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Research Article

Fuzzy Symmetric Solutions of Fuzzy Matrix Equations

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The fuzzy symmetric solution of fuzzy matrix equation $A\widetilde{X} = \widetilde{B}$, in which A is a crisp $m \times m$ nonsingular matrix and $\widetilde{B}$ is an $m \times n$ fuzzy numbers matrix with nonzero spreads, is investigated. The fuzzy matrix equation is converted to a fuzzy system of linear equations according to the Kronecker product of matrices. From solving the fuzzy linear system, three types of fuzzy symmetric solutions of the fuzzy matrix equation are derived. Finally, two examples are given to illustrate the proposed method.

1. Introduction

Linear systems always have important applications in many branches of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So, it is immensely important to develop a numerical procedure that would appropriately treat general fuzzy linear systems and solve them. The concept of fuzzy numbers and arithmetic operations with these numbers was first introduced and investigated by Zadeh [1], Dubois et al. [2], and Nahmias [3]. A different approach to fuzzy numbers and the structure of fuzzy number spaces was given by Puri and Ralescu [4], Goetschell et al. [5], and Wu and Ming [6, 7].

Since Friedman et al. [8, 9] proposed a general model for solving an $n \times n$ fuzzy linear systems whose coefficients matrix is crisp and the right-hand side is an arbitrary fuzzy numbers vector by an embedding approach in 1998, many works have been done about how to deal with some fuzzy linear systems with more advanced forms such as dual fuzzy linear systems (DFLSs), general fuzzy linear systems (GFLSs), fully fuzzy linear systems (FFLSs), dual full fuzzy linear systems (DFFLSs), and general dual fuzzy linear systems (GDFLSs). These works were performed mainly by Allahviranloo et al. [10–13], Abbasbandy et al. [14–17], Wang et al. [18, 19] and Dehghan et al. [20, 21], among others. However, for a fuzzy matrix equation which always has a wide use in control theory and control engineering, few works have been done in the past decades. In 2010, Guo et al. [22–24] investigated a class of fuzzy matrix equations $A\widetilde{X} = \widetilde{B}$ in which A is an $m \times n$ crisp matrix and the right-hand side matrix $\widetilde{B}$ is an $m \times l$ fuzzy numbers matrix by means of the block Gaussian elimination method and the undetermined coefficients method, and they studied least squares solutions of the inconsistent fuzzy matrix equation $A\widetilde{x} = \widetilde{B}$ by using the generalized inverses. In 2011, Allahviranloo and Salahshour [25] obtained fuzzy symmetric approximate solutions of fuzzy linear systems by solving a crisp system of linear equations and a fuzzified interval system of linear equations. Meanwhile, they [26] investigated the maximal and minimal symmetric solutions of full fuzzy linear systems $\widetilde{A}\widetilde{x} = \widetilde{b}$ by the same approach.

In this paper, we propose a general model for solving the fuzzy matrix equation $A\widetilde{X} = \widetilde{B}$ where A is crisp $m \times m$ nonsingular matrix and $\widetilde{B}$ is an $m \times n$ fuzzy numbers matrix with nonzero spreads. The model is proposed in this way, that is, we first convert the fuzzy matrix equation to a fuzzy system of linear equations based on the Kronecker product of matrices and then obtain three types of fuzzy symmetric solutions of the fuzzy matrix equation by solving the fuzzy linear systems. Finally, some examples are given to illustrate our method. The structure of this paper is organized as follows.

In Section 2, we recall the fuzzy number and present the concept of the fuzzy matrix equation and its fuzzy symmetric solutions. The method to solve the fuzzy matrix equation is proposed and the fuzzy symmetric solutions of the fuzzy matrix equation are obtained in detail in Section 3. Some examples are given to illustrate our method in Section 4 and the conclusion is drawn in Section 5.

2. Preliminaries

2.1. Fuzzy Numbers. There are several definitions for the concept of fuzzy numbers (see [1, 2, 4]).

Definition 1. A fuzzy number is a fuzzy set like $u : R \to I = [0, 1]$ which satisfies the following:

(1) u is upper semicontinuous,

(2) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\}$ for all $x, y \in R$, $\lambda \in [0, 1]$,

(3) u is normal, that is, there exists $x_0 \in R$ such that $u(x_0) = 1$,

(4) $\operatorname{supp} u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

Let E^1 be the set of all fuzzy numbers on R.

Definition 2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfies the requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

(3) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

A crisp number x is simply represented by $(\underline{u}(r), \overline{u}(r)) = (x, x)$, $0 \le r \le 1$. By appropriate definitions the fuzzy number space $\{(\underline{u}(r), \overline{u}(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space.

Definition 3. Let $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r)) \in E^1$, $0 \le r \le 1$, and real number $k \in R$. Then,

(1) $x = y$ iff $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(2) $x + y = (\underline{x}(r) + \underline{y}(r),\ \overline{x}(r) + \overline{y}(r))$,

(3) $x - y = (\underline{x}(r) - \overline{y}(r),\ \overline{x}(r) - \underline{y}(r))$,

(4)

$$kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases} \quad (1)$$

2.2. Kronecker Product of Matrices and Fuzzy Matrix. The following definitions and results about the Kronecker product of matrices are from [27].

Definition 4. Suppose $A = (a_{ij}) \in R^{m\times n}$, $B = (b_{ij}) \in R^{p\times q}$, the matrix in block form:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{12}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in R^{mp\times nq} \quad (2)$$

is said the Kronecker product of matrices A and B, denoted simply by $A \otimes B = (a_{ij}B)$.

Definition 5. Let $A = (a_{ij}) \in R^{m\times n}$, $a_i = (a_{1i}, a_{2i}, \ldots, a_{mi})^T$, $i = 1, \ldots, n$, the mn dimensions vector:

$$\operatorname{Vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad (3)$$

is called the extension on column of the matrix A.

Lemma 6. *Let $A = (a_{ij}) \in R^{m\times n}$, $B = (b_{ij}) \in R^{n\times s}$, and $C = (c_{ij}) \in R^{s\times t}$. Then,*

$$\operatorname{Vec}(ABC) = \left(C^T \otimes A\right) \operatorname{Vec}(B). \quad (4)$$

Definition 7. A matrix $\widetilde{A} = (\widetilde{a}_{ij})$ is called a fuzzy matrix, if each element $\widetilde{a}_{ij}$ of $\widetilde{A}$ is a fuzzy number, that is, $\widetilde{a}_{ij} = (\underline{a}_{ij}(r), \overline{a}_{ij}(r))$, $1 \le i \le m$, $1 \le j \le n$, $0 \le r \le 1$.

Definition 8. Let $\widetilde{A} = (\widetilde{a}_{ij} = (\underline{a}_{ij}(r), \overline{a}_{ij}(r)) \in E^{m\times n}$, $\widetilde{a}_i = (\widetilde{a}_{1j}, \widetilde{a}_{2j}, \ldots, \widetilde{a}_{mj})^T$, $j = 1, \ldots, n$. Then, the mn dimensions fuzzy numbers vector:

$$\operatorname{Vec}\left(\widetilde{A}\right) = \begin{pmatrix} \widetilde{a}_1 \\ \widetilde{a}_2 \\ \vdots \\ \widetilde{a}_n \end{pmatrix} \quad (5)$$

is called the extension on column of the fuzzy matrix $\widetilde{A}$.

2.3. Fuzzy Matrix Equations

Definition 9. The matrix system:

$$\begin{aligned} &\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{12} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{1n} \\ \widetilde{x}_{21} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{x}_{m1} & \widetilde{x}_{m2} & \cdots & \widetilde{x}_{mn} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \cdots & \widetilde{b}_{1n} \\ \widetilde{b}_{21} & \widetilde{b}_{12} & \cdots & \widetilde{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{b}_{m1} & \widetilde{b}_{m2} & \cdots & \widetilde{b}_{mn} \end{pmatrix}, \end{aligned} \quad (6)$$

where a_{ij}, $1 \le i, j \le m$ are crisp numbers and $\widetilde{b}_{ij}$, $1 \le i \le m$, $1 \le j \le n$ are fuzzy numbers, is called a fuzzy matrix equations (FMEs).

Using matrix notation, we have

$$A\widetilde{X} = \widetilde{B}. \tag{7}$$

A fuzzy numbers matrix:

$$\widetilde{X} = \left(\widetilde{x}_{ij}\right) = \left(\overline{x}_{ij}(r), \underline{x}_{ij}(r)\right), \quad 1 \le i \le m,\ 1 \le j \le n,\ 0 \le r \le 1, \tag{8}$$

is called a solution of the fuzzy linear matrix equation (6) if $\widetilde{X}$ satisfies

$$A\widetilde{X} = \widetilde{B}. \tag{9}$$

Clearly, Definition 9 is just for the fuzzy matrix equation and its exact solution. In this paper we will discuss its approximate fuzzy symmetric solutions.

3. Method for Solving FMEs

In this section, we will investigate the fuzzy matrix equation (7), that is, convert it to a crisp system of linear equations and a fuzzified interval system of linear equations, define three types of fuzzy approximate symmetric solution and give its solution representation to the original fuzzy matrix equation.

At first, we convert the fuzzy matrix equation (7) to a fuzzy system of linear equations based on the Kronecker product of matrices.

Theorem 10. *Let $A = (a_{ij})$ belong to $R^{m\times n}$, let $\widetilde{X} = (\widetilde{x}_{ij}) = (\overline{x}_{ij}(r), \underline{x}_{ij}(r))$ belong to $E^{n\times l}$, and let $B = (b_{ij})$ belong to $R^{l\times s}$. Then,*

$$\operatorname{Vec}\left(A\widetilde{X}B\right) = \left(B^T \otimes A\right)\operatorname{Vec}\left(\widetilde{X}\right). \tag{10}$$

Proof. Let $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)$, $\widetilde{x}_j = (\overline{x}_{ij}(r), \underline{x}_{ij}(r)) \in E^m$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, l$. $B = (b_1, b_2, \dots, b_l)$, $b_j \in R^n$, $j = 1, 2, \dots, l$. Then,

$$\operatorname{Vec}\left(A\widetilde{X}B\right) = \operatorname{Vec}\left(A\widetilde{X}b_1, A\widetilde{X}b_2, \dots, A\widetilde{X}b_l\right) = \begin{pmatrix} A\widetilde{X}b_1 \\ A\widetilde{X}b_2 \\ \vdots \\ A\widetilde{X}b_l \end{pmatrix}. \tag{11}$$

Since

$$\begin{aligned} A\widetilde{X}b_j &= (A\widetilde{x}_1, A\widetilde{x}_2, \dots, A\widetilde{x}_n)b_j = (A\widetilde{x}_1, A\widetilde{x}_2, \dots, A\widetilde{x}_l)\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{lj} \end{pmatrix} \\ &= b_{1j}A\widetilde{x}_1 + b_{2j}A\widetilde{x}_2 + \cdots + b_{lj}A\widetilde{x}_l \\ &= \left(b_{1j}A, b_{2j}A, \dots, b_{nj}A\right)\operatorname{Vec}\left(\widetilde{X}\right), \end{aligned} \tag{12}$$

we have

$$\begin{aligned} \operatorname{Vec}\left(A\widetilde{X}B\right) &= \begin{pmatrix} b_{11}A & b_{22}A & \cdots & b_{l1}A \\ b_{12}A & b_{22}A & \cdots & b_{l2}A \\ \vdots & \vdots & \vdots & \vdots \\ b_{1l}A & b_{2l}A & \cdots & b_{ls}A \end{pmatrix}\operatorname{Vec}\left(\widetilde{X}\right) \\ &= \left(B^T \otimes A\right)\operatorname{Vec}\left(\widetilde{X}\right). \end{aligned} \tag{13}$$

□

Theorem 11. *The matrix $\widetilde{X} \in E^{m\times n}$ is the solution of the fuzzy matrix equation (7) if and only if $\widetilde{x} = \operatorname{Vec}(\widetilde{X}) \in E^{mn}$ is the solution of the following linear fuzzy system:*

$$G\widetilde{x} = \widetilde{y}, \tag{14}$$

where $G = I_n \otimes A$ and $\widetilde{y} = \operatorname{Vec}(\widetilde{B})$.

Proof. Setting $B = I_n$ in (10), we have

$$\operatorname{Vec}\left(A\widetilde{X}\right) = \left(I_n \otimes A\right)\operatorname{Vec}\left(\widetilde{X}\right). \tag{15}$$

Applying the extension operation the Definition 8 to two sides of (7), we also have

$$G\widetilde{x} = \widetilde{y}, \tag{16}$$

where $G = I_n \otimes A$ is an $mn \times mn$ matrix and $\widetilde{y} = \operatorname{Vec}(\widetilde{B})$ is an mn fuzzy numbers vector. Thus, the $\widetilde{X}$ is the solution of (7) which is equivalent to that $\widetilde{x} = \operatorname{Vec}(\widetilde{X})$ which is the solution of (14).

For simplicity, we denote $p = mn$ in (7), thus

$$G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1,p} \\ g_{21} & g_{12} & \cdots & g_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ g_{p,1} & g_{p,2} & \cdots & g_{p,p} \end{pmatrix}, \quad \widetilde{y} = \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \vdots \\ \widetilde{y}_p \end{pmatrix} \tag{17}$$

in (14). □

The following definitions show what the fuzzy symmetric solutions of the fuzzy matrix equation are.

Definition 12 (see [28]). The united solution set (USS), the tolerable solution set (TSS), and the controllable solution set (CSS) for the system (14) are, respectively, as follows:

$$\begin{aligned} X_{\exists\exists} &= \{x \in R^p : Gx \cap y \neq \phi\}, \\ X_{\forall\exists} &= \{x \in R^p : Gx \subseteq y\}, \\ X_{\exists\forall} &= \{x \in R^p : Gx \supseteq y\}. \end{aligned} \tag{18}$$

Definition 13. A fuzzy vector $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_p)^{\mathsf{T}}$ given by $\widetilde{x}_i = [\underline{x}_i(r), \overline{x}_i(r)]$, $1 \le i \le p$, $0 \le r \le 1$ is called the minimal symmetric solution of the fuzzy matrix equation (7) which is placed in CSS if for any arbitrary symmetric solution $\widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_p)^{\mathsf{T}}$, which is placed in CSS, that is, $\widetilde{x}(1) = \widetilde{z}(1)$, we have

$$(\widetilde{z} \supseteq \widetilde{x}), \text{ that is}, (\widetilde{z}_i \supseteq \widetilde{x}_i), \text{ that is}, \ \sigma_{\widetilde{z}_i} \ge \sigma_{\widetilde{x}_i}, \quad i = 1, 2, \ldots, p, \tag{19}$$

where $\sigma_{\widetilde{z}_i}$ and $\sigma_{\widetilde{x}_i}$ are symmetric spreads of $\widetilde{z}_i$ and $\widetilde{x}_i$, respectively.

Definition 14. A fuzzy vector $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_p)^{\mathsf{T}}$ given by $\widetilde{x}_i = [\underline{x}_i(r), \overline{x}_i(r)]$, $1 \le i \le p$, $0 \le r \le 1$ is called the maximal symmetric solution of the fuzzy matrix equation (7) which is placed in TSS if for any arbitrary symmetric solution $\widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_p)^{\mathsf{T}}$, which is, placed in TSS, that is $\widetilde{x}(1) = \widetilde{z}(1)$, we have

$$(\widetilde{x} \supseteq \widetilde{z}), \text{ that is}, (\widetilde{x}_i \supseteq \widetilde{z}_i), \text{ that is}, \ \sigma_{\widetilde{x}_i} \ge \sigma_{\widetilde{z}_i}, \quad i = 1, 2, \ldots, p, \tag{20}$$

where $\sigma_{\widetilde{z}_i}$ and $\sigma_{\widetilde{x}_i}$ are symmetric spreads of $\widetilde{z}_i$ and $\widetilde{x}_i$, respectively.

Secondly, in order to solve the fuzzy matrix equation (7), we need to consider the fuzzy system of linear equation (14). For the fuzzy linear system (14), we can extend it into to a crisp system of linear equations and a fuzzified interval system of linear equations to obtain its fuzzy symmetric solutions.

Theorem 15 (see [25]). *The fuzzy linear system* (14) *can be extended into a $p \times p$ crisp function system of linear equations:*

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1,p} \\ g_{21} & g_{12} & \cdots & g_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ g_{p,1} & g_{p,2} & \cdots & g_{p,p} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \widetilde{y}_1(1) \\ \widetilde{y}_2(1) \\ \vdots \\ \widetilde{y}_p(1) \end{pmatrix}, \tag{21}$$

$$\begin{gathered} g_{11}(x_1 - \alpha_1(r), x_1 + \alpha_1(r)) + \cdots \\ + g_{1p}\left(x_p - \alpha_1(r), x_p + \alpha_1(r)\right) = \left(\underline{y}_1(r), \overline{y}_1(r)\right), \\ g_{21}(x_1 - \alpha_1(r), x_1 + \alpha_1(r)) + \cdots \\ + g_{2p}\left(x_p - \alpha_1(r), x_p + \alpha_1(r)\right) = \left(\underline{y}_2(r), \overline{y}_2(r)\right), \\ \vdots \\ g_{p1}\left(x_1 - \alpha_p(r), x_1 + \alpha_p(r)\right) + \cdots \\ + a_{pp}\left(x_p - \alpha_p(r), x_p + \alpha_p(r)\right) = \left(\underline{y}_p(r), \overline{y}_p(r)\right), \end{gathered} \tag{22}$$

where $\widetilde{y}_i(1) \in R$, $i = 1, 2, \ldots, p$ and $\alpha_i(r)$, $i = 1, 2, \ldots, p$ are unknown spreads.

Now, one solves the crisp linear system (21) *to obtain x_i, $i = 1, 2, \ldots, p$, that is, existed uniquely since* $\det(G) = \det(I_n \otimes A) = \det(A)^n \ne 0$ *and solve the interval equations* (22) *to obtain $\alpha_i(r)$, $i = 1, 2, \ldots, p$.*

So, without loss of generality and for simplicity to express the theory, it is assumed that the coefficients matrix G is positive. Then, ith equation of interval system (22) *is*

$$\begin{gathered} g_{i1}(x_1 - \alpha_i(r), x_1 + \alpha_i(r)) + \cdots \\ + g_{ip}\left(x_p - \alpha_i(r), x_p + \alpha_i(r)\right) = \left(\underline{y}_i(r), \overline{y}_i(r)\right), \end{gathered} \tag{23}$$

it can be rewritten in parametric form:

$$\sum_{j=1}^{p} g_{ij}\left(x_j - \alpha_i(r)\right) = \underline{y}_i(r), \quad i = 1, 2, \ldots, p, \tag{24}$$

$$\sum_{j=1}^{p} g_{ij}\left(x_j + \alpha_i(r)\right) = \overline{y}_i(r), \quad i = 1, 2, \ldots, p. \tag{25}$$

So, after some computations and replacing $\alpha_i(r)$ with $\alpha_{i1}(r)$ in (24) *and replacing $\alpha_i(r)$ with $\alpha_{i2}(r)$ in* (25), (24), *and* (25), *they are transformed, respectively, to*

$$\begin{gathered} \alpha_{i1}(r) = f_1\left(x_1, \ldots, x_p, g_{i1}, \ldots, g_{ip}, \underline{y}_i(r)\right), \quad i = 1, 2, \ldots, p, \\ \alpha_{i2}(r) = f_2\left(x_1, \ldots, x_p, g_{i1}, \ldots, g_{ip}, \overline{y}_i(r)\right), \quad i = 1, 2, \ldots, p. \end{gathered} \tag{26}$$

However, $\alpha_{i1}(r)$ is function of $x_1, \ldots, x_p, g_{i1}, \ldots, g_{ip}, \underline{y}_i(r)$, $\alpha_{i2}(r)$ is function of $x_1, \ldots, x_p, g_{i1}, \ldots, g_{ip}$, $\overline{y}_i(r)$ such that $\alpha_{i1}(r)$ and $\alpha_{i2}(r)$ are obtained spreads of ith equation in system (22). *Perhaps, $\alpha_{i1}(r)$ and $\alpha_{i2}(r)$ do not satisfy the rest of interval equations* (22). *Therefore, one should determine the reasonable spreads according to decision makers. To this end, three type of spreads are proposed as follows:*

$$\begin{gathered} \alpha_L(r) = \min\{\alpha_{i1}(r), \alpha_{i2}(r)\}, \quad i = 1, 2, \ldots, p, \ 0 \le r \le 1, \\ \alpha_U(r) = \max\{\alpha_{i1}(r), \alpha_{i2}(r)\}, \quad i = 1, 2, \ldots, p, \ 0 \le r \le 1, \\ \alpha_\lambda(r) = \lambda \alpha_U(r) + (1 - \lambda)\alpha_L(r), \quad i = 1, 2, \ldots, p, \\ 0 \le r \le 1, \ \lambda \in [0, 1]. \end{gathered} \tag{27}$$

Hence, by such computations, the fuzzy vector solution of system (7) *under proposed spreads* (27) *will be as follows. For $i = 1, 2, \ldots, p$, $0 \le r$, $\lambda \le 1$:*

$$\begin{gathered} \widetilde{X}_L = \left(\widetilde{x}_1(r), \ldots, \widetilde{x}_p(r)\right)^t, \\ \widetilde{x}_i(r) = (x_i - \alpha_L(r), x_i + \alpha_L(r)), \\ \widetilde{X}_U = \left(\widetilde{x}_1(r), \ldots, \widetilde{x}_p(r)\right)^t, \\ \widetilde{x}_i(r) = (x_i - \alpha_U(r), x_i + \alpha_U(r)), \\ \widetilde{X}_\lambda = \left(\widetilde{x}_1(r), \ldots, \widetilde{x}_p(r)\right)^t, \\ \widetilde{x}_i(r) = (x_i - \alpha_\lambda(r), x_i + \alpha_\lambda(r)). \end{gathered} \tag{28}$$

Now, it is shown that this method always gives us a fuzzy vector solution provided that the right-hand side of system (7) be a triangular fuzzy vector with nonzero left and right spreads.

Theorem 16. *Let the right-hand side of the system* (14)*, be* $\tilde{y}(r) = (\tilde{y}_1(r), \ldots, \tilde{y}_p(r))^t$*, where* $\tilde{y}_i(r) = [\underline{y}_i(1) - \sigma_i(1-r), \overline{y}_i(1) + \beta_i(1-r)]$*,* $i = 1, 2, \ldots, p$ *and let* $\alpha_L(r)$*,* $\alpha_U(r)$ *and* $\alpha_\lambda(r)$ *be defined by* (27)*, then* $\alpha_L(r)$*,* $\alpha_U(r)$*, and* $\alpha_\lambda(r)$ *are positive for all* $0 \le r, \ \lambda \le 1$*, such that*

$$\alpha_L(r) = \min\left\{\frac{\sigma_i(1-r)}{\sum\left|g_{ij}\right|}, \frac{\beta_i(1-r)}{\sum\left|g_{ij}\right|}\right\}, \tag{29}$$

$$\alpha_U(r) = \max\left\{\frac{\sigma_i(1-r)}{\sum\left|g_{ij}\right|}, \frac{\beta_i(1-r)}{\sum\left|g_{ij}\right|}\right\}, \tag{30}$$

$$\begin{aligned}\alpha_\lambda(r) = {} & \lambda \max\left\{\frac{\sigma_i(1-r)}{\sum\left|g_{ij}\right|}, \frac{\beta_i(1-r)}{\sum\left|g_{ij}\right|}\right\} \\ & + (1-\lambda)\min\left\{\frac{\sigma_i(1-r)}{\sum\left|g_{ij}\right|}, \frac{\beta_i(1-r)}{\sum\left|g_{ij}\right|}\right\}.\end{aligned} \tag{31}$$

Proof. Let us consider the ith row of interval equations (22), then by applying (24)-(25), we have

$$\begin{aligned}&\sum_{j\in H} g_{ij}\left(x_j - \alpha_{i1}(r)\right) + \sum_{j\in Q} g_{ij}\left(x_j + \alpha_{i1}(r)\right) \\ &\quad = \sum_{j=1}^{p} g_{ij}x_j - \sum_{j=1}^{p}\left|g_{ij}\right|\alpha_{i1}(r) = \underline{y}_i(1) - \sigma_i(1-r), \\ &\sum_{j\in H} g_{ij}\left(x_j + \alpha_{i2}(r)\right) + \sum_{j\in Q} g_{ij}\left(x_j - \alpha_{i2}(r)\right) \\ &\quad = \sum_{j=1}^{p} g_{ij}x_j - \sum_{j=1}^{p}\left|g_{ij}\right|\alpha_{i2}(r) = \overline{y}_i(1) + \beta_i(1-r),\end{aligned} \tag{32}$$

where H and Q include positive and negative components of coefficient matrix G, respectively. Also, it is obvious that σ_i, β_i, and the denominator are positive numbers. Therefore, $\alpha_L(r) > 0$.

Since that $\underline{y}_i(1) = \overline{y}_i(1) = \sum_{j=1}^{p} g_{ij}x_j$, it is sufficient to show (29), that is,

$$\begin{gathered}\alpha_{i1}(r) = \frac{\sigma_i(1-r)}{\sum\left|g_{ij}\right|}, \qquad \alpha_{i2}(r) = \frac{\beta_i(1-r)}{\sum\left|g_{ij}\right|}, \\ i = 1, 2, \ldots, p, \ \ 0 \le r \le 1.\end{gathered} \tag{33}$$

For results (30)-(31), the proofs are similar. □

Theorem 17. *Consider spreads* (29)–(31) *and corresponding solutions* $\tilde{X}_L$*,* $\tilde{X}_U$*, then we one gets*

(1) $\tilde{X}_L \in$ *TSS*,

(2) $\tilde{X}_U \in$ *CSS*.

In addition, one can find the maximal and minimal solutions of fuzzy linear system (7) *which are placed in TSS and CSS when the cores of compared solutions in each cases are equal.*

Theorem 18. $\tilde{X}_L$ *is maximal symmetric solution in TSS,* $\tilde{X}_U$ *is minimal symmetric solution in CSS.*

Proof. Using definitions of TSS and CSS, the proofs are obvious. □

Moreover, we could express our proposed method by algorithm as follows.

Algorithm 19.

(1) We convert the fuzzy linear matrix equation (7) to a fuzzy system of linear equations (14) based on the Kronecker product of matrices.

(2) We solve system (21) and obtain its crisp solution, that is, $x = (x_1, \ldots, x_q)$, $x_i \in R$.

(3) By applying crisp solution (solution of 1-cut), system (14) is transformed to the system of interval equations (22).

(4) The spread of all elements of fuzzy vector solution will be obtained by solving system (22), whereas, spreads are named as $\alpha_{i1}(r)$, $\alpha_{i2}(r)$, respectively, $i = 1, 2, \ldots, q$, $0 \le r \le 1$.

(5) The symmetric spreads can be assessed using (27).

(6) The fuzzy vector solutions are derived by (28).

4. Numerical Examples

Example 20. Consider the following fuzzy matrix system:

$$\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}\begin{pmatrix}\tilde{x}_{11} & \tilde{x}_{12}\\ \tilde{x}_{21} & \tilde{x}_{22}\end{pmatrix} = \begin{pmatrix}(r, 2-r) & (1+r, 3-r)\\ (4+r, 7-2r) & (4+2r, 6-2r)\end{pmatrix}. \tag{34}$$

By calculations, we know that the exact solution of above fuzzy matrix system is

$$\begin{pmatrix}\left(\frac{11}{8} + \frac{5}{8}r, \frac{23}{8} - \frac{7}{8}r\right) & \left(\frac{16}{8} + \frac{4}{8}r, \frac{24}{8} - \frac{4}{8}r\right)\\ \left(\frac{7}{8} + \frac{1}{8}r, \frac{11}{8} - \frac{3}{8}r\right) & \left(\frac{4}{8}r, \frac{8}{8} - \frac{4}{8}r\right)\end{pmatrix}, \tag{35}$$

it admits a strong fuzzy solution.

By Theorems 10 and 11, the original fuzzy matrix equation is equivalent to the following fuzzy linear system $G\tilde{x} = \tilde{y}$, that is,

$$\begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 3 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 3\end{pmatrix}\begin{pmatrix}\tilde{x}_{11}\\ \tilde{x}_{21}\\ \tilde{x}_{12}\\ \tilde{x}_{22}\end{pmatrix} = \begin{pmatrix}(r, 2-r)\\ (4+r, 7-2r)\\ (1+r, 3-r)\\ (4+2r, 6-2r)\end{pmatrix}. \tag{36}$$

Then, 1-cut of system is

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 4 \end{pmatrix}. \tag{37}$$

Therefore, the crisp solution is $x = (2, 1, 2.5, 0.5)^{\mathsf{T}}$. Now, the system of interval equations (22) is as follows:

$$\begin{aligned} &(2-\alpha_1(r), 2+\alpha_1(r)) - (1-\alpha_1(r), 1+\alpha_1(r)) \\ &\quad = (r, 2-r), \\ &(2-\alpha_2(r), 2+\alpha_2(r)) + 3(1-\alpha_2(r), 1+\alpha_2(r)) \\ &\quad = (4+r, 7-2r), \\ &(2.5-\alpha_3(r), 2.5+\alpha_3(r)) - (0.5-\alpha_3(r), 0.5+\alpha_3(r)) \\ &\quad = (1+r, 3-r), \\ &(2.5-\alpha_4(r), 2.5+\alpha_4(r)) + 3(0.5-\alpha_4(r), 0.5+\alpha_4(r)) \\ &\quad = (2+2r, 6-2r). \end{aligned} \tag{38}$$

Hence, the following results are obtained for all $r \in [0, 1]$ as

$$\begin{aligned} &\alpha_{11}(r) = \alpha_{13}(r) = \alpha_{14}(r) = \frac{1-r}{2}, \qquad \alpha_{12}(r) = \frac{1-r}{4}, \\ &\alpha_{21}(r) = \alpha_{22}(r) = \alpha_{23}(r) = \alpha_{24}(r) = \frac{1-r}{2}, \end{aligned} \tag{39}$$

and applying (27) we get for all $0 \le r,\ \lambda \le 1$,

$$\begin{aligned} &\alpha_L(r) = \frac{1-r}{4}, \qquad \alpha_U(r) = \frac{1-r}{2}, \\ &\alpha_\lambda(r) = \frac{(1-r)(1+\lambda)}{4}. \end{aligned} \tag{40}$$

Thus, the fuzzy symmetric solutions of the (14) are obtained as follows:

$$\widetilde{x}_L(r) = \begin{pmatrix} \left(2-\frac{1-r}{4}, 2+\frac{1-r}{4}\right) \\ \left(1-\frac{1-r}{4}, 1+\frac{1-r}{4}\right) \\ \left(2.5-\frac{1-r}{4}, 2.5+\frac{1-r}{4}\right) \\ \left(0.5-\frac{1-r}{4}, 0.5+\frac{1-r}{4}\right) \end{pmatrix},$$

$$\widetilde{x}_U(r) = \begin{pmatrix} \left(2-\frac{1-r}{2}, 2+\frac{1-r}{2}\right) \\ \left(1-\frac{1-r}{2}, 1+\frac{1-r}{2}\right) \\ \left(2.5-\frac{1-r}{2}, 2.5+\frac{1-r}{2}\right) \\ \left(0.5-\frac{1-r}{2}, 0.5+\frac{1-r}{2}\right) \end{pmatrix},$$

$$\widetilde{x}_\lambda(r) = \begin{pmatrix} \left(2-\frac{(1-r)(1+\lambda)}{4}, 2+\frac{(1-r)(1+\lambda)}{4}\right) \\ \left(1-\frac{(1-r)(1+\lambda)}{4}, 1+\frac{(1-r)(1+\lambda)}{4}\right) \\ \left(2.5-\frac{(1-r)(1+\lambda)}{4}, 2.5+\frac{(1-r)(1+\lambda)}{4}\right) \\ \left(0.5-\frac{(1-r)(1+\lambda)}{4}, 0.5+\frac{(1-r)(1+\lambda)}{4}\right) \end{pmatrix}. \tag{41}$$

According to Theorem 15, we know that the fuzzy approximate symmetric solutions of the original fuzzy matrix equation $A\widetilde{X} = \widetilde{B}$ are

$$\begin{aligned} \widetilde{X}_L(r) &= \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} = \begin{pmatrix} \left(2-\frac{1-r}{4}, 2+\frac{1-r}{4}\right) & \left(2.5-\frac{1-r}{4}, 2.5+\frac{1-r}{4}\right) \\ \left(1-\frac{1-r}{4}, 1+\frac{1-r}{4}\right) & \left(0.5-\frac{1-r}{4}, 0.5+\frac{1-r}{4}\right) \end{pmatrix}, \\ \widetilde{X}_U(r) &= \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} = \begin{pmatrix} \left(2-\frac{1-r}{2}, 2+\frac{1-r}{2}\right) & \left(2.5-\frac{1-r}{2}, 2.5+\frac{1-r}{2}\right) \\ \left(1-\frac{1-r}{2}, 1+\frac{1-r}{2}\right) & \left(0.5-\frac{1-r}{2}, 0.5+\frac{1-r}{2}\right) \end{pmatrix}, \\ \widetilde{X}_\lambda(r) &= \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} \\ &= \begin{pmatrix} \left(2-\frac{(1-r)(1+\lambda)}{2}, 2+\frac{(1-r)(1+\lambda)}{2}\right) & \left(2.5-\frac{(1-r)(1+\lambda)}{2}, 2.5+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(1-\frac{(1-r)(1+\lambda)}{2}, 1+\frac{(1-r)(1+\lambda)}{2}\right) & \left(0.5-\frac{(1-r)(1+\lambda)}{2}, 0.5+\frac{(1-r)(1+\lambda)}{2}\right) \end{pmatrix}, \end{aligned} \tag{42}$$

respectively.

Example 21. Consider the fuzzy matrix system:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{pmatrix} = \begin{pmatrix} (1+r,3-r) & (5+r,7-r) & (2+r,4-r) \\ (6+r,8-r) & (3+r,5-r) & (r,2-r) \end{pmatrix}. \tag{43}$$

The exact solution of above fuzzy matrix system is

$$\begin{pmatrix} (1+r,3-r) & (5+r,7-r) & (2+r,4-r) \\ (9,9) & (6,6) & (4,4) \end{pmatrix}, \tag{44}$$

and it is a weak fuzzy solution.

By Theorems 10 and 11, the original fuzzy matrix equation is equivalent to the following fuzzy linear system $G\tilde{x} = \tilde{y}$, that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \\ \tilde{x}_{13} \\ \tilde{x}_{23} \end{pmatrix} = \begin{pmatrix} (1+r,3-r) \\ (6+r,8-r) \\ (5+r,7-r) \\ (3+r,5-r) \\ (2+r,4-r) \\ (r,2-r) \end{pmatrix}. \tag{45}$$

Then, 1-cut of system is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \\ x_{13} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 6 \\ 4 \\ 3 \\ 1 \end{pmatrix}. \tag{46}$$

Therefore, the crisp solution is $x = (2, 9, 6, 10, 3, 4)^{\mathsf{T}}$. Now, the system of interval equations (22) is as follows:

$$\begin{aligned} &(2-\alpha_1(r), 2+\alpha_1(r)) = (1+r, 3-r), \\ &-(2-\alpha_2(r), 2+\alpha_2(r)) + (9-\alpha_2(r), 9+\alpha_2(r)) \\ &\qquad = (6+r, 8-r), \\ &(6-\alpha_3(r), 6+\alpha_3(r)) = (5+r, 7-r), \\ &-(6-\alpha_4(r), 6+\alpha_4(r)) + (10-\alpha_4(r), 10+\alpha_4(r)) \\ &\qquad = (3+r, 5-r), \\ &(3-\alpha_5(r), 3+\alpha_5(r)) = (2+r, 4-r), \\ &-(3-\alpha_6(r), 3+\alpha_6(r)) + (4-\alpha_6(r), 4+\alpha_6(r)) \\ &\qquad = (r, 2-r). \end{aligned} \tag{47}$$

Hence, the following results are obtained for all $r \in [0, 1]$ as

$$\begin{aligned} \alpha_{11}(r) &= \alpha_{13}(r) = \alpha_{15}(r) = 1-r, \\ \alpha_{12}(r) &= \alpha_{14}(r) = \alpha_{16}(r) = \frac{1-r}{2}, \\ \alpha_{21}(r) &= \alpha_{23}(r) = \alpha_{25}(r) = 1-r, \\ \alpha_{22}(r) &= \alpha_{24}(r) = \alpha_{26}(r) = \frac{1-r}{2}, \end{aligned} \tag{48}$$

and applying (27), we get for all $0 \le r,\ \lambda \le 1$,

$$\begin{aligned} \alpha_L(r) &= \frac{1-r}{2}, \qquad \alpha_U(r) = 1-r, \\ \alpha_\lambda(r) &= \frac{(1-r)(1+\lambda)}{2}. \end{aligned} \tag{49}$$

Thus, the fuzzy symmetric solutions of the (14) are obtained as follows:

$$\tilde{x}_L(r) = \begin{pmatrix} \left(2-\frac{1-r}{2}, 2+\frac{1-r}{2}\right) \\ \left(9-\frac{1-r}{2}, 9+\frac{1-r}{2}\right) \\ \left(6-\frac{1-r}{2}, 6+\frac{1-r}{2}\right) \\ \left(10-\frac{1-r}{2}, 10+\frac{1-r}{2}\right) \\ \left(3-\frac{1-r}{2}, 3+\frac{1-r}{2}\right) \\ \left(4-\frac{1-r}{2}, 4+\frac{1-r}{2}\right) \end{pmatrix},$$

$$\tilde{x}_U(r) = \begin{pmatrix} (2-(1-r), 2+1-r) \\ (9-(1-r), 9+(1-r)) \\ (6-(1-r), 6+(1-r)) \\ (10-(1-r), 10+(1-r)) \\ (3-(1-r), 3+(1-r)) \\ (4-(1-r), 4+(1-r)) \end{pmatrix},$$

$$\tilde{x}_\lambda(r) = \begin{pmatrix} \left(2-\frac{(1-r)(1+\lambda)}{2}, 2+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(9-\frac{(1-r)(1+\lambda)}{2}, 9+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(6-\frac{(1-r)(1+\lambda)}{2}, 6+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(10-\frac{(1-r)(1+\lambda)}{2}, 10+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(3-\frac{(1-r)(1+\lambda)}{2}, 3+\frac{(1-r)(1+\lambda)}{2}\right) \\ \left(4-\frac{(1-r)(1+\lambda)}{2}, 4+\frac{(1-r)(1+\lambda)}{2}\right) \end{pmatrix}, \tag{50}$$

respectively.

According to Theorem 11, we know that the fuzzy approximate symmetric solutions of the original fuzzy matrix equation $A\tilde{X} = \tilde{B}$ are

$$\widetilde{X}_L(r) = \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \widetilde{x}_{13} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \widetilde{x}_{23} \end{pmatrix} = \begin{pmatrix} \left(2-\frac{1-r}{2}, 2+\frac{1-r}{2}\right) & \left(6-\frac{1-r}{2}, 6+\frac{1-r}{2}\right) & \left(3-\frac{1-r}{2}, 3+\frac{1-r}{2}\right) \\ \left(9-\frac{1-r}{2}, 9+\frac{1-r}{2}\right) & \left(10-\frac{1-r}{2}, 10+\frac{1-r}{2}\right) & \left(4-\frac{1-r}{2}, 4+\frac{1-r}{2}\right) \end{pmatrix},$$

$$\widetilde{X}_U(r) = \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \widetilde{x}_{13} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \widetilde{x}_{23} \end{pmatrix} = \begin{pmatrix} (1+r, 3-r) & (5+r, 7-r) & (2+r, 4-r) \\ (8+r, 10-r) & (9+r, 11-r) & (3+r, 5-r) \end{pmatrix}, \tag{51}$$

$$\widetilde{X}_\lambda(r) = \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \widetilde{x}_{13} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \widetilde{x}_{23} \end{pmatrix} = \begin{pmatrix} \left(2-\frac{\mathcal{A}}{2}, 2+\frac{\mathcal{A}}{2}\right) & \left(6-\frac{\mathcal{A}}{2}, 6+\frac{\mathcal{A}}{2}\right) & \left(3-\frac{\mathcal{A}}{2}, 3+\frac{\mathcal{A}}{2}\right) \\ \left(9-\frac{\mathcal{A}}{2}, 9+\frac{\mathcal{A}}{2}\right) & \left(10-\frac{\mathcal{A}}{2}, 10+\frac{\mathcal{A}}{2}\right) & \left(4-\frac{\mathcal{A}}{2}, 4+\frac{\mathcal{A}}{2}\right) \end{pmatrix},$$

respectively, where $\mathcal{A}$ denotes $(1-r)(1+\lambda)$.

5. Conclusion

In this work, we presented a model for solving fuzzy matrix equations $A\widetilde{X} = \widetilde{B}$ in which A is crisp $m \times m$ nonsingular matrix and $\widetilde{B}$ is an $m \times n$ arbitrary fuzzy numbers matrix with nonzero spreads. The model was proposed in this way, that is, we converted the fuzzy linear matrix equation to a fuzzy system of linear equations based on the Kronecker product of matrices, and then we extended the fuzzy linear system into a crisp system of linear equations and a fuzzified interval system of linear equations. The fuzzy symmetric solutions of the fuzzy linear matrix equation were derived from solving the crisp systems of linear equations. Numerical examples showed that our method is feasible to solve this type of fuzzy matrix equations.

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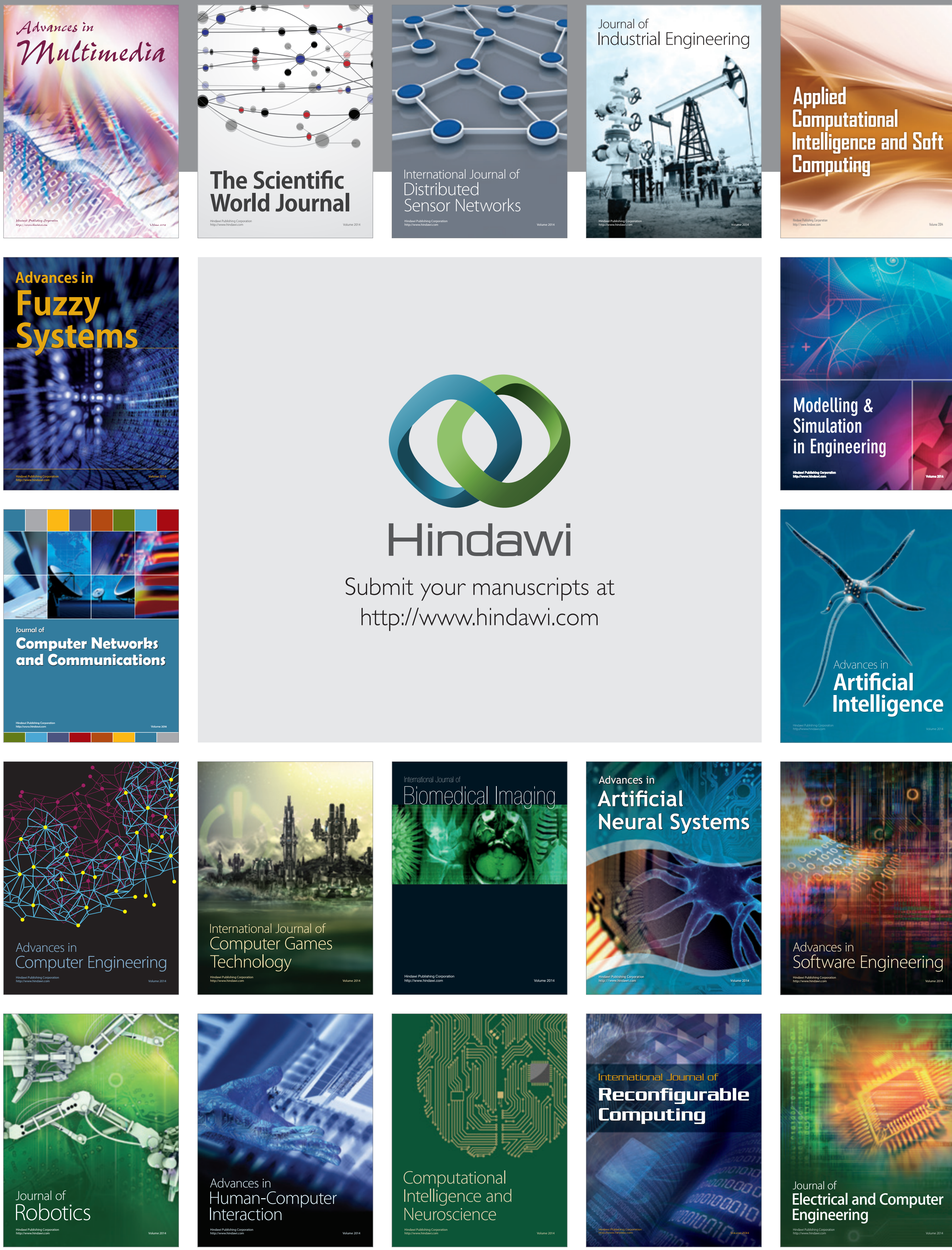
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