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Reinventing the slide rule for redshifts: the case for logarithmic wavelength shift

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ABSTRACT
Redshift is not a shift, it is defined as a fractional change in wavelength. Nevertheless, it is a fairly common misconception that $\Delta z\, c$ represents a velocity where Δz is the redshift separation between two galaxies. When evaluating large changes in a quantity, it is often more useful to consider logarithmic differences. Defining $\zeta = \ln \lambda_{\rm obs} - \ln \lambda_{\rm em}$ results in a more accurate approximation for line-of-sight velocity and, more importantly, this means that the cosmological and peculiar velocity terms become additive: $\Delta\zeta\, c$ can represent a velocity at any cosmological distance. Logarithmic shift ζ, or equivalently $\ln(1+z)$, should arguably be used for photometric redshift evaluation. For a comparative non-accelerating universe, used in cosmology, comoving distance ($D_{\rm C}$) is proportional to ζ. This means that galaxy population distributions in ζ, rather than z, are close to being evenly distributed in $D_{\rm C}$, and they have a more aesthetic spacing when considering galaxy evolution. Some pedagogic notes on these quantities are presented.

Key words: redshift, wavelength, peculiar velocity, cosmological scalefactor, frame, comoving distance

1 REDSHIFT IS NOT A SHIFT

The definition of redshift is given by

$$z \;=\; \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} \quad , \tag{1}$$

where $\lambda_{\rm obs}$ is the observed wavelength and $\lambda_{\rm em}$ is the emitted or rest-frame wavelength (e.g. eq. 7 of Hubble & Tolman 1935). For low redshifts, it is common to quote $z\,c$ for observed galaxies as a recession velocity in units of $\rm km\,s^{-1}$. This is related to the approximation

$$z_{\rm pec} \;\simeq\; \frac{v}{c} \tag{2}$$

where $z_{\rm pec}$ is the redshift (or blueshift) caused by a line-of-sight peculiar velocity (v) component. This sometimes leads to the *incorrect* assumption that the 'velocity' due to the cosmological expansion and the peculiar velocity add, or that the redshifts add. Davis & Scrimgeour (2014) show how, that even at modest redshift, the peculiar velocity can be significantly overestimated by naively subtracting the cosmological redshift from the observed redshift.

The correct formula for relating redshift terms, also incorporating the Sun's peculiar motion, can be given by

$$1+z_{\rm cmb} = (1+z_{\rm helio})(1+z_{\rm pec,\odot}) = (1+z_{\rm cos})(1+z_{\rm pec}) \quad , \tag{3}$$

where $z_{\rm cmb}$ and $z_{\rm helio}$ are the redshifts of an observed galaxy in the cosmic-microwave-background (CMB) frame and heliocentric frame, respectively, $z_{\rm pec,\odot}$ is the component caused by the motion of our Sun wrt. the CMB frame toward the observed galaxy, $z_{\rm pec}$ is caused by the peculiar velocity of the observed galaxy, and $z_{\rm cos}$ is the cosmological redshift caused by the expansion of the Universe only. This is evident from considering the definition of redshift, i.e., *'one plus redshift' has a multiplicative effect on wavelength* (Harrison 1974). Note there is also a term for gravitational redshift and the heliocentric redshift should be determined correctly from the observed redshift.

Taking the difference in redshifts between two galaxies that are at the same distance, we obtain

$$\begin{aligned}\Delta z \;=\; z_1 - z_2 \;=\; & (1+z_{\rm cos})\,(z_{1,\rm pec} - z_{2,\rm pec}) \\ \simeq\; & (1+z_{\rm cos})\,\frac{v_1 - v_2}{c}\end{aligned} \quad , \tag{4}$$

using the approximation of Eq. 2. So it appears that to estimate the velocity difference requires knowledge of the cosmological redshift, though typically one could just set $\Delta v = \Delta z\, c/(1+z_1)$, for example, or use one plus the average redshift for the denominator (Danese, de Zotti, & di Tullio 1980). This is a well known consideration when determining the velocity dispersions of galaxy clusters. A related consequence for counting galaxies in cylinders (e.g. Balogh et al. 2004) is that to allow a fixed maximum extent in *velocity* difference around a galaxy requires increasing the extent in Δz with redshift proportional to $1+z$.

Revisiting the approximation, the peculiar redshift is accurately given by the Doppler shift formula:

$$1+z_{\rm pec} \;=\; \gamma(1+\beta_{\rm los}) \tag{5}$$

where $\gamma = (1-\beta^2)^{-1/2}$ is the Lorentz factor and $\beta_{\rm los}$ is the line-of-sight velocity divided by the speed of light. Using Taylor series

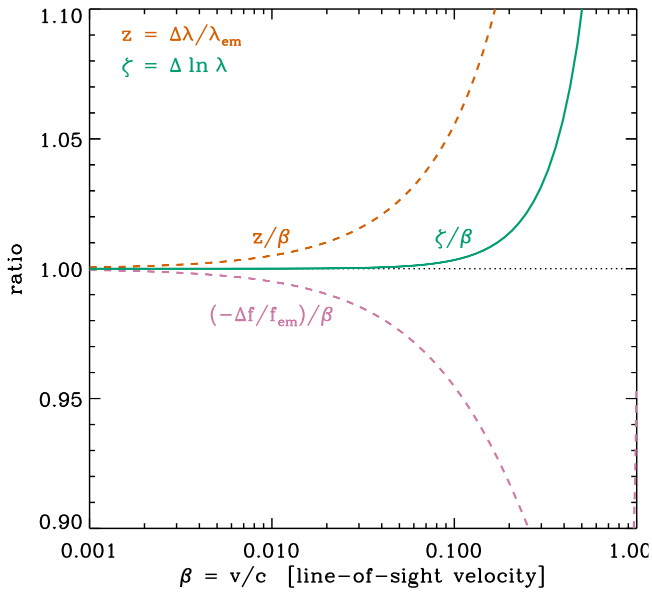

Figure 1. Comparison between approximations for recession velocity, i.e., assuming pure line-of-sight motion ($\beta_{\rm los} = \beta$). The Doppler formula is used to compute the redshift (Eq. 5), zeta (Eq. 7) and the radio definition of velocity as a function of β. Notably zeta remains an accurate approximation of recession velocity, within a percent, up to 0.1 c.

expansion, we can then simplify to:

$$z_{\rm pec} \simeq \beta_{\rm los} + \frac{1}{2}\beta^2 + \frac{1}{2}\beta^2\beta_{\rm los} \quad . \tag{6}$$

This simplifies further to $z_{\rm pec} \simeq \beta_{\rm los}$ after dropping the higher order terms. This is usually sufficiently accurate for use in astrophysics but it is worth bearing in mind that it is an approximation.

2 LOGARITHMIC SHIFT ZETA

Determining redshifts by cross correlation makes it evident that a 'redshift' or velocity measurement is actually a shift on a logarithmic wavelength scale (Tonry & Davis 1979). So arguably it is more natural to define a quantity (here called zeta) that is a logarithmic shift as

$$\zeta = \ln\lambda_{\rm obs} - \ln\lambda_{\rm em} = \ln(1+z) \quad . \tag{7}$$

First we check its approximation for velocity, using Taylor series,

$$\begin{aligned}\zeta_{\rm pec} &= -\frac{1}{2}\ln(1-\beta^2) + \ln(1+\beta_{\rm los}) \\ &\simeq \beta_{\rm los} + \frac{1}{2}(\beta^2 - \beta_{\rm los}^2) + \frac{1}{3}\beta_{\rm los}^3\end{aligned} \tag{8}$$

from the natural logarithm of Eq. 5. Such that $\zeta_{\rm pec}$ is always a more accurate approximation for $\beta_{\rm los}$ than $z_{\rm pec}$, with the quadratic term vanishing for pure line-of-sight motion. Figure 1 shows a comparison between the redshift, zeta and 'radio definition' approximations for recession velocity.

Given the improved accuracy, it is reasonable to use

$$\zeta_{\rm pec} \simeq \frac{v}{c} \tag{9}$$

for peculiar velocities. This is used implicitly when velocity dispersions of galaxies are determined from a logarithmically binned wavelength scale (Simkin 1974).

More importantly, the use of zeta means that, the equivalent of Eq. 3 for relating redshift terms becomes

$$\zeta_{\rm cmb} = \zeta_{\rm helio} + \zeta_{{\rm pec},\odot} = \zeta_{\rm cos} + \zeta_{\rm pec} \quad . \tag{10}$$

It is immediately evident that the separation in zeta between two galaxies at the same distance is related to velocity directly by

$$\Delta\zeta \simeq \frac{\Delta v}{c} \tag{11}$$

with no dependence on the choice of frame or cosmological redshift. In addition to being more accurate than Eq. 4, it is precisely symmetric when determining the separations in velocity between two or more galaxies, i.e., there is no need to pick a fiducial redshift. A velocity dispersion is given by $\sigma(\zeta)$ regardless of the frame.

Redshift measurement errors can also be addressed as follows. Spectroscopic or photometric redshifts are generally estimated by matching a template to a set of observed fluxes at different wavelengths. In order to determine the redshift, the template must be shifted in $\ln\lambda$, thus we can immediately see that:

$$\sigma(\zeta) = \sigma[\Delta\ln(\lambda)] \quad , \tag{12}$$

which is the uncertainty in the logarithmic shift between the observed and emitted wavelengths. Alternatively the redshift uncertainties are often quoted in fractional form:

$$\sigma(\zeta) \simeq \frac{\sigma(z)}{1+z} \quad . \tag{13}$$

Either can be related to a velocity uncertainty (Eq. 11), and it is thus reasonable to quote spectroscopic measurements using velocity uncertainties (Baldry et al. 2014). The concern is that some papers quote redshift errors in km/s using $\sigma(z)\,c$ (e.g. Colless et al. 2001), which does *not* represent a physical velocity uncertainty even though it has the same units.

It is appropriate to treat the evaluation of photometric redshift errors in the same way and determine the uncertainties in ζ. The typical use of quoting $\sigma(z)/(1+z_{\rm spec})$, where $z_{\rm spec}$ is a spectroscopic redshift, for the performance of photometric redshift estimates, approximates this (e.g. Brinchmann et al. 2017). This is somewhat inelegant because the uncertainties on photometric redshifts are obtained using spectroscopic redshifts in the denominator. This is no such problem using $\sigma(\zeta)$ and it is more natural since a measurement corresponds to a shift in $\ln\lambda$. This is just a recognition that fractional differences between two quantities ($1+z$ in this case) depend on a fiducial value whereas logarithmic differences are symmetric. More importantly, this strongly suggests that probability distribution functions, for example, should be assessed as a function of zeta (binning, outliers, biases, second peak offsets) rather than z. Rowan-Robinson (2003) used $\log_{10}(1+z)$, which equals $\zeta/\ln(10)$, in his analysis including plots but this is far from standard in the literature.

3 COSMOLOGICAL SCALEFACTOR

At a team meeting, I once presented a slide jokingly noting that "z is an abomination, it is neither multiplicative, additive or a shift". Of course, redshift's saving grace is that a human's computational ability is sufficient to convert z to the inverse scalefactor, add unity and you get $1+z_{\rm cos} = a^{-1}$, where a is the cosmological scalefactor with the common convention that the present-day value $a_0 = 1$.

Using the logarithmic shift ζ, the relationship is evidently $\zeta_{\rm cos} = \ln a^{-1}$. Spacing in logarithm of the scalefactor has desirable properties when considering galaxy populations or cosmology

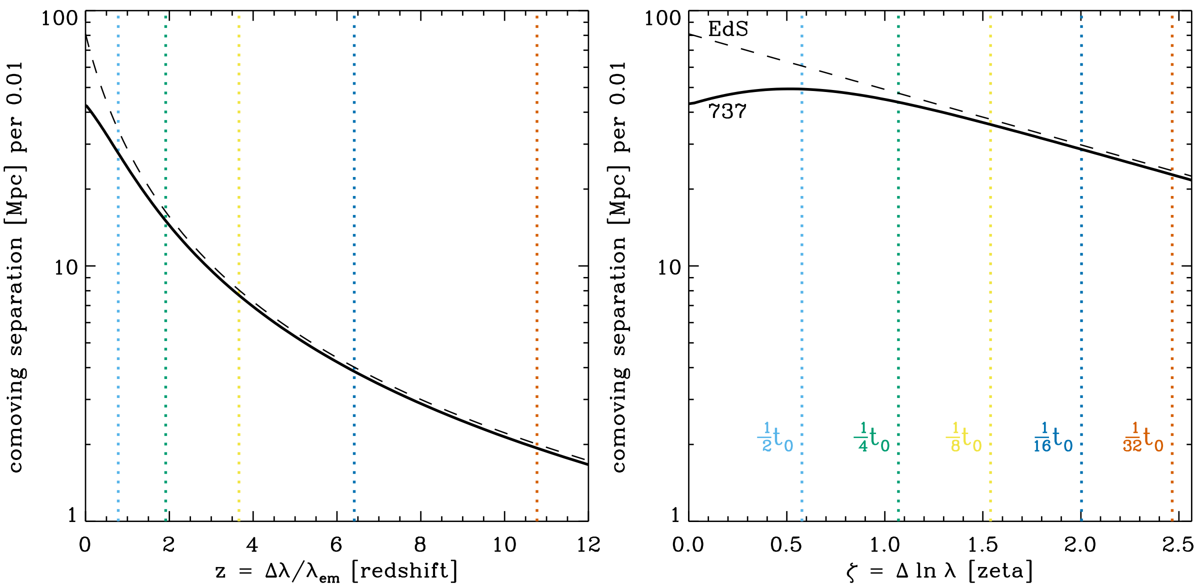

Figure 2. Comparison between spacing in redshift and zeta. The black lines show the comoving separation per 0.01 in z (left) and ζ (right) (Eq. 14). The solid lines represents the '737 cosmology' ($h = 0.7$, $\Omega_{\rm m} = 0.3$, $\Omega_\Lambda = 0.7$) while the dashed lines represent an Einstein-de-Sitter cosmology ($h = 0.37$ arbitrary, $\Omega_{\rm m} = 1$). The dotted lines show the points at which the universe was one half, one quarter, etc., of its present-day age for the 737 cosmology.

(Table 1). Figure 2 shows the separation in line-of-sight comoving distance ($D_{\rm C}$) versus redshift and zeta for two different cosmologies. The black lines show

$$S_z = 0.01 \frac{{\rm d}D_{\rm C}}{{\rm d}z} \quad \text{and} \quad S_\zeta = 0.01 \frac{{\rm d}D_{\rm C}}{{\rm d}\zeta} \qquad (14)$$

in each plot. These are inversely proportional to $\dot{a}/a$ (e.g. Hogg 1999) and $\dot{a}$, respectively. Notably S_ζ varies less, particular at $\zeta < 1$. This is a desirable property since large-scale structure is evaluated using comoving distances. Spacing in ζ corresponds to constant velocity and approximately constant comoving distance.

The turnover in S_ζ demonstrates the onset of dark energy dominating the dynamics for the '737 cosmology'. This is evident even without the comparison to the Einstein-de-Sitter (EdS) cosmology because for a non-accelerating universe ($\ddot{a} = 0$), S_ζ is constant. For the EdS model, $S_z \propto a^{3/2}$ and $S_\zeta \propto a^{1/2}$ so that

$$\ln S_\zeta = -\frac{1}{2}\zeta + \ln(0.01\, c/H_0) \quad , \qquad (15)$$

which explains why the dashed line is straight in the right plot of Figure 2. See, for example, fig. 2 of Aubourg et al. (2015) for related plots [using $\dot{a}$ and $\ln(1+z)$] comparing different models of dark energy, and Sutherland & Rothnie (2015) who advocated changing the redshift variable to $\ln(1+z)$ in analysis of luminosity distance residuals.

Also shown in Figure 2, with vertical lines, are the points at which the universe halves its age (737 cosmology), with increasing z and ζ. For z, the last half of cosmic time covers only a small fraction of the plot ($z < 0.8$), whereas for ζ, the spacing is approximately logarithmic in time. For an EdS model, it would be equally spaced in $\ln t$ because $a \propto t^{2/3}$. For the 737 cosmology, an increase in ζ of ~ 0.5 corresponds to halving the age of the universe across the epochs shown. A generic plot related to galaxy evolution shows the cosmic star-formation rate (SFR) density, logarithmically scaled, versus z but often scaled linearly in $\ln(1+z)$ (Hopkins & Beacom 2006; Madau & Dickinson 2014). This a recognition of the aesthetic of $\ln(a)$ separation.

Table 1. zeta-redshift-scalefactor lookup

ζ	z	a	note
0.1	0.105	0.905	$\sim$ present-day galaxy properties
0.5	0.649	0.607	$\sim$ transition to cosmic acceleration
1.0	1.72	0.368	$\sim$ peak of cosmic SFR density
1.5	3.48	0.223	
2.0	6.39	0.135	$\sim$ end of reionization
2.5	11.2	0.0821	
3.0	19.1	0.0498	$\sim$ first stars
7.0	1096	0.000912	$\sim$ matter-radiation decoupling

4 CLOSING REMARKS AND PERSONAL COMMENTS

In closing, redshift z started out being considered as a 'recession velocity' but is now considered as the inverse scalefactor minus unity when assuming $z = z_{cos}$, noting also that $z \sim \zeta$ at $z \ll 1$ and $z \sim a^{-1}$ at $z \gg 1$. Using the logarithmic shift ζ, the cosmological and peculiar velocity terms are additive (Eq. 10). In addition, linear spacing in ζ corresponds to logarithmic spacing in a, which is often a practical and aesthetically desirable feature for plots highlighting cosmological models and galaxy evolution. Astronomers regularly use logarithmic differences, magnitude and dex, so it would be natural to use logarithmic shift for wavelength.

Selected points are given below:

- Use of $z\,c$ for galaxy recession velocities is poor practice especially beyond a couple of thousand km/s.
- Regarding $\zeta\,c$, it is neat that the quadratic term vanishes for pure line-of-sight motion. I appreciate this is a special case for peculiar velocities but it is arguably more appropriate for 'recession velocity' out to $\zeta \sim 0.1$.
- For sources at the same distance, $\Delta z\,c$ is not a velocity, $\Delta\zeta\,c$ is a velocity other than for highly relativistic sources.
- Use of ζ, or $\ln(1+z)$, is natural for studies that deal with the combination of cosmological and velocity terms.

- Photometric redshift analysis should arguably use ζ as standard including presentation and diagnostics. These measurements are effectively analysing shifts in $\ln(\lambda)$.
- A plot of $z_{\rm phot}$ versus $z_{\rm spec}$ is inelegant on two counts: it does not relate to the logarithmic shift nature of the measurements, and the spacing is aesthetically poor.
- The Hubble-Lemaître law $v = H_0 D$ is exact for a non-accelerating universe if we use *velocity and distance definitions* $v = \zeta\, c$ and $D = D_{\rm C}$ (line-of-sight comoving distance). Thus any deviations from the 'law', in this form, reflect accelerating or decelerating expansion.

Comments on the revision history of this paper are given below:

- An earlier iteration of this paper was rejected by MNRAS (with the title "Shouldn't we be using a shift in logarithmic wavelength as standard?"). The anonymous referee noted that it was just an argument for "re-inventing the slide rule": harsh but fair. I have used this quote in the revised title.
- The same iteration was also rejected as a tutorial by PASP. The referee noted "It isn't exactly a tutorial, ... it is more a plea to established astronomers for a revision of notation. That notation is so deeply embedded in the literature that most working astronomers would not think that the small benefits of changing it would be worth the disruption and confusion that would result". I would argue that confusion, related to z and velocity, for example, already exists and will continue; I've noticed it many times. While the referee's view will be common, I think there are some uses mentioned in this paper where switching to ζ, or $\ln(1 + z)$ to avoid a new symbol, is more readily justified.
- The tone of the MNRAS submitted version was changed somewhat for arXiv v1, along with other minor changes.
- A reference and note on Hubble-Lemaître law were added, following comments from W. Sutherland, for arXiv v2.

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