $2 \leq \kappa_i \leq \aleph_0$, then a space is $\overline{\kappa}$-multiadic iff it is dyadic. This is because if $2 \leq \kappa_i < \aleph_0$ then $\alpha\kappa_i$ is a compact metric space - since every finite space is metric (you can just let distance = 1 for any two distinct points). Hence, $\alpha\kappa_i$ is an image of 2^ω, say by a map f_i. Now we can form a map f from the product of 2^ω along the index set I to the product of $\alpha\kappa_i$ by letting $f(x_0, x_1, \ldots x_i, \ldots) = (f_0(x_0), f_1(x_1), \ldots f_i(x_i), \ldots)$. The second part of the statement holds since for each i, we can define a map from $\alpha\kappa_i$ onto $\{0, 1\}$ by $0 \mapsto 0$, $\alpha\kappa_i\backslash 0 \mapsto 1$, hence $\prod_{i\in I}\alpha\kappa_i \twoheadrightarrow 2^I$.

Also if for all $i \in I$, $\kappa_i = \kappa$ then a space is $\overline{\kappa}$-multiadic iff it is κ-polyadic. Moreover, we can observe that the multiadicity implies polyadicity if we allow large enough κ, similarly like in the following example:

Example: Any $\langle\aleph_n : \ n \in \omega\rangle-$multiadic space is $\aleph_\omega$-polyadic. Let $X = (\alpha\aleph_\omega)^\omega$ and $Y = \prod_{i\in\omega}\alpha\aleph_i$. Consider the map $\Phi : \ X \ \to \ Y$ which is defined by

$$\Phi(f)(n) = \begin{cases} f(n) & \text{if } f(n) \leq \aleph_n, \\ \infty & \text{otherwise.} \end{cases}$$

It is obvious that this map is surjective. We claim that Φ is continuous. Let $U = \prod_{i\in\omega} U_i$ be a basic open set in Y. So there exist a finite set $F \subseteq \omega$ such that $U_i \neq \alpha\aleph_i$ if $i \in F$ and $U_i = \alpha\aleph_i$ otherwise. We only need to check whether the i^{th} projection of the inverse image of U when it contains ∞ in the ith position of U, is open in $\alpha\aleph_\omega$. Since $\{\infty\} \in U_i$, there exist a set $L_i \subseteq \aleph_i$ such that $\aleph_i \backslash L_i$ is finite and $U_i = L_i \cup \{\infty\}$. Therefore

$$\pi_i(\Phi^{-1}(U)) = \{\infty\} \cup (\aleph_\omega \backslash \aleph_i) \cup \pi_i(\Phi^{-1}(L_i)).$$

This set is open in $\alpha\aleph_\omega$ as $|\aleph_\omega \backslash ((\aleph_\omega \backslash \aleph_i) \cup \pi_i(\Phi^{-1}(L_i)))|$ equals the size of $\aleph_i \backslash L_i$ which is finite. Hence $Y = \prod_{i \in \omega} \alpha\aleph_i$ is $\aleph_\omega$-polyadic.

The question arises, whether X is an image of Y? Later in Section 3.2.1, we will see that a space defined analogously to X using a weakly inaccessible cardinal λ in place of $\aleph_\omega$ can't be an image of the analogously defined Y.

Corollary 2.1.2. *If X is multiadic then it is polyadic.*

Proof: Say X is an image of $\prod_{i \in I} \alpha\kappa_i$. Let $\kappa = \sup\{\kappa_i : \ i \in I\}$. For each $i \in I$ we can define a continuous map from $\alpha\kappa \twoheadrightarrow \alpha\kappa_i$ that maps $\beta \mapsto \beta$ if $\beta < \kappa$, otherwise it maps β to ∞. This gives a continuous map from $(\alpha\kappa)^I \twoheadrightarrow \prod_{i \in I} \alpha\kappa_i$, Hence X is polyadic. ■

In the beginning of this section we have observed that the class of multiadic spaces includes the polyadic and dyadic spaces also as a consequence from the previous Corollary, the multiadic spaces are polyadic. The point is to distinguish the measure of "polyadicity " and "multiadicity", i.e. the least κ such that $(\alpha\kappa)^\omega$ maps onto X versus the sequence $\langle \kappa_i : \ i \in I \rangle$ such that $\prod_{i \in I} \alpha\kappa_i$ maps onto X.

Definition 2.1.3. *We will denote by $\mu-$multiadic, a multiadic space X which is a continuous image of $\prod_{i \in I} \alpha\kappa_i$ with $\sup\{|\kappa_i| : \ i \in I\} \leq \mu$.*

Corollary 2.1.4. *Any $\mu-$multiadic space is a continuous image of $(\alpha\mu)^I$ for some I.*

The proof is analogous to the proof of Corollary 2.1.2.

2.1.1 Basic properties of multiadic spaces

Here we concentrate on some general properties of the class of multiadic compacta such as that multiadic spaces are 0-dimensional spaces and that the class of multiadic spaces is closed under continuous maps and products and finite sums.

Lemma 2.1.5. *All multiadic spaces are Boolean.*

Proof: The conclusion follows because any multiadic is polyadic. We given a direct proof for completeness. First we shall prove that for any cardinal κ with discrete topology, the one point compactification $\alpha\kappa$ is Boolean. Then since all necessarily properties for Boolean spaces are productive and they are preserved under a continuous mapping, we can finish the proof.

i. $\alpha\kappa$ is compact: Consider any open cover of $\alpha\kappa$. One of these sets contains ∞ and it is in the form $G \cup \{\infty\}$, where $\kappa \backslash G$ is finite. So we need only finitely many more of the open cover sets to cover $\kappa \backslash G$ and thus have a finite subcover for $\alpha\kappa$.
ii. $\alpha\kappa$ is Hausdorff. Let $x, y \in \alpha\kappa$. If $x, y \neq \infty$, since κ is discrete, then the singleton points $\{x\}$, $\{y\}$ are disjoint neighborhoods of x and y in κ. So the only question is whether we can separate any point $x \in \kappa$ from ∞. Let $G = \kappa \backslash \{x\}$. Its complement is closed and compact in κ. Thus $G \cup \{\infty\}$ is open in $\alpha\kappa$ which is separated from the open set $\{x\}$.
iii. $\alpha\kappa$ is 0-dimensional: It contains a basis of clopen sets namely: all singleton points together with the sets of the form $G \cup \{\infty\}$, where $\kappa \backslash G$ is finite.■

Lemma 2.1.6. *For any sequence $\langle \kappa_i : \ i \in I \rangle$ the class of $\langle \kappa_i : \ i \in I \rangle$-multiadic spaces is closed under continuous Hausdorff images.*

Proof: Suppose that X is $\langle \kappa_i : \ i \in \lambda \rangle$-multiadic. Thus $\prod\limits_{i \in \lambda} \alpha\kappa_i$ is a preimage of X under a continuous map f. Let g be a continuous function from X onto a Hausdorff space Y. Thus Y is a continuous image of $\prod\limits_{i \in \lambda} \alpha\kappa_i$ under $f \circ g$. Hence, Y is $\langle \kappa_i : \ i \in \lambda \rangle$-multiadic. ■

Lemma 2.1.7. *Suppose that $\{X_s\}_{s \in S}$ is a family of multiadic spaces and each X_s is $\langle \kappa_i : \ i \in I_s \rangle$-multiadic. Then $\prod\limits_{s \in S} X_s$ is $\langle \kappa_i^s : \ i \in I_s, \ s \in S \rangle$-multiadic.*

Proof: For each $s \in S$ there exists a continuous map f_s from the product $\prod\limits_{i \in I_s} \alpha\kappa_i^s$ of the one-point compactification of a discrete spaces κ_i^s onto the Hausdorff space X_s. Consider the Cartesian product $X = \prod\limits_{s \in S} X_s$ and define a map f as follows

$$f : \prod_{s \in S} \prod_{i \in I_s} \alpha\kappa_i^s \longrightarrow X$$

$$\{x_s\}_{s \in S} \xrightarrow{f} (f_s(x_s))_{s \in S}.$$

Clearly, f is a surjective map. Moreover, f is continuous: if O is an open basic subset of X, there exists a finite set $F \subset S$ such that $U_i \neq X_s$ only for $s \in F$. So O can be written as $O = \prod\limits_{s \in F} U_s \times \prod\limits_{s \in S \setminus F} X_s$. For such F consider $\mathcal{W} = \{ \prod\limits_{s \in F} f_s^{-1}(U_s) \times \prod\limits_{s \in S \setminus F} \prod\limits_{i \in I_s} \alpha\kappa_i^s \}$. Clearly that by the continuity and surjectivity of f_s where $s \in S$, $f_s^{-1}(U_s)$ is an open set in $\prod\limits_{i \in I_s} \alpha\kappa_i^s$ and for $s \notin F$, $f_s^{-1}(X_s) = \prod\limits_{i \in I_s} \alpha\kappa_i^s$. Since we have that $\prod\limits_{s \in F} f_s^{-1}(U_s)$ is a finite product of open sets in $\prod_{i\in I_s}\alpha\kappa_i^s$, therefore, $\mathcal{W}$ is a typical open set in Tychonoff topology $\prod_{s\in S}\prod_{i\in I_s}\alpha\kappa_i^s$ which contains $f^{-1}(O)$. Hence $\prod_{s\in S}X_s$ is $\langle\kappa_i^s:\ i\in I_s,\ s\in S\rangle$-multiadic. ■

Definition 2.1.8. *[10] Suppose that to every σ in a set Σ directed by the relation $\leq$ corresponds a topological space X_σ, and that for any $\sigma,\rho\in\Sigma$ satisfying $\rho\leq\sigma$ a continuous mapping $\pi_\rho^\sigma: X_\sigma\to X_\rho$ is defined; suppose further that $\pi_\tau^\rho\pi_\rho^\sigma=\pi_\tau^\sigma$ for any $\sigma,\rho,\tau\in\Sigma$ satisfying $\tau\leq\rho\leq\sigma$ and that $\pi_\sigma^\sigma=id_{X_\sigma}$ for every $\sigma\in\Sigma$. In this situation we say that the family $S=\{X_\sigma,\pi_\rho^\sigma,\Sigma\}$ is an inverse system of the spaces X_σ.*
An element $\{x_\sigma\}$ of the Cartesian product $\prod_{\sigma\in\Sigma}X_\sigma$ is called a thread of S if $\pi_\rho^\sigma(x_\sigma)=x_\rho$ for any $\rho,\sigma\in\Sigma$ satisfying $\rho\leq\sigma$, and the subspace of $\prod_{\sigma\in\Sigma}X_\sigma$ consisting of all threads of S is called the limit of the inverse system S and is denoted by $\varprojlim S$.

Proposition 2.1.9. *The limit of an inverse system $S=\{X_\sigma,\pi_\rho^\sigma,\Sigma\}$ of multiadic spaces X_σ is a closed subspace of the Cartesian product $\prod_{\sigma\in\Sigma}X_\sigma$.*

Proof: This follows because the same is true for any compact spaces, see Engelking [10].

2.1.2 On a measure of multiadicity

Here we prove a theorem showing to what extent it matters which sequence of cardinals we use to show that a certain space is multiadic. For example, can we replace a given sequence by a cofinal sequence with the same limit? Firstly we should mention the following definition:

Definition 2.1.10. *(Shapirovskii) For a compact space X, consider a system $\mathfrak{A} = \{\mathcal{A}_\alpha : \ \alpha \in L\}$ of families of subsets of X. Define*

$$\bigwedge \mathfrak{A} = \bigwedge \{\mathcal{A}_\alpha : \ \alpha \in L\} = \Big\{ \bigcap \{\pi_\alpha(A) : \ \alpha \in L\} : A \in \prod_{\alpha \in L} \mathcal{A}_\alpha \Big\}.$$

We say that a system $\mathfrak{A} = \big\{\mathcal{A}_\alpha : \ \alpha \in L\big\}$ is orthogonal *if $\emptyset \notin \bigwedge \mathfrak{A}$.*

For all $\alpha \in L$, let $f_\alpha : X \to Y_\alpha$ be a continuous surjective map. The diagonal map $f = \Delta\{f_\alpha : \alpha \in L\} : X \to \prod_{\alpha \in L} Y_\alpha$ is defined as the following sequence: $f(x) = (f_\alpha(x))$.

Theorem 2.1.11. (*Shapirovskii [30]*) *The diagonal map $f = \Delta\{f_\alpha : \alpha \in L\} : \ X \to \prod_{\alpha \in L} Y_\alpha$ is surjective if and only if the system $\mathfrak{F} = \{\mathcal{F}_\alpha : \alpha \in L\}$ is orthogonal, where $\mathcal{F}_\alpha = \{f_\alpha^{-1}(y) : \ y \in Y_\alpha\}, \ \alpha \in L$.*

Proof: ($\Leftarrow$) Suppose $\emptyset \in \bigwedge \mathfrak{F}$, so there are L many $\mathcal{F}_\alpha$ such that there intersection is empty (i.e $\bigcap_{\alpha \in L} \mathcal{F}_\alpha = \emptyset$). For each $\alpha \in L$ choose $y_\alpha \in Y_\alpha$ such that $f_\alpha^{-1}(y_\alpha) \in \mathcal{F}_\alpha$ and $\bigcap_{\alpha \in L} f_\alpha^{-1}(y_\alpha) = \emptyset$. Consider $y = \langle y_\alpha : \ \alpha \in L \rangle$. This y can't be in $f(X)$ as otherwise $\exists x \in X$ such that $f(x) = \langle f_\alpha(x) : \alpha \in L \rangle = \langle y_\alpha : \alpha \in L \rangle$. Hence $(\forall \alpha \in L)$ $f_\alpha(x) = y_\alpha$ which implies that $(\forall \alpha)$ $x \in f_\alpha^{-1}(y_\alpha)$ and so $x \in \bigcap_{\alpha \in L} f_\alpha^{-1}(y_\alpha)$, a contradiction.

($\Rightarrow$) Suppose f is not surjective, so $\exists y = (y_\alpha) \in \prod_{\alpha \in L} Y_\alpha$ such that $\forall x \in X, \ f(x) = (f_\alpha(x)) \neq (y_\alpha) \implies (\forall x)$, and $(\forall \alpha \in L)$ $x \notin f_\alpha^{-1}(y_\alpha)$, which means $\bigcap_{\alpha \in L} \mathcal{F}_\alpha = \emptyset$, again is a contradiction. ■

Theorem 2.1.12. *Suppose λ is a cardinal and let $\langle \lambda_i : i < i^* \rangle$ and $\langle \kappa_j : j < j^* \rangle$ be increasing sequences with limit λ such that $|i^*| \geq |j^*|$. Then $\prod_{j<j^*} \alpha\kappa_j$ is a continuous image of $\prod_{i<i^*} \alpha\lambda_i$.*

Proof: The proof is divided into two main parts. Firstly, for a fixed $\gamma < i^*$ we attempt to define a continuous map such that $\alpha\kappa_\gamma$ is an image of $\prod_{i<i^*} \alpha\lambda_i$. So fix $\gamma < j^*$. Since $\langle \lambda_i : i < i^* \rangle$ is cofinal in λ, let $i(\gamma) = \min\{i : \kappa_\gamma < \lambda_i\}$. Consider for each $\beta \in \lambda_{i(\gamma)}$, $F^\beta_{i(\gamma)}$ such that if $\beta < \kappa_\gamma$, $F^\beta_{i(\gamma)} = \{x \in \prod_{i<i^*} \alpha\lambda_i : x(i(\gamma)) = \beta\}$ and for $\beta \in (\alpha\lambda_i(\gamma) \backslash \kappa_\gamma)$ consider $F^\infty_{i(\gamma)} = \{x \in \prod_{i<i^*} \alpha\lambda_i : x(i(\gamma)) \geq \kappa_\gamma\}$. It is clear that the collection $\{F^\beta_{i(\gamma)} : \beta \in \kappa_{i(\gamma)}\} \cup F^\infty_{i(\gamma)}$ is pairwise disjoint family of closed sets in $\prod_{i<i^*} \alpha\lambda_i$ and for $\beta < \infty$, $F^\beta_{i(\gamma)}$ is also open. Define a function $\psi_\gamma : \prod_{i<i^*} \alpha\lambda_i \to \alpha\kappa_\gamma$ such that

$$\psi_\gamma \upharpoonright F^\beta_{i(\gamma)} = \beta,\ \psi_\gamma \upharpoonright F^\infty_{i(\gamma)} = \infty$$

i. **Claim 1:** ψ_γ is surjective.

Let $\xi \in \alpha\kappa_\gamma$, then any $f \in F^\xi_{i(\gamma)} \subseteq \prod_{i<i^*} \alpha\lambda_i$ is sufficient to show surjectivity, as $F^\xi_{i(\gamma)}$ is the preimage of ξ under ψ_γ.

ii. **Claim 2:** ψ_γ is continuous.

Let U be an open set in $\alpha\kappa_\gamma$. Firstly if $\infty \notin U$, then $\psi_\gamma^{-1}(U) = \bigcup_{\beta \in U} F^\beta_{i(\gamma)} = \{x \in \prod_{i<i^*} \alpha\lambda_i : x(i(\gamma)) \in U\}$, hence $\psi_\gamma^{-1}(U)$ is open. Secondly, if $\infty \in U$ then

$U = \alpha\kappa_\gamma \backslash W$ where W is finite. So

$$\psi^{-1}(U) = \{x \in \prod_{i<i^*} \alpha\lambda_i : \ x(i(\gamma)) \in (\alpha\lambda_{i(\gamma)} \backslash W)\}.$$

Since W is also finite in $\lambda_{i(\gamma)}$, $\alpha\lambda_{i(\gamma)} \backslash W$, is open in $\alpha\lambda_{i(\gamma)}$. Therefore $\psi^{-1}(U)$ is a typical open set in $\prod_{i<\kappa} \alpha\lambda_i$, and hence ψ_γ is continuous.

Secondly, we shall define a continuous map $\phi : \prod_{i<i^*} \alpha\lambda_i \twoheadrightarrow \prod_{j<j^*} \alpha\kappa_j$. Firstly by induction on j^* choose a strictly increasing sequence $\langle i(\gamma) : \ \gamma \in j \rangle$ so that for γ limit, $i(\gamma) = \min\{i < i^* : \ \kappa_\gamma < \lambda_i\}$ and for the successor step $S(\gamma) = \gamma + 1$, let $i(\gamma+1) = \min\{i < i^* : \kappa_{\gamma+1} < \lambda_i$ and $i(\gamma)) < i\})$ to avoid the occurrence of two cardinals from κ_γ's in one cardinal in λ_i. This is possible since $|i^*| \geq |j^*|$. By using Theorem 2.1.11 the map ϕ exists iff the system $\mathfrak{J} = \{\mathcal{J}_\gamma : \gamma \in j^*\}$ is orthogonal, where $\forall \gamma \in j^*$ $\mathcal{J}_\gamma = \{\psi_\gamma^{-1}(y) : \ y \in \alpha\kappa_\gamma\}$. In our case $\mathcal{J}_\gamma = \left\{F^\beta_{i(\gamma)} : \ \beta \in \lambda_{i(\gamma)}\right\} \bigcup F^\infty_{i(\gamma)}$. If $A \subseteq j^*$ is finite and $forall \gamma \in A$ take any element $x_\gamma \in \mathcal{J}_\gamma$. We claim that $\bigcap_{\gamma\in A} x_\gamma \neq \emptyset$. The reason is that $\forall \gamma \in A$, $\exists \beta_\gamma$ such that $x_n = F^{\beta_\gamma}_{i(\gamma)}$. Let f be the element in $\prod_{i<i^*} \alpha\lambda_i$ such that $\forall \gamma \in A$, $f(\gamma) = \beta_\gamma$. Therefore the intersection of any countable members of $\mathfrak{J}$ is non-empty. This shows that $\mathfrak{J}$ is orthogonal, and hence such map exists.$\blacksquare$

2.2 The topological structure of closed subsets of $\prod_{i\in\tau} \alpha\kappa_i$.

Later in Chapter 4 we will expose an example showing that the property of being multiadic is not inherited by regular closed sets. Does this argument apply for a closed G_δ of a multiadic space? In this section we shall see that the property of being multiadic is preserved by closed G_δ. First we give an example of a closed set of a multiadic space that is still multiadic, just to give us an idea about the structure of closed subsets of multiadic spaces.

Example: A closed subset of the multiadic space $X = \prod_{n\in\omega} \alpha\aleph_n$ which is multiadic.

For $\beta \in \omega$, let $L_\beta = \{f \in X : \ f(\beta) \in \{\beta, \infty\}\}$. Note that each L_β is closed. Consider the set $Y = \bigcap\{L_\beta : \ \beta \in C\}$ where C is any infinite subset of ω. Y is closed because it is an intersection of closed sets.

Moreover, Y is $\langle \aleph_n : \ n \in \omega\rangle-$multiadic as will be shown. Define a map $\Phi : \prod_{n\in\omega} \alpha\aleph_n \to Y$ such that

$$\Phi((f)(\beta)) = \begin{cases} f(\beta) & \text{if } \beta \notin C \\ f(\beta) & \text{if } \beta \in C \text{ and } f(\beta) = \beta \\ \infty & \text{if } \beta \in C \text{ and } f(\beta) \neq \beta. \end{cases}$$

It is evident that Φ is surjective.

Claim: ϕ is continuous.

Let U be an open set in Y. Then there exists a finite set $F \subseteq \omega$ such that $U_i \neq \alpha\aleph_i$, if $i \in F$ and if $i \notin F$, $U_i = \alpha\aleph_i$. We need only to check if $U_\beta = \{\infty\}$, for some $\beta \in C \cap F$, otherwise $\Phi^{-1}(U)$ is open. But $\pi_\beta(\Phi^{-1}(U)) = (\alpha\aleph_\beta)\backslash\{\beta\}$ is an open set in $\alpha\aleph_\beta$, hence Φ is continuous. ■

Theorem 2.2.3 gives the structure of the preimage of any regular closed subset C of multiadic spaces (i.e. C is the closure of its interior). Similar theorem was proved by Marty for the Polyadic spaces. Before we giving its proof we need to the following:

Definition 2.2.1. *Let* $X = \prod_{\alpha \in A} X_\alpha$. A support of a function f *defined on* $S \subseteq X$ *is defined to be a set* $B \subseteq A$ *such that* $f(x) = f(y)$ *for every* $x, y \in S$ *with* $x \upharpoonright B = y \upharpoonright B$ *(*$x \upharpoonright B$ *is the restriction of* x *to* B*). A support of a subset* $F \subseteq X$ *is defined to be a set* $B \subseteq A$ *such that if for all* $x \in F, y \in X$ *and* $x \upharpoonright B = y \upharpoonright B$ *then* $y \in F$.

Lemma 2.2.2. *[23] Let* $\{X_\alpha : \alpha \in I\}$ *be a collection of spaces each having density character at most* μ*. Then every regular closed subset o f the product* X *has a support of cardinality not exceeding* μ.

This is a generalization of a result of Ross and Stone [29], who only worked with $\mu = \aleph_0$ bu the proof for arbitrary μ is analogous.

Theorem 2.2.3. *Every regular closed set* F *in a* $\mu-$*multiadic space* X *of weight* τ *is a regular closed subset* C *of* $\prod_{i \in \tau} \alpha\kappa_i$*, having support* B *of cardinality which does not exceed the weight of* X.

Proof. : We shall prove later, in Theorem 3.2.1 that a $\langle \kappa_i : i \in I \rangle$-multiadic space of weight τ is a continuous image of $\prod_{i \in \tau} \alpha\kappa_i$ (without loss of generality let assume $\mu \leq \tau$). Hence there is a continuous map f of $\prod_{i \in \tau} \alpha\kappa_i$ onto X where τ is the weight of X. Suppose that F is a regular closed subset of X, then $F = \overline{U}$ where U is an open in X. By the compactness of X, f is a closed map. Let $C = \overline{f^{-1}(U)}$. Therefore $f(C) \subseteq U$, by the continuity of the map f and $f(C) \supseteqq U$, by the closeness of f. Hence $f(C) = \overline{U}$. Since for all i, $d(\alpha\kappa_i) \leq \mu$ and by a generalization of Ross and Stone theorem, every regular closed subset of the product $\prod_{i\in\tau} \alpha\kappa_i$ has a support of cardinality not exceeding μ, therefore $\overline{f^{-1}(U)}$ has a support B with $|B| \leq \mu \leq \tau$, as for any topological space its density character on exceed its weight. Thus $F = (F \restriction B) \times (\prod_{i\in\tau\setminus B} \alpha\kappa_i)$. ■

Before we are going to give other results regarding closed sets of type G_δ let us give the following:

Theorem 2.2.4. *Marty [23]: Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces each having density character not exceeding μ. Let F be a subset of the product X. Then F is a zero-set if and only if $F = (F \restriction B) \times (X \restriction A\setminus B)$, for some set $B \subseteq A$ such that $|B| \leq \mu$ and $F \restriction B$ is a zero-set in $X \restriction B$.*

Theorem 2.2.5. *Every closed G_δ-set F in a $\mu-$multiadic space X with $w(X) = \tau$ is a continuous image of $C \times \prod_{i\in\tau} \alpha\kappa_i$, where C is a closed G_δ-set in $\prod_{j\in I} \alpha\kappa_j$, and $|I|$ does not exceed the weight of X.*

Proof: We shall prove later, in Theorem 3.2.1 that a $\langle\kappa_i : i \in I\rangle$-multiadic space of weight τ is a continuous image of $\prod_{i\in\tau} \alpha\kappa_i$ (without loss of generality $\mu \leq \tau$.) Hence there is a continuous map f of $\prod_{i\in\tau} \alpha\kappa_i$ onto X. Since F is a closed G_δ in a compact space, F is a zero set. Thus $f^{-1}[F]$ is a zero-set. Consequently, by Theorem 2.2.4, $f^{-1}[F] = ((f^{-1}(F)) \restriction B) \times \prod_{i\in\tau\setminus B} \alpha\kappa_i$, where $|B| \leq \mu \leq \tau$, and $(f^{-1}[F]) \restriction B$ is a closed G_δ-set in $\prod_{j\in B} \alpha\kappa_j$. ■

This does not ensure that every closed G_δ-set in a multiadic space is multiadic because we do not know enough about $(f^{-1}[F]) \restriction B$, for example if it is multiadic or not. Hence this result is a weaker result than the one in the class of dyadic spaces which was obtained by Efimov [8] who proved in 1965, that:

Theorem 2.2.6. *Every closed G_δ of a dyadic space is dyadic.*

In proving his theorem he showed that for $\tau \geq \aleph_0$ every regular closed set in D^τ is of type G_δ and every regular closed subset of D^τ is homeomorphic to the whole space D^τ. However, this is not the case for $\prod_{i\in\tau} \alpha\kappa_i$ and not even for a polyadic $(\alpha\kappa)^\tau$. In 1978, a similar result was showed for polyadic spaces by Gerlits [13], who proved that a closed G_δ of a polyadic space is polyadic. Since the situation of multiadic spaces is similar to the case of the polyadic ones, we obtain analogous results for multiadic. The idea of the proof of Theorem 2.2.7 is to factorize any G_δ of a multiadic compactum into a compact G_δ of a dyadic space and another multiadic space as we shall see in the proof.

Theorem 2.2.7. *A compact G_δ-set of a $\langle \kappa_i : \ i \in I\rangle$-multiadic compactum where for all i, $\kappa_i > \aleph_1$ is $2^I \times \langle \kappa_i : \ i \in I\rangle$-multiadic.*

Proof: Let X be a Hausdorff continuous image of $\prod_{i\in I} \alpha\kappa_i$. Since the inverse image of a closed G_δ under a continuous surjective map is a G_δ too, it is enough to show that if $C \subseteq \prod_{i\in I} \alpha\kappa_i$ is a closed G_δ then C is $2^I \times \langle \kappa_i : \ i \in I\rangle$-multiadic.

Since $\prod_{i\in I} \alpha\kappa_i$ has clopen base, we can assume that C is the intersection of countably many clopen sets, i.e $C = \bigcap_{n<\omega} O_n$ where O_n is clopen in $\prod_{i\in I} \alpha\kappa_i$.

Each basic set B in $\prod_{i\in I} \alpha\kappa_i$ is of the form $B = \prod_{i\in I} b_i$ where for a finite subset I_B of I, $b_i \neq \alpha\kappa_i$ and either b_i is finite or cofinite, otherwise $b_i = \alpha\kappa_i$. Let for $i \in I_B,\ K_i(B) = b_i$ if b_i is finite and otherwise $K_i(B) = \alpha\kappa_i \backslash b_i$. $K_i(B)$ is a finite set in $\alpha\kappa_i$ and $\infty \notin K_i(B)$. Since O_n is compact then for each $n \in \omega$ the clopen set O_n is the union of a finite family of base sets, say $O_n = \bigcup_{j<k_n} B_j^n$. Consider for each $i \in I,\ K_i(O_n) \stackrel{\text{def}}{=} \bigcup\{K_i(B_j^n) :\ j < k_n\}$ and $K_i = \bigcup_{n\leq\omega} K_i(O_n)$. Clearly $\forall i \in I,\ K_i \subseteq (\alpha\kappa_i)\backslash\{\infty\}$ and $|K_i| \leq \omega < \kappa_i$. Now we shall use an idea of Gerlits [13].

Claim 1: Let $p, q \in \prod_{i\in I} \alpha\kappa_i,\ p \in C$. Assume that for each $i \in I$ if $p(i) \neq q(i)$ then $\{p(i),\ q(i)\} \cap K_i = \emptyset$. Then $q \in C$.

Proof: Suppose not; $q \notin C$. There is some n such that $p \in O_n$ and $q \notin O_n$. Since $\prod_{i\in I} \alpha\kappa_i$ is Hausdorff, there exists a basic clopen set $B \subseteq O_n$ in $\prod_{i\in I} \alpha\kappa_i$ such that $p \in B,\ q \notin B$ and an $i^* \in I$ with $p(i^*) \in \pi_{i^*}(B) = B_{i^*}, q(i^*) \notin \pi_{i^*}(B) = B_{i^*}$. One of the sets $B_{i^*},\ B_{i^*}^c$ is contained in the set K_{i^*}, hence $\{p(i), q(i)\} \cap K_{i^*} \neq \emptyset$, a contradiction. ■

Now fix an arbitrary point $z \in \prod_{i\in I} \left[\alpha\kappa_i \backslash (K_i \cup \{\infty\})\right]$, so for each $i,\ z_i \notin \{\infty\} \cup K_i$. Let $L_i = K_i \cup \{z_i\} \cup \{\infty\},\ F = C \cap \prod_{i\in I} L_i$. Since for each $i \in I, |L_i| \leq \omega$, the space $\prod_{i\in I} L_i$ is homeomorphic to 2^I (as mentioned in the introduction). Hence F is an image of 2^I by applying the Efimov result 2.2.6. Define $\phi : F \times \prod_{i\in I}(\alpha\kappa_i \backslash K_i) \longrightarrow \prod_{i\in I} \alpha\kappa_i$ so that

$$\phi(x, y)(i) = \begin{cases} y(i) & \text{if } x(i) = z_i \\ x(i) & \text{otherwise.} \end{cases}$$

Claim 2: ϕ is continuous.

Since ϕ is continuous iff for each i, $\phi_i = \pi_i \circ \phi$ is continuous, we should show that for any singleton point β of $\alpha\kappa_i$, $\phi_i^{-1}(\{\beta\})$ is a clopen set in $F \times \prod_{i\in I} \alpha\kappa_i \backslash K_i$. Let $Y = \prod_{i\in I} \alpha\kappa_i \backslash K_i$. For each $i \in I$ let $\pi_i|F$ be the projection of F into the i^{th} factor space and $\pi_i|Y$ the projection of $\prod_{i\in I} \alpha\kappa_i \backslash K_i$ into the i^{th} factor space.

If $\beta \neq z_i$ then $\phi_i^{-1}(\{\beta\}) = \left[\pi_i^{-1}|F(\{z_i\}) \cap (\pi_i^{-1}|Y(\{\beta\})\right] \cup \pi_i^{-1}|F(\{\beta\}))$ is a clopen set. Also when $\beta = z_i$, $\phi_i^{-1}(\{\beta\}) = \pi_i^{-1}|F(\{z_i\}) \cap \pi_i^{-1}|Y(\{z_i\}$ is clopen. This proves the claim. ■

Claim 3: $\phi(F \times \prod_{i\in I} \alpha\kappa_i \backslash K_i) = C$.

I. Assume that $p \in C$. Let p' be the point in $\prod_{i\in I} \alpha\kappa_i$ such that

$$p'(i) = \begin{cases} z_i & \text{if } p(i) \notin K_i \\ p(i) & \text{if } p(i) \in K_i. \end{cases}$$

For each $i \in I$ we can easily check that $p'(i) \in L_i$. Using Claim 1 on the point $q = p'$ we get $p' \in C$. Thus $p' \in C \cap \prod_{i\in I} L_i = F$. Denote by $p'' \in \prod_{i\in I} \alpha\kappa_i$ the point

$$p''(i) = \begin{cases} z_i & \text{if } p(i) \in K_i \\ p(i) & \text{if } p(i) \notin K_i. \end{cases}$$

Clearly that for each $i \in I, p''(i) \notin K_i$, hence $(p', p'') \in F \times \prod_{i\in I} \alpha\kappa_i \backslash K_i$. Now we have to show that $\phi(p', p'') = p$. If $p'(i) \notin K_i$ then $p'(i) = z_i$, $p''(i) = p(i)$ and $\phi(p', p'')(i) = p(i)$; if $p'(i) \in K_i$ then $p'(i) = p(i)$, $p''(i) = z_i$ and $\phi(p', p'')(i) = p(i)$. Since this is for all $i \in I$, we have therefore $\phi(p', p'') = p$.

II. Let us make sure that the image of any point $(p, p') \in F \times \prod_{i \in I} \alpha\kappa_i \backslash K_i$ must be in C. Let $q = \phi(p, p')$. Since $p \in F$, so $p \in C$. If $i \in I$ and $p(i) = p'(i)$ then $q(i) = p(i)$. If $p(i) \neq q(i)$ then $p(i) = z_i$ and $q(i) = p'(i)$. But $p'(i) \notin K_i$, therefore $\{p(i), q(i)\} \cap K_i = \emptyset$; by Claim 1 $q \in C$. ■

Hence C is the continuous image of multiadic space $2^I \times \prod_{i \in I} \alpha\kappa_i$. ■

2.3 Multiadic spaces are AD-compact spaces

The notion of AD compact was introduced by Plebanek [25] in 1995. In this section we show that the properties of being AD compact is satisfied by all multiadic spaces. In the beginning we give a definition of an adequate family of subsets of a nonempty set X.

Definition 2.3.1. *(Talagrand [31]) An adequate family $\mathcal{A} \subseteq \mathcal{P}(X)$ of a nonempty set X is a family that satisfies*
(i) $A \in \mathcal{A}$ and $B \subseteq A$ implies $B \in \mathcal{A}$;
(ii) given $A \subseteq X$, if every finite subset of A is in $\mathcal{A}$ then $A \in \mathcal{A}$.

For any space X, the power set $\mathcal{P}(X)$ can be identified with the Cantor cube D^X (mapping A to χ_A). Thus every subfamily $\mathcal{A}$ of $\mathcal{P}(X)$ can be treated as a subset of D^X associated with the induced topology. The space X generated by an adequate family $\mathcal{A}$ will be written as $K(\mathcal{A})$.

According to the definition due to Plebanek [25] a compact space X is called AD-compact if it is a continuous image of some adequate compact space. He showed that the space $(\alpha\kappa)^\lambda$ is AD-compact. In particular he proved that the property of being AD-compact is productive and that $\alpha\kappa$ is homeomorphic to $K(\mathcal{A})$ where $\mathcal{A} = \{A \subseteq \kappa : \ |A| \leq 1\}$. The same proof can be used to show that every κ-multiadic space of weight κ is a continuous image of $K(\mathcal{A})$ for some $\mathcal{A} \subseteq \mathcal{P}(X)$, as we now show.

Let $\mathcal{A} = \{A \subseteq \kappa : \ |A| \leq 1\}$ be identified with a subset of 2^κ. Note that $\mathcal{A}$ is an adequate family. We can see that $\mathcal{A}$ is a closed subspace of 2^κ: We shall concentrate on the case $y \in \mathcal{A}^c$ with $|y| = 2$, say $\alpha, \beta \in y$. Then $W = \{w \in 2^\kappa : \ \ w_\alpha = w_\beta = 1\}$ is an open set in 2^κ containing y which is disjoint from $\mathcal{A}$. The compact space $\alpha\kappa$ is homeomorphic to $K(\mathcal{A})$ under the continuous mapping $f : \alpha\kappa \to \ \mathcal{A}$, defined by

$f(\infty) = \chi_\emptyset, \ f(\beta) = \chi_{\{\beta\}}$ for $\beta < \kappa,$ where χ denotes characteristic function..

It follows that for a sequence $\langle \kappa_i : \ \ i \in I\rangle$ with supremum κ, the space $\prod_{i\in I} \alpha\kappa_i$ is adequate as witnessed by some family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$. As a consequence of adequacy for multiadic spaces, we can get more results for the class of multiadic compacta that follow from Plebanek's Theorem:

Theorem 2.3.2. *Plebanek [25] For an adequate compact $K = K(\mathcal{A})$, where $\mathcal{A} \subseteq X$, and for a continuous mapping g from K into a space L such that $\chi(L) \leq \tau$, there exists $Y \subseteq X$ such that $|Y| \leq \tau$ and $g(A \cap Y) = g(A)$ for all $A \in K$.*

Corollary 2.3.3. *(a) If X is a multiadic space of weight κ then there is an adequate family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that X is a continuous image of $K(\mathcal{A})$.*
(b) For every multiadic space X, $\chi(X) = w(X)$.

Proof: (a) Applying Plebanek's Theorem it is enough to recall that if $\mathcal{A}$ is an adequate family in $\mathcal{P}(X)$ and $Y \subseteq X$ then $\mathcal{A} \cap Y = \{A \cap Y : A \in \mathcal{A}\}$ is adequate.
(b) Since $w(K(\mathcal{A} \cap Y)) \leq |Y|$, and topological weight is not increased by continuous surjection of compact spaces, we have that $\chi(X) = w(X)$. ■

Corollary 2.3.4. *If K is κ-multiadic then every closed $\mathcal{G}_\delta$ subspace of K is AD-compact as witnessed by a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$.*

This follows because the same is true for any AD-compact [see Plebanek [25]].

Chapter 3

Identifying multiadic spaces by their cardinal invariants

The contents of this chapter fall into three parts. The first part concerns the difference between polyadic and multiadic spaces while the second part discusses some cardinal functions of multiadic spaces and their relationships. Finally we give a characterization of the multiadic class.

3.1 Non-homeomorphic spaces

Two cardinal invariants will be studied in this section to distinguish polyadic spaces and multiadic spaces. More precisely, we are going to consider $(\alpha\aleph_\omega)^\omega$ and $\prod\limits_{n<\omega} \alpha\aleph_n$ from the point of view of the point character $\chi(p, X)$, and the point $\sigma-$character, $\sigma\chi(p, X)$, of a point $p \in X$.

3.1.1 σ-character of polyadic and multiadic spaces.

Here we introduce a new cardinal invariant not usually considered in the literature, but needed in our proof of Theorem 3.1.2.

Definition 3.1.1. *A local σ-base of a point p in an infinite topological space X is defined to be a family $\mathcal{F}$ of nonempty closed non-singleton sets in X such that every infinite open neighborhood of p contains a member of $\mathcal{F}$.*

$$\sigma\chi(p, X) = \min\{|\mathcal{F}| \ : \ \mathcal{F} \text{ is a local } \sigma{-}\text{base of } p \in X\}$$

is the $\sigma-$character of p in X.

$$\sigma\chi(X) = \sup\{\sigma\chi(p, X) : \ p \in X\}$$

is the σ- character of X.

Theorem 3.1.2. *Let κ be a singular cardinal such that $cf(\kappa) = \omega$. Then $(\alpha\kappa)^\omega$ and $(\alpha\kappa)^\kappa$ are not homeomorphic.*

Proof: We attempt to distinguish the two spaces by the $\sigma-$character of a particular point.

Claim: If $p = \infty \in \alpha\kappa$ then $\sigma\chi(p, \alpha\kappa) = \omega$.

Proof of the Claim Let $p = \infty$ and let $\{x_i : i \in \omega\}$ be a cofinal set in κ. Consider $B = \{b_i \ : \ b_i = \alpha\kappa \backslash x_i\}$. Here b_i is closed in $\alpha\kappa$ since we deal with the discrete topology on κ. Every neighborhood of $\{\infty\}$ in $\alpha\kappa$ is of the form $U \cup \{\infty\}$, where $\kappa \backslash U$ is finite. Thus for every U there is $i \in \omega$ such that $b_i \subseteq U$. Namely, it suffices to choose x_i large enough that it is bigger than all the finitely many points of κ which are not in U. Hence B is a local $\sigma-$base of $\{\infty\}$. Therefore $\sigma\chi(\infty, \alpha\kappa) \leq \omega$.

On the other hand, we have to show that every local σ-base of a point p has size at least ω, i.e. $\sigma\chi(p, \alpha\kappa) \geq \omega$. Suppose for contradiction there is a finite local $\sigma-$base of $\{\infty\}$, say $B = \{b_i \ : \ i \leq m\}$. For each $i \leq m$ fix one point r_i such that $r_i \neq \{\infty\}$ from b_i. Consider $U' = \kappa \backslash \{r_i \ : \ i \leq m\}$. So $U = U' \cup \{\infty\}$ is a neighborhood of $\{\infty\}$ which is not a superset of any of b_i's. Hence $\sigma\chi(p, \alpha\kappa) = \omega$. This proves our claim.

Continuation of the proof. Let ∞_κ^*, ∞_ω^* be the points that contain infinity everywhere in $(\alpha\kappa)^\kappa$ and $(\alpha\kappa)^\omega$ respectively. Since in the product space the basic open sets are determined by finite partial functions, so $\sigma\chi(\infty_\kappa^*, (\alpha\kappa)^\kappa) = \max\{\omega, \kappa\} = \kappa$ and $\sigma\chi(\infty_\omega^*, (\alpha\kappa)^\omega) = \omega$.

Moreover, every point $p \in \kappa$ has $\sigma\chi(p, \alpha\kappa) = \kappa$. This is due to the fact that for the singleton point p in κ, infinite open sets containing this point are of the form $A \cup \{p\}$ for some infinite set $A \subseteq \kappa$ or of the form $\alpha\kappa \backslash F$ for some finite F and $p \notin F$. Therefore $\{H \cup \{p\} : \ H \text{ is finite } \subseteq \kappa\}$ forms a $\sigma-$basis at p and it is a local σ-base of smallest cardinality. Thus

$$\sigma\chi(p, \alpha\kappa) = \begin{cases} \omega & \text{if } p = \infty \\ \kappa & \text{if } p \in \kappa. \end{cases}$$

Now consider any $q \in (\alpha\kappa)^\kappa$. If $q = \infty_\kappa^*$ then we have proved $\sigma\chi(\infty_\kappa^*, (\alpha\kappa)^\kappa) = \kappa$. Suppose $q \neq \infty_\kappa^*$, so al least on one coordinate say i, $q(i) \neq \infty$. Hence $\sigma\chi(q(i), (\alpha\kappa)) = \kappa$. By the definition of the product space, $\sigma\chi(q, (\alpha\kappa)^\kappa) \geq \sigma\chi(q(i), \alpha\kappa) = \kappa$. Since there are κ many basic open neighborhoods of q in $(\alpha\kappa)^\kappa$, we clearly have $\sigma\chi(q, (\alpha\kappa)^\kappa) \leq \kappa$ and hence $\sigma\chi(q, (\alpha\kappa)^\kappa = \kappa$. Similarly, for $q \in (\alpha\kappa)^\omega$ and $q \neq \infty_\omega^*$, $\sigma\chi(q, (\alpha\kappa)^\omega = \kappa$

Finally, suppose for contradiction that there is a homeomorphic function $\theta : \ (\alpha\kappa)^\omega \ \rightarrow \ (\alpha\kappa)^\kappa$. Let $p = \infty_\omega^* \in (\alpha\kappa)^\omega$. So $\sigma\chi(p, (\alpha\kappa)^\omega) = \omega$ and hence

$\sigma\chi(\theta(p), (\alpha\kappa)^\kappa) = \omega$ (noting that $\sigma\chi$ is preserved by homeomorphism), which contradicts the previous result about points in $(\alpha\kappa)^\kappa$. ■

Theorem 3.1.3. *Let κ be a cardinal with $cf(\kappa) = \omega < \kappa$ and let $\langle \kappa_i : i \in \omega \rangle$ be an increasing sequence of cardinals to κ, such that $cf(\kappa_n) = \lambda_n > \omega$. Then the product $\prod_{i\in\omega} \alpha\kappa_i$ is not homeomorphic to $(\alpha\kappa)^\omega$, although the σ-character of both of them is κ.*

Proof: First we have to calculate the $\sigma-$character of each point in $\prod_{i\in\omega} \alpha\kappa_i$. Let us consider 4 cases:

1. If p contains no ∞ in its coordinates then $\sigma\chi(p, \prod_{i\in\omega} \alpha\kappa_i) = \sup\{\kappa_i : i \in \omega\} = \kappa$.
2. If p contains ∞ everywhere then $\sigma\chi(p, \prod_{i\in\omega} \alpha\kappa_i) = \sup\{\lambda_i : i \in \omega\} = \lambda_\omega$.
3. If p consists finitely many coordinates that are ∞ say the maximum one is the m^{th} coordinate, then $\sigma\chi(p, \prod_{i\in\omega} \alpha\kappa_i) = \max\{\lambda_m, \kappa_\omega\}$.
4. If p consists infinitely many coordinates that are ∞, then $\sigma\chi(p, \prod_{i\in\omega} \alpha\kappa_i) = \lambda_\omega$.

Now suppose for contradiction that there is a homeomorphism

$$\psi : (\alpha\kappa)^\omega \rightarrow \prod_{i\in\omega} \alpha\kappa_i.$$

Let $p = \infty^*_\omega \in (\alpha\kappa)^\omega$ be the point that contains ∞ everywhere. We have seen from the previous Theorem that $\sigma\chi(p, (\alpha\kappa)^\omega) = \omega$, however, there exists no point in $\prod_{i\in\omega} \alpha\kappa_i$ that has $\sigma-$character omega, which is a contradiction. ■

3.1.2 Character of polyadic and multiadic spaces.

We would like to thank István Juhász for pointing out the statements of Theorem 3.1.4 and Theorem 3.1.7, for which we supplied the proofs here. These theorems can be used to give a different proof (supplied by us) to our Theorem 3.1.3.

Theorem 3.1.4. *The character of every point in $X = (\alpha\kappa)^\omega$ is either ω or κ.*

Proof: Case 1: Let X' be the collection of points in X that do not have infinity in their coordinates. In this case we can think of every element of the space X' as a function f from ω to κ such that for every n, $f(n) \in \kappa$. $(*)$ The basic open sets then are determined by finite partial functions, i.e. functions g which have domain equal to a finite subset of ω and satisfy the condition $(*)$. We can denote by $[g]$ the set of f such that g is a subset of f, i.e. f is a function which extends g. Thus the point character of each f in X' is the smallest number of basic open sets such that their intersection is exactly $\{f\}$. In this case it will actually be countable since each f is obtained as the intersection of $[f \restriction n]$ for all n in ω.

Case 2: Suppose that $f \in X$ contains at least one coordinate that is infinity. Let $A = \{n : \ f(n) \neq \infty\}$, $B = \{n : \ f(n) = \infty\} \neq \emptyset$. The basic open sets in general for $(\alpha\kappa)^\omega$ are in the form $\prod_{n\in\omega} U_n$ where for finite set $F \subseteq \omega$, $U_n \neq \alpha\kappa$ and either U_n is open in κ if $n \in A$ or $U_n = U_n' \cup \{\infty\}$ if $n \in B$ where U_n' is open in κ and $\kappa \backslash U_n'$ is compact; and for $n \notin F$, $U_n = \alpha\kappa$. Thus there are several kinds of basic open sets around f.

1. There are countably many basic open set of the form $[g]$ where g is a finite

partial function from A to κ.

2. Let $n \in B$. Such an n exists because $B \neq \emptyset$. For each finite $H \subseteq \kappa$ consider the set $(U_n^H)' = \{g \in (\alpha\kappa)^\omega : \ g(n) \in (\kappa \backslash H) \cup \{\infty\}\}$. Then $(U_n^H)'$ is a basic open set in $(\alpha\kappa)^\omega$ containing ∞. There are κ many choices for such a set, as there are κ many sets H.

Now suppose that $\emptyset \neq F \subseteq B$ is finite. Then for any $\{H_i : \ i \in F\}$ finite subsets of κ, we have that $\prod_{i\in F}(\kappa\backslash H_i) \cup \{\infty\} \times \prod_{i \notin F} \alpha\kappa$ is a basic open set containing f, and $\{f\}$ is the intersection of all of these κ many open sets. On the other hand, $\{f\}$ is not the intersection of any subfamily of $< \kappa$ many such sets. Namely, suppose $\{\prod_{i\in F}(\kappa\backslash H_i^\gamma) \cup \{\infty\} \times \prod_{i\notin F} \alpha\kappa : \ \gamma < \gamma* < \kappa\}$ is such family. Then $|\bigcup\{H_i^\gamma : \ i \in F, \gamma < \gamma*\}| < \kappa$. Therefore there is $\varepsilon \in \kappa \backslash \bigcup\{H_i^\gamma : \ i \in F, \gamma < \gamma*\}$. Fix $i \in F$. Let $g(i) = \varepsilon$ and $g(n) = f(n)$ for $n \notin \varepsilon$. Hence $g \neq f$ but $g \in \bigcap \prod_{i\in F}(\kappa\backslash H_i^\gamma) \cup \{\infty\} \times \ prod_{i\notin F}\alpha\kappa$. This means that $\{f\} \neq \bigcap \prod_{i\in F}(\kappa\backslash H_i^\gamma) \cup \{\infty\} \times \prod_{i\notin F} \alpha\kappa$. ■

Corollary 3.1.5. *The character of the polyadic space $X = (\alpha\kappa)^\omega$ is κ.*

Corollary 3.1.6. *The point character of each point in the polyadic space $(\alpha\aleph_\omega)^\omega$ is either ω or $\aleph_\omega$.*

Theorem 3.1.7. *The character of every point in $Y = \prod_{n\in\omega} \alpha\aleph_n$ is ω, $\aleph_n$ where $n \in \omega$ or $\aleph_\omega$.*

Proof: Case 1: We apply the same logic as in case 1 in Theorem 3.1.4 when $y \in Y$ does not have infinity in any of its coordinates and thus $\chi(y, Y) = \aleph_0$.

Case 2: Suppose $y \in Y$ such that $B = \{n : \ y(n) = \infty\} \neq \emptyset$. So we have to investigate two cases. Firstly, if B is finite. Let $m = \max(B)$ and we shall prove that $\chi(y, Y) = \aleph_m$. It is clear that the form of all basic open sets containing y are in the same form as in case 2 of Theorem 3.1.4. Since for each $n \in B$, there are $\aleph_n$ many choices for a basic open set in the form $(U_n^H)'$, their intersection is ∞ in $n-$th coordinate. Thus there are $\aleph_m$ many choices for a basic open set in $\prod_{n\in\omega} \alpha\aleph_n$ so that the intersection of these $\aleph_m$ many open sets contains ∞ in all coordinates in B. Hence $\chi(y, Y) \leq \aleph_m$. By a similar argument as in the end proof of case 2 of Theorem 3.1.4, we get $\chi(y, Y) \geq \aleph_m$. Therefore, $\chi(y, Y) = \aleph_m$.

Secondly, if the cardinality of B is infinite. For simplicity, consider y to be a point that has ∞ everywhere in Y say $y = \infty^*$. For any finite $F \subseteq B$, let $l = \max(F)$. By the above argument, there are $\aleph_l$ choices for an open basic set that contains ∞ in $\aleph_n,\ n \in F$. Thus the minimum number of the open sets of that form whose intersection is only $\{\infty^*\}$ is $\aleph_\omega$. Otherwise if there are only $\aleph_k,\ k < \omega$ choices for $(U_n^H)'$ in $\aleph_n$ where $n < \omega$, then there are only $\aleph_k$ choices for $(U_{k+1}^H)'$ in $\aleph_{k+1}$; which is a contradiction. ■

Theorem 3.1.8. *The two spaces* $X = (\alpha\aleph_\omega)^\omega$ *and* $Y = \prod_{n\in\omega} \alpha\aleph_n$ *are not homeomorphic to each other.*

Proof: Suppose for contradiction that there is a homeomorphism $F : Y \to X$. Let $y \in Y$ be the point that contains only one infinity in the m^{th} position. So $\chi(y, Y) = \aleph_m$ and hence $\chi(F(y), X) = \aleph_m$ which contradicts Theorem 3.1.4 as there is no such element in X. ■

Note: This theorem is actually implied by Theorem 3.1.3.

3.2 Cardinal Invariants of Multiadic spaces

In this section we investigate some cardinal functions of the multiadic compacta and their relations. For example one result shows that a multiadic space X is μ-multiadic iff $c(X) \leq \mu$ and an analogous result concerns the density character. We also prove that if the weight of a μ-multiadic compactum is $\geq \mu$, then its weight equals to its pseudo-character.

Theorem 3.2.1. *Suppose $J \supseteq \tau$. Then every $\langle \kappa_j : j \in J \rangle$-multiadic space X, having weight τ, is a continuous image of $\prod_{i \in \tau} \alpha\kappa_i$ (i.e it is $\langle \kappa_j : j \in \tau \rangle$-multiadic).*

Proof: Suppose that f is a continuous map of $\prod_{j \in J} \alpha\kappa_j$ onto X for some cardinal J and cardinals $\langle \kappa_j : j \in J \rangle$. Since $J \supseteq \tau$, then $\tau < |J|$. Since the Tychonoff cube I^m where I denotes the interval $[0,1]$ is universal for all Tychonoff spaces of weight $m \geq \aleph_0$ [10], we can consider X as a subspace of I^τ. Let π_ξ be the projection of I^τ onto $I_\xi = I$ for every $\xi \in \tau$. Consider

$$\mathcal{F} = \{\pi_\xi \circ f : \xi \in \tau\}.$$

By a classical theorem [10] (every real-valued continuous function on a product of compact spaces has a countable support), each member of $\mathcal{F}$ has a countable support say D_ξ. Let $D = \bigcup_{\xi < \tau} D_\xi$. Here $|D| = \tau$. Consequently, f has a support D of cardinality τ, as if $f(x) \neq f(y)$ where $x, y \in X$ with $x \restriction D = y \restriction D$ then $\forall \xi < \tau$ $x \restriction D_\xi = y \restriction D_\xi$, so $(\pi_\xi \circ f)(x) = (\pi_\xi \circ f)(y)$, and hence $f(x) = f(y)$. Hence there is a continuous map of $\prod_{i \in \tau} \alpha\kappa_i$ onto X. ■

In the case where $J \subseteq \tau$ we get an analogous result to above result as $f \circ \pi_J$

is a continuous map of $\prod_{i\in\tau} \alpha\kappa_i$ onto X where π_J is the projection of $\prod_{i\in\tau} \alpha\kappa_i$, onto $\prod_{j\in J} \alpha\kappa_j$.

Another important cardinal invariant is the density character. The following proposition is proved by Engelking in [10] for $\mu = \aleph_0$ and the proof for arbitrary μ is analogous.

Proposition 3.2.2. *Engelking [10] : Let $\{X_i \ : i \in I\}$ be a collection of spaces such that $d(X_i) \leq \mu$ for all $i \in I$ and $|I| \leq 2^\mu$. Then the cardinality of every family of mutually disjoint nonempty open subsets of the product $X = \prod_{i\in I} X_i$ does not exceed μ, i.e. $c(X) \leq \mu$.*

Proof: Let $\{O_j\}_{j\in J}$ be a family of pairwise disjoint non-empty open subsets of the product $X = \prod_{i\in I} X_i$. Without loss of generality we can assume that $\{O_j\}_{j\in J}$ consists of members of the canonical base for X, i.e., that for every $j \in J$ there exist a finite set $I_j \subseteq I$ and a family $\{O_i^j\}_{i\in I}$ where O_i^j is an open subset of X_i and $O_i^j = X_i$ if $i \notin I_j$, such that $O_j = \prod_{i\in I} O_i^j$.

Assume that $|J| > \mu$; obviously, we can suppose that $|J| \leq 2^\mu$. The set $I_0 = \bigcup_{j\in J} I_j$ also has cardinality $\leq 2^\mu$, so that the Cartesian product $\prod_{i\in I_0} X_i$ contains, by the Hewitt-Marczewski-Pondiczery Theorem 1.4.20, a dense subset A of cardinality $\leq \mu$. The family $\{\prod_{i\in I_0} O_i^j\}_{j\in J}$ consists of non-empty open subsets of $\prod_{i\in I_0} X_i$. Since $O_j = \prod_{i\in I_0} O_i^j \times \prod_{i\in I/I_0} X_i$ for every $j \in J$, the members of $\{\prod_{i\in I_0} O_i^j\}_{j\in J}$ are pairwise disjoint and since every member contains an elements of A, it follows that $|J| \leq |A| \leq \mu$, which contradicts our assumption that $|J| > \mu$. ■

The following results might be known, as they relate to well known concepts. However we could not find the proofs in print, hence we include them here.

Observation 3.2.3. *There is a family of open disjoint sets in $\alpha\kappa$ of size κ.*

This can be illustrated by the family of all singleton points, since they are clopen and disjoint sets in $\alpha\kappa$ of size κ. A more general result is given in the following theorem:

Theorem 3.2.4. *Let X be a topological space. If $c(X) < \kappa$ then there is no continuous map from X onto $\alpha\kappa$.*

Proof: Suppose for a contradiction that there is a continuous map $f : X \twoheadrightarrow \alpha\kappa$. Let $\mathfrak{U} = \{\{i\} : i < \kappa\}$ be the system of all pairwise disjoint open sets of the singleton points in $\alpha\kappa$. Since f is continuous then $B_i = f^{-1}(\{i\})$ is an open set in X and hence $\mathfrak{B} = \{B_i : i < \kappa\}$ is a disjoint family of open sets of size κ. Therefore $c(X) = \sup\{|\gamma| : \gamma \text{ is a disjoint family of open sets in } X\} \geq \kappa$, which contradicts our assumption. ■

Theorem 3.2.5. *If $\kappa < \lambda$, then there is no continuous map from $(\alpha\kappa)^\omega$ onto $(\alpha\lambda)^\omega$.*

Proof: Suppose there is a continuous map f that maps $(\alpha\kappa)^\omega$ onto $(\alpha\lambda)^\omega$.

For any cardinal κ, $c((\alpha\kappa)) = \kappa$. It follows from Observation 3.2.3, $c(\alpha\kappa) \geq \kappa$, since $c(\alpha\kappa)$ is clearly $\leq \kappa$ we have $c(\alpha\kappa) = \kappa$.

Since for any infinite cardinal κ, $d(\alpha\kappa) = \kappa$ so by 3.2.2, $c((\alpha\kappa)^\omega) \leq \kappa$. Also the family $\mathcal{O} = \{O_n : n \in \omega\}$, where $O_n = \{x \in (\alpha\kappa)^\omega : x(1) = n\}$ is a disjoint open family of size κ, so $c((\alpha\kappa)^\omega) \geq \kappa$ and hence $c((\alpha\kappa)^\omega = \kappa$.

Since $\forall i \in \omega$ the projection map $\pi_i : (\alpha\lambda)^\omega \to \alpha\lambda$ defined by $(x_i)_{i\in\omega} \mapsto x_i$ is continuous and surjective, we get that the composition function $\pi_i \circ f : (\alpha\kappa)^\omega \to \alpha\lambda$ is also continuous and surjective map. By applying Theorem 3.2.3, we must have $c((\alpha\kappa)^\omega) \geq \lambda$, which contradicts above claim that $c((\alpha\kappa)^\omega) = \kappa$. ■

Observation 3.2.6. *If $\kappa < \lambda$, then there is no continuous map from $\alpha\kappa$ onto $\alpha\lambda$.*

Theorem 3.2.7. *Let $\langle \kappa_i : i \in I \rangle$ be a sequence of distinct cardinals of limit μ. Then every collection of pairwise disjoint nonempty open subsets of a μ-multiadic space X has cardinality at most μ. Hence $c(X) \leq \mu$.*

Proof: Suppose that f is a continuous map of $\prod_{i\in I} \alpha\kappa_i$ onto X for some cardinal I with $|I| \leq \mu$ and each $\kappa_i \leq \mu$. Let $\mathcal{O}$ be a collection of mutually disjoint nonempty open subsets of X with $|\mathcal{O}| > \mu$. Then $f^{-1}[\mathcal{O}] = \{f^{-1}[G] : G \in \mathcal{O}\}$ is a collection of mutually disjoint nonempty open subsets of $\prod_{i\in I} \alpha\kappa_i$ and $|f^{-1}[\mathcal{O}]| > \mu$. However, this is a contradiction with 3.2.2, since the density character of each $\alpha\kappa_i$ is at most μ. ■

Theorem 3.2.9 is an analogous result to Gerlits [12] that shows a polyadic compactum X is $\mu-$polyadic iff $c(X) \leq \mu$. Before giving the proof of our theorem we have to recall the following theorem due to him:

Theorem 3.2.8. *[12] : If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous and surjective, then there exists a compact subspace $C \subseteq X$ such that $f(C) = Y$ with $c(C) \leq c(Y)$.*

Theorem 3.2.9. *If X is μ-multiadic and $c(X) \leq \xi$ for some ξ such that $(\aleph_0 \leq \xi \leq \mu)$, then X is ξ-multiadic.*

Proof: Let f be a continuous mapping from the space $K = \prod_{i \in I} \alpha\kappa_i$ onto the space X with $\sup\{\kappa_i : i \in I\} \leq \mu$. By Gerlits's result 3.2.8, we can find a compact subspace $K' \subseteq K$, $f(K') = X, c(K') \leq \xi$. Thus for the space $K_i = \pi_i(K') \subseteq \alpha\kappa_i$, we have $c(K_i) \leq \xi$ and so there exists a set L_i, $K_i \subseteq L_i \subseteq \alpha\kappa_i$, L_i is homeomorphic to $\alpha\lambda_i$ for some $\lambda_i \leq \xi$. Naturally $K' \subseteq \prod_{i \in I} L_i$ and this shows that X is $\xi-$multiadic indeed. ■

One of the generalizing notion is of polyadicity was introduced by Gerlits in 1973. The generalized class is called ξ-adic. Denote by $W^*(\xi)$ the ordered topological space of the ordinal numbers $\leq \xi$ for an ordinal ξ. A topological space is said to be ξ-adic iff it is a continuous image of a power of the space $W^*(\xi)$. It can be seen that, for an ordinal ξ, $\alpha|\xi|$ is a continuous image of $W^*(\xi)$, so a polyadic space which is an image of $\alpha|\xi|$ is necessarily ξ-adic (corresponding for successor ordinal $\{\gamma + 1\} = (\gamma, \gamma + 1) \longmapsto \gamma$, and ∞ otherwise). Gerlits asserted that for an ordinal ξ if X is a Hausdorff ξ-adic then, $c(X) \leq \chi(X)$. By applying his result we get the following:

Corollary 3.2.10. *If X is $\mu-$multiadic and $\chi(X) \leq \xi$ such that $(\aleph_0 \leq \xi \leq \mu)$, then X is $\xi-$multiadic.*

In the following theorem we prove a result similar to an unpublished argument of Mrówka [see [23]].

Theorem 3.2.11. *Every $\mu-$multiadic space X having density character ξ is $\xi-$multiadic.*

Proof: Suppose that $\xi \geq \mu$. Since $c(X) \leq d(X)$, it is obvious that an $\mu-$multiadic space X is $\xi-$multiadic by 3.2.9. Thus let $\xi < \mu$ and let D be a dense subset of X of cardinality ξ, $\overline{D} = X$. Since X is $\mu-$multiadic, there is a continuous map f of $\prod_{i\in I} \alpha\kappa_i$ onto X for some cardinal I and cardinals $\{\kappa_i :\ i \in I\}$ satisfying $\sup\{\kappa_i :\ i \in I\} \leq \mu$. Let $S \subseteq \prod_{i\in I} \alpha\kappa_i$ so that $|S| = \xi$ and $f[S] = D$. Then $|\pi_i[S]| \leq \xi$ for every $i \in I$. Consequently, for every $i \in I$, $\pi_i[S]$ is homeomorphic to a subspace of $\alpha\lambda_i$ for some $\lambda_i \leq \xi$. Consider the product of the subspaces $\alpha\lambda_i$, $\prod_{i\in I} \alpha\lambda_i$. It is clear that $\prod_{i\in I} \alpha\lambda_i$ is a subspace of $\prod_{i\in I} \alpha\kappa_i$ with $\sup\{\lambda_i :\ i \in I\} \leq \xi \leq \mu$. Thus D is a continuous image of $S \subseteq \prod_{i\in I} \alpha\lambda_i$ with $\sup_{i\in I} \lambda_i = \xi$. Since $f[\prod_{i\in I} \alpha\lambda_i]$ is compact so it is closed. Thus $D \subseteq f[\prod_{i\in I} \alpha\lambda_i]$ and $X = f[\prod_{i\in I} \alpha\lambda_i]$. Therefore X is $\xi-$multiadic. ■

Regarding the pseudo-character of a multiadic space X, we shall now see that $w(X) = \psi\chi(X)$. This result is a similar to Esenin-Volpin's theorem for dyadic spaces and Marty's theorem for polyadic spaces.

Definition 3.2.12. The pseudo-character of a space Y *is defined to be the smallest cardinal* λ *such that for every* $y \in Y$, y *has a pseudo base of size* $\leq \lambda$. *By* pseudo base *we mean that for every* $y \in Y$ *there is a collection* $\mathcal{O}_y$ *of open subsets of* Y *for which* $\{y\} = \bigcap \mathcal{O}_y$.

The following proposition due to Marty about the pseudo-character and the weight is the main key in proving Theorem 3.2.14.

Proposition 3.2.13. *Marty [23] : Let* $\{X_i\ : i \in I\}$ *be a collection of spaces each having weight at most* μ *and let* $X = \prod_{i\in I} X_i$. *For every continuous map* f *from* X *into a space* Y *having pseudo-character* λ, *there is a subspace* X_f *of* X *of weight at most* $\max\{\mu, \lambda\}$ *and such that* $f[X_f] = f[X]$. *Moreover, if* X *is compact, then* $f[X]$ *has weight not exceeding* $\max\{\mu, \lambda\}$.

Theorem 3.2.14. *Let $\mu \leq \xi$. An $\mu-$multiadic space has pseudo-character $\leq \xi$ if and only if it has weight $\leq \xi$.*

Proof: This direction $\leftarrow$ is obvious, since the pseudo character of a space never exceeds its weight. For the other direction, let X be an $\mu-$multiadic space such that the pseudo-character of X is at most ξ. So there exists is a continuous map f from $\prod_{i\in I} \alpha\kappa_i$ onto X for some cardinals κ_i's, I with $\sup\{\kappa_i : i \in I\} \leq \mu$. Since $\prod_{i\in I} \alpha\kappa_i$ is compact and $\forall i \in I$, $\alpha\kappa_i$ has weight at most μ, we can apply Proposition 3.2.13 directly to get our desired conclusion.■

3.2.1 $\prod_{i<\lambda} \alpha\lambda_i$ is not a preimage of $(\alpha\lambda)^\lambda$ for a weakly inaccessible cardinal λ

This section is devoted to show that for a weakly inaccessible cardinal λ the polyadic space $(\alpha\lambda)^\lambda$ is not a continuous image of $\prod_{i<\lambda} \alpha\lambda_i$ where $\langle \lambda_i : i < \lambda \rangle$ is a sequence of cardinals with limit λ. Hence $(\alpha\lambda)^\lambda$ is not $\langle \lambda_i : i < \lambda \rangle$-multiadic. The key idea of the proof of this theorem is due to Gerlits [12] who relies on a cardinal function called $\hat{c}(X)$.

Definition 3.2.15. *According to Gerlits [12] the cardinal function $\hat{c}(X)$ for a topological space X is defined by*

$$\hat{c}(X) = \min\{\kappa : \textit{ if } \mathcal{U} \textit{ is a disjoint open system in } X, \textit{ then } |\mathcal{U}| < \kappa\}.$$

Recall that the cellularity of X is defined as

$$c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a disjoint open system in } X\}.$$

To clarify the difference between $c(X)$ and $\hat{c}(X)$ take the following example. If for all open disjoint families $\mathcal{U}$ in some space X, $|\mathcal{U}| \leq \aleph_n$ for some $n < \omega$ then $c(X) = \hat{c}(X) \leq \aleph_\omega$. But if there is $\mathcal{U}_\omega \subseteq X$ such that $\mathcal{U}_\omega$ is disjoint and $|\mathcal{U}_\omega| = \aleph_\omega$ then $c(X) = \aleph_\omega$ and $\hat{c}(X) = \aleph_\omega^+$. Simply, if $c(X)$ is attained i.e there is $\mathcal{U}$ a family of open disjoint sets of E of size $c(X)$, then $\hat{c}(X) = c(X)^+$. Otherwise $c(X) = \hat{c}(X)$. Hence $c(X) \leq \hat{c}(X)$.

This shows a difference between polyadic spaces and multiadic spaces. For instance, for the polyadic space $Y = (\aleph_\omega)^\omega$, $\hat{c}(Y) = \aleph_\omega^+$ and $c(Y) = \aleph_\omega$, meanwhile, for a multiadic spaces it is not necessarily that $c(X)$ and $\hat{c}(X)$ are different. Take $X = \prod_{i \in \omega} \alpha\aleph_i$, then $\hat{c}(X) = c(X) = \aleph_\omega$.

Without defining multiadic spaces, Gerlits [12] actually discusses when a multiadic space is polyadic as we can see in his following theorem:

Theorem 3.2.16. *[12] : Let $\lambda > \omega$ be a regular cardinal, a polyadic compactum X is the continuous image of a product $\prod_{i \in I} \alpha\lambda_i$ with $\lambda_i < \lambda$ $(i \in I)$ iff $\hat{c}(X) \leq \lambda$.*

Example: Suppose that $\lambda > \omega$ is a regular limit cardinal, i.e λ is weakly inaccessible. Let $\langle \lambda_i : \ i \in \lambda \rangle$ be a sequence of regular cardinals increasing to λ. By Gerlits' Theorem 3.2.16 a polyadic space X is the continuous image of a product $\prod_{i \in \lambda} \alpha\lambda_i$ iff $\hat{c}(X) \leq \lambda$. Consider $X = (\alpha\lambda)^\lambda$. For each $\gamma \in \lambda$, let $U_\gamma = \{s \in (\alpha\lambda)^\lambda : \ s(1) = \gamma\}$. Then the family $\mathcal{U} = \{U_\gamma : \ \gamma \in \lambda\}$ is a family of λ many disjoint open sets. Hence $\hat{c}(X) = \lambda^+ > \lambda$. Thus the topological space X can not be $\langle \lambda_i : \ i \in \lambda \rangle-$multiadic by Theorem 3.2.16.

Corollary 3.2.17. *For a weakly inaccessible cardinal λ, there exists no continuous map from $\prod_{i \in \lambda} \alpha\lambda_i$ onto $(\alpha\lambda)^\lambda$.*

Another version of Theorem 3.2.16 for λ which is not necessarily regular will be proved in Lemma 3.2.18.

Lemma 3.2.18. *Let $\langle \lambda_i : i \in I\rangle$ be a sequence of cardinals such that $\forall i,\ \lambda_i < \lambda$ and E is an image of $(\alpha\lambda)^I$. Then if $\hat{c}(E) \leq \lambda$ then E is $\langle \lambda_i : i \in I\rangle-$multiadic.*

Proof: Let f be a continuous mapping from the space $(\alpha\lambda)^I$ onto the space E with $\hat{c}(E) \leq \lambda$. Hence $c(E) < \lambda$. Applying Gerlits's result 3.2.8, there exists a compact subspace $K^0 \subseteq (\alpha\lambda)^I,\ f(K^0) = X$ with $c(K^0) < \lambda$. Thus for the projection map $\pi_i(K^0) = K_i \subseteq \alpha\lambda$, we have $c(K_i) < \lambda$ and so there exists a compact subspace L_i such that $K_i \subseteq L_i \subseteq \alpha\lambda$. L_i is homeomorphic to $\alpha\lambda_i$ for some $\lambda_i < \lambda$. Therefore, $K^0 \subseteq \prod_{i\in I} \alpha\lambda_i$ and this shows that X is $\langle \lambda_i : i \in I\rangle-$multiadic indeed. ■

Note that Lemma 3.2.18 is weaker than the analogue of Theorem 3.2.16, so for example we can not use it to prove that $(\alpha\aleph_\omega)^\omega$ is not an image of $\prod_{n<\omega} \alpha\aleph_n$.

Corollary 3.2.19. *For any topological space E, if $c(E) = \lambda = \hat{c}(E)$, then E is a polyadic space which is a continuous image of $(\alpha\lambda)^I$ iff E is $\langle \lambda_i : i \in I\rangle-$multiadic with $\lambda_i \leq \lambda$.*

3.3 The class of multiadic compacta

Due to Gerlits [13], the class of polyadic compacta is the smallest class containing $D(1)$, closed with respect to continuous mappings and topological products of compact spaces and such that for any system $\{R_i : i \in I\}$ of polyadic spaces there exists a polyadic space which is a compactification of the topological sum $\sum_{i\in I} R_i$. In this section we identify the class of multiadic spaces $\mathscr{MC}$ with the class of polyadic spaces! This is actually already contained in Corollary 2.1.2, but here we give an alternative proof which at the same time satisfies the family properties of the class $\mathscr{MC}$.

From the result 3.2.17 in the previous section, we split the polyadic class $\mathscr{PC}$ of fixed λ into two kinds of spaces. One that are $\langle\lambda_i:\ i\in I\rangle$-multiadic and spaces that are not $\langle\lambda_i:\ i\in I\rangle$-multiadic for a sequence of cardinals $\langle\lambda_i:\ i\in I\rangle$.

Theorem 3.3.1. *$\mathscr{MC}$ is the smallest class $\mathcal{O}$ of topological spaces such that:*
a. $D(1)\in\mathcal{O}$, where $D(1)$ is the discrete space of one point;
b. $\mathcal{O}$ is closed under arbitrary topological products;
c. $\mathcal{O}$ is closed under continuous Hausdorff images;
d. Given any system $\{R_i;\ i\in I\}\subseteq\mathcal{O}$ there exists a space $R\in\mathcal{O}$ which is a compactification of the topological sum $\sum_{i\in I} R_i$.

Proof: Firstly, $\Rightarrow$: We should prove that $\mathscr{MC}$ satisfies $(a)-(d)$. Here we only need to prove only (d) as we have proved $(b),(c)$ in Theorem 2.1.7, 2.1.6 respectively. Suppose we have a system $\{R_i;\ i\in I\}\subseteq\mathcal{O}$ such that for each $i\in I$ there is a continuous map $f_i:\prod_{j\in J_i}\alpha\lambda_j^i\to R_i$. Let $J=\bigcup\big\{J_i: i\in I\big\}$, and $\lambda_j=\sup\big\{\lambda_j^i:\ j\in J_i\big\}$ for each $j\in J$. Now for each $i\in I$:

$$\prod_{j\in J}\alpha\lambda_j\twoheadrightarrow\prod_{j\in J_i}\alpha\lambda_j^i\twoheadrightarrow R_i.$$

Denote by R the one-point compactification of the topological sum $\sum_{i\in I} R_i$, say $R=\sum_{i\in I} R_i\cup\{\infty_R\}$. Hence R is the continuous image of the product space $\alpha|I| \times \prod_{j\in J} \alpha\lambda_j$ given by the following map

$$\big(i, (x_j)_{i\in J}\big) \;\mapsto\; f_i\big((x_j)_{j\in J}\big) \;\;\&\;\; \big(\infty, (x_j)_{j\in J}\big) \;\mapsto\; \infty_R.$$

Since $\mathcal{O}$ is closed under arbitrary topological products and under continuous Hausdorff images, $R \in \mathscr{MC}$.

Secondly, $\Leftarrow$: Gerlits [13] proved that any element of $\mathcal{O}$ is in $\mathscr{PC}$ so it is in $\mathscr{MC}$. ■

From Theorem 3.3.1 we conclude that the class of multiadic compacta $\mathscr{MC}$ is the same as the class of polyadic compacta $\mathscr{PC}$ that was introduced by Gerlits [13]. This is because [13] proved that $\mathscr{PC}$ is exactly the class $\mathcal{O}$ mentioned in Theorem 3.3.1. So $\mathscr{MC} = \mathscr{PC}$. Therefore we have the following:

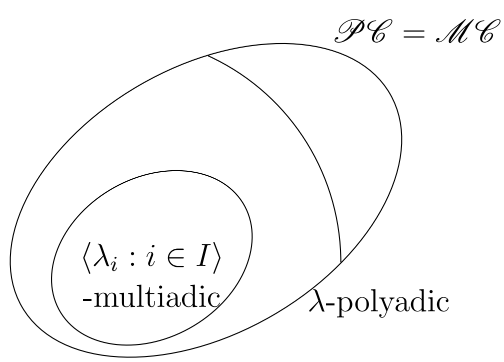

Figure 3.1: A representation of $\mathscr{MC}$

For each $X \in \mathscr{PC}$, let $\lambda = \min\big\{\lambda : \; (\alpha\lambda)^\tau \to X \text{ for some } \tau\big\}$. Suppose there is a sequence of cardinals $\langle\lambda_i : \; i \in I\rangle$ such that $\sup\langle\lambda_i : \; i \in I\rangle = \lambda$. It does not follow that X is $\langle\lambda_i : \; i \in I\rangle$-multiadic. This means that for each

λ, the class of λ-polyadic spaces is divided into 2 disjoint subclasses. First, those spaces which are $\langle \lambda_i : i \in I \rangle$-multiadic for a sequence of cardinals $\lambda_i < \lambda$, $\langle \lambda_i : i \in I \rangle$, with limit λ, while for the other one there exists no such sequence.

Chapter 4

Ramsey theoretic graphs and associated spaces

Ramsey's Theorem has been generalized in many ways, giving rise to an area of combinatorial mathematics known as Ramsey theory. In this chapter we are focusing on studying some Ramsey theoretic properties that are satisfied by multiadic compacta. The properties are called Q_λ, R_λ and property W_1. These notions were introduced by Bell [4] and Mrówka [24] respectively. We also study properties K_λ and S_λ. We give the generalizations of the Standard Sierpiński graph and use them to give examples of polyadic spaces that do not satisfy K_λ for various regular cardinals λ. Furthermore we show that under GCH the two properties S_λ and K_λ are equivalent for any topological space X and any regular cardinal $\lambda > \omega_1$.

4.1 Properties R_λ and Q_λ

Inspired by an argument by Bell in [4], we in this section prove a Ramsey-like property for multiadic spaces, called property Q_λ. This property for Boolean spaces is equivalent to property R_λ. Before starting our work, it is convenient to present some definitions and results that will be used in this section and a proof of one fundamental theorem which will be used a couple of times.

Definition 4.1.1. *For $n < \omega$, a collection $\mathcal{O}$ of sets is* n-linked *if for each $\mathcal{O}' \subseteq \mathcal{O}$ with $|\mathcal{O}'| = n$, $\bigcap \mathcal{O}' \neq \emptyset$. We abbreviate 2-linked by linked.* A Δ system *is a collection $\mathcal{O}$ of sets for which there exists a set R (called the root of the Δ-system) such that if A and B are any two distinct elements of $\mathcal{O}$, then $A \cap B = R$. A standard fact is the following lemma due to Shanin:*

Lemma 4.1.2. (Delta system Lemma) *Suppose that S is a set of finite sets such that $cf(|S|) \geq \omega_1$. Then there is a $S' \subseteq S$ such that $|S'| = |S|$ and S' is a Δ-system.*

Proof: For any element of S there is a natural number n which is the cardinality of that element. Since the cofinality of $|S|$ is at least ω_1, there must be some n and S^* with $|S^*| = |S|$ such that $a \in S^* \Rightarrow |a| = n$. By induction on n, we shall show that the lemma holds. Let $\lambda = |S|$.

The trivial case $n = 0$, when the only set in S^* is the empty set, can not happen as S^* is supposed to be uncountable.

If $n = 1$ then each element of S^* is distinct, and has no intersection with the others, so $R = \emptyset$ and $S' = S^*$.

Suppose $n > 1$. If there is some x which is in λ many of S^* then take $S^{**} = \{a \backslash \{x\} : x \in a \in S^*\}$. Obviously this has size λ and every element has

$n-1$ elements, so by the induction hypothesis there is some $S' \subseteq S^{**}$ of size λ such that the intersection of any two elements is same fixed R. Obviously $\{a \cup \{x\} : a \in S'\}$ satisfies the lemma, since the intersection of any two elements is $R \cup \{x\}$.

On the other hand, if there is no such x then we can inductively construct a sequence $\langle a_i : i < \lambda \rangle$ such that each $a_i \in S^*$ and for any $i \neq j, a_i \cap a_j = \emptyset$. Take any element for a_0. For the given sequence $\langle a_i : i < \alpha \rangle$, consider $A = \bigcup_{i<\alpha} a_i$. Since $\alpha < \lambda$, $|A| < \lambda$ and each a_i is finite. Obviously each element of A is in only $< \lambda$ elements of S^* so there are λ many of elements of S^* which are candidates for a_α and we can continue. Since the intersection of any two elements of the constructed sequence is $\emptyset$, this sequence satisfies the lemma. ■

Notation: For an infinite cardinal λ, we write $\lambda \to (l_1, \cdots, l_r)^2$ if whenever the doubletons of λ, i.e. $[\lambda]^2$, are partitioned into sets $A_1, \cdots, A_r$, then there is $1 \leq i \leq r$ and a subset C of λ with cardinality l_i which is homogeneous for A_i, i.e., $[C]^2 \subseteq A_i$. In the case $l_1 = \cdots = l_r$ we use the shorthand $\lambda \to (l)^2_r$.

The above partition calculus arrow notation is very powerful in giving a unified expression to the two following fundamental results in Combinatorics, see e.g.[22].

Theorem 4.1.3. *(Ramsey theorem) For any positive integer n, we have*

$$\omega \to (\omega)^2_n.$$

Theorem 4.1.4. *(Dushnik-Miller) For an infinite regular cardinal λ, we have*

$$\lambda \to (\lambda, \omega)^2.$$

Definition 4.1.5. *(Property R_λ) Let λ be a cardinal. We say that a space has* property R_λ *if every family of its clopen sets of cardinality λ has a subfamily of cardinality λ which is either linked or disjoint.*

The following lemma was asserted without proof by Bell in [4].

Lemma 4.1.6. *(Bell [4]) Let $cf(\lambda) \geq \omega_1$ and let $\{X_i : i \in I\}$ be a collection of Boolean spaces such that every finite product of them satisfies property R_λ. Then $\prod_{i \in I} X_i$ has property R_λ.*

Proof: Suppose that there is a collection $\mathcal{O} = \{O_\alpha : \alpha < \lambda\}$ of clopen sets in $\prod_{i \in I} X_i$ and we need to find a subcollection of $\mathcal{O}$ of cardinality λ which is either linked or disjoint. All O_α's are compact and therefore each O_α's is the finite union of basic open sets. Thus all clopen subsets of the product of Boolean spaces only depend of finitely many coordinates. Therefore for each α, there exists a finite set $F_\alpha \subseteq I$ such $O_\alpha = \prod_{i \in I} U_i^\alpha$ and for $i \in F_\alpha$ $U_i^\alpha \neq X_i$ and $U_i^\alpha = X_i$ otherwise.
Consider $\mathcal{G} = \{F_\alpha : \ \alpha < \lambda\}$. This is a collection of finite sets, so it contains a Δ-system of size λ, by Lemma 4.1.2. Say $\mathcal{D} \subseteq \mathcal{G}$ is a Δ-system of size λ with root R, and let $\mathcal{D} = \{F_\alpha : \alpha \in A\}$ for some set A of size λ. Now we will work with the projections of the O_α's to the product $\prod_{i \in R} X_i$. Clearly these projections are clopen in $\prod_{i \in R} X_i$. On this finite product $\prod_{i \in R} X_i$ we can thin out the collection of clopen sets so that we left with λ sets that are either linked or disjoint and therefore, to complete the proof we only need to go back to the full space. The corresponding subcollection of the O_α will have the desired property.

Since the family $\{\prod_{i\in R} U_i^\alpha : \alpha \in A\}$ has a subcollection of size λ which is either linked or disjoint, we have to study two cases:

Firstly, suppose that $B \subseteq A$ is of size λ and $\{\prod_{i\in R} U_i^\alpha : \alpha \in B\}$ is a linked subcollection of $\{\prod_{i\in R} U_i^\alpha : \alpha \in A\}$. So for all $\alpha \neq \beta \in B$, there is an element $z \in (\prod_{i\in R} U_i^\alpha \cap \prod_{i\in R} U_i^\beta)$. Thus we can extend the range of z to be an element of $\prod_{i\in I} X_i$ as follows:

For $i \in I\backslash R$ we have $U_i^\alpha = X_i$ or $U_i^\beta = X_i$. Certainly $U_i^\alpha \cap U_i^\beta \neq \emptyset$. Let $u(i) \in U_i^\alpha \cap U_i^\beta$ and define $y \in \prod_{i\in I} X_i$ by putting

$$y(i) = \begin{cases} z(i) & \text{if } i \in R \\ u(i) & \text{otherwise.} \end{cases}$$

So, $y \in O_\alpha \cap O_\beta$ and hence the collection $\{O_\alpha : \alpha \in B\}$ is a linked subcollection of $\prod_{i\in I} X_i$.

Secondly, suppose that for some $A^* \in [A]^\lambda$ and for any $\alpha \neq \beta$ in A^* we have $\prod_{i\in R} U_i^\alpha \cap \prod_{i\in R} U_i^\beta = \emptyset$. Since

$$\prod_{i\in R} U_i^\alpha \cap \prod_{i\in R} U_i^\beta = \emptyset \Longrightarrow \prod_{i\in I} U_i^\alpha \cap \prod_{i\in I} U_i^\beta = \emptyset \Longrightarrow O_\alpha \cap O_\beta = \emptyset;$$

therefore $\{O_\alpha : \alpha < A^*\}$ is a disjoint subcollection of contradiction with the fact that there is no disjoint subcollection of $\mathcal{O}$ of size λ. ■

Theorem 4.1.7. *For any sequence $\langle \kappa_i : i \in I\rangle$ of cardinals and any regular cardinal $\lambda > \omega$, every finite product of one-point compactification spaces from $\langle \alpha\kappa_i : i \in I\rangle$ has property R_λ.*

Proof: This result is implied by Bell's Theorem [4] that asserts every finite product of $\alpha\kappa$ has property R_λ, as R_λ is an imaging property. However, we include a proof here as the proof given by Bell is quite sketchy. For every finite subset $S \subseteq I$, we shall prove that $\prod_{s\in S} \alpha\kappa_s$ has property R_λ. Let $\mathcal{B} = \{b_\beta : \beta < \lambda\}$ be a collection of clopen sets of $\prod_{s\in S} \alpha\kappa_s$ of cardinality λ. Assume that $\mathcal{B}$ does not contain a linked subfamily of cardinality λ and we try to construct a disjoint subfamily of cardinality λ. For each $\beta < \lambda$, b_β is the union of finitely many, say $m_\beta < \omega$, basic clopen sets r_i^β in $\prod_{s\in S} \alpha\kappa_s$. Without loss of generality let us say $m_\beta = m$ is fixed. Let $n = |S|$. Since a basic clopen subset of $\alpha\kappa_s$ is either a finite or a co-finite set, b_β can be written in the following form:

$$b_\beta = \bigcup_{i<m} r_i^\beta = \bigcup_{i<m} \prod_{k<n} r_i^\beta(k)$$

where for all $i < m$ and all $k < n$ either $r_i^\beta(k)$ is a finite subset of κ_i which without loss of generality can be assumed to be of a constant size for every $\beta < \lambda$ or $r_i^\beta(k) \cap \kappa_i$ is a co-finite subset of κ_i for every $\beta < \lambda$ and $|\kappa_i \backslash r_i^\beta(k)|$ is fixed for each k.

Define an indicator function $I : m \times n \to \{0, 1\}$

$$I(i,k) = \begin{cases} 0 & \text{if } r_i^\beta(k) \text{ is a finite subset of } \kappa_k \\ 1 & \text{if the complement of } r_i^\beta(k) \text{ is a finite subset of } \kappa_k. \end{cases}$$

For both cases, we apply a Δ-system lemma for each $i < m$ and $k < n$, using the fact that λ is uncountable cofinality. Let us assume that R_{ik} is the root for $\{r_i^\beta(k) : \beta < \lambda\}$ where $I(i,k) = 0$ and if $I(i,k) = 1$, let R'_{ik} be the root for $\{\kappa \backslash r_i^\beta(k) : \beta < \lambda\}$. Before we complete this proof we have to prove the following claim:

Claim 4.1.8. *For all $i, j < m$ there exists $H \subseteq \lambda$ with cardinality λ such that $\beta < \gamma$ in H implies $r_i^\beta \cap r_j^\gamma = \emptyset$.*

Proof of the claim: First fix $i, j < m$ and then define a case function $\psi : n \to \{1, 2, 3, 6\}$ by $\psi(k) = 2^{I(i,k)} 3^{I(j,k)}$.

Let (S1) be the assumption that for all $k < n$ with $\psi(k) = 3$,

$$\neg[R_{ik} \subseteq R'_{jk} \; \& \; (\forall \beta < \lambda) \; r_i^\beta(k) = R_{ik}].$$

We can without loss of generality assume (S1), otherwise the claim holds for i, j immediately with $H = \lambda$: If not (S1) then there exist $k < n$ with $\psi(k) = 3$ such that $R_{ik} \subseteq R'_{jk}$ and $R_{ik} = r_i^\beta(k)$ $(\forall \beta < \lambda)$. Thus $r_i^\beta(k) \subseteq R'_{jk}$. Also, $r_j^\gamma(k) \cap R'_{jk} = \emptyset$ for all γ as R'_{jk} is the root of the complements of $\{r_j^\gamma(k) : \gamma < \lambda\}$. Hence $r_i^\beta \cap r_j^\gamma = \emptyset$ for all $\beta, \gamma \in \lambda$ and by putting $H = \lambda$ the claim is satisfied.

Similarly, we can assume (S2) for all $k < n$ with $\psi(k) = 2$, where (S2) is the assumption

$$\neg[R_{jk} \subseteq R'_{ik} \; \& \; (\forall \beta < \lambda) \; r_j^\beta(k) = R_{jk}].$$

Finally, we are going to do a case analysis: Define a subset $P \subseteq [\lambda]^2$ such that $\{\beta < \gamma\} \in P$ iff $r_i^\beta \cap r_j^\gamma \neq \emptyset$. We have that $\lambda \to (\lambda, \omega)^2$ by Theorem 4.1.4. Since λ is uncountable from the main hypothesis, there is no subcollection of λ of cardinality λ that is homogeneous for P. Therefore there is a countably infinite set $A \subseteq \lambda$ with $[A]^2 \cap P = \emptyset$. Since $\omega \to (\omega)^2_n$, we get $k < n$ and an infinite $B \subseteq A$ such that $\beta < \gamma$ in B implies $r_i^\beta(k) \cap r_j^\gamma(k) = \emptyset$. We are going to show that $\psi(k) = 1$ by case analysis.

1- Clearly, $\psi(k) \neq 6$: Otherwise $I(i, k)$ and $I(j, k)$ are both equal to 1 and hence $r_i^\beta(k)$ and $r_j^\gamma(k)$ are co-finite. If $r_i^\beta(k) \cap r_j^\gamma(k) = \emptyset$ then $r_j^\beta(k) \subseteq \kappa \backslash r_i^\gamma(k)$ and thus $r_j^\beta(k)$ is finite; a contradiction.

2- $\psi(k) \neq 2$: Suppose not: From our assumption $r_i^\beta(k) \cap r_j^\gamma(k) = \emptyset$. Thus means that $r_j^\gamma(k) \subseteq \kappa \backslash r_i^\beta(k)$, so we have $R_{jk} \subseteq R'_{ik}$. Let $|R'_{ik}| = r$. Choose $C = \{\beta_1, \beta_2\} \subseteq B$, $D = \{\gamma_1, \gamma_2, \cdots, \gamma_{r+1}\} \subseteq B$, such that if $\beta \in C$ and $\gamma \in D$ then $\beta < \gamma$. Thus if $\beta \in C$, we have:

$$r_j^{\gamma_1}(k) \subseteq \kappa \backslash r_i^\beta(k)$$

$$\vdots$$

$$r_j^{\gamma_{r+1}}(k) \subseteq \kappa \backslash r_i^\beta(k)$$

and hence, $\bigcup_{\gamma \in D} r_j^\gamma(k) \subseteq \kappa \backslash r_i^\beta(k)$. But this is true for all β's in C, so we have:

$$\bigcup_{\gamma \in D} r_j^\gamma(k) \subseteq \kappa \backslash r_i^{\beta_1}(k) \text{ and,}$$

$$\bigcup_{\gamma \in D} r_j^\gamma(k) \subseteq \kappa \backslash r_i^{\beta_2}(k).$$

By taking the intersection between them, we get:

$$\bigcup_{\gamma \in D} r_j^\gamma(k) \subseteq (\kappa \backslash r_i^{\beta_1}(k)) \cap (\kappa \backslash r_i^{\beta_2}(k)). \qquad (*)$$

This gives us $\bigcup_{\gamma \in D} r_j^\gamma(k) \subseteq R'_{ik}$. From (S2) since $\psi(k) = 2$ and $R_{jk} \subseteq R'_{ik}$ so not the second part of (S2). This means that there exist $\delta < \lambda$ such that $R_{jk} \neq r_j^\delta(k)$. So R_{jk} is a proper subset of $r_j^\delta(k)$. Thus $|R_{jk}| < |r_j^\delta(k)|$, i.e. there is at least one extra element in $r_j^\beta(k)$ which is not on R_{jk}. But R_{jk} is a root for all $r_j^\gamma(k)$ where $\gamma < \lambda$. Also from the assumption that all $r_j^\gamma(k)$

of constant size, we get that $|R_{jk}| < |r_j^\delta(k)| = |r_j^\gamma(k)|$ for all $\gamma < \lambda$. Hence $|\bigcup_{\gamma \in D} r_j^\gamma(k)| > r+1$ which implies that $|R'_{ik}| \geq r+1$; a contradiction with(*).

3- Using similar method with (S1) we show that $\psi(k) \neq 3$.

4- Hence, $\psi(k) = 1$ which means that both R_{ik} and R_{jk} are roots for $\{r_i^\beta(k) : \beta < \lambda\}$ and $\{r_j^\gamma(k) : \gamma < \lambda\}$ respectively. But $r_i^\beta(k) \cap r_j^\gamma(k) = \emptyset$ so, $R_{ik} \cap R_{jk} = \emptyset$. We now apply thinning to complete the proof of the claim. Since there are only finitely many β's in λ such that $r_i^\beta(k) \cap R_{jk} \neq \emptyset$, remove these β's. The remaining β's are such that $r_i^\beta(k) \cap R_{jk} = \emptyset$. For each remaining β there exist only finitely many $\gamma > \beta$ with $r_i^\beta(k) \cap r_j^\gamma(k) \neq \emptyset$. Finally produce inductively a set $H \subseteq \lambda$ of cardinality λ such that $\gamma > \beta$ in H implies $r_i^\beta(k) \cap r_j^\gamma(k) = \emptyset$. This proves the claim.

To complete the proof of this Lemma, just apply the claim m^2 times inductively to get a subset $K \subseteq \lambda$ with cardinality λ such that $\beta < \gamma$ in K implies $b_\beta \cap b_\gamma = \emptyset$. ■

Definition 4.1.9. (*Property Q_λ*) *Let λ be an uncountable regular cardinal. We say that* a compact space X satisfies property Q_λ *if for every λ and every family $\{U_\alpha, V_\alpha\}_{\alpha<\lambda}$ of open subsets of X with $\overline{U}_\alpha \subseteq V_\alpha$, there exists an $A \subseteq \lambda$ with $|A| = \lambda$, such that either $\{V_\alpha : \alpha \in A\}$ is linked or $\{U_\alpha : \alpha \in A\}$ is disjoint.*

In a Boolean space properties Q_λ and R_λ are equivalent as a clopen set B_α can be placed between any open sets U and V such that $\overline{U} \subseteq V$.

Theorem 4.1.10. *For each regular cardinal $\lambda > \omega$, property Q_λ is satisfied by all multiadic spaces.*

Proof: This follows from Bell's theorem in [4] which asserts that the property Q_λ is satisfied by all polyadic spaces and our theorem Theorem 3.3.1 which shows $\mathscr{MC} = \mathscr{PC}$. It can also be proved directly using our Theorem 4.1.7. Namely: Let $\lambda > \omega$ be a regular cardinal and X a Hausdorff space such that for some cardinals κ_i and τ, X is a continuous image of $\prod_{i\in\tau} \alpha\kappa_i$. Since property Q_λ is an imaging property [4], it suffices to show that $\prod_{i\in\tau} \alpha\kappa_i$ has property Q_λ. Since for all n, $\alpha\kappa_n$ are Boolean spaces, by Lemma 4.1.6 it suffices to show that every finite product of one-point compactification $\alpha\kappa_i$ has property R_λ. This was proved in Theorem 4.1.7 for any cardinal κ, and for any regular cardinal $\lambda > \omega$. ■

4.2 Property K_λ for X and $H(X)$

We have seen in in section 4.1 Properties Q_λ and R_λ which dealt with pairs of open sets or clopen sets. In this section we consider property K_λ that deals with a single family of open sets. We show that the family $H(X)$ of non empty closed subsets of a regular ccc space X endowed with Vietoris topology has property K_λ if $H(X)$ fulfills property Q_λ. This is a generalization of a result of Bell who proved it for $\lambda = \omega_1$.

Definition 4.2.1. (Vietoris topology): *Let (X, τ) be a topological space and let $H(X)$ be the family of the non-empty closed subsets of X.* The Vietoris topology τ_1 on $H(X)$ *depends only on the topological structure on X. Its base is defined by letting it have a basis consisting of all collections of the form*

$$\langle \mathcal{V} \rangle = \{F \in H(X) : (\forall\ V \in \mathcal{V}) \quad F \cap V \neq \emptyset \quad \wedge \quad F \subseteq \bigcup \mathcal{V}\}$$

where $\mathcal{V}$ runs over the finite families of open subsets of X.

Lemma 4.2.2. *If $H(X)$ satisfies Property Q_λ then so does X.*

proof: Let $\{U_\alpha, V_\alpha\}_{\alpha<\lambda}$ be a family of pairs of open sets in X such that $\overline{U_\alpha} \subseteq V_\alpha$. For each $\alpha < \lambda$, consider $\langle U_\alpha \rangle$ and $\langle V_\alpha \rangle$ which are basic open sets in $H(X)$

Claim: $\overline{\langle U_\alpha \rangle} \subseteq \langle V_\alpha \rangle$.
It suffices to show that $\forall F \notin \langle V_\alpha \rangle$ there is an open set contains F and disjoint from $\langle U_\alpha \rangle$. Given such F such that $F \notin \langle V_\alpha \rangle$. Consider two cases: $F \cap V_\alpha = \emptyset$ and $F \cap V_\alpha \neq \emptyset$.

1. If $F \cap V_\alpha = \emptyset$ then $F \cap \overline{U_\alpha} = \emptyset$. This means $F \subseteq X \backslash \overline{U_\alpha}$ and $F \in \langle X \backslash \overline{U_\alpha} \rangle$, hence $\langle X \backslash \overline{U_\alpha} \rangle \cap \langle U_\alpha \rangle = \emptyset$.

2. Let $F \cap V_\alpha \neq \emptyset$. Since from our assumption $F \nsubseteq V_\alpha$, we get $F \in \langle X \backslash \overline{U_\alpha},\ V_\alpha \rangle$, hence $\langle X \backslash \overline{U_\alpha},\ V_\alpha \rangle \cap \langle U_\alpha \rangle = \emptyset$. These two cases prove that $\overline{\langle U_\alpha \rangle} \subseteq \langle V_\alpha \rangle$.

Let $A \in [\lambda]^\lambda$ be a subcollection of $\{\langle U_\alpha \rangle, \langle V_\alpha \rangle\}_{\alpha<\lambda}$ that has for each $\alpha, \beta \in A$ with $\alpha \neq \beta$ either $\langle U_\alpha \rangle \cap \langle U_\beta \rangle = \emptyset$ or $\langle V_\alpha \rangle \cap \langle V_\beta \rangle \neq \emptyset$. Suppose $\{\langle U_\alpha \rangle\}_{\alpha \in A}$ are pairwise disjoint. We shall show that the family $\{U_\alpha\}_{\alpha \in A}$ is pairwise disjoint in X. Suppose not, if $\alpha \neq \beta \in A$ and $U_\alpha \cap U_\beta \neq \emptyset$ then $U_\alpha \cap U_\beta$ is non-empty open set. Let $x \in U_\alpha \cap U_\beta$, hence $\{x\}$ is closed and so $\{x\} \in H(X)$. We can easily see that $\{x\} \in \langle U_\alpha \rangle \cap \langle U_\beta \rangle$, which a contradiction. So $\{U_\alpha\}_{\alpha \in A}$ are pairwise disjoint.

Now suppose that $\{\langle V_\alpha \rangle\}_{\alpha \in A}$ are linked. Let $\alpha \neq \beta$ and $\langle V_\alpha \rangle \cap \langle V_\beta \rangle \neq \emptyset$. Let $F \in \langle V_\alpha \rangle \cap \langle V_\beta \rangle$, hence $F \subseteq V_\alpha, F \subseteq V_\beta$, so $F \subseteq V_\alpha \cap V_\beta,\ F \neq \emptyset$. Hence $V_\alpha \cap V_\beta \neq \emptyset$. Therefore $\{V_\alpha\}_{\alpha \in A}$ is linked. ■

Definition 4.2.3. (Hyper-extendible): *Property P is called* hyper-extendible *if it transfers from a Hausdorff space to its Vietoris hyperspace of all non-empty closed subsets, i.e. if X has property P then so does $H(X)$.*

Definition 4.2.4. (Property K_λ) *Let λ be an uncountable regular cardinal. A Hausdorff space has* property K_λ*, the property of Knaster, if every collection of its open sets of cardinality λ contains a linked subcollection of size λ.*

Lemma 4.2.5. *If X is a Hausdorff space and λ is a cardinal with $cf(\lambda) > \omega$ then Property K_λ is hyper-extendible.*

Proof: Suppose that a Hausdorff space X has property K_λ and let us prove that so does $H(X)$. Let $\mathcal{O} = \{O_\alpha : \alpha < \lambda\}$ be a collection of open sets in $H(X)$ of size λ. We have to find a subcollection of $\mathcal{O}$ of size λ such that each two have non-empty intersection. We can without loss of generality shrink O_α's to smaller sets. Let us assume that each O_α is a basic open set $\langle V_i^\alpha : i < n_\alpha \rangle$. Moreover, we can thin out and assume that all n_α are the same value n. Here each V_i^α is an open set from X.

By induction on $i < n$ we choose A_i's each of λ size such that $A_0 \supset A_1 \supset \cdots A_{n-1}$ and for all $i < n$, the family $\{V_i^\alpha : \alpha \in A_i\}$ satisfies that each two have a nonempty intersection. Each time we simply apply the property K_λ of X.

At the end we claim that $\{O_\alpha : \alpha \in A_{n-1}\}$ is a family of pairwise non-disjoint sets in $H(X)$. So let α and β be from A_{n-1}. Hence for each $i < n$ we have $V_i^\alpha \cap V_i^\beta$ is nonempty, say has a point x_i. Then $F = \{x_i : i < n\}$ is a nonempty closed set in X as every finite set in a Hausdorff space is closed. Thus F is an element of $H(X)$ and it is easy to see that F is in $O_\alpha \cap O_\beta$. ■

Lemma 4.2.6. *For any cardinal λ such that $cf(\lambda) > \omega$, a regular ccc Hausdorff space X with Property Q_λ has Property K_λ.*

Proof: Let $\mathcal{C} = \{V_\alpha : \ \alpha < \lambda\}$ be a collection of open sets in X of size λ. Since X is a regular space then for each $\alpha < \lambda$ there exist an open set U_α such that $\overline{U_\alpha} \subseteq V_\alpha$. Since X satisfies property Q_λ, there exists an $A \subseteq \lambda$ with $|A| = \lambda$, such that either $\{V_\alpha : \alpha \in A\}$ is linked or $\{U_\alpha : \alpha \in A\}$ is disjoint. But X is *ccc* so, the cardinality of $\{U_\alpha \ : \alpha \in A\}$ is of most countable and therefore the first possibility must occur. Hence, $\{V_\alpha : \alpha \in A\}$ is a linked subcollection of $\mathcal{C}$, which proves that X has Property K_λ. ■

Corollary 4.2.7. *For any cardinal λ such that $cf(\lambda) > \omega$, if X is a regular ccc space and $H(X)$ has property Q_λ then $H(X)$ has property K_λ.*

Since $H(X)$ has property Q_λ so by Lemma 4.2.2 X has property Q_λ. But a regular *ccc* space with Property Q_λ has Property K_λ (Lemma 4.2.6), X has Property K_λ. Since Property K_λ is hyper-extendible (Lemma 4.2.5), $H(X)$ has Property K_λ. ■

Corollary 4.2.8. *Assume that X and $H(X)$ are as in corollary 4.2.7. Then $H(X)$ has λ-cc, where λ is a cardinal such that $cf(\lambda) > \omega$.*

Remark 4.2.9. *In the case where $\lambda = \omega_1$, Corollary 4.2.8 implies that $H(X)$ is ccc.*

4.3 Property W_1

In this section we address the existence of compact spaces that are not multiadic. Since we have shown $\mathscr{PC} = \mathscr{MC}$, in fact we only to discuss if all compact spaces are polyadic. This is not the case by results of Mrówka in [24] who gave examples of compact spaces which are not polyadic. For completeness we include his proofs as they also fit in general context of applying combinatorial properties of sequences to make conclusion about topological spaces. Mrówka's proof depended on a property called K_1. To avoid a confusion between K_λ for a regular cardinal λ and K_1 we shall denote it by W_1.

Definition 4.3.1. *Let X be a topological space. Given a subset $A \subseteq X$ of a space X,* the sequential closure $[A]_{seq}$ *is the set*

$$[A]_{seq} = \{x \in X : \exists \text{ a sequence } \{a_n\} \to x, a_n \in A\}$$

that is, the set of all points $x \in X$ for which there is a sequence in A that converges to x.

Definition 4.3.2. *(Property W_1) A topological space X satisfies Property W_1 if the closure of the union of arbitrarily many G_δ sets of X coincides with its sequential closure.*

Proposition 4.3.3. *Suppose that X is a compact space with property W_1 and Y is a continuous image of X, then Y has property W_1.*

Proof: Suppose that $f : X \to Y$ is a continuous surjection, in particular, it is a closed map by the compactness of X. Let $\mathcal{A}$ be a family of G_δ sets in Y, by continuity, $\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$ is a family of G_δ sets. Let B^* be the union of $\mathcal{B}$, by the assumption the closure C of B^* is equal to its sequential closure. By the closeness of the map f, $f(C)$ is the closure of the union of $\mathcal{A}$. Call this union A^*. If y is in $f(C)$, then $f^{-1}(y)$ is in C and hence there is a sequence $(x_n)_n$ from $B*$converging to $f^{-1}(y)$. Then $f(x_n)$ is a sequence from A^* converging to y, hence $f(C)$ the closure of A^* is equal to its sequential closure. ■

By applying the above proposition and Mrówka's theorem [24] that states the product of compact spaces $\prod_{i \in I} \alpha\kappa_i$ verifies property W_1, we get the following theorem.

Theorem 4.3.4. *Property W_1 is satisfied by all multiadic spaces.*

The above assertion can be used to provide some examples of various spaces that are not multiadic or polyadic. We first introduce the following definition and lemma which are needed to explain our examples.

Definition 4.3.5. *Let X be a linearly ordered set by $<$, the order topology on X is generated by the subbase of "open rays"*

$$(a, \infty) = \{x \mid a < x\}$$

$$(-\infty, b) = \{x \mid x < b\}$$

for all $a, b \in X$. This is equivalent to saying that the open intervals

$$(a, b) = \{x \mid a < x < b\}$$

together with the above rays form a base for the ordered topology. The open sets in X are the sets that are a union of (possibly infinitely many) such open intervals and rays.
An element $u \in X$ is called the least upper bound of a subset $A \subseteq X$, $u = supA$, if $x \leq u$ for every $x \in A$ and if any $v \in X$ satisfying $x \leq v$ for every $x \in A$ also satisfies the inequality $u \leq v$. The greatest lower bound of a subset $A \subseteq X$ is defined analogously.

Lemma 4.3.6. *A map from the hyperspace $H(X)$ of a compact ordered space X defined by $A \mapsto \sup(A)$ is well defined.*

Proof: For each $A \in H(X)$, suppose that A has no maximal element. That means for every $a \in A$, there exists $b \in A$ such that $b > a$. The collection $\{(\infty, a) : \ a \in A\}$ covers A. Since a closed set of a compact space is compact, A is compact. Therefore there is a finite set $C \subseteq A$ such that $\bigcup_{a \in C} (\infty, a) = A$. Take $c = \max C$ and hence $A \subseteq (\infty, c)$; a contradiction because $c \in A$. Therefore $\exists \ a \in A$ such that $\forall \ b \in A, \ b \leq a$. Hence A has a maximal element and $\sup(A) = \max(A)$ is well defined. ■

Lemma 4.3.7. *If X is a compact ordered space, then X is a continuous image of $H(X)$.*

Proof: Define a function Φ that maps a closed subset A of X to $\sup(A)$. So $\Phi : \ H(X) \to X$. This is well defined by Lemma 4.3.6. To show that Φ is continuous, let (a, b) be an open set in X. Then we have to assert that $\Phi^{-1}((a, b))$ is open in $H(X)$. Let C be a closed in X such that $C \in \Phi^{-1}((a, b))$ and $\Phi(C) = \gamma$. So $\gamma \in (a, b)$. Hence $C \cap (a, b) \neq \emptyset$. Let

$$\mathcal{A} = \{A : \ A \text{ is open in } X, \ a < \sup A < b \ \& \ C \cap A \neq \emptyset\}.$$

We claim that $\mathcal{A}$ is an open cover for C in X:

If there is an ε in X such that $\gamma < \varepsilon < b$ then $(-\infty, \varepsilon) \in \mathcal{A}$ and hence $C \subseteq (-\infty, \varepsilon)$, otherwise if there is no such an ε then $b = \gamma + 1$ and hence $(-\infty, \gamma] = (-\infty, b)$. If $x \in C, \ x < \gamma$ then $x \in (-\infty, \gamma)$, also $\gamma \in (-\infty, \gamma] = (-\infty, b)$, which is open in X, hence we prove our claim.

Since C is closed in a compact space X, C is compact. Therefore there exists a finite set $\{A_1, \cdots, \ A_n\} \subseteq \mathcal{A}$ such that $C \subseteq \bigcup_{i \leq n} A_i$. Without loss of generality $\forall i \leq n, \ C \cap A_i \neq \emptyset$, since otherwise we can throw away the A_i.

Now consider

$$\langle A_1, \cdots, A_n\rangle = \{F \subseteq X : F \text{ closed and } (\forall i \le n),\ F \cap A_i \ne \emptyset \ \&\ F \subseteq \bigcup_{i\le n} A_i\}.$$

Hence for any closed set $F \in \langle A_1, \cdots, A_n\rangle$ we have that

$$F \subseteq \bigcup_{i\le n} A_i \Rightarrow \sup(F) \le \sup(\bigcup_{i\le n} A_i) < b.$$

This is an open basic set in $H(X)$ which contains C. To check that

$$\langle A_1, \cdots, A_n\rangle \subseteq \Phi^{-1}((a, b)) :$$

we need to check that for any $F \in \langle A_1, \cdots, A_n\rangle$, $\sup F \in (a, b)$, but this is true by our choice of $\mathcal{A}$. ■

Examples: Property W_1 is not verified by ordered compact spaces that are not first countable.

If X is not first countable then there is a point $x_0 \in X$ such that $\chi(x_0, X) > \omega$. Hence either $\sup\{x \in X : x < x_0\} = x_0$ and for any countable set A subset of $\{x \in X : x < x_0\}$, $\sup(A) < x_0$ or $\inf\{x \in X : x_0 < x\} = x_0$ and for any countable set A subset of $\{x \in X : x > x_0\}$, $\inf(A) > x_0$. To prove this consider the following 4 cases. Suppose that $\sup\{x \in X : x < x_0\} = y$ and $\inf\{x \in X : x_0 < x\} = z$. Firstly, $y < x_0$ and $z > x_0$. This implies that $\{x_0\} = (y, z)$ and hence $\chi(x_0, X) = \omega$, a contradiction. Secondly, $y < x_0$ and there exist a countable set A subset of $\{x \in X : x > x_0\}$, such that $\inf(A) = x_0$. Then $x_0 = \bigcap_{a\in A}(y, a)$ and so $\chi(x_0, X) = \omega$; a contradiction. Similarly for the third case when we have a countable set A subset of $\{x \in X : x < x_0\}$, such that $\sup(A) = x_0$ and $z > x_0$, $x_0 = \bigcap_{a\in A}(a, z)$. Finally, if there exist two countable A_1, A_2 subset of $\{x \in X : x < x_0\}$ and $\{x \in X : x > x_0\}$, respectively such that $\sup(A_1) = x_0 = \inf(A_2)$, then $\bigcap_{a\in A_1}(a, z) = x_0 = \bigcap_{a\in A_2}(y, a)$. Hence $\chi(x_0, X) = \omega$.

Now consider the first case where $\sup\{x \in X : x < x_0\} = x_0$ and for any countable set A subset of $\{x \in X : x < x_0\}$, $\sup(A) < x_0$. Here $x_0 \in cl(\{x \in X : x < x_0\})$, and hence $\{x \in X : x < x_0\}$ is not closed. Also if A is a countable subset of $\{x \in X : x < x_0\}$ then

$$cl(A) = A \cup \{\sup(A)\} \cup \{\inf(A)\} \subseteq \{x \in X : x < x_0\}.$$

Hence $\{x \in X : x < x_0\}$ is a sequentially closed non closed subset of X and $\{x \in X : x < x_0\}$ is a union of G_δ sets. So X does not coincide with its sequential closure. Hence X does not satisfies property W_1. Similarly for the second case where $\inf\{x \in X : x_0 < x\} = x_0$ and and for any countable set A subset of $\{x \in X : x > x_0\}$, $\inf(A) > x_0$.

Corollary 4.3.8. *If X is an ordered compact space that is not first countable, then neither X nor $H(X)$ are multiadic.*

Proof: We shall prove non-multiadicity only for $H(X)$, since it is clear for X from last example. Assume that X is compact ordered, then X is a continuous image of $H(X)$ as in Lemma 4.3.7. Let g be a surjective map defined on $H(X)$ onto X. Suppose that $H(X)$ is multiadic, so let $f : \prod_{i\in I}\alpha\kappa_i \to H(X)$ be a continuous surjective function. Then $g \circ f : \prod_{i\in I}\alpha\kappa_i \to X$ is continuous and surjective, which contradicts the assumption on X. ■

Corollary 4.3.9. *If X is a compact ordered space and $H(X)$ is multiadic, then X is metrizable.*

Proof: From the last example we have shown that if X is a compact ordered space with $H(X)$ multiadic, then X must be first countable. So, $\chi(X) = \omega$. By invoking Theorem 3.2.14 that states $\chi(X) = w(X)$ for a multiadic space X, it follows that $w(X) = \omega$ and therefore X must be metrizable. ■

4.4 Generalizations of the Sierpiński graph

In the beginning of this section we recall what Standard Sierpiński graph means. After that we give generalizations of the Standard Sierpiński graph using ordering defined by Hausdorff [15] [see chapter 9 of [28]]. Then we consider another property called S_λ that also deals with a single family of open sets. We show that under GCH these properties are equivalent. Some applications of such properties among multiadic spaces will be provided. Finally, we use the generalized Sierpiński graphs to give an example of a regular closed subsets of multiadic spaces of arbitrarily large weight that are not multiadic.

Definition 4.4.1. (Standard Sierpiński graph): *Let $\mathbb{R}$ be the set of real numbers, let $A \subseteq \mathbb{R}$ be of cardinality ω_1, let $<$ denote the usual ordering on A and let $\prec$ denote a well-ordering on A. We say $<$ and $\prec$* agree *on $\{x, y\}$ if $x < y \Leftrightarrow x \prec y$. Otherwise, we say that they* disagree *on $\{x, y\}$.*

Notation: Let A be defined as in the above definition. Define $G \subseteq [A]^2$ by $\{x, y\} \in G$ iff $<$ and $\prec$ agree on $\{x, y\}$. For $x \in A$, we denote the subset of A where $\{x, y\} \in G$ by J_x, that is $J_x = \{y \in A : \{x, y\} \in G\}$.

The key property of the Standard Sierpiński graph G is the following remark.

Lemma 4.4.2. *There exists no uncountable $A' \subseteq A$ on which either $<$ and $\prec$ agree for all of $[A']^2$ or on which $<$ and $\prec$ disagree for all of $[A']^2$.*

The proof of this Lemma is similar to one given later (Theorem 4.4.8). Now we may assume the following:

Claim 4.4.3. *For $x \in A$, J_x and $A \backslash J_x$ are both uncountable.*

Proof: Let $B = \{x \in A : J_x \text{ or } A\backslash J_x \text{ is countable}\}$. We have to show that B is countable and then we will define a set $A^+ \subseteq A$ to be $A^+ = A\backslash B$. Clearly it is exactly in the form:

$$A^+ = \{x \in A : J_x \text{ and } A\backslash J_x \text{ are uncountable}\}.$$

with $|A^+| = \omega_1$. This would complete the proof of the claim.

Suppose for contradiction that B is uncountable. For all $x \in B$ either J_x is countable or $A\backslash J_x$ is countable. So, either there exist an uncountable B_0 subset of B such that for all $x \in B_0$, J_x is countable or there exist an uncountable B_1 subset of B such that for all $x \in B_1$, $A\backslash J_x$ is countable.

Firstly, if $x \in B_0$ then for any $\alpha < \omega_1$ choose by induction

$$x_\alpha \in B_0 \backslash \bigcup_{\beta<\alpha} \left(J_{x_\beta} \cup \{x_\beta\}\right).$$

If $\beta < \alpha$, $x_\alpha \notin J_{x_\beta}$ so $\{x_\alpha, x_\beta\} \notin G$. Hence $<, \prec$ disagree on $\{x_\alpha, x_\beta\}$. This can not happen by the second part of Lemma 4.4.2. Therefore B_0 is not uncountable. Secondly, in an analogous fashion, we get there is no such B_1. Thus, B is countable. ■

Definition 4.4.4. *[15] Given an ordinal α, a linear ordering A is said to be an μ_α-ordering if, given any two subsets $X, Y \subseteq A$ each of cardinality less than $\aleph_\alpha$ and such that $X < Y$* (that is, for all $x \in X$ and all $y \in Y$ we have $x < y$), there is an $a \in A$ such that $X < a < Y$ (that is, $x < a < y$ for all $x \in X$ and all $y \in Y$).

Let A_α consist of all ω_α-sequences of 0 and 1, ordered lexicographically, where ω_α is the ordinal $\aleph_\alpha$, i.e. if $a = \langle a_\xi : \ \xi < \omega_\alpha \rangle$ and $b = \langle b_\xi : \ \xi < \omega_\alpha \rangle$ are two such sequences, $a < b$ if at the first ξ with $a_\xi \neq b_\xi$ we have $a_\xi < b_\xi$. So $|A_\alpha| = 2^{\aleph_\alpha}$.

Let $\mathbb{Q}_\alpha$ consists of those elements $a \in A_\alpha$ for which there exists an ordinal $\gamma < \omega_\alpha$ such that $a_\gamma = 1$ and $a_\xi = 0$ for all $\xi > \gamma$; thus a sequence is in $\mathbb{Q}_\alpha$ if and only if it has a last 1. Thus $\mathbb{Q}_0$ consists of those ω-sequences in which 1 occurs a finite but nonzero number of times.

Theorem 4.4.5. *Hausdorff [15]: If $\aleph_\alpha$ is a regular cardinal number then $\mathbb{Q}_\alpha$ is an μ_α-ordering.*

The cardinality of $\mathbb{Q}_\alpha$: For each $\xi < \omega_\alpha$, let $\mathbb{Q}_\alpha(\xi)$ consist of all those elements $a \in \mathbb{Q}_\alpha$ such that ξ is the last ordinal such that $a_\xi = 1$; thus $\{\mathbb{Q}_\alpha(\xi) : \xi < \omega_\alpha\}$ is a partition of $\mathbb{Q}_\alpha$. The number of elements of $\mathbb{Q}_\alpha(\xi)$ is the same as the number of subsets of ξ, by the function mapping each element of $\mathbb{Q}_\alpha(\xi)$ to the set of ordinals less than ξ at which its value is 1. Thus if the cardinality of ξ is $\aleph_\beta$, $\mathbb{Q}_\alpha(\xi)$ has $2^{\aleph_\beta}$ elements. Since for each $\beta < \alpha$, there are $\aleph_{\beta+1}$ ordinals of cardinality $\aleph_\beta$, $\mathbb{Q}_\alpha$ has a total of $\sum\{\aleph_{\beta+1} \cdot 2^{\aleph_\beta} : \ \beta < \alpha\} = \sum\{2^{\aleph_\beta} : \ \beta < \alpha\}$ elements since $\aleph_{\beta+1} \leq 2^{\aleph_\beta}$. Now if $\alpha = \gamma + 1$ is a successor ordinal, then $\mathbb{Q}_{\ gamma+1}$ has exactly $2^{\aleph_\gamma}$ elements. So if GCH holds, $|\mathbb{Q}_\alpha| = |\mathbb{Q}_{\gamma+1}| = \aleph_{\gamma+1} = \aleph_\alpha$, and if α is a limit then $|\mathbb{Q}_\alpha| = \sup\{\aleph_{\beta+1} : \ \beta < \alpha\} = \aleph_\alpha$. In any case $|\mathbb{Q}_\alpha| < 2^{\aleph_\alpha}$.

Define $\mathbb{R}_\alpha$ to be the subordering of A_α containing all $\omega_\alpha-$sequences except those which are eventually 1. The number of sequences which are eventually 1 is $\sup\{2^{\aleph_\beta} : \ \beta < \alpha\}$. Hence under GCH, $|\mathbb{R}_\alpha| = 2^{\aleph_\alpha}$. It is clear that $\mathbb{Q}_\alpha$ is dense in $\mathbb{R}_\alpha$ as follows. Let p, q are two sequences in $\mathbb{R}_\alpha$, with $p < q$. There exists a unique ξ such that $p_\gamma = q_\gamma$ for all $\gamma < \xi$ but $p_\xi = 0$ and $q_\xi = 1$. Define $r \in \mathbb{Q}_\alpha$ such that $r_\gamma = q_\gamma$ for $\gamma \leq \xi$ and $q_\gamma = 0$ for $\gamma > \xi$. Hence $p \leq r \leq q$. By using the density of $\mathbb{Q}_\alpha$ in $\mathbb{R}_\alpha$ we can get the following result:

Lemma 4.4.6. *Assume GCH holds. Then for any $\alpha \geq 0$, in $\mathbb{R}_\alpha$ there is no increasing or decreasing sequence of length $\omega_{\alpha+1}$.*

Proof: We have $|\mathbb{Q}_\alpha| = \aleph_\alpha$, and $|\mathbb{R}_\alpha| = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Since $\mathbb{Q}_\alpha$ is dense in $\mathbb{R}_\alpha$, it impossible to find a strictly increasing sequence in $\mathbb{R}_\alpha$ of order type $\omega_{\alpha+1}$.

Suppose otherwise, there is a strictly increasing sequence $\langle x_\beta : \beta < \omega_{\alpha+1} \rangle$ in $\mathbb{R}_\alpha$. Since $\mathbb{Q}_\alpha$ is dense in $\mathbb{R}_\alpha$, for each $\beta < \gamma$ with $x_\beta < x_\gamma$ there exists $q_\lambda \in \mathbb{Q}_\alpha$ such that $x_\beta \leq q_\lambda \leq x_\gamma$. Hence $\langle q_\lambda : \lambda \in \omega_{\alpha+1} \rangle$ is an increasing sequence in $\mathbb{Q}_\alpha$ of order type $\omega_{\alpha+1}$; a contradiction as $|\mathbb{Q}_\alpha| = \aleph_\alpha$. ■

In the following definition we give a generalizations of the Standard Sierpiński graph. Now we can use the same notation in the generalized graph as we did in the Standard Sierpiński graph.

Definition 4.4.7. (Generalizations of the Standard Sierpiński graph): *Let $\mathbb{R}_\alpha$ be the set of cardinality $2^{\aleph_\alpha}$, let $A_\alpha \subseteq \mathbb{R}_\alpha$ be of cardinality $2^{\aleph_\alpha}$, let $<$ denote the usual ordering on A_α and let $\prec$ denote a well-ordering on $\mathbb{A}_\alpha$. We say $<$ and $\prec$* agree *on $\{x, y\}$ if $x < y \Leftrightarrow x \prec y$. Otherwise, we say that they* disagree *on $\{x, y\}$.*

Notation: Let A_α be defined as the definition 4.4.7. Define $G \subseteq [A_\alpha]^2$ by $\{x, y\} \in G$ iff $<$ and $\prec$ agree on $\{x, y\}$. For $x \in A_\alpha$, we denote the subset of $\mathbb{A}_\alpha$ where $\{x, y\} \in G$ by J_x, that is $J_x = \{y \in A : \{x, y\} \in G\}$. Assume that for $x \in \mathbb{A}$, J_x and $\mathbb{A}$

Theorem 4.4.8. *Assume GCH holds. There exists no $A'_\alpha \subseteq \mathbb{R}_\alpha$ of size $\omega_{\alpha+1}$ on which either $<$ and $\prec$ agree for all of $[A']^2$ or on which $<$ and $\prec$ disagree for all of $[A'_\alpha]^2$.*

Proof: Suppose not, so there is $A'_\alpha \subseteq \mathbb{R}_\alpha$ of size $\omega_{\alpha+1}$ such that for all of $[A'_\alpha]^2$ either $<$ and $\prec$ are agree or $<$and $\prec$ are disagree.

Firstly, in the case, where for all $x, y \in A'_\alpha$, $x < y$ iff $x \prec y$, we can construct an increasing sequence $\langle x_\gamma \rangle \in (A'_\alpha, \prec)$ of length $\omega_{\alpha+1}$. Since $<$ and $\prec$ are agree, then $\langle x_\gamma \rangle$ an increasing sequence in $(A'_\alpha, <)$ of length $\omega_{\alpha+1}$ which contradicts Lemma 4.4.6.

Secondly, when $<$ and $\prec$ are disagree, if we introduce an increasing sequence $\langle x_\gamma \rangle \in (A'_\alpha, \prec)$ of length $\omega_{\alpha+1}$, we get a decreasing sequence $\langle x_\gamma \rangle$ in $(A'_\alpha, >)$ of length $\omega_{\alpha+1}$, absurd since there is no decreasing subset of $\mathbb{R}_\alpha$ of the order type $\omega_{\alpha+1}$. ■

Analogously to the proof of Bell in [4] we can prove that the property of being κ-multiadic is not regular closed hereditary as in the following theorem.

Theorem 4.4.9. *Suppose GCH holds. There is a closed subset of the polyadic space $(\alpha\omega_{\mu+1})^2$ which is not multiadic.*

Proof: Let G be the generalized Sierpiński graph on the set $\omega_{\mu+1}$. Put $U = \{(\gamma, \beta) : \{\gamma, \beta\} \in G\} \subseteq (\alpha\omega_{\mu+1})^2$. Note that for any $\gamma, \beta < \omega_{\mu+1}$, $(\gamma, \beta) \in U$ iff $(\gamma, \beta) \in \overline{U}$ since (γ, β) is an isolated point in $(\alpha\omega_{\mu+1})^2$. For each $\delta < \omega_{\mu+1}$, put $B_\delta = (\{\delta\} \times \alpha\omega_{\mu+1}) \cup (\alpha\omega_{\mu+1} \times \{\delta\})$ and put $U_\delta = B_\delta \cap \overline{U}$. It is clear that U_δ is a clopen set in $\overline{U}$.

Now, $U_\delta \cap U_\gamma \neq \emptyset \Leftrightarrow \{\delta, \gamma\} \in G$. Indeed, suppose that $(x, y) \in U_\delta \cap U_\gamma$, so $(x, y) \in B_\delta \cap \overline{U}$ and $(x, y) \in B_\gamma \cap \overline{U}$. This implies that $(x, y) \in (\{\delta\} \times \alpha\omega_{\mu+1}) \cup$

$(\alpha\omega_{\mu+1} \times \{\delta\})$ and $(x, y) \in (\{\gamma\} \times \alpha\omega_{\mu+1}) \cup (\alpha\omega_{\mu+1} \times \{\gamma\})$. This holds iff $(x, y) = (\delta, \gamma)$ or $(x, y) = (\gamma, \delta)$. Since $(x, y) \in U$, so $\{x, y\} = \{\delta, \gamma\} \in G$. Thus the collection $\mathcal{F} = \{U_\alpha : \alpha < \omega_{\mu+1}\}$ of subsets of $\overline{U}$ does not have neither linked nor disjoint subfamily of size $\omega_{\mu+1}$, i.e it violates property $R_{\omega_{\mu+1}}$, as from the key property of the generalized Sierpiński graph, there is no subcollection A' of G of size $\omega_{\mu+1}$ on which that $\{U_\alpha : \alpha \in A'\}$ is linked or disjoint. But property $R_{\omega_{\mu+1}}$ is equivalent to property $Q_{\omega\alpha+1}$ for the Boolean space $(\alpha\omega_{\mu+1})^2$, therefore $\overline{U}$ is not multiadic by Theorem 4.1.10. ■

Corollary 4.4.10. *For any regular cardinal κ there is a polyadic space of weight at least κ with a regular closed subspace which is not κ-multiadic.*

Corollary 4.4.11. *The property of being κ-multiadic is not regular closed hereditary.*

Now for a regular cardinal λ we give another definition of a Ramsey-theoretic property called property S_λ and show analogously to Bell that this property is equivalent to K_λ .

Definition 4.4.12. (Property S_λ) *For any regular cardinal λ a space X has* property S_λ *if every collection of size λ of its open sets contains a subcollection of the same size which is either linked or pairwise disjoint.*

Similarly to the proof of Bell in [4] we can prove the following proposition.

Proposition 4.4.13. *Property $S_{\omega_{\alpha+1}}$ and property $K_{\omega_{\alpha+1}}$ are equivalent for any space X and any α.*

Proof: Property $K_{\omega_{\alpha+1}}$ clearly implies property $S_{\omega_{\alpha+1}}$, so we only need to prove the other implication. Assume X has property $S_{\omega_{\alpha+1}}$. Let $\mathcal{O}$ be an open collection of sets in X of cardinality $\omega_{\alpha+1}$. We need to show it has property $K_{\omega_{\alpha+1}}$ by showing that $\mathcal{O}$ does not contain a disjoint subfamily of size $\omega_{\alpha+1}$. Suppose not, and let $\mathcal{O}'$ be a disjoint subfamily of $\mathcal{O}$ with $|\mathcal{O}'| = \omega_{\alpha+1}$. Let G be the generalized Sierpiński graph on $\mathbb{R}_\alpha$ so by GCH $|G| = \omega_{\alpha+1}$. Let ψ be a bijection $\psi : [\mathbb{R}_\alpha]^2 \to \mathcal{O}'$. For each $x \in \mathbb{R}_\alpha$, put $U_x = \bigcup_{\{x,y\}\in G} \psi(\{x, y\})$. Then, for each $x \in \mathbb{R}_\alpha$, U_x is open and, furthermore, the collection $\{U_x : x \in \mathbb{R}_\alpha\}$ has property that $U_x \cap U_y \neq \ emptyset$ iff $\{x, y\} \in G$. Thus property $S_{\omega_{\alpha+1}}$ is violated by the collection $\{U_x : x \in \mathbb{R}_\alpha\}$ as there is no a linked subcollection nor a disjoint subcollection of size $\omega_{\alpha+1}$. Hence, $\mathcal{O}$ must contain a linked subfamily of size $\omega_{\alpha+1}$ and therefore X has property $K_{\omega_{\alpha+1}}$. ■

Lemma 4.4.14. *Under GCH, the polyadic space* $(\alpha\omega_{\gamma+1})^\omega$ *has Property* $Q_{\omega_{\gamma+1}}$ *but does not have Property* $S_{\omega_{\gamma+1}}$.

Proof: From proposition 4.4.13, we only need to show that the polyadic space $(\alpha\omega_{\gamma+1})^\omega$ does not satisfy Property $K_{\omega_{\gamma+1}}$. The collection $\mathcal{O} = \{O_i : i < \omega_{\gamma+1}\}$ of open sets in $(\alpha\omega_{\gamma+1})^\omega$ where $O_i = \{x \in (\alpha\omega_{\gamma+1})^\omega : x(1) = i\}$ is a disjoint family of open sets of size $\omega_{\gamma+1}$ which violates Property $K_{\omega_{\gamma+1}}$. ■

Inspired by Avilés [2], we can prove the following theorem.

Theorem 4.4.15. *Assume GCH. Let* $\Gamma \subseteq \mathbb{R}_\alpha$ *be a set of size* $\omega_{\alpha+1}$ *and* $1 < p < \infty$. *There exists an equivalent norm on* $l_p(\Gamma)$ *whose unit ball does not satisfy Property* $Q_{\omega_{\alpha+1}}$ *and hence it is not* $\omega_{\alpha+1}$*-multiadic.*

Proof: Consider $\omega_{\alpha+1}$ as a subset of Γ and let $\phi : \omega_{\alpha+1} \to \Gamma$ be a one-to-one map because $|\Gamma| = \aleph_{\alpha+1}$. Let G be the generalized Sierpiński graph on the set $\omega_{\alpha+1}$ written in the following form

$$G = \{(\gamma, \beta) \in \omega_{\alpha+1} \times \omega_{\alpha+1} \ : \phi(\gamma) < \phi(\beta) \Longleftrightarrow \gamma \prec \beta\}.$$

Then define an equivalent norm on $l_p(\Gamma) \times l_p(\Gamma) \sim l_p(\Gamma)$ by

$$\|(x,y)\|' = \sup\{\|x\|_p, \|y\|_p, |x_\alpha| + |y_\beta| : (\gamma, \beta) \in G\}.$$

It is clear that $\|(x,y)\| = \sup\{\|x\|_p, \|y\|_p\} \leq \|(x,y)\|'$ and since

$$\begin{aligned}
\sup\{|x_\gamma| + |y_\beta| : (\gamma,\beta) \in G\} &\leq \sup\{|x_\gamma| + |y_\beta| : (\gamma,\beta) \in \Gamma \times \Gamma\} \\
&\leq \sup\{|x_\gamma| : \gamma \in \Gamma\} + \sup\{|y_\beta| : \beta \in \Gamma\} \\
&\leq (\sum_{\gamma \in \Gamma} |x_\gamma|^p)^{1/p} + (\sum_{\beta \in \Gamma} |y_\beta|^p)^{1/p} \\
&= \|x\|_p + \|y\|_p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(x,y)\|' &\leq \sup\{\|x\|_p, \|y\|_p, \|x\|_p + \|y\|_p\} \\
&\leq \|x\|_p + \|y\|_p \\
&\leq C \cdot \sup\{\|x\|_p, \|y\|_p,\} \text{ where } C \geq 2 \\
&= C \cdot \|(x,y)\|.
\end{aligned}$$

This shows that $\|.\|'$ is equivalent to $\|x\|$. Now let K be the unit ball of $l_p(\Gamma)$ considered in its weak topology and norm $\|\cdot\|'$. Fix numbers $1 < \xi_1 < \xi_2 < 2^{1-\frac{1}{p}}$. The families

$$U_\beta = \{(x,y) \in K : |x_\beta| + |y_\beta| > \xi_2\},\ \beta < \omega_{\alpha+1}$$

$$V_\beta = \{(x,y) \in K : |x_\beta| + |y_\beta| > \xi_1\},\ \beta < \omega_{\alpha+1}$$

are open sets because all the functionals $f : l_p(\Gamma) \times l_p(\Gamma) \to \mathbb{R}$ are continuous in the weak topology. For instance, define f to be $f(\overline{x}, \overline{y}) = |x_\gamma| + |y_\gamma|$. This is a functional which is continuous in the original topology by the norm, so it is continuous in the weak topology. Thus the inverse image of any open set is open. But $f^{-1}(\xi_2, \infty) = \{(\overline{x}, \overline{y}) : f(\overline{x}, \overline{y}) > \xi_2\} = U_\beta$, hence U_β is open. Similarly for V_β. The two families $\{U_\beta\}_{\beta<\omega_{\alpha+1}}, \{V_\beta\}_{\beta<\omega_{\alpha+1}}$ satisfy that $\overline{U_\beta} \subseteq V_\beta$. Moreover, for any $\beta, \gamma \in \omega_{\alpha+1}$, $U_\beta \cap U_\gamma = \emptyset$ if and only if $(\beta, \gamma) \in G$ if and only if $V_\beta \cap V_\gamma = \emptyset$. Namely, if there is some $(x, y) \in V_\beta \cap V_\gamma$, then

$$|x_\beta| + |y_\beta| + |x_\gamma| + |y_\gamma| > \xi_1 + \xi_1 > 2$$

and therefore either $|x_\beta| + |x_\gamma| > 1$ or $|y_\beta| + |y_\gamma| > 1$ and this implies that if $(\beta, \gamma) \in G$ then $\|(x, y)\|' > 1$ (Contradiction since $(x, y) \in K$). Hence $(\beta, \gamma) \notin G$. On the other hand, if $(\beta, \gamma) \notin G$ then the element $(x, y) \in l_p(\Gamma) \times l_p(\Gamma\ )$ which has all coordinates zero except $x_\beta = x_\gamma = y_\beta = y_\gamma = 2^{-\frac{1}{p}}$ lies in $U_\beta \cap U_\gamma$. Thus from the fact that G is the generalized Sierpiński graph and GCH hold, there is no $A \subseteq \omega_{\alpha+1}$ of size $\omega_{\alpha+1}$ on which $\{U_\beta :\ \beta \in A\}$ is disjoint or on which $\{V_\beta :\ \beta \in A\}$ is linked. Therefore, K does not have property $(Q_{\omega_{\alpha+1}})$ and hence, it is not $\omega_{\alpha+1}$-multiadic. ■

Chapter 5

Some Consistency Proofs

The aim of this chapter is to prove consistently that there can be a singular cardinal λ such that $(\alpha\lambda)^\lambda$ is not an image of $\prod_{i<I} \alpha\lambda_i$ for any sequence $\langle \lambda_i : i \in I \rangle$ cofinal in λ. We remind the reader that in Corollary 3.2.17, we have proved the analogous result for a weakly inaccessible cardinal λ.

5.1 Non equivalent

Recall from example 3.2.1 that if λ is an inaccessible cardinal then $\hat{c}((\alpha\lambda)^\lambda) = \lambda^+$ so by Gerlits' Theorem 3.2.16 we got the following:

Remark 5.1.1. *For a weakly inaccessible cardinal λ, $(\alpha\lambda)^\lambda$ is not a continuous image of $\prod_{i<\lambda} \alpha\lambda_i$.* (*)

Our aim is to show that in a forcing extension in which we start with a cardinal λ which is at least inaccessible in V, and make it singular in $V[G]$, the formula from (*) still holds. Of course, for such an argument we assume that V is a universe of ZFC in which there is a cardinal λ with the required properties (inaccessible, measurable etc).

Let us suppose that we are in such a forcing situation, for example λ is measurable and G is Prikry generic, hence in $V[G]$, λ is singular with $cf(\lambda) = \aleph_0$. Let $\mathcal{F}$ be a disjoint family of λ many open subsets of $(\alpha\lambda)^\lambda$ in V. From example 3.2.1 we know that we can take

$$\mathcal{F} = \{\{x \in (\alpha\lambda)^\lambda : x(1) = \beta\} : \beta < \lambda\}.$$

Lemma 5.1.2. *$\mathcal{F}$ is still a disjoint family of λ many open subsets of $(\alpha\lambda)^\lambda$ in $V[G]$.*

Proof: Follows by definition of $\mathcal{F}$. ■

Now work in $V[G]$ and suppose that $\langle\lambda_i : i < i^*\rangle$ is a cofinal sequence in λ. By Gerlits' Theorem 3.2.16, there is no surjective map from $\prod_{i<\lambda} \alpha\lambda_i$ to $(\alpha\lambda)^\lambda$. Therefore we have proved:

Theorem 5.1.3. *Assume the consistency of a measurable cardinal. Then it is consistent that there is a singular cardinal λ such that $(\alpha\lambda)^\lambda$ is not a continuous image of any $\prod_{i<i^*} \alpha\lambda_i$ where $i^* = cf(\lambda)$ and $\langle\lambda_i : i < i^*\rangle$ are ordinals or cardinals smaller than λ and with limit λ.*

Theorem 5.1.4. *Assume the consistency of a measurable cardinal κ with the Mitchell order $o(\kappa) = \kappa^{++}$. Then it is consistent that $(\alpha\aleph_\omega)^{\aleph_\omega}$ is not a continuous image of any $\prod_{i<i^*} \alpha\lambda_i$ where $\langle\lambda_i : i < i^*\rangle$ are ordinals or cardinals smaller than $\aleph_\omega$ and with limit $\aleph_\omega$.*

Proof: For Theorem 5.1.3, use the Prikry extension [27] on a measurable cardinal. In this forcing we start with a model V where λ is measurable and in the extension $V[G]$ it becomes singular of $cf(\lambda) = \omega$. In $V[G]$ we apply Lemma 5.1.2.

For Theorem 5.1.4 use a Gitik's extension [14] over a measurable cardinal κ with $o(\kappa) = \kappa^{++}$. The proof here is the same but the assumption are that κ in V is measurable with $o(\kappa) = \kappa^{++}$.

Remark 5.1.5. *$o(\kappa) = \kappa^{++}$ means that the Mitchell order of κ is κ^{++}. These notions are not explained here because they require a lot of background going outside of the scope of this thesis.*

5.2 Prikry Forcing

Since we have seen the relevance of Prikry- like extensions in the previous section, we shall now explore the concrete simplest extension of that type, namely the Prikry forcing. We devote this section to a review of its properties. The material in this section can be found in [17].

Definition 5.2.1. *A partial order, $\mathbb{P} = \langle P, \leq \rangle$, consists of a set P together with a relation that is transitive, reflexive and anti-symmetric.* A forcing notion *is a partial order $(P, \leq)$ with the greatest element $1_{\mathbb{P}}$. Elements of a forcing notion are called conditions. We will often abuse notation by writing $p \in \mathbb{P}$ rather than $p \in P$.*

Definition 5.2.2. *Let $\langle \mathbb{P}, \leq \rangle$ be a forcing notion. A chain in $\mathbb{P}$ is a set $C \subseteq \mathbb{P}$ such that $\forall\ p, q \in C$ $(p \leq q$ or $q \leq p)$. p and q are compatible iff*

$$\exists r \in \mathbb{P}(r \leq p \wedge r \leq q);$$

otherwise they are incompatible $(p \perp q)$*. An antichain in* $\mathbb{P}$ *is a subset* $A \subseteq \mathbb{P}$ *such that* $\forall p, q \in A(p \neq q \to p \perp q)$*. A partial order* $\langle \mathbb{P}, \leq \rangle$ *has the* θ *chain condition* θ*.c.c. iff every antichain in* $\mathbb{P}$ *has size* $< \theta$*.*

Definition 5.2.3. *A subset* D *of* $\mathbb{P}$ *is called* dense *if for every* $p \in \mathbb{P}$ *there is some* $q \in D$ *with* $q \leq p$*.*

Definition 5.2.4. *A collection* $\mathcal{F}$ *of non-empty subsets of a partially ordered set* $\mathbb{P}$ *is* a filter on $\mathbb{P}$ *if:*
i. $\mathcal{F}$ *is closed upwards. That is if* $p \leq q$ *and* $p \in \mathcal{F}$*, then* $q \in \mathcal{F}$*;*
ii. If $p, q \in \mathcal{F}$*, then there exists* $r \in \mathcal{F}$ *such that* $r \leq p$ *and* $r \leq q$*.*

Definition 5.2.5. *Suppose that* M *is a countable transitive model of a sufficient amount* ZFC^* *of* ZFC[1]*. A set of conditions* $G \subseteq \mathbb{P}$ *is generic over* M *if*
i. G *is a filter on* $\mathbb{P}$*;*
ii. If D *is dense in* $\mathbb{P}$ *and* $D \in M$*, then* $G \cap D \neq \emptyset$*.*
We also say that G *is* M*-generic, or* P*-generic (over* M*), or just generic.*

Definition 5.2.6. *By induction we define objects that are* $\mathbb{P}$*-names:*
$\dot{\tau}$ *is a* $\mathbb{P}$*-name iff for all* $(\dot{\sigma}, p) \in \dot{\tau}$ *for some* $p \in \mathbb{P}$ *and* $\dot{\sigma}$ *is a* $\mathbb{P}$*-name.*
For a $\mathbb{P}$*-name and a filter* G*, let* $\dot{\tau}_G = \{\dot{\sigma}_G : \exists p \in G((\dot{\sigma}, p) \in \dot{\tau})\}$*. Again this is a recursive definition. We also set*

$$V[G] = \{\dot{\tau}_G : \dot{\tau} \in V \text{ is a } \mathbb{P}\text{-name}\}.$$

Theorem 5.2.7. *Let* $V \models ZFC^*$ *and* $\mathbb{P}$ *be a notion of forcing. If* G *is* $\mathbb{P}$*-generic filter over* V*, then* $V[G] \models ZFC^*$ *and* $G \in V[G]$*.*

[1] Of course ZFC is used to denote the usual Zermelo-Fraenkel axioms of set theory with Axiom of choice.

The forcing relation $\Vdash$ is defined as follows:

Definition 5.2.8. *Let $\dot{\tau}_1, \dot{\tau}_2, \cdots, \dot{\tau}_n$ be $\mathbb{P}$-names and $\phi(\dot{\tau}_1, \dot{\tau}_2, \cdots, \dot{\tau}_n)$ be a sentence in the language of set theory. Then for a condition $p \in \mathbb{P}$, $p \Vdash$ "$\phi(\dot{\tau}_1, \dot{\tau}_2, \cdots, \dot{\tau}_n)$" iff for any generic filter G such that $p \in G$, we have $V[G] \vDash \phi((\dot{\tau}_1)_G, (\dot{\tau}_2)_G, \cdots, (\dot{\tau}_n)_G)$.*

Definition 5.2.9. *A partially ordered set $\langle \mathbb{P}, \leq \rangle$ is* separative *if for all $p, q \in \mathbb{P}$ with $p \not\leq q$, there exists an r such that $r \leq p$ that is incompatible with q.*

This property is needed to prove that the generic filter G of $\mathbb{P}$ is not in V.

Proposition 5.2.10. *Suppose that $\mathbb{P}$ is a separative forcing notion and G is $\mathbb{P}$-generic over M, for some M a transitive model of ZFC^*. Then $G \notin M$.*

Proof: Suppose otherwise. Let $D = \mathbb{P} \backslash G$. We claim that D is a dense subset of $\mathbb{P}$. So, given $p \in \mathbb{P}$, as $\mathbb{P}$ is separative, we can find q and r such that $q \perp r$ and both extending p. Not both q, r can be in G, as G is a filter, so at least one has to be in D, proving that D is dense. Since $G \in M$ and M is a model of ZFC, D is in M-as we can evaluate it in M using the operations $\cup, \backslash$ which are expressible in M and absolute by transitivity, and the parameter G which is in M. Since $D \in M$ and is dense, we must have $G \cap D \neq \emptyset$ by G being generic, a contradiction. ■

Now we shall give some definitions relating to measures.

Definition 5.2.11. *Let S be a nonempty set. A measure on S is a function $\mu : \mathcal{P}(S) \to [0, 1]$ such that*
(a) $\mu(\emptyset) = 0,\ \mu(S) = 1.$
(b) $\mu(\{a\}) = 0$ *for every* $a \in S$.

(d) If $\{X_n : n \in \omega\}$ is a collection of mutually disjoint subsets of S, then

$$\mu(\bigcup_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} \mu(X_n).$$

A consequence of (b) and (d) is that every at most countable subset of S has measure 0. Hence if there is a measure on S, then S is uncountable.

Definition 5.2.12. A filter $\mathcal{F}$ *on a set S is a subset of $\mathcal{P}(S)$ with the following properties:*
i. S is in $\mathcal{F}$, and if A and B are in $\mathcal{F}$, then so is their intersection.
ii. The empty set is not in $\mathcal{F}$.
iii. If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, for all subsets B of S.
The first two properties imply that a filter on a set has the finite intersection property.

A collection $\mathcal{U}$ of subsets of $\mathcal{P}(S)$ is an ultrafilter *if is a filter, and whenever $A \subseteq \mathcal{P}(S)$ then either $A \in \mathcal{U}$ or $S \backslash A \in \mathcal{U}$. Equivalently, an ultrafilter on S is a maximal filter on S.*

An ultrafilter is κ-complete *if it closed under all intersections of fewer than κ sets.*

Definition 5.2.13. A normal measure over a cardinal κ *is a $\kappa-$ complete ultrafilter $\mathcal{U}$ such that for any sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of elements of $\mathcal{U}$ its diagonal intersection*

$$\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \ \xi \in \bigcap_{\alpha<\xi} X_\alpha\} \in \mathcal{U}.$$

Equivalently, if $f : \kappa \to \kappa$ is such that $f(\alpha) < \alpha$ for most $\alpha < \kappa$, then there is a $\beta < \kappa$ such that $f(\alpha) = \beta$ for an ultrafilter many $\alpha < \kappa$.

Definition 5.2.14. *A cardinal is called* measurable *if it has a normal measure.*

The existence of such a cardinal can't be prove in ZFC.

Definition 5.2.15. *Let κ be a measurable cardinal and let U be a normal measure over κ. Prikry forcing is the poset $\mathbb{P}_U$ consisting of pairs $\langle s, A\rangle$ such that s is finite increasing sequence of ordinals less than κ and $A \subseteq \kappa$ belongs to U. A condition $\langle t, B\rangle$ is stronger than $\langle s, A\rangle$ if t is an initial segment of s and $A \cup (s - t) \subseteq B$, i.e*

$$\langle t, B\rangle \geq \langle s, A\rangle \leftrightarrow s \text{ extends } t,\ A \cup (s - t) \subseteq B.$$

We immediately note that if $\langle s, A\rangle$ and $\langle t, B\rangle$ are compatible, then t is an initial segment of s or vice versa. We also note that any two conditions $p = \langle s, A\rangle$, $q = \langle s, B\rangle$ with the same first coordinate are compatible. Hence any antichain if P_U has size at most κ (i.e. P_U has the κ^+-*c.c.*).

Proposition 5.2.16. *[20] If κ is a cardinal of a countable transitive model of ZFC^*, M, $\mathbb{P} \in M$ satisfies κ^+-cc in M, then $\mathbb{P}$ preserves regular cardinals $\geq \kappa^+$, and also preserves cofinalities $\geq \kappa$. If also κ is regular in M, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*

This shows that all cofinalities and cardinals above and including κ are preserved by $\mathbb{P}_U$.

Note that $\mathbb{P}_U$ is not separative, because for some $p \not\leq q$ there is no $r \leq q$ such that $r \perp p$. Let $p = \langle s, A\rangle, q = \langle s, B\rangle$ be such that $B \subsetneq A$. Then $q \leq p$ so p, q are compatible. For all $r = \langle t, C\rangle$ with $r \leq q$, s is an initial segment of t and $C \cup (t - s) \subseteq B \subsetneq A$. Hence $r < p$ and therefore r, p are compatible.

To make $\mathbb{P}_U$ separative, one more restriction is required on conditions $\langle s, A\rangle$ which is $A \cap (\max(s)+1) = \emptyset$. The new notion of the Prikry forcing $\mathbb{P}_U$ is dense in $\mathbb{P}_U$, so they give the same generic.

Definition 5.2.17. *A p.o. P is* $(<\kappa)$-closed*, if for every decreasing sequence of $<\kappa$ conditions in the forcing $p_0 \geq p_1 \geq \cdots$, there is a condition that is below all of them.*

$\mathbb{P}_U$ is not $(<\kappa)$-closed: Consider the conditions $\langle s_i, \kappa\rangle$ where $s_i = \langle 0, \cdots, i\rangle$ for $i \in \omega$. This is a sequence of ω-many conditions, and clearly $\langle s_{i+1}, \kappa\rangle \leq \langle s_i, \kappa\rangle$, but there is obviously no single condition below all the $\langle s_i, \kappa\rangle$: where $\langle s, A\rangle$ such a condition, then s would have to be infinite, which contradicts the definition of condition. Thus to know that $\mathbb{P}_U$ preserves cardinals below κ we can not apply the theorem [20] that says if $\mathbb{P} \in M$, λ is a cardinal in M and $\mathbb{P}$ is $(<\lambda)$-closed then $\mathbb{P}$ preserves cofinalities and cardinals $\leq \lambda$. The cardinals $\leq \kappa$ are preserved by Prikry forcing because of the Prikry Lemma (Theorem 5.2.19).

Lemma 5.2.18. *If G is $\mathbb{P}_U$-generic and $x = \bigcup\{s : \exists A \ \langle s, A\rangle \in G\}$, then x is an unbounded subset of κ of ordertype ω and $V[x] = V[G]$.*

An ω-sequence, x, is called a Prikry sequence for U. The generic set G can easily reconstructed from x by:

$$G_x = \{\langle s, A\rangle \in P_U : s \text{ is an initial segment of } x \text{ and } x \backslash (\max(s)+1) \subseteq A\}.$$

So, $V[G] = V[x]$.

Theorem 5.2.19. *Prikry[19] : Suppose that κ is measurable and U is a normal ultrafilter over κ. Then for any $\langle s, A\rangle \in P_U$ and a formula φ in the* *forcing language, there is a* $B \subseteq A$ *with* $B \in U$ *such that* $\langle s, B\rangle \parallel \varphi$ (i.e. $\langle s, B\rangle \Vdash \varphi \vee \langle s, B\rangle \Vdash \neg\varphi$).

This implies Lemma 5.2.20 that shows no new bounded subsets are added to κ after the forcing with $\mathbb{P}_U$.

Lemma 5.2.20. *Every bounded set in* $V[G]$ *must be in* V.

proof: Let $A \in V[G],\ A \subseteq \kappa$ with $\sup(A) < \kappa$. Let $\dot{\tau}_G$ be a name and p a condition in $\mathbb{P}_U$ such that $\dot{\tau}_G = A$ and assume

$$p \Vdash \dot{\tau} \subseteq \kappa,\ \ \sup \dot{\tau} < \kappa.$$

Let $q \leq p,\ \alpha < \kappa$ such that $q \Vdash \dot{\tau} \subseteq \alpha$. Without loss of generality assume that α is a limit ordinal. By induction on $\beta < \alpha$ we construct a decreasing sequence $q = q_0 \geq^* q_1 \geq^* \cdots$, such that

i. $q_{\beta+1} \parallel \beta \in \dot{\tau}$, given q_β, use the Prikry lemma to find $q_{\beta+1}$.
ii. For the limit β, $q_\beta \leq^* \{q_\gamma :\ \gamma < \beta\}$ of length $< \kappa$. At the end let $q_\alpha \leq^* q_\beta (\beta < \alpha)$ and $\forall \beta < \alpha,\ q_\alpha \parallel \beta \in \dot{\tau}$. Thus the set $\{\beta < \alpha :\ q_\alpha \parallel \beta \in \dot{\tau}\}$ is bounded and it is in V. Hence cardinals $\leq \kappa$ are preserved. ■

Theorem 5.2.21. *Prikry[19]: Suppose that* κ *is measurable and* U *is a normal ultrafilter over* κ. *If* G *is* $\mathbb{P}$*-generic, then* $\mathbb{P}$ *preserves cardinals (i.e. the cardinals of* V *and* $V[G]$ *coincide) yet* $\mathrm{cf}^{V[G]}(\kappa) = \omega$.

5.3 More results

Let λ be a measurable cardinal in V. Let G be a Prikry generic and let $\langle \kappa_n : n < \omega\rangle$ be a cofinal sequence of λ in $V[G]$. Recall that it follows from Theorem 5.1.3 that

Theorem 5.3.1. *In $V[G]$, there is no continuous map from $\prod_{n<\omega} \alpha\kappa_n$ onto $(\alpha\lambda)^\lambda$.*

We would like to explore how low we can go on the exponent of $\alpha\lambda$ and still get an analogue of Theorem 5.3.1. That is, for which θ is it true that there is no continuous map from $\prod_{n<\omega} \alpha\kappa_n$ onto $(\alpha\lambda)^\theta$. The following Lemma 5.3.3 shows that for $\theta = 1$ is too small. First we will start with the following Lemma:

Lemma 5.3.2. *If $\langle \kappa_n : \ n < \omega \rangle$ is a Prikry sequence then there exists a surjective map*

$$f : \prod_{n<\omega} \alpha\kappa_n \twoheadrightarrow \alpha\omega.$$

Proof: Fix a sequence $\langle x_n : \ n < \omega \rangle \in \prod_{n<\omega} \alpha\kappa_n$ such that $\forall n, \ x_n \neq 0$. For each $n \in \omega$, let O_n be the set that consists of all points in $\prod_{n<\omega} \alpha\kappa_n$ which contains $(n-1)$ zeros in the first $(n-1)$ coordinates and x_n in n^{th} coordinate.

$$O_n = \{(a_n) \in \prod_{n<\omega} \alpha\kappa_n : \ \langle a_n : \ n < \omega \rangle = \langle 0, 0, \cdots, 0 \rangle^\frown x_n \underline{\triangle} x\}.$$

It is clear that for $n \neq m, \ O_n \cap O_m = \emptyset$. Then we can define $f : \prod_{n<\omega} \alpha\kappa_n \twoheadrightarrow \alpha\omega$ such that:

$$f(\langle a_n \rangle_{n<\omega}) = \begin{cases} n & \text{if } (a_n) \in O_n \\ \infty & \text{otherwise.} \end{cases}$$

We should show it is continuous especially for any open set around ∞. Let $U = \alpha\omega \backslash W$ where W is finite. That means W is closed. Since $\psi^{-1}(W) =$

$\bigcup_{m\in W} O_m$ is closed in $\prod\limits_{n<\omega} \alpha\kappa_n$, its complement is open and it contains $\psi^{-1}(U)$. Hence f is as required. ■

Lemma 5.3.3. *If $\langle \kappa_n : n < \omega\rangle$ is a Prikry sequence then there exist a surjective map*

$$f : \prod_{n<\omega} \alpha\kappa_n \twoheadrightarrow \alpha\lambda.$$

Proof: Fix $n \in \omega$. For each $\alpha \in \kappa_n$ consider the set F_n^β that consists of all points in $\prod\limits_{n<\omega} \alpha\kappa_n$ which contains $(n-1)$ zeros in the first $(n-1)$ coordinates and then β in the rest of other coordinates where $\beta \in \kappa_n$. Obviously, $|F_n^\beta| = |\kappa_n|$. If $\mathcal{F} = \bigcup\limits_n F_n^\beta$, $|\mathcal{F}| = \lambda$. Let

$$O_n^\beta = \big\{(a_n) \in \prod_{n<\omega} \alpha\kappa_n : \ \langle a_n : n < \omega\rangle = \langle 0, 0, \cdots, 0\rangle^\frown\langle\beta\rangle\underline{\triangle}x\big\}.$$

O_n^β is open and it contains F_n^β. Also the collection $\mathcal{O} = \bigcup\limits_n O_n^\beta$ is a pairwise disjoint family. Therefore there is a bijection between $\mathcal{O}$ and λ. Enumerate $\mathcal{O}$ as $\{O_\beta : \beta < \lambda\}$. Define the required map $f : \prod\limits_{n<\omega} \alpha\kappa_n \twoheadrightarrow \alpha\lambda$ as the following:

$$f(\langle a_n\rangle_{n<\omega}) = \begin{cases} \beta & \text{if } (a_n) \in O_\beta \\ \infty & \text{otherwise.} \end{cases}$$

By applying the same argument as in proving the continuity of f in Theorem 5.3.2, we get our conclusion. ■

We would like to know if $\theta = \omega$ would work, that is if in this extension there is a continuous map from $\prod\limits_{n<\omega} \alpha\kappa_n$ to $(\alpha\lambda)^\omega$. However, for the moment, we could not resolve this problem.

The reader might be tempted to think that to obtain a continuous map from $\prod\limits_{n<\omega} \alpha\kappa_n$ onto $(\alpha\lambda)^\omega$, all we need to do is to take a diagonal of the mapping just mentioned. Note, however, that this idea will not work because the requirement in Shapirovskii's Theorem 2.1.11 that the family be orthogonal, is not satisfied. Consider $g : \prod\limits_{n<\omega} \alpha\kappa_n \twoheadrightarrow (\alpha\lambda)^\omega$ given by $g(a) = (g_1(a), g_2(a), \cdots)$. Suppose $\mathcal{G} = \{g_n : \ n \in \omega\}$, where $g_n = \{g_n^{-1}(y) : \ y \in \alpha\lambda\}$. If we want to prove that this is orthogonal, it is sufficient to show that for any $m < \omega$, $\{g_n : \ n < m\}$ is orthogonal. Given $n_1 < n_2 < m$ and $A_1 \in g_{n_1}$, $A_2 \in g_{n_2}$. Then $A_1 = g_{n_1}^{-1}(s)$ for some $s \in \prod\limits_{n<n_1} \alpha\kappa_n$, $A_2 = g_{n_2}^{-1}(t)$ for some $t \in \prod\limits_{n<n_2} \alpha\kappa_n$. This implies that $g_{n_1}^{-1}(s) = \{a \in \ displaystyle\prod_{n<\omega} \alpha\kappa_n : (f_1(a), f_1(a), \cdots, f_{n_1}) = s\}$, $g_{n_2}^{-1}(t) = \{a \in \prod\limits_{n<\omega} \alpha\kappa_n : (f_2(a), f_2(a), \cdots, f_{n_2}) = t\}$. Hence, $g_{n_1}^{-1}(s) \cap g_{n_2}^{-1}(t) \neq \emptyset$ iff $s \trianglelefteq t$. Clearly that this is not always true in our case.

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