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Research Article

On Approximate Solutions for Fractional Logistic Differential Equation

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A new approximate formula of the fractional derivatives is derived. The proposed formula is based on the generalized Laguerre polynomials. Global approximations to functions defined on a semi-infinite interval are constructed. The fractional derivatives are presented in terms of Caputo sense. Special attention is given to study the convergence analysis and estimate an error upper bound of the presented formula. The new spectral Laguerre collocation method is presented for solving fractional Logistic differential equation (FLDE). The properties of Laguerre polynomials approximation are used to reduce FLDE to solve a system of algebraic equations which is solved using a suitable numerical method. Numerical results are provided to confirm the theoretical results and the efficiency of the proposed method.

1. Introduction

Ordinary and partial fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering [1]. Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary noninteger order. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. Most FDEs do not have exact solutions, so approximate and numerical techniques [2–8] must be used. Several numerical and approximate methods to solve FDEs have been given such as variational iteration method [9–12], homotopy perturbation method [13], Adomian's decomposition method [14, 15], and collocation method [16, 17].

The fractional Logistic model can be obtained by applying the fractional derivative operator on the Logistic equation. The model is initially published by Pierre Verhulst in 1838 [18, 19]. The continuous Logistic model is described by first-order ordinary differential equation. The discrete Logistic model is simple iterative equation that reveals the chaotic property in certain regions [20]. There are many variations of the population modeling [19, 21]. The Verhulst model is the classic example to illustrate the periodic doubling and chaotic behavior in dynamical system [20]. The model which is described the population growth may be limited by certain factors like population density [18, 19, 21].

Applications of Logistic Equation. A typical application of the Logistic equation is a common model of population growth. Another application of Logistic curve is in medicine, where the Logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above-mentioned use in the framework of ecology. Denoting by $u(t)$ the size of the tumor at time t.

The solution of Logistic equation is explained the constant population growth rate which not includes the limitation on food supply or spread of diseases [19]. The solution curve of the model is increasing exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity [19], $dN/dt = \rho N(1 - (N/K))$ where N is the population size with respect to time, ρ is the rate of maximum population growth, and K is the carrying capacity.

The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^{\rho t}$ where N_0 is the initial population [22].

In this paper, we consider FLDE of the form

$$D^{\nu} u(t) = \rho u(t)(1 - u(t)), \quad t > 0, \ \rho > 0, \tag{1}$$

here, the parameter ν refers to the fractional order of time derivative with $0 < \nu \le 1$.

We also assume an initial condition

$$u(0) = u_0, \quad u_0 > 0. \tag{2}$$

For $\nu = 1$, (1) is the standard Logistic differential equation

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t)). \tag{3}$$

The exact solution to this problem is $u(t) = u_0/((1-u_0)e^{-\rho t} + u_0)$.

The existence and the uniqueness of the proposed problem (1) are introduced in details in [23, 24].

The main aim of the presented paper is concerned with an extension of the previous work on FDEs and derive an approximate formula of the fractional derivative of the Laguerre polynomials and then we apply this approach to obtain the numerical solution of FLDE. Also, we present study of the convergence analysis and estimate an error upper bound of the proposed formula.

The structure of this paper is arranged in the following way: in Section 2, we introduce some basic definitions about Caputo fractional derivatives and properties of the Laguerre polynomials. In Section 3, we give an approximate formula of the fractional derivative of Laguerre polynomials and its convergence analysis. In Section 4, we implement the proposed method for solving FLDE to show the accuracy of the presented method. Finally, in Section 5, the paper ends with a brief conclusion and some remarks.

2. Preliminaries and Notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

2.1. The Caputo Fractional Derivative

Definition 1. The Caputo fractional derivative operator D^{ν} of order ν is defined in the following form:

$$D^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}} dt, \quad \nu > 0, \ x > 0, \tag{4}$$

where $m - 1 < \nu \le m$, $m \in \mathbb{N}$.

Similar to integer-order differentiation, Caputo fractional derivative operator is linear

$$D^{\nu}(\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x), \tag{5}$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^{\nu} C = 0, \quad C \text{ is a constant}, \tag{6}$$

$$D^{\nu} x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0, \ n < \lceil \nu \rceil; \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for } n \in \mathbb{N}_0, \ n \ge \lceil \nu \rceil. \end{cases} \tag{7}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Recall that for $\nu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and their properties see [1, 25–28].

2.2. The Definition and Properties of the Generalized Laguerre Polynomials. Spectral collocation methods are efficient and highly accurate techniques for numerical solution of nonlinear differential equations. The basic idea of the spectral collocation method is to assume that the unknown solution $u(t)$ can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for a problem under consideration. In [16], Khader introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials. In [29] the generalized Laguerre polynomials were used to compute a spectral solution of a nonlinear boundary value problems. The generalized Laguerre polynomials constitute a complete orthogonal sets of functions on the semi-infinite interval $[0, \infty)$. Convolution structures of Laguerre polynomials were presented in [30]. Also, other spectral methods based on other orthogonal polynomials are used to obtain spectral solutions on unbounded intervals [31].

The generalized Laguerre polynomials $[L_n^{(\alpha)}(x)]_{n=0}^{\infty}$, $\alpha > -1$ are defined on the unbounded interval $(0, \infty)$ and can be determined with the aid of the following recurrence formula:

$$\begin{aligned} &(n+1) L_{n+1}^{(\alpha)}(x) + (x - 2n - \alpha - 1) L_n^{(\alpha)}(x) \\ &\quad + (n+\alpha) L_{n-1}^{(\alpha)}(x) = 0, \quad n = 1, 2, \ldots, \end{aligned} \tag{8}$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$.

The analytic form of these polynomials of degree n is given by

$$\begin{aligned} L_n^{(\alpha)}(x) &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k \\ &= \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}, \end{aligned} \tag{9}$$

$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight function $w(x) = (1/\Gamma(1+\alpha))x^{\alpha}e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} x^{\alpha}e^{-x}L_m^{(\alpha)}(x)\,L_n^{(\alpha)}(x)\,dx = \binom{n+\alpha}{n}\delta_{mn}. \tag{10}$$

Also, they satisfy the differentiation formula

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x), \quad k = 0,1,\ldots,n. \tag{11}$$

Any function $u(x)$ belongs to the space $L_w^2[0,\infty)$ of all square integrable functions on $[0,\infty)$ with weight function $w(x)$ can be expanded in the following Laguerre series:

$$u(x) = \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{12}$$

where the coefficients c_i are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}\int_0^{\infty} x^{\alpha}e^{-x}L_i^{(\alpha)}(x)\,u(x)\,dx, \quad i = 0,1,\ldots. \tag{13}$$

Consider only the first $(m+1)$ terms of generalized Laguerre polynomials, so we can write

$$u_m(x) = \sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{14}$$

For more details on Laguerre polynomials, its definitions, and properties, see [29, 31, 32].

3. An Approximate Fractional Derivative of $L_n^{(\alpha)}(x)$ and Its Convergence Analysis

The main goal of this section is to introduce the following theorems to derive an approximate formula of the fractional derivatives of the generalized Laguerre polynomials and study the truncating error and its convergence analysis.

Lemma 2. *Let $L_n^{(\alpha)}(x)$ be a generalized Laguerre polynomial then*

$$D^{\nu} L_n^{(\alpha)}(x) = 0, \quad n = 0,1,\ldots,\lceil\nu\rceil - 1, \ \nu > 0. \tag{15}$$

Proof. This lemma can be proved directly by applying (6)-(7) on (9). □

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 3. *Let $u(x)$ be approximated by the generalized Laguerre polynomials as* (14) *and also suppose $\nu > 0$; then its approximated fractional derivative can be written in the following form:*

$$D^{\nu}(u_m(x)) \cong \sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i w_{i,k}^{(\nu)}\,x^{k-\nu}, \tag{16}$$

where $w_{i,k}^{(\nu)}$ is given by

$$w_{i,k}^{(\nu)} = \frac{(-1)^k}{\Gamma(k+1-\nu)}\binom{i+\alpha}{i-k}. \tag{17}$$

Proof. Since the Caputo's fractional differentiation is a linear operation, we obtain

$$D^{\nu}(u_m(x)) = \sum_{i=0}^{m} c_i D^{\nu}\left(L_i^{(\alpha)}(x)\right). \tag{18}$$

Also, from (9) we can get

$$D^{\nu} L_i^{(\alpha)}(x) = 0, \quad i = 0,1,\ldots,\lceil\nu\rceil - 1, \ \nu > 0. \tag{19}$$

Therefore, for $i = \lceil\nu\rceil, \lceil\nu\rceil+1,\ldots,m$, and by using (6)-(7) in (9), we get

$$\begin{aligned} D^{\nu} L_i^{(\alpha)}(x) &= \sum_{k=0}^{i}\frac{(-1)^k}{k!}\binom{i+\alpha}{i-k}D^{\nu}x^k \\ &= \sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^k}{\Gamma(k+1-\nu)}\binom{i+\alpha}{i-k}x^{k-\nu}. \end{aligned} \tag{20}$$

A combination of (18)–(20) leads to the desired result (16) and ends the proof of the theorem. □

Test Example. Consider the function $u(x) = x^3$ with $m = 3$, $\nu = 1.5$, and $\alpha = -0.5$, the generalized Laguerre series of x^3 is

$$x^3 = 1.875L_0^{(\alpha)}(x) - 11.25L_1^{(\alpha)}(x) + 15L_2^{(\alpha)}(x) - 6L_3^{(\alpha)}(x). \tag{21}$$

Now, by using formula (16), we obtain

$$D^{1.5}x^3 = \sum_{i=2}^{3}\sum_{k=2}^{i} c_i w_{i,k}^{(1.5)}x^{k-1.5}, \tag{22}$$

where $w_{2,2}^{(1.5)} = 1.12838$, $w_{3,2}^{(1.5)} = 2.82095$, $w_{3,3}^{(1.5)} = -0.752253$, therefore,

$$\begin{aligned} D^{1.5}x^3 &= c_2 w_{2,2}^{(1.5)}x^{0.5} + c_3 w_{3,2}^{(1.5)}x^{0.5} + c_3 w_{3,3}^{(1.5)}x^{1.5} \\ &= \frac{\Gamma(4)}{\Gamma(2.5)}x^{1.5}, \end{aligned} \tag{23}$$

which agrees with the exact derivative (7).

Theorem 4. *The Caputo fractional derivative of order ν for the generalized Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form:*

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=\lceil\nu\rceil}^{i}\sum_{j=0}^{k-\lceil\nu\rceil}\Omega_{ijk}\,L_j^{(\alpha)}(x), \quad i = \lceil\nu\rceil, \lceil\nu\rceil+1,\ldots,m, \tag{24}$$

where

$$\Omega_{ijk} = \sum_{r=0}^{j}\frac{(-1)^{r+k}(\alpha+i)!\,(j)!\,(k+\alpha-\nu+r)!}{(k-\nu)!\,(i-k)!\,(\alpha+k)!\,r!\,(j-r)!\,(\alpha+r)!}. \tag{25}$$

Proof. From the properties of the generalized Laguerre polynomials [33] and expanding $x^{k-\nu}$ in (20) in the following form:

$$x^{k-\nu} = \sum_{j=0}^{k-\lceil\nu\rceil} c_{kj} L_j^{(\alpha)}(x), \tag{26}$$

where c_{kj} can be obtained using (13), where $u(x) = x^{k-\nu}$, then

$$\begin{aligned} c_{kj} &= \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} \int_0^\infty x^{k+\alpha-\nu} e^{-x} L_j^{(\alpha)}(x)\,dx \\ &= \sum_{r=0}^{j} \frac{(-1)^r (j)!(k-\nu+\alpha+r)!}{r!(j-r)!(\alpha+r)!}, \\ &\qquad j = 0,1,\ldots, \end{aligned} \tag{27}$$

this by substituting from (9) and using the definition of Gamma function. Now, we can write (26) in the following form:

$$x^{k-\nu} = \sum_{j=0}^{k-\lceil\nu\rceil} \sum_{r=0}^{j} \frac{(-1)^r (j)!(k-\nu+\alpha+r)!}{r!(j-r)!(\alpha+r)!} L_j^{(\alpha)}(x). \tag{28}$$

Therefore, the Caputo fractional derivative $D^\nu L_i^{(\alpha)}(x)$ in (20) can be rewritten in the following form:

$$\begin{aligned} D^\nu L_i^{(\alpha)}(x) &= \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \sum_{r=0}^{j} \Big((-1)^{r+k} (\alpha+i)!(j)!(k-\nu+\alpha+r)! \\ &\quad \times \big((k-\nu)!(i-k)!(\alpha+k)!r!(j-r)! \\ &\quad \times (\alpha+r)!\big)^{-1} \Big) L_j^{(\alpha)}(x), \end{aligned} \tag{29}$$

for $i = \lceil\nu\rceil, \lceil\nu\rceil+1, \ldots, m$. Equation (29) leads to the desired result (24) and this completes the proof of the theorem. □

Theorem 5. *For the Laguerre polynomials $L_n^{(\alpha)}(x)$, one has the following global uniform bounds estimates:*

$$\left|L_n^{(\alpha)}(x)\right| \le \begin{cases} \dfrac{(\alpha+1)_n}{n!} e^{x/2}, & \text{for } \alpha \ge 0,\ x \ge 0, \\ & n = 0,1,\ldots; \\ \left(2 - \dfrac{(\alpha+1)_n}{n!}\right) e^{x/2}, & \text{for } -1 < \alpha \le 0, \\ & x \ge 0,\ n = 0,1,\ldots. \end{cases} \tag{30}$$

Proof. These estimates were presented in [33–35]. □

Theorem 6. *The error in approximating $D^\nu u(x)$ by $D^\nu u_m(x)$ is bounded by*

$$\begin{aligned} \left|E_T(m)\right| &\le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j) \frac{(\alpha+1)_j}{j!} e^{x/2}, \\ &\qquad \alpha \ge 0,\ x \ge 0,\ j = 0,1,\ldots, \\ \left|E_T(m)\right| &\le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j) \left(2 - \frac{(\alpha+1)_j}{j!}\right) e^{x/2}, \\ &\qquad -1 < \alpha \le 0,\ x \ge 0,\ j = 0,1,\ldots, \end{aligned} \tag{31}$$

where $|E_T(m)| = |D^\nu u(x) - D^\nu u_m(x)|$ *and* $\Pi_\nu(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \Omega_{ijk}$.

Proof. A combination of (12), (14), and (24) leads to

$$\begin{aligned} \left|E_T(m)\right| &= \left|D^\nu u(x) - D^\nu u_m(x)\right| \\ &\le \sum_{i=m+1}^{\infty} c_i \Pi_\nu(i,j) \left|L_j^{(\alpha)}(x)\right|, \end{aligned} \tag{32}$$

using (30) and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. □

4. Implementation of Laguerre Spectral Method for Solving FLDE

In this section, we introduce a numerical algorithm using Laguerre spectral method for solving the fractional Logistic differential equation of the form (1).

The procedure of the implementation is given by the following steps.

(1) Approximate the function $u(t)$ using the formula (14) and its Caputo fractional derivative $D^\nu u(t)$ using the presented formula (16) with $m = 5$, then FLDE (1) is transformed to the following approximated form:

$$\begin{aligned} &\sum_{i=1}^{5} \sum_{k=1}^{i} c_i w_{i,k}^{(\nu)} t^{k-\nu} - \rho \left(\sum_{i=0}^{5} c_i L_i^{(\alpha)}(t) \right) \\ &\quad \times \left(1 - \sum_{i=0}^{5} c_i L_i^{(\alpha)}(t) \right) = 0, \end{aligned} \tag{33}$$

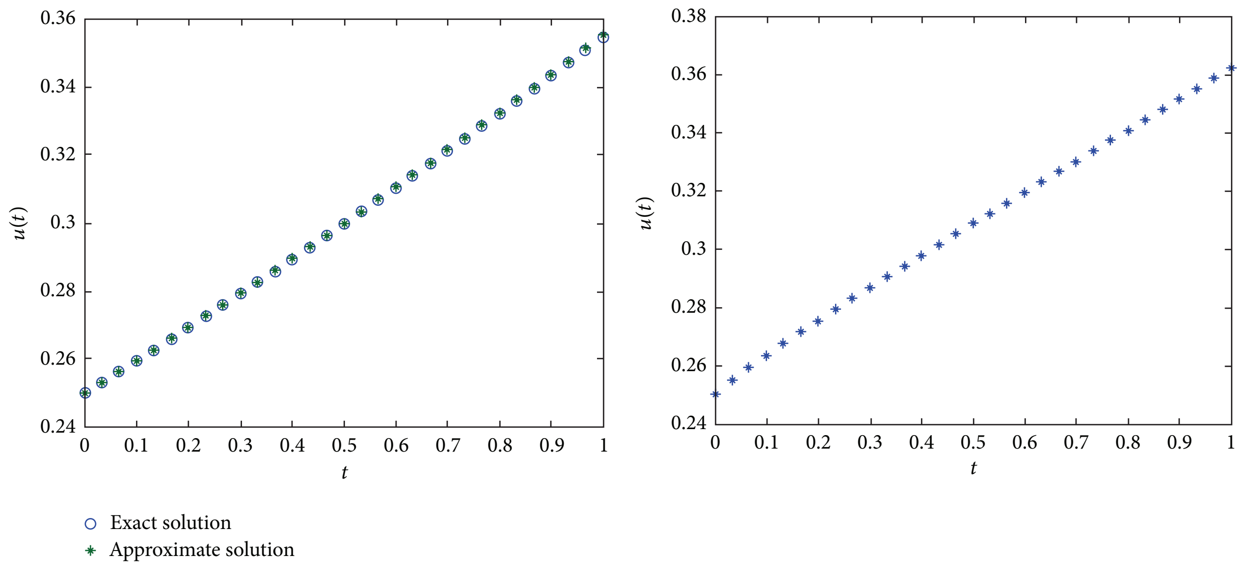
where $w_{i,k}^{(\nu)}$ is defined in (17).

We now collocate (33) at $(m+1-\lceil\nu\rceil)$ points t_p, $p = 0,1,\ldots,m-\lceil\nu\rceil$ as

$$\begin{aligned} &\sum_{i=1}^{5} \sum_{k=1}^{i} c_i w_{i,k}^{(\nu)} t_p^{k-\nu} - \rho \left(\sum_{i=0}^{5} c_i L_i^{(\alpha)}(t_p) \right) \\ &\quad \times \left(1 - \sum_{i=0}^{5} c_i L_i^{(\alpha)}(t_p) \right) = 0. \end{aligned} \tag{34}$$

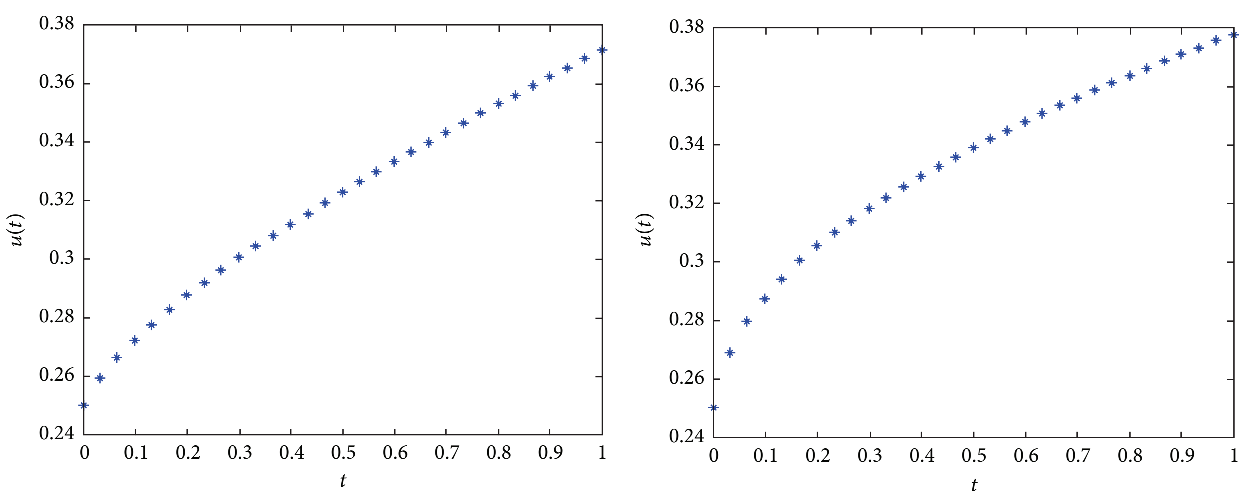
(2) From the initial condition (2) we obtain the following equation:

$$\sum_{i=0}^{5} c_i L_i^{(\alpha)}(0) = u_0. \tag{35}$$



(a) (b)

Figure 1: A comparison between the approximate solution and the exact solution at $\nu = 1$ (a). The behavior of the approximate solution using the proposed method at $\nu = 0.85$ (b).



(a) (b)

Figure 2: The behavior of the approximate solution using the proposed method at $\nu = 0.65$ (a) and at $\nu = 0.45$ (b).

Equations (34)-(35) represent a system of nonlinear algebraic equations which contains six equations for the unknowns c_i, $i = 0, 1, \ldots, 5$.

(3) Solve the resulting system using the Newton iteration method to obtain the unknowns c_i, $i = 0, 1, \ldots, 5$. Therefore, the approximate solution will take the form

$$u(t) = c_0 L_0^{(\alpha)}(t) + c_1 L_1^{(\alpha)}(t) + c_2 L_2^{(\alpha)}(t) + c_3 L_3^{(\alpha)}(t) + c_4 L_4^{(\alpha)}(t) + c_5 L_5^{(\alpha)}(t). \tag{36}$$

The numerical results of the proposed problem (1) are given in Figures 1 and 2 with different values of ν in the interval $[0, 1]$ with $\rho = 0.5$ and $u_0 = 0.25$. Where in Figure 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at $\nu = 1$ (Figure 1(a)), and the behavior of the approximate solution using the proposed method at $\nu = 0.85$ (Figure 1(b)). But, in Figure 2, we presented the behavior of the approximate solution with different values of ν ($\nu = 0.65$ (Figure 2(a)) and $\nu = 0.45$ (Figure 2(b))).

5. Conclusion and Remarks

In this paper, we introduced a new spectral collocation method based on Laguerre polynomials for solving FLDE.

We have introduced an approximate formula for the Caputo fractional derivative of the generalized Laguerre polynomials in terms of generalized Laguerre polynomials themselves. In the proposed method we used the properties of the Laguerre polynomials to reduce FLDE to solve a system of algebraic equations. The error upper bound of the proposed approximate formula is stated and proved. The obtained numerical results show that the proposed algorithm converges as the number of m terms is increased. The solution is expressed as a truncated Laguerre series and so it can be easily evaluated for arbitrary values of time using any computer program without any computational effort. From illustrative examples, we can conclude that this approach can obtain very accurate and satisfactory results. Comparisons are made between the approximate solution and the exact solution to illustrate the validity and the great potential of the technique. All computations are done using Matlab.

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