Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2014, Article ID 161756, 8 pages
http://dx.doi.org/10.1155/2014/161756

Research Article

Multifractal Structure of the Divergence Points of Some Homogeneous Moran Measures

JiaQing Xiao and YouMing He

Wuhan College, Zhongnan University of Economics and Law, Wuhan 430070, China

Correspondence should be addressed to JiaQing Xiao; xiaojq@whut.edu.cn

Received 18 January 2014; Accepted 3 March 2014; Published 7 April 2014

Academic Editor: Christian Maes

The point x for which the limit $\lim_{r\to 0}(\log \mu(B(x,r))/\log r)$ does not exist is called divergence point. Recently, multifractal structure of the divergence points of self-similar measures has been investigated by many authors. This paper is devoted to the study of some Moran measures with the support on the homogeneous Moran fractals associated with the sequences of which the frequency of the letter exists; the Moran measures associated with this kind of structure are neither Gibbs nor self-similar and than complex. Such measures possess singular features because of the existence of so-called divergence points. By the box-counting principle, we analyze multifractal structure of the divergence points of some homogeneous Moran measures and show that the Hausdorff dimension of the set of divergence points is the same as the dimension of the whole Moran set.

1. Introduction and Statement of Results

1.1. Moran Set. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and let $\{r_k\}$ be a sequence of positive real number with $n_k r_k < 1$ for any $k \in \mathbb{N}$. Define $D_o = \phi$, and for any $k \geq 1$, set $D_{m,k} = \{(i_m, i_{m+1}, \ldots, i_k);\ 1 \leq i_j \leq n_j,\ m \leq j \leq k\}$, $D_k = D_{1,k}$, and

$$D = \bigcup_{k\geq 0} D_k. \tag{1}$$

If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in D_k$, $\tau = (\tau_1, \ldots, \tau_{m-k}) \in D_{k+1,m}$, let $\sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_{m-k})$. And for $1 \leq l \leq k$, remark $\sigma \mid l = (\sigma_1, \ldots, \sigma_l)$.

Definition 1. Suppose J is a closed interval of length 1. The collection $\mathcal{F} = \{J_\sigma;\ \sigma \in D\}$ of closed subintervals of J is said to have a homogeneous Moran structure, if it satisfies the following conditions (MSC):

(i) $J_\phi = J$;

(ii) for all $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ and satisfy that $J^\circ_{\tau*i} \bigcap J^\circ_{\tau*j} = \phi$ $(i \neq j)$, where A° denotes the interior of A;

(iii) for any $k \geq 1$ and $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$,

$$\frac{|J_{\sigma*j}|}{|J_\sigma|} = r_k, \tag{2}$$

where $|A|$ denotes the diameter of A.

Suppose that $\mathcal{F}$ is a collection of closed subintervals of J having homogeneous Moran structure, and set

$$E_k = \bigcup_{\sigma\in D_k} J_\sigma, \qquad E = \bigcap_{k\geq 0} E_k. \tag{3}$$

It is ready to see that E is a nonempty compact set. The set $E := E(\mathcal{F})$ is called the homogeneous Moran set associated with the collection $\mathcal{F}$.

Let $\mathcal{F}_k = \{J_\sigma;\ \sigma \in D_k\}$, and let $\mathcal{F} = \bigcup_{k\geq 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called the basic elements of order k of the homogeneous Moran set E and the elements of $\mathcal{F}$ are called the basic elements of the homogeneous Moran set E.

Remark 2. If $\lim_{n\to\infty} \sup_{\sigma\in D_n} |J_\sigma| > 0$, then E contains interior points. Thus, the measure and dimension properties will be trivial. We assume therefore that

$$\lim_{n\to\infty} \sup_{\sigma\in D_n} |J_\sigma| = 0. \tag{4}$$

Proposition 3 (see [1, Proposition 3.1]). *For a homogeneous Moran set E defined as above, suppose furthermore that*

$$\lim_{k\to\infty}\frac{\log r_k}{\log r_1 r_2\cdots r_{k-1}}=0. \quad (5)$$

Then we have

$$\dim_H E=\liminf_{k\to\infty} s_k, \quad (6)$$

where s_k satisfies the equation $\sum_{\sigma\in D_k} r_\sigma^{s_k}=1$ for each k.

Let $A=\{a_1,a_2,\ldots,a_m\}$, and let $\omega=s_1s_2\cdots s_k\cdots$ be a sequence over A, $s_i\in A$. For $k\geq 1$, write $\omega_k=\omega|_k=s_1s_2\cdots s_k$; then $|\omega_k|=k$. We denote by $|\omega_k|_{a_i}$ the number of occurrences of the letter a_i in ω_k. If for any $a_i\in A$, $\lim_{k\to\infty}(|\omega_k|_{a_i}/k)=\eta_i>0$, then we say that the sequence ω has the frequency vector $\eta=(\eta_1,\eta_2,\ldots,\eta_m)$. It is easy to see that $\sum_{i=1}^m|\omega_k|_{a_i}=k$ and $\sum_{j=1}^m\eta_j=1$. For $\eta=(\eta_1,\eta_2,\ldots,\eta_m)$, let

$$A_\eta^{\mathbb{N}}=\left\{\omega=\{s_k\}_{k\geq1}; s_k\in A, \lim_{k\to\infty}\frac{|\omega_k|_{a_i}}{k}=\eta_i, 1\leq i\leq m\right\}. \quad (7)$$

For $1\leq i\leq m$, let $m_i\in\mathbb{N}$ and let c_i be a positive real number with $m_ic_i\leq 1$. For $\omega\in A^{\mathbb{N}}$, in the homogeneous Moran construction above, for any $k\geq 1$ if $s_k=a_i$ take $n_k=m_i$, $r_k=c_i$. Then we construct the homogeneous Moran set relating to $\omega\in A_\eta^{\mathbb{N}}$ and denote it by $E(\omega)=\{J,\{n_k\},\{r_k\}\}$.

Remark 4. In this paper, we assume that $J_\sigma\in\mathcal{F}_k$ $(k\geq 1)$, let $J_{\sigma*1},J_{\sigma*2},\ldots,J_{\sigma*n_{k+1}}$ be the n_{k+1} basic intervals of order $k+1$ contained in J_σ arranged from left to right. For all $1\leq j\leq n_{k+1}-1$, let $d(J_{\sigma*j},J_{\sigma*(j+1)})\geq\Delta_k|J_\sigma|$, where $\{\Delta_k\}$ is a sequence of positive real number. Let $\Delta=\inf_{k\geq1}\Delta_k$. In this paper we suppose $\Delta>0$.

1.2. Moran Measure. Let $P_{a_i}=(p_{i1}p_{i2}\cdots p_{im_i})$ $(1\leq i\leq m)$ be probability vectors; that is, $p_{ij}>0$ and $\sum_{j=1}^{m_i}p_{ij}=1$ $(1\leq i\leq m)$. For any $k\geq 1$, $\sigma\in D_k$, from Section 1.1, we know $\sigma=\sigma_1\sigma_2\cdots\sigma_k\in D_k$ where $\sigma_k\in\{1,2,\ldots,m_i\}$, if $s_k=a_i$. For $\sigma=\sigma_1\sigma_2\cdots\sigma_k$, define $\sigma(a_i)$ as follows: let $\omega_k=s_1\cdots s_k$, $e_1<e_2<\cdots e_{|\omega_k|_{a_i}}$ be the occurrences of the letter a_i in ω_k; then $\sigma(a_i)=\sigma_{e_1}\sigma_{e_2}\cdots\sigma_{e_{|\omega_k|_{a_i}}}$. For convenience we will write $\sigma(a_i)=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{|\omega_k|_{a_i}}}$, where $\sigma_{i_j}\in\{1,2,\ldots,m_i\}$ $(1\leq i\leq m)$. In fact, $\sigma_{(a_1)}*\sigma_{(a_2)}*\cdots*\sigma_{(a_m)}$ is a rearrangement of $\sigma=\sigma_1\cdots\sigma_k$. We make the convention that $\sigma_{(a_i)}=\phi$ if $|\omega_k|_{a_i}=0$.

Now define

$$p_{\sigma(a_i)}=p_{i\sigma_{i_1}}\cdots p_{i\sigma_{i_{|\omega_k|_{a_i}}}},\quad 1\leq i\leq m. \quad (8)$$

It is obvious that $\sum_{\sigma\in D_k}\prod_{i=1}^m p_{\sigma(a_i)}=1$ for any $k\geq 1$. We make the convention that $p_{\sigma(a_i)}=1$ if $\sigma(a_i)=\phi$.

Let μ be a mass distribution on $E(\omega)$, such that for any $J_\sigma\in\mathcal{F}_k$, $\sigma\in D_k$,

$$\mu(J_\sigma)=p_{\sigma(a_1)}p_{\sigma(a_2)}\cdots p_{\sigma(a_m)}, \quad (9)$$

and $\mu(\sum_{\sigma\in D_k}J_\sigma)=1$. Since μ is related to ω, we denote it by $\mu(\omega)$. Here $\mu(\omega)$ is a homogeneous Moran measure on $E(\omega)$, and it is an extension of the self-similar measure by Hutchinson [2].

1.3. Main Results. From now on, we assume that $E(\omega)$ is a homogeneous Moran fractal defined in Section 1.1, and $\mu(\omega)$ is a probability measure introduced in Section 1.2. The notationa D, D_k, J_σ, $\mu(J_\sigma)$, $P_{\sigma(a_i)}$ $(1\leq i\leq m)$ are as above; in the following, if $\sigma\in D_k$, $\tau\in D_{k+1,k+n}$, let $r_\sigma=|J_\sigma|$, $p_\sigma=\mu(J_\sigma)$ and $r_\tau=|J_{\sigma*\tau}|/|J_\sigma|$, $p_\tau=\mu(J_{\sigma*\tau})/\mu(J_\sigma)$. Now we define an auxiliary function $\beta(q)$ as follows. For each $q\in\mathbb{R}$ and $k\geq 1$, there is a unique number $\beta_k(q)$ such that

$$\sum_{\sigma\in D_k}p_\sigma^q r_\sigma^{\beta_k(q)}=1. \quad (10)$$

By simple calculation, we get

$$\lim_{k\to\infty}\beta_k(q)=\beta(q)=\frac{\sum_{i=1}^m\eta_i\log\left(\sum_{j=1}^{m_i}p_{ij}^q\right)}{\sum_{i=1}^m\eta_i\log c_i}. \quad (11)$$

Proposition 5 (see [3, Proposition 2.3]). *For all $q\in\mathbb{R}$, $\beta(q)$ defined by* (10) *satisfies the following:*

(i) $\beta(0)=\dim E$;

(ii) *$\beta(q)$ is strictly decreasing, and* $\lim_{q\to\mp\infty}\beta(q)=\pm\infty$;

(iii) *$\beta(q)$ is convex in q, and $\beta(q)$ is strictly convex if and only if $\log p_{ij}/\log c_i$ is not the same for all $1\leq j\leq m_i$, $i=1,2,\ldots,m$.*

Let μ be a Borel probability measure on $\mathbb{R}^d$; let

$$\Theta(q;r)=\sup\sum_i\mu(B(x_i,r))^q,\quad r>0,\ q\in\mathbb{R}, \quad (12)$$

where the supremum is taken over all families of disjoint closed balls $\{B(x_i,r)\}_i$ with $x_i\in\operatorname{supp}\mu$. If $\lim_{r\to0}(\log\Theta(q;r)/\log r)$ exists, we call that the L^q-spectrum $\tau(q)$ of μ exists; that is,

$$\tau(q)=\lim_{r\to0}\frac{\log\Theta(q;r)}{-\log r}. \quad (13)$$

Peres and Solomyak [4] give alternative definition of L^q-spectrum. Let μ be a Borel probability measure on $\mathbb{R}^d$. Let $\mathbf{D}_n$ be the partition of $\mathbb{R}^d$ into grid boxes $\prod_{i=1}^d[k_i2^{-n},(k_i+1)2^{-n})$ with $k_i\in\mathbb{Z}$. For $q>0$, denote $\tau_n^{(q)}(\mu)=\sum_{Q\in\mathbf{D}_n}(\mu Q)^q$. If $\lim_{n\to0}(\log\tau_n^{(q)}(\mu)/-n\log 2)$ exists, we call that the L^q-spectrum $\tau_0(q)$ of μ exists; that is,

$$\tau_0(q)=\lim_{n\to0}\frac{\log\tau_n^{(q)}(\mu)}{-n\log 2}. \quad (14)$$

Peres and Solomyak [4] prove that $\tau(q)=\tau_0(q)$.

Proposition 6 (see [5]). *Let μ be a Moran measure supported on the homogeneous Moran fractals E; then*

$$\tau(q)=\tau_0(q)=\beta(q). \quad (15)$$

The Legendre transform of β is the function $\beta^* : (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ defined by

$$\beta^*(\alpha) = \inf_{q\in\mathbb{R}} \{\beta(q) + \alpha q\}, \tag{16}$$

where $\alpha_{\min} = \inf_{q\in\mathbb{R}} \beta'(q)$, $\alpha_{\max} = \sup_{q\in\mathbb{R}} \beta'(q)$.

Let X be a complete separable metric space and μ a finite Borel measure on X. In the multifractal analysis one is interested in the size of the following level sets:

$$X_\alpha = \left\{x \in X : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha\right\}, \quad -\infty \le \alpha \le +\infty. \tag{17}$$

The space X has the following natural decomposition:

$$X = \bigcup_{-\infty\le\alpha\le+\infty} X_\alpha \cup X^o, \tag{18}$$

where

$$X^o = \left\{x \in X : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist}\right\}. \tag{19}$$

The set X^o is called the set of divergence points and the point x for which the limit $\lim_{r\to 0}(\log \mu(B(x,r))/\log r)$ does not exist is called divergence point. Recently, multifractal structure of the divergence points of self-similar measures has been investigated by a large number of authors. Barreira and Schmeling [6] and Chen and Xiong [7] have shown that for self-similar measures satisfying the SSC the set of divergence points typically has the Hausdorff dimension as the support K. Furthermore, Olsen and Winter [8] analyse its structure and give a decomposition of this set for the case that the SSC satisfies. However, with only the OSC satisfied, we cannot do most of the work on a symbolic space and then transfer the results to the subsets of $\mathbb{R}^d$, which makes things more difficult. By the box-counting principle we (2011) [9] show that the set of divergence point has still the same Hausdorff dimension as the support K for self-similar measures satisfying the OSC. Li et al. [10] further analyse its structure and give a decomposition of this set for the case that the OSC satisfies. This paper is devoted to the study of some Moran measures with the support on the homogeneous Moran fractals associated with the sequences of which the frequency of the letter exists; such measures possess singular features because of the existence of so-called divergence points. By the box-counting principle, we analyze Multifractal structure of the divergence points of some homogeneous Moran measures and show that the Hausdorff dimension of the set of divergence points is the same as the dimension of the whole Moran set. It should be pointed out that the Moran measures associated with this kind of structure are neither Gibbs nor self-similar and more than complex.

Theorem 7. *Let μ be a Moran measure supported on the homogeneous Moran fractals $E(\Omega)$ associated with the sequences Ω of which the frequency of the letter exists as above. Set*

$$E_{q_1,q_2} = \left\{x \in E(\Omega) \mid \beta'(q_1) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \beta'(q_2)\right\}. \tag{20}$$

Then

$$\dim E_{q_1,q_2} = \beta^*\left(\beta'(q_1)\right). \tag{21}$$

By Theorem 7 and Proposition 5, we easily obtain that the Hausdorff dimension of the set of divergence points is the same as the dimension of the whole Moran set.

2. Several Lemma

Lemma 8. *Suppose that $\alpha \in \mathbb{R}$ is such that $\alpha = -\beta'(q)$ for some $q \in \mathbb{R}$. Then for any $\delta > 0$, $\rho > 0$,*

(I) *there exist $d \in (0,\rho)$, $\ell \ge d^{-\beta^*(\alpha)+\delta(q+1)}$, an integral number N, and $u_1, \ldots, u_\ell \in D_N$ satisfying the following properties:*

(a) $d^{1+\delta} \le r_{u_i} \le d^{1-\delta}$ *for all* $1 \le i \le \ell$,

(b) $d^{\alpha+\delta} \le p_{u_i} \le d^{\alpha-\delta}$ *for all* $1 \le i \le \ell$;

(II) *there exist an integral number n_0 such that for any integer $k > n_0$, there exist $d \in (0,\rho)$, $\ell \ge d^{-\beta^*(\alpha)+\delta(q+1)}$, an integral number N, and $u_1, \ldots, u_\ell \in D_{k+1,N+1}$ satisfying the following properties:*

(a) $d^{1+\delta} \le r_{u_i} \le d^{1-\delta}$ *for all* $1 \le i \le \ell$;

(b) $d^{\alpha+\delta} \le p_{u_i} \le d^{\alpha-\delta}$ *for all* $1 \le i \le \ell$.

Proof. For the given $\delta > 0$, we choose a small $0 < \epsilon < 1$ such that

$$\left(\alpha - \frac{\delta}{2}\right)\epsilon \le \beta(q) - \beta(q+\epsilon) \le \left(\alpha + \frac{\delta}{2}\right)\epsilon, \tag{22}$$

$$\left(\alpha - \frac{\delta}{2}\right)\epsilon \le \beta(q-\epsilon) - \beta(q) \le \left(\alpha + \frac{\delta}{2}\right)\epsilon. \tag{23}$$

Using (10), we can pick $0 < \gamma < \min\{\epsilon\delta/6, 1\}$ and an integral number n_0 such that for any integer $n > n_0$,

$$r_1 r_2 \cdots r_n < \min\left\{\rho, 3^{-1/\gamma}\right\}, \tag{24}$$

$$r_\sigma^{-\beta(q)+\gamma} \le \sum_{\sigma\in D_n} p_\sigma^q \le r_\sigma^{-\beta(q)-\gamma}, \tag{25}$$

$$r_\sigma^{-\beta(q+\epsilon)+\gamma} \le \sum_{\sigma\in D_n} p_\sigma^{q+\epsilon} \le r_\sigma^{-\beta(q+\epsilon)-\gamma}. \tag{26}$$

Set

$$\mathbb{B}_1 = \left\{\sigma \in D_n : p_\sigma \geq r_\sigma^{\alpha-\delta}\right\}, \quad \mathbb{B}_2 = \left\{\sigma \in D_n : p_\sigma \geq r_\sigma^{\alpha-\delta}\right\}, \\ \mathbb{B}_3 = \left\{\sigma \in D_n : r_\sigma^{\alpha+\delta} < p_\sigma < r_\sigma^{\alpha-\delta}\right\}. \tag{27}$$

Then

$$\sum_{\sigma\in\mathbb{B}_1} p_\sigma^q = \sum_{\sigma\in\mathbb{B}_1} p_\sigma^{q+\epsilon} p_\sigma^{-\epsilon} \leq r_\sigma^{-\beta(q+\epsilon)-\gamma-\epsilon(\alpha-\delta)} \leq r_\sigma^{-\beta(q)+\epsilon\delta/2-\gamma}. \tag{28}$$

Similarly, we have

$$\sum_{\sigma\in\mathbb{B}_2} p_\sigma^q = \sum_{\sigma\in\mathbb{B}_2} p_\sigma^{q-\epsilon} p_\sigma^{\epsilon} \leq r_\sigma^{-\beta(q-\epsilon)-\gamma+\epsilon(\alpha+\delta)} \leq r_\sigma^{-\beta(q)+\epsilon\delta/2-\gamma}. \tag{29}$$

These two inequalities together with (22)–(26) imply

$$\begin{aligned}\sum_{\sigma\in\mathbb{B}_3} p_\sigma^q &= \left(\sum_{\sigma\in D_n} - \sum_{\sigma\in\mathbb{B}_1} - \sum_{\sigma\in\mathbb{B}_2}\right) p_\sigma^q \\ &\geq r_\sigma^{-\beta(q)+\gamma} - 2r_\sigma^{-\beta(q)+\epsilon\delta/2-\gamma} \\ &= r_\sigma^{-\beta(q)+2\gamma}\left(r_\sigma^{-\gamma} - 2r_\sigma^{(\epsilon\delta/2)-3\gamma}\right) \\ &\geq r_\sigma^{-\beta(q)+2\gamma}\left(r_\sigma^{-\gamma} - 2\right) \quad \left(\text{by } 0 < \gamma < \min\left\{\frac{\epsilon\delta}{6}, 1\right\}\right) \\ &\geq r_\sigma^{-\beta(q)+2\gamma} \quad (\text{by } (24)).\end{aligned} \tag{30}$$

Note that for each $\sigma \in \mathbb{B}_3$, $p_\sigma \leq \max\{r_\sigma^{(\alpha\pm\delta)q}\} = r_\sigma^{\alpha q-|\delta|q} = (r_1 r_2 \cdots r_n)^{\alpha q-|\delta|q}$. Hence,

$$\sum_{\sigma\in\mathbb{B}_3} p_\sigma^q \leq (\#\mathbb{B}_3)(r_1 r_2 \cdots r_n)^{\alpha q-|\delta|q} \tag{31}$$

which combining with (30) yields

$$\begin{aligned}\#\mathbb{B}_3 &\geq (r_1 r_2 \cdots r_n)^{-\beta(q)-\alpha q+\delta|q|+2\gamma} \\ &= (r_1 r_2 \cdots r_n)^{-\beta^*(\alpha)+\delta|q|+2\gamma} \geq (r_1 r_2 \cdots r_n)^{-\beta^*(\alpha)+\delta(|q|+1)}.\end{aligned} \tag{32}$$

We suppose N is an integral number satisfying $N > n_0$ and $d = r_1 r_2 \cdots r_N$. This completes the proof of (I).

Next we prove (II). Note that $\sum_{\sigma\in D_k} p_\sigma^q r_\sigma^{\beta_k(q)} = 1$; we get

$$\begin{aligned}&\beta_k(q) \\ &= \frac{\log \sum_{\sigma\in D_k} p_\sigma^q}{-\log r_\sigma} \\ &= \frac{\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij}^q + \sum_{i=1}^m \left(\left(|\omega_k|_{\alpha_i}/k\right) - \eta_i\right) \log \sum_{j=1}^{m_i} p_{ij}^q}{-\left(\sum_{i=1}^m \eta_i \log c_i + \sum_{i=1}^m \left(\left(|\omega_k|_{\alpha_i}/k\right) - \eta_i\right) \log c_i\right)}, \\ &\frac{\log \sum_{\sigma\in D_{k+1,k+n}} p_\sigma^q}{-\log r_\sigma} \\ &= \Bigg(\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_1} p_{ij}^q \\ &\quad + \sum_{i=1}^m \Big(\Big((k+n)\left(\left(|\omega_{k+n}|_{\alpha_i}/(k+n)\right) - \eta_i\right) \\ &\qquad - k\left(\left(|\omega_k|_{\alpha_i}/k\right) - \eta_i\right)\Big) \times (n)^{-1}\Big) \\ &\quad \times \log \sum_{j=1}^{m_i} p_{ij}^q\Bigg) \\ &\quad \times \Bigg(-\Bigg(\sum_{i=1}^m \eta_i \log c_i \\ &\qquad + \sum_{i=1}^m \Big(\Big((k+n)\left(\left(|\omega_{k+n}|_{\alpha_i}/(k+n)\right) - \eta_i\right) \\ &\qquad\quad - k\left(\left(|\omega_k|_{\alpha_i}/k\right) - \eta_i\right)\Big) \times (n)^{-1}\Big) \\ &\qquad \times \log c_i\Bigg)\Bigg)^{-1}.\end{aligned} \tag{33}$$

Note that $\lim_{n\to\infty}\beta_n(q) = \beta(q)$ and $\lim_{k\to\infty}(|\omega_k|_{a_i}/k) = \eta_i$ for $1 \leq i \leq m$ and using (33), we can choose an integral N_1 such that for any integral $k, n > N_1$, the following properties are satisfied; that is,

$$\begin{aligned}&\beta(q) - \gamma \leq \frac{\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij}^q}{-\sum_{i=1}^m \eta_i \log c_i} \leq \beta(q) + \gamma, \\ &\frac{\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij}^q}{-\sum_{i=1}^m \eta_i \log c_i} - \gamma \leq \frac{\log \sum_{\sigma\in D_{k+1,k+n}} p_\sigma^q}{-\log r_\sigma} \\ &\qquad \leq \frac{\sum_{i=1}^m \eta_i \log \sum_{j=1}^{m_i} p_{ij}^q}{-\sum_{i=1}^m \eta_i \log c_i} + \gamma.\end{aligned} \tag{34}$$

Combining (34) we get

$$\beta(q) - 2\gamma \leq \frac{\log \sum_{\sigma\in D_{k+1,k+n}} p_\sigma^q}{-\log r_\sigma} \leq \beta(q) + 2\gamma. \tag{35}$$

Therefore

$$r_\sigma^{-(\beta(q)-2\gamma)} \le \sum_{\sigma\in D_{k+1,k+n}} p_\sigma^q < r_\sigma^{-(\beta(q)+2\gamma)}. \quad (36)$$

Using the same method, we can choose an integral N_2 such that for any integral $k, n > N_2$,

$$r_\sigma^{-(\beta(q+\epsilon)-2\gamma)} \le \sum_{\sigma\in D_{k+1,k+n}} p_\sigma^{q+\epsilon} \le r_\sigma^{-(\beta(q+\epsilon)+2\gamma)}. \quad (37)$$

Set $n_0 = \max\{N_1, N_2\}$, and (36) and (37) satisfy simultaneously. We suppose N is an integral number satisfying $N > n_0$ and

$$\left(\prod_{i=1}^{m}(c_i^{\eta_i})\right)^N < \rho,$$

$$\begin{aligned}\prod_{i=1}^{m}(c_i^{\eta_i})^\delta &\le \prod_{i=1}^{m} c_i^{((k+N)((|\omega_{k+N}|_{\alpha_i}/(k+N))-\eta_i)-k((|\omega_k|_{\alpha_i}/k)-\eta_i))/N} \\ &\le \prod_{i=1}^{m}(c_i^{\eta_i})^{-\delta}.\end{aligned} \quad (38)$$

Set $d = (\prod_{i=1}^m c_i^{\eta_i})^N$. Replace (25) and (26) with (36) and (37); we can prove (II) by the same method with (I). □

Lemma 9. *Let μ be a Moran measure supported on the homogeneous Moran fractals $E(\Omega)$ associated with the sequences Ω of which the frequency of the letter exists as above. Set*

$$E_{q_1,q_2} = \left\{x \in E(\Omega) \mid \beta'(q_1) = \liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} < \limsup_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} = \beta'(q_2)\right\}. \quad (39)$$

Then

$$\dim E_{q_1,q_2} \le \beta^*\left(\beta'(q_1)\right). \quad (40)$$

Proof. Let $f = \dim E_{q_1,q_2}$, then for any $0 < \varepsilon < f$, $\mathbf{H}^{f-\varepsilon}(E_{q_1,q_2}) = \infty$; Using (39), there exist $E^\circ_{q_1,q_2} \subset E_{q_1,q_2}$ and a number sequence $\{r_i\} \uparrow 0$ such that

$$\mathbf{H}^{f-\varepsilon}\left(E^\circ_{q_1,q_2}\right) > 1, \quad (41)$$

$$\mu(B(x,r_i)) \ge 3^{1/q_1} r_i^{\beta'(q_1)+\varepsilon}, \quad (42)$$

for any $x \in E^\circ_{q_1,q_2}$. We can choose $0 < \delta \le (1/2)r_0$ such that $\mathbf{H}_\delta^{f-\varepsilon}(E^\circ_{q_1,q_2}) \ge 1$. For any $r_i \le \delta$ and $2^{-(n_i+1)} \le r_i < 2^{-n_i}$, we consider grid boxes $[k2^{-n_i}, (k+1)2^{-n_i})$; there exist 3 adjacent grid boxes A, A_1, A_2 such that $B(x, r_i) \subset A \cup A_1 \cup A_2$, and there $x \in A$ and A_1, A_2 are neighbours with A. Therefore there exists $A_0 \in \{A, A_1, A_2\}$ such that $\mu(A_0) \ge r_i^{\beta'(q_1)+\varepsilon} \ge 2^{-n_i(\beta'(q_1)+\varepsilon)}$ (by $\beta'(q_1) < 0$). On the other hand, notice that $\#\{C \in \mathbf{D}_{n_i} \mid C \cap E^\circ_{q_1,q_2} \ne \varnothing\} \ge 2^{-n_i(\varepsilon-f)}\mathbf{H}_\delta^{f-\varepsilon}(E^\circ_{q_1,q_2}) \ge 2^{-n_i(\varepsilon-f)}$ (by (41)). Thus

$$\#\left\{C \in \mathbf{D}_k \mid \mu(A_0) \ge 2^{-n_i(\beta'(q_1)+\varepsilon)}\right\} \ge \frac{1}{3}2^{-n_i(\varepsilon-f)}. \quad (43)$$

Thus

$$\tau_{n_i}^{(q_1)}(\mu) = \sum_{Q\in\mathbf{D}_{n_i}}(\mu Q)^{q_1} \ge 2^{-n_i q_1(\beta'(q_1)+\varepsilon)}2^{-n_i(\varepsilon-f)}. \quad (44)$$

Therefore

$$\lim_{i\to\infty}\frac{\log\tau_{n_i}^{(q_1)}(\mu)}{-\log 2^{n_i}} \ge -q_1\left(\beta'(q_1)+\varepsilon\right) + f - \varepsilon. \quad (45)$$

Using (14) and (15), notice that $\lim_{n\to\infty}(\log\tau_n^{(q_1)}(\mu)/-\log 2^n) = \beta(q_1)$; we have

$$f \le \beta(q_1) + q_1\left(\beta'(q_1)+\varepsilon\right) + \varepsilon. \quad (46)$$

Thus

$$\dim E_{q_1,q_2} \le \beta^*\left(\beta'(q_1)\right). \quad (47)$$

□

3. Proof of Theorem 7

Lemma 10. *Suppose $\Delta > 0$ (Δ is from Remark 4), for all $r > 0$, and $x \in J_\sigma \in F$, and choose $k, l \in \mathbb{N}$ such that*

$$\begin{aligned}\left|J_{\sigma|k+1}(x)\right| &\le r < \left|J_{\sigma|k}(x)\right|, \\ \Delta\left|J_{\sigma|l+1}(x)\right| &\le r < \Delta\left|J_{\sigma|l}(x)\right|,\end{aligned} \quad (48)$$

where $J_{\sigma|l+1}(x)$ denote the basic elements of order $l+1$ that contains the point x. Then

$$\begin{aligned}&(i)\ J_{\sigma|k+1}(x) \subseteq B(x,r),\ E\cap B(x,r) \subseteq J_{\sigma|l+1}(x), \\ &(ii)\ \exists\Delta_0 > 0 \text{ such that } k-l \le \Delta_0.\end{aligned} \quad (49)$$

Proof. It is obvious that $J_{\sigma|k+1}(x) \subseteq B(x,r)$. Now let $y \in E \cap B(x,r)$, but $y \bar{\in} J_{\sigma|l+1}(x)$; then there exists $j < l+1$ such that $\tau|j = \sigma|j$, and $\tau|j+1 \ne \sigma|j+1$, and $x \in J_{\sigma|j+1}(x)$, $y \in J_{\tau|j+1}(y)$. Therefore,

$$\begin{aligned}|y-x| &\ge d\left(J_{\tau|j+1}(x), J_{\sigma|j+1}(y)\right) \\ &\ge \Delta_j\left|J_{\sigma|j}(x)\right| \ge \Delta\left|J_{\sigma|l}(x)\right| > r,\end{aligned} \quad (50)$$

which is a contradiction since $y \in E \cap B(x,r)$; therefore $E \cap B(x,r) \subseteq J_{\sigma|l+1}(x)$.

Since $\Delta < 1$ and $k \ge l$. (48) imply that

$$1 \le \frac{\left|J_{\sigma|k}(x)\right|}{\Delta\left|J_{\sigma|l+1}(x)\right|} \le \frac{(\max\{c_i\})^{k-l-1}}{\Delta}, \quad (51)$$

which yields

$$k - l \le 1 + \frac{\log\Delta}{\log\max\{c_i\}} \triangleq \Delta_0. \quad (52)$$

□

Proof of Theorem 7. For any $q_2 < q_1$, define a number sequence $\{\alpha_i\}_{i=1}^{\infty}$ in the following manner:

$$q_i = \begin{cases} q_1, & i \text{ is odd number;} \\ q_2, & i \text{ is even number,} \end{cases} \quad (53)$$

$$\alpha_i = \begin{cases} \beta'(q_1), & i \text{ is odd number;} \\ \beta'(q_2), & i \text{ is even number.} \end{cases} \quad (54)$$

We choose a positive sequence $\{\delta_i\}_{i=1}^{\infty} \downarrow 0$. For $\{\alpha_i\}$ and $\{\delta_i\}$, there exist an integral number sequence $\{n_0^{(i)}\}_{i=1}^{\infty}$ ($n_0^{(i)}$ corresponds to n_0 in Lemma 8) and a real number $\{d_i\}_{i=1}^{\infty}$ ($1 > d_1 > d_2 > \cdots$) such that the follow properties are satisfied.

For $\alpha_1 = \beta'(q_1)$ and δ_1, using Lemma 8 we can pick $u_{11}^{(1)}, u_{12}^{(1)}, \ldots, u_{1\ell^{(1)}}^{(1)} \in D_{N^{(1)}}, u_{21}^{(1)}, u_{22}^{(1)}, \ldots, u_{2,\ell^{(1)}}^{(1)} \in D_{N^{(1)}+1,2N^{(1)}}, \ldots, u_{M_1 1}^{(1)}, u_{M_1 2}^{(1)} \ldots, u_{M_1 \ell^{(1)}}^{(1)} \in D_{(M_1-1)N^{(1)}+1, M_1 N^{(1)}}$ satisfying $(d_1)^{\alpha_1+\delta_1} \le p_{u_{jl}^{(1)}} \le (d_1)^{\alpha_1-\delta_1}$, $(d_1)^{1+\delta_1} \le r_{u_{jl}^{(1)}} \le (d_1)^{1-\delta_1}$ for all $1 \le j \le M_1$; $1 \le l \le \ell^{(1)}$ and $\ell^{(1)} \ge (d_1)^{-\beta^*(\alpha_1)+\delta_1(q+1)}$.

For $1 \le j \le M_1$, remark $\mathbf{B}_j^{(1)} = \{u_{j1}^{(1)}, u_{j2}^{(1)}, \ldots, u_{j\ell^{(1)}}^{(1)}\}$.

By the same step, for α_i and δ_i, we also construct $\mathbf{B}_j^{(i)} = \{u_{j1}^{(i)}, u_{j2}^{(i)} \ldots, u_{j\ell^{(i)}}^{(i)}\}$ ($1 \le j \le M_i$) such that

(a) $\ell^{(i)} \ge (d_i)^{-\beta^*(\alpha_i)+\delta_i(q+1)}$;

(b) $u_{j1}^{(i)}, u_{j2}^{(i)}, \ldots, u_{j,\ell^{(i)}}^{(i)} \in D_{\sum_{k=1}^{i-1} M_i N^{(k)}+(j-1)N^{(i)}+1, \sum_{k=1}^{i-1} M_i N^{(k)}+jN^{(i)}}$;

(c) $d_i^{1+\delta_i} \le r_{u_{jl}^{(i)}} \le d_i^{1-\delta_i}$ for $1 \le j \le M_i$, $1 \le l \le \ell^{(i)}$;

(d) $d_i^{\alpha_i+\delta_i} \le p_{u_{jl}^{(i)}} \le d_i^{\alpha_i-\delta_i}$ for $1 \le j \le M_i$, $1 \le l \le \ell^{(i)}$.

Also we let $\{M_i\}_{i=1}^{\infty}$ be a sequence of integers large enough such that

(e) $M_i N^{(i)} > n_0^{(i)}$;

(f) $d_i^{M_i} < (d_{i+1})^{2^i}$ for each $i \in \mathbb{N}$;

(g) $\lim_{k\to\infty}(\sum_{i=1}^{k} M_i \log d_i / M_{k+1} \log d_{k+1}) = 0$.

Now we define a sequence of subsets of symbol set Σ^* in the following manner:

$$\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_{M_1}^{(1)}, \mathbf{B}_1^{(2)}, \ldots, \mathbf{B}_{M_2}^{(2)}, \ldots, \mathbf{B}_1^{(i)}, \ldots, \mathbf{B}_{M_i}^{(i)}, \ldots \quad (55)$$

and relabel them as $\{\mathbf{B}_n^*\}_{n=1}^{\infty}$. Let

$$F_{q_1,q_2} = \bigcap_{n=1}^{\infty} \bigcup_{v_1 \in B_1^*, \ldots, v_n \in B_n^*} J_{v_1 v_2 \cdots v_n}. \quad (56)$$

It is easy to check that F_{q_1,q_2} is the homogeneous *Moran set* which is a subset of the homogeneous *Moran set* $E(\Omega)$.

Next we show that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \beta'(q_1),$$
$$\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \beta'(q_2), \quad \forall x \in F_{q_1,q_2}, \quad (57)$$
$$\dim_H F_{q_1,q_2} \ge \beta^*(\beta'(q_1)).$$

Let $x \in F_{q_1,q_2}$, then there exist $v_i \in \mathbf{B}_i^*$ ($i = 1, 2, \ldots$) such that

$$\{x\} = \lim_{n\to\infty} J_{v_1 \cdots v_n}. \quad (58)$$

Let $n_k = M_1 + M_2 + \cdots + M_{2k} - 1$; there exists r_{n_k} such that

$$\left|J_{v_1 \cdots v_{n_k+1}}\right| \le r_{n_k} < \left|J_{v_1 \cdots v_{n_k}}\right|. \quad (59)$$

Using Lemma 10, we attain

$$\frac{\log p_{v_1 \cdots v_{n_k}} + \Delta_0 \log \min_{1\le i\le m, 1\le j\le m_i}\{p_{i,j}\}}{\log r_{v_1 \cdots v_{n_k+1}}} \le \frac{\log \mu(B(x, r_{n_k}))}{\log r_{n_k}} \le \frac{\log p_{v_1 \cdots v_{n_k+1}}}{\log r_{v_1 \cdots v_{n_k}}}. \quad (60)$$

Thus to calculate $\lim_{k\to\infty}(\log \mu(B(x, r_{n_k}))/\log r_{n_k})$, we need to estimate $p_{v_1 \cdots v_M}$ and $r_{v_1 \cdots v_M}$ for $M = n_k, n_k + 1$. By (c) and (d), we obtain

$$\prod_{i=1}^{2k-1} d_i^{M_i(\alpha_i+\delta_i)} d_{2k}^{M_{2k}(\alpha_{2k}+\delta_{2k})} \le p_{v_1 \cdots v_{n_k+1}} \le p_{v_1 \cdots v_{n_k}} \le \prod_{i=1}^{2k-1} d_i^{M_i(\alpha_i-\delta_i)} d_{2k}^{(M_{2k}-1)(\alpha_{2k}-\delta_{2k})}, \quad (61)$$

$$\prod_{i=1}^{2k-1} d_i^{M_i(1+\delta_i)} d_{2k}^{M_{2k}(1+\delta_{2k})} \le r_{v_1 \cdots v_{n_k+1}} \le r_{v_1 \cdots v_{n_k}} \le \prod_{i=1}^{2k-1} d_i^{M_i(1-\delta_i)} d_{2k}^{(M_{2k}-1)(1-\delta_{2k})}. \quad (62)$$

By (54), (60)–(62), and (g), we obtain

$$\lim_{k\to\infty} \frac{\log \mu(B(x, r_{n_k}))}{\log r_{n_k}} = \beta'(q_2). \quad (63)$$

Let $n'_k = M_1 + M_2 + \cdots + M_{2k+1} - 1$; there exists $r_{n'_k}$ such that

$$\left|J_{v_1 \cdots v_{n'_k+1}}\right| \le r_{n'_k} < \left|J_{v_1 \cdots v_{n'_k}}\right|. \quad (64)$$

By the same method, we obtain

$$\lim_{k\to\infty}\frac{\log\mu\left(B\left(x,r_{n_k'}\right)\right)}{\log r_{n_k'}}=\beta'\left(q_1\right). \tag{65}$$

Next we show $\beta'(q_1) \leq \liminf_{r\to 0}(\log\mu(B(x,r))/\log(r)) < \limsup_{r\to 0}(\log\mu(B(x,r))/\log(r)) \leq \beta'(q_2)$. For $r>0$ small enough, there is a unique large integer n such that

$$\left|J_{v_1\cdots v_{n+1}}\right|\leq r<\left|J_{v_1\cdots v_n}\right|. \tag{66}$$

Using the same method as above, we attain

$$\begin{aligned}\frac{\log p_{v_1\cdots v_n}+\Delta_0\log\min_{1\leq i\leq m,1\leq j\leq m_i}\left\{p_{i,j}\right\}}{\log r_{v_1\cdots v_{n+1}}}&\\ \leq\frac{\log\mu\left(B\left(x,r\right)\right)}{\log r}&\\ \leq\frac{\log p_{v_1\cdots v_{n+1}}}{\log r_{v_1\cdots v_n}}.&\end{aligned} \tag{67}$$

Now we estimate $p_{v_1\cdots v_M}$ and $r_{v_1\cdots v_M}$ for $M=n,n+1$. For large n, write $n=\sum_{i=1}^k M_i+p$ with $1\leq p\leq M_{k+1}$. In the case that $1\leq p\leq M_{k+1}-1$, by (c) and (d), we obtain

$$\begin{aligned}&\prod_{i=1}^k d_i^{M_i(\alpha_i+3\delta_i)}d_{k+1}^{(p+1)(\alpha_{k+1}+\delta_{k+1})}\\ &\quad\leq p_{v_1\cdots v_{n+1}}\leq p_{v_1\cdots v_n}\leq\prod_{i=1}^k d_i^{M_i(\alpha_i-\delta_i)}d_{k+1}^{p(\alpha_{k+1}-\delta_{k+1})},\end{aligned} \tag{68}$$

$$\begin{aligned}&\prod_{i=1}^k d_i^{M_i(1+\delta_i)}d_{k+1}^{(p+1)(1+\delta_{k+1})}\\ &\quad\leq r_{v_1\cdots v_{n+1}}\leq r_{v_1\cdots v_n}\leq\prod_{i=1}^k d_i^{M_i(1-\delta_i)}d_{k+1}^{p(1-\delta_{k+1})}.\end{aligned} \tag{69}$$

In the other case $p=M_{k+1}$, we have the similar inequalities where the lower bounds for $p_{v_1\cdots v_n}$ and $r_{v_1\cdots v_n}$ in (68) and (69) are replaced, respectively, by

$$\begin{aligned}&\left(\prod_{i=1}^{k+1}d_i^{M_i(\alpha_i+\delta_i)}\right)d_{k+2}^{(\alpha_{k+2}+\delta_{k+2})}\leq p_{v_1\cdots v_{n+1}}\\ &\quad<p_{v_1\cdots v_n}\leq\prod_{i=1}^{k+1}d_i^{M_i(\alpha_i+\delta_i)},\end{aligned} \tag{70}$$

$$\begin{aligned}&\left(\prod_{i=1}^{k+1}d_i^{M_i(1+\delta_i)}\right)d_{k+2}^{(1+\delta_{k+2})}\leq r_{v_1\cdots v_{n+1}}\\ &\quad<r_{v_1\cdots v_n}\leq\prod_{i=1}^{k+1}d_i^{M_i(1+\delta_i)}.\end{aligned} \tag{71}$$

By (54), using the inequalities (67)–(71), (f), and (g), we obtain

$$\begin{aligned}\beta'\left(q_1\right)&\leq\liminf_{r\to 0}\frac{\log\mu\left(B\left(x,r\right)\right)}{\log r}\\ &<\limsup_{r\to 0}\frac{\log\mu\left(B\left(x,r\right)\right)}{\log r}\leq\beta'\left(q_2\right).\end{aligned} \tag{72}$$

Combining (63), (65), and (72), we obtain

$$\begin{aligned}&\liminf_{r\to 0}\frac{\log\mu\left(B\left(x,r\right)\right)}{\log r}=\beta'\left(q_1\right),\\ &\limsup_{r\to 0}\frac{\log\mu\left(B\left(x,r\right)\right)}{\log r}=\beta'\left(q_2\right),\quad\forall x\in F_{q_1,q_2}.\end{aligned} \tag{73}$$

To prove $\dim_H F_{q_1,q_2}\geq\beta^*(q_1)$, recall that F_{q_1,q_2} is the homogeneous *Moran set* which is a subset of the homogeneous *Moran set* $E(\Omega)$. For large n, write $n=\sum_{i=1}^k M_i+p$ with $1\leq p\leq M_{k+1}$. By (a) and (c), we have

$$\prod_{s=1}^n\#\mathbf{B}_s^*\geq\prod_{i=1}^k d_i^{M_i(-\beta^*(\alpha_i)+\delta_i(q_i+1))}d_{k+1}^{p(-\beta^*(\alpha_{k+1})+\delta_{k+1}(q_{k+1}+1))} \tag{74}$$

and for any $v_1\in\mathbf{B}_1^*,\ldots,v_n\in\mathbf{B}_n^*$

$$r_{v_n}\geq d_{k+1}^{1+\delta_{k+1}},\quad r_{v_1\cdots v_n}\leq\sum_{i=1}^k d_i^{M_i(1-\delta_i)}d_{k+1}^{p(1-\delta_{k+1})}. \tag{75}$$

Using (75) and (f), we have

$$\lim_{n\to\infty}\frac{\log r_{v_n}}{\log r_{v_1\cdots v_n}}=0. \tag{76}$$

This implies the condition (5) in Proposition 3; we have $\dim_H F_{q_1,q_2}=\liminf_{n\to\infty}s_n$, where s_n satisfies the equation

$$\sum_{v_1\in B_1^*\cdots v_n\in B_n^*}\left(r_{v_1\cdots v_n}\right)^{s_n}=1. \tag{77}$$

It follows that

$$\dim_H F_{q_1,q_2}\geq\liminf_{n\to\infty}\frac{\log\left(\prod_{s=1}^n\#B_s^*\right)}{-\log r_{v_1\cdots v_n}}. \tag{78}$$

This, together with (74)-(75), yields

$$\begin{aligned}
&\dim_H F_{q_1,q_2} \\
&\geq \liminf_{k\to\infty}\left(\left(\sum_{i=1}^{k} M_i\left(\beta^*\left(\alpha_i\right)-\delta_i\left(q_i+1\right)\right)\log d_i\right.\right.\\
&\qquad\left.+p\left(\beta^*\left(\alpha_{k+1}\right)-\delta_{k+1}\left(q_{k+1}+1\right)\right)\log d_{k+1}\right)\\
&\qquad\left.\times\left(\sum_{i=1}^{k} M_i\left(1-\delta_i\right)\log d_i+p\left(1-\delta_{k+1}\right)\log d_{k+1}\right)^{-1}\right)\\
&\geq \liminf_{k\to\infty}\left(\left(\sum_{i=1}^{k} M_i\left(\beta^*\left(\beta'\left(q_1\right)\right)-\delta_i\left(q_i+1\right)\right)\log d_i\right.\right.\\
&\qquad\left.+p\left(\beta^*\left(\beta'\left(q_1\right)\right)-\delta_{k+1}\left(q_{k+1}+1\right)\right)\log d_{k+1}\right)\\
&\qquad\left.\times\left(\sum_{i=1}^{k} M_i\left(1-\delta_i\right)\log d_i+p\left(1-\delta_{k+1}\right)\log d_{k+1}\right)^{-1}\right)\\
&\geq \beta^*\left(\beta'\left(q_1\right)\right)\\
&\left(\text{by (54) and } \beta^*\left(\beta'\left(q_1\right)\right)<\beta^*\left(\beta'\left(q_2\right)\right)\right.\\
&\quad\left.\text{since } \beta^* \text{ is concave (see [3])}\right).
\end{aligned} \tag{79}$$

Using Lemma 9, we finish proof of Theorem 7. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are supported by Natural Science Foundation of China (11271140) and Teaching Research Project of Wuhan College, Zhongnan University of Economics and Law.

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