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Semisimple types for p-adic classical groups

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Abstract We construct, for any symplectic, unitary or special orthogonal group over a locally compact nonarchimedean local field of odd residual characteristic, a type for each Bernstein component of the category of smooth representations, using Bushnell–Kutzko's theory of covers. Moreover, for a component corresponding to a cuspidal representation of a maximal Levi subgroup, we prove that the Hecke algebra is either abelian, or a generic Hecke algebra on an infinite dihedral group, with parameters which are, at least in principle, computable via results of Lusztig. In an appendix, we make a correction to the proof of a result of the second author: that every irreducible cuspidal representation of a classical group as considered here is irreducibly compactly-induced from a type.

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1 Introduction

The study of the irreducible smooth (complex) representations of p-adic groups G has seen much progress over the last fifty years, inspired especially by the (local) Langlands programme. A basic approach, due to Harish–Chandra, is: first classify all the irreducible representations which do *not* arise as quotients of representations

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parabolically induced from representations of a proper Levi subgroup (these are called *cuspidal*); then classify all quotients of representations parabolically induced from a *cuspidal* representation of a Levi subgroup. Because parabolic induction does not preserve irreducibility, and because its reducibility is related to the poles and zeros of L-functions, in following this approach it is both necessary and interesting to study the full (abelian) category of smooth representations $\mathfrak{R}(G)$.

A fundamental general result, for G a connected reductive p-adic group, is the Bernstein decomposition [1], which splits $\mathfrak{R}(G)$ into *blocks* (indecomposable abelian summands) $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$. These are indexed by (equivalence classes of) pairs $\mathfrak{s} = [\mathrm{M}, \tau]_{\mathrm{G}}$, with M a Levi subgroup of G and τ a cuspidal irreducible representation of M, while the irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$ are precisely the irreducible quotients of the parabolically induced representations $\mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{G}}\tau\chi$, for P any parabolic subgroup with Levi factor M, and χ any character (1-dimensional representation) of M trivial on every compact subgroup (an *unramified* character).

Bushnell and Kutzko [11] give a strategy for understanding any block $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$: one seeks to construct a pair (J, λ) (called an $\mathfrak{s}$-*type*), consisting of a compact open subgroup J of G and an irreducible (smooth) representation λ of J, which characterizes the block in the sense that the irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$ are exactly the irreducible representations π of G such that $\mathrm{Hom}_{\mathrm{J}}(\pi, \lambda) \neq 0$. (We say that π *contains* λ.) Then the block $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$ is equivalent to the category of modules over the spherical Hecke algebra $\mathscr{H}(\mathrm{G}, \lambda) = \mathrm{End}_{\mathrm{G}}(\text{c-Ind}_{\mathrm{J}}^{\mathrm{G}}\lambda)$, so we are reduced to computing $\mathscr{H}(\mathrm{G}, \lambda)$ and its modules. Moreover, Bushnell–Kutzko's theory of *covers*, which we recall below, gives a technique for trying to construct types (and their Hecke algebras) for general M from those in the cuspidal case (that is, when $\mathrm{M} = \mathrm{G}$).

This programme has been carried out in its entirety for the groups GL_N [9,12] and its inner forms [25,26], SL_N [10,16,17], and, when the residual characteristic p is odd, $\mathrm{U}(2,1)$ [3] and Sp_4 [2,4]. It has also been completed for an arbitrary connected reductive group for *level zero* blocks, that is, for $[\mathrm{M}, \tau]_{\mathrm{G}}$ where τ contains the trivial representation of the pro-p-radical of some parahoric subgroup [23,24]. For inner forms of GL_N, the Hecke algebras which arise are all tensor products of generic Hecke algebras of type A; for SL_N one gets a similar algebra tensored with the group algebra of a finite group, but twisted by a cocycle.

In this paper, we largely complete the programme for an arbitrary classical group G when the residual characteristic is odd. More precisely, let F_{o} be a locally compact nonarchimedean local field with residue field of odd cardinality q_{o}, and let G be the group of rational points of a symplectic, special orthogonal or unitary group defined over F_{o}. Our first main result is:

Theorem 1.1 *Let* M *be a Levi subgroup of* G*, let* τ *be a cuspidal irreducible representation of* M*, and put* $\mathfrak{s} = [\mathrm{M}, \tau]_{\mathrm{G}}$. *There is an* $\mathfrak{s}$-*type* (J, λ) *which is, moreover, a cover of the* $\mathfrak{s}_{\mathrm{M}}$-*type* $(\mathrm{J} \cap \mathrm{M}, \lambda | \mathrm{J} \cap \mathrm{M})$.

At present, we are only able to determine the Hecke algebra in the case of a *maximal* proper Levi subgroup (though see the comments below for some implications in other cases); it turns out that the Hecke algebras which arise are as for the group $\mathrm{Sp}_4(\mathrm{F})$ [4], although there are more possibilities for the parameters. We also remark that this case of a maximal Levi subgroup is the most interesting in terms of implications on poles and zeros of L-functions; in particular, it is possible to use the results here to compute explicitly the cuspidal representations in an L-packet.

Theorem 1.2 *In the situation of Theorem 1.1, suppose moreover that* M *is a maximal proper Levi subgroup of* G *and write* $\mathrm{N_G}(\mathfrak{s}_\mathrm{M})$ *for the set of* $g \in \mathrm{G}$ *such that* g *normalizes* M *and* ${}^g\tau$ *is equivalent to* $\tau\chi$*, for some unramified character* χ *of* M.

(i) *If* $\mathrm{N_G}(\mathfrak{s}_\mathrm{M}) = \mathrm{M}$ *then the Hecke algebra* $\mathscr{H}(\mathrm{G}, \lambda)$ *is abelian, isomorphic to* $\mathbb{C}\left[X^{\pm 1}\right]$.
(ii) *If* $\mathrm{N_G}(\mathfrak{s}_\mathrm{M}) \neq \mathrm{M}$ *then the Hecke algebra* $\mathscr{H}(\mathrm{G}, \lambda)$ *is a generic Hecke algebra on an infinite dihedral group; that is, it is generated by* T_0, T_1*, each invertible and supported on a single double coset, with relations*

$$(T_i - q_i)(T_i + 1) = 0,$$

for some integer $q_i \in q_\mathrm{o}^{\mathbb{Z}}$.

Moreover, in Sect. 6, we give a recipe which reduces the calculation of the parameters q_i in this Hecke algebra to the computation of a certain quadratic character (which is sometimes known to be trivial) and of the parameters in two finite Hecke algebras, which are computable through the work of Lusztig [20]. We explore certain cases of this further in work in progress, though we emphasise that the computation of the quadratic character appears, in general, to be a very subtle matter: see the work of Blondel [6] for more on this.

We also remark that, for symplectic groups, the propagation results of Blondel [5] together with our Theorem 1.2 now give the Hecke algebra when $\mathrm{M} \simeq \mathrm{GL}_r(\mathrm{F})^s \times \mathrm{Sp}_{2N}(\mathrm{F})$ and $\tau = \widetilde{\tau}^{\otimes s} \otimes \tau_0$. Whether the results there and here could be pushed to give a description of the Hecke algebra in the general case is not clear.

We now describe the proofs so we suppose we are in the situation of Theorem 1.1. The class $\mathfrak{s} = [\mathrm{M}, \tau]_\mathrm{G}$ determines a (cuspidal) class $\mathfrak{s}_\mathrm{M} = [\mathrm{M}, \tau]_\mathrm{M}$ for M, which gives us a block $\mathfrak{R}^{\mathfrak{s}_\mathrm{M}}(\mathrm{M})$ of the category of smooth representations of M. An $\mathfrak{s}_\mathrm{M}$-type $(\mathrm{J_M}, \lambda_\mathrm{M})$ was constructed by the second author in [29] (though we take the opportunity here to correct some inaccuracies in the proof of [29, Theorem 7.14]—see the Appendix). We say that a pair (J, λ) is *decomposed* over $(\mathrm{J_M}, \lambda_\mathrm{M})$ if, for any parabolic subgroup $\mathrm{P} = \mathrm{MU}$ with Levi factor M,

(i) J has an Iwahori decomposition with respect to (M, P) and $\mathrm{J} \cap \mathrm{M} = \mathrm{J_M}$; and
(ii) λ restricts to λ_M on $\mathrm{J_M}$, and to a multiple of the trivial representation on $\mathrm{J} \cap \mathrm{U}$.

If a further technical condition on the Hecke algebra $\mathscr{H}(\mathrm{J}, \lambda)$ is satisfied (it contains an invertible element supported only on the double coset of a strongly positive element of the centre of M) then (J, λ) is a *cover* of $(\mathrm{J_M}, \lambda_\mathrm{M})$, in which case it is also an $\mathfrak{s}$-type. Moreover, one gets an embedding of Hecke algebras $\mathscr{H}(\mathrm{M}, \lambda_\mathrm{M}) \hookrightarrow \mathscr{H}(\mathrm{G}, \lambda)$ and, in certain circumstances, one can also deduce the rank (and other structure) of $\mathscr{H}(\mathrm{G}, \lambda)$ as an $\mathscr{H}(\mathrm{M}, \lambda_\mathrm{M})$-module.

To construct a cover, we do *not* in fact start with the type $(\mathrm{J_M}, \lambda_\mathrm{M})$ but rather construct (J, λ) directly, then observing that it is a cover of its restriction to M, which is indeed an $\mathfrak{s}_\mathrm{M}$-type. To this end, the starting point is a result of Dat [14], building on work of the second author in [28]. In the latter paper, so-called *semisimple characters* of certain compact open subgroups of G were constructed, generalizing constructions of Bushnell and Kutzko [9]. These come in families indexed by a semisimple element β of the Lie algebra of G and a lattice sequence Λ, which can be interpreted as a point in the building of the centralizer G_β of β via [8].

Dat proved that, given $\mathfrak{s} = [M, \tau]_G$ as above, there is a *self-dual semisimple character* θ of a compact open subgroup H^1 of G such that (H^1, θ) is a decomposed pair over $(H^1 \cap M, \theta|_{H^1\cap M})$ and τ contains $\theta|_{H^1\cap M}$. There is considerable flexibility here; in particular, the associated lattice sequence Λ may be chosen so that the parahoric subgroup it defines in G_β (that is, the stabilizer of the point it defines in the building) also has an Iwahori decomposition with respect to any parabolic subgroup with Levi factor M. (Indeed, this is generically the case.) The element β also has a Levi subgroup L attached to it (the minimal Levi subgroup containing G_β) and we have $M \subseteq L$.

Our first task is to extend the constructions of [28,29] to the self-dual case, in particular the notion of *(standard) β-extension κ* and its realization as an induced representation $\mathrm{Ind}_{J_P}^{J} \kappa_P$, for P a parabolic subgroup with Levi component M. The main property here is that the representation $\kappa_M := \kappa_P|_{J_P \cap M}$ of $J_M := J_P \cap M$ is a (standard) β-extension in M, in the sense of [29], with extra compatibility properties coming from conjugation in L; indeed, it is ensuring these compatibilities which would make it difficult to start with a type in M and build a cover from it.

Now our cuspidal representation τ of M contains a representation of J_M of the form $\lambda_M = \kappa_M \otimes \rho_M$, for ρ_M the inflation of a cuspidal representation of the (possibly disconnected) finite reductive quotient J_M/J_M^1, and (J_M, λ_M) is an $\mathfrak{s}$-type. Since $J_P/J_P^1 \simeq J_M/J_M^1$, we can also form the representation $\lambda_P = \kappa_P \otimes \rho_M$ and the claim is then that (J_P, λ_P) is a cover of (J_M, λ_M). There is a small but important subtlety here: it is in fact the inverse image J_M° of the connected component of J_M/J_M^1 that we work with, along with a representation $\lambda_M^\circ = \kappa_M \otimes \rho_M^\circ$ contained in λ_M, and we prove that $(J_P^\circ, \lambda_P^\circ)$ is a cover of $(J_M^\circ, \lambda_M^\circ)$. That (J_P, λ_P) is also a cover follows from a result of Morris: this phenomenon already arises for level zero representations.

The proof uses transitivity of covers, showing that $(J_P^\circ, \lambda_P^\circ)$ is a cover of $(J_P^\circ \cap M', \lambda_P^\circ|_{J_P^\circ \cap M'})$ for a chain of Levi subgroups M' ending with M. The first step is with $M' = L$, which is straightforward by consideration of intertwining; indeed, the embedding of Hecke algebras in this case is an isomorphism. This reduces us to the case $L = G$, which is the case of a *skew* semisimple character considered in [29], and the rest of the argument is essentially contained there. By intertwining arguments, we reduce to the case in which there is no proper Levi subgroup of G containing the normalizer of $\rho_M^\circ|_{J_M^\circ}$. Finally, we pull off the remaining blocks of M one at a time; that is, we go in steps with $M' = GL_r(F) \times G^0$ a maximal proper Levi subgroup of G containing $M = GL_r(F) \times M^0$, with M^0 a Levi subgroup of the classical group G^0. (In the case of even special orthogonal groups we must sometimes remove blocks in pairs.)

The final step is achieved by producing Hecke algebra embeddings $\mathscr{H}(\mathscr{G}_i, \rho_M^\circ \chi_i) \hookrightarrow \mathscr{H}(G, \lambda_P^\circ)$, for $i = 0, 1$, where $\mathscr{G}_i$ is a finite reductive group having J_M°/J_M^1 as a maximal proper Levi subgroup, and χ_i is a quadratic character. Each of these finite Hecke algebras is two-dimensional, generated by an element T_i which is supported on a single double-coset and satisfies a quadratic relation. It is a power of the product of the images of T_i in $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ})$ which gives the required invertible element of the Hecke algebra.

In the case that M is maximal and $\mathrm{N}_{\mathrm{G}}(\mathfrak{s}_{\mathrm{M}}) \neq \mathrm{M}$, the same argument allows one to describe the Hecke algebra of the cover completely: the images of the two embeddings together generate $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ})$ and there are no further relations by support considerations. Again, there are some additional complications arising from the fact that the finite groups $\mathscr{G}_i$ (which are the reductive quotients of non-connected parahoric subgroups in G_β) need not be connected; some care is needed in dealing with these.

Finally we summarize the contents of the various sections. The basic objects involved in the construction are recalled in Sect. 2, while Sect. 3 extends the various constructions from the skew case in [29] to the case of a self-dual semisimple character. In Sect. 4 we recall the construction of types in the cuspidal case, before constructing the cover and proving Theorem 1.1 in Sect. 5. Finally, the computation of the Hecke algebra is given in Sect. 6. In the appendix we make the necessary corrections to the proof of [29, Theorem 7.14].

2 Notation and preliminaries

Let F be a nonarchimedean locally compact field of odd residual characteristic. Let $\lambda \mapsto \overline{\lambda}$ denote a (possibly trivial) galois involution on F with fixed field $\mathrm{F_o}$. For K a finite extension of $\mathrm{F_o}$, we denote by $\mathfrak{O}_{\mathrm{K}}$ its ring of integers, by $\mathfrak{p}_{\mathrm{K}}$ the maximal ideal of $\mathfrak{O}_{\mathrm{K}}$, by k_{K} its residue field and by q_{K} the cardinality of k_{K}. We also denote by $e(\mathrm{K}/\mathrm{F_o})$ and $f(\mathrm{K}/\mathrm{F_o})$ the ramification index and residue class degree of $\mathrm{K}/\mathrm{F_o}$ respectively, and put $\varepsilon_{\mathrm{F}} = (-1)^{e(\mathrm{F}/\mathrm{F_o})+1}$.

We fix ϖ_{F} a uniformizer of F such that $\overline{\varpi_{\mathrm{F}}} = \varepsilon_{\mathrm{F}}\varpi_{\mathrm{F}}$, and put $\varpi_{\mathrm{o}} = \varpi_{\mathrm{F}}^{e(\mathrm{F}/\mathrm{F_o})}$, a uniformizer of $\mathrm{F_o}$. We also fix ψ_{o}, a character of the additive group of $\mathrm{F_o}$ with conductor $\mathfrak{p}_{\mathrm{F_o}}$; then we put $\psi_{\mathrm{F}} = \psi_{\mathrm{o}} \circ \mathrm{tr}_{\mathrm{F}/\mathrm{F_o}}$, a character of the additive group of F with conductor $\mathfrak{p}_{\mathrm{F}}$. We also denote by $f \mapsto \overline{f}$ the involution induced on the polynomial ring $\mathrm{F}[X]$.

For u a real number, we denote by $\lceil u \rceil$ the smallest integer which is greater than or equal to u, and by $\lfloor u \rfloor$ the greatest integer which is smaller than or equal to u, that is, its integer part.

All representations considered here are smooth and complex.

The material of this section is essentially a summary of necessary definitions and basic results. More details can be found in [9,28].

2.1 Let $\varepsilon = \pm 1$ and let V be a finite-dimensional F-vector space equipped with a nondegenerate ε-hermitian form h: thus

$$\lambda h(v, w) = h(\lambda v, w) = \varepsilon\overline{h(w, \lambda v)}, \qquad v, w \in \mathrm{V},\ \lambda \in \mathrm{F}.$$

Put $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$, an F-split simple central F-algebra equipped with the adjoint anti-involution $a \mapsto \overline{a}$ defined by

$$h(av, w) = h(v, \overline{a}w), \qquad v, w \in \mathrm{V};$$

this anti-involution coincides with the galois involution on the naturally embedded copy of F in A.

2.2 Set $\widetilde{\mathrm{G}} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$ and let σ be the involution given by $g \mapsto \overline{g}^{-1}$, for $g \in \widetilde{\mathrm{G}}$. We also have an action of σ on the Lie algebra A given by $a \mapsto -\overline{a}$, for $a \in \mathrm{A}$. We put $\Sigma = \{1, \sigma\}$, where 1 acts as the identity on both $\widetilde{\mathrm{G}}$ and A.

Put $\mathrm{G}^+ = \widetilde{\mathrm{G}}^{\Sigma} = \{g \in \widetilde{\mathrm{G}} : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$, the F_{o}-points of a unitary, symplectic or orthogonal group $\mathbf{G}^+$ over F_{o}. Let G be the F_{o}-points of the connected component $\mathbf{G}$ of $\mathbf{G}^+$, so that $\mathrm{G} = \mathrm{G}^+$ except in the orthogonal case. Put $\mathrm{A}_- = \mathrm{A}^{\Sigma}$, the Lie algebra of G. In general, for S a subset of A, we will write S_- or S^- for $\mathrm{S} \cap \mathrm{A}_-$, and, for $\widetilde{\mathrm{H}}$ a subgroup of $\widetilde{\mathrm{G}}$, we will write H for $\widetilde{\mathrm{H}} \cap \mathrm{G}$.

If $\mathrm{F} = \mathrm{F}_{\mathrm{o}}, \varepsilon = +1, \dim_{\mathrm{F}} \mathrm{V} = 2$ and h is isotropic, then $\mathrm{G} \simeq \mathrm{SO}(1,1)(\mathrm{F}) \simeq \mathrm{GL}_1(\mathrm{F})$ so is well-understood. Consequently, we exclude this case. In particular, the centre of G^+ is the naturally embedded copy of $\mathrm{F}^1 := \{\lambda \in \mathrm{F} : \lambda\overline{\lambda} = 1\}$, which is compact.

2.3 An $\mathcal{O}_{\mathrm{F}}$-*lattice sequence* on V is a map

$$\Lambda : \mathbb{Z} \to \{\mathcal{O}_{\mathrm{F}} - \text{ lattices in V}\}$$

which is decreasing (that is, $\Lambda(k) \supseteq \Lambda(k+1)$ for all $k \in \mathbb{Z}$) and such that there exists a positive integer $e = e(\Lambda|\mathcal{O}_{\mathrm{F}})$ satisfying $\Lambda(k+e) = \mathfrak{p}_{\mathrm{F}}\Lambda(k)$, for all $k \in \mathbb{Z}$. This integer is called the $\mathcal{O}_{\mathrm{F}}$-*period* of Λ. If $\Lambda(k) \supsetneq \Lambda(k+1)$ for all $k \in \mathbb{Z}$, then the lattice sequence Λ is said to be *strict*. If $\dim_{k_{\mathrm{F}}} \Lambda(k)/\Lambda(k+1)$ is independent of k, we say that the lattice sequence is *regular*.

Associated with an $\mathcal{O}_{\mathrm{F}}$-lattice sequence Λ on V, we have an $\mathcal{O}_{\mathrm{F}}$-lattice sequence on A defined by

$$k \mapsto \mathfrak{P}_k(\Lambda) = \{a \in \mathrm{A} : a\Lambda(i) \subseteq \Lambda(i+k),\ i \in \mathbb{Z}\}, \quad k \in \mathbb{Z}.$$

The lattice $\mathfrak{A}(\Lambda) = \mathfrak{P}_0(\Lambda)$ is a hereditary $\mathcal{O}_{\mathrm{F}}$-order in A, and $\mathfrak{P}(\Lambda) = \mathfrak{P}_1(\Lambda)$ is its Jacobson radical; these two lattices depend only on the set $\{\Lambda(k) : k \in \mathbb{Z}\}$.

We denote by $\mathfrak{K}(\Lambda)$ the $\widetilde{\mathrm{G}}$-*normalizer* of Λ: that is, the subgroup of $\widetilde{\mathrm{G}}$ made of all elements g for which there is an integer $n \in \mathbb{Z}$ such that $g(\Lambda(k)) = \Lambda(k+n)$ for all $k \in \mathbb{Z}$. Given $g \in \mathfrak{K}(\Lambda)$, such an integer is unique: it is denoted $\upsilon_\Lambda(g)$ and called the Λ-*valuation* of g. This defines a group homomorphism υ_Λ from $\mathfrak{K}(\Lambda)$ to $\mathbb{Z}$. Its kernel, denoted $\widetilde{\mathrm{P}}(\Lambda)$, is the group of invertible elements of $\mathfrak{A}(\Lambda)$. We set $\widetilde{\mathrm{P}}_0(\Lambda) = \widetilde{\mathrm{P}}(\Lambda)$ and, for $k \geqslant 1$, we set $\widetilde{\mathrm{P}}_k(\Lambda) = 1 + \mathfrak{P}_k(\Lambda)$.

2.4 Given Λ an $\mathcal{O}_{\mathrm{F}}$-lattice sequence, the *affine class* of Λ is the set of all $\mathcal{O}_{\mathrm{F}}$-lattice sequences on V of the form:

$$a\Lambda + b : k \mapsto \Lambda(\lceil (k-b)/a \rceil),$$

with $a, b \in \mathbb{Z}$ and $a \geqslant 1$. The $\mathcal{O}_F$-period of $a\Lambda + b$ is a times the period $e(\Lambda|\mathcal{O}_F)$ of Λ. Note that

$$\mathfrak{P}_k(a\Lambda + b) = \mathfrak{P}_{\lceil k/a \rceil}(\Lambda)$$

so that changing Λ in its affine class only changes $\mathfrak{P}_k(\Lambda)$ in its affine class, indeed only by a scale in the indices; similarly, $\widetilde{\mathrm{P}}_k(\Lambda)$ is only changed by a scale in the indices, while $\mathfrak{K}(a\Lambda + b) = \mathfrak{K}(\Lambda)$.

2.5 We call an $\mathcal{O}_F$-lattice sequence Λ *self-dual* if there exists $d \in \mathbb{Z}$, such that $\{v \in \mathrm{V} : h(v, \Lambda(k)) \subseteq \mathfrak{p}_F\} = \Lambda(d-k)$ for all $k \in \mathbb{Z}$. By changing a self-dual $\mathcal{O}_F$-lattice sequence in its affine class, we may and do normalize all self-dual lattice sequences so that $d = 1$ and $e(\Lambda|\mathcal{O}_F)$ is even.

For Λ a self-dual lattice sequence, the $\mathcal{O}_F$-lattices $\mathfrak{P}_k(\Lambda)$ are stable under the involution σ (on A). Similarly, the groups $\widetilde{\mathrm{P}}_k$ are fixed by σ (on $\widetilde{\mathrm{G}}$) and we put $\mathrm{P}^+ = \mathrm{P}^+(\Lambda) = \widetilde{\mathrm{P}} \cap \mathrm{G}^+$, a compact open subgroup of G^+, and $\mathrm{P} = \mathrm{P}(\Lambda) = \mathrm{P}^+ \cap \mathrm{G}$. We have a filtration of $\mathrm{P}(\Lambda)$ by normal subgroups $\mathrm{P}_k = \mathrm{P}_k(\Lambda) = \widetilde{\mathrm{P}}_k^{\Sigma} = \widetilde{\mathrm{P}}_k \cap \mathrm{G}$, for $k > 0$. We also have, for $k > 0$, a bijection $\mathfrak{P}_k^-(\Lambda) \to \mathrm{P}_k$ given by the Cayley map $x \mapsto (1 + \frac{x}{2})(1 - \frac{x}{2})^{-1}$, which is equivariant under conjugation by P.

The quotient group $\mathscr{G} = \mathrm{P}/\mathrm{P}_1$ is (the group of rational points of) a reductive group over the finite field k_{F_0}. However, it is not, in general, connected. We denote by $\mathrm{P}^\circ = \mathrm{P}^\circ(\Lambda)$ the inverse image in P of (the group of rational points of) the connected component $\mathscr{G}^\circ$ of $\mathscr{G}$; then P° is a parahoric subgroup of G.

2.6 A *stratum* in A is a quadruple $[\Lambda, n, m, \beta]$ made of an $\mathcal{O}_F$-lattice sequence Λ on V, two integers m, n such that $0 \leqslant m \leqslant n$, and an element $\beta \in \mathfrak{P}_{-n}(\Lambda)$. Two strata $[\Lambda, n, m, \beta_i]$, for $i = 1, 2$, in A are said to be *equivalent* if $\beta_2 - \beta_1 \in \mathfrak{P}_{-m}(\Lambda)$. A stratum $[\Lambda, n, m, \beta]$ is called *null* if it is equivalent to $[\Lambda, n, m, 0]$, that is, if $\beta \in \mathfrak{P}_{-m}(\Lambda)$.

A stratum $[\Lambda, n, m, \beta]$ is called *self-dual* if Λ is self-dual and $\beta \in \mathrm{A}_-$. (Note that this notion has been called *skew* in previous papers; here we reserve the term skew for a more precise situation—see Sect. 3.)

For $n \geq m \geq \frac{n}{2} > 0$, an equivalence class of strata corresponds to a character of $\widetilde{\mathrm{P}}_{m+1}(\Lambda)$, by

$$[\Lambda, n, m, \beta] \mapsto (\widetilde{\psi}_\beta : x \mapsto \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1)), \quad \text{for } x \in \widetilde{\mathrm{P}}_{m+1}(\Lambda)),$$

while an equivalence class of self-dual strata corresponds to a character of $\mathrm{P}_{m+1}(\Lambda)$, by

$$[\Lambda, n, m, \beta] \mapsto \psi_\beta = \widetilde{\psi}_\beta|_{\mathrm{P}_{m+1}(\Lambda)}.$$

A null stratum corresponds to the trivial character.

2.7 For $[\Lambda, n, m, \beta]$ a stratum in A, we set

$$y = y(\beta, \Lambda) = \varpi_F^{n/g} \beta^{e/g},$$

where $e = e(\Lambda|\mathcal{O}_F)$ and $g = \gcd(n, e)$. The characteristic polynomial of $y + \mathfrak{P}_1(\Lambda)$ (considered as an element of $\mathfrak{A}(\Lambda)/\mathfrak{P}_1(\Lambda)$) is called the *characteristic polynomial* $\varphi_\beta(X) \in k_F[X]$ *of the stratum* $[\Lambda, n, m, \beta]$. The stratum $[\Lambda, n, m, \beta]$ is said to be *split* if $\varphi_\beta(X)$ has (at least) two distinct irreducible factors.

If $[\Lambda, n, m, \beta]$ is self-dual then we have $\overline{y} = \varepsilon_\beta y$, where $\varepsilon_\beta = \varepsilon_F^{n/g}(-1)^{e/g}$, and thus $\varphi_\beta(X) = \overline{\varphi}_\beta(\varepsilon_\beta X)$. We say that the stratum is G-*split* if $\varphi_\beta(X)$ has an irreducible factor $\psi(X)$ such that $\psi(X), \overline{\psi}(\varepsilon_\beta X)$ are coprime.

2.8 Let E be a finite extension of F contained in A. An $\mathcal{O}_F$-lattice sequence Λ on V is said to be E-*pure* if it is normalized by $E^\times$, in which case it is also an $\mathcal{O}_E$-lattice sequence. Denote by $B = \mathrm{End}_E(V)$ the centralizer of E in A and by $\Lambda_{\mathcal{O}_E}$ the lattice sequence Λ considered as an $\mathcal{O}_E$-lattice sequence.

2.9 Given a stratum $[\Lambda, n, m, \beta]$ in A, we denote by E the F-algebra generated by β. This stratum is said to be *pure* if E is a field, if Λ is E-pure and if $\upsilon_\Lambda(\beta) = -n$. Given a pure stratum $[\Lambda, n, m, \beta]$, we denote by B the centralizer of E in A. For $k \in \mathbb{Z}$, we set:

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}(\Lambda) \mid \beta x - x\beta \in \mathfrak{P}_k(\Lambda)\}.$$

The smallest integer $k \geqslant \upsilon_\Lambda(\beta)$ such that $\mathfrak{n}_{k+1}(\beta, \Lambda)$ is contained in $\mathfrak{A}(\Lambda) \cap B + \mathfrak{P}(\Lambda)$ is called the *critical exponent* of the stratum $[\Lambda, n, m, \beta]$, denoted $k_0(\beta, \Lambda)$.

The stratum $[\Lambda, n, m, \beta]$ is said to be *simple* if it is pure and if we also have $m < -k_0(\beta, \Lambda)$.

Given $n \geqslant 0$ and Λ an $\mathcal{O}_F$-lattice sequence, there is another stratum which plays a very similar role to simple strata, namely the *zero stratum* $[\Lambda, n, n, 0]$. (Note that this was called a null stratum in [28,29].)

2.10 Let $[\Lambda, n, m, \beta]$ be a stratum in A and suppose we have a decomposition $V = \bigoplus_{i \in I} V^i$ into F-subspaces. Let Λ^i be the lattice sequence on V^i given by $\Lambda^i(k) = \Lambda(k) \cap V^i$ and put $\beta_i = \mathbf{e}^i \beta \mathbf{e}^i$, where $\mathbf{e}^i$ is the projection onto V^i with kernel $\bigoplus_{j \neq i} V^j$. We use the block notation $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$.

We say that $V = \bigoplus_{i \in I} V^i$ is a *splitting* for $[\Lambda, n, m, \beta]$ if $\Lambda(k) = \bigoplus_{i \in I} \Lambda^i(k)$, for all $k \in \mathbb{Z}$, and $\beta = \sum_{i \in I} \beta_i$.

Suppose $V = \bigoplus_{i \in I} V^i$ and $V = \bigoplus_{j \in J} W^j$ are two decompositions of V. We say that $\bigoplus_{i \in I} V^i$ is a *refinement* of $\bigoplus_{j \in J} W^j$ (or $\bigoplus_{j \in J} W^j$ is a *coarsening* of $\bigoplus_{i \in I} V^i$) if, for each $i \in I$, there exists $j \in J$ such that $V^i \subseteq W^j$.

2.11 A stratum $[\Lambda, n, m, \beta]$ in A is called *semisimple* if either it is a zero stratum or $\beta \notin \mathfrak{P}_{1-n}(\Lambda)$ and there is a splitting $V = \bigoplus_{i \in I} V^i$ for the stratum such that

(i) for $i \in \mathrm{I}$, $[\Lambda^i, q_i, m, \beta_i]$ is a simple or zero stratum in A^{ii}, where $q_i = m$ if $\beta_i = 0$, $q_i = -\upsilon_{\Lambda^i}(\beta_i)$ otherwise; and
(ii) for $i, j \in \mathrm{I}$, $i \neq j$, the stratum $[\Lambda^i \oplus \Lambda^j, q, m, \beta_i + \beta_j]$ is not equivalent to a simple or zero stratum, with $q = \max\{q_i, q_j\}$.

In this case, the splitting is uniquely determined (up to ordering) by the stratum and we put $\mathscr{L}_\beta = \bigoplus_{i\in\mathrm{I}} \mathrm{A}^{ii}$. We put $\mathrm{E} = \mathrm{F}[\beta] = \bigoplus_{i\in\mathrm{I}} \mathrm{E}_i$, where $\mathrm{E}_i = \mathrm{F}[\beta_i]$. We will sometimes write "Λ is an $\mathcal{O}_\mathrm{E}$-lattice sequence" to mean that $\Lambda = \bigoplus_{i\in\mathrm{I}} \Lambda^i$ and each Λ^i is an $\mathcal{O}_{\mathrm{E}_i}$-lattice sequence on V^i.

Let $\mathrm{B} = \mathrm{B}_\beta$ denote the A-centralizer of β, so that $\mathrm{B} = \bigoplus_{i\in\mathrm{I}} \mathrm{B}_i$, where B_i is the centralizer of β_i in A^{ii}. We write $\widetilde{\mathrm{G}}_\mathrm{E} = \mathrm{B}^\times$, $\widetilde{\mathrm{G}}^i = \mathrm{Aut}_\mathrm{F}(\mathrm{V}^i)$ and, put $\widetilde{\mathrm{G}}_{\mathrm{E}_i} = \mathrm{B}_i^\times = \widetilde{\mathrm{G}}^i \cap \widetilde{\mathrm{G}}_\mathrm{E}$, so that $\widetilde{\mathrm{L}}_\beta = \mathscr{L}_\beta^\times = \prod_{i\in\mathrm{I}} \widetilde{\mathrm{G}}^i$ is a Levi subgroup of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}_\mathrm{E} = \prod_{i\in\mathrm{I}} \widetilde{\mathrm{G}}_{\mathrm{E}_i} \subseteq \widetilde{\mathrm{L}}_\beta$. Each $\widetilde{\mathrm{G}}_{\mathrm{E}_i}$ is (the group of F_o-points of) the restriction of scalars to F_o of a general linear group over E_i, provided E_i/F is separable; in any case, $\widetilde{\mathrm{G}}_{\mathrm{E}_i}$ is isomorphic to some $\mathrm{GL}_{m_i}(\mathrm{E}_i)$. We also write $\mathfrak{P}_k(\Lambda_{\mathcal{O}_\mathrm{E}}) = \mathfrak{P}_k(\Lambda) \cap \mathrm{B}$, for $k \in \mathbb{Z}$, which gives the filtration induced on B by thinking of Λ as an $\mathcal{O}_\mathrm{E}$-lattice sequence, and $\widetilde{\mathrm{P}}_k(\Lambda_{\mathcal{O}_\mathrm{E}}) = \widetilde{\mathrm{P}}_k(\Lambda) \cap \mathrm{B}$, for $k \geq 0$.

2.12 Let $[\Lambda, n, m, \beta]$ be a semisimple stratum in A. The *affine class* of the stratum $[\Lambda, n, m, \beta]$ is the set of all (semisimple) strata of the form

$$[\Lambda', n', m', \beta],$$

where $\Lambda' = a\Lambda + b$ is in the affine class of Λ, $n' = an$ and m' is any integer such that $\lfloor m'/a \rfloor = m$.

In the course of the paper, there will be several objects associated to a semisimple stratum $[\Lambda, n, m, \beta]$, in particular semisimple characters (see Sect. 3). By a straightforward induction (cf. [7, Lemma 2.2]), these objects depend only on the affine class of the stratum.

3 Self-dual semisimple characters

In this section we recall the notion of *self-dual* semisimple strata and characters from [14], generalizing the *skew* semisimple case from [28]. We also develop the theory of β-extensions in the self-dual situation. The results here are the expected generalizations of the results in the skew case from [28,29]. Moreover, most of the proofs follow by taking fixed points under the involution σ so are essentially identical to those in the skew case; we will only give details when new phenomena arise.

Self-dual semisimple strata

3.1 Let $[\Lambda, n, m, \beta]$ be a semisimple stratum and denote by $\mathrm{V} = \bigoplus_{i\in\mathrm{I}} \mathrm{V}^i$ the associated splitting and use all the notations introduced in Sect. 2. If $\Psi_i(X) \in \mathrm{F}[X]$ denotes the minimum polynomial of β_i then, by [28, Remark 3.2(iii)], we have $\mathrm{V}^i = \ker \Psi_i(\beta)$.

If $[\Lambda, n, m, \beta]$ is also self-dual then, for each $i \in \mathrm{I}$, there is a unique $j = \sigma(i) \in \mathrm{I}$ such that $\overline{\beta_i} = -\beta_j$. Moreover, $\overline{\Psi}_i(X) = \Psi_{\sigma(i)}(-X)$, whence $(\mathrm{V}^i)^\perp = \bigoplus_{j \neq \sigma(i)} V^j$. Then, using the usual block notation in A, the action of the involution $^-$ on A is such that $\overline{\mathrm{A}^{ij}} = \mathrm{A}^{\sigma(j)\sigma(i)}$.

We set $\mathrm{I}_0 = \{i \in \mathrm{I} \mid \sigma(i) = i\}$ and choose a set of representatives I_+ for the orbits of σ in $\mathrm{I}\backslash\mathrm{I}_0$. Then we will write $\mathrm{I}_- = \sigma(\mathrm{I}_+)$ so that $\mathrm{I} = \mathrm{I}_- \cup \mathrm{I}_0 \cup \mathrm{I}_+$ (disjoint union) and

$$\mathrm{V} = \bigoplus_{i \in \mathrm{I}_+} (\mathrm{V}^i \oplus \mathrm{V}^{\sigma(i)}) \oplus \bigoplus_{i \in \mathrm{I}_0} \mathrm{V}^i.$$

It will sometimes be useful to place on ordering on I_+, in which case we will write $\mathrm{I}_+ = \{1, \ldots, l\}$ and put $\sigma(i) = -i \in \mathrm{I}_-$, for $i \in \mathrm{I}_+$; in this case we will write $\mathrm{V}^0 = \bigoplus_{i \in \mathrm{I}_0} \mathrm{V}^i$ so that $\mathrm{V} = \bigoplus_{i=-l}^{l} \mathrm{V}^i$, which we call the *self-dual decomposition* associated to $[\Lambda, n, m, \beta]$. We will also put $\beta_0 = \mathbf{e}^0 \beta \mathbf{e}^0$, where $\mathbf{e}^0$ is the projection onto V^0 with kernel $\bigoplus_{j \neq 0} \mathrm{V}^j$.

3.2 Let $[\Lambda, n, m, \beta]$ be a self-dual semisimple stratum and $\mathrm{V} = \bigoplus_{i=-l}^{l} \mathrm{V}^i$ as above, with $\mathrm{V}^0 = \bigoplus_{i \in \mathrm{I}_0} \mathrm{V}^i$. We put $\widetilde{\mathrm{G}}^i = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}^i)$, $\mathrm{L}_\beta^+ = (\prod_{i=-l}^{l} \widetilde{\mathrm{G}}^i) \cap \mathrm{G}^+$ and $\mathrm{L}_\beta = \mathrm{L}_\beta^+ \cap \mathrm{G}$, which is a Levi subgroup of G. We have $\mathrm{L}_\beta = \mathrm{G}^0 \times \prod_{i=1}^{l} \widetilde{\mathrm{G}}^i$, where G^0 is the unitary, symplectic or special orthogonal group fixing the nondegenerate form $h|_{\mathrm{V}^0 \times \mathrm{V}^0}$.

Put $\widetilde{\mathrm{G}}_E = \mathrm{B}^\times$, the centralizer of β, as in Sect. 2. We put $\mathrm{G}_\mathrm{E}^+ = \widetilde{\mathrm{G}}_E \cap \mathrm{G}^+$ and $\mathrm{G}_\mathrm{E} = \widetilde{\mathrm{G}}_E \cap \mathrm{G}$, so that $\mathrm{G}_\mathrm{E} \subseteq \mathrm{L}_\beta$. For $i \in \mathrm{I}_0$, the involution on F extends to each E_i and we write $\mathrm{E}_{i,\mathrm{o}}$ for the subfield of fixed points; it is a subfield of index 2 except in the case $\mathrm{E}_i = \mathrm{F} = \mathrm{F}_\mathrm{o}$ (so that $\beta_i = 0$).

We have $\mathrm{G}_\mathrm{E} = \mathrm{G}_{\mathrm{E}_0} \times \prod_{i=1}^{l} \widetilde{\mathrm{G}}_{\mathrm{E}_i}$ and $\mathrm{G}_{\mathrm{E}_0} = \prod_{i \in \mathrm{I}_0} \mathrm{G}_{\mathrm{E}_i}$, where, for $i \in \mathrm{I}_0$, each $\mathrm{G}_{\mathrm{E}_i}$ is the group of points of a unitary, symplectic or special orthogonal group over $\mathrm{E}_{i,\mathrm{o}}$. (For each $i \in \mathrm{I}_0$, there is a nondegenerate $\mathrm{E}_i/\mathrm{E}_{i,\mathrm{o}}$ ε-hermitian form f_i on V^i such that the notions of lattice duality for $\mathfrak{O}_{\mathrm{E}_i}$-lattices in V^i given by $h|_{\mathrm{V}^i \times \mathrm{V}^i}$ and by f_i coincide; then $\mathrm{G}_{\mathrm{E}_i}$ is the group determined by this form.)

For $k \geq 0$, we write $\mathrm{P}_k(\Lambda_{\mathfrak{O}_\mathrm{E}}) = \mathrm{P}_k(\Lambda) \cap \mathrm{G}_\mathrm{E} = \widetilde{\mathrm{P}}_k(\Lambda_{\mathfrak{O}_\mathrm{E}}) \cap \mathrm{G}$ and denote by $\mathrm{P}^\mathrm{o}(\Lambda_{\mathfrak{O}_\mathrm{E}})$ the inverse image in $\mathrm{P}(\Lambda_{\mathfrak{O}_\mathrm{E}}) = \mathrm{P}_0(\Lambda_{\mathfrak{O}_\mathrm{E}})$ of the connected component of the reductive quotient $\mathrm{P}(\Lambda_{\mathfrak{O}_\mathrm{E}})/\mathrm{P}_1(\Lambda_{\mathfrak{O}_\mathrm{E}})$.

3.3 The following two results are straightforward generalizations of results from [28].

Lemma 3.1 (cf. [28, Proposition 3.4]) *Let* $[\Lambda, n, 0, \beta]$ *be a self-dual semisimple stratum in* A, *with associated splitting* $\mathrm{V} = \bigoplus_{i \in \mathrm{I}} \mathrm{V}^i$. *For* $0 \leq m \leq n$, *there is a self-dual semisimple stratum* $[\Lambda, n, m, \gamma]$ *equivalent to* $[\Lambda, n, m, \beta]$ *such that* $\gamma \in \mathscr{L}_\beta^-$; *in particular, its associated splitting is a coarsening of* $\bigoplus_{i \in \mathrm{I}} \mathrm{V}^i$.

Let $[\Lambda, n, m, \beta]$ be a self-dual semisimple stratum in A and, for $i \in \mathrm{I}_+ \cup \mathrm{I}_0$, let $s_i : \mathrm{A}^{ii} \to \mathrm{B}_i$ be a *tame corestriction* relative to E_i/F (see [9, §1.3] for the definition); for $i \in \mathrm{I}_0$ we may and do assume s_i commutes with the involution.

Lemma 3.2 (cf. [28, Lemma 3.5]) *Let* $[\Lambda, n, m, \beta]$ *be a self-dual semisimple stratum in* A*, with associated splitting* $V = \bigoplus_{i \in I} V^i$. *For* $i \in I_+ \cup I_0$*, let* $b_i \in \mathfrak{P}_{-m}(\Lambda) \cap A^{ii}$ *be such that* $[\Lambda^i_{\mathfrak{O}_{E_i}}, m, m-1, s_i(b_i)]$ *is equivalent to a semisimple stratum, and assume that* $b_i \in A_-$ *for* $i \in I_0$. *Put* $b_i = -\overline{b}_{-i}$, *for* $i \in I_-$, *and* $b = \sum_{i \in I} b_i$. *Then* $[\Lambda, n, m-1, \beta + b]$ *is equivalent to a self-dual semisimple stratum, whose associated splitting is a refinement of* $\bigoplus_{i \in I} V^i$.

The point of these lemmas is that now all objects associated to a self-dual semisimple stratum may be defined inductively with all intermediate strata also self-dual semisimple. In particular, all the objects will be stable under the involution σ.

3.4 A self-dual semisimple stratum $[\Lambda, n, m, \beta]$ is called *skew* if its associated splitting $V = \bigoplus_{i \in I} V^i$ is orthogonal; equivalently, in the notation above, if $I = I_0$.

Lemma 3.3 *Let* $[\Lambda, n, 0, \beta']$ *be a self-dual semisimple stratum in* A *and suppose that* $[\Lambda, n, m, \beta']$ *is equivalent to a self-dual semisimple stratum* $[\Lambda, n, m, \beta]$ *with* $\beta \in \mathscr{L}_{\beta'}$. *Write* $V = \bigoplus_{i \in I} V^i$ *for the splitting associated to* $[\Lambda, n, m, \beta]$*, which is a coarsening of that for* $[\Lambda, n, 0, \beta']$.

(i) For each $i \in I_+ \cup I_0$*, the derived stratum* $[\Lambda^i_{\mathfrak{O}_{E_i}}, m, m-1, s_i(\beta'_i - \beta_i)]$ *is either null or equivalent to a semisimple stratum.*

(ii) Suppose $0 < m \leq n$ *is minimal such that* $[\Lambda, n, m, \beta']$ *is equivalent to a skew semisimple stratum. Then* $[\Lambda, n, m, \beta]$ *is skew and there is an* $i \in I = I_0$ *such that the derived stratum* $[\Lambda^i_{\mathfrak{O}_{E_i}}, m, m-1, s_i(\beta'_i - \beta_i)]$ *is* G*-split.*

Proof

(i) Write $V = \bigoplus_{j \in I'} V^j$ for the splitting associated to $[\Lambda, n, 0, \beta']$ and $\mathbf{e}^j$ for the associated idempotents; then, for each $j \in I'$, there is a unique index $i \in I$ such that $V^j \subseteq V^i$. Now, applying [9, Theorem 2.4.1] to the simple stratum $[\Lambda^j, n, m, \mathbf{e}^j \beta \mathbf{e}^j]$ and the pure stratum $[\Lambda^j, n, m, \mathbf{e}^j \beta' \mathbf{e}^j]$, we see that $[\Lambda^j_{\mathfrak{O}_{E_i}}, m, m-1, \mathbf{e}^j \left(s_i(\beta'_i - \beta_i)\right) \mathbf{e}^j]$ is either null or equivalent to a simple stratum. The result follows since any direct sum of simple or null strata is equivalent to a semisimple stratum.

(ii) If $[\Lambda^i_{\mathfrak{O}_{E_i}}, m, m-1, s_i(\beta'_i - \beta_i)]$ is not G-split then it is skew; thus, if no $[\Lambda^i_{\mathfrak{O}_{E_i}}, m, m-1, s_i(\beta'_i - \beta_i)]$ is G-split then, by [28, Lemma 3.5], the stratum $[\Lambda, n, m-1, \beta']$ is equivalent to a skew semisimple stratum, contradicting the minimality of m. ☐

Self-dual semisimple characters and Heisenberg extensions

3.5 Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in A. Associated to this we have certain orders $\widetilde{\mathfrak{H}} = \widetilde{\mathfrak{H}}(\beta, \Lambda)$ and $\widetilde{\mathfrak{J}} = \widetilde{\mathfrak{J}}(\beta, \Lambda)$ in A (see [28, §3.2]), along with compact groups with filtration

$$\widetilde{H} = \widetilde{H}(\beta, \Lambda) = \widetilde{\mathfrak{H}} \cap \widetilde{P}(\Lambda), \qquad \widetilde{H}^n = \widetilde{H}^n(\beta, \Lambda) = \widetilde{H} \cap \widetilde{P}_n(\Lambda), \text{ for } n \geq 1,$$

and similarly for $\widetilde{\mathrm{J}}$. For each $m \geq 0$ there is also a set $\mathcal{C}(\Lambda, m, \beta)$ of *semisimple characters* of the group $\widetilde{\mathrm{H}}^{m+1}$ (see [28, Definition 3.13]) with nice properties, some of which we recall in Lemma 3.4 below.

Recall that, given a representation ρ of a subgroup $\widetilde{\mathrm{K}}$ of $\widetilde{\mathrm{G}}$ and $g \in \widetilde{\mathrm{G}}$, the *g-intertwining space* of ρ is

$$\mathrm{I}_g(\rho) = \mathrm{I}_g(\rho \mid \widetilde{\mathrm{K}}) = \mathrm{Hom}_{\widetilde{\mathrm{K}} \cap {}^g\widetilde{\mathrm{K}}}(\rho, {}^g\rho),$$

where ${}^g\rho$ is the representation of ${}^g\widetilde{\mathrm{K}} = g\widetilde{\mathrm{K}}g^{-1}$ given by ${}^g\rho(gkg^{-1}) = \rho(k)$, and the $\widetilde{\mathrm{G}}$-*intertwining* of ρ is

$$\mathrm{I}_{\widetilde{\mathrm{G}}}(\rho) = \mathrm{I}_{\widetilde{\mathrm{G}}}(\rho \mid \widetilde{\mathrm{K}}) = \{g \in \widetilde{\mathrm{G}} : \mathrm{I}_g(\rho) \neq \{0\}\}.$$

Lemma 3.4 ([28, Theorem 3.22, Corollary 3.25]) *Let $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$. Then*

(i) *the intertwining of $\widetilde{\theta}$ is given by $\mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\theta}) = \widetilde{\mathrm{J}}^1\widetilde{\mathrm{G}}_{\mathrm{E}}\widetilde{\mathrm{J}}^1$;*
(ii) *there is a unique irreducible representation $\widetilde{\eta}$ of $\widetilde{\mathrm{J}}^1$ which contains $\widetilde{\theta}$; moreover, $\mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\eta}) = \widetilde{\mathrm{J}}^1\widetilde{\mathrm{G}}_{\mathrm{E}}\widetilde{\mathrm{J}}^1$.*

3.6 Now suppose $[\Lambda, n, 0, \beta]$ is a self-dual semisimple stratum and retain the notation of the previous paragraph. The associated orders and groups are invariant under the action of the involution σ and we put $\mathrm{H} = \widetilde{\mathrm{H}} \cap \mathrm{G}$ etc., as usual. The set $\mathcal{C}_-(\Lambda, m, \beta)$ of *self-dual semisimple characters* is the set of restrictions to H^{m+1} of the semisimple characters $\theta \in \mathcal{C}(\Lambda, m, \beta)^{\Sigma}$; this can also be described in terms of the Glauberman correspondence (cf. [28, §3.6]). The next lemma now follows exactly as in [28, Proposition 3.27, Proposition 3.31].

Lemma 3.5 *Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$. Then*

(i) *the intertwining of θ is given by $\mathrm{I}_{\mathrm{G}}(\theta) = \mathrm{J}^1\mathrm{G}_{\mathrm{E}}\mathrm{J}^1$;*
(ii) *there is a unique irreducible representation η of J^1 which contains θ; moreover, if $\theta = \widetilde{\theta}|_{\mathrm{H}^1}$, for $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)^{\Sigma}$ and $\widetilde{\eta}$ is the corresponding representation of $\widetilde{\mathrm{J}}^1$, then η is the Glauberman transfer of $\widetilde{\eta}$.*

Transfer

3.7 Let $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta]$ be semisimple strata in A. Then (see [28, Proposition 3.26]) there is a canonical bijection (called the *transfer*)

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda, 0, \beta) \to \mathcal{C}(\Lambda', 0, \beta)$$

such that, for $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$, the character $\widetilde{\theta}' := \tau_{\Lambda,\Lambda',\beta}(\widetilde{\theta})$ is the unique semisimple character in $\mathcal{C}(\Lambda', 0, \beta)$ such that $\widetilde{\mathrm{G}}_{\mathrm{E}} \cap \mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\theta}, \widetilde{\theta}') \neq \emptyset$. Indeed, $\widetilde{\mathrm{G}}_{\mathrm{E}} \subseteq \mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\theta}, \widetilde{\theta}')$.

If the semisimple strata are self-dual then the bijection $\tau_{\Lambda,\Lambda',\beta}$ commutes with the involution (cf. [28, Proposition 3.32]) so induces a bijection $\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}_-(\Lambda, 0, \beta) \to \mathcal{C}_-(\Lambda', 0, \beta)$.

Since, by Lemma 3.5, for each $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ there is a unique Heisenberg extension η, we will also write $\tau_{\Lambda,\Lambda',\beta}(\eta)$ for the Heisenberg extension η' of the semi-simple character $\theta' := \tau_{\Lambda,\Lambda',\beta}(\theta)$.

3.8 Now suppose $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta]$ are self-dual semisimple strata with the additional property that $\mathfrak{A}(\Lambda_{\mathcal{O}_E}) \subseteq \mathfrak{A}(\Lambda'_{\mathcal{O}_E})$. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, denote by η the Heisenberg representation given by Lemma 3.5, and put $\theta' = \tau_{\Lambda,\Lambda',\beta}(\theta)$ and $\eta' = \tau_{\Lambda,\Lambda',\beta}(\eta)$. We form the group $J^1_{\Lambda,\Lambda'} = P_1(\Lambda_{\mathcal{O}_E})J^1(\beta, \Lambda')$. As in [29, Propositions 3.7, 3.12, Corollary 3.11] (see also [6, Proposition 1.2]), we have:

Proposition 3.6 *There is a unique irreducible representation $\eta_{\Lambda,\Lambda'}$ of $J^1_{\Lambda,\Lambda'}$ such that*

(i) $\eta_{\Lambda,\Lambda'}|_{J^1(\beta,\Lambda')} = \eta'$, *and*
(ii) *for any self-dual semisimple stratum $[\Lambda'', n'', 0, \beta]$ such that $\mathfrak{A}(\Lambda_{\mathcal{O}_E}) = \mathfrak{A}(\Lambda''_{\mathcal{O}_E})$ and $\mathfrak{A}(\Lambda'') \subseteq \mathfrak{A}(\Lambda')$, we have that $\eta_{\Lambda,\Lambda'}$ and $\tau_{\Lambda,\Lambda'',\beta}(\eta)$ induce equivalent irreducible representations of $P_1(\Lambda'')$.*

The intertwining of $\eta_{\Lambda,\Lambda'}$ is given by

$$\dim I_g(\eta_{\Lambda,\Lambda'}) = \begin{cases} 1 & \text{if } g \in J^1_{\Lambda,\Lambda'} G_E^+ J^1_{\Lambda,\Lambda'}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $\mathfrak{A}(\Lambda_{\mathcal{O}_E})$ is a minimal self-dual $\mathcal{O}_E$-order contained in $\mathfrak{A}(\Lambda'_{\mathcal{O}_E})$ then $\eta_{\Lambda,\Lambda'}$ is the unique extension of η' to $J^1_{\Lambda,\Lambda'}$ which is intertwined by all of G_E.

Standard β-extensions

3.9 We continue with the notation of the previous paragraph so $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and η is the Heisenberg representation, while θ', η' are their transfers to the self-dual semisimple stratum $[\Lambda', n', 0, \beta]$, with $\mathfrak{A}(\Lambda_{\mathcal{O}_E}) \subseteq \mathfrak{A}(\Lambda'_{\mathcal{O}_E})$. We form the groups $J^+ = \widetilde{J}(\beta, \Lambda) \cap G^+$ and $J^+_{\Lambda,\Lambda'} = P^+(\Lambda_{\mathcal{O}_E})J^1(\beta, \Lambda')$.

Lemma 3.7 ([29, Lemma 4.3]) *In this situation, there is a canonical bijection $\mathfrak{B}_{\Lambda,\Lambda'}$ from the set of extensions κ of η to J^+ to the set of extensions κ' of η' to $J^+_{\Lambda,\Lambda'}$.*

If $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Lambda')$ then $\kappa' = \mathfrak{B}_{\Lambda,\Lambda'}(\kappa)$ is the unique extension of η' such that κ, κ' induce equivalent irreducible representations of $P^+(\Lambda_{\mathcal{O}_E})P_1(\Lambda)$.

3.10 For $[\Lambda, n, 0, \beta]$ a self-dual semisimple stratum, we define a related self-dual $\mathcal{O}_E$-lattice sequence $\mathfrak{M}_\Lambda$ as follows. Recall that we have the decomposition $V = \bigoplus_{i \in I} V^i$ and $I = I_- \cup I_0 \cup I_+$. For $i \in I$, $r \in \mathbb{Z}$ and $s = 0, 1$, we put

$$\mathfrak{M}^i_\Lambda(2r + s) = \begin{cases} \mathfrak{p}^r_{E_i} \Lambda^i(0) & \text{if } i \in I_+, \\ \mathfrak{p}^r_{E_i} \Lambda^i(s) & \text{if } i \in I_0, \\ \mathfrak{p}^r_{E_i} \Lambda^i(1) & \text{if } i \in I_-. \end{cases}$$

Then $\mathfrak{M}_\Lambda := \bigoplus_{i \in I} \mathfrak{M}^i_\Lambda$ is a self-dual $\mathcal{O}_E$-lattice sequence on V with the property that $\mathfrak{A}(\mathfrak{M}_\Lambda) \cap B_\beta$ is a maximal self-dual $\mathcal{O}_E$-order in B_β.

Now we can define the notion of a standard β-extension.

Definition 3.8 ([29, Definition 4.5]) Let $[\Lambda, n, 0, \beta]$ be a self-dual semisimple stratum, let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and let η be the Heisenberg representation containing θ.

(i) Suppose $\mathfrak{A}(\Lambda_{\mathcal{O}_E})$ is a maximal self-dual $\mathcal{O}_E$-order in B. Then a representation κ of J^+ is called a (*standard*) β*-extension of* η if, for Λ^{m} any self-dual $\mathcal{O}_E$-lattice sequence such that $\mathfrak{A}(\Lambda^{\mathsf{m}}_{\mathcal{O}_E})$ is a minimal self-dual $\mathcal{O}_E$-order contained in $\mathfrak{A}(\Lambda_{\mathcal{O}_E})$, it is an extension of the representation $\eta_{\Lambda^{\mathsf{m}},\Lambda}$ of Proposition 3.6.
(ii) In general, a representation κ of J^+ is called a *standard* β*-extension of* η if there is a β-extension $\kappa_{\mathfrak{M}}$ of $\eta_{\mathfrak{M}} = \tau_{\Lambda,\mathfrak{M}_\Lambda,\beta}(\eta)$ such that $\mathfrak{B}_{\Lambda,\mathfrak{M}_\Lambda}(\kappa) = \kappa_{\mathfrak{M}}|_{J^+_{\Lambda,\mathfrak{M}_\Lambda}}$. In this case we say that $\kappa_{\mathfrak{M}}$ is *compatible with* κ.

We will often say that κ is a *standard* β *-extension of* θ, since η is determined by θ. We will also say that the restriction to J (respectively J°) of a standard β-extension κ is a *standard* β*-extension of* θ *to* J (respectively J°).

We also remark that β-extensions of a semisimple character $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ for $\widetilde{G}$ may be defined in the same way. This generalizes the construction for simple characters and strict lattice sequences in [9, §5.2].

Iwahori decompositions

3.11 Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in A with associated splitting $V = \bigoplus_{i\in I} V^i$ and let $V = \bigoplus_{j=1}^m W_j$ be a decomposition into subspaces which is *properly subordinate* to $[\Lambda, n, 0, \beta]$ in the sense of [29, Definition 5.1]: that is, each $W_j \cap V^i$ is an E_i-subspace of V^i and $W_j = \bigoplus_{i\in I}(W_j \cap V^i)$, we have

$$\Lambda(r) = \bigoplus_{j=1}^{m}(\Lambda(r) \cap W_j), \quad \text{for all } r \in \mathbb{Z},$$

and, for each $r \in \mathbb{Z}$ and $i \in I$, there is at most one j such that

$$(\Lambda(r) \cap W_j \cap V^i) \supsetneq (\Lambda(r+1) \cap W_j \cap V^i).$$

Denote by $\widetilde{M}$ the Levi subgroup of $\widetilde{G}$ which is the stabilizer of the decomposition $V = \bigoplus_{j=1}^m W_j$ and let $\widetilde{P}$ be any parabolic subgroup with Levi component $\widetilde{M}$ and unipotent radical $\widetilde{U}$.

By [29, Proposition 5.2], the groups $\widetilde{J}, \widetilde{J}^1$ and $\widetilde{H}^1$ have Iwahori decompositions with respect to $(\widetilde{M}, \widetilde{P})$ and we put

$$\widetilde{H}^1_{\widetilde{P}} = \widetilde{H}^1(\widetilde{J}^1 \cap \widetilde{U}), \quad \widetilde{J}^1_{\widetilde{P}} = \widetilde{H}^1(\widetilde{J}^1 \cap \widetilde{P}), \quad \text{and} \quad \widetilde{J}_{\widetilde{P}} = \widetilde{H}^1(\widetilde{J} \cap \widetilde{P}).$$

For $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ we define the character $\widetilde{\theta}_{\widetilde{P}}$ of $\widetilde{H}^1_{\widetilde{P}}$ by

$$\widetilde{\theta}_{\widetilde{P}}(hj) = \widetilde{\theta}(h), \quad \text{for } h \in \widetilde{H}^1,\ j \in \widetilde{J}^1 \cap \widetilde{U}.$$

This is well-defined.

Lemma 3.9 ([29, Corollary 5.7, Lemma 5.8]) *Let $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ and let $\widetilde{\eta}$ be the corresponding representation of $\widetilde{\mathrm{J}}^1$. Then*

(i) *the intertwining of $\widetilde{\theta}_{\widetilde{\mathrm{P}}}$ is given by $\mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\theta}_{\widetilde{\mathrm{P}}}) = \widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}\widetilde{\mathrm{G}}_{\mathrm{E}}\widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}$;*

(ii) *there is a unique irreducible representation $\widetilde{\eta}_{\widetilde{\mathrm{P}}}$ of $\widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}$ which contains $\widetilde{\theta}_{\widetilde{\mathrm{P}}}$; moreover, $\mathrm{I}_{\widetilde{\mathrm{G}}}(\widetilde{\eta}_{\widetilde{\mathrm{P}}}) = \widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}\widetilde{\mathrm{G}}_{\mathrm{E}}\widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}$ and $\widetilde{\eta} \simeq \mathrm{Ind}_{\widetilde{\mathrm{J}}^1_{\widetilde{\mathrm{P}}}}^{\widetilde{\mathrm{J}}^1}\widetilde{\eta}_{\widetilde{\mathrm{P}}}$.*

3.12 Now suppose $[\Lambda, n, 0, \beta]$ is a self-dual semisimple stratum and $\mathrm{V} = \bigoplus_{j=-m}^{m} \mathrm{W}_j$ is a properly subordinate *self-dual* decomposition, that is, the orthogonal complement of W_j is $\bigoplus_{k\neq -j} \mathrm{W}_k$, for each j. (We allow the possibility that $\mathrm{W}_0 = \{0\}$.) We use the notation of the previous paragraph and put $\mathrm{M} = \widetilde{\mathrm{M}} \cap \mathrm{G}$, a Levi subgroup of G, and, choosing $\widetilde{\mathrm{P}}$ to be a σ-stable parabolic subgroup of $\widetilde{\mathrm{G}}$, put $\mathrm{P} = \widetilde{\mathrm{P}} \cap \mathrm{G} = \mathrm{MU}$, a parabolic subgroup of G. Then H^1 has an Iwahori decomposition with respect to (M, P), while $\widetilde{\mathrm{H}}^1_{\widetilde{\mathrm{P}}}$ is stable under the involution, and we put $\mathrm{H}^1_{\mathrm{P}} = \widetilde{\mathrm{H}}^1_{\widetilde{\mathrm{P}}} \cap \mathrm{G} = \mathrm{H}^1\left(\mathrm{J}^1 \cap \mathrm{U}\right)$. Similarly, we have $\mathrm{J}^1_{\mathrm{P}}$, J_{P} and $\mathrm{J}^+_{\mathrm{P}}$, as well as $\mathrm{J}^{\circ}_{\mathrm{P}} = \mathrm{H}^1(\mathrm{J}^{\circ} \cap \mathrm{P})$.

For $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, define the character θ_{P} of $\mathrm{H}^1_{\mathrm{P}}$ by

$$\theta_{\mathrm{P}}(hj) = \theta(h), \quad \text{for } h \in \mathrm{H}^1,\ j \in \mathrm{J}^1 \cap \mathrm{U};$$

thus, if $\theta = \widetilde{\theta}|_{\mathrm{H}^1}$ for some $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)^{\Sigma}$, then $\theta_{\mathrm{P}} = \widetilde{\theta}_{\widetilde{\mathrm{P}}}|_{\mathrm{H}^1_{\mathrm{P}}}$. Exactly as in [29, Lemma 5.12], we get:

Lemma 3.10 *Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and let η be the corresponding representation of J^1. Then*

(i) *the intertwining of θ_{P} is given by $\mathrm{I}_{\mathrm{G}}(\theta_{\mathrm{P}}) = \mathrm{J}^1_{\mathrm{P}}\mathrm{G}_{\mathrm{E}}\mathrm{J}^1_{\mathrm{P}}$;*

(ii) *there is a unique irreducible representation η_{P} of $\mathrm{J}^1_{\mathrm{P}}$ which contains θ_{P}; moreover, if $\theta = \widetilde{\theta}|_{\mathrm{H}^1}$, for $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)^{\Sigma}$ and $\widetilde{\eta}$ is the corresponding representation of $\widetilde{\mathrm{J}}^1$, then η_{P} is the Glauberman transfer of $\widetilde{\eta}_{\widetilde{\mathrm{P}}}$;*

(iii) *with η_{P} as in (ii), we have $\eta \simeq \mathrm{Ind}_{\mathrm{J}^1_{\mathrm{P}}}^{\mathrm{J}^1}\eta_{\mathrm{P}}$ and*

$$\dim \mathrm{I}_g(\eta_{\mathrm{P}}) = \begin{cases} 1 & \textit{if } g \in \mathrm{J}^1_{\mathrm{P}}\mathrm{G}^+_E\mathrm{J}^1_{\mathrm{P}}, \\ 0 & \textit{otherwise.} \end{cases}$$

3.13 We continue with the notation of the previous paragraph. Let κ be a standard β-extension of η to J^+. We form the natural representation κ_{P} of $\mathrm{J}^+_{\mathrm{P}}$ on the space of $\mathrm{J} \cap \mathrm{U}$-fixed vectors in κ; then κ_{P} is an extension of η_{P} and $\mathrm{Ind}_{\mathrm{J}^+_P}^{\mathrm{J}^+}\kappa_{\mathrm{P}} \simeq \kappa$. Similar results apply to the restriction of κ_{P} to J_{P} and to $\mathrm{J}^{\circ}_{\mathrm{P}}$.

We can also make the same construction for a β-extension $\widetilde{\kappa}$ of a semisimple character $\widetilde{\theta}$ for $\widetilde{\mathrm{G}}$, thus obtaining a representation $\widetilde{\kappa}_{\widetilde{\mathrm{P}}}$ of $\widetilde{\mathrm{J}}_{\widetilde{\mathrm{P}}}$.

3.14 Suppose $[\Lambda, n, 0, \beta]$ is a self-dual semisimple stratum, with associated Levi subgroup $\mathrm{L} = \mathrm{L}_{\beta}$ as in paragraph 3.2, which we identify with $\mathrm{G}^0 \times \prod_{i=1}^{l} \widetilde{\mathrm{G}}^i$. Note that the associated decomposition $\mathrm{V} = \bigoplus_{i=-l}^{l} \mathrm{V}^i$ is properly subordinate to the stratum.

Let Q be a parabolic subgroup of G with Levi component L. We write $\mathrm{H}^1_{\mathrm{L}} = \mathrm{H}^1_{\mathrm{Q}} \cap \mathrm{L} = \mathrm{H}^1 \cap \mathrm{L}$; then $\mathrm{H}^1_{\mathrm{L}} = \mathrm{H}^1(\beta_0, \Lambda_0) \times \prod_{i=1}^{l} \widetilde{\mathrm{H}}(\beta_i, \Lambda_i)$. Similarly we have $\mathrm{J}^1_{\mathrm{L}}$, etc.

For $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ a semisimple character we put $\theta_{\mathrm{L}} = \theta|_{\mathrm{H}^1_{\mathrm{L}}}$. Then θ_{L} is of the form $\theta_0 \otimes \bigotimes_{i=1}^{l} \widetilde{\theta}_i$, with θ_0 a skew semisimple character in $\mathcal{C}_-(\Lambda_0, 0, \beta_0)$, and $\widetilde{\theta}_i$ a *simple* character in $\mathcal{C}(\Lambda_i, 0, 2\beta_i)$.

By a *standard β-extension of* θ_{L}, we mean a representation κ_{L} of $\mathrm{J}^+_{\mathrm{L}}$ (or $\mathrm{J}_{\mathrm{L}}, \mathrm{J}^{\circ}_{\mathrm{L}}$) of the form $\kappa_{\mathrm{L}} = \kappa_0 \otimes \bigotimes_{i=1}^{l} \widetilde{\kappa}_i$, with κ_0 a standard β_0-extension of θ_0 and $\widetilde{\kappa}_i$ a $2\beta_i$-extension of $\widetilde{\theta}_i$.

3.15 We continue with notation of the previous paragraph.

Let $\mathrm{V} = \bigoplus_{j=-m}^{m} \mathrm{W}_j$ be another self-dual decomposition properly subordinate to $[\Lambda, n, 0, \beta]$ and M the Levi subgroup of G stabilizing the decomposition. We suppose also that $\mathrm{M} \subseteq \mathrm{L}$ and let $\mathrm{P} = \mathrm{MU} \subseteq \mathrm{Q}$ be a parabolic subgroup of G with Levi component M. Then $\mathrm{P} \cap \mathrm{L} = \mathrm{M}(\mathrm{U} \cap \mathrm{L})$ is a parabolic subgroup of L with Levi component M.

Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a semisimple character and let κ be a standard β-extension to J. We form the representation κ_{P} of J_{P} as above, and also the representation κ_{Q} of J_{Q}. Note that, since $\mathrm{M} \subseteq \mathrm{L}$, we have $\mathrm{J}_{\mathrm{P}} \subseteq \mathrm{J}_{\mathrm{Q}}$, and κ_{P} can be viewed as the natural representation on the $\mathrm{J} \cap \mathrm{U}$-fixed vectors in κ_{Q}.

We also have $\mathrm{J}_{\mathrm{Q}} \cap \mathrm{L} = \mathrm{J} \cap \mathrm{L}$, since $\mathrm{G}_{\mathrm{E}} \subseteq \mathrm{L}$, and we can consider the natural representation of $\mathrm{J}_{\mathrm{P}} \cap \mathrm{L}$ on the $\mathrm{J} \cap \mathrm{L} \cap \mathrm{U}$-fixed vectors in $\kappa_{\mathrm{Q}}|_{\mathrm{J}\cap\mathrm{L}}$. This is naturally isomorphic to the restriction $\kappa_{\mathrm{P}}|_{\mathrm{J}_{\mathrm{P}}\cap\mathrm{L}}$. We will need the following compatibility result.

Proposition 3.11 *In the situation above, the restriction $\kappa_{\mathrm{P}}|_{\mathrm{J}_{\mathrm{P}}\cap\mathrm{L}}$ takes the form $\kappa'_{\mathrm{P}\cap\mathrm{L}}$, where $\kappa' := \kappa_{\mathrm{Q}}|_{\mathrm{J}\cap\mathrm{L}}$ is a standard β-extension of θ_{L} to $\mathrm{J}_{\mathrm{L}} = \mathrm{J} \cap \mathrm{L}$.*

Proof We need to check that $\kappa' := \kappa_{\mathrm{Q}}|_{\mathrm{J}\cap\mathrm{L}}$ is a standard β-extension. If $\mathfrak{A}(\Lambda_{\mathcal{O}_{\mathrm{E}}})$ is a maximal self-dual order in B (in which case L = M) then this follows from [29, Proposition 6.3].

For the general case, denote by $\kappa_{\mathfrak{M}}$ the unique β-extension of $\mathrm{J}_{\mathfrak{M}} = \mathrm{J}(\beta, \mathfrak{M}_{\Lambda})$ compatible with κ; then $\kappa_{\mathfrak{M},\mathrm{Q}}|_{\mathrm{J}_{\mathfrak{M}}\cap\mathrm{L}}$ is a (standard) β-extension by the previous case, and $\kappa_{\mathrm{Q}}|_{\mathrm{J}\cap\mathrm{L}}$ is compatible with $\kappa_{\mathfrak{M},\mathrm{Q}}|_{\mathrm{J}_{\mathfrak{M}}\cap\mathrm{L}}$, by [6, Proposition 1.17]. Thus κ' is indeed a standard β-extension.

4 Cuspidal types

In this section we recall the notions of cuspidal types from [9,29], correcting along the way a mistake in the definition in [29] pointed out by Laure Blasco and Corinne Blondel.

4.1 We recall from [9] the definition of a *simple type* and of a *maximal simple type* for $\widetilde{\mathrm{G}}$; we call the latter a *cuspidal type*. The generalizations to the case of lattice sequences come from [26].

Definition 4.1 A *simple type* for $\widetilde{\mathrm{G}}$ is a pair $(\widetilde{\mathrm{J}}, \widetilde{\lambda})$, where $\widetilde{\mathrm{J}} = \widetilde{\mathrm{J}}(\beta, \Lambda)$ for some simple stratum $[\Lambda, n, 0, \beta]$ such that

- $\widetilde{\mathrm{P}}(\Lambda_{\mathcal{O}_\mathrm{E}})/\widetilde{\mathrm{P}}_1(\Lambda_{\mathcal{O}_\mathrm{E}}) \simeq \mathrm{GL}_f(k_\mathrm{E})^e$, for some positive integers f, e,

and $\widetilde{\lambda} = \widetilde{\kappa} \otimes \widetilde{\tau}$, for $\widetilde{\kappa}$ a β-extension of some simple character $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ and $\widetilde{\tau}$ the inflation of an irreducible cuspidal representation $\widetilde{\tau}_0^{\otimes e}$ of $\widetilde{\mathrm{J}}/\widetilde{\mathrm{J}}^1 \simeq \mathrm{GL}_f(k_\mathrm{E})^e$.

A *cuspidal type* for $\widetilde{\mathrm{G}}$ is a simple type for which $\widetilde{\mathrm{P}}(\Lambda_{\mathcal{O}_\mathrm{E}})$ is a maximal parahoric subgroup of $\widetilde{\mathrm{G}}_\mathrm{E}$; that is, $e = 1$ in the notation above.

Every irreducible cuspidal representation $\widetilde{\pi}$ of $\widetilde{\mathrm{G}}$ contains a cuspidal type $(\widetilde{\mathrm{J}}, \widetilde{\lambda})$. Then $\widetilde{\pi}$ is irreducibly compactly induced from a representation of $\mathrm{E}^\times\widetilde{\mathrm{J}}$ containing $\widetilde{\lambda}$ and the cuspidal type $(\widetilde{\mathrm{J}}, \widetilde{\lambda})$ is a $[\widetilde{\mathrm{G}}, \widetilde{\pi}]_{\widetilde{\mathrm{G}}}$-type.

The following proposition can be extracted from the results in [9, §§7–8] (see also [25, Proposition 5.15, Corollaire 5.20]).

Proposition 4.2 *Let $[\Lambda, n, 0, \beta]$ be a simple stratum, $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ a simple character, and $\widetilde{\kappa}$ a β-extension. Let $\widetilde{\tau}$ be (the inflation to $\widetilde{\mathrm{J}}$ of) an irreducible representation of $\widetilde{\mathrm{P}}(\Lambda_{\mathcal{O}_\mathrm{E}})/\widetilde{\mathrm{P}}_1(\Lambda_{\mathcal{O}_\mathrm{E}})$. Suppose a cuspidal representation $\widetilde{\pi}$ of $\widetilde{\mathrm{G}}$ contains $\widetilde{\theta}$ and $\widetilde{\kappa} \otimes \widetilde{\tau}$. Then $\widetilde{\mathrm{P}}(\Lambda_{\mathcal{O}_\mathrm{E}})$ is a maximal parahoric subgroup of $\widetilde{\mathrm{G}}_\mathrm{E}$, and $\widetilde{\tau}$ is cuspidal; that is, $(\widetilde{\mathrm{J}}, \widetilde{\kappa} \otimes \widetilde{\tau})$ is a cuspidal type.*

4.2 Now we recall from [29] the (corrected) definition of a maximal simple type for G, which we again call a cuspidal type. Recall that we have assumed that G is not itself a split two-dimensional special orthogonal group; thus its centre is compact.

Definition 4.3 A *cuspidal type* for G is a pair (J, λ), where $\mathrm{J} = \mathrm{J}(\beta, \Lambda)$ for some skew semisimple stratum $[\Lambda, n, 0, \beta]$ such that

- G_E has compact centre and
- $\mathrm{P}^\circ(\Lambda_{\mathcal{O}_\mathrm{E}})$ a maximal parahoric subgroup of G_E,

and $\lambda = \kappa \otimes \tau$, for κ a β-extension and τ the inflation of an irreducible cuspidal representation of $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{P}(\Lambda_{\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\Lambda_{\mathcal{O}_\mathrm{E}})$.

By [29, Theorem 7.14], whose proof is corrected in the appendix, every irreducible cuspidal representation of G contains a cuspidal type. Moreover, the proof in the appendix shows that we have the following analogue of Proposition 4.2.

Proposition 4.4 *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum, $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ a semisimple character, and κ a standard β-extension. Let τ be (the inflation to* J *of) an irreducible representation of $\mathrm{P}(\Lambda_{\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\Lambda_{\mathcal{O}_\mathrm{E}})$. Suppose a cuspidal representation π of* G *contains θ and $\kappa \otimes \tau$. Then G_E has compact centre, $\mathrm{P}^\circ(\Lambda_{\mathcal{O}_\mathrm{E}})$ is a maximal parahoric subgroup of G_E, and τ is cuspidal; that is, $(\mathrm{J}, \kappa \otimes \tau)$ is a cuspidal type.*

5 Semisimple types

In this section we will prove Theorem 1.1 of the introduction, explaining how to construct a type for each Bernstein component, via the theory of covers.

5.1 We suppose given a Levi subgroup M of G, which is the stabilizer of the self-dual decomposition

$$\mathrm{V} = \mathrm{W}_{-m} \oplus \cdots \oplus \mathrm{W}_m; \qquad (*)$$

thus, putting $\widetilde{\mathrm{G}}_j = \mathrm{Aut}_{\mathrm{F}}(\mathrm{W}_j)$ and $\mathrm{G}_0 = \mathrm{Aut}_{\mathrm{F}}(\mathrm{W}_0) \cap \mathrm{G}$, we have $\mathrm{M} = \mathrm{G}_0 \times \prod_{j=1}^m \widetilde{\mathrm{G}}_j$. Let τ be a cuspidal irreducible representation of M, which we write $\tau = \tau_0 \otimes \bigotimes_{j=1}^m \widetilde{\tau}_j$.

Also let $\mathscr{M}$ denote the stabilizer of the decomposition $(*)$ in A; thus $\mathscr{M}_-$ is the Lie algebra of M. For $-m \leq j \leq m$, we denote by $\mathbf{e}_j$ the idempotent given by projection onto W_j.

For each $j > 0$, let $[\Lambda_j, n_j, 0, \beta_j]$ be a simple stratum in A_j and let $\widetilde{\theta}_j$ in $\mathcal{C}(\Lambda_j, 0, \beta_j)$ be such that $\widetilde{\tau}_j$ contains $\widetilde{\theta}_j$; let also $[\Lambda_0, n_0, 0, \beta_0]$ be a skew semisimple stratum in A_0 and let $\theta_0 \in \mathcal{C}_-(\Lambda_0, 0, \beta_0)$ be such that τ_0 contains θ_0.

Proposition 5.1 ([14, Proposition 8.4]) *There are a self-dual semisimple stratum* $[\Lambda, n, 0, \beta]$ *with* $\beta \in \mathscr{M}$*, and a self-dual semisimple character* θ *of* $\mathrm{H}^1 = \mathrm{H}^1(\beta, \Lambda)$ *such that:*

(i) *The decomposition* $(*)$ *is properly subordinate to* $[\Lambda, n, 0, \beta]$*;*
(ii) $\mathrm{H}^1(\beta, \Lambda) \cap \mathrm{M} = \mathrm{H}^1(\beta_0, \Lambda_0) \times \prod_{j=1}^m \widetilde{\mathrm{H}}^1(\beta_j, \Lambda_j)$*; and*
(iii) $\theta|_{\mathrm{H}^1(\beta,\Lambda)\cap\mathrm{M}} = \theta_0 \otimes \bigotimes_{j=1}^m \widetilde{\theta}_j$.

Proof (ii) and (iii) are given by [14, Proposition 8.4], and (i) by the comments following its statement.

For $j \neq 0$, we note that (iii) implies that $\widetilde{\theta}_j$ is a simple character for the simple stratum $[\Lambda_j, n_j, 0, 2\mathbf{e}_j\beta\mathbf{e}_j]$; likewise, θ_0 is a skew semisimple character for $[\Lambda_0, n_0, 0, \mathbf{e}_0\beta\mathbf{e}_0]$. Thus we may, and do, assume that $\beta_j = 2\mathbf{e}_j\beta\mathbf{e}_j$, for $j > 0$, and $\beta_0 = \mathbf{e}_0\beta\mathbf{e}_0$. Similarly, we may and do assume that the lattice sequence Λ_j is equal to $\Lambda \cap \mathrm{W}_j$.

Remark 5.2 The property in Proposition 5.1 that $[\Lambda, n, 0, \beta]$ is semisimple is strictly stronger than the property that each stratum $[\Lambda_j, n_j, 0, \mathbf{e}_j\beta\mathbf{e}_j]$ is (semi)simple. In general, the direct sum of (semi)simple strata need not be semisimple.

Let $\mathrm{V} = \bigoplus_{i=-l}^{l} \mathrm{V}^i$ be the self-dual decomposition associated to the stratum $[\Lambda, n, 0, \beta]$ and let $\mathrm{L} = \mathrm{L}_\beta$ be the G-stabilizer of this decomposition. Since $\beta \in \mathscr{M}$, this is a coarsening of the decomposition $(*)$: that is, each V^i is a sum of certain W_j, with $\mathrm{W}_0 \subseteq \mathrm{V}^0$, so that $\mathrm{L} \supseteq \mathrm{M}$.

We will abbreviate $\mathrm{H}^1 = \mathrm{H}^1(\beta, \Lambda)$, and similarly $\mathrm{J}^1, \mathrm{J}^\circ, \mathrm{J}$. By Proposition 5.1(i), all these groups have Iwahori decompositions with respect to (M, P), for any parabolic subgroup $\mathrm{P} = \mathrm{MU}$ with Levi component M; thus we may form the groups $\mathrm{H}^1_{\mathrm{P}}, \mathrm{J}^1_{\mathrm{P}}, \mathrm{J}^\circ_{\mathrm{P}}, \mathrm{J}_{\mathrm{P}}$ as in Sect. 3.

Write G_{E} for the centralizer of β in G, so $\mathrm{G}_{\mathrm{E}} \subseteq \mathrm{L}$. We note that, by Propositions 4.2 and 4.4, the group $\mathrm{J}^\circ \cap \mathrm{G}_{\mathrm{E}} \cap \mathrm{M}$ is a maximal parahoric subgroup of $\mathrm{G}_{\mathrm{E}} \cap \mathrm{M}$. In particular, the decomposition $(*)$ is *exactly subordinate* to $[\Lambda, n, 0, \beta]$, in the language of [29, Definition 6.5]. (In fact, the definition of exactly subordinate in *loc. cit.* should have required that $\mathrm{P}^\circ(\Lambda_{\mathcal{O}_{\mathrm{E}}}) \cap \mathrm{M}$ be a maximal parahoric subgroup of $\mathrm{G}_{\mathrm{E}} \cap \mathrm{M}$.)

Let η be the unique irreducible representation of J^1 containing θ, and choose a standard β-extension κ of θ. Denote by κ_P the natural representation of J_P on the $(J \cap U)$-fixed vectors in κ, by η_P its restriction to J^1_P, and by θ_P the character of H^1_P which extends θ and is trivial on $J^1 \cap U$.

Since the decomposition $(*)$ is exactly subordinate to $[\Lambda, n, 0, \beta]$, by [29, Proposition 6.3] the restriction $\kappa_M = \kappa_P|_{J\cap M}$ is a standard β-extension of $\eta_M = \eta_P|_{J^1\cap M}$, which is itself the unique irreducible representation of $J^1 \cap M$ containing $\theta_M = \theta|_{H^1\cap M}$; this means that $\kappa_M = \kappa_0 \otimes \bigotimes_{j=1}^m \widetilde{\kappa}_j$, where $\widetilde{\kappa}_j$ is a β_j-extension containing $\widetilde{\theta}_j$ and κ_0 is a standard β_0-extension containing θ_0.

Since τ contains θ_M, it also contains η_M, and hence some representation of $J^\circ \cap M$ of the form $\lambda^\circ_M = \kappa_M \otimes \rho^\circ_M$, with ρ°_M the inflation to $J^\circ \cap M$ of an irreducible representation of the connected reductive group $P^\circ(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E})$. Moreover, by Propositions 4.2 and 4.4, the representation ρ°_M is necessarily cuspidal. We write $\rho^\circ_M = \rho^\circ_0 \otimes \bigotimes_{j=1}^m \widetilde{\rho}_j$, where $\widetilde{\rho}_j$ is a cuspidal representation of $\widetilde{P}(\Lambda_{j,\mathcal{O}_{E_j}})/\widetilde{P}_1(\Lambda_{j,\mathcal{O}_{E_j}})$, for $j \geq 1$, and ρ°_0 is a cuspidal representation of $P^\circ(\Lambda_{0,\mathcal{O}_{E_0}})/P_1(\Lambda_{0,\mathcal{O}_{E_0}})$.

Now we have an isomorphism $J^\circ_P/J^1_P \simeq (J^\circ \cap M)/(J^1 \cap M)$ so we can also regard ρ°_M as a representation of J°_P by inflation. Thus we can form the representation $\lambda^\circ_P = \kappa_P \otimes \rho^\circ_M$ of J°_P. The main result is then:

Theorem 5.3 *The pair* $(J^\circ_P, \lambda^\circ_P)$ *is a cover of* $(J^\circ_P \cap M, \lambda^\circ_M)$.

Remark 5.4 Certainly, the pair $(J^\circ_P, \lambda^\circ_P)$ is a decomposed pair above $(J^\circ_P \cap M, \lambda^\circ_M)$, in the sense of [11, Definition 6.1]. Moreover, putting $\lambda^\circ = \mathrm{Ind}_{J^\circ_P}^{J^\circ} \lambda^\circ_P = \kappa \otimes \lambda^\circ_M$ (with λ°_M regarded as a representation of J° trivial on J^1), we have a support-preserving isomorphism

$$\mathscr{H}(G, \lambda^\circ_P) \simeq \mathscr{H}(G, \lambda^\circ),$$

as in [29, Lemma 6.1]. In particular, the condition on the Hecke algebra which needs to be checked to prove that $(J^\circ_P, \lambda^\circ_P)$ is a cover is independent of the choice of parabolic subgroup P with Levi component M. Thus we can, and will, change our choice of P where necessary.

The proof of Theorem 5.3 will occupy the next few paragraphs. Let us see how this implies Theorem 1.1 of the introduction.

Proof of Theorem 1.1 Since τ contains λ°_M, it contains some irreducible representation λ_M of $J \cap M = J_P \cap M$ which contains λ°_M; more precisely, we can write $\lambda_M = \kappa_M \otimes \rho_M$, with ρ_M the inflation of an irreducible representation of $P(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E})$ which contains ρ°_M. Thus $(J_P \cap M, \lambda_M)$ is a cuspidal type in M, which is an $[M, \tau]_M$-type.

We put $\lambda_P = \kappa_P \otimes \rho_M$, so that $\lambda_P|_{J_P\cap M} = \lambda_M$. Then certainly (J_P, λ_P) is a decomposed pair above $(J_P \cap M, \lambda_M)$, while $(J^\circ_P, \lambda^\circ_P)$ is a cover of $(J^\circ_P \cap M, \lambda^\circ_M)$, by Theorem 5.3. Thus, by [23, Lemma 3.9], (J_P, λ_P) is also a cover of $(J_P \cap M, \lambda_M)$. Since $(J_P \cap M, \lambda_M)$ is an $[M, \tau]_M$-type, we conclude from [11, Theorem 8.3] that (J_P, λ_P) is an $[M, \tau]_G$-type. Since the pair (M, τ) was arbitrary, we have a type for every Bernstein component.

5.2 The proof of Theorem 5.3 proceeds by transitivity of covers [11, Proposition 8.5]. Putting $\lambda_{\mathrm{L}}^{\circ} = \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}}$, the first step is to show:

Lemma 5.5 *The pair* $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ *is a cover of* $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}, \lambda_{\mathrm{L}}^{\circ})$. *Moreover, there is a support-preserving Hecke algebra isomorphism*

$$\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ}) \simeq \mathscr{H}(\mathrm{L}, \lambda_{\mathrm{L}}^{\circ}).$$

Proof Since $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ is a decomposed pair above $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{M}, \lambda_{\mathrm{M}}^{\circ})$ and $\mathrm{M}\subseteq\mathrm{L}\subseteq\mathrm{G}$, it is certainly also a decomposed pair above $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}, \lambda_{\mathrm{L}}^{\circ})$.

Now the support of the Hecke algebra $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ})$ is the intertwining of $\lambda_{\mathrm{L}}^{\circ}$, which is contained in the intertwining of θ_{P}. By Lemma 3.10(i), this intertwining is $\mathrm{J}_{\mathrm{P}}\mathrm{G}_{\mathrm{E}}\mathrm{J}_{\mathrm{P}} \subseteq \mathrm{J}_{\mathrm{P}}^{\circ}\mathrm{L}\mathrm{J}_{\mathrm{P}}^{\circ}$. The result now follows from [11, Theorem 7.2].

5.3 Lemma 5.5 reduces us to proving that $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}, \lambda_{\mathrm{L}}^{\circ})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{M}, \lambda_{\mathrm{M}}^{\circ})$. By Proposition 3.11, we have $\lambda_{\mathrm{L}}^{\circ} = \kappa'_{\mathrm{P}\cap\mathrm{L}} \otimes \rho_{\mathrm{M}}^{\circ}$, where $\kappa' = \kappa_{\mathrm{Q}}|_{\mathrm{J}^{\circ}\cap\mathrm{L}}$ is a standard β-extension of $\theta|_{\mathrm{H}^1\cap\mathrm{L}}$, and we think of $\rho_{\mathrm{M}}^{\circ}$ as a representation of $\left(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}\right)/\left(\mathrm{J}_{\mathrm{P}}^{1}\cap\mathrm{L}\right) \simeq (\mathrm{J}^{\circ}\cap\mathrm{M})/\left(\mathrm{J}^{1}\cap\mathrm{M}\right)$. The first step is to describe $\kappa'_{\mathrm{P}\cap\mathrm{L}}$, whence $\lambda_{\mathrm{L}}^{\circ}$, more carefully.

Recall that $V = \bigoplus_{i=-l}^{l} \mathrm{V}^i$ is the self-dual decomposition associated to the semi-simple stratum $[\Lambda, n, 0, \beta]$, and that, for each i, we have $\mathrm{V}^i = \bigoplus_{j\in J_i} \mathrm{W}_j$, for some subset J_i of $\{-m, \dots, m\}$. Writing $\mathbf{e}^i$ for the projection onto V^i as usual, and $\Lambda^i = \Lambda\cap\mathrm{V}^i$, the stratum $[\Lambda^i, n_i, 0, \mathbf{e}^i\beta\mathbf{e}^i]$ is

- skew semisimple, for $i = 0$,
- simple, for $i \neq 0$,

where $n_i = -\upsilon_{\Lambda}(\mathbf{e}^i\beta\mathbf{e}^i) = -\upsilon_{\Lambda^i}(\mathbf{e}^i\beta\mathbf{e}^i)$.

We write $\mathrm{L} = \mathrm{G}^0 \times \prod_{i=1}^{l} \widetilde{\mathrm{G}}^i$, where $\widetilde{\mathrm{G}}^i = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}^i)$. We have $\theta|_{\mathrm{H}^1\cap\mathrm{L}} = \theta'_0 \otimes \bigotimes_{i=1}^{l} \widetilde{\theta}'_i$, where θ'_0 is a skew semisimple character in $\mathcal{C}_-(\Lambda^0, 0, \mathbf{e}^0\beta\mathbf{e}^0)$, and $\widetilde{\theta}'_i$ is a simple character in $\mathcal{C}(\Lambda^i, 0, 2\mathbf{e}^i\beta\mathbf{e}^i)$. Then the standard β-extension κ' takes the form $\kappa' = \kappa'_0 \otimes \bigotimes_{i=1}^{l} \widetilde{\kappa}'_i$, for κ'_0 a standard $\mathbf{e}^0\beta\mathbf{e}^0$-extension of θ'_0, and $\widetilde{\kappa}'_i$ a $2\mathbf{e}^i\beta\mathbf{e}^i$-extension of $\widetilde{\theta}'_i$.

Since $\mathrm{M}\subseteq\mathrm{L}$, we have $\mathrm{P}\cap\mathrm{L} = \mathrm{P}^0 \times \prod_{i=1}^{l} \widetilde{\mathrm{P}}^i$, with P^i a parabolic subgroup of G^i. We put $\rho'_0 = \rho_0^{\circ} \otimes \bigotimes_{j\in J_0,\, j>0} \widetilde{\rho}_j$, and $\widetilde{\rho}'_i = \bigotimes_{j\in J_i} \widetilde{\rho}_j$, for $i > 0$. Then we put $\lambda_0^{\circ} = \kappa'_0 \otimes \rho'_0$, and $\widetilde{\lambda}'_i = \widetilde{\kappa}'_i \otimes \widetilde{\rho}'_i$, for $i > 0$.

Now $\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L} = \mathrm{J}_{\mathrm{P}^0}^{\circ} \times \prod_{i=1}^{l} \widetilde{\mathrm{J}}_{\widetilde{\mathrm{P}}^i}$ (with the obvious notation) and we have $\kappa'_{\mathrm{P}\cap\mathrm{L}} \simeq \kappa'_{0,\mathrm{P}^0} \otimes \bigotimes_{i=1}^{l} \widetilde{\kappa}'_{i,\widetilde{\mathrm{P}}^i}$. In particular, we also get $\lambda_{\mathrm{L}}^{\circ} \simeq \lambda_{0,\mathrm{P}^0}^{\circ} \otimes \bigotimes_{i=1}^{l} \widetilde{\lambda}'_{i,\widetilde{\mathrm{P}}^i}$.

Finally, we write $\mathrm{M}^0 = \mathrm{M}\cap\mathrm{G}^0$ and $\widetilde{\mathrm{M}}^i = \mathrm{M}\cap\widetilde{\mathrm{G}}^i$, for $i > 0$, so that $\mathrm{M} = \mathrm{M}^0 \times \prod_{i=1}^{l} \widetilde{\mathrm{M}}^i$. Then, in order to prove that $(\mathrm{J}_{\mathrm{P}}^{\circ}\cap\mathrm{L}, \lambda_{\mathrm{L}}^{\circ})$ is a cover of $(\mathrm{J}^{\circ}\cap\mathrm{M}, \lambda_{\mathrm{M}}^{\circ})$ we need to show:

- $(\mathrm{J}_{\mathrm{P}^0}^{\circ}, \lambda_{0,\mathrm{P}^0}^{\circ})$ is a cover of $(\mathrm{J}_{\mathrm{P}^0}^{\circ}\cap\mathrm{M}^0, \lambda_{0,\mathrm{P}^0}^{\circ}|_{\mathrm{J}_{\mathrm{P}^0}\cap\mathrm{M}^0})$; and
- $(\widetilde{\mathrm{J}}_{\widetilde{\mathrm{P}}^i}, \widetilde{\lambda}'_{i,\widetilde{\mathrm{P}}^i})$ is a cover of $(\widetilde{\mathrm{J}}_{\widetilde{\mathrm{P}}^i}\cap\widetilde{\mathrm{M}}^i, \widetilde{\lambda}'_{i,\widetilde{\mathrm{P}}^i}|_{\widetilde{\mathrm{J}}_{\widetilde{\mathrm{P}}^i}\cap\widetilde{\mathrm{M}}^i})$, for $i > 0$.

The latter is given by [26, Proposition 8.1]: since the underlying stratum is simple, it is a *homogeneous semisimple type*, in the sense of [12,26]. On the other hand, the former is the case of a *skew* semisimple stratum; that is, we have reduced the proof of Theorem 5.3 to the case $\mathrm{L} = \mathrm{G}$, and we are in the situation of [29, §7]. Indeed it is possible to extract the proof that we get a cover here from the results in *loc. cit.*, which we will do in the following paragraphs.

5.4 We are now in the situation of Theorem 5.3 in the special case $\mathrm{L} = \mathrm{G}$, so that $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum. In [29, §6.3], an involution σ_j is defined on $\widetilde{\mathrm{G}}_j$, for $j > 0$, coming from the composition of the involution σ on G and a Weyl group element which exchanges W_j with W_{-j}. By [29, Lemma 6.9, Corollary 6.10], the group $\widetilde{\mathrm{J}}(\beta_j, \Lambda_j)$ is stable under this involution, and $\widetilde{\kappa}_j \simeq \widetilde{\kappa}_j \circ \sigma_j$.

Recall that we have $\rho_{\mathrm{M}}^{\circ} = \rho_0^{\circ} \otimes \bigotimes_{j=1}^{m} \widetilde{\rho}_j$. For $j > 0$ we put $\widetilde{\rho}_{-j} = \widetilde{\rho}_j \circ \sigma_j$.

We suppose first that there is an index $k > 0$ such that $\widetilde{\rho}_k \not\simeq \widetilde{\rho}_{-k}$. We put

$$J_1 = \{-m \le j \le m \mid \widetilde{\rho}_j \simeq \widetilde{\rho}_k\}, \quad J_0 = \{j \mid \pm j \notin J_1\}, \quad J_{-1} = \{-j \mid j \in J_1\},$$

and set $\mathrm{Y}^i = \bigoplus_{j \in J_i} \mathrm{W}_j$, for $i = -1, 0, 1$. We have $\mathrm{V} = \mathrm{Y}^{-1} \oplus \mathrm{Y}^0 \oplus \mathrm{Y}^1$, since $\widetilde{\rho}_k \not\simeq \widetilde{\rho}_{-k}$. Let M' be the Levi subgroup of G stabilizing this decomposition and let $\mathrm{P}' = \mathrm{M}'\mathrm{U}'$ be a parabolic subgroup containing P. (Note that one may need to change the choice of the parabolic subgroup P in order to achieve this.) We have $\mathrm{M}' = \mathrm{G}^0 \times \widetilde{\mathrm{G}}^1$, where $\mathrm{G}^0 = \mathrm{Aut}_{\mathrm{F}}\mathrm{Y}^0 \cap \mathrm{G}$ and $\widetilde{\mathrm{G}}^1 = \mathrm{Aut}_{\mathrm{F}}\mathrm{Y}^1$, and write $\mathrm{M} = \mathrm{M}^0 \times \widetilde{\mathrm{M}}^1$ also.

By [29, Proposition 7.10] and its proof we have:

Lemma 5.6 *The pair* $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ *is a cover of* $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$ *and there is a support-preserving isomorphism of Hecke algebras* $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ}) \simeq \mathscr{H}(\mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$.

Now we have $\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}' = (\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0) \times (\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1)$ and, as in the previous paragraph, we need to prove:

- $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}^0, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}^0})$; and
- $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{M}}^1, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{M}}^1})$.

Again as in the previous paragraph, the latter is a cover by [26, Proposition 6.7]; it is a simple type. The former is again the case of a skew semisimple stratum, but with fewer indices j such that $\widetilde{\rho}_j \not\simeq \widetilde{\rho}_{-j}$. In particular, by repeating the process in this paragraph, we can reduce to the case where $\widetilde{\rho}_j \simeq \widetilde{\rho}_{-j}$, for all j.

5.5 We suppose now that G_{E} does not have compact centre. This implies that G is a special orthogonal group, that $\beta_k = 0$ for a unique $k > 0$, and that $\dim_{\mathrm{F}} \mathrm{W}_k = 1$. In this case set $\mathrm{Y}^1 = \mathrm{W}_k$, $\mathrm{Y}^{-1} = \mathrm{W}_{-k}$, and $\mathrm{Y}^0 = \bigoplus_{j \ne \pm k} \mathrm{W}_j$, let M' be the Levi subgroup stabilizing the decomposition $\mathrm{V} = \mathrm{Y}^{-1} \oplus \mathrm{Y}^0 \oplus \mathrm{Y}^1$, and let $\mathrm{P}' = \mathrm{M}'\mathrm{U}'$ be a parabolic subgroup containing P. (Again, this may require the choice of P to be changed.) We have $\mathrm{G}_{\mathrm{E}} \subseteq \mathrm{M}'$ and, by [29, Corollary 6.16], $\mathrm{I}_{\mathrm{G}}(\lambda_{\mathrm{P}}^{\circ}) \subseteq \mathrm{J}_{\mathrm{P}}^{\circ}\mathrm{M}'\mathrm{J}_{\mathrm{P}}^{\circ}$. In particular we get:

Lemma 5.7 *The pair* $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ *is a cover of* $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$ *and there is a support-preserving isomorphism of Hecke algebras* $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ}) \simeq \mathscr{H}(\mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$.

As in previous paragraphs, this reduces us to the case where G_{E} has compact centre.

5.6 We have finally reduced to the case where $\widetilde{\rho}_j \simeq \widetilde{\rho}_{-j}$, for all j and G_E has compact centre; this is exactly the situation of [29, §7.2.2]. Moreover, by changing P if necessary, we may assume the parabolic subgroup is the same one as in *loc. cit.* In [29, §7.2.2], two auxiliary $\mathcal{O}_E$-lattice sequences $\mathfrak{M}_t$, $t = 0, 1$, are defined, along with Weyl group elements $s_t \in P(\mathfrak{M}_{t,\mathcal{O}_E})$, which we describe below, along with some auxiliary elements. We have $G_E = \prod_{i \in I_0} G_{E_i}$ and we will write $I_0 = \{1, \ldots, l\}$, to match the notation of [29, §7.2.2]; then $W^{(m)} \subset V^\ell$, with $1 \le \ell \le l$ maximal such that V^ℓ contains some $W^{(j)}$, and $\beta_i \ne 0$ for $i > 1$.

We put $W^{(\ell,0)} = V^\ell \cap W_0$ and denote by $\Lambda^{(\ell,0)}$ the $\mathcal{O}_{E_\ell}$-lattice sequence $\Lambda \cap W^{(\ell,0)}$. Let $p_\Lambda \in P(\Lambda^{(\ell,0)}_{\mathcal{O}_{E_\ell}})$ be an element of order at most 2 such that the quotient $P(\Lambda^{(\ell,0)}_{\mathcal{O}_{E_\ell}})/P^\circ(\Lambda^{(\ell,0)}_{\mathcal{O}_{E_\ell}})$ (which has order 1 or 2) is generated by the image of p_Λ. Then also $P(\Lambda^\ell_{\mathcal{O}_{E_\ell}})/P^\circ(\Lambda^\ell_{\mathcal{O}_{E_\ell}})$ is generated by the image of p_Λ. We split into cases.

(i) Suppose either that G_{E_ℓ} is *not* an orthogonal group, or that $\dim_{E_\ell} W_m$ is even. Then s_0, s_1 are the elements denoted s_m, s_m^ϖ respectively in *loc. cit.*. Note that p_Λ commutes with both s_0 and s_1.
In this situation, it is straightforward to check, using the definitions of the elements in [29, §6.2], that $s_t \in P^\circ(\mathfrak{M}_{t,\mathcal{O}_E})$ unless $E_\ell/E_{\ell,\circ}$ is ramified, m is odd (so $m = 1$) and $\varepsilon = (-1)^t$. Moreover, if $s_t \notin P^\circ(\mathfrak{M}_{t,\mathcal{O}_E})$ then either $p_\Lambda s_t \in P^\circ(\mathfrak{M}_{t,\mathcal{O}_E})$ or else $P^\circ(\mathfrak{M}_{t,\mathcal{O}_E}) = P^\circ(\Lambda_{\mathcal{O}_E})$, in which case $P(\mathfrak{M}^\ell_{t,\mathcal{O}_E})/P_1(\mathfrak{M}^\ell_{t,\mathcal{O}_E})$ has the form $O(1,1)(k_{E_\ell}) \times \mathcal{G}$, for $\mathcal{G}$ some product of connected finite reductive groups, while $P(\Lambda^\ell_{\mathcal{O}_E})/P_1(\Lambda^\ell_{\mathcal{O}_E})$ has the form $SO(1,1)(k_{E_\ell}) \times \mathcal{G}$.

(ii) If G_{E_ℓ} is (special) orthogonal (so that $E_\ell = F$ and $\varepsilon = +1$) and $\dim_F W_m$ is odd, the choice of P and the property that $\widetilde{\rho}_j \simeq \widetilde{\rho}_{-j}$, for all j, mean that $\ell = 1$ and $\dim_F W_k = 1$, for all $k > 0$, and there are two cases.
  (a) If $G^+_{E_1} \cap \mathrm{Aut}_F(W_0) \ne 1$, then $P^\circ(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E}) \simeq GL_1(k_F) \times \mathcal{G}^\circ_0 \times \mathcal{G}^\circ_1 \times \mathcal{G}^\circ$, where each $\mathcal{G}^\circ_t$ is a special orthogonal group over k_F (one of which may be trivial) and $\mathcal{G}^\circ$ is some product of connected finite reductive groups. If $\mathcal{G}^\circ_t$ is non-trivial, then there is an element $p_t \in P^+(\Lambda^{(1,0)}) \backslash P(\Lambda^{(1,0)})$ such that $p_t^2 = 1$, which commutes with both s_m, s_m^ϖ, and whose image in $P^+(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E}) \simeq GL_1(k_F) \times \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}$ lies in the orthogonal group $\mathcal{G}_t$ (whose connected component is $\mathcal{G}^\circ_t$). If $\mathcal{G}^\circ_t$ is trivial, we put $p_t = p_{1-t}$; in any case, p_0, p_1 commute. Moreover, we can assume that $p_\Lambda = p_0 p_1$.
  If exactly one of p_0, p_1 normalizes the representation ρ°_M, viewed as a representation of $P^\circ(\Lambda_{\mathcal{O}_E})$ trivial on $P_1(\Lambda_{\mathcal{O}_E})$, then we set p to be this element; if both or neither normalize, then we arbitrarily choose p to be one of them. Then s_0, s_1 are the elements ps_m, ps_m^ϖ respectively, which lie in G_E.
  Note that $s_t \in P^\circ(\mathfrak{M}_{t,\mathcal{O}_E})$ precisely when $\mathcal{G}^\circ_t$ is non-trivial and $p = p_t$. If $\mathcal{G}^\circ_t$ is trivial, then $P^\circ(\mathfrak{M}_{t,\mathcal{O}_E})/P_1(\mathfrak{M}_{t,\mathcal{O}_E}) \simeq SO(1,1)(k_F) \times \mathcal{G}^\circ_{1-t} \times \mathcal{G}^\circ$.
  (b) Otherwise, s_0, s_1 are the elements denoted $s_m s_{m-1}, s_m^\varpi s_{m-1}^\varpi$ respectively in [29, §7.2.2]. Note that in this case $m \ge 2$, since G_{E_1} has compact centre so we cannot have $G_{E_1} \simeq SO(1,1)(F)$.

In all cases but case (ii)(b), we set $\mathrm{Y}^1 = \mathrm{W}_m$, $\mathrm{Y}^{-1} = \mathrm{W}_{-m}$ and $\mathrm{Y}^0 = \sum_{j \neq \pm m} \mathrm{W}_j$; in the exceptional case we set $\mathrm{Y}^1 = \mathrm{W}_m \oplus \mathrm{W}_{m-1}$, $\mathrm{Y}^{-1} = \mathrm{W}_{-m} \oplus \mathrm{W}_{1-m}$ and $\mathrm{Y}^0 = \sum_{j \neq \pm m, m-1} \mathrm{W}_j$. Denote by M' the Levi subgroup stabilizing the decomposition $\mathrm{V} = \mathrm{Y}^{-1} \oplus \mathrm{Y}^0 \oplus \mathrm{Y}^1$, and let $\mathrm{P}' = \mathrm{M}'\mathrm{U}'$ be a parabolic subgroup containing P. We deal with an easy case first.

Lemma 5.8 *Suppose we are in case (ii)(a) and neither p_0 nor p_1 normalizes $\rho_{\mathrm{M}}^{\circ}$. Then the pair $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$ and there is a support-preserving isomorphism of Hecke algebras $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ}) \simeq \mathscr{H}(\mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$.*

Proof By [29, Corollary 6.16], we have $\mathrm{I}_{\mathrm{G}}(\lambda_{\mathrm{P}}^{\circ}) \subseteq \mathrm{J}_{\mathrm{P}}^{\circ}\mathrm{M}'\mathrm{J}_{\mathrm{P}}^{\circ}$, and the result follows as usual, as in Lemma 5.5.

Now suppose we are not in the case of Lemma 5.8. For $t = 0, 1$, denote by κ_t a β-extension of η compatible with some standard β-extension of $\mathrm{J}^{\circ}(\beta, \mathfrak{M}_t)$. By [29, Corollary 6.13], we have

$$\kappa_t \simeq \mathrm{Ind}_{\mathrm{J}_{\mathrm{P}}^{\circ}}^{\mathrm{J}^{\circ}} \kappa_{\mathrm{P}} \otimes \chi_t,$$

for some self-dual character χ_t. We write $\rho_t^{\circ} = \rho_M^{\circ} \otimes \chi_t^{-1}$, which is still a self-dual cuspidal representation. Moreover, by [29, (7.3)], there is a support-preserving injective algebra map

$$\mathscr{H}(\mathrm{P}(\mathfrak{M}_{t,\mathfrak{O}_{\mathrm{E}}}), \rho_t^{\circ}) \hookrightarrow \mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ}), \tag{$\dagger$}$$

where ρ_t° is being regarded as a cuspidal representation of $\mathrm{P}(\Lambda_{\mathfrak{O}_{\mathrm{E}}})$. By [22, Theorem 7.12], there is an invertible element in $\mathscr{H}(\mathrm{P}(\mathfrak{M}_{t,\mathfrak{O}_{\mathrm{E}}}), \rho_t^{\circ})$ with support $\mathrm{P}(\Lambda_{\mathfrak{O}_{\mathrm{E}}}) s_t \mathrm{P}(\Lambda_{\mathfrak{O}_{\mathrm{E}}})$, and we denote by T_t its image in $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ})$.

By [29, Lemmas 7.11,7.12], for a suitable integer e, the element $(T_0 T_1)^e$ is an invertible element of $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{\circ})$ supported on the double coset of a strongly $(\mathrm{P}, \mathrm{J}_{\mathrm{P}}^{\circ})$-positive element of the centre of M'. Indeed, we have:

Proposition 5.9 ([29, Proposition 7.13]) *The pair $(\mathrm{J}_{\mathrm{P}}^{\circ}, \lambda_{\mathrm{P}}^{\circ})$ is a cover of the pair $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}', \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}'})$.*

As in previous paragraphs, we have $\mathrm{M}' = \mathrm{G}^0 \times \widetilde{\mathrm{G}}^1$, where $\mathrm{G}^0 = \mathrm{Aut}_{\mathrm{F}} \mathrm{Y}^0 \cap \mathrm{G}$ and $\widetilde{\mathrm{G}}^1 = \mathrm{Aut}_{\mathrm{F}} \mathrm{Y}^1$, and we write $\mathrm{M} = \mathrm{M}^0 \times \widetilde{\mathrm{M}}^1$. Then $\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}' = (\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0) \times (\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1)$ and we need to prove:

- $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{G}^0})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}^0, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \mathrm{M}^0})$; and
- $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{G}}^1})$ is a cover of $(\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{M}}^1, \lambda_{\mathrm{P}}^{\circ}|_{\mathrm{J}_{\mathrm{P}}^{\circ} \cap \widetilde{\mathrm{M}}^1})$.

Again as previously, the latter is a cover by [26, Proposition 6.7]; it is a simple type. (In fact, except in case (ii)(b) above, we have $\widetilde{\mathrm{M}}^1 = \widetilde{\mathrm{G}}^1$.)

The former is again the case of a skew semisimple stratum, but with smaller m. In particular, by repeating the process in this paragraph, we reduce to the case $m = 0$, in which case $\mathrm{M}^0 = \mathrm{G}^0$ and there is nothing left to do.

6 Hecke algebras

In this section we prove Theorem 1.2 of the introduction: that is, we describe the Hecke algebra of a cover in the case that τ is a cuspidal irreducible representation of a *maximal* proper Levi subgroup M of G (so M is the stabilizer of a self-dual decomposition $V = W_{-1} \oplus W_0 \oplus W_1$), up to the computation of some parameters. We will also explain how, in principle, these parameters can be computed.

As in the introduction, we write $\mathfrak{s} = [M, \tau]_G$ and $\mathfrak{s}_M = [M, \tau]_M$ and put

$$N_G(\mathfrak{s}_M) = \{g \in N_G(M) : {}^g\tau \text{ is inertially equivalent to } \tau\}.$$

We also put $\mathbf{W}_{\mathfrak{s}} = N_G(\mathfrak{s}_M)/M$, a subgroup of the group $N_G(M)/M$ of order 2.

We denote by (J_P, λ_P) the $\mathfrak{s}$-type constructed in the previous section, and we put $J_M = J_P \cap M$ and $\lambda_M = \lambda_P|_{J_P \cap M}$, so that (J_P, λ_P) is a cover of the $\mathfrak{s}_M$-type (J_M, λ_M).

6.1 Before giving the proof, we give a brief explanation of the dichotomy $|\mathbf{W}_{\mathfrak{s}}| = 1$ or 2 and of the implications of the results here.

The case $|\mathbf{W}_{\mathfrak{s}}| = 1$ occurs precisely when the (normalized) parabolically induced representations $\mathrm{Ind}_{M,P}^G \tau\chi$ are irreducible *for all unramified characters* χ *of* M; thus, with arithmetic applications in mind, this case is rather uninteresting.

The case $|\mathbf{W}_{\mathfrak{s}}| = 2$ is much more interesting. Here, if we write $M = G_0 \times \widetilde{G}_1$ and $\tau = \tau_0 \otimes \widetilde{\tau}_1$ as in the previous section (so that $\widetilde{G}_1 = \mathrm{Aut}_F(W_1)$ is a general linear group), the condition is that $\widetilde{\tau}_1$ is equivalent to an unramified twist of the $\mathrm{Gal}(F/F_o)$-conjugate of its contragredient. In this case, replacing τ by an unramified twist, we can assume that $\widetilde{\tau}_1$ is equivalent to its conjugate-contragredient. Now determining those complex s for which $\mathrm{Ind}_{M,P}^G \tau_0 \otimes \widetilde{\tau}_1 |\det(\cdot)|^s$ is reducible is of great arithmetic interest; in particular, if we find that there is a real $s > \frac{1}{2}$ for which the corresponding representation is reducible, this should imply a transfer relation between τ_0 and $\widetilde{\tau}_1$ via the Langlands correspondence (namely, the Langlands parameter for $\widetilde{\tau}_1$ should appear in that of τ_0, viewed via the natural embedding of the L-group of G_0 into the L-group of some general linear group—see [21] for more on this).

On the other hand, the theory of types and covers gives us a commutative diagram

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}}(G) & \longleftrightarrow & \mathscr{H}(G, \lambda_P)\text{-Mod} \\ \uparrow{\scriptstyle \mathrm{Ind}_{M,P}^G} & & \uparrow{\scriptstyle (t_P)_*} \\ \mathfrak{R}^{\mathfrak{s}_M}(M) & \longleftrightarrow & \mathscr{H}(M, \lambda_M)\text{-Mod} \end{array}$$

where -Mod denotes the category of left modules, map $(t_P)_*$ is Hom-induction given via an embedding $t_P : \mathscr{H}(M, \lambda_M) \hookrightarrow \mathscr{H}(G, \lambda_P)$, and the horizontal arrows are equivalences of categories. The algebra $\mathscr{H}(M, \lambda_M)$ is abelian, isomorphic to $\mathbb{C}[X^{\pm 1}]$. Using this diagram, together with the embedding of Hecke algebras, one can in principle compute those s for which $\mathrm{Ind}_{M,P}^G \tau_0 \otimes \widetilde{\tau}_1 |\det(\cdot)|^s$ is reducible. For example, if the two parameters for the Hecke algebra are $q_F^{f_i}$, for $i = 0, 1$, (so that f_i may be a half-integer if F/F_o is unramified) then imaginary part of s must be a multiple of $\pi i / \log q_F$. Blondel shows in [6, Proposition 3.12] that the real part of s is

$$\pm\frac{(f_1 \pm f_2)}{2t(\widetilde{\tau}_1)},$$

where the signs are independent and $t(\widetilde{\tau}_1)$ denotes the *unramified twist number* of $\widetilde{\tau}_1$: the number of unramified characters χ such that $\widetilde{\tau}_1\chi \simeq \widetilde{\tau}_1$.

The parameters $q_F^{f_i}$ are almost computable from results of Lusztig [20]. The problem is the presence of the quadratic character χ_0 of paragraph 5.6, which (at least in some cases) must first be computed. We discuss this further in paragraph 6.5 below.

6.2 We proceed with the proof of Theorem 1.2. Suppose first that $N_G(\mathfrak{s}_M) = M$, so that $\mathbf{W}_{\mathfrak{s}}$ is trivial. In this case, by [13, Theorem 1.5], we have an isomorphism

$$\mathscr{H}(M, \lambda_M) \to \mathscr{H}(G, \lambda_P).$$

Since $\mathscr{H}(M, \lambda_M)$ is isomorphic to $\mathbb{C}[X^{\pm 1}]$, the result follows.

6.3 Now suppose that $N_G(\mathfrak{s}_M) \neq M$, so that $\mathbf{W}_{\mathfrak{s}}$ has order 2. We note first that, in this situation, we cannot have a support-preserving isomorphism $\mathscr{H}(M, \lambda_M) \to \mathscr{H}(G, \lambda_P)$ since the induced representation $\mathrm{Ind}_P^G \tau \otimes \chi$ reduces for some unramified character χ of M. This implies that we also do not have a support-preserving isomorphism $\mathscr{H}(M, \lambda_M^{\circ}) \to \mathscr{H}(G, \lambda_P^{\circ})$.

We now proceed through the construction of Sect. 5 and we use all the notation from there. Note that we must have $L = G$, or else we would have $L = M$ and Lemma 5.5 would give us an isomorphism $\mathscr{H}(M, \lambda_M^{\circ}) \to \mathscr{H}(G, \lambda_P^{\circ})$. Similarly, we cannot be in the situation of paragraph 5.4 or paragraph 5.5, by Lemmas 5.6, 5.7.

Thus we are in the situation of paragraph 5.6, whose notation we adopt. Further, we are not in the exceptional case (ii)(b), since M is a maximal Levi subgroup, nor in the case of Lemma 5.8 since we do not have an isomorphism $\mathscr{H}(M, \lambda_M^{\circ}) \to \mathscr{H}(G, \lambda_P^{\circ})$. The lattice sequence $\mathfrak{M}_1$ is just the standard lattice sequence $\mathfrak{M}_\Lambda$ used to define the standard β-extension κ. In particular, the character χ_1 is trivial. Moreover, as in [15, §2.3], by changing κ_0 if necessary, we may assume that χ_0 is a quadratic character.

Recall the element $p_\Lambda \in P(\Lambda^{(l,0)}_{\mathfrak{O}_{E_\ell}})$ defined in paragraph 5.6: its image generates the quotient $P(\Lambda^{l}_{\mathfrak{O}_{E_\ell}})/P^{\circ}(\Lambda^{l}_{\mathfrak{O}_{E_\ell}})$. We define $J_P^* = P(\Lambda^{l}_{\mathfrak{O}_{E_\ell}})J_P^{\circ}$, which contains J_P° with index at most 2. We fix $t \in \{0, 1\}$ and split according to the cases of paragraph 5.6, which we further subdivide.

(i) Suppose either that G_{E_ℓ} is *not* an orthogonal group, or that $\dim_{E_\ell} W_1$ is even.
   (a) Assume first that $s_t \in P^{\circ}(\mathfrak{M}_{t,\mathfrak{O}_E})$. We denote by $\mathscr{G}_t$ the connected finite reductive group $P^{\circ}(\mathfrak{M}_{t,\mathfrak{O}_E})/P_1(\mathfrak{M}_{t,\mathfrak{O}_E})$, and regard the representation $\rho_M^{\circ}\chi_t$ as the inflation to the parabolic subgroup $\mathscr{P}_t = P^{\circ}(\Lambda_{\mathfrak{O}_E})/P_1(\mathfrak{M}_{t,\mathfrak{O}_E})$ of a cuspidal representation of the Levi subgroup $P^{\circ}(\Lambda_{\mathfrak{O}_E})/P_1(\Lambda_{\mathfrak{O}_E})$. From (†), we get an injection of Hecke algebras

$$\mathscr{H}(\mathscr{G}_t, \rho_M^{\circ}\chi_t) \hookrightarrow \mathscr{H}(G, \lambda_P^{\circ}).$$

The element denoted T_t in paragraph 5.6 is the image of an invertible element $\bar{T}_t$ in $\mathscr{H}(\mathscr{G}_t, \rho^{\circ}_{\mathrm{M}}\chi_t)$ which satisfies a quadratic relation. This quadratic relation is given explicitly (in principle) by [20, Theorem 8.6]. By scaling T_t if necessary, we may assume that the relation takes the form

$$(T_t - q_t)(T_t + 1) = 0,$$

and then, by [20, Theorem 8.6], q_t is a power of $q_{\mathrm{o}} := q_{\mathrm{F_o}}$; indeed, by [18, Theorem 4.14], it can also be described as the quotient of the dimensions of the two irreducible components of $\mathrm{Ind}_{\mathscr{P}_t}^{\mathscr{G}_t} \rho^{\circ}_{\mathrm{M}}\chi_t$.

Now we induce to $\mathrm{J}^*_{\mathrm{P}}$. Let λ^*_{P} be an irreducible component of $\mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}}$ contained in λ_{P}. If $\lambda^*_{\mathrm{P}}|_{\mathrm{J}^{\circ}_{\mathrm{P}}}$ is reducible (equivalently, if p_Λ does not normalize $\lambda^{\circ}_{\mathrm{M}}$) then $\lambda^*_{\mathrm{P}} \simeq \mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}}$. Then, by [9, (4.1.3)], we have a support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda^{\circ}_{\mathrm{P}}) \simeq \mathscr{H}(\mathrm{G}, \lambda^*_{\mathrm{P}}),$$

and we denote by T^*_t the image of T_t under this isomorphism, which satisfies the same quadratic relation.

Otherwise, $\lambda^*_{\mathrm{P}}|_{\mathrm{J}^{\circ}_{\mathrm{P}}}$ is irreducible, p_Λ normalizes $\lambda^{\circ}_{\mathrm{M}}$, and $\mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}}$ has two inequivalent irreducible components λ^*_{P} and λ'_{P}. We can identify $\mathscr{H}(G, \lambda^*_{\mathrm{P}})$ and $\mathscr{H}(G, \lambda'_{\mathrm{P}})$ as subalgebras of $\mathscr{H}(\mathrm{G}, \mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}})$, canonically since $\mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}}$ is multiplicity free. Note also that $(\mathrm{J}^*_{\mathrm{P}}, \lambda^*_{\mathrm{P}})$ is a cover of $(\mathrm{J}^*_{\mathrm{P}} \cap \mathrm{M}, \lambda^*_{\mathrm{P}}|_{\mathrm{J}^*_{\mathrm{P}} \cap \mathrm{M}})$, by [23, Lemma 3.9], and the same applies to λ'_{P}. Finally, since s_t normalizes both restrictions $\lambda^*_{\mathrm{P}}|_{\mathrm{J}^*_{\mathrm{P}} \cap \mathrm{M}}$ and $\lambda'_{\mathrm{P}}|_{\mathrm{J}^*_{\mathrm{P}} \cap \mathrm{M}}$, the image of T_t under the support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda^{\circ}_{\mathrm{P}}) \simeq \mathscr{H}(\mathrm{G}, \mathrm{Ind}_{\mathrm{J}^{\circ}_{\mathrm{P}}}^{\mathrm{J}^*_{\mathrm{P}}} \lambda^{\circ}_{\mathrm{P}})$$

decomposes as $T^*_t + T'_t$, with $T^*_t \in \mathscr{H}(G, \lambda^*_{\mathrm{P}})$ and $T'_t \in \mathscr{H}(G, \lambda'_t)$ each satisfying the same relation as T_t.

In either case, when $s_t \in \mathrm{P}^{\circ}(\mathfrak{M}_{t,\mathcal{O}_{\mathrm{E}}})$, we end with an invertible element $T^*_t \in \mathscr{H}(G, \lambda^*_{\mathrm{P}})$ supported on $\mathrm{J}^*_{\mathrm{P}} s_t \mathrm{J}^*_{\mathrm{P}}$ and satisfying a quadratic relation of the required form, with computable parameter q_t.

(b) Now suppose that $s_t \notin \mathrm{P}^{\circ}(\mathfrak{M}_{t,\mathcal{O}_{\mathrm{E}}})$. If p_Λ normalizes $\lambda^{\circ}_{\mathrm{M}}$ and $p_\Lambda s_t \in \mathrm{P}^{\circ}(\mathfrak{M}_{t,\mathcal{O}_{\mathrm{E}}})$, we can replace s_t by $p_\Lambda s_t$ and argue exactly as in the previous case to get an element $T^*_t \in \mathscr{H}(G, \lambda^*_{\mathrm{P}})$ as required.

(c) Suppose now that $p_\Lambda s_t \in \mathrm{P}^{\circ}(\mathfrak{M}_{t,\mathcal{O}_{\mathrm{E}}})$ but p_Λ does not normalize $\lambda^{\circ}_{\mathrm{M}}$. We write $\mathrm{P}^*(\Lambda_{\mathcal{O}_{\mathrm{E}}})$ for the group generated by p_Λ and $\mathrm{P}^{\circ}(\Lambda_{\mathcal{O}_{\mathrm{E}}})$, so that $\mathrm{J}^*_{\mathrm{P}} = \mathrm{P}^*(\Lambda_{\mathcal{O}_{\mathrm{E}}})\mathrm{J}^1_{\mathrm{P}}$. Then, the quotient group $\mathrm{P}^*(\Lambda_{\mathcal{O}_{\mathrm{E}}})/\mathrm{P}_1(\Lambda_{\mathcal{O}_{\mathrm{E}}})$ has the form $\mathrm{GL}_1(k_{\mathrm{E}_\ell}) \times \mathscr{G}_t \times \mathscr{G}^{\circ}$, for $\mathscr{G}_t$ some orthogonal group over k_{E_ℓ} and $\mathscr{G}^{\circ}$ a product of connected finite reductive groups, while $\mathrm{P}^{\circ}(\Lambda_{\mathcal{O}_{\mathrm{E}}})/\mathrm{P}_1(\Lambda_{\mathcal{O}_{\mathrm{E}}}) \simeq \mathrm{GL}_1(k_{\mathrm{E}_\ell}) \times \mathscr{G}^{\circ}_t \times \mathscr{G}^{\circ}$, where $\mathscr{G}^{\circ}_t$ is the connected component of $\mathscr{G}_t$.

We also set $\mathrm{P}^*(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}}) = \mathrm{P}^*(\Lambda_{\mathcal{O}_\mathrm{E}})\mathrm{P}^\circ(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})$. Then $\mathrm{P}^*(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}}) \simeq \mathscr{G}_{1,t} \times \mathscr{G}^\circ$, where $\mathscr{G}_{1,t}$ is an orthogonal group over k_{E_ℓ} with Levi subgroup $\mathrm{GL}_1(k_{\mathrm{E}_\ell}) \times \mathscr{G}_t$, and $\mathrm{P}^\circ(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}}) \simeq \mathscr{G}^\circ_{1,t} \times \mathscr{G}^\circ$, where $\mathscr{G}^\circ_{1,t}$, the connected component of $\mathscr{G}_{1,t}$, is a special orthogonal group over k_{E_ℓ} with Levi subgroup $\mathrm{GL}_1(k_{\mathrm{E}_\ell}) \times \mathscr{G}^\circ_t$.
We write the image of $\mathrm{P}^*(\Lambda_{\mathcal{O}_\mathrm{E}})$ in $\mathrm{P}^*(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})$ as $\mathscr{P}_t \times \mathscr{G}^\circ$, where $\mathscr{P}_t$ is a parabolic subgroup of $\mathscr{G}_{1,t}$ with Levi component $\mathrm{GL}_1(k_{\mathrm{E}_\ell}) \times \mathscr{G}_t$. Similarly, we write the image of $\mathrm{P}^\circ(\Lambda_{\mathcal{O}_\mathrm{E}})$ as $\mathscr{P}^\circ_t \times \mathscr{G}^\circ$. We have the following picture:

$$\begin{array}{ccc} \mathscr{P}^\circ_t \times \mathscr{G}^\circ & \xrightarrow{\mathrm{Ind}} & \mathscr{G}^\circ_{1,t} \times \mathscr{G}^\circ \\ \Big\downarrow{\scriptstyle \mathrm{Ind}} & & \Big\downarrow{\scriptstyle \mathrm{Ind}} \\ \mathscr{P}_t \times \mathscr{G}^\circ & \xrightarrow[\mathrm{Ind}]{} & \mathscr{G}_{1,t} \times \mathscr{G}^\circ \end{array}$$

Since (the image of) p_Λ does not normalize λ°_M, but does normalize the β-extension κ_t, it also does not normalize $\rho^\circ_\mathrm{M}\chi_t$ and hence $\rho^*_\mathrm{M} := \mathrm{Ind}_{\mathscr{P}^\circ_t \times \mathscr{G}^\circ}^{\mathscr{P}_t \times \mathscr{G}^\circ} \rho^\circ_\mathrm{M}\chi_t$ is irreducible. Similarly, since $s_t \notin \mathrm{P}^\circ(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})$, the induced representation $\mathrm{Ind}_{\mathscr{P}^\circ_t \times \mathscr{G}^\circ}^{\mathscr{G}^\circ_{1,t} \times \mathscr{G}^\circ} \rho^\circ_\mathrm{M}\chi_t$ is also irreducible. On the other hand, since s_t intertwines $\rho^\circ_\mathrm{M}\chi_t$, the induced representation $\mathrm{Ind}_{\mathscr{P}_t \times \mathscr{G}^\circ}^{\mathscr{G}_{1,t} \times \mathscr{G}^\circ} \rho^*_\mathrm{M}$ is reducible. By restricting back to $\mathscr{G}^\circ_{1,t} \times \mathscr{G}^\circ$, we see that it must reduce as a direct sum of two inequivalent irreducible representations of the same dimension. Thus there is an element $\bar{T}^*_t \in \mathscr{H}(\mathscr{G}_{1,t} \times \mathscr{G}^\circ, \rho^*_\mathrm{M})$ satisfying $(\bar{T}^*_t)^2 = 1$. Finally, by [29, (7.3)], there is again a support-preserving injective algebra map

$$\mathscr{H}(\mathrm{P}(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}}), \rho^*_\mathrm{M}) \hookrightarrow \mathscr{H}(\mathrm{G}, \lambda^*_\mathrm{P}),$$

and we find an invertible element $T^*_t \in \mathscr{H}(G, \lambda^*_\mathrm{P})$ satisfying a quadratic relation

$$(T^*_t - 1)(T^*_t + 1) = 0,$$

and $q_t = q^0_\mathrm{o} = 1$.

(d) Finally, suppose that $s_t \notin \mathrm{P}^\circ(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})$ but $p_\Lambda = 1$, so that $\mathrm{J}^*_\mathrm{P} = \mathrm{J}^\circ_\mathrm{P}$. The argument here is very similar. In this case we have $\mathrm{P}(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}})/\mathrm{P}_1(\mathfrak{M}_{t,\mathcal{O}_\mathrm{E}}) \simeq \mathrm{O}(1,1)(k_{\mathrm{E}_\ell}) \times \mathscr{G}$, for some product of (possibly non-connected) finite reductive groups, and the image of s_t lies in $\mathrm{O}(1,1)(k_{\mathrm{E}_\ell})$. We denote by $\mathscr{G}_t$ the non-connected group $\mathrm{O}(1,1)(k_{\mathrm{E}_\ell}) \times \mathscr{G}^\circ$, where $\mathscr{G}^\circ$ is the connected component of $\mathscr{G}$. The image $\mathscr{P}_t$ of $\mathrm{P}^\circ(\Lambda_{\mathcal{O}_\mathrm{E}})$ in $\mathscr{G}_t$ is $\mathrm{SO}(1,1)(k_{\mathrm{E}_\ell}) \times \mathscr{G}^\circ$, which is normalized by the image of s_t. Since image of s_t normalizes $\rho^\circ_\mathrm{M}\chi_t$, the induced representation $\mathrm{Ind}_{\mathscr{P}_t}^{\mathscr{G}_t} \rho^\circ_\mathrm{M}\chi_t$ decomposes into two pieces of equal dimension and the argument is exactly as in previous cases, with $\bar{T}^2_t = 1$. Thus, letting $T^*_t = T_t$

be the image of $\bar{T}_t$, again we have an invertible element $T_t^* \in \mathscr{H}(G, \lambda_P^*)$ satisfying a quadratic relation

$$(T_t^* - 1)(T_t^* + 1) = 0,$$

and $q_t = q_o^0 = 1$.

This ends the first case, so we move on to the second.

(ii) Suppose that G_{E_ℓ} is an orthogonal group and $\dim_{E_\ell} W_1 = 1$.

As in case (i) above, there are four possible situations. The details are almost identical to those in case (i) so we omit them.

(a) Suppose first that $\mathscr{G}_t^o$ is non-trivial and that $p = p_t$ normalizes ρ_M^o. In this case $s_t \in P^o(\mathfrak{M}_{t,\mathfrak{O}_E})$ and the argument proceeds exactly as in case (i)(a) to give $T_t^* \in \mathscr{H}(G, \lambda_P^*)$ as required.

(b) Similarly, if $\mathscr{G}_t^o$ is non-trivial and $p_t \neq p$ normalizes ρ_M^o, we can replace s_t by $p_\Lambda s_t$ to get the same conclusion.

(c) Now suppose $\mathscr{G}_t^o$ is non-trivial and p_t does not normalize ρ_M^o (in which case we have $p_t \neq p$, since p normalizes λ_M^o). In this case, $p_\Lambda = pp_t$ does not normalize $\rho_M^o \chi_t$, and we can copy the argument in case (i)(c) to obtain $T_t^* \in \mathscr{H}(G, \lambda_P^*)$ such that $(T_t^*)^2 = 1$.

(d) Finally suppose $\mathscr{G}_t^o$ is trivial, in which case

$$P^o(\mathfrak{M}_{t,\mathfrak{O}_E})/P_1(\mathfrak{M}_{t,\mathfrak{O}_E}) \simeq SO(1,1)(k_F) \times \mathscr{G}_{1-t}^o \times \mathscr{G}^o \simeq P^o(\Lambda_{\mathfrak{O}_E})/P_1(\Lambda_{\mathfrak{O}_E}).$$

The argument is now exactly as in case (i)(d).

6.4 We continue in the situation of the previous paragraph. In all cases, we have two elements $T_t^* \in \mathscr{H}(G, \lambda_P^*)$, supported on $J_P^* s_t J_P^*$, which satisfy quadratic relations of the required form. The same proof as that of [4, Théorème 1.11] now shows that $\mathscr{H}(G, \lambda_P^*)$ is a convolution algebra on $(\mathsf{W}, \{s_0, s_1\})$, where W is the infinite dihedral group generated by s_0, s_1.

Finally, we must see that the Hecke algebra $\mathscr{H}(G, \lambda_P)$ has the same form. For this, we revisit the argument of [23, Lemma 3.9], which was used in deducing that (J_P, λ_P) is a cover. (In fact, we will be repeating the argument in some of the cases above.) We note that we are in a particularly simple situation here, as J_P/J_P^* is a product of cyclic groups of order 2.

Put $J_M^* = J_P^* \cap M$ and $\lambda_M^* = \lambda_P^*|_{J_M^*}$. Then, since the difference between J_M^* and J_M is only in the blocks V^i with $i < l$, the element s_t normalizes each irreducible constituent of $\mathrm{Ind}_{J_M^*}^{J_M} \lambda_M^*$.

We choose a chain of normal subgroups

$$J_P^* = K_0 \subset K_1 \subset \cdots \subset K_r = J_P,$$

such that each quotient K_i/K_{i-1} is cyclic of order 2. We will prove, inductively on i, that, for each irreducible constituent λ_i of $\mathrm{Ind}_{J_P^*}^{K_i} \lambda_P^*$, there is a support-preserving Hecke algebra isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{*}) \simeq \mathscr{H}(\mathrm{G}, \lambda_i).$$

The case $i = 0$ is vacuous so suppose $i \geq 1$.

If $\lambda_i|_{\mathrm{K}_{i-1}}$ is reducible then $\lambda_i \simeq \mathrm{Ind}_{\mathrm{K}_{i-1}}^{\mathrm{K}_i} \lambda_{i-1}$, for some irreducible constituent λ_{i-1} of $\mathrm{Ind}_{\mathrm{J}_{\mathrm{P}}^{*}}^{\mathrm{K}_{i-1}} \lambda_{\mathrm{P}}^{*}$. Then, by [9, (4.1.3)], we have a support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda_{i-1}) \simeq \mathscr{H}(\mathrm{G}, \lambda_i),$$

and the claim follows by the inductive hypothesis.

Otherwise, $\lambda_{i-1} := \lambda_i|_{\mathrm{K}_{i-1}}$ is irreducible and $\mathrm{Ind}_{\mathrm{K}_{i-1}}^{\mathrm{K}_i} \lambda_{i-1}$ has two irreducible components $\lambda_i = \lambda_i^{(1)}$ and $\lambda_i^{(2)}$, which are not equivalent. Note that $(\mathrm{K}_i, \lambda_i^{(j)})$ is a cover of $(\mathrm{K}_i \cap \mathrm{M}, \lambda_i^{(j)}|_{\mathrm{K}_i \cap \mathrm{M}})$, for $j = 1, 2$, by [23, Lemma 3.9]. We denote by T_t^i the image of T_t^* under the support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{*}) \simeq \mathscr{H}(\mathrm{G}, \lambda_{i-1}) \simeq \mathscr{H}(\mathrm{G}, \mathrm{Ind}_{\mathrm{K}_{i-1}}^{\mathrm{K}_i} \lambda_{i-1})$$

given by the inductive hypothesis and [9, (4.1.3)]. We can also identify each $\mathscr{H}(G, \lambda_i^{(j)})$ as a subalgebra of $\mathscr{H}(\mathrm{G}, \mathrm{Ind}_{\mathrm{K}_{i-1}}^{\mathrm{K}_i} \lambda_{i-1})$, canonically since $\mathrm{Ind}_{\mathrm{K}_{i-1}}^{\mathrm{K}_i} \lambda_{i-1}$ is multiplicity free. Then, since s_t normalizes the restrictions $\lambda_i^{(j)}|_{\mathrm{K}_i \cap \mathrm{M}}$, it follows that $T_t^i = T_t^{(1)} + T_t^{(2)}$, with $T_t^{(j)} \in \mathscr{H}(G, \lambda_i^{(j)})$ satisfying the same relation as T_t^*. Thus we get a support-preserving isomorphism $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{*}) \simeq \mathscr{H}(\mathrm{G}, \lambda_i^{(1)}) = \mathscr{H}(\mathrm{G}, \lambda_i)$.

In particular, taking $\lambda_r = \lambda_{\mathrm{P}}$, we deduce that $\mathscr{H}(G, \lambda_{\mathrm{P}})$, isomorphic to $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{P}}^{*})$, as required.

6.5 This completes the proof of Theorem 1.2. Note also that the computation of the parameters q_i then comes down to computing the quadratic character χ_0 of paragraph 6.3 and the parameters in the two finite Hecke algebras $\mathscr{H}(\mathscr{G}_0, \rho_{\mathrm{M}}^{\circ}\chi_0)$ and $\mathscr{H}(\mathscr{G}_1, \rho_{\mathrm{M}}^{\circ})$. As mentioned above, these parameters can be computed using work of Lusztig [20]. Examples can be found in the work of Kutzko and Morris [19] on level zero types for the Siegel Levi; note that, for level zero representations, the β-extensions are just trivial representations so the character χ_0 is trivial.

For positive level representations the situation is much more subtle. There are many cases where the character χ_0 makes no difference to the values of the parameters; that is, we find the same reducibility points whether χ_0 is trivial or not. However, in the interesting cases it is crucial.

That χ_0 can sometimes be non-trivial can already be seen when inducing from the Siegel Levi subgroup [15]. If $\mathrm{M} = \mathrm{GL}_N(\mathrm{F})$ is viewed as the Siegel Levi subgroup of G, which is either $\mathrm{Sp}_{2N}(\mathrm{F})$ or $\mathrm{SO}_{2N+1}(\mathrm{F})$, and $\widetilde{\tau}$ is a self-dual cuspidal representation of M then $\mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{SO}_{2N+1}(\mathrm{F})} \widetilde{\tau}$ is irreducible if and only if $\mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{Sp}_{2N}(\mathrm{F})} \widetilde{\tau}$ is reducible, by [27, Theorem 6.3]. However, if a skew stratum $[\Lambda, n, 0, \beta]$ used to construct a self-dual simple character in $\widetilde{\tau}$ is such that $\mathrm{E} = \mathrm{F}[\beta]$ is of degree N over F and $\mathrm{E}/\mathrm{E}_{\mathrm{o}}$ is ramified, then one can check that one gets reducibility of $\mathrm{Ind}_{\mathrm{M,P}}^{\mathrm{G}} \widetilde{\tau} |\det(\cdot)|^s$ either when the real part of s is 0, or when it is $\pm\frac{1}{2}$, depending on whether χ_0 is trivial or not. Thus χ_0 must be non-trivial for exactly one of $\mathrm{Sp}_{2N}(\mathrm{F})$ and $\mathrm{SO}_{2N+1}(\mathrm{F})$.

The nature of the character χ_0 has also been examined more closely in the work of Blondel [6], which looks at the case when $L = G$ and $V^i = W_{-1} \oplus W_1$, for some $i \in I_0$ (that is, the simple character in $\widetilde{\tau}_1$ is "completely different" from the semisimple character in τ_0). The group $\widetilde{G}_1$ can also be viewed as the Siegel Levi subgroup in the (smaller) classical group $G^i \subset \mathrm{Aut}_F(V^i)$ and one finds another quadratic character χ_0^i here, by making the same construction. It turns out that the characters χ_0 and χ_0^i differ by the signature characters of certain (explicit) permutation representations (see [6, Théorème 2.35]).

6.6 We finish with some remarks on the Hecke algebra of a cover in the general case of a non-maximal Levi subgroup. Firstly, one interesting case is now resolved: if $M \simeq GL_r(F)^s \times Sp_{2N}(F)$ is a Levi subgroup of $Sp_{2(N+rs)}(F)$ and λ_M takes the form $\widetilde{\lambda}^{\otimes s} \otimes \lambda_0$, with $\widetilde{\lambda}$ self-dual cuspidal, then Blondel [5] has given a description of the Hecke algebra, contingent on a suitable description of the Hecke algebra in the case $s = 1$ (which was already known when $N = 0$). Given Theorem 1.2, Blondel's result can now be used in full generality.

It seems likely that the methods of [5] could equally well be applied to other classical groups. However, it is not clear to the authors whether the methods used here and in [5] could together be pushed to allow a description of the Hecke algebra in a completely general case.

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Appendix A: Correction to the proof of [29, Theorem 7.14]

In this appendix, we give the correction to the proof of the main result of [29] (Theorem 7.14), with the newly corrected definition of cuspidal type from Definition 4.3, which we repeat here, recalling that G has compact centre:

A *cuspidal type* for G is a pair (J, λ), where $J = J(\beta, \Lambda)$ for some skew semisimple stratum $[\Lambda, n, 0, \beta]$ such that

- G_E has compact centre and
- $P^\circ(\Lambda_{\mathcal{O}_E})$ a maximal parahoric subgroup of G_E,

and $\lambda = \kappa \otimes \tau$, for κ a β-extension and τ the inflation of an irreducible cuspidal representation of $J/J^1 \simeq P(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E})$.

Remark A.1 *We thank Laure Blasco, Corinne Blondel and Van-Dinh Ngo for pointing out the problem with the definition in [29, Definition 6.17]. There, the two conditions on the stratum* $[\Lambda, n, 0, \beta]$ *in Definition 4.3 are replaced by the (insufficient) condition that* $\mathfrak{A}(\Lambda_{\mathcal{O}_E})$ *be a maximal self-dual* $\mathcal{O}_E$*-order in* B.

Firstly, this is not enough to guarantee that $P^\circ(\Lambda_{\mathcal{O}_E})$ *be a maximal parahoric subgroup of* G_E*: for example, if* G_E *is a quasi-split ramified unitary group in* 2 *vari-*

ables then, for one of the two (up to conjugacy) maximal self-dual $\mathcal{O}_E$-orders, the corresponding parahoric subgroup is an Iwahori subgroup, so not maximal.

Secondly, even if $P^\circ(\Lambda_{\mathcal{O}_E})$ *is a maximal parahoric subgroup, it can still happen that its normalizer in* G_E *is not compact: this happens precisely when* G_E *has a factor isomorphic to the split torus* $SO(1,1)(F)$*, which can only happen when* G *is an even-dimensional orthogonal group and* $\beta_i = 0$*,* $\dim_F V^i = 2$*, for some* $i \in I_0$*. The condition that* G_E *have compact centre rules out exactly this possibility.*

In particular, with the definition of cuspidal type (J, λ) *given here, the proof of [29, Proposition 6.18] is valid, and* $\text{c-Ind}_J^G \lambda$ *is an irreducible cuspidal representation of* G.

A.1 In this paragraph we indicate the minor changes that must be made to [29, §7.2] in order to correct the proof of the main result there [29, Theorem 7.14]: every irreducible cuspidal representation of G contains a cuspidal type. This paragraph should be read alongside that paper and we will make free use of notations from there.

Suppose π is an irreducible representation of G and suppose that there is a pair $([\Lambda, n, 0, \beta], \theta)$, consisting of a skew semisimple stratum $[\Lambda, n, 0, \beta]$ and a semisimple character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, such that π contains θ. Suppose moreover that, for fixed β, we have chosen a pair for which the parahoric subgroup $P^\circ(\Lambda_{\mathcal{O}_E})$ is *minimal amongst such pairs*. If κ is a standard β-extension then π also contains a representation $\vartheta = \kappa \otimes \rho$ of J°, for ρ an irreducible representation of $J^\circ/J^1 \simeq P^\circ(\Lambda_{\mathcal{O}_E})/P_1(\Lambda_{\mathcal{O}_E})$. By [29, Lemma 7.4], the minimality of $P^\circ(\Lambda_{\mathcal{O}_E})$ implies that the representation ρ is cuspidal.

We suppose that either the parahoric subgroup $P^\circ(\Lambda_{\mathcal{O}_E})$ *is not maximal in* G_E *or* G_E *does not have compact centre* and will find a non-zero Jacquet module. (This assumption takes the place of hypothesis (H) in [29, §7.2].) Most of [29, §7.2] now goes through essentially unchanged, with two small changes in the cases called (i) and (ii) in §7.2.2 (page 350).

In case (i), the change happens when the element p *cannot* be chosen to normalize the representation ρ, interpreted as a representation of $P^\circ(\Lambda_{\mathcal{O}_E})$ trivial on $P_1(\Lambda_{\mathcal{O}_E})$. (Note that $p \in P^+(\Lambda_{\mathcal{O}_E})$ so it does normalize the group $P^\circ(\Lambda_{\mathcal{O}_E})$.) In this case, $N_\Lambda(\rho) \subseteq M'$, where M' is the Levi subgroup of *loc. cit.* (Note, however, that this would not be the case if we were working in the non-connected group G^+, rather than G.) Thus, by [29, Corollary 6.16], we have $I_G(\vartheta_P) \subseteq J_P^\circ M' J_P^\circ$ and, as in the proof of [29, Proposition 7.10], (J_P°, ϑ_P) is a cover of $(J_P^\circ \cap M', \vartheta_P|_{J_P^\circ \cap M'})$. (See also Lemma 5.8.)

In case (ii), the change happens when $m = 1$. In this case $P^\circ(\Lambda_{\mathcal{O}_E})$ is a maximal parahoric subgroup but G_E does not have compact centre; indeed $G_{E_1} \simeq SO(1,1)(F)$ and we have $G_E \subseteq M'$. As in case (i) above, we get that $I_G(\vartheta_P) \subseteq J_P^\circ M' J_P^\circ$ and (J_P°, ϑ_P) is a cover of $(J_P^\circ \cap M', \vartheta_P|_{J_P^\circ \cap M'})$.

In particular, in all cases, the representation π containing ϑ cannot be cuspidal and we have proved the first two assertions of Proposition 4.4; the third assertion follows from [29, Lemma 7.4].

In particular, since by [28, Theorem 5.1] every irreducible cuspidal representation of G does contain a semisimple character, and hence a representation of J of the form $\kappa \otimes \tau$, this also proves [29, Theorem 7.14].

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