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THE ORIENTED GRAPH OF MULTI-GRAFTINGS IN THE FUCHSIAN CASE

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Abstract: We prove the connectedness and compute the diameter of the oriented graph of multi-graftings associated to exotic $\mathbb{CP}^1$-structures on a compact surface S with a given holonomy representation of Fuchsian type.

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1. Introduction

Let Γ_g be the fundamental group of a compact oriented surface S of genus $g \geq 2$, and $\rho\colon \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ be a *Fuchsian* representation, namely a faithful and discrete one. A marked surface of genus g is the data of a simply connected cover $\widetilde{S}$ of S together with a free discontinuous action of Γ_g. A $\mathbb{CP}^1$-structure (sometimes referred to as a projective structure) with holonomy ρ on the marked surface is a local diffeomorphism $D\colon \widetilde{S} \to \mathbb{CP}^1$ called developing map which is ρ-equivariant. We denote by $P(\rho)$ the set of equivalence classes of marked $\mathbb{CP}^1$-structures on a surface of genus g with holonomy ρ, where two projective structures $(\widetilde{S}_i, D_i)$, $i = 1, 2$ are equivalent if there exists a Γ_g-equivariant diffeomorphism $\Phi\colon \widetilde{S_1} \to \widetilde{S_2}$ such that $D_1 = D_2 \circ \Phi$. This definition of projective structure coincides with the classical one because there is no ambiguity in the choice of developing map when the holonomy representation is non-elementary, see [**2**, Lemma 12.10].

This article deals with the study of a surgery operation called *grafting* that produces, given an element in $P(\rho)$, new elements in the same set. Grafting consists in cutting a surface equipped with a $\mathbb{CP}^1$-structure along a particular type of simple closed curve called *graftable curve*, and gluing a Hopf annulus, namely the quotient of a simply connected

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domain of the Riemann sphere invariant by the (loxodromic) holonomy of the graftable curve. This operation produces a new element of $P(\rho)$.

Grafting was used by Hejhal [**5**, Theorem 4] and Thurston (unpublished) to produce examples of projective structures with holonomy ρ that are different from the uniformizing structure $\sigma_u = \rho(\Gamma_g)\backslash\mathbb{H}^2$. Such structures are called *exotic*. The importance of grafting comes from the fact that it allows to define coordinates on $\mathcal{P}(\rho)$ when ρ is a Fuchsian representation: Goldman proved that any $\mathbb{CP}^1$-structure with holonomy ρ is obtained from the uniformizing one by grafting a collection of disjoint graftable simple closed curves (see [**4**]). Such an operation will be called a *multi-grafting*.

The goal of this note is to improve Goldman's result in the following way.

Theorem 1.1. *Let σ_1 and σ_2 be two exotic projective structures sharing the same Fuchsian holonomy. Then σ_2 can be obtained from σ_1 by a sequence of two multi-graftings.*

A consequence of this result is that there exist positive cycles of graftings, namely finite sequences of marked $\mathbb{CP}^1$-structures $\sigma_0, \dots, \sigma_r = \sigma_0$ such that for each $i = 1, \dots, r$, σ_i is a grafting of σ_{i-1}. The integer r is then called the period of the cycle. Observe that an immediate corollary of the theorem is that any couple of exotic $\mathbb{CP}^1$-structures are contained in such a positive cycle of period bounded by 4. We will see (Corollary 4.2) that indeed there are such cycles of period 2.

Let $MG(\rho)$ be the oriented graph whose vertices are elements of $\mathcal{P}(\rho)$ and two vertices σ_1, σ_2 are joined by an oriented edge from σ_1 to σ_2 if σ_2 is obtained from σ_1 by a multi-grafting. Theorem 1.1 can be restated by saying that the oriented graph of multi-graftings $MG(\rho) \setminus \sigma_u$ is a connected graph of radius 2. As a consequence we also get that the fundamental group of $MG(\rho)$ is not finitely generated.

To prove the results we will use some surgery operations on multi-curves introduced by Luo [**7**] and later developed by Ito [**6**]. Our results and methods are closely related to Thompson's, see [**8**], but he considers the case of Schottky representations instead of Fuchsian ones. We observe that our argument extends *stricto sensu* to the case of quasi-Fuchsian representations.

2. Graftable curves

In this section we introduce the action of grafting on $\mathcal{P}(\rho)$ and define the graph of multi-graftings.

2.1. Definition. Recall that a *multi-curve* on a surface S is a finite disjoint union of simple closed curves none of which is homotopically trivial. Let σ be a marked projective structure on a compact orientable surface S. A multi-curve is said to be *graftable* (in σ) if all of its components have loxodromic holonomy and the developing map is injective when restricted to a lift of any of those components in $\widetilde{S}$. The condition is independent of the choice of representative in the class $[\sigma] \in \mathcal{P}(\rho)$.

2.2. Grafting along graftable curves. If $\alpha = \{\alpha_i\}_{i \in I}$ is a graftable multi-curve, one can produce another marked projective structure, called the grafting along α, and denoted $\mathrm{Gr}(\sigma, \alpha)$. We recall the construction here. We cut the surface $\widetilde{S}$ along the lifts $\widetilde{\alpha_i}$'s of the curves α_i's, and glue to each of them a copy of $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})}$ using the developing map for the gluing. We then obtain a new surface denoted by $\widetilde{S}'$, together with a new map $D' \colon \widetilde{S}' \to \mathbb{CP}^1$ which is defined by D on $\widetilde{S} \setminus \pi^{-1}(\bigcup_i \alpha_i)$ and by the identity on the spherical domains $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})}$. The Γ_g-action on $\widetilde{S}$ induces a Γ_g-action on $\widetilde{S}'$ which is free and discontinuous, and the map D' is obviously ρ-equivariant. Hence, this defines a new marked projective structure $\mathrm{Gr}(\sigma, \alpha)$ with holonomy ρ: the grafting of σ over the graftable multi-curve α.

As α_i has loxodromic holonomy, it acts freely and properly discontiuosly on $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})}$, and its quotient is a cylinder equipped with a projective structure. Therefore, the grafting can be viewed as a cut-and-paste procedure directly in S, which cuts S along each α_i and glues back the cylinder $\langle \alpha_i \rangle \backslash (\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})})$.

2.3. Isotopy class of graftable curves. It is an easy fact to verify that if α and α' belong to the same connected component of the set of *graftable* multi-curves (for the compact open topology), then the resulting projective structures $\mathrm{Gr}(\sigma, \alpha)$ and $\mathrm{Gr}(\sigma, \alpha')$ are equivalent. However, we will see that it can happen that α and α' are two graftable multi-curves that are isotopic as multi-curves by an isotopy that leaves the space of graftable multi-curves, and such that their corresponding graftings are not equivalent (see Remark 3.4).

2.4. The graph of multi-graftings. Let ρ be a representation from Γ_g to $\mathrm{PSL}(2, \mathbb{C})$. Let us define the graph of multi-graftings $MG(\rho)$ in the following way. The vertices are the elements of $\mathcal{P}(\rho)$ and two of them (S_1, σ_1) and (S_2, σ_2) are the connected by a positive segment from σ_1 to σ_2 if there exists a graftable multi-curves α in S_1 such that $\mathrm{Gr}(\sigma_1, \alpha) = \sigma_2$.

3. Fuchsian case: construction of graftable curves

Recall that a representation $\rho\colon \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian if it is discrete and faithful. In the sequel ρ will always be assumed to be Fuchsian.

3.1. Goldman's parametrization of $MG(\rho)$. We will denote by σ_u the uniformizing structure on the surface $S_u := \rho(\Gamma_g)\backslash\mathbb{H}^2$, which is obtained by taking the quotient of $\mathbb{H}^2$ by the ρ-action of Γ_g on $\mathbb{H}^2$. For this structure, the developing map is just the identity when identifying the universal cover of S_u with $\mathbb{H}^2$, and in particular is injective. Hence, any simple closed curve on S_u is a graftable curve. Hence in this case the space of graftable multi-curves and the space of multi-curves are the same. By the discussion in §2.3 the grafting $\mathrm{Gr}(\sigma_u,\alpha)$ depends only on the isotopy class of α as a *multi-curve*.

Goldman proved in [**4**] that every marked projective structure σ with holonomy ρ is obtained by grafting the structure σ_u along a multi-curve $\alpha = \{\alpha_i\}_i$. Moreover, this family is unique, and can be reconstructed from σ in the following way. For a Fuchsian projective structure σ, denote by $S^{\mathbb{R}}$ (resp. $S^{\pm}$) the quotient of $D^{-1}(\mathbb{RP}^1)$ (resp. $D^{-1}(\mathbb{H}^{\pm})$) by the covering group Γ_g. Since ρ is Fuchsian, it preserves the decomposition $\mathbb{CP}^1 = \mathbb{H}^+ \cup \mathbb{RP}^1 \cup \mathbb{H}^-$, and thus $S^{\mathbb{R}}$ is an analytic real submanifold of S separating S in domains which are either positive or negative according they belong to S^+ or S^-. Goldman proved that the components of S^- are necessarily annuli. The set of annuli is homotopic to a unique multi-loop α satisfying $\sigma = \mathrm{Gr}(\sigma_u,\alpha)$. To abridge notations we define $\mathrm{Gr}_\alpha := \mathrm{Gr}(\sigma_u,\alpha)$.

3.2. Homotopically transverse multi-curves. Let $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ be two multi-curves. They are homotopically transverse if the following conditions hold:

- for each $i \in I$ and $j \in J$, the curves α_i and β_j are not homotopic,
- they are transverse in the usual sense, and
- the complement of $(\bigcup \alpha_i) \cup (\bigcup \beta_j)$ in S has no bi-gon component.

3.3. Construction of graftable multi-curves. Given a multi-curve $\alpha = \{\alpha_i\}_{i\in I}$, a set of turning directions for α is an assignment to each curve α_i of a turning direction $T_i \in \{R, L\}$ ("Right" or "Left") in such a way that any two parallel curves have the same turning direction.

In this paragraph we provide a construction that, given two homotopically transverse multi-curves $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$, and a set $T = \{T_i\}$ of turning directions for α, produces a multi-curve β_T which is graftable in Gr_α and isotopic to β.

We begin by assuming that there are no parallel curves in the families α and β. In this case we can assume that the components of α and β are simple closed geodesics in the uniformizing structure σ_u.

Recall that Gr_α is obtained by gluing $S_u \setminus \alpha$ with some grafting annuli. We will explain the construction of β_T in each piece of this decomposition separately, beginning with the intersection of β_T with $S_u \setminus \alpha$, and then construct the intersection of β_T with the grafting annuli glued to $S_u \setminus \alpha$ to obtain Gr_α.

The boundary of $S_u \setminus \alpha$ consists of two copies α_i' and α_i'' of each curve α_i, and for each component C of $S_u \setminus \alpha$, its boundary is a union of such components. We fix a small positive number ε, and for each $p \in \alpha_i \cap \beta \in \partial C$, we consider the point $p_T \in \partial C$ lying at distance ε from p to the side of p indicated by T_i with respect to the orientation induced on α_i by C. If we do this for all components of $S_u \setminus \alpha$, we get for each point $p \in \alpha_i \cap \beta$ a couple of distinct points $p' \in \alpha_i'$ and $p'' \in \alpha_i''$ lying at distance ε from p (as seen as a point in α_i or α_i'' under the natural identifications $\alpha_i \simeq \alpha_i' \simeq \alpha_i''$).

Now, $\beta \cap C$ is a union of geodesic segments $[p, q]$ joining points of ∂C. We define β_T in $S_u \setminus \alpha \subset S$ to be the union of the segments $[p_T, q_T]$ with p_T and q_T constructed as above. Observe that if we move the points p, q a little bit, then the segments $[p_T, q_T]$ are disjoint in the component C, but also in the whole surface S.

Then, one has to define the curve β_T in the grafting annuli in a graftable way. The continuation should start from the point p' above and end at p''. (In Figure 1 we depicted the case $T_i = L$.)

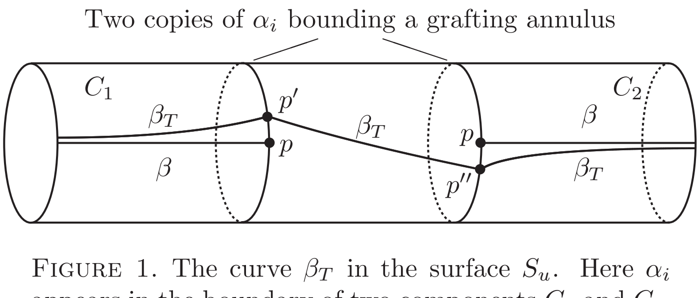


Figure 1. The curve β_T in the surface S_u. Here α_i appears in the boundary of two components C_1 and C_2. In the picture, we used $T_i = L$.

To be sure that β_T is graftable and in the isotopy class of β, we need some care. First, we suppose that β intersects α_i once. Figure 2 provides a sketch of the construction in the universal cover (we used the convention that $\mathbb{H}^2 = \mathbb{H}^+$ is the upper half-plane).

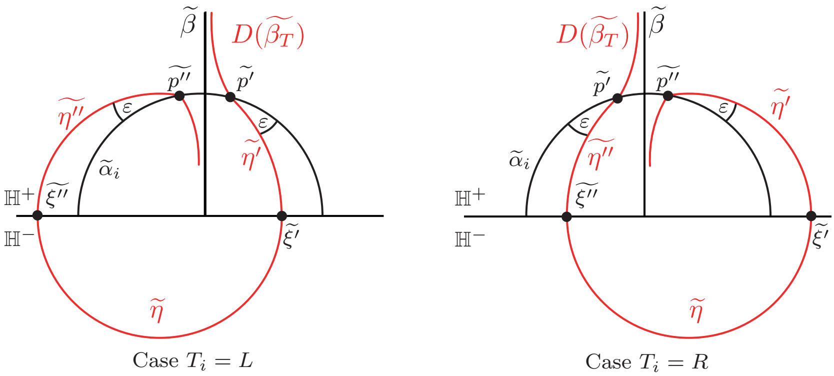


FIGURE 2. The portion of β_T in the universal cover of the grafting annulus.

When the path β_T enters in the grafting, it means that any lift $\widetilde{\beta_T}$ enters in the subset $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$ that we have glued to $\widetilde{S_u}$ to obtain $\widetilde{\mathrm{Gr}_\alpha}$. It enters at the point $\widetilde{p'}$ and needs to get out at the point $\widetilde{p''}$ by a path in $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$. For this it has to turn around the segment $\overline{\widetilde{\alpha_i}}$ in the sphere. Since we want a graftable curve we need to avoid creating self-intersection points of the developed image of $\widetilde{\beta_T}$. An example of such a curve can be constructed as follows. Consider two semi-infinite geodesics $\widetilde{\eta'}$ and $\widetilde{\eta''}$ starting from $\widetilde{p'}$ and $\widetilde{p''}$ and forming and angle ε with $\widetilde{\alpha_i}$ as in Figure 2. Such geodesics meet the real line (i.e. the boundary of $\mathbb{H}^+$) at two points $\widetilde{\xi'}$, $\widetilde{\xi''}$.

When $\widetilde{\beta_T}$ meets $\widetilde{\alpha_i}$ at the point $\widetilde{p'}$, we continue it by $\widetilde{\eta'}$, then in $\mathbb{H}^-$ by the geodesic $\widetilde{\eta}$ between $\widetilde{\xi'}$ and $\widetilde{\xi''}$, and finally with $\widetilde{\eta''}$. (See Figure 2.)

The path $\widetilde{\eta'} * \widetilde{\eta} * \widetilde{\eta''}$ takes values in the set $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$. Such path remains embedded when quotienting $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$ by the action of α_i and provides the path β_T in the grafting annulus. Moreover, since $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$ is a disc, any two paths joining two points in the boundary are homotopic. This shows that β_T is indeed isotopic to β. (See also Figure 7.)

Let us do the construction when β intersects α_i in more than one point. What we need to describe is the part of β_T in the grafting annulus. Again, we work in the universal cover. In Figure 3 we sketched the case of two points of intersection.

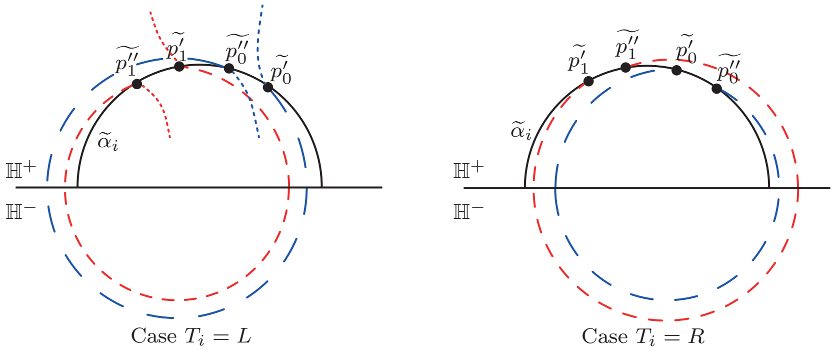


Figure 3. The case of two intersection points. In the case $T_i = L$ we depicted two lifts of β_T, in the case $T_i = R$ we depicted only the segments in the grafting region.

Let $\{p_j\}$ be the set of points of intersection between α_i and β, and form the points p_j' and p_j'' as before (choosing ε small enough). If $\widetilde{\alpha_i}$ is a lift of α_i, we see lifts $\widetilde{p_j'}$ and $\widetilde{p_j''}$ of such points. We remark that for $j \neq k$, the point $\widetilde{p_j'}$ correspond to a lift of β different from that of $\widetilde{p_k'}$. This is because α and β are homotopically transverse. It is worth noting at this point that it happens that the developed images of two such lifts intersect, but this is not a problem for our construction. Indeed, for β_T to be graftable in Gr_α, we only need that any single lift of β_T is developed injectively. In Figure 3 we have drawn in red (small dashed line) and blue (big dashed line) two different lifts of β_T entering in the same grafting region $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$. The intersections of the two lifts with the grafting region are two disjoint segments, and it is clear that such segments remain disjoint and embedded when projecting to the grafting annulus. Thus, β_T is embedded and homotopic to β also when multiple intersections arise.

Let us check that any lift of β_T develops injectively. We choose a lift of β and the corresponding lift of β_T. Say the red (small dashed) lift. Since α and β are homotopically transverse, the red lift of β intersects any lift of any component α_i of α at most once. Thus, when the red $\widetilde{\beta_T}$ enters the grafting region $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$, the situation is exactly that of Figure 2. By construction, the developed image of the red $\widetilde{\beta_T}$ stay close to $\widetilde{\alpha_i}$ and its analytic prolongation to $\mathbb{H}^-$. Since the lift $\widetilde{\alpha_i}$ is disjoint from the other lifts of α_i and from the lifts of different components of α, for ε small enough the developed image of the red $\widetilde{\beta_T}$ is embedded.

We now explain the variation of the construction when some α_i appears with multiplicity d_i. As was said before, it is then very important that parallel curves have the same turning directions. In this case the grafting regions are branched coverings of $\mathbb{CP}^1$. More precisely, the universal cover of the surface Gr_α is obtained by cutting $\widetilde{S_u}$ along the lifts $\widetilde{\alpha}_i$ and then by gluing back a branched covering of $\mathbb{CP}^1$ of degree d_i, branched at the endpoints of $\widetilde{\alpha}_i$, and cut along a pre-image of $\widetilde{\alpha}_i$.

For any intersection point between α_i and β, we consider a sequence of points $p_0 = p', p_1, \ldots, p_{d_i} = p''$ in $\widetilde{\alpha}_i$ increasing from p' to p'', and we iterate a construction similar to that of the case of multiplicity 1. (See Figure 4 for the situation in S_u and Figure 5 for the situation in the universal cover.)

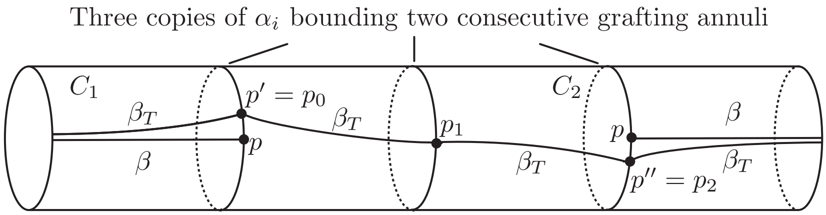


Figure 4. The curve β_T in the surface S_u when α_i has multiplicity 2. Here $T_i = L$.

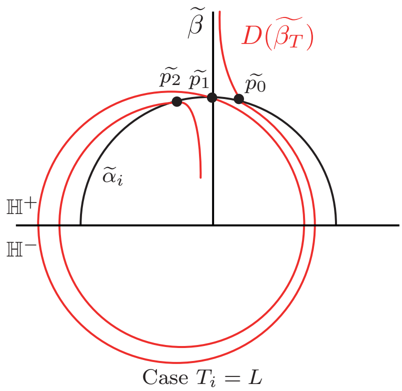


Figure 5. The case where α_i has multiplicity two. The grafting region is a branched covering of degree two, and β_T must complete two laps before exiting the region.

Finally, if some component β_j of β comes with multiplicity e_j, then we do the construction above for one copy of β_j and then we replace the result with e_j parallel copies of the corresponding component of β_T.

Remark 3.1. Note that in particular, we proved that, if σ is a projective structure on a marked surface S with Fuchsian holonomy, and β is *any* multi-curve without component homotopic to a point, then it is possible to find a multi-curve which is graftable in σ and isotopic to β. It would be interesting to find conditions on a multi-curve β that generalize the statement for a general projective structure (not necessarily with Fuchsian holonomy).

Remark 3.2. There are other ways of finding graftable curves in the isotopy class of β, obtained by fixing a letter to each equivalence class of parallel curves of the multi-curve β, instead of α. However, this construction of multi-curve will not be discussed here.

3.4. The operation $*_T$ on homotopically transverse multi-curves. Given the data (α, β, T) as in §3.3, we produce a new isotopy class of a multi-curve γ in the hyperbolic surface S_u in the following way: at each point of intersection $p \in \alpha_i \cap \beta_j$ choose a disc D_p centered at p. After an isotopy we can suppose that this disc is parametrized by an orientation preserving map of the unit disc in the plane to S_u and the image of α_i corresponds to the horizontal axis and that of β_j to the vertical axis.

On $S_u \setminus \bigcup D_p$ the multi-curve γ has the same components as $\alpha \cup \beta$. To get a multi-curve we need to join the endpoints by paths on $\bigcup \partial D_p$ by the rule given by T. As we approach an endpoint of $\alpha_i \cap \partial D_p$ from outside D_p we choose the segment of ∂D_p lying on the side of α_i given by T_i between the chosen endpoint and the next point of $\beta_j \cap \partial D_p$ (see Figure 6 for the two possibilites).

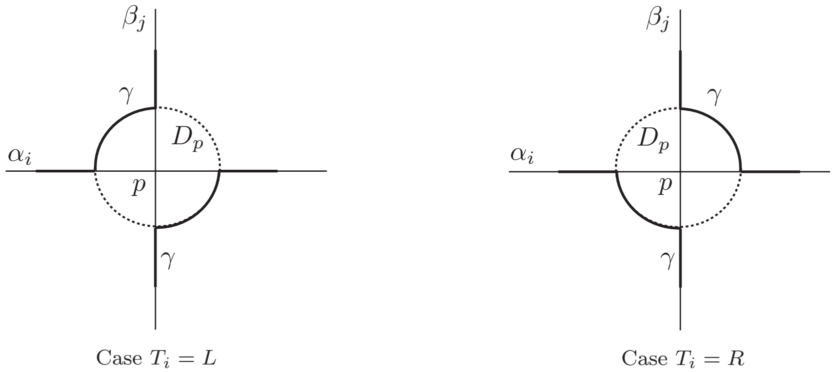


FIGURE 6. Construction of γ around a point of intersection between α_i and β_j.

This produces a family of disjoint simple closed curves γ in S_u. The transversality condition guarantees that none of its components is homotopically trivial in S_u and hence γ is a multi-curve (see references [**6, 7**]). In the sequel, for any (α, β, T) we will denote by $\alpha *_T \beta$ the resulting multi-curve: $\alpha *_T \beta := \gamma$.

3.5. Computation of grafting annuli. Recall that for a graftable multi-curve α in S_u we use the notation $\mathrm{Gr}_\alpha = \mathrm{Gr}(\sigma_u, \alpha)$.

Proposition 3.3. *Given two homotopically transverse multi-curves α and β, and a set of turning directions T for α, let β_T denote the graftable multi-curve constructed in §3.3, and $\gamma = \alpha *_T \beta$. Then*

$$\mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_T) = \mathrm{Gr}_\gamma\,.$$

Proof: We have to compute the negative annuli for the structure $\sigma' = \mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_T)$ given by Goldman's theorem (see §3.1). To this end, we will construct a curve γ_j in each negative annulus, and then show that the collection of the constructed curves $\bigcup \gamma_j$ is isotopic to the (graftable) multi-curve γ. By the discussion on §3.1 we conclude that $\sigma' = \mathrm{Gr}_\gamma$.

First of all, note that by arguing inductively on the number of components of β, we can reduce to the case where β is a simple loop.

To begin with, we orient β, we choose one of its lifts $\widetilde{\beta}$, and we number the lifts of the components of α that meet $\widetilde{\beta}$ in order of intersection with $\widetilde{\beta}$ as $\{\widetilde{\alpha}_i : i \in \mathbb{Z}\}$. So $\widetilde{\beta}$ meets $\widetilde{\alpha}_i$, then $\widetilde{\alpha_{i+1}}$, and so on.

If (S, σ) denotes the projective surface corresponding to the structure $\sigma = \mathrm{Gr}_\alpha$, $\widetilde{S}$ is constructed by gluing to $\widetilde{S_u} \setminus \bigcup \widetilde{\alpha}$ the grafting regions $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha}}$ (here $\widetilde{\alpha}$ varies among all lifts of all components of α). Such sets will be referred to as *bubbles*. See Figure 7.

Note that in case some component of α has multiplicity, then the corresponding bubbles are adjacent (this case is not depicted in the picture).

In each bubble, let $\widetilde{\alpha_i^-}$ be the geodesic in $\mathbb{H}^-$ which is the continuation of the geodesic $\widetilde{\alpha}_i$ as a round circle of the Riemann sphere (the dotted lines in Figure 7). The curve $\widetilde{\beta_T}$ intersects these geodesics successively. For each n, we denote by $\widetilde{r_n}$ the point of intersection of $\widetilde{\beta_T}$ and of $\widetilde{\alpha_n^-}$. Recall that $\gamma = \alpha *_T \beta$ and note that by construction $\widetilde{\gamma}$ is equivariantly homotopic to $\widetilde{\alpha} *_T \widetilde{\beta_T}$. On the other hand $\widetilde{\alpha}_i$ is homotopic to $\widetilde{\alpha_i^-}$. A local argument shows that $\widetilde{\alpha} *_T \widetilde{\beta_T}$ is equivariantly homotopic to $\widetilde{\alpha^-} *_T \widetilde{\beta_T}$. If we show that this multi-curve is homotopic to a union of curves $\bigcup \gamma_j$ contained in the negative part of σ', and such that each connected component of the negative part contains one of the γ_j's we will be done. Let

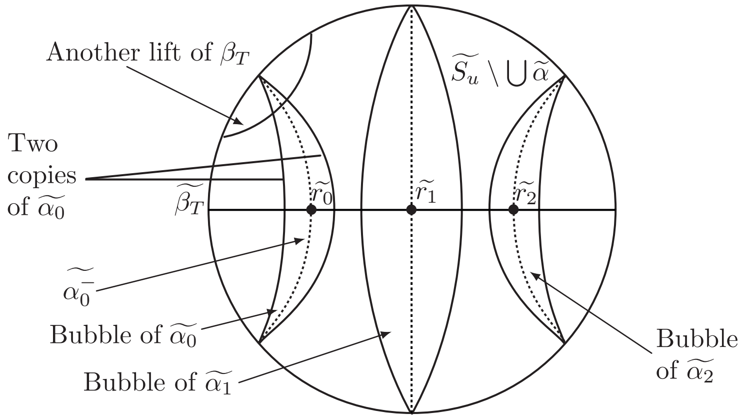


FIGURE 7. The curve β_T in $\widetilde{S}$. The bubbles corresponding to three consecutive lifts of components of α are depicted as "banana" sectors.

us analyze the structure $\mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_T)$ in detail. To obtain it we have to cut $\widetilde{S}$ along $\widetilde{\beta_T}$ and glue back a copy of $\mathbb{CP}^1 \setminus D(\widetilde{\beta_T})$, where D is the developing map for σ. Once we have cut, we have two copies $\widetilde{\beta_T^R}$ and $\widetilde{\beta_T^L}$ of $\widetilde{\beta_T}$: $\widetilde{\beta_T^R}$ is the boundary component that has the bubble of $\widetilde{\beta_T}$ on its right. In other words, $\widetilde{\beta_T^L}$ is the component which is oriented according to the orientation of $\partial(\mathbb{CP}^1 \setminus D(\widetilde{\beta_T}))$. Let $\widetilde{r_n^R}$ and $\widetilde{r_n^L}$ be the points corresponding to $\widetilde{r_n}$ lying in $\widetilde{\beta_T^R}$ and $\widetilde{\beta_T^L}$ respectively. See these objects in Figure 8.

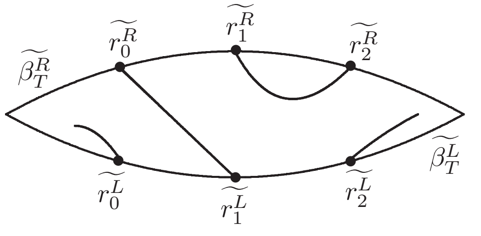


FIGURE 8. The bubble of $\widetilde{\beta_T}$. Here the segments $\widetilde{r_0^R}\widetilde{r_1^L}$ and $\widetilde{r_i^R}\widetilde{r_2^R}$ correspond to those constructed along the proof, contained in the negative part in the particular case $T_0 = L$, $T_1 = L$, $T_2 = R$.

The union of curves $\bigcup \gamma_j$ that we are going to describe in the negative part of σ' is a concatenation of two types of geodesic segments with respect to the hyperbolic metric in the negative part: segments contained in α_i^- and geodesic segments contained in the bubble of β_T joining a point $\widetilde{r_n^L}$ (resp. $\widetilde{r_n^R}$) with one of $\widetilde{r_{n+1}^L}$, $\widetilde{r_{n+1}^R}$, $\widetilde{r_{n-1}^L}$, $\widetilde{r_{n-1}^R}$. The choice will be uniquely defined by the sequence of turnings described by T along β_T. Some examples are sketched on Figure 8. These segments are most easily defined by using the developed image of $\widetilde{\beta_T}$ by the developing map D of σ. As the developed image of the points $\widetilde{r_n}$ lie in the lower half plane, we can consider the geodesic segments joining $D(\widetilde{r_n})$ with $D(\widetilde{r_{n+1}})$ for all n. Now as we cut $\mathbb{CP}^1$ along the oriented curve $D(\widetilde{\beta_T})$ we realize that the pairs of points corresponding to each $D(\widetilde{r_n})$ on each side of the cut are connected by the constructed segments. It is clear that for each n one of the points in the corresponding pair is joined by a segment to one of the points in the pair corresponding to $D(\widetilde{r_{n+1}})$ and the other to one of the points corresponding to $D(\widetilde{r_{n-1}})$. The actual correspondence depends on the sequence of turnings. If $T_n = R$ (resp. $T_n = L$) then it is $\widetilde{r_n^L}$ (resp. $\widetilde{r_n^R}$) that is joined to one of $\widetilde{r_{n+1}^L}$, $\widetilde{r_{n+1}^R}$, and this information is enough to determine which segments appear. Namely, if $T_n = T_{n+1}$, then the segment corresponding to $D(\widetilde{r_n})D(\widetilde{r_{n+1}})$ describes a segment joining the two *different* sides of the cut along $D(\widetilde{\beta_T})$. If $T_n \neq T_{n+1}$, the segment joins two points on the same side of the cut. The different possibilities before cutting $D(\widetilde{\beta_T})$ are sketched in Figure 9.

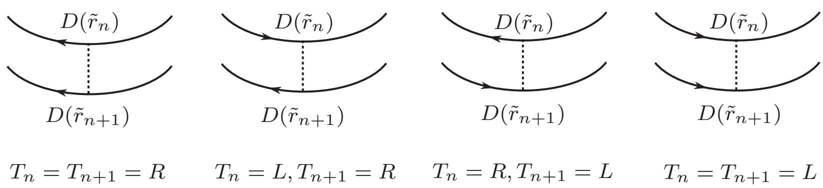


Figure 9. The oriented lines represent segments of $D(\widetilde{\beta_T})$ before cutting. The dashed lines segments of geodesic in the negative part.

After cutting $\mathbb{CP}^1$ along $D(\widetilde{\beta_T})$ we get a disc bounded by the two sides of the cut, that we identify with $\widetilde{\beta_T^L}$ and $\widetilde{\beta_T^R}$. Apart from that we have produced a union of disjoint segments in the disc each having one endpoint in $\{\widetilde{r_n^L}, \widetilde{r_n^R}\}$ and the other in $\{\widetilde{r_{n+1}^L}, \widetilde{r_{n+1}^R}\}$ (see Figure 8 for an example of the segments obtained after the cut). The constructed segments produce by concatenation with those of $\widetilde{\alpha_n^-}$ a union of curve $\bigcup \gamma_j$ contained in the negative part. To construct a homotopy with $\alpha^- *_T \widetilde{\beta_T}$, for each n we choose a_n^L and a_n^R points on α_n^- lying close to $\widetilde{r_n^L}$ and $\widetilde{r_n^R}$ respectively. Remark that a segment in γ_j joining two consecutive points of the a_n's has the property that either it cuts a single side of the cut (if the R, L-labels of $\widetilde{r_n}$ and $\widetilde{r_{n+1}}$ are different) or it cuts both sides. If it intersects only one side of the cut, we can homotope it with fixed endpoints to a segment that does not intersect the cut. Otherwise, we are obliged to intersect it. In fact this property characterizes the homotopy type with fixed endpoints of the segment. On the other hand γ has the property that a segment between two consecutive a_n's either cuts β once (if $T_n = T_{n+1}$) or it is homotopic to a segment that does not intersect β (if $T_n \neq T_{n+1}$). Therefore the segments between two consecutive points among the a_n's of $\bigcup \gamma_j$ and γ are homotopic with fixed endpoints. On the other parts of γ_j they are equal. Therefore we can construct a homotopy between $\bigcup \gamma_j$ and $\widetilde{\alpha^-} *_T \widetilde{\beta_T}$ and the result follows. ☐

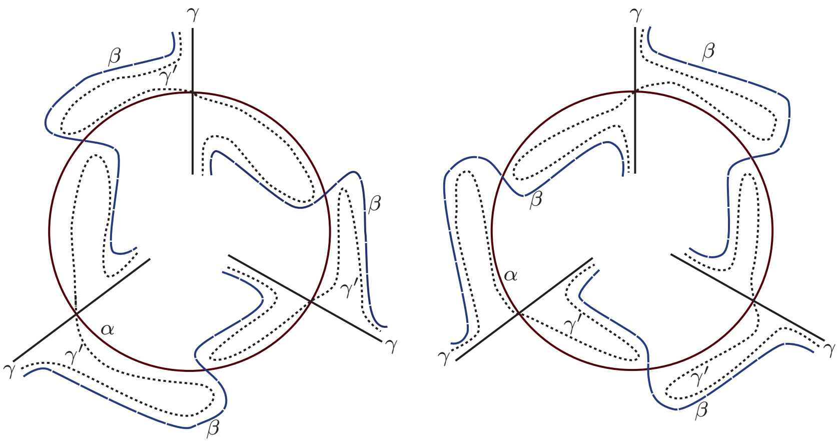
Remark 3.4. Note that a corollary of Proposition 3.3 is that there exist graftable curves that are isotopic as curves but that produce different structures when grafted. Indeed, let α and β two simple geodesics in the uniformizing structure such that they intersect only in one point. Then, β_R and β_L are isotopic curves (both are isotopic to β) and both graftable in Gr_α. By Proposition 3.3 we have that $\mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_R) = \mathrm{Gr}_{\alpha *_R \beta}$ and $\mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_L) = \mathrm{Gr}_{\alpha *_L \beta}$, which are different exotic structures because $\alpha *_R \beta$ and $\alpha *_L \beta$ are not isotopic (they are positive and negative Dehn twist of β along α). As the referee of this paper observed, this phenomenon was already present in Ito's work (see [**6**, Theorem 1.3]).

4. Positive connectedness

In this section we prove Theorem 1.1. We begin by the following lemma, which shows that the operation $*_T$ is invertible.

Lemma 4.1. *Let α and γ be two multi-curves in S intersecting transversally in the sense of §3.3. Suppose that every component of α intersects γ and vice versa. Let T be a set of turning directions for α. Then there exists a multi-curve β intersecting α transversally in the sense of §3.3 and such that the multi-curve $\alpha *_T \beta$ is isotopic to γ.*

Proof: The proof is done by first constructing a multi-curve γ' isotopic to the multi-curve γ which almost self-intersects in a suitable way. More precisely, for each component α_i of α, deform γ in a small annular neighborhood of α_i as indicated in Figure 10, depending on the specified turning direction. Then define the multi-curve β as indicated in Figure 10. It has the required properties. □



Constructing β from γ and α when turning right (R)

Constructing β from γ and α when turning left (L)

Figure 10. Constructing β.

Corollary 4.2. *There exists a cycle of length* 2 *in the graph of multi-graftings.*

Proof: Two symmetric applications of Lemma 4.1 produces curves β_1 and β_2 so that $\mathrm{Gr}_\gamma = \mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_1)$ and $\mathrm{Gr}_\alpha = \mathrm{Gr}(\mathrm{Gr}_\gamma, \beta_2)$, proving the existence of oriented cylces of length two. □

We are now in a position to prove Theorem 1.1. Let (S_i, σ_i), $i = 1, 2$, be projective structures with holonomy ρ, both different from the uniformizing structure σ_u. We denote by α_1 and α_2 the two multi-curves coding the negative annuli of σ_1 and σ_2 (that we think as a multi-geodesic with multiplicities) so that $\sigma_i = \mathrm{Gr}_{\alpha_i}$. Consider a simple closed geodesic γ cutting all components of α_1 and all components of α_2, and denote $(S_3, \sigma_3) = \mathrm{Gr}_\gamma$. By two applications of Proposition 3.3 and Lemma 4.1, there exist a multi-curve $\hat{\beta}_1 \subset S_1$ and a multi-curve $\hat{\beta}_2 \subset$

S_3 such that $\mathrm{Gr}(\sigma_1, \hat{\beta}_1) = \sigma_3$ and $\mathrm{Gr}(\sigma_3, \hat{\beta}_2) = \sigma_2$. This proves the theorem. □

References

[1] S. Baba, 2π-graftings and complex projective structures I. arXiv: 1011.5051.

[2] G. Calsamiglia, B. Deroin, and S. Francaviglia, Branched projective structures with Fuchsian holonomy, *Geom. Topol.*, to appear in 2014. arXiv: 1203.6038.

[3] D. Gallo, M. Kapovich, and A. Marden, The monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. of Math. (2)* **151(2)** (2000), 625–704. DOI: 10.2307/121044.

[4] W. M. Goldman, Projective structures with Fuchsian holonomy, *J. Differential Geom.* **25(3)** (1987), 297–326.

[5] D. A. Hejhal, Monodromy groups and linearly polymorphic functions, *Acta Math.* **135(1)** (1975), 1–55. DOI: 10.1007/BF02392015.

[6] K. Ito, Exotic projective structures and quasi-Fuchsian space. II, *Duke Math. J.* **140(1)** (2007), 85–109. DOI: 10.1215/S0012-7094-07-14013-4.

[7] F. Luo, Some applications of a multiplicative structure on simple loops in surfaces, in: *"Knots, braids, and mapping class groups"*-papers dedicated to Joan S. Birman (New York, 1998), AMS/IP Stud. Adv. Math. **24**, Amer. Math. Soc., Providence, RI, 2001, pp. 123–129.

[8] J. Thompson, Some graftings of complex projective structures with Schottky holonomy, *Geom. Dedicata* **166(1)** (2013), 203–232. DOI: 10.1007/s10711-012-9792-3.

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