Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 264874, 16 pages
doi:10.1155/2012/264874

Research Article

A Stochastic Dynamic Model of Computer Viruses

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Received 8 June 2012; Accepted 16 July 2012

Academic Editor: Bimal Kumar Mishra

A stochastic computer virus spread model is proposed and its dynamic behavior is fully investigated. Specifically, we prove the existence and uniqueness of positive solutions, and the stability of the virus-free equilibrium and viral equilibrium by constructing Lyapunov functions and applying Ito's formula. Some numerical simulations are finally given to illustrate our main results.

1. Introduction

A generalized computer virus, including the narrowly defined virus and the worm, is a kind of computer program that can replicate itself and spread from one computer to another. Viruses mainly attack the file system and worms use system vulnerability to search and attack computers. As hardware and software technology developed and computer networks became widespread, computer virus has come to be one major threat to our daily life. Consequently, in order to deal with the threat, the trial on better understanding the computer virus propagation dynamics is an important matter. Similar to the biological virus, there are two ways to study this problem: microscopic and macroscopic. Following a macroscopic approach, since [1, 2] took the first step towards modeling the spread behavior of computer virus, much effort has been done in the area of developing a mathematical model for the computer virus propagation [3–13]. These models provide a reasonable qualitative understanding of the conditions under which viruses spread much faster than others.

In [13], the authors investigated a differential $SEIR$ model by making the following assumptions.

(H_1) The total population of computers is divided into four groups: susceptible, exposed, infected, and recovered computers. Let S, E, I, and R denote the numbers of susceptible, exposed, infected, and recovered computers, respectively. N denotes the total number of computers.

(H_2) New computers are attached to the computer network with rate μN.

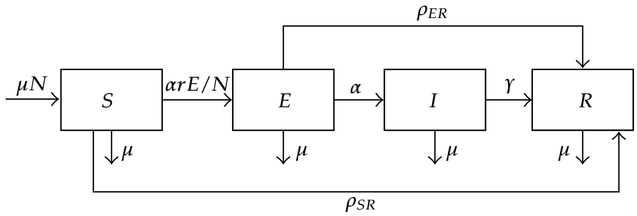

Figure 1

(H_3) Computers are disconnected to the computer network with constant rate μ.

(H_4) S computers become E computers with rate $\alpha r/N$, where r denotes the averaged number of neighbor nodes (with various states) that are directly connected; α is the transition rate from E to I. S computers become R computers with rate ρ_{SR}.

(H_5) E computers become I computers with constant rate α; E computers become R computers with constant rate ρ_{SR}; I computers become R computers with constant rate γ.

According to the above assumptions, the following model (see Figure 1) is derived:

$$\begin{aligned}
\dot{S}(t) &= \mu N - \frac{\alpha r}{N} E(t)S(t) - \rho_{SR}S(t) - \mu S(t),\\
\dot{E}(t) &= \frac{\alpha r}{N} E(t)S(t) - (\alpha + \rho_{ER} + \mu)E(t),\\
\dot{I}(t) &= \alpha E(t) - (\gamma + \mu)I(t),\\
\dot{R}(t) &= \rho_{SR}S(t) + \rho_{ER}E(t) - \gamma I(t) - \mu R(t).
\end{aligned} \tag{1.1}$$

Notably, the first three equations in (1.1) do not depend on the fourth equation, since $\dot{S}(t) + \dot{E}(t) + \dot{I}(\mathrm{t}) + \dot{R}(t) = 1$. Therefore, the forth equation can be omitted and the model (1.1) can be rewritten as

$$\begin{aligned}
\dot{S}(t) &= \mu N - \frac{\alpha r E(t)S(t)}{N} - \rho_{SR}S(t) - \mu S(t),\\
\dot{E}(t) &= \frac{\alpha r E(t)S(t)}{N} - (\alpha + \rho_{ER} + \mu)E(t),\\
\dot{I}(t) &= \alpha E(t) - (\gamma + \mu)I(t).
\end{aligned} \tag{1.2}$$

In [13], authors have proved the virus-free equilibrium $\mathrm{EQ}_{\mathrm{vf}} = ((\mu/(\rho_{SR}+\mu))N, 0, 0)$ is globally asymptotically stable if $R_0 = (\alpha r\mu/(\alpha r\mu)(\rho_{SR} + \mu)) \ \leq 1$, and the viral equilibrium $\mathrm{EQ}_{\mathrm{ve}}$ is globally asymptotically stable if $R_0 > 1$, where

$$\mathrm{EQ}_{\mathrm{ve}} = \left(\frac{(\alpha+\rho_{ER}+\mu)}{\alpha r}N,\ \frac{\mu N}{(\alpha+\rho_{ER}+\mu)} - \frac{(\rho_{ER}+\mu)N}{\alpha r},\ \frac{\alpha}{\gamma+\mu}\left[\frac{\mu N}{(\alpha+\rho_{ER}+\mu)} - \frac{(\rho_{ER}+\mu)N}{\alpha r}\right]\right). \tag{1.3}$$

However, in the real world, systems are inevitably affected by environmental noise. Hence the deterministic approach has some limitations in mathematically modeling the transmission of an infectious disease, and it is quite difficult to predict the future dynamics of the system accurately. This happens due to the fact that deterministic models do not incorporate the effect of a fluctuating environment. Stochastic differential equation models play a significant role in various branches of applied sciences, including infectious dynamics, as they provide some additional degree of realism compared to their deterministic counterpart. In this paper, we introduce a noise into (1.2) and we transform the deterministic problem into a corresponding stochastic problem.

In this paper, we introduce randomness into the model by replacing the parameters μ, μ and μ by $\mu \rightarrow \mu+\sigma_1\dot{B}_1(t), \mu \rightarrow \mu+\sigma_2\dot{B}_2(t)$, and $\mu \rightarrow \mu+\sigma_3\dot{B}_3(t)$, where $\dot{B}_1(t)$, $\dot{B}_2(t)$, and $\dot{B}_3(t)$ are mutual independent standard Brownian motions with $B_1(0) = 0$, $B_2(0) = 0$, and $B_3(0) = 0$, and intensity of white noise $\sigma_1^2 \geq 0$, $\sigma_2^2 \geq 0$ and $\sigma_3^2 \geq 0$, respectively. Then the stochastic system is

$$\begin{gathered}\dot{S}(t) = \mu N - \frac{\alpha r E(t)S(t)}{N} - \rho_{SR}S(t) - \mu S(t) - \sigma_1 S(t)\dot{B}_1(t), \\ \dot{E}(t) = \frac{\alpha r E(t)S(t)}{N} - \left(\alpha + \rho_{ER} + \mu\right)E(t) - \sigma_2 E(t)\dot{B}_1(t), \\ \dot{I}(t) = \alpha E(t) - \left(\gamma + \mu\right)I(t) - \sigma_3 I(t)\dot{B}_1(t).\end{gathered} \tag{1.4}$$

The organization of this paper is as follows. In Section 2, we prove the existence and the uniqueness of the nonnegative solution of (1.3). In Section 3, if $R_0 \leq 1$, we show that the solution is oscillating around the virus-free equilibrium of (1.3). Section 4 focuses on the persistence of the virus. By choosing appropriate Lyapunov function, we show that there is a stationary distribution for (1.3) and that it is persistent if $R_0 > 1$. Some numerical simulations are performed in Section 5. In Section 6, a brief conclusion is given.

Throughout this paper, consider the n-dimensional stochastic differential equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), \quad \text{on } t \geq t_0, \tag{1.5}$$

with the initial value $x(t_0) = x_0 \in R^n$. $B(t)$ denotes n-dimensional standard Brownian motion defined on the above probability space. Define the differential operator L associated with (1.4) by

$$L = \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,j=1}^{n}\left[g^T(x,t)g(x,t)\right]\frac{\partial^2}{\partial x_k \partial x_j}. \tag{1.6}$$

If L acts on a function V, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\operatorname{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right], \tag{1.7}$$

where $V_t = \partial V/\partial_t$, $V_x = (\partial V/\partial x_1, \dots, \partial V/\partial x_n)$, $V_{xx} = (\partial^2 V/\partial x_k \partial x_k)_{n*n}$.

By Ito's formula, if $x(t) \in R^n$, then for (1.4), assume that $f(0,t) = 0, g(0,t) = 0$ for all $t \geq t_0$. So $x(t) \equiv 0$ is a solution of (1.4), called the trivial solution or equilibrium position.

2. Existence and Uniqueness of the Nonnegative Solution

To investigate the dynamical behavior of a population model, the first concern is whether the solution is positive or not and whether it has the global existence or not. Hence, in this section, we mainly use the Lyapunov analysis method to show that the solution of system (1.3) is positive and global.

Theorem 2.1. *Let $(S_0, E_0, I_0) \in \Delta$, then the system* (1.2) *admits a unique solution $(S(t), E(t), I(t))$ on $t \geq 0$, and this solution remains in R_+^3 with probability 1.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value (S_0, E_0, I_0) there is a unique local solution $(S(t), E(t), I(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [2, 13]. To show this solution is global, we need to show that $\tau_e = \infty$ a. s. Let $k_0 > 0$ be sufficiently large so that every component of x_0 lies within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time,

$$\tau_k = \inf\left\{t \in [0, \tau_e) : S(t) \notin \left(\frac{1}{k}, k\right) \text{ or } E(t) \notin \left(\frac{1}{k}, k\right) \text{ or } I(t) \notin \left(\frac{1}{k}, k\right)\right\}, \tag{2.1}$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$, whence $\tau_\infty \leq \tau_e$ a. s. If we can show that $\tau_\infty = \infty$ a. s., then $\tau_e = \infty$ and $(S(t), E(t), I(t))$ a. s. for all $t \geq 0$. In other words, to complete the proof we need to show that $\tau_\infty = \infty$ a. s. For if this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P(\tau_\infty \leq T) > \varepsilon. \tag{2.2}$$

Hence, there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon \quad \forall k > k_1. \tag{2.3}$$

Define a C^2-function V for $X(S, E, I) \in R_+^3$ by

$$V(X) = \left(S - a - \log\frac{S}{a}\right) + (E - 1 - \log E) + (I - 1 - \log I). \tag{2.4}$$

The nonnegativity of this function can be seen from $\mu + 1 - \log\mu \geq 0$, for all $\mu > 0$. Using Ito's formula we get

$$\begin{aligned} dV(X) &= \left(a - \frac{a}{S}\right)dS + \frac{a}{2S^2}(dS)^2 + \left(1 - \frac{1}{E}\right)dE + \frac{1}{2E^2}(dE)^2 + \left(1 - \frac{1}{I}\right)dI + \frac{1}{2I^2}(dI)^2 \\ &\doteq LV\,dt - \left[\sigma_1(S - a)\dot{B}_1(t) + \sigma_2(E - 1)\dot{B}_2(t) + \sigma_3(I - 1)\dot{B}_3(t)\right], \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} LV &= \left(1-\frac{a}{S}\right)\left[\mu N - \frac{\alpha r E(t) S(t)}{N} - \rho_{SR} S(t) - \mu S(t)\right] + \frac{a\sigma_1^2}{2} \\ &\quad + \left(1-\frac{1}{E}\right)\left[\frac{\alpha r E(t) S(t)}{N} - (\alpha + \rho_{SR} + \mu) E(t)\right] + \frac{\sigma_1^2}{2} \\ &\quad + \left(1-\frac{1}{I}\right)\left[\alpha E(t) - (\gamma+\mu) I(t)\right] + \frac{\sigma_3^2}{2} \\ &= \left(\mu N + a\rho_{SR} + \mu a + \alpha + \rho_{SR} + \mu + \gamma + \mu + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}\right) \\ &\quad + \left[\frac{\alpha r a}{N} E - \rho_{SR} S - \mu S - \frac{a}{S}\mu N - \rho_{SR} E - \mu E - \frac{\alpha r}{N} S - \gamma I - \mu I - \frac{\alpha}{I} E\right] \\ &\le \mu N + a\rho_{SR} + \mu a + \alpha + \rho_{SR} + \mu + \gamma + \mu + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\alpha r a}{N} E - \rho_{ER} E - \mu E. \end{aligned} \tag{2.6}$$

By choosing $a = (\rho_{ER} + \mu) N / \alpha r$, then

$$LV \le \mu N + a\rho_{SR} + \mu a + \alpha + \rho_{SR} + \mu + \gamma + \mu + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \doteq \dot{M}. \tag{2.7}$$

Therefore,

$$\begin{gathered} \int_0^{\tau_m \wedge T} dV(X) \le \int_0^{\tau_m \wedge T} \dot{M} dt - \int_0^{\tau_m \wedge T} \left[\sigma_1 (S-a) dB_1(t) + \sigma_2 (E-1) dB_2(t) + \sigma_3 (I-1) dB_3(t)\right], \\ EV(X(\tau_m \wedge T)) \le V(X(0)) + E\left[\int_0^{\tau_m \wedge T} \dot{M} dt\right] \le V(X(0)) + \dot{M} T. \end{gathered} \tag{2.8}$$

Setting $\Omega_m = \{\tau_m \le T\}$ for $m \ge m_1$, then by (2.3), we know that $P(\Omega_m) \ge \varepsilon$. Note that for every $\omega \in \Omega_m$, there is at least one of $S(\Omega_m, \omega)$, $E(\Omega_m, \omega)$, and $I(\Omega_m, \omega)$ that equals either m or $1/m$. Then

$$V(X(\tau_m)) \ge (m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 + \log m\right) \wedge \left(m - a - a \log \frac{m}{a}\right) \wedge \left(\frac{1}{m} - a + a \log am\right), \tag{2.9}$$

where $1\Omega_m(\omega)$ is the indicator function of Ω_m. Let $m \to \infty$ lead to the contradiction that $\infty > V(X(0)) + \dot{M}T = \infty$. So $\tau_\infty = \infty$ is necessary. The proof of Theorem 2.1 is completed. □

3. Stability of Virus-Free Equilibrium

It is clear that $EQ_{vf} = (\mu N/(\rho_{SR}+\mu), 0, 0)$ is the virus-free equilibrium of system (1.3), which has been mentioned above, and EQ_{vf} is globally stable if $R_0 \leq 1$, which means that the virus will die out after some period of time. Since there is no virus-free equilibrium of system (1.3), in this section, we show that the solution is oscillating in a small neighborhood of EQ_{vf} if the white noise is small.

Theorem 3.1. *If $\rho_{SR}+\mu > \sigma_1^2$, $3\alpha^2+2\rho_{SR}+2\mu > \sigma_2^2$, $2\gamma+2\mu-\alpha > \sigma_3^2$ and $R_0 \leq 1$, then the solution $X(t)$ of system* (1.3) *with initial value $X(0) \in R_+^3$ has the property*

$$\begin{aligned} \lim_{x\to\infty}\sup \frac{1}{t}E\int_0^t \Bigg[&(1+b)\left(\rho_{SR}+\mu-\sigma_1^2\right)\mu^2(s) + \left(\frac{1}{2}\alpha+\rho_{SR}+\mu-\frac{1}{2}\sigma_2^2\right)\sigma^2(s) \\ &+\left(\gamma+\mu-\frac{\alpha}{2}-\frac{1}{2}\sigma_3^2 w^2(s)\right)ds \leq (1-b)\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)\Bigg], \end{aligned} \tag{3.1}$$

where b is positive constants, defined as in the proof.

Proof. For simplicity, let $u(t) = S(t) - \mu N/(\rho_{SR}+\mu)$, $v(t) = E(t)$, $w(t) = I(t)$, system (1.3) can be written as

$$\begin{aligned} \dot{u}(t) &= -\frac{\alpha r v(t)}{N}\left[u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right] - (\rho_{SR}+\mu)u(t) - \sigma_1\left(u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right)\dot{B}(t), \\ \dot{v}(t) &= \frac{\alpha r v(t)}{N}\left[u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right] - (\alpha+\rho_{SR}+\mu)v(t) - \sigma_2 v(t)\dot{B}(t), \\ \dot{w}(t) &= \alpha v(t) - (\gamma+\mu)w(t) - \sigma_3 w(t)\dot{B}_3(t). \end{aligned} \tag{3.2}$$

Let

$$\begin{aligned} V(x) &= \frac{1}{2}(u+v)^2 + \frac{1}{2}bu^2 + \frac{1}{2}bv + \frac{1}{2}w^2 \\ &= V_1 + bV_2 + bV_3 + V_4, \end{aligned} \tag{3.3}$$

then b is positive constants to be determined later. By Ito's formula, we compute

$$dV_1 \doteq LV_1 dt - (u(t)+v(t))\left[\sigma_1\left(u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right)\dot{B}(t) + \sigma_2 v(t)\dot{B}_2(t)\right],$$

$$
\begin{aligned}
LV_1 &= (u(t)+v(t))\left[-(\rho_{SR}+\mu)u(t)-(\alpha+\rho_{SR}+\mu)v(t)\right]+\frac{1}{2}\sigma_1^2\left(u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right)^2 \\
&\quad+\frac{1}{2}\sigma_2^2 v^2(t) \\
&\le (u(t)+v(t))\left[-(\rho_{SR}+\mu)u(t)-(\alpha+\rho_{SR}+\mu)v(t)\right]+\sigma_1^2u^2(t)+\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2 \\
&\quad+\frac{1}{2}\sigma_2^2 v^2(t) \\
&= -\Bigg[\left(\rho_{SR}+\mu-\sigma_1^2\right)u^2(t)+\left(\alpha+\rho_{SR}+\mu-\frac{1}{2}\sigma_2^2\right)v^2(t)+(\alpha+2\rho_{SR}+2\mu)u(t)v(t) \\
&\qquad-\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\Bigg],
\end{aligned}
$$

$$
dV_2 \doteq LV_2dt-\sigma_1u(t)\left(u(t)+\frac{\mu}{\rho_{SR}+\mu}\right)\dot{B}(t),
$$

$$
\begin{aligned}
LV_2 &= u(t)\left\{-\frac{\alpha r v(t)}{N}\left[u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right]-(\rho_{SR}+\mu)u(t)\right\} \\
&\quad+\frac{1}{2}\sigma_1^2\left(u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right)^2 \\
&\le u(t)\left\{-\frac{\alpha r v(t)}{N}\left[u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right]-(\rho_{SR}+\mu)u(t)\right\}+\sigma_1^2u^2(t) \\
&\quad+\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2 \\
&= -\Bigg[\left(\rho_{SR}+\mu-\sigma_1^2\right)u^2(t)+\frac{\alpha r\mu}{\rho_{SR}+\mu}u(t)v(t)+\frac{\alpha r}{N}v(t)u^2(t) \\
&\qquad+\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\Bigg] \\
&\le -\left[\left(\rho_{SR}+\mu-\sigma_1^2\right)u^2(t)+\frac{\alpha r\mu}{\rho_{SR}+\mu}u(t)v(t)+\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
dV_3 &= \frac{\alpha r v(t)}{N}\left[u(t)+\frac{\mu}{\rho_{SR}+\mu}N\right]dt-(\alpha+\rho_{SR}+\mu)v(t)dt-\sigma_2v(t)\dot{B}(t) \\
&= \left[\frac{\alpha r}{N}v(t)u(t)+\left(\frac{\alpha r\mu}{\rho_{SR}+\mu}-(\alpha+\rho_{SR}+\mu)\right)v(t)\right]dt-\sigma_2v(t)\dot{B}(t) \\
&\doteq LV_3-\sigma_2v(t)\dot{B}(t),
\end{aligned}
$$

$$
\begin{aligned}
dV_4 &= \left[w(t)\left(\alpha v(t) - (\gamma+\mu)w(t)\right) + \frac{1}{2}\sigma_3^2 w^2(t)\right]dt - \sigma_3 w^2(t)\dot{B}(t) \\
&= \left[\alpha v(t)w(t) - (\gamma+\mu)w^2(t) + \frac{1}{2}\sigma_3^2 w^2(t)\right]dt - \sigma_3 w^2(t)\dot{B}(t) \\
&\le \left[\frac{\alpha}{2}\left(v^2(t) + w^2(t)\right) - (\gamma+\mu)w^2(t) + \frac{1}{2}\sigma_3^2 w^2(t)\right]dt - \sigma_3 w^2(t)\dot{B}(t) \\
&= \left[\left(\frac{\alpha}{2} - \gamma - \mu + \frac{1}{2}\sigma_3^2\right)w^2(t) + \frac{\alpha}{2}v^2(t)\right]dt - \sigma_3 w^2(t)\dot{B}(t) \\
&\doteq LV_4 dt - \sigma_3 w^2(t)\dot{B}(t), \\
LV &= LV_1 + bLV_2 + bLV_3 + LV_4 \\
&= -\Bigg[\left(\rho_{SR} + \mu - \sigma_1^2\right)u^2(t) + \left(\alpha + \rho_{SR} + \mu - \frac{1}{2}\sigma_2^2\right)v^2(t) \\
&\qquad + (\alpha + 2\rho_{SR} + 2\mu)u(t)v(t) - \sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\Bigg] \\
&\quad - b\left[\left(\rho_{SR} + \mu - \sigma_1^2\right)u^2(t) + \frac{\alpha r\mu}{\rho_{SR}+\mu}u(t)v(t) + \sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\right] \\
&\quad + b\left[\frac{\alpha r}{N}v(t)u(t) + \left(\frac{\alpha r\mu}{\rho_{SR}+\mu} - (\alpha + \rho_{SR} + \mu)\right)v(t)\right] \\
&\quad + \left[\left(\frac{\alpha}{2} - \gamma - \mu + \frac{1}{2}\sigma_3^2\right)w^2(t) + \frac{\alpha}{2}v^2(t)\right].
\end{aligned} \tag{3.4}
$$

Choosing $b = (N(\alpha + 2\rho_{SR} + 2\mu)(\rho_{SR} + \mu))/(\alpha r(\rho_{SR} + \mu - N\mu))$, then we get

$$
\begin{aligned}
LV &= -(1+b)\left(\rho_{SR} + \mu - \sigma_1^2\right)u^2(t) - \left(\frac{1}{2}\alpha + \rho_{SR} + \mu - \frac{1}{2}\sigma_2^2\right)v^2(t) - \left(\gamma + \mu - \frac{\alpha}{2} - \frac{1}{2}\sigma_3^2\right)w^2(t) \\
&\quad - b\left((\alpha + \rho_{SR} + \mu) - \frac{\alpha r\mu}{\rho_{SR}+\mu}\right)v(t) + (1-b)\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}\right)^2, \\
dV &\le -(1+b)\left(\rho_{SR} + \mu - \sigma_1^2\right)u^2(t) - \left(\frac{1}{2}\alpha + \rho_{SR} + \mu - \frac{1}{2}\sigma_2^2\right)v^2(t) - \left(\gamma + \mu - \frac{\alpha}{2} - \frac{1}{2}\sigma_3^2\right)w^2(t) \\
&\quad + (1-b)\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2 - (u(t) + v(t))\left[\sigma_1\left(u(t) + \frac{\mu}{\rho_{SR}+\mu}N\right)\dot{B}_1(t) + \sigma_2 v(t)\dot{B}_2(t)\right] \\
&\quad - \sigma_1 u(t)\left(u(t) + \frac{\mu}{\rho_{SR}+\mu}N\right)\dot{B}(t) - \sigma_2 v(t)\dot{B}_2(t) - \sigma_3 w^2(t)\dot{B}_3(t).
\end{aligned} \tag{3.5}
$$

Integrating this from 0 to t and taking the expectation, we have

$$\begin{aligned} &E[V(t)] - V(0) \\ &\le -E\int_0^t \Bigg[(1+b)\left(\rho_{SR}+\mu-\sigma_1^2\right)u^2(s) + \left(\frac{1}{2}\alpha+\rho_{SR}+\mu-\frac{1}{2}\sigma_2^2\right)v^2(s) \\ &\qquad + \left(\gamma+\mu-\frac{\alpha}{2}-\frac{1}{2}\sigma_3^2\right)w^2(s) - (1-b)\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2\Bigg]ds. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned} &\lim_{x\to\infty}\sup\frac{1}{t}E\int_0^t \Bigg[(1+b)\left(\rho_{SR}+\mu-\sigma_1^2\right)u^2(s) + \left(\frac{1}{2}\alpha+\rho_{SR}+\mu-\frac{1}{2}\sigma_2^2\right)v^2(s) \\ &\qquad + \left(\gamma+\mu-\frac{\alpha}{2}-\frac{1}{2}\sigma_3^2\right)w^2(s)\Bigg]ds \le (1-b)\sigma_1^2\left(\frac{\mu}{\rho_{SR}+\mu}N\right)^2. \end{aligned} \tag{3.7}$$

□

Remark 3.2. Theorem 3.1 shows that the solution of system (1.3) would oscillate around the virus-free equilibrium of system (1.1) if some conditions are satisfied, and the intensity of fluctuation is proportional to σ_1^2, which is the intensity of the white noise $\dot{B}_1(t)$. In a biological interpretation, if the stochastic effect on S is small, the solution of system (1.3) will be close to the virus-free equilibrium of system (1.1) most of the time.

4. Permanence

When studying epidemic dynamical systems, we are interested in when the computer viruses will persist in network. For a deterministic model, this is usually solved by showing that the viral equilibrium is a global attractor or is globally asymptotically stable. But, for system (1.3), there is no viral equilibrium. In this section, we show that there is a stationary distribution, which reveals that the computer viruses will persist.

Lemma 4.1 (see [14, 15]). *Assumption B: there exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties.*

(B.1) *In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.*

(B.2) *If $x \in E_l/U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$. If (B) holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\bullet)$. Let $f(\bullet)$ be a function integrable with respect to the measure μ. Then*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_l} f(x)\mu d(x)\right\} = 1, \quad \forall x \in E_l. \tag{4.1}$$

Lemma 4.2 (see [14, 15]). *Let $X(t)$ be a regular temporally homogeneous Markov process in E_l. If $X(t)$ is recurrent relative to some bounded domain U, then it is recurrent relative to any nonempty domain in E_l.*

Theorem 4.3. *If $\sigma_1^2 < (\rho_{SR}+\mu)(1+(\alpha r/S^*N))(S^*/(S^*-1)), \sigma_2^2 < (\alpha/2)+\rho_{SR}+\mu\sigma_3^2 < \gamma+\mu-(\alpha/2)$, and $R_0 > 1$, then, for any initial value $X(0) \in R_+^3$, there is a stationary distribution $\mu(\bullet)$ for system (1.3), and it has an ergodic property, where a, c are defined as in the proof, $Q_{ve} = (S^*, E^*, I^*)$ is the viral equilibrium of system.*

Proof. When $R_0 > 1$, there is an viral equilibrium EQ_{ve} of system (1.3). Then

$$\mu N = \frac{\alpha r E^* S^*}{N} - \rho_{SR} S^* + \mu S^*, \qquad \frac{\alpha r E^* S^*}{N} = (\alpha + \rho_{SR} + \mu)E^*, \qquad \alpha E^* = (\gamma + \mu)I^*. \tag{4.2}$$

Define

$$\begin{aligned} V(x) &= a\left(S - S^* - S^* \log \frac{S}{S^*} + E - E^* - E^* \log \frac{E}{E^*}\right) + \left(E - E^* - E^* \log \frac{E}{E^*}\right) \\ &\quad + \frac{1}{2}(S - S^* + E - E^*)^2 + \frac{1}{2}c(S - S^*)^2 + \frac{1}{2}(I - I^*)^2 \\ &= aV_1 + V_2 + V_3 + cV_4 + V_5, \end{aligned} \tag{4.3}$$

where a, c, are positive constants to be determined later. Then V is positive definite. By Ito's formula, we compute

$$\begin{aligned} dV_1 &= \left(1 - \frac{S^*}{S}\right)\left[\left(\mu N - \frac{\alpha r E S}{N} - \rho_{SR} S - \mu S\right)dt - \sigma_1 S \dot{B}_1(t)\right] \\ &\quad + \left(1 - \frac{E^*}{E}\right)\left[\left(\frac{\alpha r E S}{N} - (\alpha + \rho_{SR} + \mu)E\right)dt - \sigma_2 E \dot{B}_2(t)\right] \\ &\quad + \frac{1}{2}S^*\sigma_1^2 dt + \frac{1}{2}E^*\sigma_2^2 dt \\ &= LV_1 dt - \left(1 - \frac{S^*}{S}\right)\sigma_1 S(t)\dot{B}_1(t) - \left(1 - \frac{E^*}{E}\right)\sigma_2 E(t)\dot{B}_2(t), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} LV_1 &= \left(1 - \frac{S^*}{S}\right)\left[\left(\mu N - \frac{\alpha r E S}{N} - \rho_{SR} S - \mu S\right)dt\right] \\ &\quad + \left(1 - \frac{E^*}{E}\right)\left[\left(\frac{\alpha r E S}{N} - (\alpha + \rho_{SR} + \mu)E\right)dt\right] \\ &\quad + \frac{1}{2}S^*\sigma_1^2 dt + \frac{1}{2}E^*\sigma_2^2 dt \\ &= \left(1 - \frac{S^*}{S}\right)\left[\frac{\alpha r}{N}(E^*S^* - ES)dt + (\rho_{SR} + \mu)(S^* - S)dt\right] \end{aligned}$$

$$
\begin{aligned}
&+\left(1-\frac{E^*}{E}\right)\left[\left(\frac{\alpha r E S}{N}-(\alpha+\rho_{SR}+\mu)E\right)dt\right]\\
&+\frac{1}{2}S^*\sigma_1^2dt+\frac{1}{2}E^*\sigma_2^2dt\\
&=-\frac{(S-S^*)^2}{S}(\rho_{SR}+\mu)+\frac{\alpha r}{N}E^*S^*+E^*(\alpha+\rho_{SR}+\mu)\\
&+\left(\frac{\alpha r}{N}S^*-\alpha-\rho_{SR}-\mu\right)E-\frac{\alpha r}{N}E^*S^*-\frac{\alpha r}{N}\frac{E^*S^{*2}}{S}\\
&+\frac{1}{2}S^*\sigma_1^2+\frac{1}{2}E^*\sigma_2^2\\
&\leq-\frac{(S-S^*)^2}{S}(\rho_{SR}+\mu)\frac{1}{2}S^*\sigma_1^2+\frac{1}{2}E^*\sigma_2^2,\\
dV_2&=\left(1-\frac{E^*}{E}\right)\left[\left(\frac{\alpha r E S}{N}-(\alpha+\rho_{SR}+\mu)E\right)dt-\sigma_2E\dot{B}_2(t)\right]+\frac{1}{2}E^*\sigma_2^2dt\\
&=LV_2dt-\left(1-\frac{E^*}{E}\right)\sigma_2E(t)\dot{B}_2(t).
\end{aligned}
\tag{4.5}
$$

Let $\widehat{B}=(\alpha r/N)E^*S^*=(\alpha+\rho_{SR}+\mu)E^*$ and $\alpha-1-\log\alpha>0$, for all α

$$
\begin{aligned}
LV_2&=\left(1-\frac{E^*}{E}\right)\left(\frac{\alpha r E S}{N}-(\alpha+\rho_{SR}+\mu)E\right)dt+\frac{1}{2}E^*\sigma_2^2dt\\
&=\left(1-\frac{E^*}{E}\right)\left(\widehat{B}\frac{ES}{E^*S^*}-\widehat{B}\frac{E}{E^*}\right)+\frac{1}{2}E^*\sigma_2^2\\
&=\left(\widehat{B}\frac{ES}{E^*S^*}-\widehat{B}\frac{E}{E^*}+\widehat{B}\frac{S}{S^*}+\widehat{B}\right)+\frac{1}{2}E^*\sigma_2^2\\
&\leq\widehat{B}\left[\frac{ES}{E^*S^*}-\frac{E}{E^*}-\left(1+\log\frac{S}{S^*}\right)+1\right]+\frac{1}{2}E^*\sigma_2^2\\
&\leq\widehat{B}\left[\frac{ES}{E^*S^*}-\frac{E}{E^*}+\left(\frac{S}{S^*}-2\right)+1\right]+\frac{1}{2}E^*\sigma_2^2\\
&=\widehat{B}\left(\frac{E}{E^*}-1\right)\left(\frac{S}{S^*}-1\right)+\widehat{B}\left(\frac{S^*}{S}+\frac{S}{S^*}-2\right)+\frac{1}{2}E^*\sigma_2^2\\
&=\frac{\alpha r}{N}(E-E^*)(S-S^*)+\frac{\alpha r}{N}E^*\frac{(S-S^*)^2}{S}+\frac{1}{2}E^*\sigma_2^2,\\
dV_3&=(S-S^*+E-E^*)\left[\mu N-(\rho_{SR}+\mu)S-(\alpha+\rho_{SR}+\mu)E\right]dt\\
&-(S-S^*+E-E^*)\left[\sigma_1S\dot{B}_1(t)+\sigma_2E\dot{B}_2(t)\right]+\left(\frac{\sigma_1^2}{2}S^2+\frac{\sigma_2^2}{2}E^2\right)dt,
\end{aligned}
$$

$$
\begin{aligned}
LV_3 &= (S - S^* + E - E^*)\left[-(\rho_{SR} + \mu)(S - S^*) - (\alpha + \rho_{SR} + \mu)(E - E^*)\right] \\
&\quad + \frac{\sigma_1^2}{2}S^2 + \frac{\sigma_2^2}{2}E^2 \\
&\le -(\rho_{SR} + \mu)(S - S^*)^2 - (\alpha + \rho_{SR} + \mu)(E - E^*)^2 \\
&\quad - (\alpha + \rho_{ER} + \rho_{SR} + 2\mu)(S - S^*)(E - E^*) \\
&\quad + \sigma_1^2\left[(S - S^*) + S^{*2}\right] + \sigma_2^2\left[(E - E^*) + E^{*2}\right] \\
&\le \left(\sigma_1^2 - \rho_{SR} - \mu\right)(S - S^*)^2 + \left(\sigma_2^2 - \alpha - \rho_{ER} - \mu\right)(E - E^*)^2 + \sigma_1^2 S^{*2} + \sigma_2^2 E^{*2}, \\
dV_4 &= (S - S^*)\left(\mu N - \frac{\alpha r}{N}ES - (\rho_{SR} + \mu)S\right) - S(S - S^*)\sigma_1 \dot{B}(t) + \frac{1}{2}\sigma_1^2 S^2, \\
LV_4 &= (S - S^*)\left(\mu N - \frac{\alpha r}{N}ES - (\rho_{SR} + \mu)S\right) + \frac{1}{2}\sigma_1^2 S^2 \\
&= (S - S^*)\left[-\frac{\alpha r}{N}(ES - E^*S^*) - (\rho_{SR} + \mu)(S - S^*)\right] + \frac{1}{2}\sigma_1^2 S^2 \\
&= -\frac{\alpha r}{N}S^*(S - S^*)(E - E^*) - \frac{\alpha r}{N}(S - S^*)^2 E - \frac{\alpha r}{N}(\rho_{SR} + \mu)(S - S^*)^2 + \frac{1}{2}\sigma_1^2 S^2 \\
&\le -\frac{\alpha r}{N}S^*(S - S^*)(E - E^*) - \left(\frac{\alpha r}{N}(\rho_{SR} + \mu) - \sigma_1^2\right)(S - S^*)^2 + \sigma_1^2 S^{*2}, \\
dV_5 &= (I - I^*)\left[\alpha E - (\gamma + \mu)I\right] + \frac{\sigma_3^2}{2}I^2 \\
&= LV_5 dt - \sigma_3 I(I - I^*)\dot{B}_3, \\
LV_5 &= (I - I^*)\left[\alpha E - (\gamma + \mu)I\right] + \frac{\sigma_3^2}{2}I^2 \\
&= (I - I^*)\left[\alpha(E - E^*) - (\gamma + \mu)(I - I^*)\right] + \frac{\sigma_3^2}{2}I^2 \\
&\le \alpha(E - E^*)(I - I^*) - \left(\gamma + \mu - \sigma_3^2\right)(I - I^*)^2 + \sigma_3^2 I^{*2} \\
&\le \frac{\alpha}{2}(E - E^*)^2 - \left(\gamma + \mu - \sigma_3^2 - \frac{\alpha}{2}\right)(I - I^*)^2 + \sigma_3^2 I^{*2}.
\end{aligned}
\tag{4.6}
$$

Choosing $a = (\alpha r/(\rho_{SR} + \mu)N)E^*$, then

$$
\begin{aligned}
aV_1 + V_2 &= a\left[-\frac{(S - S^*)^2}{S}(\rho_{SR} + \mu) + \frac{1}{2}S^*\sigma_1^2 + \frac{1}{2}E^*\sigma_2^2\right] \\
&\quad + \left[\frac{\alpha r}{N}(E - E^*)(S - S^*) + \frac{\alpha r}{N}E^*\frac{(S - S^*)^2}{S} + \frac{1}{2}E^*\sigma_2^2\right] \\
&= \frac{\alpha r}{N}(E - E^*)(S - S^*) + \frac{1}{2}aS^*\sigma_1^2 + \frac{1}{2}E^*\sigma_2^2.
\end{aligned}
\tag{4.7}
$$

Choosing $c = 1/S^*$, then

$$\begin{aligned}
aV_1 &+ V_2 + V_3 + V_4 + V_5 \\
&\le \frac{\alpha r}{N}(E - E^*)(S - S^*) + \frac{1}{2}aS^*\sigma_1^2 + \frac{1}{2}(a+1)E^*\sigma_2^2 \\
&\quad + \left[\left(\sigma_1^2 - \rho_{SR} - \mu\right)(S - S^*)^2 + \left(\sigma_2^2 - \alpha - \rho_{SR} - \mu\right)(E - E^*)^2 + \sigma_1^2 S^{*2} + \sigma_2^2 E^{*2}\right] \\
&\quad + c\left[-\frac{\alpha r}{N}S^*(S - S^*)(E - E^*) - \left(\frac{\alpha r}{N}(\rho_{SR} + \mu) - \sigma_1^2\right)(S - S^*)^2 + \sigma_1^2 S^{*2}\right] \\
&\quad + \left[\frac{\alpha}{2}(E - E^*)^2 - \left(\gamma + \mu - \sigma_3^2 - \frac{\alpha}{2}\right)(I - I^*)^2 + \sigma_3^2 I^{*2}\right] \\
&= \frac{1}{2}a\sigma_1^2 S^* + \sigma_1^2 S^* + \sigma_1^2 S^{*2} + \frac{1}{2}(a+1)\sigma_2^2 E^* + \sigma_2^2 E^{*2} + \sigma_3^2 I^{*2} \\
&\quad + \left[\left(\sigma_1^2 - \rho_{SR} - \mu\right) - c\left(\frac{\alpha r}{N}(\rho_{SR} + \mu) - \sigma_1^2\right)\right](S - S^*)^2 \\
&\quad + \left(\sigma_2^2 - \frac{\alpha}{2} - \rho_{SR} - \mu\right)(E - E^*)^2 - \left(\gamma + \mu - \sigma_3^2 - \frac{\alpha}{2}\right)(I - I^*)^2 \\
&\doteq \delta - \left[\left(1 + \frac{\alpha r}{S^* N}\right)(\rho_{SR} + \mu) - \sigma_1^2\left(1 + \frac{1}{S^*}\right)\right](S - S^*)^2 \\
&\quad - \left(-\frac{\alpha}{2} + \rho_{SR} + \mu - \sigma_2^2\right)(E - E^*)^2 - \left(\gamma + \mu - \sigma_3^2 - \frac{\alpha}{2}\right)(I - I^*)^2.
\end{aligned} \tag{4.8}$$

Then the ellipsoid

$$\delta - \left[\left(1 + \frac{\alpha r}{S^* N}\right)(\rho_{SR} + \mu) - \sigma_1^2\left(1 + \frac{1}{S^*}\right)\right](S - S^*)^2 - \left(-\frac{\alpha}{2} + \rho_{SR} + \mu - \sigma_2^2\right)(E - E^*)^2 = 0 \tag{4.9}$$

lies entirely in R_+^3. We can take U to be a neighborhood of the ellipsoid with $U \subset R_+^3$, so, for $x \in U/R_+^3$, $LV \le -K$ (K is a positive constant), which implies that condition (B.2) in Lemma 4.1 is satisfied. Hence, the solution $X(t)$ is recurrent in the domain U, which, together with Lemma 4.2, implies that $X(t)$ is recurrent in any bounded domain $D \subset R_+^3$. Besides, for all D, there is an

$$M = \min\left\{\sigma_1^2 S^2, \sigma_2^2 E^2, \sigma_3^2 I^2 \in D\right\} > 0, \tag{4.10}$$

such that $\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 E^2 \xi_2^2 + \sigma_3^2 I^2 \xi_3^2 \ge M\|\xi^2\|$ for all $X \in D, \xi \in R^3$ which implies that condition (B.1) is also satisfied. Therefore, the stochastic system (1.3) has a stationary distribution $\mu(*)$ and it is ergodic. This completes the proof. □

5. Numerical Simulations

In this section, we have performed some numerical simulations to show the geometric impression of our results. To demonstrate the global stability of infection-free solution of

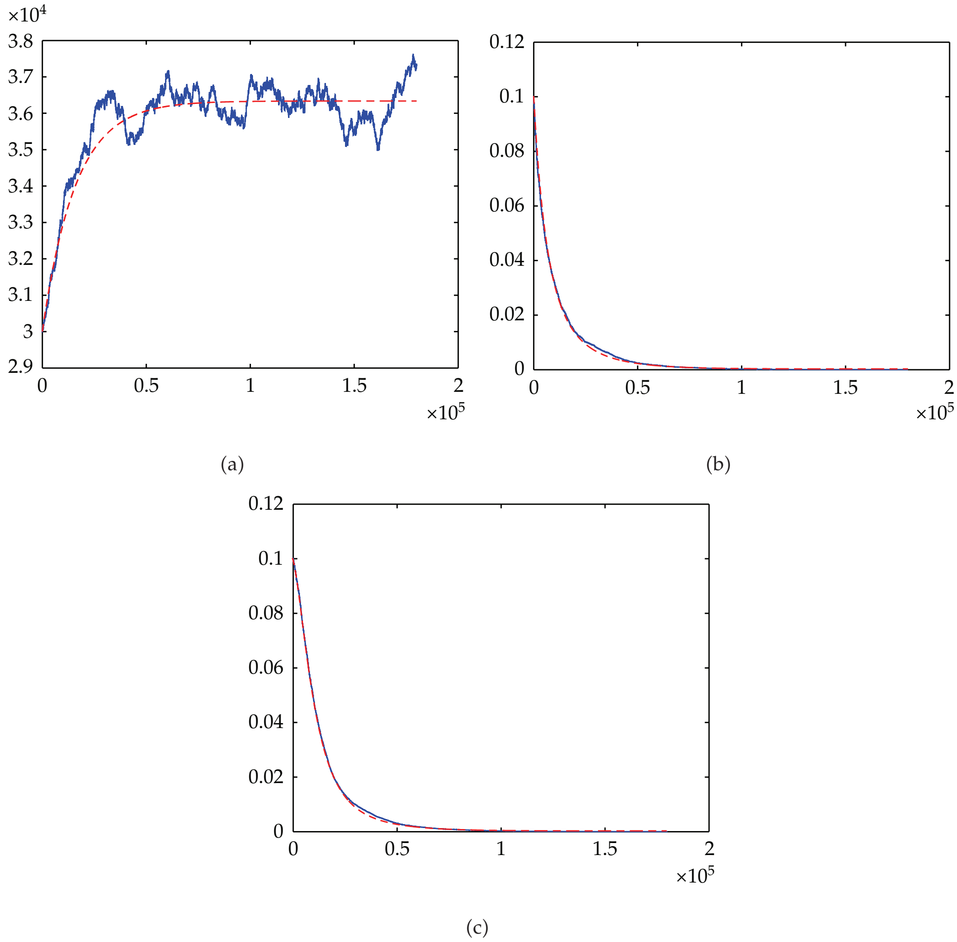

Figure 2: Deterministic and stochastic trajectories around infection-free solution.

system (1.3) we take following set parameter values: $\mu = 1/4380, N = 100000$, $\alpha = 1/500$, $r = 7$, $\rho_{SR} = 1/2500$, $\rho_{ER} = 1/300$, $\gamma = 1/500, \sigma_1^2 = 0.0006$, $\sigma_2^2 = 0.001$, $\sigma_3^2 = 0.002$. In this case, we have $R_0 = 0.9147 < 1$. In Figures 2(a), 2(b), and 2(c), we have displayed, respectively, the susceptible, infected and recovered computer of system (1.4) with initial conditions: $S(0) = 3$, $E(0) = 0.1$ and $I(0) = 0.1$.

To demonstrate the permanence of system (1.4), we take the following set parameter values: $\mu = 1/4380$, $N = 100000$, $\alpha = 1/500$, $r = 30$, $\rho_{SR} = 1/2500$, $\rho_{ER} = 1/300$, $\gamma = 1/500$, $\sigma_1^2 = 0.0006$, $\sigma_2^2 = 0.001$, $\sigma_3^2 = 0.002$. In this case, we have $R_1 = 3.9201 > 1$. In Figures 3(a), 3(b), and 3(c), we have displayed, respectively, the susceptible and infected population of system (1.4) with initial conditions: $S(0) = 15000, E(0) = 2000$ and $I(0) = 2000$.

6. Conclusion

In this paper, a stochastic computer virus spread model has been proposed and analyzed. First, we prove the existence and uniqueness of positive solutions. Then, by constructing

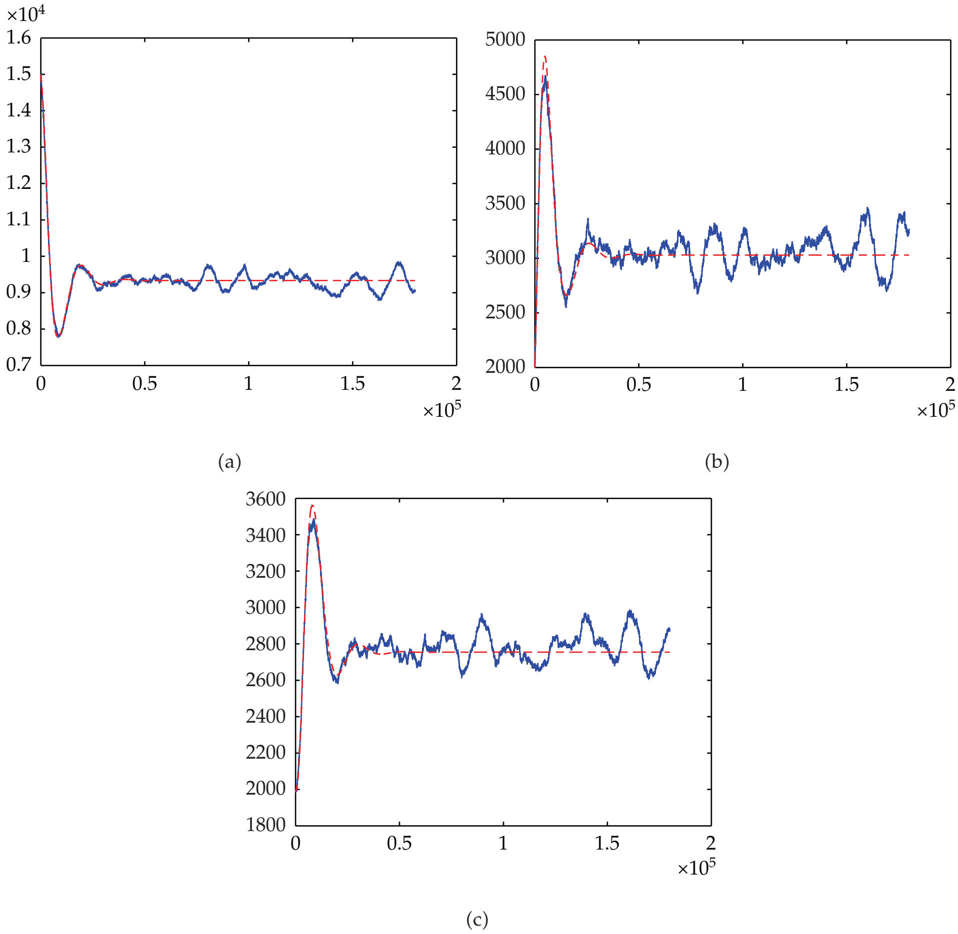

Figure 3: Deterministic and stochastic trajectories around virus endemic equilibrium.

Lyapunov functions and applying Ito's formula, the stability of the virus-free equilibrium and viral equilibrium is studied.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China (no. 61170320), the Natural Science Foundation of Guangdong Province (no. S2011040002981) and the Scientific Research Foundation of Guangdong Medical College (no. KY1048).

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