On the Stratifications of 2-Qubits X-State Space

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Abstract. The 7-dimensional family $\mathfrak{P}_X$ of the so-called mixed X-states of 2-qubits is considered. Two versions of stratifications of $\mathfrak{P}_X$, i.e., its decomposition into strata according to orbit types of the adjoint actions of two groups, are described. The first stratification is due to the action of global unitary group $G_X \subset SU(4)$, while the second one corresponds to the action of the local unitary group $LG_X \subset G_X$. The equations and inequalities in the invariants of the corresponding groups, determining each stratification component, are given.

1 Introduction

The understanding of a symmetry that a physical system possesses, as well as this symmetry breaking pattern allows us to explain uniquely a wide variety of phenomena in many areas of physics, including elementary particle physics and condensed matter physics [1]. The mathematical formulation of symmetries related to the Lie group action consists of the detection of the stratification of the representation space of the corresponding symmetry group. Dealing with closed quantum systems the symmetries are realized by the unitary group actions and the quantum state space plays the role of the symmetry group representation space. Below, having in mind these observations, we will outline examples of the stratifications occurring for a quantum system composed of a pair of 2-level systems, two qubits. We will analyze symmetries associated with two subgroups of the special unitary group $SU(4)$. More precisely, we will consider the 7-dimensional subspace $\mathfrak{P}_X$ of a generic 2-qubit state space, the family of X-states (for definition see [2], [3] and references therein) and reveal two types of its partition into sets of points having the same symmetry type. The primary stratification originates from the action of the invariance group of X-states, named the global unitary group $G_X \subset SU(4)$, whereas the secondary one is due to the action of the so-called local group $LG_X \subset G_X$ of the X-states.

2 X-states and their symmetries

The mixed 2-qubit X-states can be defined based on the purely algebraic consideration. The idea is to fix the subalgebra $\mathfrak{g}_X := \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \in \mathfrak{su}(4)$ of the algebra $\mathfrak{su}(4)$ and define the density matrix

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of the X-states as

$$\varrho_X = \frac{1}{4}\left(I + i\mathfrak{g}_X\right) . \tag{1}$$

In order to coordinatize the X-state space we use the tensorial basis for the $\mathfrak{su}(4)$ algebra, $\sigma_{\mu\nu} = \sigma_\mu \otimes \sigma_\nu,\ \mu,\nu = 0,1,2,3$. It consists of all possible tensor products of two copies of Pauli matrices and a unit 2×2 matrix, $\sigma_\mu = (I, \sigma_1, \sigma_2, \sigma_3)$, which we order as follows (see details in [3]):

$$\lambda_1, \dots, \lambda_{15} = \frac{i}{2}\left(\sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z}, \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\right) . \tag{2}$$

In this basis the 7-dimensional subalgebra $\mathfrak{g}_X$ is generated by the subset $\alpha_X = (\lambda_3, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, -\lambda_{11}, \lambda_{15})$, and thus the unit norm X-state density matrix is given by the decomposition:

$$\varrho_X = \frac{1}{4}\left(I + 2i \sum_{\lambda_k \in \alpha_X} h_k \lambda_k\right) . \tag{3}$$

The real coefficients h_k are subject to the polynomial inequalities ensuring the semi-positivity of the density matrix, $\varrho_X \geq 0$:

$$\mathfrak{P}_X = \left\{h_i \in \mathbb{R}^7 \mid (h_3 \pm h_6)^2 + (h_8 \pm h_{10})^2 + (h_7 \pm h_{11})^2 \leq (1 \pm h_{15})^2\right\} . \tag{4}$$

Using the definition (1) one can conclude that the $X-$state space $\mathfrak{P}_X$ is invariant under the 7-parameter group, $G_X := \exp(\mathfrak{g}_X) \in SU(4)$:

$$g\varrho_X g^\dagger \in \mathfrak{P}_X , \qquad \forall g \in G_X . \tag{5}$$

Group G_X plays the same role for the X-states as the special unitary group $SU(4)$ plays for a generic 4-level quantum system, and thus is termed the *global unitary group* of the $X-$states. According to [3], group G_X admits the representation:

$$G_X = P_\pi \left(\begin{array}{c|c} e^{-i\omega_{15}} SU(2) & 0 \\ \hline 0 & e^{i\omega_{15}} SU(2)' \end{array}\right) P_\pi , \quad \text{with} \quad P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{6}$$

Correspondingly, the *local unitary group* of the X-states is

$$LG_X = P_\pi \exp\left(\imath \frac{\phi_1}{2}\right) \otimes \exp\left(\imath \frac{\phi_2}{2}\right) P_\pi \ \subset G_X . \tag{7}$$

3 Global orbits and state space decomposition

Now we give a classification of the global G_X-orbits according to their dimensionality and isotropy group. Every density matrix ϱ_X can be diagonalized by some element of the global G_X group. In other words, all global G_X-orbits can be generated from the density matrices, whose eigenvalues form the partially ordered simplex $\underline{\Delta}_3$, depicted on Figure 1.

The tangent space to the G_X-orbits is spanned by the subset of linearly independent vectors, built from the vectors: $t_k = [\lambda_k, \varrho_X],\ \lambda_k \in \alpha_X$. The number of independent vectors t_k determines the dimensionality of the G_x-orbits and is given by the rank of the 7×7 Gram matrix:

$$\mathcal{G}_{kl} = \frac{1}{2} \operatorname{Tr}(t_k t_l) . \tag{8}$$

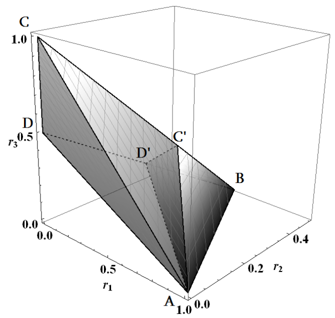

Figure 1. The tetrahedron $ABCD$ describes the partially ordered simplex $\underline{\Delta}_3 := \{ \sum_{i=1}^4 r_i = 1\,,\ \{1 \geq r_1 \geq r_2 \geq 0\} \cup \{1 \geq r_3 \geq r_4 \geq 0\}\}$ of the density matrix eigenvalues, while the tetrahedron $ABC'D'$ inside it corresponds to a 3D simplex with the following complete order: $\{ \sum_{i=1}^4 r_i = 1\,,\ 1 \geq r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0 \}$.

The Gram matrix (8) has three zero eigenvalues and two double multiplicity eigenvalues:

$$\mu_{\pm} \quad = \quad -\frac{1}{8}\left((h_3 \pm h_6)^2 + (h_8 \pm h_{10})^2 + (h_7 \pm h_{11})^2\right) . \tag{9}$$

Correspondingly, the G_X-orbits have dimensionality of either 4, 2 or 0. The orbits of maximal dimensionality, $\dim(O)_{\text{Gen}} = 4$, are characterized by non-vanishing $\mu_{\pm} \neq 0$ and consist of the set of density matrices with a generic spectrum, $\sigma(\varrho_X) = (r_1, r_2, r_3, r_4)$. If the density matrices obey the equations

$$h_6 = \pm h_3\,,\ h_{10} = \pm h_8\,,\ \ h_{11} = \pm h_7\,, \tag{10}$$

they belong to the so-called degenerate orbits, $\dim(O)_{\pm} = 2$. The latter are generated from matrices which have a double degenerate spectrum of the form, $\sigma(\varrho_X) = (p, p, r_3, r_4)$ and $\sigma(\varrho_X) = (r_1, r_2, q, q)$ respectively. Finally, there is a single orbit $\dim(O)_0 = 0$, corresponding to the maximally mixed state $\varrho_X = \frac{1}{4}I$.

Considering the diagonal representative of the generic G_X-orbit one can be convinced that its *isotropy group* is

$$H = P_\pi \left(\begin{array}{c|c} e^{i\omega} \exp i\frac{\gamma_1}{2}\sigma_3 & 0 \\ \hline 0 & e^{-i\omega} \exp i\frac{\gamma_2}{2}\sigma_3 \end{array} \right) P_\pi\,, \tag{11}$$

while for a diagonal representative with a double degenerate spectrum the isotropy group is given by one of two groups:

$$H_+ = P_\pi \left(\begin{array}{c|c} e^{i\omega} SU(2) & 0 \\ \hline 0 & e^{-i\omega} \exp i\frac{\gamma_2}{2}\sigma_3 \end{array} \right) P_\pi\,, \quad H_- = P_\pi \left(\begin{array}{c|c} e^{i\omega} \exp i\frac{\gamma_1}{2}\sigma_3 & 0 \\ \hline 0 & e^{-i\omega} SU(2)' \end{array} \right) P_\pi\,.$$

For the single zero dimensional orbit the isotropy group H_0 coincides with the whole invariance group, $H_0 = G_X$. Therefore, the isotropy group of any element of G_X-orbits belongs to one of these conjugacy classes: $[H]$, $[H_\pm]$ or $[H_0]$. Moreover, a straightforward analysis shows that $[H_+] = [H_-]$. Hence, any point $\varrho \in \mathfrak{P}_X$ belongs to one of three above-mentioned types of G_X-orbits[1], denoted afterwards as $[H_t]\,, t = 1, 2, 3$. For a given H_t, the associated *stratum* $\mathfrak{P}_{[H_t]}$, defined as the set of all points whose stabilizer is conjugate to H_t:

$$\mathfrak{P}_{[H_t]} := \{y \in \mathfrak{P}_X | \text{ isotropy group of } y \text{ is conjugate to } H_t\}$$

determines the sought-for decomposition of the state space $\mathfrak{P}_X$ into strata according to the orbit types:

$$\mathfrak{P}_X = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_i]}\,. \tag{12}$$

[1]The *orbit type* $[\varrho]$ of a point $\varrho \in \mathfrak{P}_X$ is given by the conjugacy class of the isotropy group of point ϱ, i.e., $[\varrho] = [G_{\varrho_X}]$.

The strata $\mathfrak{P}_{(H_i)}$ are determined by this set of equations and inequalities:

$$\text{(1)}\quad \mathfrak{P}_{[H]} := \{h_i \in \mathfrak{P}_X \mid \mu_+ > 0,\, \mu_- > 0\}\,, \tag{13}$$

$$\text{(2)}\quad \mathfrak{P}_{[H_+]} \cup \mathfrak{P}_{[H_-]} := \{h_i \in \mathfrak{P}_X \mid \mu_+ = 0,\, \mu_- > 0\} \cup \{h_i \in \mathfrak{P}_X \mid \mu_+ > 0,\, \mu_- = 0\}\,, \tag{14}$$

$$\text{(3)}\quad \mathfrak{P}_{[H_0]} := \{h_i \in \mathfrak{P}_X \mid \mu_+ = 0,\, \mu_- = 0\}\,. \tag{15}$$

4 Local orbits and state space decomposition

Analogously, one can build up the X-state space decomposition associated with the local group LG_x action. For this action the dimensionality of LG_x-orbits is given by the rank of the corresponding 2×2 Gram matrix constructed out of vectors t_3 and t_6. Since its eigenvalues read:

$$\mu_1 = -\frac{1}{8}\left((h_8 + h_{10})^2 + (h_7 + h_{11})^2\right), \qquad \mu_2 = -\frac{1}{8}\left((h_8 - h_{10})^2 + (h_7 - h_{11})^2\right), \tag{16}$$

the LG_X-orbits are either generic ones with the dimensionality of $\dim(O_L)_{\text{Gen}} = 2$, or degenerate $\dim(O_L)_\pm = 1$, or exceptional ones, $\dim(O_L)_0 = 0$. The LG_X-orbits can be collected into the strata according to their orbit type. There are three types of strata associated with the "local" isotropy subgroups of $LH \in LG_X$. Correspondingly, one can define the following "local" strata of state space:

- the generic stratum, $\mathfrak{P}^L_{[I]}$, which has a trivial isotropy type, $[I]$, and is represented by the inequalities: $\mathfrak{P}^L_{[I]} := \{h_i \in \mathfrak{P}_X \mid \mu_1 > 0,\, \mu_2 > 0\}$,
- the degenerate stratum, $\mathfrak{P}^L_{[H_L^\pm]}$, collection of the orbits whose type is $[H_L^\pm]$, with the subgroup either $H_L^+ = I \times \exp(iu\sigma_3)$, or $H_L^- = \exp(iv\sigma_3) \times I$. The stratum defining equations read respectively:

$$h_{10} = \pm h_8\,, \qquad h_{11} = \pm h_7\,, \tag{17}$$

- the exceptional stratum, $\mathfrak{P}_{[LG_X]}$ of the type $[LG_X]$, determined by the equations: $h_{11} = h_{10} = h_8 = h_7 = 0$.

Therefore, the local group action prescribes the following stratification of 2-qubit $X-$state space:

$$\mathfrak{P}_X = \mathfrak{P}_{[I]} \cup \mathfrak{P}_{[H_L^+]} \cup \mathfrak{P}_{[H_L^-]} \cup \mathfrak{P}_{[LG_X]}\,. \tag{18}$$

5 Concluding remarks

In the present article two stratifications of quantum state space associated with the global and local unitary symmetries of two qubits in the X-states are described. Stratifications encode information on system's physical characteristics. The first one, due to the global unitary symmetry, is related to the properties of a system as a whole, while the second stratification, according to the local symmetries, comprises information on the entanglement, cf. [4]. In an upcoming publication, based on the introduced stratifications of state space, we plan to analyze an interplay between these two symmetries and particularly to determine the entanglement/separability characteristics of every stratum.

References

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