Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 7023696, 12 pages
https://doi.org/10.1155/2018/7023696

Research Article

Near-Integrability of Low-Dimensional Periodic Klein-Gordon Lattices

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Received 30 September 2017; Accepted 24 December 2017; Published 22 January 2018

Academic Editor: Alkesh Punjabi

The low-dimensional periodic Klein-Gordon lattices are studied for integrability. We prove that the periodic lattice with two particles and certain nonlinear potential is nonintegrable. However, in the cases of up to six particles, we prove that their Birkhoff-Gustavson normal forms are integrable, which allows us to apply KAM theory in most cases.

1. Introduction

In this article we deal with the periodic Klein-Gordon (KG) lattice (see, e.g., [1] and references therein) described by the Hamiltonian

$$H = \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \left[\frac{p_j^2}{2} + \frac{C}{2}\left(q_{j+1} - q_j\right)^2 + V\left(q_j\right) \right], \qquad p_j = \dot{q}_j. \tag{1}$$

The constant $C > 0$ measures the interaction to nearest neighbor particles (with unit masses) and $V(x)$ is a nonlinear potential.

We study the integrability of (1). When $C = 0$ the Hamiltonian is separable and, hence, integrable. There exist plenty of periodic or quasi-periodic solutions in the dynamics of (1). It is natural to investigate whether this behavior persists for C small enough (see, e.g., [2]). Here we assume that C is neither very small nor too large.

We are interested in the behavior at low energy; that is why the following main assumptions are in order:

$$V(x) = \frac{a}{2}x^2 + \frac{b}{2}x^4, \tag{2}$$

$$a > 0 \quad \text{irrational}. \tag{3}$$

Remark 1. Such types of potentials are frequently used in the literature [1]. As it can be seen below the choice of a simplifies considerably the calculations.

We can also assume that $C = 1$ which can be achieved by rescaling of t. Then our Hamiltonian takes the form

$$H = \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \left[\frac{p_j^2}{2} + \frac{1}{2}\left(q_{j+1} - q_j\right)^2 + \frac{a}{2}\left(q_j\right)^2 + \frac{b}{2}\left(q_j\right)^4 \right], \qquad p_j = \dot{q}_j. \tag{4}$$

Our first result concerns the Hamiltonian with two degrees of freedom ($q_2 = q_0$); that is,

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(2q_1^2 - 4q_1q_2 + 2q_2^2\right) + \frac{a}{2}\left(q_1^2 + q_2^2\right) + \frac{b}{2}\left(q_1^4 + q_2^4\right). \tag{5}$$

It simply says that the corresponding Hamiltonian system is integrable only when it is linear.

Theorem 2. *The periodic KG lattice with $n = 2$ is nonintegrable unless $b = 0$.*

The above result suggests that generically the periodic KG lattice is nonintegrable for $n \geq 2$. Still there might be some special cases of integrability, while the opposite is not proven rigorously.

Motivated by the works of Rink and Verhulst [3, 4], who presented the periodic FPU chain as a perturbation of an integrable and KAM nondegenerated system, namely, the truncated Birkhoff-Gustavson normal form of order 4 in the neighborhood of an equilibrium, our aim is to verify whether this can be done for the low-dimensional KG lattices.

One should note that Rink and Verhulst's result is due to the special symmetry and resonance properties of the FPU chain and should not be expected for lower-order resonant Hamiltonian systems (see, e.g., [5]).

We summarize our second result concerning the integrability of truncated resonant normal forms of the periodic KG lattices up to six particles in the following.

Theorem 3. *The truncated normal form* $\overline{H} = H_2 + \overline{H}_4$ *of the periodic KG lattice is*

(i) *completely integrable and KAM nondegenerated for* $n = 2, 3, 4$;

(ii) *completely integrable and KAM nondegenerated when* $n = 5$ *for all a but one;*

(iii) *completely integrable for* $n = 6$.

As a consequence from this result, we may conclude for the low-dimensional KG lattices when KAM theory applies that there exist many quasi-periodic solutions of small energy on a long time scale (see Section 2 and for more detailed explanation [3]) and chaotic orbits are of small measure.

The paper is organized as follows. In Section 2 some notions and facts used in the paper are given. We discuss also the appearance of some additional resonant relations. In Section 3 we calculate the Birkhoff-Gustavson normal forms for the cases up to six particles and show that they are integrable in most cases. We finish with some concluding remarks as well as some possible lines of further study.

The proof of Theorem 2 is based on the Ziglin-Morales-Ramis theory and since it is more algebraic in nature, it is carried out in the Appendix.

2. Resonances and Normalization

In this section we recall briefly some notions and facts about integrability of Hamiltonian systems, action-angle variables, perturbation of integrable systems, and normal forms. More complete exposition can be found in [6–8].

Let H be an analytic Hamiltonian defined on a $2n$-dimensional symplectic manifold. The corresponding Hamiltonian system is

$$\dot{x} = X_H(x). \tag{6}$$

It is said that a Hamiltonian system is completely integrable if there exist n independent integrals $F_1 = H, F_2, \ldots, F_n$ in involution, namely, $\{F_i, F_j\} = 0$ for all i and j, where $\{,\}$ is the Poisson bracket. On a neighborhood U of the connected compact level sets of the integrals $M_c = \{F_j = c_j,\ j = 1, \ldots, n\}$ by Liouville-Arnold theorem one can introduce a special set of symplectic coordinates, I_j, φ_j, called action-angle variables. Then, the integrals $F_1 = H, F_2, \ldots, F_n$ are functions of action variables only and the flow of X_H is simple

$$\dot{I}_j = 0,$$
$$\dot{\varphi}_j = \frac{\partial H}{\partial I_j}, \tag{7}$$
$$j = 1, \ldots, n.$$

Therefore, near M_c, the phase space is foliated with X_{F_i} invariant tori over which the flow of X_H is quasi-periodic with frequencies $(\omega_1(I), \ldots, \omega_n(I)) = (\partial H/\partial I_1, \ldots, \partial H/\partial I_n)$.

The map

$$(I_1, I_2, \ldots, I_n) \longrightarrow \left(\frac{\partial H}{\partial I_1}, \frac{\partial H}{\partial I_2}, \ldots, \frac{\partial H}{\partial I_n}\right) \tag{8}$$

is called frequency map.

Consider a small perturbation of an integrable Hamiltonian H_0. According to Poincaré the main problem of mechanics is to study the perturbation of quasi-periodic motions in the system given by the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi), \quad \varepsilon \ll 1. \tag{9}$$

KAM theory [9–11] gives conditions on the integrable Hamiltonian H_0 which ensures the survival of the most of the invariant tori. The following condition, usually called Kolmogorov's condition, is that the frequency map should be a local diffeomorphism, or equivalently

$$\det\left(\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right) \neq 0 \tag{10}$$

on an open and dense subset of U. We should note that the measure of the surviving tori decreases with the increase of both perturbation and the measure of the set where above Hessian is too close to zero.

In a neighborhood of the equilibrium $(q, p) = (0, 0)$ we have the following expansion of H:

$$H = H_2 + H_3 + H_4 + \cdots,$$
$$H_2 = \sum \omega_j \left(q_j^2 + p_j^2\right), \quad \omega_j > 0. \tag{11}$$

We assume that H_2 is a positively defined quadratic form. The frequency $\omega = (\omega_1, \ldots, \omega_n)$ is said to be in resonance if there exists a vector $k = (k_1, \ldots, k_n)$, $k_j \in \mathbb{Z}$, $j = 1, \ldots, n$, such that $(\omega, k) = \sum k_j \omega_j = 0$, where $|k| = \sum |k_j|$ is the order of resonance.

With the help of a series of canonical transformations close to the identity, H simplifies. In the absence of resonances the simplified Hamiltonian is called Birkhoff normal form and, otherwise, Birkhoff-Gustavson normal form, which may contain combinations of angles arising from resonances.

Often to detect the behavior in a small neighborhood of the equilibrium, instead of the Hamiltonian H one considers the normal form truncated to some order

$$\overline{H} = H_2 + \cdots + \overline{H}_m. \tag{12}$$

TABLE 1

n	Ω_k
2	$\sqrt{a+4},\ \sqrt{a}$
3	$\sqrt{a+3},\ \sqrt{a+3},\ \sqrt{a}$
4	$\sqrt{a+2},\ \sqrt{a+4},\ \sqrt{a+2},\ \sqrt{a}$
5	$\sqrt{a+(5-\sqrt{5})/2},\ \sqrt{a+(5+\sqrt{5})/2},\ \sqrt{a+(5+\sqrt{5})/2},\ \sqrt{a+(5-\sqrt{5})/2},\ \sqrt{a}$
6	$\sqrt{a+1},\ \sqrt{a+3},\ \sqrt{a+4},\ \sqrt{a+3},\ \sqrt{a+1},\ \sqrt{a}$

It is known that the truncated to any order Birkhoff normal form is integrable [8]. The truncated Birkhoff-Gustavson normal form has at least two integrals—H_2 and $\overline{H}$. Therefore, the truncated normal form of two-degree-of-freedom Hamiltonian is integrable.

In order to obtain estimates of the approximation by normalization in a neighborhood of an equilibrium point we scale $q \to \varepsilon\tilde{q},\ p \to \varepsilon\tilde{p}$. Here ε is a small positive parameter and ε^2 is a measure for the energy relative to the equilibrium energy. Then, dividing by ε^2 and removing tildes we get

$$\overline{H} = H_2 + \varepsilon\overline{H}_3 + \cdots + \varepsilon^{m-2}\overline{H}_m. \tag{13}$$

Provided that $\omega_j > 0$ it is proven in [12] that $\overline{H}$ is an integral for the original system with error $O(\varepsilon^{m-1})$ and H_2 is an integral for the original system with error $O(\varepsilon)$ for the whole time interval. If we have more independent integrals, then they are integrals for the original Hamiltonian system with error $O(\varepsilon^{m-2})$ on the time scale $1/\varepsilon$.

The first integrals for the normal form $\overline{H}$ are approximate integrals for the original system; that is, if the normal form is integrable, then the original system is *near integrable* in the above sense.

Returning to the Hamiltonian of the periodic KG lattice (4) we see that its quadratic part H_2 is not in diagonal form:

$$H_2 = \frac{1}{2}p^T p + \frac{1}{2}q^T L_n q. \tag{14}$$

Here L_n is the following $n \times n$ matrix:

$$L_n := \begin{pmatrix} 2+a & -1 & & & -1 \\ -1 & 2+a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+a & -1 \\ -1 & & & -1 & 2+a \end{pmatrix}. \tag{15}$$

The eigenvalues of L_n are of the form $\Omega_k^2 = a + \omega_k^2$, $\omega_k = 2\sin(k\pi/n)$. In order to obtain the corresponding eigenvectors y^k, following [3] we define

$$y^n := \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^T \tag{16}$$

and if n is even,

$$y^{n/2} := \frac{1}{\sqrt{n}}(1, -1, 1, -1, \dots, -1)^T. \tag{17}$$

Further, for $1 \le k < n/2$, we define y^k and y^{n-k} via their coordinates

$$\begin{aligned} y_j^k &:= \sqrt{\frac{2}{n}}\cos\left(\frac{2kj\pi}{n}\right), \\ y_j^{n-k} &:= \sqrt{\frac{2}{n}}\sin\left(\frac{2kj\pi}{n}\right). \end{aligned} \tag{18}$$

It is easily checked that $\{y^1, \dots, y^n\}$ is an orthonormal basis of $\mathbb{R}^n$, consisting of eigenvectors of L_n. Let Y be the $n \times n$ matrix formed by the vectors y^k as columns; then $Y^T Y = Id$, $Y^{-1}L_n Y = \Omega := \operatorname{diag}(\Omega_1^2, \dots, \Omega_n^2)$. The symplectic Fourier-transformation $q = Y\overline{q}$, $p = Y\overline{p}$ brings H_2 in diagonal form

$$H_2 = \frac{1}{2}p^T p + \frac{1}{2}q^T \Omega q. \tag{19}$$

The variables $(\overline{q}, \overline{p})$ are known as *phonons*.

We need one more definition.

Definition 4 (see [3]). It is said that $\omega \in \mathbb{R}^n$ satisfies the property of *internal resonance* if for any $k \in \mathbb{Z}^n$ with $(k, \omega) = 0$, $|k| = 4$, we have $k_j = -k_{n-j}$ when $1 \le j < n/2$.

In Table 1 we list the frequencies Ω_k of some low-dimensional periodic KG lattice.

As it is seen from the table we almost always have internal resonances. In general, $\Omega_k = \Omega_j$ holds if only if $j = n - k$. There are plenty of resonances when $a \in \mathbb{Q}$.

Nevertheless, the assumption on a (3) does not prevent the appearance of more complicated resonances. Indeed, in the case of 4 particles there exists unique $a > 0$ irrational, for which the following resonant relations are fulfilled ($\Omega_1 = \Omega_3$):

$$\begin{aligned} \Omega_2 - \Omega_3 - \Omega_4 &= 0, \\ \Omega_2 - \Omega_1 - \Omega_4 &= 0. \end{aligned} \tag{20}$$

There are many like them and in particular for $n = 5$ and $n = 6$. Luckily, these resonances are of no importance for our considerations, because they produce third-order resonant terms and none of them appears due to the assumption on $V(x)$ (2).

The resonances we have to worry about are the forth-order ones. Trivial inspection shows that for a in the domain (3), there are no such resonances when $n = 2, 3, 4, 6$. However, there are some of that kind when $n = 5$ ($\Omega_1 = \Omega_4$, $\Omega_2 = \Omega_3$); namely,

(1) for $a_1 = (5 - \sqrt{5})/16$ the following resonant relations are fulfilled:

$$\Omega_1 - 3\Omega_5 = 0,\quad \Omega_4 - 3\Omega_5 = 0; \tag{21}$$

(2) for $a_2 = (5 + \sqrt{5})/16$ the following resonant relations are fulfilled:

$$\Omega_2 - 3\Omega_5 = 0,\quad \Omega_3 - 3\Omega_5 = 0, \tag{22}$$

(3) for $a_3 = 5(3\sqrt{5} - 5)/16$ the following resonant relations are fulfilled:

$$2\Omega_1 - \Omega_2 - \Omega_5 = 0,\quad 2\Omega_4 - \Omega_2 - \Omega_5 = 0, \tag{23}$$

$$2\Omega_1 - \Omega_3 - \Omega_5 = 0,\quad 2\Omega_4 - \Omega_3 - \Omega_5 = 0. \tag{24}$$

We will see how these additional resonances affect $\overline{H}_4$ in the end of Section 3.4.

3. Low-Dimensional Lattices

In this section we calculate the normal forms for the periodic KG lattices with particles up to six and in this way we prove Theorem 3. We do not use ε in the fourth-degree expression, but keep in mind that we are close to the equilibrium. Of course, it is assumed that $b \neq 0$.

3.1. Two Particles. This case is easy. It is well known that the truncated to any order normal form of a two-degree-of-freedom Hamiltonian is integrable. It remains only to verify the KAM condition.

We have already brought the quadratic part of the Hamiltonian in diagonal form. It is important that it is written in the phonons $(\overline{q}, \overline{p})$.

$$H = \frac{1}{2}\left(\overline{p}_1^2 + \overline{p}_2^2\right) + \frac{1}{2}\left[(4 + a)\,\overline{q}_1^2 + a\overline{q}_2^2\right] + \frac{b}{4}\left(\overline{q}_1^4 + 6\overline{q}_1^2\overline{q}_2^2 + \overline{q}_2^4\right). \tag{25}$$

Further, we perform a scaling

$$\overline{q}_1 \longrightarrow \frac{1}{\sqrt[4]{a+4}}\overline{q}_1,\quad \overline{p}_1 \longrightarrow \sqrt[4]{a+4}\,\overline{p}_1,\quad \overline{q}_2 \longrightarrow \frac{1}{\sqrt[4]{a}}\overline{q}_2,\quad \overline{p}_2 \longrightarrow \sqrt[4]{a}\,\overline{p}_2, \tag{26}$$

which preserves the symplectic form. The Hamiltonian (25) becomes

$$H = \frac{\sqrt{a+4}}{2}\left(\overline{p}_1^2 + \overline{q}_1^2\right) + \frac{\sqrt{a}}{2}\left(\overline{p}_2^2 + \overline{q}_2^2\right) + \frac{b}{4}\left(\frac{\overline{q}_1^4}{a+4} + \frac{6\overline{q}_1^2\overline{q}_2^2}{\sqrt{a\,(a+4)}} + \frac{\overline{q}_2^4}{a}\right). \tag{27}$$

Usually at this place one makes the following change of variables:

$$\overline{q}_j = \frac{1}{2}\left(z_j + w_j\right),\quad \overline{p}_j = \frac{1}{2i}\left(z_j - w_j\right),\quad j = 1, 2. \tag{28}$$

Since the frequencies $\Omega_1 = \sqrt{a+4}$, $\Omega_2 = \sqrt{a}$ are incommensurable, the only resonant terms which remain are

$$z_j w_j,\ (z_j w_j)^2,\ j = 1, 2,\ z_1 w_1 z_2 w_2.$$

The other terms can be removed via symplectic near-identity change. Therefore, the normal form of (25) up to order four $\overline{H} = H_2 + \overline{H}_4$ is

$$\overline{H} = \frac{\sqrt{a+4}}{2} z_1 w_1 + \frac{\sqrt{a}}{2} z_2 w_2 + \frac{3b}{32}\left[\frac{(z_1 w_1)^2}{a+4} + 4\frac{z_1 w_1 z_2 w_2}{\sqrt{a\,(a+4)}} + \frac{(z_2 w_2)^2}{a}\right] \tag{29}$$

or, returning to the $(\overline{q}, \overline{p})$ coordinates, we get

$$\overline{H} = \frac{\sqrt{a+4}}{2}\left(\overline{p}_1^2 + \overline{q}_1^2\right) + \frac{\sqrt{a}}{2}\left(\overline{p}_2^2 + \overline{q}_2^2\right) + \frac{3b}{32}\left[\frac{\left(\overline{p}_1^2 + \overline{q}_1^2\right)^2}{a+4} + 4\frac{\left(\overline{p}_1^2 + \overline{q}_1^2\right)\left(\overline{p}_2^2 + \overline{q}_2^2\right)}{\sqrt{a\,(a+4)}} + \frac{\left(\overline{p}_2^2 + \overline{q}_2^2\right)^2}{a}\right]. \tag{30}$$

This normal form is clearly integrable with quadratic first integrals $I_j = \overline{p}_j^2 + \overline{q}_j^2$, $j = 1, 2$.

Finally, introducing symplectic polar coordinates, which are action-angle variables

$$\overline{q}_j = \sqrt{2I_j}\cos\varphi_j,\quad \overline{p}_j = \sqrt{2I_j}\sin\varphi_j,\quad j = 1, 2 \tag{31}$$

we get

$$\overline{H} = \sqrt{a+4}I_1 + \sqrt{a}I_2 + \frac{3b}{8}\left[\frac{I_1^2}{a+4} + 4\frac{I_1 I_2}{\sqrt{a(a+4)}} + \frac{I_2^2}{a}\right]. \quad (32)$$

It can easily be checked that Kolmogorov's condition is valid.

3.2. Three Particles. First, we make use of the phonons $(\overline{q}, \overline{p})$ (as explained in Section 2) to transform the quadratic part of the Hamiltonian (4) $n = 3$ in diagonal form

$$\begin{aligned} H = {} & \frac{1}{2}\left(\overline{p}_1^2 + \overline{p}_2^2 + \overline{p}_3^2\right) + \frac{a+3}{2}\left(\overline{q}_1^2 + \overline{q}_2^2\right) + \frac{a}{2}\overline{q}_3^2 \\ & + \frac{b}{18}\left[\frac{9}{2}\left(\overline{q}_1^4 + \overline{q}_2^4\right) + 3\overline{q}_3^4 + 6\sqrt{2}\overline{q}_1^3\overline{q}_3 + 9\overline{q}_1^2\overline{q}_2^2\right. \\ & \left. + 18\left(\overline{q}_1^2 + \overline{q}_2^2\right)\overline{q}_3^2 - 18\sqrt{2}\overline{q}_1\overline{q}_2^2\overline{q}_3\right]. \end{aligned} \quad (33)$$

Further, the scaling

$$\begin{aligned} \overline{q}_{1,2} &\longrightarrow \frac{1}{\sqrt[4]{a+3}}\overline{q}_{1,2}, \\ \overline{p}_{1,2} &\longrightarrow \sqrt[4]{a+3}\,\overline{p}_{1,2}, \\ \overline{q}_3 &\longrightarrow \frac{1}{\sqrt[4]{a}}\overline{q}_3, \\ \overline{p}_3 &\longrightarrow \sqrt[4]{a}\,\overline{p}_3 \end{aligned} \quad (34)$$

results in

$$\begin{aligned} H = {} & \frac{\sqrt{a+3}}{2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_2^2 + \overline{q}_2^2\right) + \frac{\sqrt{a}}{2}\left(\overline{p}_3^2 + \overline{q}_3^2\right) \\ & + \frac{b}{18}\left[\frac{9\left(\overline{q}_1^4 + \overline{q}_2^4\right)}{2(a+3)} + \frac{3\overline{q}_3^4}{a} + \frac{6\sqrt{2}\overline{q}_1^3\overline{q}_3}{\sqrt[4]{a(a+3)^3}} + \frac{9\overline{q}_1^2\overline{q}_2^2}{a+3}\right. \\ & \left. + \frac{18\left(\overline{q}_1^2 + \overline{q}_2^2\right)\overline{q}_3^2}{\sqrt{a(a+3)}} - \frac{18\sqrt{2}\overline{q}_1\overline{q}_2^2\overline{q}_3}{\sqrt[4]{a(a+3)^3}}\right]. \end{aligned} \quad (35)$$

Passing to the variables (z_j, w_j), $j = 1, 2, 3$ (28), we notice that there is an internal resonance between the frequencies Ω_1 and Ω_2. Therefore, the generators of the normal form are

$$z_j w_j, \; j = 1, 2, 3, \text{ and } z_1 w_2, z_2 w_1.$$

After removing the nonresonant terms, the normal form of the (33) up to order four $\overline{H} = H_2 + \overline{H}_4$ is

$$\begin{aligned} \overline{H} = {} & \frac{\sqrt{a+3}}{2}\left(z_1 w_1 + z_2 w_2\right) + \frac{\sqrt{a}}{2} z_3 w_3 \\ & + \frac{b}{18}\left[\frac{27\left(z_1 w_1 + z_2 w_2\right)^2}{16(a+3)} + \frac{9\left(z_3 w_3\right)^2}{8a}\right. \\ & \left. + \frac{9\left(z_1 w_1 + z_2 w_2\right) z_3 w_3}{2\sqrt{a(a+3)}} + \frac{9\left(z_1 w_2 - z_2 w_1\right)^2}{16(a+3)}\right], \end{aligned} \quad (36)$$

or, returning to $(\overline{q}, \overline{p})$, we get

$$\begin{aligned} \overline{H} = {} & \frac{\sqrt{a+3}}{2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_2^2 + \overline{q}_2^2\right) + \frac{\sqrt{a}}{2}\left(\overline{p}_3^2 + \overline{q}_3^2\right) \\ & + \frac{b}{2}\left[\frac{3\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_2^2 + \overline{q}_2^2\right)^2}{16(a+3)} + \frac{\left(\overline{p}_3^2 + \overline{q}_3^2\right)^2}{8a}\right. \\ & + \frac{\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_2^2 + \overline{q}_2^2\right)\left(\overline{p}_3^2 + \overline{q}_3^2\right)}{2\sqrt{a(a+3)}} \\ & \left. - \frac{\left(\overline{p}_1\overline{q}_2 - \overline{q}_1\overline{p}_2\right)^2}{4(a+3)}\right]. \end{aligned} \quad (37)$$

This normal form is integrable with the following quadratic first integrals:

$$\begin{aligned} F_1 &\coloneqq \overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_2^2 + \overline{q}_2^2, \\ G_1 &\coloneqq \overline{p}_1\overline{q}_2 - \overline{q}_1\overline{p}_2, \\ I_3 &\coloneqq \overline{p}_3^2 + \overline{q}_3^2. \end{aligned} \quad (38)$$

In order to introduce action-angle variables, we need to find the set of regular values of the energy momentum map

$$\mathrm{EM} : (\overline{q}, \overline{p}) \longrightarrow (F_1, G_1, I_3). \quad (39)$$

In fact, this is done in [4]. Denote by $U_r = \{(F_1, G_1, I_3) \in \mathbb{R}^3,\ F_1 > 0,\ |G_1| < F_1,\ I_3 > 0\}$. Then for all $(F_1, G_1, I_3) \in U_r$ the level sets of $\mathrm{EM}^{-1}(F_1, G_1, I_3)$ are diffeomorphic to 3-tori.

Let $\arg : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}/2\pi\mathbb{Z}$ be the argument function $\arg(r\cos\Phi, r\sin\Phi) \to \Phi$. Define the following set of variables $(F_1, G_1, I_3, \phi_1, \psi_1, \varphi_3)$ F_1, G_1, I_3 as above and

$$\begin{aligned} \phi_1 &\coloneqq \frac{1}{2}\arg\left(-\overline{p}_2 - \overline{q}_1, \overline{p}_1 - \overline{q}_2\right) + \frac{1}{2}\arg\left(\overline{p}_2 - \overline{q}_1, \overline{p}_1 + \overline{q}_2\right), \\ \psi_1 &\coloneqq \frac{1}{2}\arg\left(-\overline{p}_2 - \overline{q}_1, \overline{p}_1 - \overline{q}_2\right) - \frac{1}{2}\arg\left(\overline{p}_2 - \overline{q}_1, \overline{p}_1 + \overline{q}_2\right), \\ \varphi_3 &\coloneqq \arctan\frac{\overline{p}_3}{\overline{q}_3}. \end{aligned} \quad (40)$$

Using the formula $d\arg(x, y) = (x\,dy - y\,dx)/(x^2 + y^2)$, one can verify that $(F_1, G_1, I_3, \phi_1, \psi_1, \varphi_3)$ are indeed canonical coordinates $\sum d\overline{p}_j \wedge \overline{q}_j = dF_1 \wedge d\phi_1 + dG_1 \wedge d\psi_1 + dI_3 \wedge d\varphi_3$.

Then the truncated up to order 4 normal form as a function of actions is

$$\overline{H} = \frac{\sqrt{a+3}}{2}F_1 + \frac{\sqrt{a}}{2}I_3 + \frac{b}{2}\left[\frac{3F_1^2}{16(a+3)} + \frac{F_1 I_3}{2\sqrt{a(a+3)}} - \frac{G_1^2}{4(a+3)} + \frac{I_3^2}{8a}\right]. \quad (41)$$

Then one can easily check up that Kolmogorov's condition is valid.

Remark 5. If we set $a = 1$, we get

$$\begin{aligned}\overline{H} &= \frac{2}{2}\left(\overline{p}_1^2+\overline{q}_1^2+\overline{p}_2^2+\overline{q}_2^2\right)+\frac{1}{2}\left(\overline{p}_3^2+\overline{q}_3^2\right)\\&+\frac{b}{2}\left[\frac{3\left(\overline{p}_1^2+\overline{q}_1^2+\overline{p}_2^2+\overline{q}_2^2\right)^2}{64}+\frac{\left(\overline{p}_3^2+\overline{q}_3^2\right)^2}{8}\right.\\&+\frac{\left(\overline{p}_1^2+\overline{q}_1^2+\overline{p}_2^2+\overline{q}_2^2\right)\left(\overline{p}_3^2+\overline{q}_3^2\right)}{4}\\&\left.-\frac{\left(\overline{p}_1\overline{q}_2-\overline{q}_1\overline{p}_2\right)^2}{16}\right],\end{aligned} \tag{42}$$

which is an example of an integrable KAM nondegenerate normal form of 2 : 2 : 1 (or 1 : 2 : 2) Hamiltonian resonance (see, e.g., [13]).

3.3. Four Particles. It turns out that the normal form of the periodic KG lattice in the case of four particles is surprisingly simple, no matter the internal resonance.

After transforming the quadratic part in diagonal form and scaling

$$\begin{aligned}\overline{q}_{1,3} &\longrightarrow \frac{1}{\sqrt[4]{a+2}}\overline{q}_{1,3},\\ \overline{p}_{1,3} &\longrightarrow \sqrt[4]{a+2}\,\overline{p}_{1,3},\\ \overline{q}_2 &\longrightarrow \frac{1}{\sqrt[4]{a+4}}\overline{q}_2,\\ \overline{p}_2 &\longrightarrow \sqrt[4]{a+4}\,\overline{p}_2,\\ \overline{q}_4 &\longrightarrow \frac{1}{\sqrt[4]{a}}\overline{q}_4,\\ \overline{p}_4 &\longrightarrow \sqrt[4]{a}\,\overline{p}_4\end{aligned} \tag{43}$$

the Hamiltonian (4) $n = 4$ takes the form

$$\begin{aligned}H &= \frac{\sqrt{a+2}}{2}\left(\overline{p}_1^2+\overline{q}_1^2\right)+\frac{\sqrt{a+4}}{2}\left(\overline{p}_2^2+\overline{q}_2^2\right)\\&+\frac{\sqrt{a+2}}{2}\left(\overline{p}_3^2+\overline{q}_3^2\right)+\frac{\sqrt{a}}{2}\left(\overline{p}_4^2+\overline{q}_4^2\right)\\&+\frac{b}{8}\left[\frac{2\left(\overline{q}_1^4+\overline{q}_3^4\right)}{(a+2)}+\frac{\overline{q}_2^4}{a+4}+\frac{\overline{q}_4^4}{a}\right.\\&+\frac{12\overline{q}_2\overline{q}_3^2\overline{q}_4}{\sqrt[4]{a(a+4)(a+2)^2}}-\frac{12\overline{q}_1^2\overline{q}_2\overline{q}_4}{\sqrt[4]{a(a+4)(a+2)^2}}\\&+\frac{6\overline{q}_2^2\left(\overline{q}_1^2+\overline{q}_3^2\right)}{\sqrt{(a+2)(a+4)}}+\frac{6\overline{q}_4^2\left(\overline{q}_1^2+\overline{q}_3^2\right)}{\sqrt{a(a+2)}}\\&\left.+\frac{6\overline{q}_2^2\overline{q}_4^2}{\sqrt{a(a+4)}}\right].\end{aligned} \tag{44}$$

There is an internal resonance between the frequencies Ω_1 and Ω_3. In variables (z_j, w_j), $j = 1, 2, 3, 4$, the generators of the normal form are

$$z_j w_j, \text{ and } z_1 w_3, z_3 w_1.$$

However, as it is seen from (44), the variables $\overline{q}_1$ and $\overline{q}_3$ do not couple, so the last two generators do not appear in the normal form as if there are no resonances. After removing the nonresonant terms, the normal form of (44) up to order four $\overline{H} = H_2 + \overline{H}_4$ reads

$$\begin{aligned}\overline{H} &= \frac{\sqrt{a+2}}{2}z_1w_1+\frac{\sqrt{a+4}}{2}z_2w_2+\frac{\sqrt{a+2}}{2}z_3w_3\\&+\frac{\sqrt{a}}{2}z_4w_4+\frac{b}{8}\left[\frac{3\left(z_1w_1\right)^2}{4(a+2)}+\frac{3\left(z_2w_2\right)^2}{8(a+4)}\right.\\&+\frac{3\left(z_3w_3\right)^2}{4(a+2)}+\frac{3\left(z_4w_4\right)^2}{8a}+\frac{3z_2w_2\left(z_1w_1+z_3w_3\right)}{\sqrt{2(a+2)(a+4)}}\\&\left.+\frac{3z_4w_4\left(z_1w_1+z_3w_3\right)}{\sqrt{2a(a+2)}}+\frac{3z_2w_2z_4w_4}{\sqrt{2a(a+4)}}\right],\end{aligned} \tag{45}$$

or, in $(\overline{q}, \overline{p})$ variables,

$$\begin{aligned}\overline{H} &= \frac{\sqrt{a+2}}{2}\left(\overline{p}_1^2+\overline{q}_1^2\right)+\frac{\sqrt{a+4}}{2}\left(\overline{p}_2^2+\overline{q}_2^2\right)\\&+\frac{\sqrt{a+2}}{2}\left(\overline{p}_3^2+\overline{q}_3^2\right)+\frac{\sqrt{a}}{2}\left(\overline{p}_4^2+\overline{q}_4^2\right)\\&+\frac{b}{8}\left[\frac{3\left(\overline{p}_1^2+\overline{q}_1^2\right)^2}{4(a+2)}+\frac{3\left(\overline{p}_2^2+\overline{q}_2^2\right)^2}{8(a+4)}+\frac{3\left(\overline{p}_3^2+\overline{q}_3^2\right)^2}{4(a+2)}\right.\\&+\frac{3\left(\overline{p}_4^2+\overline{q}_4^2\right)^2}{8a}+\frac{3\left(\overline{p}_2^2+\overline{q}_2^2\right)\left(\overline{p}_1^2+\overline{q}_1^2+\overline{p}_3^2+\overline{q}_3^2\right)}{\sqrt{2(a+2)(a+4)}}\\&+\frac{3\left(\overline{p}_4^2+\overline{q}_4^2\right)\left(\overline{p}_1^2+\overline{q}_1^2+\overline{p}_3^2+\overline{q}_3^2\right)}{\sqrt{2a(a+2)}}\\&\left.+\frac{3\left(\overline{p}_2^2+\overline{q}_2^2\right)\left(\overline{p}_4^2+\overline{q}_4^2\right)}{\sqrt{2a(a+4)}}\right].\end{aligned} \tag{46}$$

This normal form is integrable with integrals $I_j = \overline{p}_j^2 + \overline{q}_j^2$, $j = 1, 2, 3, 4$ and the action-angle variables are clear. The truncated normal form up to order 4 as a function of the action variables is

$$\begin{aligned}\overline{H} = {} & \frac{\sqrt{a+2}}{2} I_1 + \frac{\sqrt{a+4}}{2} I_2 + \frac{\sqrt{a+2}}{2} I_3 + \frac{\sqrt{a}}{2} I_4 \\ & + \frac{b}{8}\left[\frac{3\left(I_1^2+I_3^2\right)}{4(a+2)} + \frac{3I_2^2}{8(a+4)} + \frac{3I_4^2}{8a}\right. \\ & + \frac{3I_2\left(I_1+I_3\right)}{\sqrt{2(a+2)(a+4)}} + \frac{3I_4\left(I_1+I_3\right)}{\sqrt{2a(a+2)}} \\ & \left. + \frac{3I_2 I_4}{\sqrt{2a(a+4)}}\right],\end{aligned} \tag{47}$$

and the verification of the validity of the Kolmogorov's condition is straightforward.

3.4. Five Particles. There is a close resemblance between the two cases of tree and five particles.

We transform the quadratic part of the Hamiltonian (4) $n = 5$ using the phonons $(\overline{q}, \overline{p})$ and scale in the usual way. Further, we use the notations Ω_j for short. Recall that

$$\begin{aligned}\Omega_1 &= \Omega_4 = \sqrt{a + \frac{\left(5-\sqrt{5}\right)}{2}}, \\ \Omega_2 &= \Omega_3 = \sqrt{a + \frac{\left(5+\sqrt{5}\right)}{2}}, \\ \Omega_5 &= \sqrt{a};\end{aligned} \tag{48}$$

that is, Ω contains two internal resonances. The generators of the normal form in (z, w) variables are

$$z_j w_j,\ j = 1, \ldots, 5, \text{ and } z_1 w_4, z_4 w_1, z_2 w_3, z_3 w_2.$$

Then the truncated up to order 4 normal form $\overline{H} = H_2 + \overline{H}_4$ for (4) $n = 5$ reads

$$\begin{aligned}\overline{H} = {} & \frac{\Omega_1}{2}\left(z_1 w_1 + z_4 w_4\right) + \frac{\Omega_2}{2}\left(z_2 w_2 + z_3 w_3\right) + \frac{\Omega_5}{2} \\ & \cdot z_5 w_5 + \frac{b}{10}\left[\frac{9}{16\Omega_1^2}\left(z_1 w_1 + z_4 w_4\right)^2\right. \\ & + \frac{9}{16\Omega_2^2}\left(z_2 w_2 + z_3 w_3\right)^2 + \frac{3}{8\Omega_5^2}\left(z_5 w_5\right)^2 \\ & + \frac{3}{16\Omega_1^2}\left(z_1 w_4 - z_4 w_1\right)^2 + \frac{3}{16\Omega_2^2}\left(z_2 w_3 - z_3 w_2\right)^2 \\ & + \frac{3}{2\Omega_1\Omega_2}\left(z_1 w_1 + z_4 w_4\right)\left(z_2 w_2 + z_3 w_3\right) \\ & + \frac{3}{2\Omega_1\Omega_5}\left(z_1 w_1 + z_4 w_4\right) z_5 w_5 \\ & \left. + \frac{3}{2\Omega_2\Omega_5}\left(z_2 w_2 + z_3 w_3\right) z_5 w_5\right],\end{aligned} \tag{49}$$

or, written in the coordinates $(\overline{q}, \overline{p})$, it is

$$\begin{aligned}\overline{H} = {} & \frac{\Omega_1}{2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right) + \frac{\Omega_2}{2}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2\right. \\ & \left. + \overline{q}_3^2\right) + \frac{\Omega_5}{2}\left(\overline{p}_5^2 + \overline{q}_5^2\right) \\ & + \frac{b}{10}\left[\frac{9}{16\Omega_1^2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)^2\right. \\ & + \frac{9}{16\Omega_2^2}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right)^2 + \frac{3}{8\Omega_5^2}\left(\overline{p}_5^2 + \overline{q}_5^2\right)^2 \\ & - \frac{3}{4\Omega_1^2}\left(\overline{p}_1\overline{q}_4 - \overline{q}_1\overline{p}_4\right)^2 - \frac{3}{4\Omega_2^2}\left(\overline{p}_2\overline{q}_3 - \overline{q}_2\overline{p}_3\right)^2 \\ & + \frac{3}{2\Omega_1\Omega_2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right) \\ & + \frac{3}{2\Omega_1\Omega_5}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)\left(\overline{p}_5^2 + \overline{q}_5^2\right) \\ & \left. + \frac{3}{2\Omega_2\Omega_5}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right)\left(\overline{p}_5^2 + \overline{q}_5^2\right)\right].\end{aligned} \tag{50}$$

As it is seen, the above normal form is integrable with the following first integrals:

$$\begin{aligned}F_1 &:= \overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2, \\ F_2 &:= \overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2, \\ G_1 &:= \overline{p}_1\overline{q}_4 - \overline{q}_1\overline{p}_4, \\ G_2 &:= \overline{p}_2\overline{q}_3 - \overline{q}_2\overline{p}_3, \\ I_5 &:= \overline{p}_5^2 + \overline{q}_5^2.\end{aligned} \tag{51}$$

In a similar way as in [4] and above treated case of tree particles, in the domain where the first integrals are independent, we can introduce action-angle variables

$$(\overline{q}, \overline{p}) \longrightarrow \left(F_1, F_2, G_1, G_2, I_5, \phi_1, \phi_2, \psi_1, \psi_2, \varphi_5\right), \tag{52}$$

where F_j, G_j, $j = 1, 2$ are as above and

$$\begin{aligned}\phi_j &:= \frac{1}{2}\arg\left(-\overline{p}_{5-j} - \overline{q}_j, \overline{p}_j - \overline{q}_{5-j}\right) \\ &\quad + \frac{1}{2}\arg\left(\overline{p}_{5-j} - \overline{q}_j, \overline{p}_j + \overline{q}_{5-j}\right), \quad j = 1, 2, \\ \psi_j &:= \frac{1}{2}\arg\left(-\overline{p}_{5-j} - \overline{q}_j, \overline{p}_j - \overline{q}_{5-j}\right) \\ &\quad - \frac{1}{2}\arg\left(\overline{p}_{5-j} - \overline{q}_j, \overline{p}_j + \overline{q}_{5-j}\right), \quad j = 1, 2, \\ \varphi_5 &:= \arctan\frac{\overline{p}_5}{\overline{q}_5}.\end{aligned} \tag{53}$$

As above one can verify that $(F_1, F_2, G_1, G_2, I_5, \phi_1, \phi_2, \psi_1, \psi_2, \varphi_5)$ are canonical coordinates. Then the truncated up to order 4 normal form as a function of the action variables is

$$\overline{H} = \frac{\Omega_1}{2}F_1 + \frac{\Omega_2}{2}F_2 + \frac{\Omega_5}{2}I_5 + \frac{b}{10}\left[\frac{9}{16\Omega_1^2}(F_1)^2 + \frac{9}{16\Omega_2^2}(F_2)^2 + \frac{3}{8\Omega_5^2}(I_5)^2 - \frac{3}{4\Omega_1^2}(G_1)^2 - \frac{3}{4\Omega_2^2}(G_2)^2 + \frac{3}{2\Omega_1\Omega_2}F_1F_2 + \frac{3}{2\Omega_1\Omega_5}F_1I_5 + \frac{3}{2\Omega_2\Omega_5}F_2I_5\right]. \quad (54)$$

It is straightforward to check up whether the Kolmogorov's condition is valid.

Remark 6. Now, let us study the normal forms for the exceptional cases, listed in the end of Section 2. Surprisingly, the resonant terms corresponding to the resonant relations (21), (22), and (24) are missing in H_4. Solely the resonant terms corresponding to the resonant relations (23) appear in H_4 in the following way:

$$\frac{15\sqrt{2}}{8\Omega_1\sqrt{\Omega_2\Omega_5}}\left(z_1^2w_2w_5 + w_1^2z_2z_5 - w_4^2z_2z_5 - z_4^2w_2w_5\right). \quad (55)$$

Therefore, the truncated normal form $\overline{H} = H_2 + \overline{H}_4$ for (4) $n = 5$ and $a = a_3$ is

$$\begin{aligned}\overline{H} &= \frac{\Omega_1}{2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right) + \frac{\Omega_2}{2}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right) + \frac{\Omega_5}{2}\left(\overline{p}_5^2 + \overline{q}_5^2\right) + \frac{b}{10}\left[\frac{9}{16\Omega_1^2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)^2 + \frac{9}{16\Omega_2^2}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right)^2 + \frac{3}{8\Omega_5^2}\left(\overline{p}_5^2 + \overline{q}_5^2\right)^2 - \frac{3}{4\Omega_1^2}\left(\overline{p}_1\overline{q}_4 - \overline{q}_1\overline{p}_4\right)^2 - \frac{3}{4\Omega_2^2}\left(\overline{p}_2\overline{q}_3 - \overline{q}_2\overline{p}_3\right)^2 \right.\\ &\quad + \frac{3}{2\Omega_1\Omega_2}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right) + \frac{3}{2\Omega_1\Omega_5}\left(\overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_4^2 + \overline{q}_4^2\right)\left(\overline{p}_5^2 + \overline{q}_5^2\right) + \frac{3}{2\Omega_2\Omega_5}\left(\overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_3^2 + \overline{q}_3^2\right)\left(\overline{p}_5^2 + \overline{q}_5^2\right)\\ &\quad \left. + \frac{15\sqrt{2}}{4\Omega_1\sqrt{\Omega_2\Omega_5}}\left[2\left(q_2p_5 + p_2q_5\right)\left(q_1p_1 - q_4p_4\right) + \left(q_2q_5 - p_2p_5\right)\left(q_1^2 - p_1^2 + p_4^2 - q_4^2\right)\right]\right].\end{aligned} \quad (56)$$

It can be easily checked that the expressions (51) are no longer first integrals for this normal form. The known integrals are H_2 and $\overline{H}$ and that makes the integrability of this particular resonant normal form unclear.

3.5. Six Particles. In the diagonalized Hamiltonian system (4) $n = 6$ we scale

$$\begin{aligned}\overline{q}_{1,5} &\longrightarrow \frac{1}{\sqrt[4]{a+1}}\overline{q}_{1,5},\\ \overline{p}_{1,5} &\longrightarrow \sqrt[4]{a+1}\,\overline{p}_{1,5},\\ \overline{q}_{2,4} &\longrightarrow \frac{1}{\sqrt[4]{a+3}}\overline{q}_{2,4},\\ \overline{p}_{2,4} &\longrightarrow \sqrt[4]{a+3}\,\overline{p}_{2,4},\\ \overline{q}_{3} &\longrightarrow \frac{1}{\sqrt[4]{a+4}}\overline{q}_{3},\\ \overline{p}_{3} &\longrightarrow \sqrt[4]{a+4}\,\overline{p}_{3},\\ \overline{q}_{6} &\longrightarrow \frac{1}{\sqrt[4]{a}}\overline{q}_{6},\\ \overline{p}_{6} &\longrightarrow \sqrt[4]{a}\,\overline{p}_{6}.\end{aligned} \quad (57)$$

Recall that $\Omega_1 = \Omega_5 = \sqrt{a+1}$, $\Omega_2 = \Omega_4 = \sqrt{a+3}$, $\Omega_3 = \sqrt{a+4}$, $\Omega_6 = \sqrt{a}$; that is, in these cases we again have two independent 1:1 resonances. In the variables (z_j, w_j), $j = 1, \ldots, 6$, the generators of the normal form are

$$z_jw_j \text{ and } z_1w_5, w_1z_5, z_2w_4, z_4w_2.$$

We normalize and find that the normal form $\overline{H} = H_2 + \overline{H}_4$ up to order 4 is

$$\begin{aligned}\overline{H} &= \frac{\Omega_1}{2}\left(z_1w_1 + z_5w_5\right) + \frac{\Omega_2}{2}\left(z_2w_2 + z_4w_4\right) + \frac{\Omega_3}{2}\cdot z_3w_3 + \frac{\Omega_6}{2}z_6w_6 + \frac{b}{72}\left[\frac{27}{8\Omega_1^2}\left(z_1w_1 + z_5w_5\right)^2 + \frac{27}{8\Omega_2^2}\left(z_2w_2 + z_4w_4\right)^2 + \frac{9}{4\Omega_3^2}\left(z_3w_3\right)^2 \right.\\ &\quad + \frac{9}{4\Omega_6^2}\left(z_6w_6\right)^2 + \frac{9\left(z_3w_3\right)\left(z_6w_6\right)}{\Omega_3\Omega_6} + \frac{9z_3w_3}{\Omega_1\Omega_3}\left(z_1w_1 + z_5w_5\right) + \frac{9z_3w_3}{\Omega_2\Omega_3}\left(z_2w_2 + z_4w_4\right) + \frac{9z_6w_6}{\Omega_1\Omega_6}\left(z_1w_1 + z_5w_5\right) + \frac{9z_6w_6}{\Omega_2\Omega_6}\left(z_2w_2 + z_4w_4\right)\\ &\quad + \frac{9}{8}\left(\frac{z_1w_5 - w_1z_5}{\Omega_1} - \frac{z_2w_4 - w_2z_4}{\Omega_2}\right)^2 + \frac{9}{2\Omega_1\Omega_2}\left(z_1w_1z_4w_4 + z_2w_2z_5w_5 + 3z_1w_1z_2w_2 + 3z_4w_4z_5w_5 - \left(z_1w_5 + w_1z_5\right)\left(z_2w_4 + w_2z_4\right)\right.\\ &\quad \left.\left. + \frac{1}{2}\left(z_1w_5 - w_1z_5\right)\left(z_2w_4 - w_2z_4\right)\right)\right].\end{aligned} \quad (58)$$

Before returning to the variables $(\overline{q}, \overline{p})$ we denote

$$\begin{aligned} F_1 &:= \overline{p}_1^2 + \overline{q}_1^2 + \overline{p}_5^2 + \overline{q}_5^2, \\ F_2 &:= \overline{p}_2^2 + \overline{q}_2^2 + \overline{p}_4^2 + \overline{q}_4^2, \\ I_3 &:= \overline{p}_3^2 + \overline{q}_3^2, \\ I_6 &:= \overline{p}_6^2 + \overline{q}_6^2. \end{aligned} \tag{59}$$

Then the truncated normal form $\overline{H} = H_2 + \overline{H}_4$ becomes

$$\begin{aligned} \overline{H} = {} & \frac{\Omega_1}{2} F_1 + \frac{\Omega_2}{2} F_2 + \frac{\Omega_3}{2} I_3 + \frac{\Omega_6}{2} I_6 + \frac{b}{8} \left[\frac{3F_1^2}{8\Omega_1^2} \right. \\ & + \frac{3F_2^2}{8\Omega_2^2} + \frac{I_3^2}{4\Omega_3^2} + \frac{I_6^2}{4\Omega_6^2} + \frac{F_1 F_2}{2\Omega_1 \Omega_2} + \frac{I_3 I_6}{\Omega_3 \Omega_6} + \frac{I_3 F_1}{\Omega_1 \Omega_3} \\ & + \frac{I_3 F_2}{\Omega_2 \Omega_3} + \frac{I_6 F_1}{\Omega_1 \Omega_6} + \frac{I_6 F_2}{\Omega_2 \Omega_6} - \frac{1}{2} \left(\frac{\overline{p}_1 \overline{q}_5 - \overline{q}_1 \overline{p}_5}{\Omega_1} \right. \\ & \left. - \frac{\overline{p}_2 \overline{q}_4 - \overline{q}_2 \overline{p}_4}{\Omega_3} \right)^2 + \frac{1}{\Omega_1 \Omega_2} \left(\left(\overline{p}_1^2 + \overline{q}_1^2 \right) \left(\overline{p}_2^2 + \overline{q}_2^2 \right) \right. \\ & + \left(\overline{p}_4^2 + \overline{q}_4^2 \right) \left(\overline{p}_5^2 + \overline{q}_5^2 \right) \\ & - 2 \left(\overline{q}_1 \overline{q}_5 + \overline{p}_1 \overline{p}_5 \right) \left(\overline{q}_2 \overline{q}_4 + \overline{p}_2 \overline{p}_4 \right) \\ & \left. \left. - \left(\overline{p}_1 \overline{q}_5 - \overline{q}_1 \overline{p}_5 \right) \left(\overline{p}_2 \overline{q}_4 - \overline{q}_2 \overline{p}_4 \right) \right) \right]. \end{aligned} \tag{60}$$

This normal form is integrable: the independent first integrals are

$F_1, F_2, I_3, I_6, G := (\overline{p}_1 \overline{q}_5 - \overline{q}_1 \overline{p}_5) - (\overline{p}_2 \overline{q}_4 - \overline{q}_2 \overline{p}_4)$ and $\overline{H}_4$.

However, the construction of all action-angle variables is unclear so far.

This finishes the proof of Theorem 3.

4. Concluding Remarks

This paper presents partial results on integrability of normal forms of the periodic KG lattices.

We study the normal forms because the original systems are nonintegrable. This is proven rigorously in the case of two degrees of freedom (Theorem 2) and that is one of the differences with the FPU chain. One can carry out the nonintegrability proof for $n = 3$ in the same line, but with more efforts. There is a technical difficulty to carry through that proof in the higher-dimensions, however. The variational equation (VE) does not split nicely and one needs either a tool to deal with higher dimensional (NVE) or another particular solution with the VE along it suitable enough. Nevertheless, we claim that the periodic KG lattice is nonintegrable for all $n \geq 2$.

The calculation of the normal forms goes in the standard way, because we treat only low dimensions. It is also facilitated by the assumption an a. The result in Theorem 3 allows us in most cases to view the periodic KG Hamiltonian (4) as a perturbation of a nondegenerate Liouville integrable Hamiltonian, namely, the truncated up to order four Birkhoff-Gustavson normal form. One can also verify the other KAM condition, known as Arnold-Moser's condition [6].

As a consequence we have found an integrable KAM nondegenerate normal form of $1:2:2$ Hamiltonian resonance (see Remark 5).

The outcome of Theorem 3 suggests that when one considers the integrability of the normal forms in higher-dimensional periodic KG lattices, certain finite set of values of a even in the domain (3) has to be ruled out initially. For the values of this set there probably exist additional fourth-order resonant relations, the corresponding normal forms are probably more complicated than the generic ones, and, hence, more difficult to study.

In any case, the results of this paper serve to understand the lattices with many particles. We intend to study them with the approach of Rink [4], based on using the symmetry properties, which we also enjoy here, to construct suitable normal forms.

Finally, we do not address here the symmetric invariant manifolds in the KG lattices because they can be retrieved from [14].

Appendix

Nonintegrability of Periodic KG Lattice with n=2

In this Appendix we give the proof of Theorem 2. It is based on Ziglin-Morales-Ruiz-Ramis theory. The main result of this theory merely says that if a Hamiltonian system is completely integrable, then the identity component of the Galois group of the variational equation along certain particular solution is abelian.

In the applications if one finds out that the identity component of the Galois group is noncommutative, then this implies nonintegrability. However, if this component turns out to be abelian, one needs additional steps to prove nonintegrability as it is carried out below.

The necessary facts and results about differential Galois theory and its relations with the integrability of Hamiltonian systems, enough for our purposes, are written succinctly in Section 2 in [15] and we do not repeat them here. We refer the reader to [16–19] for a more detailed exposition.

The proof goes in the following lines. We obtain a particular solution and write the variational equation along this solution. It appears that the identity component of its differential Galois group is abelian. In order to obtain an obstacle to the integrability, we study the higher variational equations. Their differential Galois groups are in principle solvable. One possible way to show that some of them is not abelian is to find a logarithmic term in the corresponding solution (see [17, 18]). We obtain such a logarithmic term in the solution of the second variational equation when $b \neq 0$. Then the nonintegrability of the Hamiltonian system follows.

Proof. Suppose $b \neq 0$. First we bring the quadratic part into diagonal form. For this purpose we perform a symplectic change of variables in (5) $q = Y\overline{q}$, $p = Y\overline{p}$, $q = (q_1, q_2)$, $p = (p_1, p_2)$ with $Y = (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (see Section 2). In the new coordinates the Hamiltonian reads (we skip the bars for simplicity here)

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left[(4+a)\,q_1^2 + aq_2^2\right] + \frac{b}{4}\left(q_1^4 + 6q_1^2q_2^2 + q_2^4\right). \tag{A.1}$$

Remark A.1. The Hamiltonian (A.1) is of the form

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left[\alpha_1 q_1^2 + \alpha_2 q_2^2\right] + \frac{b}{4}\left(q_1^4 + 6q_1^2q_2^2 + q_2^4\right), \tag{A.2}$$

which is known to be integrable when $\alpha_1 = \alpha_2$ (see, e.g., [20] and the references therein). However, this is not the case here. We did not succeed in finding a nonintegrability proof for the case $\alpha_1 \neq \alpha_2$; that is why we present it here.

It is also assumed that throughout this Appendix all variables are complex: $t \in \mathbb{C}$, $q_j \in \mathbb{C}$, $p_j \in \mathbb{C}$, $j = 1, 2$. The following proposition is immediate.

Proposition A.2. *The Hamiltonian system corresponding to (A.1) admits a particular solution*

$$q_1^0(t) = \operatorname{sn}\left(\sqrt{a + 4 + \frac{b}{2}}\,t, k\right), \quad p_1^0(t) = \frac{d}{dt}q_1^0(t), \quad q_2^0(t) = p_2^0(t) = 0, \tag{A.3}$$

where sn *is the Jacobi elliptic function with the module* $k = \sqrt{(-b/2)/(4 + a + b/2)}$. □

It is straightforward that $T_1 = 4K/\sqrt{a + 4 + b/2}$ and $T_2 = 2iK'/\sqrt{a + 4 + b/2}$ are the periods of (A.3). Here K, K' are the complete elliptic integrals of the first kind. In the parallelogram of the periods, the solution (A.3) has two simple poles

$$t_1 = \frac{iK'}{\sqrt{a + 4 + b/2}}, \quad t_2 = \frac{2K + iK'}{\sqrt{a + 4 + b/2}}. \tag{A.4}$$

Denoting by $\xi_j^{(1)} = dq_j$, $\eta_j^{(1)} = dp_j$, $j = 1, 2$, the variational equations (VE) (written as second-order equations) are

$$\ddot{\xi}_1^{(1)} = -\left[(4+a) + 3b\left(q_1^0(t)\right)^2\right]\xi_1^{(1)}, \tag{A.5}$$

$$\ddot{\xi}_2^{(1)} = -\left[a + 3b\left(q_1^0(t)\right)^2\right]\xi_2^{(1)}. \tag{A.6}$$

Since $dH = p_1^0(t)\eta_1^{(1)} + (4 + a)q_1^0(t)\xi_1^{(1)} + b(q_1^0(t))^3\xi_1^{(1)}$ does not depend on $(\xi_2^{(1)}, \eta_2^{(1)})$, then (A.6) stands for normal variational equation (NVE); to be more specific:

$$\ddot{\xi}_2^{(1)} + \left[a + 3b\,\operatorname{sn}^2\left(\sqrt{4 + a + \frac{b}{2}}\,t, k\right)\right]\xi_2^{(1)} = 0. \tag{A.7}$$

This equation has regular singularities at $t_{1,2}$; that is, it is a Fuchsian one.

From the expansion of the sn in the neighborhood of the pole t_1, we have

$$q_1^0(t) = \frac{1}{\sqrt{-b/2}}\left[\frac{1}{t - t_1} + \frac{4+a}{6}(t - t_1) + c(t - t_1)^3 + d(t - t_1)^5 + \cdots\right], \tag{A.8}$$

where c is an arbitrary constant and

$$d = \frac{1}{14}\left[\frac{16}{27} + 4c + \frac{4a}{9} + ac + \frac{a^2}{9} + \frac{a^3}{108}\right]. \tag{A.9}$$

It is not difficult to see from (A.7) and (A.8) that the indicial equation [21] at $t = t_1$

$$r(r-1) - 6 = 0 \tag{A.10}$$

has roots $r_{1,2} = -2, 3$ (compare with the expansions below). Therefore, the monodromy around t_1 is trivial and can not serve as an obstacle to integrability [22, 23] (similarly for the monodromy around t_2).

In fact, we can say more about the identity component of the Galois group of (VE) (A.5), (A.6). Notice that each of the variational equations (VE) is a Lamé equation in Jacobi form.

Proposition A.3. *The identity component of the differential Galois group of (VE) (A.5), (A.6) is abelian.*

Proof. The analysis is facilitated by the fact that (VE) is split into two second-order differential equations with the coefficients in the field of elliptic functions.

Let us start with the first equation (A.5) (the consideration of the second equation is similar). It is straightforward that one solution of (A.5) is $\xi_{1,1}^{(1)} = \dot{q}_1^0(t)$. This solution belongs to the field of the coefficients. The other linearly independent solution is

$$\xi_{1,2}^{(1)} = \xi_{1,1}^{(1)}\int\frac{dt}{\left(\xi_{1,1}^{(1)}\right)^2}, \tag{A.11}$$

which does not belong in general to the coefficient field. Then the identity component of its Galois group is isomorphic to $\begin{pmatrix} 1 & 0 \\ \nu_1 & 1 \end{pmatrix}$. Therefore, the identity component of the Galois group of (VE) is isomorphic to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \nu_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \nu_2 & 1 \end{pmatrix}, \quad \nu_1, \nu_2 \in \mathbb{C} \tag{A.12}$$

and it is clearly abelian. □

In what follows we need the expansions around t_1 of the fundamental systems of solutions with unit Wronskians for (A.5) and (A.6). For the expansions of (A.5) we get

$$\begin{aligned}\xi_{1,1}^{(1)} &= \frac{1}{(t-t_1)^2} - \frac{4+a}{6} - 3c(t-t_1)^2 \\ &\quad - \left(\frac{8+6a}{27} + 2c + \frac{ac}{2} + \frac{a^2}{18} + \frac{a^3}{216}\right)(t-t_1)^4 \\ &\quad + \cdots, \\ \xi_{1,2}^{(1)} &= \frac{1}{5}(t-t_1)^3 + \cdots\end{aligned} \tag{A.13}$$

and similarly for (A.5)

$$\begin{aligned}\xi_{2,1}^{(1)} &= \frac{1}{(t-t_1)^2} - \frac{8+a}{6} - \frac{a+6-9c}{3}(t-t_1)^2 \\ &\quad + \left(\frac{56+11a}{27} - \frac{91c+9ac}{18} - \frac{a^2}{54} - \frac{a^3}{216}\right) \\ &\quad \cdot (t-t_1)^4 + \cdots, \\ \xi_{2,2}^{(1)} &= \frac{1}{5}(t-t_1)^3 + \cdots.\end{aligned} \tag{A.14}$$

One can see that these expansions are in fact convergent since t_1 is a regular singular point (cf. [21]). Hence, the fundamental matrix $X(t)$ of (VE) is

$$\begin{aligned}X(t) &= \begin{pmatrix} \xi_{1,1}^{(1)} & \xi_{1,2}^{(1)} & 0 & 0 \\ \dot{\xi}_{1,1}^{(1)} & \dot{\xi}_{1,2}^{(1)} & 0 & 0 \\ 0 & 0 & \xi_{2,1}^{(1)} & \xi_{2,2}^{(1)} \\ 0 & 0 & \dot{\xi}_{2,1}^{(1)} & \dot{\xi}_{2,2}^{(1)} \end{pmatrix}, \\ X^{-1}(t) &= \begin{pmatrix} \dot{\xi}_{1,2}^{(1)} & -\xi_{1,2}^{(1)} & 0 & 0 \\ -\dot{\xi}_{1,1}^{(1)} & \xi_{1,1}^{(1)} & 0 & 0 \\ 0 & 0 & \dot{\xi}_{2,2}^{(1)} & -\xi_{2,2}^{(1)} \\ 0 & 0 & -\dot{\xi}_{2,1}^{(1)} & \xi_{2,1}^{(1)} \end{pmatrix}.\end{aligned} \tag{A.15}$$

Now, let us consider the higher variational equations along the particular solution (A.3). We put

$$\begin{aligned}q_1 &= q_1^0(t) + \varepsilon\xi_1^{(1)} + \varepsilon^2\xi_1^{(2)} + \varepsilon^3\xi_1^{(3)} + \cdots, \\ p_1 &= \dot{q}_1, \\ q_2 &= 0 + \varepsilon\xi_2^{(1)} + \varepsilon^2\xi_2^{(2)} + \varepsilon^3\xi_2^{(3)} + \cdots, \\ p_2 &= \dot{q}_2,\end{aligned} \tag{A.16}$$

where ε is a formal parameter and substitute these expressions into the Hamiltonian system governed by (A.1). Comparing the terms with the same order in ε we obtain consequently the variational equations up to any order.

The first variational equation is, of course, (A.5), (A.6). For the second variational equation we have

$$\begin{aligned}\dot{\xi}_1^{(2)} &= \eta_1^{(2)}, \\ \dot{\eta}_1^{(2)} &= -\left[4 + a + 3b\left(q_1^0(t)\right)^2\right]\xi_1^{(2)} + K_1, \\ \dot{\xi}_2^{(2)} &= \eta_2^{(2)}, \\ \dot{\eta}_2^{(2)} &= -\left[a + 3b\left(q_1^0(t)\right)^2\right]\xi_2^{(2)} + K_2,\end{aligned} \tag{A.17}$$

where

$$\begin{aligned}K_1 &= -3bq_1^0(t)\left[\left(\xi_1^{(1)}\right)^2 + \left(\xi_2^{(1)}\right)^2\right], \\ K_2 &= -6bq_1^0(t)\,\xi_1^{(1)}\xi_2^{(1)}.\end{aligned} \tag{A.18}$$

In this way we can obtain a chain of linear nonhomogeneous differential equations

$$\dot{\xi}^{(k)} = A(t)\,\xi^{(k)} + f_k\left(\xi^{(1)}, \dots, \xi^{(k-1)}\right), \quad k = 1, 2, \dots, \tag{A.19}$$

where $A(t)$ is the linear part of the Hamiltonian equations along the particular solution and $f_1 = 0$. The above equation is called kth variational equation (VE_k). If $X(t)$ is a fundamental matrix of (VE_1), then the solutions of (VE_k), $k > 1$, can be found by

$$\xi^{(k)} = X(t)\,c(t), \tag{A.20}$$

where

$$\dot{c} = X^{-1}(t)\,f_k. \tag{A.21}$$

Let us study the local solutions of (VE_2). In our case $f_2 = (0, K_1, 0, K_2)^T$ and from (A.15) we get

$$X^{-1}(t)\,f_2 = \left(-\xi_{1,2}^{(1)}K_1, \xi_{1,1}^{(1)}K_1, -\xi_{2,2}^{(1)}K_2, K_2\right)^T. \tag{A.22}$$

We are looking for a component of $X^{-1}(t)f_2$ with a nonzero residuum at $t = t_1$. This would imply the appearance of a logarithmic term. Indeed, the residue at $t = t_1$ of $\xi_{2,1}^{(1)}K_2$ with the specific representatives is

$$\begin{aligned}&\operatorname{Res}_{t=t_1}\left(-6b\xi_{2,1}^{(1)}q_1^0(t)\,\xi_{1,1}^{(1)}\xi_{2,1}^{(1)}\right) \\ &\quad = \frac{6b}{\sqrt{-b/2}}\left[\frac{a^3}{252} + \frac{a^2}{21} + \frac{4a}{21} + \frac{3ac}{7} + \frac{73c+16}{63}\right].\end{aligned} \tag{A.23}$$

Since c is an arbitrary parameter, we choose it in such a way that the expression in the square brackets does not vanish for $a > 0$. There are many such values of c, say $c = 1$. Recall that by assumption $b \neq 0$, we have obtained a nonzero residuum at $t = t_1$, which implies the appearance of a logarithmic term in the solutions of (VE_2). Then its Galois group is solvable but not abelian. Hence, we conclude the nonintegrability of the Hamiltonian system (5).

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

This work is supported by Bulgarian Fund "Scientific Research" (Grant DN 02-5).

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