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Quantum and affine Schubert calculus and Macdonald polynomials

A Thesis

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Dedications

To my grandfather

Chunilal Atmaram Dalal

August 21, 1921 - March 20, 2010

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Abstract

Quantum and affine Schubert calculus and Macdonald polynomials

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This thesis is on the theory of symmetric functions and quantum and affine Schubert calculus. Namely, it establishes that the theory of symmetric Macdonald polynomials aligns with quantum and affine Schubert calculus using a discovery that distinguished weak chains can be identified by chains in the strong (Bruhat) order poset on the type-A affine Weyl group. Through this discovery, there is a construction of two one-parameter families of functions that respectively transition positively with Hall-Littlewood polynomials and Macdonald's P-functions. Furthermore, these functions specialize to the representatives for Schubert classes of homology and cohomology of the affine Grassmannian. This shows that the theory of symmetric Macdonald polynomials connects with affine Schubert calculus.

There is a generalization of the discovery of the strong order chains. This generalization connects the theory of Macdonald polynomials to quantum Schubert calculus. In particular, the approach leads to conjecture that all elements in a defining set of 3-point genus 0 Gromov-Witten invariants for flag manifolds can be formulated as strong covers.

Chapter 1
Introduction

The Macdonald polynomial basis for the ring Λ of symmetric functions is found at the root of many exciting projects spanning topics such as double affine Hecke algebras, quantum relativistic systems, diagonal harmonics, and Hilbert schemes on points in the plane. The transition matrix between Macdonald and Schur functions has been intensely studied from a combinatorial, representation theoretic, and algebraic geometric perspective since the time Macdonald conjectured [Mac88] that the coefficients in the expansion

$$H_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)\, s_\lambda \tag{1.1}$$

are positive sums of monomials in q and t – that is, $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

The Kostka-Foulkes polynomials are the special case $K_{\lambda\mu}(0,t)$. These appear in contexts such as Hall-Littlewood polynomials [Gre55], affine Kazhdan-Lusztig theory [Lus81], and affine tensor product multiplicities [NY97]. Kostka-Foulkes polynomials also encode the dimensions of bi-graded S_n-modules [GP92]. They were combinatorially characterized by Lascoux and Schützenberger [LS78] who associated a non-negative integer statistic called cocharge to each semi-standard Young tableau and proved that

$$K_{\lambda\mu}(0,t) = \sum_{T \in SSYT(\lambda,\mu)} t^{\mathrm{cocharge}(T)}\,, \tag{1.2}$$

summing over tableaux of shape λ and weight μ. Despite the prevelance of concrete results for the $K_{\lambda\mu}(0,t)$, it was a big effort even to establish polynomiality for general $K_{\lambda\mu}(q,t)$ and the geometry of Hilbert schemes was ultimately needed to prove positivity [Hai01]. A formula in the spirit of (1.2) still remains a complete mystery.

One study of Macdonald polynomials [LLM03] uncovered a manifestly t-Schur positive construction for polynomials $A^{(k)}_\mu(x;t)$, conjectured to be a basis for the subspace $\Lambda^t_{(k)}$ in a filtration $\Lambda^t_{(1)} \subset \Lambda^t_{(2)} \subset \cdots \subset \Lambda^t_{(\infty)}$ of Λ with the compelling feature that for every partition μ where $\mu_1 \leq k$,

$$H_\mu(x;q,t) = \sum_{\lambda:\lambda_1 \leq k} K^k_{\lambda\mu}(q,t)\, A^{(k)}_\lambda(x;t) \quad \text{for some} \quad K^k_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]\,.$$

Assuming this conjecture, since $A^{(k)}_\lambda(x;t)$ is a t-positive sum of Schur functions, the Macdonald/Schur transition matrices factor over $\mathbb{N}[q,t]$. The construction of $A^{(k)}_\mu(x;t)$ is extremely intricate and the conjectures remain unproven as a consequence. Nevertheless, their study inspired discoveries in representation theory [Hai08] and suggested a connection between the theory of Macdonald polynomials and quantum and affine Schubert calculus.

The affine Grassmannian of $G = SL(n,\mathbb{C})$ is given by $\mathrm{Gr} = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$, where $\mathbb{C}[[t]]$ is the ring of formal power series and $\mathbb{C}((t)) = \mathbb{C}[[t]][t^{-1}]$ is the ring of formal Laurent series. Quillen (unpublished) and Garland and Raghunathan [GR75] showed that Gr is homotopy-equivalent to the group $\Omega SU(n,\mathbb{C})$ of based loops into $SU(n,\mathbb{C})$. The homology $H_*(\mathrm{Gr})$ and cohomology $H^*(\mathrm{Gr})$ thus have dual Hopf algebra structures which, using results of [Bot58], can be explicitly identified by a subring $\Lambda^{(n)}$ and a quotient $\Lambda_{(n)}$ of Λ.

On one hand, the algebraic nil-Hecke ring construction of Kostant and Kumar [KK86] and the work of Peterson [Pet97] developed the study of Schubert bases associated to Schubert cells of Gr in the Bruhat decomposition of $G(\mathbb{C}((t)))$,

$$\{\xi^w \in H^*(\mathrm{Gr},\mathbb{Z}) \mid w \in \tilde{S}^0_n\} \quad \text{and} \quad \{\xi_w \in H_*(\mathrm{Gr},\mathbb{Z}) \mid w \in \tilde{S}^0_n\}\,,$$

indexed by Grassmannian elements of the affine Weyl group $\tilde{A}_{n-1}$. On the other, inspired by an empirical study of the polynomials $A^{(k)}_\lambda(x;t)$ when $t = 1$, distinguished bases for $\Lambda^{(n)}$ and $\Lambda_{(n)}$ that refine the Schur basis for Λ were introduced and connected to the quantum cohomology of Grassmannians in [LM08, LM07]. The two approaches merged when Lam proved in [Lam08] that these *k-Schur bases* are sets of representatives for the Schubert classes of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ (where $k = n-1$). Moreover, the k-Schur functions for $\Lambda_{(k)} = \Lambda_{(k)}^{t=1}$ were conjectured [LM05] to be the parameterless $\{A_\lambda^{(k)}(x;1)\}$, suggesting a link from the theory of Macdonald polynomials to quantum and affine Schubert calculus.

Here, we circumvent the problem that the characterization for $A_\mu^{(k)}(x;t)$ lacks in mechanism for proof and definitively establish this link. Our work relies on a remarkable connection between chains in the strong and the weak order poset on the type-A affine Weyl group. From this connection, we are able to construct one parameter families of symmetric functions that transition positively with $H_\mu(x;0,t)$ and Macdonald's P-functions and that specialize to the Schubert representatives for $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ when $t = 1$. The same approach leads also to a strong-cover formulation for all elements in a defining set of Gromov-Witten invariants for flag manifolds.

Our presentation begins with a fresh look at the product structure of the rings $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$. Specifically, each ring is determined by multiplication of an arbitrary class with a simple class. Explicit Pieri rules for these products were given in [LM07, LLMS10]; the homology rule is framed using saturated chains in the weak order on elements in $\tilde{A}_{n-1}$ and the cohomology rule is in terms of strong order saturated chains. We distinguish a subset of these strong chains by imposing a translation and a horizontality condition on Ferrers shapes. We prove that this subset newly characterizes the homology rule, providing a cohesive framework for the structure of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$. For $w \in \tilde{S}_n^0$ and $c_{0,m} = s_{m-1}\cdots s_1 s_0$,

$$\xi_{c_{0,m}}\,\xi_w = \sum_{u\in\tilde{S}_n^0} \xi_u,$$

where $(\mathfrak{a}(w), \mathfrak{a}(u))$ is a horizontal strong $(n-1-m)$-strip. In essence, a horizontal strong strip is a saturated chain in the Bruhat order from u to the translation of w by $s_{x-1}\cdots s_{x+1}$ (see Definition 120).

From the horizontal strong strips, we derive a new combinatorial tool called affine Bruhat countertableaux (or ABC's). We prove that their generating functions are representatives for the Schubert basis of $H^*(\mathrm{Gr})$ and by associating a non-negative integer statistic called n-cocharge to each ABC, we refine the Kostka-Foulkes polynomials. The family of *weak Kostka-Foulkes polynomials* are defined, for partitions μ and λ with parts smaller than n, by

$$K^n_{\lambda\mu}(t) = \sum_{A \in ABC(\lambda,\mu)} t^{n\text{-cocharge}(A)}\,, \tag{1.3}$$

summing over all ABC's of shape λ and weight μ. A new family of symmetric functions over $\mathbb{Q}(t)$ that reduces to Schubert representatives for the cohomology of Gr when $t = 1$ can then be drawn by

$$\mathfrak{S}^{(n)}_\lambda(x;t) = \sum_\mu K^n_{\lambda\mu}(t)\,\tilde{P}_\mu(x;t)\,, \tag{1.4}$$

where $\{\tilde{P}_\mu(x;t)\}$ are a deformation of Macdonald's P-functions. A basis that reduces to the Schubert representatives for $H_*(\mathrm{Gr})$ when $t = 1$ and whose transition matrix with Macdonald polynomials $H_\mu(x;0,t)$ has entries in $\mathbb{N}[t]$ is then given by the dual basis

$$\left\{ s^{(n)}_\lambda(x;t) \right\}\,,$$

with respect to the Hall-inner product on Λ. These are conjecturally the $A^{(n-1)}_\lambda(x;t)$.

Another advantage of the strongly formulated homology rule is that it allies with the combinatorial backdrop of quantum Schubert calculus. The quantum cohomology ring of the complete flag manifold FL_n (chains of vector spaces in $\mathbb{C}^n$) decomposes into Schubert cells indexed by permutations $w \in S_n$. As a linear space, the quantum cohomology is $QH^*(FL_n) = H^*(FL_n) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}]$ for parameters $q_1, \ldots, q_{n-1}$, and the appeal lies in its rich multiplicative structure. The quantum product

$$\sigma_u * \sigma_v = \sum_w \sum_d q_1^{d_1} \ldots q_{n-1}^{d_{n-1}} \langle u, v, w\rangle_d\, \sigma_{w_0 w} \tag{1.5}$$

is defined by 3-point Gromov-Witten invariants of genus 0 which count equivalence classes of certain rational curves in FL_n. The study of Gromov-Witten invariants is ongoing. Many attempts to gain direct combinatorial access to the structure constants have been made, but formulas are still being pursued even in the simplest case when $q_1 = \cdots = q_{n-1} = 0$. In this case, if u and v are permutations with one descent, the invariants reduce to the well-understood Littlewood-Richardson coefficients [LR34, MPS77]. For generic u and v in S_n, the invariants are the structure constants of Schubert polynomials [LS82], mysterious even when u has only one descent.

Although the construction is not manifestly positive, all the Gromov-Witten invariants of (1.5) can be calculated from the subset

$$\{\langle s_r, v, w\rangle_d : 1 \leq r < n \text{ and } v, w \in S_n\} . \tag{1.6}$$

Fomin, Gelfand, and Postnikov [FGP97] use quantum Schubert polynomials to characterize this set as a generalization of Monk's formula. Here, we approach the study by way of the affine Grassmannian. Peterson asserted that $QH^*(G/P)$ of a flag variety is a quotient of the homology $H_*(\mathrm{Gr}_G)$ up to localization (detailed and proven in [LS12]). As a by-product, the three-point genus zero Gromov-Witten invariants (1.5) are structure constants of the Schubert basis for $H_*(\mathrm{Gr})$. A precise identification of $\langle u, v, w\rangle_d$ with coefficients $c^{\nu}_{\mu,\lambda}$ in

$$\xi_\mu \, \xi_\lambda = \sum_{\nu} c^{\nu}_{\mu,\lambda} \, \xi_\nu \tag{1.7}$$

is made in [LM] (where μ, ν, λ are certain Ferres shapes associated to elements of $\tilde{S}^0_n$) and the defining set (1.6) of Gromov-Witten invariants is determined to be a subset of

$$\{c^{\nu}_{R'_r,\lambda} : \ 1 \leq r < n \text{ and } \lambda, \nu \in C^n\} ,$$

where R'_r is the rectangular Ferrers shape (r^{n-r}) with its unique corner removed.

In this context, the set can conjecturally be characterized simply as strong covers under a rectangular translation; that is,

$$\xi_{R_r'}\,\xi_\lambda = \sum_{\nu \lessdot_B R(r,\lambda)} \xi_\nu\,,$$

where $\nu_i < R(r,\lambda)_i$ for some i such that $(\lambda \cup R_r)_i = r$ and the q-parameters are readily extracted from the shape ν. We extend the definition of horizontal strong strips to a larger distinguished subset of strong order chains characterized by a condition involving ribbon shapes (see Definition 135). The ribbon strong strips are inspired by the expansion of $\xi_\mu \xi_\lambda$, where μ is $(r^{n-r-1}, r-a)$ for $1 \leq a < r < n$.

Another motivation for this approach is in its application to an open problem in the study of $H^*(\mathrm{Gr})$. The problem of expanding a Schubert homology class in the affine Grassmannian of $G = SL_{n-1}$ into Schubert homology classes in Gr was settled in [LLMS12], but the cohomological picture requires a deeper understanding of the intricacies of strong strips. The ribbon strong strips point the study towards converting between weak and strong chains so that the existing work on the homology problem can be applied to the cohomology problem (see [LLMS12] for details).

Chapter 2
Related work

The parameterless k-Schur function structure constants contain all the Schubert structure constants in the quantum cohomology of flag varieties [LS12, LM]. The search for formulas for these constants is tied to many exciting projects.

The quantum comology of the Grassmannian can be accessed [BKT03] from the ordinary cohomology of two-step flags, in which case the Schubert structure constants can be computed by an iterative algorithm of Coskun [Cos09] or by Knutson-Tao puzzles [KT03] (proved in forthcoming work [BKPT, Buc]). The constants also match the structure constants of the Verlinde fusion alegbra for WZW models [Ver88, TUY89], efficiently computed by the Kac-Walton formula [Kac90, Wal90] and combinatorially attempted by [Tud00, KS10, MS12] among others.

Formulas in the quantum cohomology of flag varieties have been derived only in special cases such as the quantum Monk formula [FGP97] and quantum Pieri formula [Pos99]. These special constants were connected in [LM] to the k-Schur expansion of $s_\mu^{(k)} s_\lambda^{(k)}$, where μ is a rectangle minus part of a row. The k-Pieri rule was given in [LM07] and a more general result appears in [BBPZ12, BSSb, BSSa]. The problem currently excites many perspectives including the Fomin-Kirillov algebra [MPP], the affine nil-Coxeter algebra [BBTZ12], Fomin-Greene monoids [BB96], residue tables [FK13], and crystal bases [MS].

The inclusion of a generic t-parameter has so far been met with limited success. Most notably, Lapointe and Pinto [PL] introduced a statistic on weak tableaux and proved that it matches the weight on the poset in [LLMS12] that describes the expansion of a Schubert homology class in Gr into Schubert homology classes in the affine Grassmannian of $G = SL(n+1, \mathbb{C})$. In [DM12], we prove that these match the statistic on ABC's. Closely related is work expressing k-Schur functions in terms of Kirillov-Reshetikhin crystals for type A_n [MS].

Jing [Jin91] introduced vertex operators B_r with the property that

$$B_r H_\mu(x;0,t) = H_{r,\mu}(x;0,t)\,.$$

These play a central role [HMZ12] in the developing theory of diagonal harmonics [GH96a]. Zabrocki [Zab98b] determined the action of B_r on a Schur function, giving a new proof of the cocharge formulation for Kostka-Foulkes polynomials. In fact, our approach to the homology Pieri rule using the strong instead of the weak order came out of a study of his work and the action of B_r on a k-atom $A_\lambda^{(k)}(x;t)$. A deeper understanding of the operators will shed light on open problems in diagonal harmonics and their connection to affine Schubert calculus.

Chapter 3
Preliminaries

In this chapter we lay down the preliminaries and definitions required for this thesis. We begin with a section on partitions.

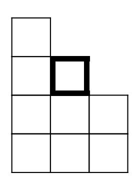
3.1 Partitions

Definition 1. *We say that a finite sequence of positive integers* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ *is a* ***partition*** *of* $n \in \mathbb{N}$, *denoted* $\lambda \vdash n$, *if* $n = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$ *and* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$. *We set* $|\lambda| = \sum_{i=1}^{\ell} \lambda_i$ *and the length of* λ *as* $\ell(\lambda) = \ell$, *the number of parts of* λ.

When λ is a sequence of positive integers which is not necessarily decreasing, then λ is called a **composition**. When λ is a partition, it corresponds uniquely to a diagram.

Definition 2. *Suppose* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$. *The* ***Ferrers diagram*** *of* λ *is an array of* n *squares having* ℓ *left justified rows with row* i *containing* λ_i *boxes for* $1 \leq i \leq \ell$. *A* **cell** (i, j) *of the partition* λ *is a square in row* i *from the bottom and column* j *from the left of the Ferrers shape of* λ.

Example 3. *If* $\lambda = (3, 3, 2, 1)$, *then the corresponding Ferrers diagram is*



Note the cell $(3, 2)$ *of* λ, *indicated by the thick frame in the Ferrers shape.*

For partitions λ and μ, where $\ell(\mu) \leq \ell(\lambda)$, we say $\mu \subset \lambda$ when $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\mu)$. There is also a partial ordering on the set of all partitions which plays an important role in the theory of symmetric functions.

Definition 4. *We define the* **dominance order** *on the set of partitions as*

$$\lambda \rhd \mu \iff \lambda_1 + \ldots + \lambda_j \geq \mu_1 + \ldots + \mu_j \quad \textit{for all } j \textit{ and all partitions } \mu, \lambda.$$

Intuitively, $\lambda \rhd \mu$ if the Ferrers diagram of λ is wider than the one for μ. Dominance order is a partial order on the set of partitions. If we consider the partitions $\lambda = (3, 3)$ and $\mu = (4, 1, 1)$, then we see that these two partitions are not comparable under the dominance order. Sometimes we need a total order on partitions, and that's where the *lexicographic* or *dictionary* order becomes useful.

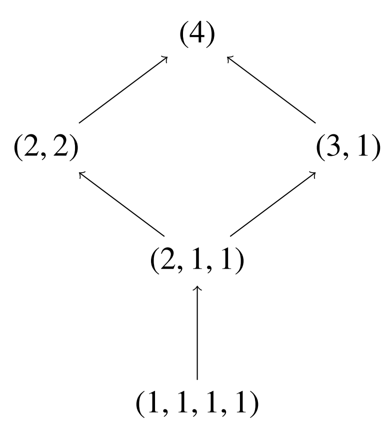
Definition 5. *We define the* **lexicographic order** *or* **dictionary order** *over the set of partitions as*

$$\mu < \lambda \iff \textit{for some } i, \mu_j = \lambda_j \textit{ for all } j < i, \textit{ and } \mu_i < \lambda_i\,.$$

Example 6. *The lexicographic order on the partitions of 4 is*

$$(1, 1, 1, 1) < (2, 1, 1) < (2, 2) < (3, 1) < (4)\,.$$

To see the dominance order on the partitions of 4*, we can put these partitions into a structure called a Hasse diagram. This is a graph with partitions as vertices and we draw an arrow from* μ *to* λ *if* $\mu \lhd \lambda$.



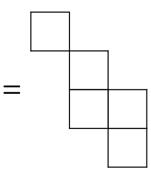
Definition 7. *Given any partition λ, we define its* ***conjugate*** λ' *as the partition that corresponds to the Ferrers diagram of λ flipped along the diagonal $y = x$.*

Example 8. *If $\lambda = (3, 3, 2, 1)$, then from the previous Ferrers diagram we see that $\lambda' = (4, 3, 2)$.*

Definition 9. *If $\mu \subseteq \lambda$, then the* ***skew shape*** *are the cells*

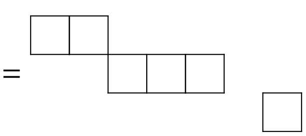
$$\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

Example 10. *If $\lambda = (3, 3, 2, 1)$ and $\mu = (2, 1, 1)$, then the skew shape*

$\lambda/\mu =$ 

Definition 11. *A skew shape λ/μ with m cells is a* **horizontal** m**-strip** *if there is at most one cell in any column.*

Example 12. *The skew shape*

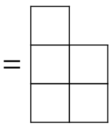
$(7, 5, 2)/(6, 2) =$ 

of 6 cells is a horizontal 6-strip. The skew shape λ/μ of 5 cells of Example 10 is not a horizontal 5-strip.

While most of the cells of a skew shape are not relevant, there are few cells of it which will be useful.

Definition 13. *The cell of a horizontal 1-strip λ/μ is called a* **removable corner** *of λ and an* **addable corner** *of μ. A cell (i, j) of a partition λ with $(i + 1, j + 1) \notin \lambda$ is called an* **extremal cell**.

Example 14. *For the partition*

(2, 2, 1) = 

The cells (3, 1)*,* (2, 2) *are removable cells of* (2, 2, 1)*, and the cells* (4, 1), (3, 2), (1, 3) *are addable cells of* (2, 2, 1)*, and the cell* (2, 1) *along with the removable cells of* (2, 2, 1) *form the extremal cells of* (2, 2, 1).

There are two simple functions on partitions which are quite useful.

Definition 15. *For a given partition λ, set*

$$n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i \qquad \& \qquad z_\lambda = \prod_{i=1}^{\ell(\lambda)} i^{n_i}(n_i!)\,,$$

where n_i is the number of parts of λ equal to i.

Example 16. *For $\lambda = (2, 2, 1)$,*

$$n(\lambda) = (1)(2) + (3)(1) = 5 \quad \& \quad z_\lambda = (1^1(1!))(2^2(2!))(3^0(0!)) = 16\,.$$

3.2 The tableau

Using partitions and their corresponding Ferrers shapes, we are now ready to define a central object in the area of algebraic combinatorics called the semi-standard tableau.

Definition 17. *A* **semi-standard tableau***, or semi-standard Young tableau, of composition weight $\mu = (\mu_1, \ldots, \mu_r)$ is a nested sequence of partitions*

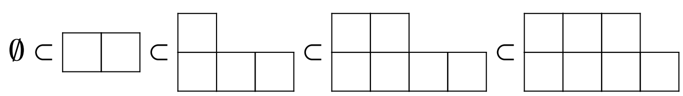
$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} \tag{3.1}$$

such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal μ_i-strip. It is generally represented with a filling of shape $\lambda^{(r)}$

by placing i in the cells of the skew $\lambda^{(i)}/\lambda^{(i-1)}$. When the weight of a tableau is $(1, 1, \ldots, 1)$ it is called a **standard tableau**. *$SSYT(\lambda, \mu)$ denotes the set of semi-standard tableaux of shape λ and composition weight μ and the union over all weights is $SSYT(\lambda)$. Similarly, $SYT(\lambda)$ denotes the set of standard tableaux of shape λ. For any $T \in SSYT(\lambda)$, we denote $wt(T)$ as the* **weight** *of T.*

A more conventional way to think of semi-standard tableau of shape λ is a filling of the Ferrers shape of λ with positive integers, where the rows are weakly increasing and the columns are strictly increasing. For the purposes of this thesis, we will work with Definition 17. Let's consider three examples of semi-standard tableaux.

Example 18. *The nested sequence of shapes*



is the semi-standard tableau

$$T = \begin{array}{|c|c|c|c} \hline 2 & 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

of weight $wt(T) = (2, 2, 2, 1)$.

Example 19. *For the partition $\lambda = (3, 2)$, all of the standard tableaux of weight $(1, 1, 1) = (1^3)$ are*

$$\begin{array}{|c|c} \hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \& \quad \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}.$$

Example 20. *The semi-standard tableau*

$$T = \begin{array}{|c|c|cc} \hline 3 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

has $wt(T) = (1, 2, 2, 1)$.

Definition 21. *For any semi-standard tableau T, the* **reading word** *of T, denoted $read(T)$, is* *obtained by reading the entries of T from left to right in the top row of T, then continuing left to right in the second row from the top of T, etc.*

Example 22. *For the semi-standard tableau*

$$T = \begin{array}{|c|c|cc} \hline 3 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array},$$

$read(T) = [3, 4, 1, 2, 2, 3]$.

Observe that if T is a standard tableau, then the reading word of T is a *permutation* of S_n, where n is the largest number in T.

3.3 Permutations

In this section we recall some of the basic definitions and properties of the symmetric group and the affine symmetric group.

3.3.1 Symmetric group

Definition 23. *For a positive integer n, the* **symmetric group**, *S_n, is the set of all bijections from $\{1, 2, \ldots, n\}$ to itself where composition is multiplication. The generators of this group are $\{s_1, \ldots, s_{n-1}\}$, where s_i is the map fixing all the entries except i and $i+1$ which are interchanged. Furthermore, these generators satisfy the relations*

$$s_i^2 = 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$s_i s_j = s_j s_i \quad \textit{for } i - j \not\equiv 1, n-1 \pmod{n}$$

with all indices related mod n.

For the permuation $\sigma \in S_n$, we will use the one line notation $\sigma_1\sigma_2\cdots\sigma_n$ or $[\sigma_1, \sigma_2, \ldots, \sigma_n]$, where i is sent to σ_i. More generally, a word (or multiset permutation) $\sigma_1\sigma_2\cdots\sigma_n$ is a linear list of the elements of some multiset of non-negative integers. We now define some combinatorial rules on the elements of S_n.

Definition 24. *For any $\sigma \in S_n$, a* **descent** *of σ is an integer i, for which $\sigma_i > \sigma_{i+1}$, for $1 \le i \le n-1$. The set of such i is called the* **descent set** *of σ, and it is denoted $Des(\sigma)$. An* **inversion** *of σ is a pair (i, j), $1 \le i < j \le n$ such that $\sigma_i > \sigma_j$. The* **inversion statistic***, $inv(\sigma)$, is the number of inversions of σ. The* **major index statistic***, $maj(\sigma)$, is the sum of the descents of σ. Namely,*

$$inv(\sigma) = \sum_{\substack{i<j \\ \sigma_i > \sigma_j}} 1\,, \qquad maj(\sigma) = \sum_{\substack{i \\ \sigma_i > \sigma_{i+1}}} i\,.$$

Example 25. *For the permutation $\sigma = [3, 2, 4, 1, 5] \in S_5$, the $Des(\sigma) = \{1, 3\}$ and the set of inversions of σ are $\{(1, 2), (1, 4), (2, 4), (3, 4)\}$. Thus we have,*

$$inv(\sigma) = 4 \quad \text{and} \quad maj(\sigma) = 1 + 3 = 4.$$

For more on the major index, inversion and descent statistics, please see [Hag05].

3.3.2 Affine symmetric group

Definition 26. *The type-A affine Weyl group is realized as the* **affine symmetric group** $\tilde{S}_n$ *given by generators $\{s_0, s_1, \ldots, s_{n-1}\}$ satisfying the relations*

$$s_i^2 = 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$s_i s_j = s_j s_i \quad \text{for } i - j \not\equiv 1, n-1 \pmod{n}$$

with all indices related mod n.

If $w = s_{i_1} \cdots s_{i_\ell} \in \tilde{S}_n$ *and* ℓ *is minimal among all such expressions for* w*, then* $s_{i_1} \cdots s_{i_\ell}$ *is called a* **reduced word** *for* w *and the* **length** *of* w *is defined by* $\ell(w) = \ell$*. The* **weak order** *on* $\tilde{S}_n$ *is defined by the covering relation*

$$w \lessdot z \iff z = s_i w \text{ and } \ell(z) = \ell(w) + 1 \,.$$

Alternatively, $\tilde{S}_n$ is the group of permutations of $\mathbb{Z}$ with the property that $w \in \tilde{S}_n$ acts by $w(i + rn) = w(i) + rn$, for all $r \in \mathbb{Z}$ and $\sum_{i=1}^{n} w(i) = \binom{n+1}{2}$. For $0 \leq i < n$, the elements $s_i \in \tilde{S}_n$ act on $\mathbb{Z}$ by $s_i(i + rn) = i + 1 + rn$, $s_i(i + 1 + rn) = i + rn$, and $s_i(j) = j$ for $j \not\equiv i, i + 1 \pmod{n}$.

Although the simple reflections s_i generate the group, there is also the notion of a transposition $\tau_{i,j}$ defined by its action $\tau_{i,j}(i + rn) = j + rn$ and $\tau_{i,j}(x) = x$ for $x \not\equiv i, j \pmod{n}$. Take integers $i < j$ with $i \not\equiv j \pmod{n}$ and $v = \lfloor (j - i)/n \rfloor$, then $\tau_{i,i+1} = s_i$ and for $j - i > 1$, and

$$\tau_{i,j} = s_i s_{i+1} s_{i+2} \cdots s_{j-v-2} s_{j-v-1} s_{j-v-2} s_{j-v-3} \cdots s_{i+1} s_i$$

where the indices of the simple reflections are taken mod n. For $j > i$, we set $\tau_{i,j} = \tau_{j,i}$.

Definition 27. *The* **strong (Bruhat) order** *is defined by the covering relation*

$$w \lessdot_B u \quad \text{when} \quad u = \tau_{i,j} w \quad \text{and} \quad \ell(u) = \ell(w) + 1 \,. \tag{3.2}$$

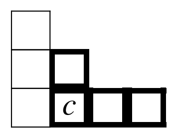
The symmetric group S_n can be viewed as the parabolic subgroup of $\tilde{S}_n$ generated by $\{s_1, s_2, \ldots, s_{n-1}\}$. The left cosets of $\tilde{S}_n / S_n$ are called **affine Grassmannian elements** and they are identified by the set $\tilde{S}_n^0 \subset \tilde{S}_n$ of minimal length coset representatives. One characterization of $\tilde{S}_n^0$ is the set of all reduced words of $\tilde{S}_n$ beginning in s_o. We will see that through this characterization, elements of $\tilde{S}_n^0$ can be conveniently represented by a subset of the set of partitions.

It is the subset of shapes C^n, called n-cores, that are in bijection with affine Grassmannian permutations. To understand C^n, we need the *hook-length* of cells in a Ferrers shape.

Definition 28. *Given a partition λ, the* **hook-length** *of any cell c of λ is the number of cells directly north of c plus the number of cells directly east of c plus one.*

Definition 29. *An n-***core** *is a partition that has no cell whose hook-length is n. Furthermore, the* **content** *of any cell (i, j) is $j - i$ and its n-***residue** *is $j - i \pmod n$.*

Example 30. *The partition $\lambda = (4, 2, 1)$ has Ferrers shape*



and the cell c in this Ferrers shape has hook-length 4. Observe that none of the cells of λ have hook-length 5, and that's because this is a 5-core. The 5-residue of the cell c is $1 \equiv (2-1) \mod 5$.

There is a left action of the generator $s_i \in \tilde{S}_n$ on an n-core λ with at least one addable corner of residue i; it is defined by letting $s_i\lambda$ be the shape where all corners of residue i have been added to λ. This extends to a bijection [Las99, LM05]

$$\mathfrak{a} : \tilde{S}_n^0 \longrightarrow C^n ,$$

where $\lambda = \mathfrak{a}(w) = s_{i_1} \cdots s_{i_\ell}\emptyset$ for any reduced word $i_1 \cdots i_\ell$ of w. We use w_λ to denote the preimage of λ under $\mathfrak{a}$.

Definition 31. *The* **degree** *of an n-core λ, $deg(\lambda)$, is $\ell = \ell(w_\lambda)$.*

An n-core λ has an addable corner of residue i if and only if

$$\ell(w_{s_i\lambda}) = \ell(w_\lambda) + 1 . \tag{3.3}$$

Property 32. *[LM05] For an n-core λ with extremal cells c and c' of the same n-residue, given that c is weakly north-west of c', if c is at the end of its row, then so is c'. If c has a cell above it, then so does c'.*

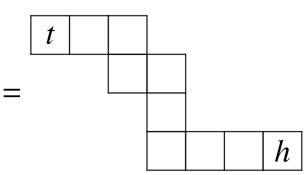
The strong order on the subset $\tilde{S}_n^0$ is characterized on elements of C^n by the containment of diagrams and its covering relation is

$$\mu \lessdot_B \lambda \iff \mu \subset \lambda \text{ and } deg(\lambda) = deg(\mu) + 1\,.$$

Given a pair $\mu \lessdot_B \lambda$, the shape λ/μ can be described by ribbons.

Definition 33. *For $r \geq 0$, an r-**ribbon** R is a skew diagram λ/μ consisting of r rookwise connected cells such that there is no 2×2 shape contained in R. In a ribbon, the southeasternmost cell is called its **head** and the northwesternmost cell is its **tail**. The **height of a ribbon** is the number of rows it occupies.*

Example 34. *The skew shape*

$(7, 4, 4, 3)/(3, 3, 2) =$ 

is a 10-ribbon of height 4. The tail is labeled t and the head is labeled h of the 10-ribbon.

Lemma 35. *[LLMS10] If $w \lessdot_B \tau_{r,s}w$ in $\tilde{S}_n^0$, then the skew $\mathfrak{a}(\tau_{r,s}w)/\mathfrak{a}(w)$ is made up of copies of one fixed ribbon such that the head of each copy has residue $s-1$ and the tail has residue r.*

Lemma 36. *Given $w \lessdot_B \tau_{r,s}w$ in $\tilde{S}_n^0$,*

$$\mathfrak{a}(\tau_{r,s}w) = \mathfrak{a}(w) + \textit{ all addable ribbons with a head of residue } r-1 \textit{ and tail of residue } s\,.$$

Proof. Since $\tau_{r,s} = s_r s_{r+1} \cdots s_{s-1} \cdots s_r$, any addable ribbon of $\mathfrak{a}(w)$ with a head of residue $s-1$ and a tail of residue r is added to $\mathfrak{a}(w)$ under multiplication by $\tau_{r,s}$. The result then follows from Lemma 35. □

Chapter 4
Ring of symmetric functions

Let R to be a commutative ring with $x_1, \ldots, x_n$ a set of n indeterminants. The set of all vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers form a monoid under addition. Furthermore, this monoid is isomorphic to the monoid of all monomials x^α under multiplicaiton:

$$x^\alpha x^\beta = x^{\alpha+\beta} \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We call the corresponding algebra the **ring of polynomials**, denoted $R[x_1, \ldots, x_n]$, which consists of all polynomials in n variables with coefficients in R.

We set $deg(x^\alpha) = |\alpha| = \alpha_1 + \cdots + \alpha_n$. An element

$$f(x_1, \ldots, x_n) = \sum_{\substack{\alpha \\ |\alpha|=d}} c_\alpha x^\alpha.$$

of $R[x_1, \ldots, x_n]$ is called a **homogenous polynomial** of degree d. If $R[x_1, \ldots, x_n]^d$ is the set of all polynomials of degree d, then we have a grading of the ring of polynomials

$$R[x_1, \ldots, x_n] = \bigoplus_{0 \leq d} R[x_1, \ldots, x_n]^d.$$

There is a natural S_n **action** on polynomials which is degree-preserving, where elements of the symmetric group S_n act by permuting the variables. Namely, for $\sigma \in S_n$ and $P(x) \in R[x_1, \ldots, R_n]$,

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Example 37. *If* $\sigma = [3, 2, 1]$ *is the permutation of* S_3 *in one-line notation, then*

$$\sigma(x_1^2 + x_2^3 + x_1 x_3) = x_3^2 + x_1^3 + x_3 x_2.$$

Definition 38. *A polynomial* $P(x) \in R[x_1, \ldots, x_n]$ *is said to be a* **symmetric polynomial** *if* $\sigma P(x) = P(x)$ *for all* $\sigma \in S_n$. *The set of symmetric polynomials of degree d in n-variables is denoted* Λ_n^d *and*

$$\Lambda_n = \bigoplus_d \Lambda_n^d$$

is the **ring of symmetric polynomials**.

4.1 Monomial symmetric functions

We now consider a particular set of symmetric polynomials constructed by symmetrizing the monomial term x^λ, for a partition λ.

Definition 39. *The* **monomial symmetric function** *indexed by a partition* λ *is*

$$m_\lambda = \sum_{\substack{\beta \\ \beta^* = \lambda}} x^\beta,$$

over all distinct β *where* β^* *is the partition rearrangements of* β.

Example 40. *Suppose we consider the ring of symmetric polynomials in* $n = 4$ *variables. If* $\lambda = (2, 2, 1)$, *then all the rearrangements of* λ *are* $(2, 2, 1, 0), (2, 2, 0, 1), (2, 0, 2, 1), (2, 0, 1, 2), (1, 2, 2, 0), (1, 2, 0, 2), (1, 0, 2, 2), (2, 1, 2, 0), (2, 1, 0, 2), (0, 2, 2, 1), (0, 2, 1, 2)(0, 1, 2, 2)$. *This tells us that*

$$m_{(2,2,1)} = x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + x_1^2 x_3^2 x_4 + x_1^2 x_3 x_4^2 + x_1 x_2^2 x_3^2 + \\ + x_1 x_2^2 x_4^2 + x_1 x_3^2 x_4^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2 x_4^2 + x_2^2 x_3^2 x_4 + x_2^2 x_3 x_4^2 + x_2 x_3^2 x_4^2. \quad (4.1)$$

It is clear that the m_λ's are independent, thus the set $\{m_\lambda \mid |\lambda| = d\}$ spans Λ_n^d. Note that if λ has more than n parts, then we don't have enough variables. To avoid this issue, we work in the vector space spanned by all the m_λ.

Definition 41. *The* **ring of symmetric functions** *is the vector space over the commutative ring R spanned by all* m_λ*:*

$$\Lambda = R[m_\lambda].$$

If Λ^d is the space spanned by all m_λ of degree d, then Λ forms the graded ring

$$\Lambda = \bigoplus_{0\leq d} \Lambda^d.$$

Theorem 42. *[Sag00]* $\{m_\lambda \mid |\lambda| = d\}$ *forms a basis for* Λ^d*.*

4.2 Elementary symmetric functions

In this section we begin by defining another set of symmetric functions.

Definition 43. *For each partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$*, the* **elementary symmetric function** *is the function* $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$*, where*

$$e_r(x_1, \ldots, x_n) = \sum_{1\leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad \text{and} \quad e_0 = 1, \quad e_r = 0 \ \forall r > n.$$

Example 44. *If* $\lambda = (2, 1)$*, then since* $e_1 = x_1 + x_2 + \cdots$*, and* $e_2 = x_1x_2 + x_1x_3 + \cdots$*, we get* $e_{(2,1)} = (x_1 + x_2 + \cdots)(x_1x_2 + x_1x_3 + \cdots)$*. Note also that if we have the partition* $(1^r) = (1, \ldots, 1)$*, where there are r ones, then* $m_{(1^r)} = e_r$*. Since* $deg(e_r) = r$*, then* $deg(e_\lambda) = |\lambda|$*.*

Expanding the elementary symmetric functions in terms of the monomial symmetric functions shows us that the elementary symmetric functions are indeed symmetric and that they form a basis for Λ.

Theorem 45. *[Sag00]* $\{e_\lambda \mid |\lambda| = d\}$ *forms a basis for* Λ^d.

4.3 Homogeneous symmetric functions

In this section we begin by defining the homogeneous symmetric functions.

Definition 46. *For each partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$*, the* **homogeneous symmetric function** *is the function* $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$*, where*

$$h_r(x_1, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad \textit{and} \quad h_0 = 1, \quad e_r = 0 \ \forall r > n.$$

Sometimes, h_λ is referred to as the *complete symmetric function*.

Example 47. *If* $\lambda = (2, 1)$*, then since* $h_1 = x_1 + x_2 + \cdots$*, and* $h_2 = x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots$*, we get* $h_{(2,1)} = (x_1 + x_2 + \cdots)(x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots)$.

Theorem 48. *[Sag00] For any partition* λ*, the function* h_λ *is a symmetric function, and the set* $\{h_\lambda \mid |\lambda| = d\}$ *forms a basis for* Λ^d.

Using the definition of the homogeneous and monomial symmetric functions, we can define an inner product on the space Λ.

Definition 49. *The* **Hall-inner product** *on the space* Λ *is defined by setting*

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

where $\delta_{\lambda\mu} = 0$ *when* $\lambda \neq \mu$ *and is 1 otherwise.*

4.4 Power symmetric functions

In this section we begin by defining the power symmetric functions.

Definition 50. *For each partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, the* **power symmetric function** *is the function $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, where*

$$p_r = \sum_i x_i^r \quad and \quad p_0 = 1, \quad p_r = 0 \; \forall r > n.$$

Example 51. *For the partition* (1)*, $p_{(1)} = x_1 + x_2 + \cdots = e_1 = h_1$. In general, $p_{(r)} = x_1^r + x_2^r + \cdots = m_{(r)}$.*

Theorem 52. *[Sag00] For any partition λ, the function p_λ is a symmetric function, and the set $\{p_\lambda \mid |\lambda| = d\}$ forms a basis for Λ^d.*

4.5 Schur functions

We next study an important set of symmetric functions called the *Schur functions*.

Definition 53. *Given a partition λ, we define the corresponding* ***Schur function****, s_λ, as*

$$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^{wt(T)}.$$

The Schur functions have been tied to irreducible representations of the symmetric group S_n, and play an important role as Schubert classes of the Grassmannian variety. Let's consider some examples of Schur functions.

Example 54. *For the partition $\lambda = (2, 1)$, we have the set of*

$$SSYT(\lambda) = \left\{ \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}, \ldots, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 4 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{4} \\ \hline \end{array}, \ldots \right\}.$$

This gives us the Schur function

$$\begin{aligned} s_{(2,1)}(x) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 2x_1x_2x_3 + 2x_1x_2x_4 + \cdots \\ &= m_{(2,1)} + 2m_{(1,1,1)}. \end{aligned}$$

Observe that in this Example we do not restrict the number of variables.

Example 55. *Suppose $\lambda = (1^k)$, the partition whose parts are all ones and of length k. For this special case, the Schur function is*

$$s_{(1^k)}(x) = \sum_{T \in SSYT((1^k))} x^{wt(T)} = e_k.$$

To see this, note that the set $SSYT((1^k))$ contains only semi-standard tableaux that are vertical strips of length k. Since we know that each one of these tableaux has increasing columns, then we see that each $T \in SSYT((1^k))$ is unique. Thus, any term of x^T in the sum must be a monomial with variables having exponent either zero or one. Namely each term is of the form $x_{i_1}x_{i_2}\ldots x_{i_k}$ for $1 \leq i_1 < i_2 < \ldots < i_k$. These are precisely the terms of e_k.

Example 56. *Suppose now $\lambda = (k)$. In this case we have*

$$s_{(k)}(x) = \sum_{T \in SSYT((k))} x^T = h_{(k)}.$$

To see this, note that the set $SSYT((k))$ contains only semi-standard tableaux that look like horizontal strips of length k. Since we know that each one of these tableaux has weakly increasing rows, then we see that each $T \in SSYT((k))$ contains any k subset of n. For instance, T can contain k ones or it can contain exactly the set $\{1, \ldots, k\}$. Thus we see that any term, x^T, is a monomial in variables to any power. Namely each term is of the form $x_{i_1}x_{i_2}\ldots x_{i_k}$ for $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k$. These are precisely the terms of h_k.

Before we show that the Schur functions form a basis for Λ^n, we must show that they are symmetric.

Proposition 57. *[Sag00] For any partition λ, The Schur function $s_\lambda(x)$ is a symmetric function.*

Proof. One way to write the Schur functions is

$$s_\lambda = \sum_\alpha K_{\lambda\alpha} x^\alpha,$$

where $K_{\lambda\alpha}$ is the number of semi-standard tableaux of shape λ and weight α. Furthermore we know that any element of the symmetric group can be written as a product of transpositions. Thus it suffices to show that $K_{\lambda\sigma_i(\alpha)} = K_{\lambda\alpha}$, where σ_i is a transposition. To show this, we find an involution ϕ_i on the set $SSYT(\lambda, \alpha)$. Namely, we define the Bender-Knuth involution

$$\phi_i = SSYT(\lambda, \alpha) \longrightarrow SSYT(\lambda, \sigma_i(\alpha))$$

as follows. Given $T \in SSYT(\lambda, \alpha)$, each column contains either an $i, i+1$ pair; exactly one of $i, i+1$; or neither. Call the pairs fixed and all other occurrences of i or $i+1$ free. Define $\phi_i(T)$ by switching the number of free i's and $(i+1)$'s in each row. To illustrate, if $i = 2$ and

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 3 \\ \hline 2 & 2 & 3 & 3 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 \\ \hline \end{array}$$

then the 2's and 3's in columns two through four and seven through ten are free. So

$$\phi_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 3 \\ \hline 2 & 2 & 2 & 3 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ \hline \end{array}$$

It is easy to show that $\phi_i(T)$ is a semi-standard tableau. Since the fixed i's and $(i+1)$'s come in pairs, this map has the desired exchange property. It is also clear that ϕ_i is an involution. Thus s_λ is a symmetric function. □

Since we know that s_λ is symmetric then it can be written as a linear combination of the monomial symmetric functions, m_μ. In particular if $\mu \vdash n$, then

$$s_\lambda = \sum_{\substack{\mu \\ \mu \vdash |\lambda|}} K_{\lambda\mu} m_\mu$$

where the **Kostka number** $K_{\lambda\mu}$ is the number of tableau of shape λ and weight μ. For a given $n > 0$, we get the following linear system.

$$\begin{bmatrix} s_{\lambda^{(1)}} \\ s_{\lambda^{(2)}} \\ \vdots \end{bmatrix} = \begin{bmatrix} K_{\lambda^{(1)}\lambda^{(1)}} & K_{\lambda^{(1)}\lambda^{(2)}} & \dots \\ K_{\lambda^{(2)}\lambda^{(1)}} & K_{\lambda^{(2)}\lambda^{(2)}} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} m_{\lambda^{(1)}} \\ m_{\lambda^{(2)}} \\ \vdots \end{bmatrix}$$

Call the coefficient matrix of this linear system the **Kostka matrix**, $[K_{\lambda\mu}]_{\lambda,\mu\vdash n}$.

Theorem 58. *[Sag00] The Kostka matrix* $[K_{\lambda\mu}]_{\lambda,\mu\vdash n}$ *is invertible.*

Corollary 59. $\{s_\lambda : \lambda \vdash n\}$ *is a basis for* Λ^n.

Example 60. *The set of all partitions of* 3 *are* $S = \{(1,1,1),(2,1),(3)\}$. *Theorem 58 gives us the system*

$$\begin{bmatrix} s_{(3)}(x) \\ s_{(2,1)}(x) \\ s_{(1,1,1)}(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{(3)}(x) \\ m_{(2,1)}(x) \\ m_{(1,1,1)}(x) \end{bmatrix}$$

where the coefficient matrix is $[K_{\lambda\mu}]_{\lambda,\mu\vdash 3}$.

Chapter 5

Symmetric functions over $\mathbb{Q}(q,t)$

In this chapter we consider symmetric functions with additional parameters t and q. Specifically, we will first consider the Hall-Littlewood polynomials which are symmetric functions with coefficients in $\mathbb{Q}(t)$, the set of rational functions in t. We will then consider the famous Macdonald polynomials which are also symmetric functions with coefficients in $\mathbb{Q}(q,t)$, the set of rational functions in q and t.

5.1 Hall-Littlewood polynomials

The Hall-Littlewood polynomials were originally defined as a basis for the algebra of symmetric functions depending on a parameter t. The motivation for the Hall-Littlewood polynomials comes from problems in group theory that led to the definition of the Hall algebra [Lit61]. The Hall-Littlewood polynomials have also been known to have applications to character theory of finite linear groups [Gre55], representations of symmetric groups [Mor65][Sch11], affine Hecke algebras [Lus81], statistical mechanics [KR88] and representations of quantum affine algebras and affine crystals [LLT97] [LS07].

From the symmetric functions view point, the Hall-Littlewood polynomials have a definition which is similar to the Schur functions. The Hall-Littlewood polynomials are a basis of the ring of symmetric functions over a field containing a parameter t. Namely, for the field of fractions over the polynomials in t, $\mathbb{Q}((t))$, we set

$$\Lambda_t = \mathbb{Q}((t))[h_1, h_2, \ldots].$$

In Λ_t, we have the following definition of the Hall-Littlewood polynomials.

Definition 61. *The* **Hall-Littlewood polynomials**, *$H_\lambda(x;t)$, are defined as the symmetric functions satisfying*

$$H_\lambda(x;t) = s_\lambda + \text{terms of the form } r_{\lambda\mu}(t)\, s_\mu, \tag{5.1}$$

for $\mu > \lambda$ (under the lexicographic or dictionary order), and

$$\langle H_\lambda(x;t), H_\mu(x;t)\rangle_t = 0$$

if $\lambda \neq \mu$. Here, the scalar product $\langle \cdot, \cdot \rangle_t$ is defined so that

$$\langle p_\lambda, p_\mu\rangle_t = z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i}).$$

We can use this definition to compute the Hall-Littlewood polynomials.

Example 62. *In this example we will show how to compute Hall-Littlewood polynomials indexed by partitions of 3. Immediately, Equation 5.1 gives us that $H_{(3)}(x;t) = s_{(3)}(x)$. Next, to compute $H_{(2,1)}(x;t)$, Equation 5.1 tells us that*

$$H_{(2,1)}(x;t) = s_{(2,1)}(x) + r_{(2,1),(3)}(t)\, s_{(3)}(x)\,. \tag{5.2}$$

This tells us that

$$\langle H_{(2,1)}(x;t), H_{(3)}(x;t)\rangle_t = \langle s_{(2,1)}(x), H_{(3)}(x;t)\rangle_t + r_{(2,1),(3)}(t)\langle s_{(3)}(x;t), s_{(3)}(x;t)\rangle_t\,. \tag{5.3}$$

Solving for $r_{(2,1),(3)}(t)$ in Equation 5.3, and substituting into Equation 5.2, we get

$$H_{(2,1)}(x;t) = s_{(2,1)} - \frac{\langle s_{(2,1)}, H_{(3)}\rangle_t}{\langle H_{(3)}, H_{(3)}\rangle_t} H_{(3)}.$$

By caclulating the expansion of Schur functions in terms of power symmetric functions, we get that

$\langle s_{(2,1)}, s_{(3)}\rangle_t = t^2 - t$ and $\langle s_{(3)}, s_{(3)}\rangle_t = 1 - t$. Thus, we have $H_{(2,1)}(x;t) = s_{(2,1)}(x) + ts_{(3)}(x)$.

A similar method can be used to compute $H_{(1,1,1)}(x;t)$. Namely,

$$H_{(1,1,1)}(x;t) = s_{(1,1,1)} - \frac{\langle s_{(1,1,1)}, H_{(2,1)}\rangle_t}{\langle H_{(2,1)}, H_{(2,1)}\rangle_t} H_{(2,1)} - \frac{\langle s_{(1,1,1)}, H_{(3)}\rangle_t}{\langle H_{(3)}, H_{(3)}\rangle_t} H_{(3)}.$$

Using the previous example, we get that the scalar product $\langle s_{(1,1,1)}, s_{(2,1)}\rangle_t = t^2 - t$, $\langle s_{(1,1,1)}, s_{(3)}\rangle_t = t^2 - t^3$, and $\langle s_{(2,1)}, s_{(2,1)}\rangle_t = (1-t)(1-t+t^2)$. From these calculations, we find that

$$H_{(1,1,1)}(x;t) = s_{(1,1,1)}(x) + (t+t^2)s_{(2,1)}(x) + t^3 s_{(3)}(x).$$

There are many other ways to compute the Hall-Littlewood polynomials. For instance, they can be defined by means of the creation operators [Jin91]. Another way to compute the Hall-Littlewood polynomials is through the *Kostka-Foulkes polynomials* and the Schur functions. For given partitions λ and μ, the Kostka-Foulkes polynomials, $K_{\lambda,\mu}(t)$, are families of polynomials that generalize the Kostka numbers. Specifically, $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. It was the work of Lascoux and Schützenberger in 1978, [LS78], which beautifully characterized the Kostka-Foulkes polynomials and gave the transition matrix between Hall-Littlewood and Schur polynomials. Their work showed an intrinsically positive formula for the Kostka-Foulkes polynomials by associating a statistic (non-negative integer) called *cocharge* to each semi-standard tableaux.

5.1.1 Cocharge of a tableau

Lascoux and Schützenberger found an intrinsically positive formula for the Kostka-Foulkes polynomials by associating a statistic (non-negative integer) called cocharge to each semi-standard tableaux and proving that

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{\text{cocharge}(T)}. \tag{5.4}$$

Definition 63. *The* **cocharge** *of a standard tableau T is the sum of the entries in the index vector*

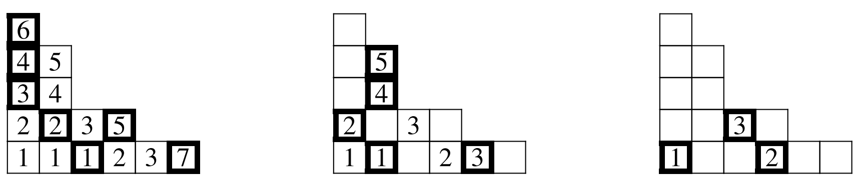
$I(T) = [0, I_2, \ldots, I_m]$ which is defined by setting $I_r = I_{r-1}$ when the content of r is larger than the content of $r-1$ and otherwise setting $I_r = I_{r-1} + 1$.

The notion is extended to give the cocharge of a semi-standard tableau with generic weight by successively computing the index of an appropriate subset of i cells containing the letters $1, 2, \ldots, i$.

Definition 64. *From a specific x in cell c of a tableau T, the desired choice of $x+1$ is the southeasternmost one lying above c. If there are none above c, the choice is the southeasternmost $x+1$ in all of T.*

Consider now any semi-standard tableau T with partition weight. Starting from the rightmost 1 in T, use Definition 64 to distinguish a standard sequence of i cells containing $1, 2, \ldots, i$. Compute the index and then delete all cells in this sequence. Repeat the process on the remaining cells. The total *cocharge* is defined to be the sum of all the index vectors.

Example 65. *The* cocharge *of the following tableau is 25:*



$I = [0, 1, 2, 3, 3, 4, 4]$ $\qquad$ $I = [0, 1, 1, 2, 3]$ $\qquad$ $I = [0, 0, 1]$

Theorem 66. *[LS78] For partitions λ and μ,*

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{cocharge(T)}.$$

The Kostka-Foulkes polynomials appear in other contexts including affine Kazhdan-Lusztig theory [Lus81] and affine tensor product multiplicities [NY97]. Furthermore, the Kostka-Foulkes polynomials encode the dimensions of certain bigraded S_n-modules [GP92].

5.1.2 Charge of a tableau

While one can state the results of Lascoux and Schützenberger using the cocharge statistic on semi-standard tableaux, it is sometimes convenient to state the results in terms of the *charge* statistic on them. The charge statistic is computed in a similar spirit to that of the cocharge, except this time we redefine the index vector over choice of the standard sequence of the tableau.

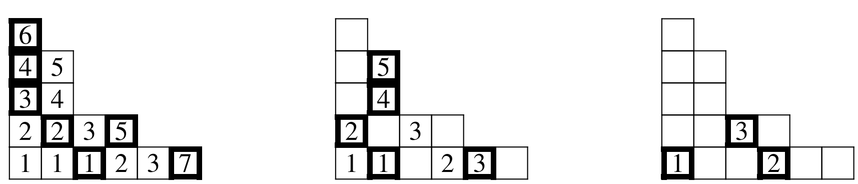
Definition 67. *The* **charge** *of a standard tableau T is the sum of the entries in the index vector $I(T) = [0, I_2, \ldots, I_m]$ which is defined by setting $I_r = I_{r-1}$ when the content of r is smaller than the content of $r-1$ and otherwise setting $I_r = I_{r-1} + 1$.*

The notion is extended to give the charge of a semi-standard tableau with generic weight by successively computing the index of an appropriate subset of i cells containing the letters $1, 2, \ldots, i$.

Algorithm 68. *From a specific x in cell c of a tableau T, the desired choice of $x + 1$ is the south-easternmost one lying above c. If there are none above c, the choice is the south-easternmost $x + 1$ in all of T.*

Consider now any semi-standard tableau T with partition weight. Starting from the rightmost 1 in T, use Algorithm 68 to distinguish a standard sequence of i cells containing $1, 2, \ldots, i$. Compute the index and then delete all cells in this sequence. Repeat the process on the remaining cells. The total *charge* is defined to be the sum of all the index vectors.

Example 69. *The* charge *of the following tableau is 9:*



$I = [0, 0, 0, 0, 1, 1, 2]$ $\qquad$ $I = [0, 0, 1, 1, 1]$ $\qquad$ $I = [0, 1, 1]$

We can now define the charge analog of the Kostka-Foulkes polynomials. We then show how these polynomials are related to the Kostka-Foulkes polynomials.

Definition 70. *For partitions λ and μ, let*

$$\tilde{K}_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{charge(T)}$$

We can now use the following fact to relate the charge and cocharge statistic over semi-standard tableaux.

Theorem 71. *[Hag05] Given partitions λ and μ, if $T \in SSYT(\lambda, \mu)$, then*

$$\text{charge}(T) = n(\mu) - \text{cocharge}(T)\,.$$

Theorem 71 gives us the following Corollary relating K and $\tilde{K}$.

Corollary 72. *For partitions λ and μ,*

$$K_{\lambda,\mu}(t) = t^{n(\mu)}\tilde{K}_{\lambda,\mu}(1/t)\,.$$

The work of Lascoux and Schützenberger on the Kostka-Foulkes polynomials was in the Coxeter group of type A. However, no generalization of this formula to other types is known. There are other applications of the type A Hall-Littlewood polynomials that extend to arbitrary types. Some of the applications are related to fermionic multiplicity formulas [AK07] and affine crystals [LS07]. For more on Hall-Littlewood polynomials of type A and other arbitrary types, we refer the reader to [JDT94] [Mac95], [NR03], [Ste05].

5.2 Macdonald polynomials

In this section we discuss a set of famous symmetric functions over $\mathbb{Q}(q,t)$ known as the Macdonald polynomials. During the 1980's, it was discovered that Selberg's integral, [Sel44], had a number of extensions. One of these extensions, due to Kadell, involved inserting symmetric functions which depended on a partition, a set of n variables in x_i, and another parameter. These are now known as the Jack symmetric functions, first studied by H. Jack [Jac70]. A problem of finding a q-analogue of the Jack symmetric functions which are an extension of Selberg's integral was posed by Kadell, [Kad88]. This was soon solved afterwords by Macdonald, [Mac88], and these q-analogues of the Jack symmetric functions are now famously referred to as the *Macdonald polynomials*, $P_\lambda(x;q,t)$. More on the connection of $P_\lambda(x;q,t)$ to Kadell's generalization of Selberg's integral can be found in [Mac95].

The Macdonald polynomials $P_\lambda(x;q,t)$ are symmetric functions with coefficients in $\mathbb{Q}(q,t)$, the set of rational functions in q and t. These Macdonald polynomials form a basis of the space of symmetric functions with two parameters

$$\Lambda_{q,t} = \mathbb{Q}((q,t))[h_1, h_2, \ldots].$$

Macdonald's construction of $P_\lambda(x;q,t)$, involves the dominance order over partitions and an extension of the Hall-inner product.

Definition 73. *The* q,t **extension of the Hall-inner product** *is given by*

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

With this definition in mind, the following conditions uniquely define a family of symmetric functions $\{P_\lambda(x;q,t)\}$ parameterized by partitions λ, and these functions have coefficients in $\mathbb{Q}(q,t)$.

Definition 74. *The* **Macdonald polynomial** $P_\lambda(x; q, t)$ *is defined uniquely by the condition*

$$P_\lambda(x; q, t) = \sum_{\mu \lhd \lambda} c_{\lambda,\mu} m_\mu,$$

where $c_{\lambda,\mu} \in \mathbb{Q}[q, t]$ *and* $c_{\lambda,\lambda} = 1$*, and the condition*

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \text{ if } \lambda \neq \mu.$$

Macdonald proved that when the parameters q and t are fixed or set equal in $P_\lambda(x; q, t)$, then $P_\lambda(x; q, t)$ relates to the symmetric function bases of Section 4.

Proposition 75. *[Mac88] For a partition* λ*,*

1. $P_\lambda(x; t, t) = s_\lambda(x)$
2. $P_\lambda(x, q, 1) = m_\lambda(x)$
3. $P_\lambda(x, 1, t) = e_{\lambda'}(x)$
4. *when* $\lambda = (1^n)$, $P_{(1^n)}(x; q, t) = e_n(x) = s_{(1^n)}(x)$.

5.3 The q, t-Kostka polynomials

Macdonald's study of $P_\lambda(x; q, t)$ found some very interesting properties. To see some of these properties, we first modify $P_\lambda(x; q, t)$. Let $J_\mu(x; q, t)$ denote the so-called Macdonald integral form, defined in [Hag05]. Expanding J_μ in terms of $s_\lambda(x(1 - t))$ gives us

$$J_\mu(x; q, t) = \sum_{\lambda \vdash |\mu|} K_{\lambda,\mu}(q, t)\ s_\lambda(x(1 - t)),$$

for some $K_{\lambda,\mu}(q,t) \in \mathbb{Q}[q,t]$. The famous Macdonald positivity conjecture stated that

$$K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]\,.$$

The work of M. Haiman on the geometry of Hilbert schemes was ultimately needed to prove the positivity conjecture [Hai01]. Macdonald posed a refinement of his positivity conjecture which is still open today.

Conjecture 76. *For given partitions λ, μ, there are statistics q-stat(T,μ) and t-stat(T,μ) given by some combinatorial rule such that*

$$K_{\lambda,\mu} = \sum_{\lambda \in SYT(\lambda)} q^{qstat(T,\mu)} t^{tstat(T,\mu)}.$$

The work of A. Garsia and M. Haiman [GH96b] shows it is more natural to work with the polynomials

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,1/t).$$

These polynomials are connected to the Kostka-Foulkes polynomials.

$$\tilde{K}_{\lambda,\mu}(0,t) = K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{cocharge(T)}.$$

Macdonald found a statistical description for $\tilde{K}_{\lambda,\mu}(q,t)$ when $\lambda = (n-k, 1^k)$ is a hook-shape [Mac95]. A combinatorial description of $\tilde{K}_{\lambda,\mu}(q,t)$ when μ is any hook-shape was proven by J. Stembridge [Ste94]. Stembridge's result can be stated using a modification of the major index, the *co-major index*, of a permutation σ.

Definition 77. *For a permutation $\sigma \in S_n$,*

$$coma j(\sigma) = \sum_{\substack{i \\ \sigma_i > \sigma_{i+1}}} (n-i)\,.$$

With the co-major index, we can express Stembridge's result of the q, t-Kostka polynomials.

Theorem 78. *[Ste94] For partition λ and $\mu = (n-k, 1^k)$,*

$$\tilde{K}_{\lambda,\mu} = \sum_{T \in SYT(\lambda)} q^{ma j(T,\mu)} t^{coma j(T, rev(\mu'))},$$

where $rev(\eta) = (\eta_\ell, \eta_{\ell-1}, \ldots, \eta_1)$ for any composition η into ℓ parts.

Until the mid 1990's, there were no combinatorial results for $\tilde{K}_{\lambda,\mu}$ when μ has more than one column. In 1995, the first statistic for $\tilde{K}_{\lambda,\mu}$ when μ has two columns was obtained by S. Fischel [Fis95]. Soon after, L. Lapointe and J. Morse [LM98] and M. Zabrocki [Zab98a] independently found alternate descriptions for the two column case of μ in $\tilde{K}_{\lambda,\mu}$. A paper by A. Garsia and J. Remmel [GR96] details a recursive formula for the $\tilde{K}_{\lambda,\mu}$ when λ is a hook plus the square $(2,2)$. In the late 1990's, A. Garsia and G. Tesler [GT96] proved that $\tilde{K}_{\lambda,\mu}$ is a polynomial with positive integer coefficients when λ is a hook plus the square $(2,2)$ plus the cell in position $(2,3)$ or $(3,2)$.

A more recent combinatorial result in 2005 comes from the work by J. Haglund, M. Haiman and N. Loehr using the inversion and the major statistics. To see this result we need the property of *Yamanouchi* of a word.

Definition 79. *A word $w \in \mathbb{Z}_+^n$ is* **Yamanouchi** *if each of its final segments $w_k, w_{k+1}, \ldots, w_n$ has partition weight. Set $Yam(\lambda)$ to be the set of Yamanouchi words with weight $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}$.*

Example 80. *For the partition $(2,2,1)$, the set*

$$Yam((2,2,1)) = \{32211, 32121, 23121, 21321, 23211\}\,.$$

Theorem 81. *[HHL05] For a partition μ with $\mu_1 \leq 2$,*

$$\tilde{K}_{\lambda,\mu} = \sum_{\substack{\sigma:\mu\to\mathbb{Z}_+ \\ w(\sigma)\in Yam(\lambda)}} q^{inv(\sigma)} t^{maj(\sigma)}.$$

Chapter 6
Affine Schubert calculus

Recall that the ring of symmetric functions Λ is generated over $\mathbb{Z}$ by the homogeneous symmetric functions $h_1, h_2, \ldots$. The ring Λ is equipped with an algebra involution $\omega : \Lambda \to \Lambda$ given by $\omega(h_i) = e_i$, where e_i is the elementary symmetric function. We use the involution ω on Λ to give it the structure of a Hopf algebra.

The ring Λ has a **coproduct** $\Delta : \Lambda \to \Lambda \otimes_{\mathbb{Z}} \Lambda$ given by

$$\begin{aligned} \Delta : \Lambda &\longrightarrow \Lambda \otimes_{\mathbb{Z}} \Lambda \\ h_i &\longmapsto \sum_{0 \le j \le i} h_j \otimes h_{i-j} \end{aligned}$$

where $h_0 := 1$. The coproduct together with the Hall-inner product gives Λ the structure of a self-dual commutative and cocommutative Hopf algebra. There is a sub-Hopf algebra and a quotient of it that is of most interest.

Definition 82. *For any positive integer n, set*

$$\Lambda^{(n)} = \Lambda / \langle m_\lambda \mid \lambda_1 \ge n \rangle \quad \textit{and} \quad \Lambda_{(n)} = \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}].$$

We see that $\Lambda^{(n)}$ is a quotient of the Hopf algebra of Λ, and $\Lambda_{(n)}$ is a sub-Hopf algebra of Λ. Furthermore, the Hall-inner product, $\langle \cdot, \cdot \rangle$, gives $\Lambda^{(n)}$ and $\Lambda_{(n)}$ the structures of dual Hopf algebras. Possible candidates for dual bases are $\{m_\lambda \mid \lambda_1 \le n-1\}$ for $\Lambda^{(n)}$ and $\{h_\lambda \mid \lambda_1 \le n-1\}$ for $\Lambda_{(n)}$. The involution ω of Λ restricts to an involution on $\Lambda_{(n)}$. By duality, there is an involution ω^+ of $\Lambda^{(n)}$ characterized by $\langle f, g \rangle_\Lambda = \langle \omega(f), \omega^+(g) \rangle_\Lambda$, for $f \in \Lambda_{(n)}$ and $g \in \Lambda^{(n)}$.

6.1 Affine Grassmannian Gr

The **affine Grassmannian** of $G = SL(n, \mathbb{C})$ is given by $\mathrm{Gr} = G(\mathbb{C}((t))/G(\mathbb{C}[[t]]$, where $\mathbb{C}[[t]]$ is the ring of formal power series and $\mathbb{C}((t)) = \mathbb{C}[[t]][t^{-1}]$ is the ring of formal Laurent series. The algebraic nil-Hecke ring construction of Kostant and Kumar [KK86] and the work of Peterson [Pet97] developed the study of Schubert bases associated to Schubert cells of Gr in the Bruhat decomposition of $G(\mathbb{C}((t)))$,

$$\{\xi^w \in H^*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}_n^0\} \quad \text{and} \quad \{\xi_w \in H_*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}_n^0\},$$

indexed by affine Grassmannian permutations. The cap product yields a pairing

$$\langle \cdot\, , \cdot \rangle_{\mathrm{Gr}} : H^*(\mathrm{Gr}) \times H_*(\mathrm{Gr}) \longrightarrow \mathbb{Z},$$

under which the Schubert bases $\{\xi^w \in H^*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}_n^0\}$ and $\{\xi_w \in H_*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}_n^0\}$ are dual.

Quillen (unpublished) and Garland and Raghunathan [GR75] showed that the space Gr is homotopy-equivalent to the group $\Omega SU(n, \mathbb{C})$ of based loops into $SU(n, \mathbb{C})$. Thus $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$ are endowed with the structures of dual commutative and co-commutative Hopf algebras. In [Bot58], R. Bott explicitly calculated these Hopf algebras. By identifying the generators, there are isomorphisms $H^*(\mathrm{Gr}) \cong \Lambda^{(n)}$ and $H_*(\mathrm{Gr}) \cong \Lambda_{(n)}$ such that the following diagram

$$\begin{array}{ccc} H^*(\mathrm{Gr}) \times H_*(\mathrm{Gr}) & \xrightarrow{\langle \cdot\, , \cdot \rangle_{\mathrm{Gr}}} & \\ \downarrow & & \mathbb{Z} \\ \Lambda^{(n)} \times \Lambda_{(n)} & \xrightarrow{\langle \cdot\, , \cdot \rangle} & \end{array}$$

commutes.

A natural problem was to identify the Schubert classes ξ_w and ξ^w with symmetric functions. Inspired by an empirical study of the polynomials $A_\lambda^{(k)}(x;t)$ when $t = 1$, distinguished bases for $\Lambda^{(n)}$ and $\Lambda_{(n)}$ that refine the Schur basis for Λ were introduced and connected to the quantum cohomology of Grassmannians in [LM08, LM07]. In 2008, confirming a conjecture of M. Shimozono, T. Lam proved in [Lam08] that the Schubert classes ξ_w and ξ^w are represented respectively by these bases, the *k-Schur functions* and the *dual k-Schur functions* (also called *affine Schur functions*).

Theorem 83. *[Lam08] Under the isomorphism $H^*(Gr) \cong \Lambda^{(n)}$, the Schubert class ξ^w is sent to the dual k-Schur function $\mathfrak{S}_w \in \Lambda^{(n)}$, and under the isomorphism $H_*(Gr) \cong \Lambda_{(n)}$, the Schubert class ξ_w is sent to the k-Schur function $s_w^{(k)}$, where $k = n - 1$.*

The dual k-Schur functions $\mathfrak{S}_w^{(k)}$ are weight generating functions of combinatorial objects known as *weak k-tableaux* and they were first introduced by L. Lapointe and J. Morse in [LM05]. On the other hand, the parameterless k-Schur functions $s_w^{(k)}$ are conjectured to be the $t = 1$ specializations of $A_\lambda^{(k)}(x;t)$ first introduced by A. Lascoux, L. Lapointe and J. Morse in [LLM03] as the *k-Atoms*, in their study of the Macdonald polynomials. This suggests that there is a link from the theory of Macdonald polynomials to quantum and affine Schubert calculus.

6.2 The *k*-Schur functions

In 2003, L. Lapointe, A. Lascoux and J. Morse were studying the q,t-Kostka coefficients when they discovered the existence of the *k-Schur functions*, $A_\lambda^{(k)}(x,t)$. Their version of the Macdonald symmetric function, $H_\lambda(x;q,t)$, is defined so that it's a unique basis which has the property that

$$H_\lambda(x;q,t) = r_\lambda(q,t)s_\lambda(x/(1-q)) + \text{terms of the form } r_{\lambda\mu}(q,t)s_\mu(x/(1-q))$$

with $\mu > \lambda$ and

$$\langle H_\lambda(x;q,t), H_\mu(x;q,t)\rangle_{q,t} = 0 \text{ if } \lambda \neq \mu\,.$$

Lapointe, Lascoux and Morse's empirical study of $H_\mu(x;q,t)$ led to a refinement of Macdonald's positivity conjecture. The version of the Macdonald polynomial, $H_\lambda(x;q,t)$, can be achieved through the one in Definition 74. For more on this please see [Mac95].

Conjecture 84. *[LLM03] For any fixed integer $k > 0$, and $\lambda \in \mathcal{P}^k$ (a partition where $\lambda_1 \leq k$),*

$$H_\lambda(x;q,t) = \sum_{\mu \in \mathcal{P}^k} K^{(k)}_{\mu\lambda}(q,t)\; A^{(k)}_\mu(x;t)\,,$$

where $K^{(k)}_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$ for some family of polynomials defined by certain sets of tableaux $\mathcal{A}^k_\mu$ as

$$A^{(k)}_\mu(x;t) = \sum_{T \in \mathcal{A}^k_\mu} t^{charge(T)}\; s_{shape(T)}\,.$$

Lapointe, Lascoux and Morse made further conjectures about $A^{(k)}_\mu(x;t)$, strengthening the ties of their work to related fields of algebraic combinatorics.

Conjecture 85. *[LLM03] For any fixed integer $k > 0$, the set $\{A^{(k)}_\lambda(x;t)\}_{\lambda \in \mathcal{P}^k}$ exists and forms a basis for*

$$\Lambda^t_{(k)} = span\{H_\lambda(x;q,t) \mid \lambda \in \mathcal{P}^k\}\,.$$

Furthermore, for any $k' > k$,

$$A^{(k)}_\lambda(x;t) = \sum_\mu B^{(k,k')}_{\lambda\mu}(t)\; A^{(k')}_\lambda(x;t) \quad where\ B^{(k,k')}_{\lambda\mu}(t) \in \mathbb{N}[t]\,,$$

and $A^{(k)}_\lambda(x;t) = s_\lambda$ for $k \geq |\lambda|$.

Conjectures 84 and 85 strengthens Macdonald's conjecture 76. Pursuant work of Lapointe and Morse, [LM03a, LM03b, LM04, LM05, LM07, LM08], led to connections of the k-Schur functions with geometry, physics and representation theory. A matter of determining an algorithm or a formula for computing $A^{(k)}_\lambda(x;t)$ was discovered by Lapointe, Lascoux and Morse which first appeared in [LM03a]. Subsequently, various conjecturally equivalent definitions have arisen, each of them having different benefits and view points. For more on this see [LLM+13].

6.3 The k-Schur functions at $t = 1$

In this section we present a construction for the k-Schur functions from the previous Section 6.2 when the parameter t is set to one. The k-Schur functions, $s_\lambda^{(k)}$, we consider will form a basis for the space

$$\Lambda_{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]\,.$$

This Hopf algebra $\Lambda_{(k)}$ is dual to the quotient

$$\Lambda^{(k)} = \Lambda/\langle m_\lambda \mid \lambda > k \rangle\,.$$

The *dual k-Schur functions*, $\mathfrak{S}_\lambda^{(k)}$, will form a basis for the dual space $\Lambda^{(k)}$. We will index the k-Schur functions and their dual elements by $(k+1)$-cores. To define the dual k-Schur functions we need the *weak tableaux*.

6.3.1 Weak k-tableaux

Definition 86. *Let λ be a $(k+1)$-core, and $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a composition of $|\lambda|$ with no part larger than k. A* **weak k-tableau** *of weight α is a semi-standard filling of shape λ with letters $1, \ldots, d$ such that the collection of cells filled with letter i is labeled by exactly α_i distinct $(k+1)$-residues.*

Example 87. *For $k = 6$, the weak k-tableaux of weight $(4, 3, 1)$ are*

$$\begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-3} 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\cline{1-3} 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|}\cline{1-3} 2 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{llllll} 3 & & & & & \\ 2 & 2 & & & & \\ 1 & 1 & 1 & 1 & 2 & 3 \end{array} \qquad \begin{array}{lllllll} 3 & & & & & & \\ 2 & & & & & & \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} \qquad \begin{array}{llllll} 2 & 3 & & & & \\ 1 & 1 & 1 & 1 & 2 & 3 \end{array}\,.$$

The weak k-tableaux are a generalization of the semi-standard tableaux because for one reason they generalize the Kostka numbers.

Definition 88. *For any $(k+1)$-core λ and $\mu \in \mathcal{P}^k$, the* **weak Kostka numbers**, *$K^{(k)}_{\lambda\mu}$, are the number of weak k-tableaux of shape λ and weight μ.*

The weak Kostka numbers satisfy an important property. Let $[K^{(k)}_{\lambda\mu}]$ denote the matrix whose elements are the weak Kostka numbers over all $(k+1)$-cores λ and $\mu \in \mathcal{P}^k$. The matrix $[K^{(k)}_{\lambda\mu}]$ is unitriangular and thus invertible [LM05]. With this in mind, the k-Schur functions were characterized in [LM07] by the system obtained from

$$h_\mu = \sum_{\lambda \in C^{k+1}} K^{(k)}_{\lambda\mu}\ s^{(k)}_\lambda\,, \tag{6.1}$$

for all $\mu \in \mathcal{P}^k$. This system defines the k-Schur functions because the elements h_λ for $\lambda \in \mathcal{P}^k$ forms a basis for the space $\Lambda_{(k)}$ and the transition matrix is invertible.

Example 89. *For $k = 6$, the weak tableaux in Example 87 gives us*

$$h_{(4,3,1)} = s^{(6)}_{(4,3,1)} + s^{(6)}_{(4,4)} + 2s^{(6)}_{(5,3)} + s^{(6)}_{(6,2,1)} + s^{(6)}_{(7,1,1)} + s^{(6)}_{(8,2)}\,.$$

We can now use duality to produce the second basis $\mathfrak{S}^{(k)}_\lambda$ for the algebra $\Lambda^{(k)}$. We will need the pairing

$$\langle\cdot\,,\cdot\rangle : \Lambda_{(k)} \times \Lambda^{(k)} \longrightarrow \mathbb{Q}\,,$$

where $h_\mu \in \Lambda_{(k)}$ and $m_\lambda \in \Lambda^{(k)}$ are dual elements from the Hall-inner product. We define the dual

k-Schur functions in terms of the monomial functions:

$$\begin{aligned}\mathfrak{S}_\lambda^{(k)} &= \sum_{\mu:\mu_1\le k} \langle h_\mu \,, \mathfrak{S}_\lambda^{(k)}\rangle \, m_\mu \\ &= \sum_{\mu:\mu_1\le k} \sum_{\gamma\in C^{k+1}} K_{\gamma\mu}^{(k)} \, \langle s_\gamma^{(k)} \,, \mathfrak{S}_\lambda^{(k)}\rangle \, m_\mu \\ &= \sum_{\mu:\mu_1\le k} K_{\lambda\mu}^{(k)} \, m_\mu \,.\end{aligned}$$

There is an involution on the set of weak k-tableaux of a fixed shape λ and weight α which sends a tableau to another tableau of shape λ and weight that is a permutation of α in [LM07]. This gives us that the dual k-schur functions are weight generating functions for weak k-tableaux.

Theorem 90. *[LM07] For $\lambda \in C^{k+1}$,*

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{T:\text{weak } k\text{-tableau} \\ \text{shape}(T)=\lambda}} x^{\text{weight}(T)} \,.$$

Example 91. *To compute the dual k-Schur function $\mathfrak{S}_{(5,2,1)}^{(3)}$, we extract the weights of each weak k-tableau of shape* $(5,2,1)$ *from Example 87 to get*

$$\mathfrak{S}_{(5,2,1)}^{(3)} = m_{(3,2,1)} + 2m_{(3,1,1,1)} + m_{(2,2,2)} + 2m_{(2,2,1,1)} + 3m_{(2,1,1,1,1)} + 4m_{(1,1,1,1,1,1)} \,.$$

The weak Kostka numbers are a result of the *weak Pieri rule* of [LM05], which is given in terms of weak horizontal chains in the $(k+1)$-core realization of the weak poset. There are other realizations of the k-Schur functions and the dual k-Schur functions. The k-Schur functions can be realized as weight generating functions for *strong k-tableaux* resulting from the *strong Pieri rule*. The language of the affine symmetric group shows us another way to characterize the dual k-Schur functions, which we consider in the next chapter.

6.3.2 Cocharge of a weak k-tableau

The k-cocharge statistic on k-tableaux is first described in standard case as it is in these terms that we will define it for semi-standard k-tableaux. Important to the definition is the number of diagonals between two cells of a specific residue.

Definition 92. *Given two cells c_1 and c_2 of a $(k+1)$-core, let $diag(c_1, c_2)$ be the number of diagonals of reside r that are strictly between c_1 and c_2 where r is the residue of the lower cell.*

When it is well-defined to do so, functions defined with a cell as input can instead take a letter as input. In particular, for standard k-tableaux it is natural to discuss the residue of a specific letter (since any cell containing that letter has the same residue) instead of the residue of a specific cell.

Example 93. *For $k = 4$, a standard k-tableau of weight (1^{10}) is*

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 10_1 \\ \cline{1-1} 8_2 \\ \cline{1-2} 5_3 & 7_4 \\ \cline{1-3} 4_4 & 6_0 & 10_1 \\ \hline 1_0 & 2_1 & 3_2 & 5_3 & 7_4 & 9_0 & 10_1 \\ \hline \end{array}$$

For this T, we see that $diag(4_4, 3_2) = 0$, and $diag(8_2, (1, 5)) = 1$.

Definition 94. *Given a standard k-tableau T of weight (1^m), place a bar on the lowest occurrence of the letter i, for $1 \le i \le m$. Define the index vector $I(T) = [I_1, I_2, \ldots, I_m]$ recursively by setting $I_1 = 0$, and*

$$I_i = \begin{cases} I_{i-1} + 1 + diag(\bar{i}, \overline{i-1}) & \text{if } \overline{i-1} \text{ is strictly below } \bar{i} \\ I_{i-1} - diag(\bar{i}, \overline{i-1}) & \text{otherwise} \end{cases}$$

for $2 \le i \le m$. The k-cocharge of T is the sum of the entries of $I(T)$,

$$kcocharge(T) = \sum_{i=1}^{m} I_i \,.$$

Example 95. *For the k-tableau T of Example 93 , Table 6.1 shows us that the kcocharge*$(T) = 10$.

i	$\text{diag}(\bar{i}, \overline{i-1})$	I_i
2	0	$0 - 0 = 0$
3	0	$0 - 0 = 0$
4	0	$0 + 1 + 0 = 1$
5	0	$1 - 0 = 1$
6	0	$1 + 1 + 0 = 2$
7	0	$2 - 0 = 2$
8	1	$2 + 1 + 1 = 4$
9	1	$4 - 1 = 3$
10	0	$3 - 0 = 3$

Table 6.1: k-cocharge of T from Example 93

One observation of Definition 94 is that the k-cocharge of a standard k-tableau is not always positive, unlike the Lascoux-Schutzenberger cocharge on a tableau. For this reason, it is sometimes useful to employ a different formulation of the k-cocharge. To see this alternate form, we first let $T_{\leq i}$ denote the subtableau obtained by deleting all the letters larger than i of the standard k-tableau T. We also need to define a *residue order*.

Definition 96. *Given a k-tableau T, the T-residue order of* $\{0, 1, \ldots, k\}$ *is defined by*

$$r > r + 1 > \cdots > k > 0 > 1 > \cdots > r - 1\,,$$

where r is the residue of the lowest addable cell of T.

Example 97. *For* $k = 4$*, consider the standard k-tableau*

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 6_3 & \multicolumn{3}{c}{} \\ \cline{1-2} 3_4 & 5_0 & \multicolumn{2}{c}{} \\ \hline 1_0 & 2_1 & 4_2 & 6_3 \\ \hline \end{array}$$

The T-residue order is $4 > 0 > 1 > 2 > 3$. The $T_{\leq 4}$-residue order is $3 > 4 > 0 > 1 > 2$, which is also the $T_{\leq 5}$-residue order.

We now define a index vector over a given standard k-tableau. This index vector will give us another formulation of the k-cocharge statistic over standard k-tableau.

Definition 98. *Given a standard k-tableau T of weight (1^m), define the index vector $M(T) = [M_1, \ldots, M_m]$ recursively by setting $M_1 = 0$, and*

$$M_i = \begin{cases} M_{i-1} + 1 & \text{if } res(i) > res(i-1) \\ M_{i-1} & \text{otherwise} \end{cases}$$

for $2 \leq i \leq m$.

Conjecture 99. *For a standard k-tableau T of weight (1^m),*

$$kcocharge(T) = \sum_{i=1}^{m} \left(M_i + diag(c_i, c_{(i)}) \right) ,$$

where c_i is the lowest cell containing an i in $T_{\leq i}$, and $c_{(i)}$ is the lowest addable cell of $T_{\leq i}$.

Example 100. *For $k = 4$, recall that the standard k-tableau of weight (1^{10}) from Example 93 is*

$$T = \begin{array}{lllllll} 10_1 & & & & & & \\ 8_2 & & & & & & \\ 5_3 & 7_4 & & & & & \\ 4_4 & 6_0 & 10_1 & & & & \\ 1_0 & 2_1 & 3_2 & 5_3 & 7_4 & 9_0 & 10_1 \end{array}$$

Table 6.2 shows us that the $kcocharge(T) = 10$.

The Lascoux and Schützenberger cocharge statistic on a semi-standard tableau is an extension of the cocharge statistic on a standard tableau. Similarly, the k-cocharge on a semi-standard k-tableau will be an extension of the k-cocharge statistic on a standard k-tableau. The trick is to

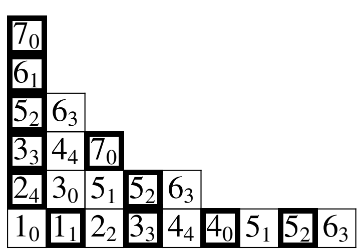
i	$T_{\le i}$-residue order	M_i	diag$(c_i, c_{(i)})$
2	$2 > 3 > 4 > 0 > 1$	0	0
3	$3 > 4 > 0 > 1 > 2$	0	0
4	$3 > 4 > 0 > 1 > 2$	1	0
5	$4 > 0 > 1 > 2 > 3$	1	0
6	$4 > 0 > 1 > 2 > 3$	2	0
7	$0 > 1 > 2 > 3 > 4$	2	0
8	$0 > 1 > 2 > 3 > 4$	3	1
9	$1 > 2 > 3 > 4 > 0$	3	0
10	$2 > 3 > 4 > 0 > 1$	3	0

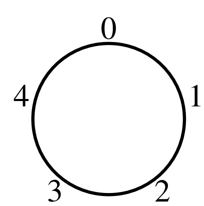
Table 6.2: k-cocharge of T from Example 100

introduce a method for making an appropriate choice of standard sequences on the semi-standard k-tableau.

Definition 101. *From an i, of some residue r, in a semi-standard k-tableau T, the appropriate choice of $i + 1$ will be determined by choosing its residue from the set, A, of all $(k + 1)$-residues labeling $(i + 1)$'s. Reading counter-clockwise from r, this choice is the closest $j \in A$ on a circle labelled clockwise with $0, 1, \ldots, k$.*

Example 102. *For $k = 4$, the k-tableau of weight $(2, 2, 2, 2, 2, 2, 2, 1)$ is*

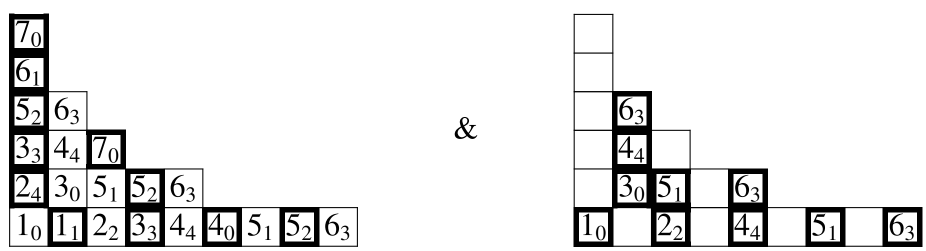
$T =$ 



The bold cells of T shows the first standard sequence using Definition 101. The set of residues labeling 5 in T is $A = \{1, 2\}$, and the residue of 4 in the first standard sequence is 0. Therefore, the choice of 2 from A is made because 2 is closer to 0 than 1 when reading counter-clockwise on the *above circle labeled with all the residues.*

For semi-standard k-tableaux, we must be careful about the residue order of Definition 96. We can first think that the letters i that appear in a semi-standard k-tableau are ordered with respect to the standard subsequences they belong to under Definition 101. Specifically, the i in the first standard sequence is larger than the i from the second standard sequence, etc. This ordering is well defined since each i has its own distinct residue r in the semi-standard k-tableau. Henceforth, the index vector M of Definition 98 is computed with respect to the $T_{\le i_r}$-residue order when dealing with the letter i_r.

Example 103. *For the k-tableau T of Example 102, we have the two standard sequences by Definition 101*



Observe that

$$T_{\le 5_1} = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-2} 3_3 & 4_4 \\ \cline{1-3} 2_4 & 3_0 & 5_1 \\ \hline 1_0 & 1_1 & 2_2 & 3_3 & 4_4 & 4_0 & 5_1 \\ \hline \end{array}$$

The lowest addable cell of $T_{\le 5_1}$ has residue 2, so the $T_{\le 5_1}$-residue order is $2 > 3 > 4 > 0 > 1$. Conjecture 99 along with Table 6.3 tells us that the kcocharge$(T) = 16$.

6.3.3 Charge of a weak k-tableau

Equation 6.1 gives us a characterization of k-Schur functions (without parameter t) by inverting

$$h_\mu = \sum_{\lambda \in C^{k+1}} K^{(k)}_{\lambda\mu} \, s^{(k)}_\lambda \, ,$$

i_r	$T_{\leq i_r}$-residue order	M_i	$\mathrm{diag}(c_{i_r}, c_{(i_r)})$	I_i	$\mathrm{diag}(\bar{i}_r, \overline{(i-1)}_r)$
2_4	$2>3>4>0>1$	1	0	1	0
3_3	$4>0>1>2>3$	1	0	1	0
4_0	$1>2>3>4>0$	1	0	1	0
5_2	$3>4>0>1>2$	1	0	1	0
6_1	$3>4>0>1>2$	2	2	4	2
7_0	$3>4>0>1>2$	3	1	4	0
2_2	$3>4>0>1>2$	0	0	0	0
3_0	$3>4>0>1>2$	1	0	1	0
4_4	$0>1>2>3>4$	1	0	1	0
5_1	$2>3>4>0>1$	1	0	1	0
6_3	$4>0>1>2>3$	1	0	1	0

Table 6.3: k-cocharge of T from Example 102

where the weak Kostka numbers, $K^{(k)}_{\lambda\mu}$, count weak k-tableaux. In this subsection we present a generalization of the weak Kostka numbers, which are polynomials in $\mathbb{N}[t]$. Namely, these *weak Kostka-Foulkes polynomials* are defined by refining the charge statistic of Subsection 5.1.2 to a statistic which associates a non-negative integer called the *k-charge* to each k-tableau. In the spirit of Equation 6.1, the Hall-Littlewood polynomials

$$H_\mu(x;t) = \sum_{\lambda \in \mathcal{P}^n} K^{(k)}_{\lambda\mu}(t)\ \tilde{s}^{(k)}_\lambda(x;t),$$

characterize the functions $\{\tilde{s}^{(k)}_\lambda(x;t)\}$, where $\mu \in \mathcal{P}^n$. Setting the coefficients

$$K^{(k)}_{\lambda\mu}(t) = \sum_{\substack{shape(T)=\mathfrak{c}(\lambda) \\ weight(T)=\mu}} t^{kcharge(T)},$$

gives us that $K^{(k)}_{\lambda\lambda}(t) = 1$, and since there are no weak k-tableaux of shape $\mathfrak{c}(\lambda)$ and weight μ when $\mu > \lambda$ in lexicographic order, then the k-charge matrix $[K^{(k)}_{\lambda\mu}(t)]$ is unitriangular.

We now give a formulation of the k-charge statistic defined directly on k-tableaux by Lapointe and Pinto [PL]. This will be described on standard weak k-tableaux, and the semi-standard case will be defined in these terms.

Definition 104. *Given a $(k+1)$-core λ with cells c_1 and c_2, the number* **diag**(c_1, c_2) *is the number of diagonals of residue x that are strictly between c_1 and c_2 where x is the residue of the lower cell in λ.*

When it is well-defined to do so, the function $diag(c_1, c_2)$ can be defined with inputs which are letters in cells c_1 and c_2 when we have a given weak k-tableau.

Definition 105. *Given a standard k-tableau T on m letters, put a bar on the topmost occurrence of letter i, for each $i = 1, 2, \ldots, m$. Define the* **index** *of T, starting from $I_1 = 0$, by*

$$I_i = \begin{cases} I_{i-1} + 1 + diag(\bar{i}, \overline{i-1}) & \text{if } \bar{i} \text{ is east of } \overline{i-1} \\ I_{i-1} - diag(\bar{i}, \overline{i-1}) & \text{otherwise}. \end{cases}$$

for $i = 2, \ldots, m$. The k-charge of T, denoted **kcharge(T)**, *is the sum of the entries in $I(T) = [I_1, \ldots, I_m]$.*

Example 106. *For $k = 3$, a standard k-tableau of weight (1^6) is*

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 4_2 \\ \cline{1-2} 2_3 & 6_0 \\ \hline 1_0 & 3_1 & 4_2 & 5_3 & 6_0 \\ \hline \end{array}$$

Table 6.4 shows us that the kcharge$(T) = 8$.

It is not clear that the k-charge of Definition 105 is a non-negative integer. As a result, it is sometimes useful to use a formulation of the k-charge, derived by J. Morse from [DM13]. For any tableau T, let $T_{\leq x}$ denote the subtableau obtained by deleting all letters larger than x from T.

i	$\text{diag}(\bar{i}, \overline{i-1})$	I_i
2	0	$0-0=0$
3	0	$0+1+0=1$
4	0	$1-0=1$
5	1	$1+1+1=3$
6	0	$3-0=3$

Table 6.4: k-charge of T from Example 106

Definition 107. *Given a k-tableau T, the* **T-residue order** *of* $\{0,\ldots,k\}$ *is defined by*

$$r > r-1 > \cdots > 0 > k > \cdots > r+2 > r+1\,,$$

where r is the residue of the highest addable corner of T.

Example 108. *For* $k=3$,

$$T = \begin{array}{|c|c|c|c|}\hline 4_3 \\ \cline{1-1} 1_0 & 2_1 & 3_2 & 4_3 \\ \hline\end{array} \quad \textit{and} \quad T_{\le 3} = \begin{array}{|c|c|c|}\hline 1_0 & 2_1 & 3_2 \\ \hline\end{array}\,.$$

The T-residue order is $2 > 1 > 0 > 3$ *and the* $T_{\le 3}$*-residue order is* $3 > 2 > 1 > 0$.

Just as the previous definition of k-charge, we define another index vector which is used in computing another k-charge.

Definition 109. *Given a standard k-tableau T on m letters, let the* **index** $J(T) = [J_1,\ldots,J_m]$, *starting from* $J_1 = 0$, *by setting for* $i = 2,\ldots,m$,

$$J_i = \begin{cases} J_{i-1}+1 & \textit{if } res(i) > res(i-1) \\ J_{i-1} & \textit{otherwise}\,. \end{cases}$$

under $T_{\le i}$*-residue order.*

We now define another k-charge statistic over standard k-tableaux and conjecture that it is equivalent to the k-charge statistic in Definition 105.

Conjecture 110. *For a standard k-tableau T of weight* (1^m)*,*

$$kcharge(T) = \sum_{i=1}^{m} \left(J_i(T) + diag(c_i, c^{(i)}) \right) ,$$

where c_i *is the highest cell containing an i and* $c^{(i)}$ *is the highest addable cell of* $T_{\leq i}$*.*

Example 111. *For* $k = 3$*, recall the that the standard k-tableau of weight* (1^6) *from Example 106 is*

$$T = \begin{array}{|c|c|c|c|c|} \hline 4_2 \\ \cline{1-2} 2_3 & 6_0 \\ \hline 1_0 & 3_1 & 4_2 & 5_3 & 6_0 \\ \hline \end{array}$$

Table 6.5 shows us that the $kcharge(T) = 8$*.*

i	$T_{\leq i}$-residue order	J_i	diag$(c_i, c_{(i)})$
2	$2 > 1 > 0 > 3$	0	0
3	$2 > 1 > 0 > 3$	1	0
4	$1 > 0 > 3 > 2$	1	0
5	$1 > 0 > 3 > 2$	2	1
6	$1 > 0 > 3 > 2$	3	0

Table 6.5: k-charge of T from Example 111

A consequence of Conjecture 110 is that Definitions 105 and 109 reduce to the charge of Lascoux and Schützenberger of Definition 67.

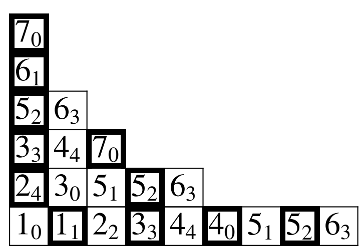
Corollary 112. *Let T be a standard k-tableau of shape λ and weight (1^m). If $k > \lambda_1 + \ell(\lambda) - 1$, then*

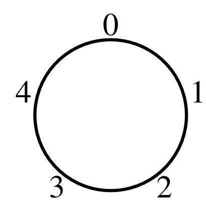
$$kcharge(T) = charge(T) = \sum_{i=1}^{m} J_i(T)\,.$$

We extend the definition of the k-charge to semi-standard k-tableaux by successively computing on an appropriate choice of standard sequences over the k-tableau.

Definition 113. *From an i (of some residue r) in a semi-standard k-tableau T, the appropriate choice of $i + 1$ will be determined by choosing its residue from the set S of all $(k + 1)$-residues labeling $(i + 1)$'s. Reading counter-clockwise from r, this choice is the closest $j \in S$ on a circle labelled clockwise with $0, 1, \ldots, k$.*

Example 114. *For $k = 4$, the k-tableau of weight $(2, 2, 2, 2, 2, 2, 1)$ is*

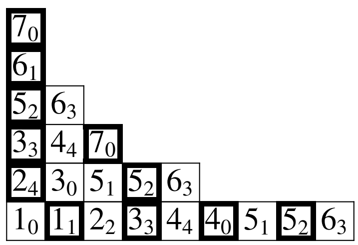
$T =$ 



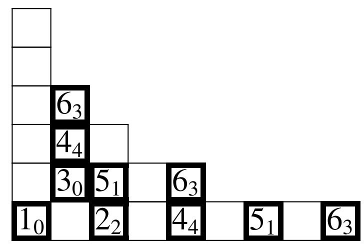
The bold cells of T shows the first standard sequence using Definition 113. The set of residues labeling 5 in T is $S = \{1, 2\}$, and the residue of 4 in the first standard sequence is 0. Therefore, the choice of 2 from S is made because 2 is closer to 0 than 1 when reading counter-clockwise on the above circle labeled with all the residues.

For semi-standard k-tableaux, we must be careful about the residue order of Definition 107. We can first think that the letters i that appear in a semi-standard k-tableau are ordered with respect to the standard subsequences they belong to under Definition 113. Specifically, the i in the first standard sequence is larger than the i from the second standard sequence, etc. This ordering is well defined since each i has its own distinct residue r in the semi-standard k-tableau. Henceforth, the index vector J of Definition 109 is computed with respect to the $T_{\leq i_r}$-residue order when dealing with the letter i_r.

Example 115. *For the k-tableau T of Example 114, we have the two standard sequences by Definition 113*



&



Observe that

$$T_{\leq 6_3} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 5_2 & 6_3 \\ \cline{1-2} 3_3 & 4_4 \\ \cline{1-5} 2_4 & 3_0 & 5_1 & 5_2 & 6_3 \\ \hline 1_0 & 1_1 & 2_2 & 3_3 & 4_4 & 4_0 & 5_1 & 5_2 & 6_3 \\ \hline \end{array}$$

The highest addable cell of $T_{\leq 6_3}$ has residue 1*, so the $T_{\leq 6_3}$-residue order is* $1 > 0 > 4 > 3 > 2$*. Conjecture 110 along with Table 6.6 tells us that the kcharge$(T) = 12$.*

i_r	$T_{\leq i_r}$-residue order	J_i	$\text{diag}(c_{i_r}, c_{(i_r)})$	I_i	$\text{diag}(\bar{i}_r, \overline{(i-1)}_r)$
2_4	$3 > 2 > 1 > 0 > 4$	0	0	0	0
3_3	$2 > 1 > 0 > 4 > 3$	0	0	0	0
4_0	$2 > 1 > 0 > 4 > 3$	1	1	2	1
5_2	$1 > 0 > 4 > 3 > 2$	1	0	1	1
6_1	$0 > 4 > 3 > 2 > 1$	1	0	1	0
7_0	$4 > 3 > 2 > 1 > 0$	1	0	1	0
2_2	$4 > 3 > 2 > 1 > 0$	1	0	1	0
3_0	$3 > 2 > 1 > 0 > 4$	1	0	1	0
4_4	$2 > 1 > 0 > 4 > 3$	1	0	1	0
5_1	$2 > 1 > 0 > 4 > 3$	2	0	2	0
6_3	$1 > 0 > 4 > 3 > 2$	2	0	2	0

Table 6.6: k-charge of T from Example 114

There are several other descriptions of k-charge not just on k-tableaux but also on other combinatorial objects. Subsection 10.2 introduces a new combinatorial object, *affine Bruhat countertableaux*, over which a new n-charge statistic is defined, for $n = k + 1$. In [LLM$^+$13] a new set of combinatorial objects called the *strong k-tableaux* are defined. For these strong k-tableaux a *spin* statistic is defined and used to expand k-Schur functions (with the parameter t) in terms of monomial functions. There are many other ways to define the k-Schur functions, and each of these definitions are conjectured to be equivalent. For more on this, the reader is encouraged to look at [LLM$^+$13].

Chapter 7
Affine Pieri rules

The main discovery in this thesis is that there is a fundamental connection between weak order chains from u to v in $\tilde{S}_n^0$ and strong order chains from v to a translation of u. We start by addressing the case that applies to Pieri rules, in which case the translation of an element $u_\lambda \in \tilde{S}_n^0$ amounts to $s_{x-1}s_{x-2}\cdots s_{x+1}u_\lambda$ where $x = \lambda_1 - 1 \pmod{n}$.

7.1 Strong and weak Pieri rules

The Pieri rules for the $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ are given in [LM07, LLMS10]. The affine homology rule is framed using saturated chains in the weak order on $\tilde{S}_n^0$, whereas the cohomology rule is in terms of strong order saturated chains.

Definition 116. *A word $a_1a_2\cdots a_\ell$ with letters in $\mathbb{Z}/n\mathbb{Z}$ is called* **cyclically decreasing** *if each letter occurs at most once and $i+1$ precedes i whenever i and $i+1$ both occur in the word. An affine permutation is called cyclically decreasing if it has a cyclically decreasing reduced word.*

The affine homology Pieri rule for $H_*(\mathrm{Gr})$ is given, for $w \in \tilde{S}_n^0$ and $c_{0,m} = s_{m-1}\cdots s_1 s_0 \in \tilde{S}_n^0$, by

$$\xi_{c_{0,m}}\,\xi_w = \sum_v \xi_{vw}, \tag{7.1}$$

over all cyclically decreasing v of length m such that $vw \in \tilde{S}_n^0$ and $\ell(vw) = \ell(w) + m$. Thus, for $u \in \tilde{S}_n^0$, the term ξ_u occurs in the product of $\xi_{c_{0,m}}\xi_w$ only when uw^{-1} is cyclically decreasing and there is a saturated chain

$$w = w^{(0)} \lessdot w^{(1)} \lessdot \cdots \lessdot w^{(m)} = u\,.$$

Alternatively, the rule can be formulated in the language of shapes using the action of the s_i-generators on n-cores.

Lemma 117. *[LLMS10] For $u, w \in \tilde{S}_n^0$ where $\ell(uw^{-1}) = m$, uw^{-1} is cyclically decreasing with reduced word $j_1 \cdots j_m$ if and only if $\mathfrak{a}(u)/\mathfrak{a}(w)$ is a horizontal strip such that the set of residues labelling its cells is $\{j_1, \ldots, j_m\}$.*

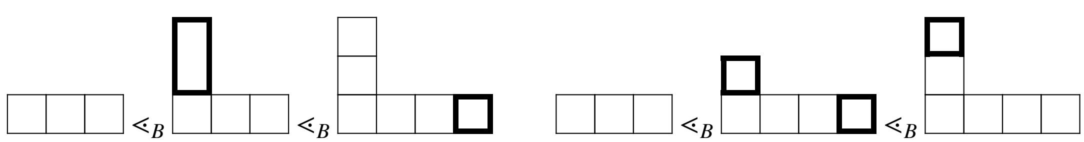
While the weak order determines the affine homology rule for $H_*(\mathrm{Gr})$, the affine cohomology Pieri rule is given as the sum over certain multisets of chains in the Bruhat (strong) order. The multisets arise by imposing a marking on strong covers $\rho \lessdot_B \gamma$ in C^n.

Definition 118. *We say (γ, c) is a* **marked strong cover** *of ρ if $\rho \lessdot_B \gamma$ and c is the content of the head of a ribbon in γ/ρ (recall that Lemma 35 assures the skew shape is made up of ribbons). Then, for $0 \leq m < n$ and n-cores ν and γ, a* **strong *m*-strip** *from ν to γ is a saturated chain of cores*

$$\nu = \gamma^{(0)} \lessdot_B \gamma^{(1)} \lessdot_B \ldots \lessdot_B \gamma^{(m)} = \gamma\,,$$

together with an increasing "content vector" $c = (c_1, c_2, \cdots, c_m)$, such that $(\gamma^{(i)}, c_i)$ is a marked strong cover of $\gamma^{(i-1)}$ for $1 \leq i \leq m$.

Example 119. *For $n = 4$, there are 2 saturated chains from $\nu = (3)$ to $\gamma = (4, 1, 1)$,*



(7.2)

The first chain with content vector $c = (-1, 3)$ is thus the only strong 2-strip from ν to γ.

The affine cohomology Pieri rule is

$$\xi^{c_{0,m}} \xi^{w} = \sum_{S} \xi^{z}\,, \tag{7.3}$$

where the sum runs over strong m-strips S from $\mathfrak{a}(w)$ to $\mathfrak{a}(z)$. In contrast to the Pieri rule for $H_*(\mathrm{Gr})$, a given term ξ^z here may occur with multiplicity greater than 1.

7.2 Horizontal strong strips

As with the affine Pieri rule for the cohomology $H^*(\mathrm{Gr})$, the Pieri rule [BS98] and the quantum Pieri rule [FGP97] for (quantum) cohomology of the flag manifold are also determined by chains in the Bruhat order (see (8.3)). However, it is the homology of Gr, not the cohomology, that is algebraically tied to the quantum cohomology of the flag manifold (detailed in § 8). To align the combinatorics with the algebra, we introduce a distinguished subclass of strong order chains that characterize the affine homology Pieri rule. The fundamental observation is that the translation of an n-core λ to the n-core $R(n-1,\lambda) = (\lambda_1 + n - 1, \lambda)$ plays a crucial role.

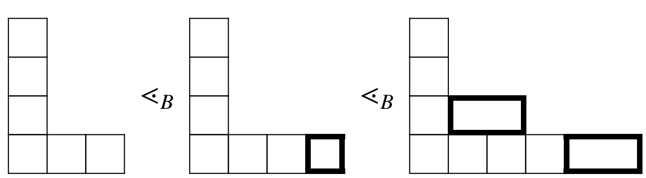
Definition 120. *A pair of n-cores* (λ, ν) *is a* **horizontal strong m-strip** *if* $\lambda \subset \nu$ *and there is a saturated chain of cores*

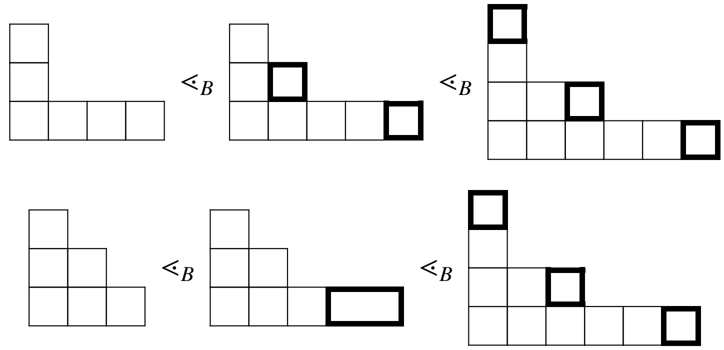
$$\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n-1,\lambda) \tag{7.4}$$

such that the bottom row of $\nu^{(i)}$ *is longer than the bottom row of* $\nu^{(i-1)}$*, for* $1 \leq i \leq m$ *where* $m = n - 1 + deg(\lambda) - deg(\nu)$*.*

Example 121. *For* $n = 4$*,* $\lambda = (1,1)$*, and* $\nu = (3)$*,* (λ, ν) *is not a horizontal strong strip since neither of the strong chains from* ν *to* $R(n-1,\lambda)$ *shown in* (7.2) *have strictly growing bottom rows.*

For $\lambda = (3,1,1)$*, the* 4*-cores* ν *such that* (λ,ν) *is a horizontal strong* 2*-strip begin with the chains to* $R(3,\lambda)$*:*





We have chosen the terminology horizontal strong strip because, although not immediately obvious, there always exists a strong strip from ν to $R(n-1,\lambda)$ of shapes that differ by ribbons of height one when (λ,ν) is a horizontal strong strip. The following lemma associates horizontal strong strips to the horizontality condition and we then connect to strong strips.

Lemma 122. *[DM13] Given n-cores $\lambda \subset \nu$ and a saturated chain of shapes* (7.4) *whose bottom rows strictly increase, there are adjacent ribbons $S^1,\ldots,S^m$ in the bottom row of $R(n-1,\lambda)/\lambda$ such that the shape $\nu^{(j)}/\nu^{(j-1)}$ is comprised of all copies of S^j that can be removed from $\nu^{(j)}$, for each $1 \le j \le m$.*

Proof. Consider $\lambda \subset \nu$ and a chain of n-cores (7.4) where the bottom rows increase in size. Let S^j denote the lowest ribbon in $\nu^{(j)}/\nu^{(j-1)}$. Since the bottom row of $\nu^{(j)}$ is strictly longer than the bottom of $\nu^{(j-1)}$, the head of S^j lies in the bottom row of $\nu^{(j)}$. Moreover, S^j has height one since $\lambda \subset \nu^{(j)} \subset (n-1+\lambda_1,\lambda)$ and $(n-1+\lambda_1,\lambda)/\lambda$ is a horizontal strip. Therefore, S^j is a removable ribbon lying entirely in the bottom row of $\nu^{(j)}$. Lemma 36 then implies that $\nu^{(j)}/\nu^{(j-1)}$ consists of all copies of S^j that can be removed from $\nu^{(j)}$. □

Proposition 123. *[DM13] For n-cores $\lambda \subset \nu$, the pair (λ,ν) is a horizontal strong m-strip if and only if there exists a strong m-strip from ν to $R(n-1,\lambda)$ whose content vector c satisfies $c_1 \ge \lambda_1$.*

Proof. Given any horizontal strong m-strip (λ,ν), we have a chain (7.4) that is characterized by ribbons $S^1,\ldots,S^m$ lying in the bottom row of $R(n-1,\lambda)/\nu$ by Lemma 122. We can obtain a strong strip by associating it to the content vector $(c_1, \ldots, c_m)$, where c_i is the content of the head of ribbon S^i. Then $c_1 \geq \lambda_1$ since $\lambda \subset \nu$.

On the other hand, consider cores $\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n-1, \lambda)$ such that the head h_i of a ribbon in $\nu^{(i)}/\nu^{(i-1)}$ has content c_i and $\lambda_1 \leq c_1 < \cdots < c_m$. The last $n-1$ cells in the bottom row of $R(n-1, \lambda)$ lie at the top of their column and therefore they are the only cells with content greater than $\lambda_1 - 1$. Therefore, h_i must lie in the bottom row of $\nu^{(i)} \subset R(n-1, \lambda)$ implying that bottom rows are strictly growing. □

Remark 124. *The proof of Theorem 128 will establish a claim stronger than Proposition 123: each horizontal strong strip (λ, ν) corresponds uniquely to a strong strip from ν to $R(n-1, \lambda)$ with $c_1 \geq \lambda_1$.*

Example 125. *For $n = 4$ and $\lambda = (3, 1, 1)$, the n-cores ν such that (λ, ν) is a horizontal strong 2-strip are given in Example 121 and each corresponds to a unique strong 2-strip from ν to $R(3, \lambda)$ with $c_1 \geq 3$: their content vectors are $(3, 5)$, $(4, 5)$, and $(4, 5)$, respectively.*

Horizontal strong strips in hand, we now discuss their correspondence with weak order cyclically decreasing chains. For a fixed $x \in [n] = \{0, \ldots, n-1\}$ and $y \in \{0, \ldots, n-1\} \backslash \{x\}$, it will be inferred that $x + 1 \leq y \leq x - 1$ is taken with respect to the total order defined by

$$x + 1 < x + 2 < \cdots < 0 < n - 1 < \cdots < x - 1.$$

For n-cores λ and ν, a simple construction produces a cyclically decreasing word for $w_\nu w_\lambda^{-1}$ from a relevant strong chain from ν to $R(n-1, \lambda)$. For $x = \lambda_1 - 1 \pmod{n}$, define the map

$$\psi : \nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n-1, \lambda) \quad \longmapsto \quad s_{j_1} \cdots s_{j_{n-1-m}}$$

where the elements $j_1 > \cdots > j_{n-1-m}$ of $\{x - 1, \ldots, x + 1\} \backslash \{a_1, \ldots, a_m\}$ are obtained by taking a_{m-i} to be the residue of the leftmost cell in the bottom row of $\nu^{(i+1)}/\nu^{(i)}$, for $0 \leq i < m$. In reverse, a strong chain arises from a reduced expression for $w_\nu w_\lambda^{-1}$ with the map

$$\phi : s_{j_1} \cdots s_{j_{n-1-m}} \quad \longmapsto \quad \nu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(m)} = R(n-1,\lambda)\,,$$

where $\nu^{(i)}$ is obtained from $\nu^{(i+1)}$ by deleting all removable copies of the ribbon whose tail has residue a_{m-i} and lies in the bottom row, where $x+1 \le a_m < \cdots < a_1 \le x-1$ are the elements of $\{x-1,\ldots,x+1\}\backslash\{j_1,\ldots,j_{n-1-m}\}$.

Several lemmas are first needed to prove that ϕ and ψ give the desired bijection. Horizontal strong strips (λ,ν) are defined on the level of cores where the key idea is to study strong chains from ν to the n-translation of λ defined by $R(n-1,\lambda)$. A preliminary result puts the idea of this translation into the framework of the affine Weyl group.

Lemma 126. *[DM13] For $w_\lambda \in \tilde{S}_n^0$, the length $\ell(w_{R(n-1,\lambda)}) = n-1+\ell(w_\lambda)$ and*

$$w_{R(n-1,\lambda)} = s_{x-1} \cdots s_{x+1}\, w_\lambda\,,$$

where $x = \lambda_1 - 1 \pmod n$.

Proof. It suffices to prove that $R(n-1,\lambda) = \mathfrak{a}(s_{x-1}\cdots s_{x+1} w)$. Since the lowest addable corner of λ has residue $x+1$, s_{x+1} acts on λ by adding all corners of residue $x+1$. Similarly, s_{x+2} adds corners of residue $x+2$ and by iteration, the degree of λ increases by $n-1$ under the action of $s_{x-1}\cdots s_{x+1}$. Since $s_{x-1}\cdots s_{x+1}$ is cyclically decreasing, Lemma 117 implies that it acts on λ by adding a horizontal strip. The result follows by noting that $R(n-1,\lambda)$ is the unique core obtained by adding a horizontal strip to λ and increasing degree by $n-1$. □

For $x \in \{0,1,\ldots,n-1\}$, let $S_{\hat{x}} = \langle s_0,\ldots,\hat{s}_x,\ldots,s_{n-1}\rangle \subset \tilde{S}_n^0$ be the subgroup generated by all simple reflections except s_x.

Lemma 127. *[DM13] Given $w_\lambda, u \in \tilde{S}_n^0$ where uw_λ^{-1} is a cyclically decreasing permutation and $\ell(uw_\lambda^{-1}) = \ell(u) - \ell(w_\lambda)$, then $uw_\lambda^{-1} \in S_{\hat{x}}$ for $x = \lambda_1 - 1 \pmod n$.*

Proof. Let $v = s_{j_1} \cdots s_{j_m}$ be a reduced expression for uw_λ^{-1}. By the definition of $\mathfrak{a}$, the residues labelling the cells in $D = \mathfrak{a}(vw_\lambda)/\lambda$ come from the set $\{j_1, \ldots, j_m\}$. In fact, since $\ell(vw_\lambda) = \ell(w_\lambda) + m$, the cells of D are labelled by precisely the set $\{j_1, \ldots, j_m\}$. Since v is cyclically decreasing, we also have that D is a horizontal strip by Lemma 117. Therefore, an extremal cell of residue j_t that does not lie at the end of its row occurs in λ for every $1 \leq t \leq m$. Since x is the residue of the last cell in the bottom row of λ, Property 32 implies that every extremal cell of λ with residue x lies at the end of its row. In particular, $x \neq j_t$ and we have $v \in S_{\hat{x}}$. □

Theorem 128. *[DM13] For n-cores λ and ν, (λ, ν) is a horizontal strong strip if and only if $w_\nu w_\lambda^{-1}$ is a cyclically decreasing permutation where $\ell(w_\nu) = \ell(w_\lambda) + \ell(w_\nu w_\lambda^{-1})$.*

Proof. From a horizontal strong m-strip (λ, ν), Lemma 122 gaurantees us a chain $\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n-1, \lambda)$, such that a ribbon S^{i+1} of $\nu^{(i+1)}/\nu^{(i)}$ has height one and is a removable ribbon in the bottom row of $\nu^{(i+1)}$. It suffices to prove that the image $s_{j_1} \cdots s_{j_t}$ of this chain under ψ is a cyclically decreasing word for $w_\nu w_\lambda^{-1}$ of length $n-1-m$ since the definition of horizontal m-strip implies that $\ell(w_\nu) = n-1-m+\ell(w_\lambda)$.

The definition of ψ uses a_{m-i} to denote the residue of the tail of S^{i+1} and thus the residue of the head of S^i must be $a_{m-i} - 1$. By Lemma 35, we have $w_{\nu^{(i)}} = \tau_{a_{m-i}, a_{m-i-1}} w_{\nu^{(i+1)}}$ for $0 \leq i < m$, where $a_0 = \lambda_1 - 1 \pmod n$. In particular, $w_\nu = \tau_{a_m, a_{m-1}} \cdots \tau_{a_1, a_0} w_{R(n-1,\lambda)}$. Since $\lambda \subset \nu$, we have that $\lambda_1 \leq a_m$ and therefore $x + 1 \leq a_m < \cdots < a_1 \leq x - 1$ for $x = \lambda_1 - 1 \pmod n$. It follows from Lemma 126 that

$$w_\nu w_\lambda^{-1} = \tau_{a_m, a_{m-1}} \cdots \tau_{a_1, x} (s_{x-1} \cdots s_{x+1}) ,$$

or $w_\nu w_\lambda^{-1} = s_{j_1} \cdots s_{j_{n-1-m}}$ where $j_1 > \cdots > j_{n-1-m}$ are the elements of $\{x-1, \ldots, x+1\} \backslash \{a_1, \ldots, a_m\}$. Since these are $n-1-m$ distinct elements, the expression is reduced.

Before proving the reverse direction, note that $s_{j_1} \cdots s_{j_{n-1-m}}$ is the unique reduced expression for $w_\nu w_\lambda^{-1}$ that is ordered by $x - 1 \geq j_1 > \cdots > j_{n-1-m} \geq x + 1$ and it is determined uniquely from ribbon tails in the given chain. Since a given chain under consideration is determined uniquely by its ribbon tails, the uniqueness claim of Remark 124 follows.

Suppose now that $j_1 \cdots j_{n-1-m}$ is a reduced word for a cyclically decreasing permutation $w_\nu w_\lambda^{-1}$ where $\ell(w_\nu) = \ell(w_\lambda) + n - 1 - m$. By Lemma 127, $w_\nu w_\lambda^{-1} \in S_{\hat{x}}$ for $x = \lambda_1 - 1 \pmod n$ and therefore there are m elements $x - 1 \geq a_1 > a_2 > \cdots > a_m \geq x + 1$ in the set $\{x-1, \ldots, x+1\}/\{j_1, \ldots, j_{n-1-m}\}$. The $n-1$ removable cells in the bottom row of $R(n-1, \lambda)$, of residues $x-1, \ldots, x+1$ from right to left, can thus be tiled uniquely into ribbons whose tails are $a_1, \ldots, a_m$, from right to left. Therefore, the shapes in the image of $j_1 \cdots j_{n-1-m}$ under ϕ

$$\nu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(m)} = R(n-1, \lambda)$$

have increasing bottom rows. We claim this is a strong saturated chain and $\lambda \subset \nu$.

Let $\eta^{(m)} = \nu^{(m)}$ so that by Lemma 126, $w_{\eta^{(m)}} = (s_{x-1} \cdots s_{x+1}) w_\lambda$. For $1 \leq i \leq m$, define

$$w_{\eta^{(m-i)}} = \tau_{a_i, a_{i-1}} w_{\eta^{(m-i+1)}} = (s_{x-1} \cdots \hat{s}_{a_1} \cdots \hat{s}_{a_2} \cdots \cdots \hat{s}_{a_i} \cdots s_{x+1}) w_\lambda \,,$$

where $a_0 = x$. Since $w_{\eta^{(0)}} = s_{j_1} \cdots s_{j_{n-1-m}} w_\lambda$, we have that $\lambda \subset \eta^{(0)}$ by Lemma 117. If $w_{\eta^{(m-i)}} \lessdot_B w_{\eta^{(m-i+1)}}$, then $\eta^{(m-i)} = \nu^{(m-i)}$ by Lemma 36 and the claim follows. To ensure that $w_{\eta^{(m-i)}} \lessdot_B w_{\eta^{(m-i+1)}}$, it suffices to show that $w_{\eta^{(m-i+1)}}$ has length $n - i + \ell(w_\lambda)$. Note that $\ell(w_{\eta^{(0)}}) = n - 1 - m + \ell(w_\lambda)$ and consider $w_{\eta^{(m-i)}}$ of length $n - 1 - i + \ell(w_\lambda)$. By commuting relations, $w_{\eta^{(m-i)}} = (\hat{s}_{a_{i-1}} \cdots \hat{s}_{a_i}) w_\mu$ for $w_\mu = (s_{x-1} \cdots \hat{s}_{a_1} \cdots \cdots \hat{s}_{a_{i-1}})(\hat{s}_{a_i} \cdots s_{x+1}) w_\lambda$. Since the lowest addable corner of λ has residue $x + 1$, the lowest addable corner of μ has residue a_i. Therefore, $w_{\eta^{(m-i+1)}} = (\hat{s}_{a_{i-1}} \cdots s_{a_i+1} s_{a_i}) w_\mu$ has length $n - i + \ell(w_\lambda)$. □

Corollary 129. *[DM13] For $1 \leq m < n$ and $w \in \tilde{S}_n^0$,*

$$\xi_{c_{0,m}} \xi_w = \sum_{u \in \tilde{S}_n^0} \xi_u \,, \tag{7.5}$$

where the sum is over u such that $(\mathfrak{a}(w), \mathfrak{a}(u))$ *is a horizontal strong* $(n-1-m)$*-strip.*

Proof. A term $v = w_\nu w_\lambda^{-1}$ occurs in the summand of (7.1) if and only if it is cyclically decreasing of length m and $\ell(w_\nu) = \ell(w_\lambda) + m$. That is, if and only if (λ, ν) is a horizontal strong $n-1-m$-strip by Theorem 128. □

Example 130. *The expansion* $\xi_{c_{0,2}}\xi_{(3,1,1)} = \xi_{(3,1,1,1)} + \xi_{(4,1,1)} + \xi_{(3,2,1)}$ *follows from Example 121 and Corollary 129.*

Chapter 8
Quantum cohomology of flags

Here we show that the strong formulation of the Pieri rule for $H_*(\mathrm{Gr})$ given in Corollary 129 can be applied to the problem of computing intersections in the (small) quantum cohomology of a flag variety. The examination leads to a distinguished family of strong chains defined by a notion of translation that generalizes $R(n-1,\lambda)$.

In this section, we switch to representing the indices of Schubert basis elements by partitions $\lambda \in \mathcal{P}^n$. Recall that $R(n-1,\lambda)$ was defined to be (λ_1+n-1,λ) when λ is an n-core in § 7.2. Here, we abuse notation and instead define $R(n-1,\eta) = (\mathfrak{c}(\eta)_1+n-1,\mathfrak{c}(\eta))$ for $\eta \in \mathcal{P}^n$.

8.1 An affine Monk formula

The quantum cohomology ring $QH^*(X)$ is defined for any Kähler algebraic manifold X, but we consider only the complete flag manifold $X = Fl_n$ of chains of vector spaces in $\mathbb{C}^n$. The quantum cohomolgy ring is simply $QH^*(Fl_n) = H^*(Fl_n)\otimes\mathbb{Z}[q_1,\ldots,q_{n-1}]$ for parameters $q_1,\ldots,q_{n-1}$ as a linear space but the multiplicative structure is much richer than the specialization of $q_1 = \cdots = q_{n-1} = 0$. Cells in the Schubert decomposition of $QH^*(Fl_n)$ are indexed by permutations $w \in S_n$, and the quantum product is defined by

$$\sigma_u * \sigma_v = \sum_w \sum_d q_1^{d_1}\ldots q_{n-1}^{d_{n-1}} \langle u,v,w\rangle_d\, \sigma_{w_0 w}\,, \tag{8.1}$$

where the structure constants are 3-point Gromov-Witten invariants of genus 0 which count equivalence classes of certain rational curves in Fl_n. The understanding and computation of Gromov-Witten invariants is a widely studied problem.

Although the construction is not manifestly positive, all 3-point, genus zero Gromov-Witten invariants $\langle u, w, v\rangle_d$ in (8.1) can be computationally obtained from the subset

$$\{\langle s_r, w, v\rangle_d : 1 \leq r < n \text{ and } w, v \in S_n\} \tag{8.2}$$

since the quantum cohomology is generated by the codimension one Schubert classes. By defining a family of *quantum Schubert polynomials*, Fomin, Gelfand, and Postnikov were able to prove that there is a simple combinatorial characterization for this set that generalizes the classical Monk formula [Mon59].

Theorem 131. *[FGP97] (Quantum Monk formula) For $w \in S_n$ and $1 \leq r < n$, the quantum product of the Schubert classes σ_{s_r} and σ_w is given by*

$$\sigma_{s_r} * \sigma_w = \sum \sigma_{w\tau_{a,b}} + \sum q_c q_{c+1} \cdots q_{d-1} \sigma_{w\tau_{c,d}} \tag{8.3}$$

where the first sum is over all transpositions $\tau_{a,b}$ such that $a \leq r < b$ and $\ell(w\tau_{a,b}) = \ell(w) + 1$, and the second sum is over all transpositions $\tau_{c,d}$ such that $c \leq r < d$ and $\ell(w\tau_{c,d}) = \ell(w) - \ell(\tau_{c,d}) = \ell(w) - 2(d - c) + 1$.

We have found that the idea of horizontal strong strips extends to include combinatorics of the flag Gromov-Witten invariants and the quantum Monk rule. Peterson asserted that $QH^*(G/P)$ of a flag variety is, up to localization, a quotient of the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian Gr_G of G (proven in [LS12]). As a consequence, the Gromov-Witten invariants arise as homology Schubert structure constants of $H_*(\mathrm{Gr}_G)$. The identification of $\langle u, v, w\rangle_d$ with the coefficients in

$$\xi_\mu \, \xi_\lambda = \sum_\nu c^\nu_{\mu,\lambda} \, \xi_\nu \tag{8.4}$$

was made explicit in [LM]. The identification hinges on a correspondence between permutations in S_n and certain partitions defined by

$$\mathrm{sh} : w \mapsto \lambda \quad \text{for} \quad \lambda'_i = \binom{n-i}{2} + inv_i(w_0 w)\,, \tag{8.5}$$

where λ' is the partition obtained by reflecting the shape of λ about the line $y = x$, the inversion $inv_i(u)$ is the number of $u_j < u_i$ for $i < j$, and $w_0 = [n, n-1, \ldots, 1]$ is the permutation of maximal length in S_n.

Theorem 132. *[LM] For $u, v, w \in S_n$ and $d \in \mathbb{N}^{n-1}$,*

$$\langle u, w, v\rangle_d = c^{\eta}_{sh(u),sh(w)}\,, \tag{8.6}$$

where η is obtained by adding $\binom{n+1-i}{2} - (n-i+1)d_i + (n-i)d_{i-1}$ cells to column i of $sh(v)$, for $1 \leq i < n$.

The image of S_n under the map sh is the set of $n!$ partitions $\mathcal{P}^n_{\square} = \{\lambda : \square/\lambda = \text{vertical strip}\}$, where the partition $\square = (n-1, n-2^2, \ldots, 1^{n-1})$. This foreshadows that $(n-1)$**-rectangles**, the shapes $R_r = (r^{n-r})$ with $n-r$ rows of length r, play a role in the combinatorics of quantum cohomology of flag varieties as they have in various contexts of affine Schubert calculus (e.g. [LLM03, LM03a, Mag07, LS12, BBTZ12]). At the root, for any partition $\lambda \in \mathcal{P}^n$ and $1 \leq r < n$,

$$\xi_{\lambda\cup R_r} = \xi_{R_r}\xi_\lambda\,. \tag{8.7}$$

Since a defining subset of Gromov-Witten invariants is given by (8.2), looking closely at $\mathrm{sh}(s_r)$ reveals the role of these shapes in the combinatorics of quantum cohomology. To be precise, it was shown in [LM] that for any $v, w \in S_n$ and $1 \leq r < n$,

$$\langle s_r, v, w\rangle_d = c^{\eta\cup \mathrm{sh}(v)\cup R_r}_{R'_r,\mathrm{sh}(w)}\,, \tag{8.8}$$

where the ith column of η is $(n-i)d_{i-1} - (n+1-i)d_i$ and R'_r is the shape obtained by deleting the corner box from R_r. In particular, Monk's classical formula is determined by $c^{\mathrm{sh}(v)\cup R_r}_{R'_r,\mathrm{sh}(w)}$. Therefore, for any $u, v, w \in S_n$, all Gromov-Witten invariants $\langle u, v, w\rangle_d$ can be computed from the set

$$\left\{c^{\nu}_{R'_r,\lambda} : 1 \le r < n,\ \lambda \in \mathcal{P}^n_{\square},\ \text{and}\ \nu_1 < n\right\}. \tag{8.9}$$

The $\eta \cup \mathrm{sh}(v) \cup R_r$ in (8.8) suggests that the formula for these invariants is related to elements covered by the generic translation of λ defined by

$$R(r,\lambda) = \mathfrak{c}(\lambda \cup R_r)\,.$$

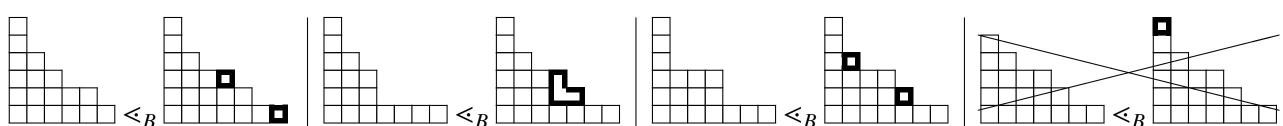
Conjecture 133. *[DM13] (Affine Monk formula) For $1 \le r < n$ and partition λ with $\lambda_1 < n$,*

$$\xi_{R'_r}\,\xi_\lambda = \sum_{\mathfrak{c}(\nu) \lessdot_B R(r,\lambda)} \xi_\nu\,, \tag{8.10}$$

where $\mathfrak{c}(\nu)_i < R(r,\lambda)_i$ for some i such that $(\lambda \cup R_r)_i = r$.

Note that the expansion (8.10) can be derived from results in [BSSb] that determine the expansion of a *non-commutative k-Schur function* indexed by R'_r in terms of words in the affine nilCoxeter algebra $\mathbb{A}$.

Example 134. *For $n = 5$, $\lambda = (3,2,1,1)$, and $R'_3 = (3,2)$, term ξ_ν occurs in the expansion of $\xi_{R'_3}\xi_\lambda$ when ν is a partition where $\mathfrak{c}(\nu) \lessdot_B R(3,\lambda)$ and $\mathfrak{c}(\nu)_i < R(3,\lambda)_i$ for some $i \in \{1,2,3\}$.*



$\nu = (3,3,2,2,1,1)$ | $\nu = (4,2,2,2,1,1)$ | $\nu = (3,3,3,1,1,1)$

Relation (8.7) *then gives the terms in the expansion of* $\xi_{R'_3}\xi_{\lambda\cup R_r}$ *for any* R_r*. In particular,*

$$\xi_{R'_3}\,\xi_{\lambda\cup R_2} = \xi_{(3,3,2,2,1,1)\cup R_2} + \xi_{(4,2,2,2,1,1)\cup R_2} + \xi_{(3,3,3,1,1,1)\cup R_2}\,. \tag{8.11}$$

Since $sh([4,2,5,3,1]) = \lambda\cup R_2 \in \mathcal{P}^n_{\square}$*, this matches the quantum Monk expansion by Equation* (8.8)*:*

$$\sigma_{s_3} * \sigma_{[4,2,5,3,1]} = \sigma_{[4,3,5,2,1]} + q_3\,\sigma_{[4,2,3,5,1]} + q_3 q_4\,\sigma_{[4,2,1,3,5]}\,.$$

8.2 Ribbon strong strips

When $r = n-1$, Conjecture 133 reduces to a special case of the expansion given in Corollary 129. The terms are defined by horizontal strong 1-strips, which are in fact the elements ν covered by $R(n-1,\lambda)$ where $\mathfrak{c}(\nu)_1 < R(n-1,\lambda)_1$. Horizontal strong strips of generic length $1 \le b < n-1$ describe the expansion of $\xi_{(n-1-b)}\xi_\lambda$. We thus turn to the more general expansion, for $1 \le b < r < n$ and partition $\lambda \in \mathcal{P}^n$,

$$\xi_{(r^{n-1-r},r-b)}\,\xi_\lambda = \sum_{\nu\in\mathcal{B}_{r,b,\lambda}} \xi_\nu\,, \tag{8.12}$$

as a guide to characterize a larger family of strong strips associated to the general translation $R(r,\lambda)$.

While horizontal strong strips are certain shapes differing from $R(n-1,\lambda)$ by a horizontal strip, the more general picture involves shapes that differ from $R(r,\lambda)$ by a *horizontal ribbon strip*, a sequences of shapes $\nu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(m)}$ such that $\nu^{(i)}/\nu^{(i-1)}$ is comprised of ribbons whose heads lie above a cell in ν (or in the bottom row), for all $1 \le i \le m$. Rather than requiring that bottom row lengths are increasing as we did for horizontal strong strip, we now require that a ribbon tail lies in a specified set of columns. For $1 \le r < n$ and a partition λ with $\lambda_1 < n$, let $\eta = \lambda \cup R_r$. Let m be the highest row of η that has length r and denote the set of r columns containing the last r cells in row m of $\mathfrak{c}(\eta)$ by $col_r(\lambda)$.

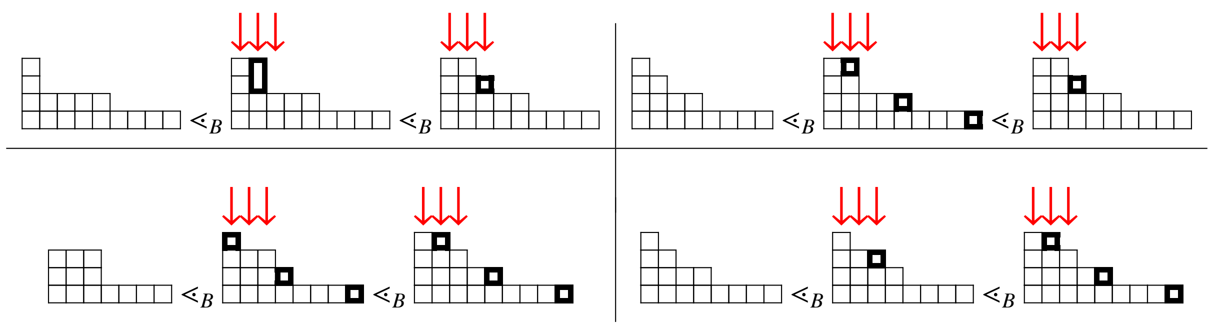
Definition 135. *Given n-cores λ and ν and $1 \le r < n$, the pair (λ, ν) is a* **ribbon strong strip** *with respect to r if there is a horizontal ribbon strip*

$$\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(r, \lambda)$$

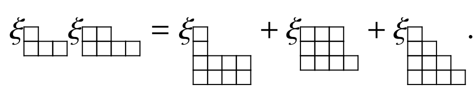
with a ribbon tail of $\nu^{(i)}/\nu^{(i-1)}$ lying in $col_r(\lambda)$ for all $i > 0$. Its length is defined to be m.

The expansion (8.12) can be derived from results in [BSSa] that determine the expansion of a *non-commutative k-Schur function* indexed by (r^{n-1-r}, b) in terms of words in the affine nilCoxeter algebra $\mathbb{A}$. We instead conjecture that the expansion is simply the sum over ν such that $(\mathfrak{c}(\lambda), \mathfrak{c}(\nu))$ is a ribbon strong strip of length b with respect to r.

Example 136. *For $n = 5$ and $\lambda = (4, 2)$, the ribbon strong strips of length 2 with respect to $r = 3$ are*



Conjecturally, this gives the expansion



Conjecture 137. *[DM13] Given n-cores λ and ν and $1 \le r < n$, the pair (λ, ν) is a ribbon strong strip with respect to r if and only if there exists a strong strip*

$$\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(r, \lambda) \tag{8.13}$$

with a ribbon tail of $\nu^{(i)}/\nu^{(i-1)}$ lying in $col_r(\lambda)$ for all $i > 0$.

Proposition 138. *[DM13] Given n-cores λ and ν, (λ, ν) is a horizontal strong strip if and only if (λ, ν) is a ribbon strong strip with respect to $n - 1$.*

Proof. Let p be the number of rows of length $n - 1$ in $\mathfrak{p}(\lambda)$ and thus the bottom $p + 1$ rows of $\mathfrak{p}(\lambda) \cup R_{n-1}$ have length $n - 1$. Note by definition of $\mathfrak{c}$ that the last $n - 1$ cells in rows $1, \ldots, p + 1$ of $\mathfrak{c}(\mathfrak{p}(\lambda) \cup R_{n-1})$ lie at the top of a column and have residues $\lambda_1, \lambda_1 + 1, \ldots, \lambda_1 + (n - 2) \pmod{n}$. Further, $col_{n-1}(\lambda)$ is defined by taking the last $n - 1$ columns in row $p + 1$.

Assume (λ, ν) is a horizontal strong strip. Lemma 122 implies that $\nu^{(j)}/\nu^{(j-1)}$ consists of all copies of S^j that can be removed from $\nu^{(j)}$ where S^j is a removable ribbon in the bottom row of $\nu^{(j)}$. For $j = m$, the discussion in the previous paragraph implies that a copy of S^m lies in row $p + 1$. By iteration, a copy of S^j (and in particular, its tail) lies in the last $n - 1$ columns of row $p + 1$ for all $j = 1, \ldots, m$.

On the other hand, consider a chain of n-cores $\nu = \nu^{(0)} \lessdot_B \nu^{(1)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n - 1, \lambda)$ where the tail of a ribbon S^j in $\nu^{(j)}/\nu^{(j-1)}$ lies in one of the last $n - 1$ columns in row $p + 1$ of $R(n - 1, \lambda)$. Since the number of cell in S^j is smaller than n and there are $n - 1$ cells at the top of a column in row p, S^m must have height one. Therefore, it can be removed from every row $1, \ldots, p + 1$ of $R(n - 1, \lambda)$. By iteration, there is a copy of S^j in the bottom row of $(n - 1 + \lambda_1, \lambda)$ for $j = 1, \ldots, m$. Since the tail of S^1 is on of the last $n - 1$ cells in row $p + 1$ of residue $\lambda_1, \ldots, \lambda_1 + (n - 2) \mod n$, $\lambda \subset \nu$. □

Proposition 139. *[DM13] For $1 \leq r < n$ and for n-cores λ and ν, a ribbon strong strip (λ, ν) with respect to r has length one if and only if $\nu \lessdot_B R(r, \lambda)$ and $\nu_i < R(r, \lambda)_i$ for some i such that $(\lambda \cup R_r)_i = r$.*

Proof. For $\eta \in \mathcal{P}^n$, let $\lambda = \mathfrak{c}(\eta)$ and let m be the highest row of length r in $R_r \cup \eta$. Note that the highest cell of $R(r, \lambda)$ in the leftmost column of $col_r(\lambda)$ lies in a row no higher than row $m + (n - 1 - r)$ and that rows $m, m - 1, \ldots, m - (n - 1 - r)$ of $R_r \cup \eta$ have length r.

Given (λ, ν) is a ribbon strong strip with respect to r of length one, $\nu \lessdot_B R(r, \lambda)$ and the tail of a ribbon $S \subset R(r, \lambda)/\nu$ lies in a column of $col_r(\lambda)$. Suppose the head of S lies in row a. If $a \leq m$, then $\nu \lessdot_B R(r, \lambda)$ and $\nu_i < R(r, \lambda)_i$ for some i such that $(\lambda \cup R_r)_i = r$. When $a > m$, all cells of S lie in colums of $col_r(\lambda)$ and therefore in rows between $m + 1$ and $m + n - 1 - r$. Proposition 9 [LM04] ensures that a removable copy of S also lies in rows $m - (n - 1 - r), \ldots, m - 1, m$, and the forward direction thus follows from Lemma 36.

On the other hand, consider n-cores λ and ν such that $\nu_i \lessdot_B R(r, \lambda)_i$ for some i where $(\lambda \cup R_r)_i = r$. In particular, there is a ribbon $S \subset R(r, \lambda)/\nu$ containing at least one cell in row i. If the tail t of S is not in a column of $col_r(\lambda)$ then t lies in row $i < m$. By Proposition 9 [LM04], there is an extremal cell $\tilde{t}$ of the same residue as t that lies in a column of $col_r(\lambda)$. Consider the subset $\tilde{S}$ of extremal cells in $R(r, \lambda)$ that is formed by taking all extremal cells between $\tilde{t}$ (as the highest) and a cell $\tilde{h}$ of the same residue as the head of S. Since S can be removed from $R(r, \lambda)$, its head lies at the end of its row. Property 32 then implies that $\tilde{h}$ is at the end of its row and thus $\tilde{S}$ is a removable ribbon. By Lemma 36, $\tilde{S} \subset R(r, \lambda)/\nu$ proving the claim. □

Chapter 9

Explicit representatives for Schubert classes

The perspective of horizontal strong strips applies to the study of the (co)homology classes of the affine Grassmannian. Here we derive a new combinatorial object with which to study the representatives for Schubert classes of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$.

9.1 Polynomial realization of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$

Quillen (unpublished) and Garland and Raghunathan [GR75] showed that Gr is homotopy-equivalent to the group $\Omega\, SU(n,\mathbb{C})$ of based loops into $SU(n,\mathbb{C})$. Results from [Bot58] can be used to obtain a polynomial identification of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ inside the ring of symmetric functions $\Lambda = \mathbb{Z}[h_1, h_2, \ldots,]$.

Traditionally, bases for the space of symmetric function are indexed by partitions. Descriptions of the homology and cohomology ring are most natural in terms of the functions h_λ and the monomial symmetric functions m_λ. The homology $H_*(\mathrm{Gr})$ is identified by the subring $\Lambda_{(n)}$ of Λ and the cohomology $H^*(\mathrm{Gr})$ can be identified by the quotient $\Lambda^{(n)}$ where

$$\Lambda_{(n)} = \mathbb{Z}[h_1, \ldots, h_{n-1}] \quad \text{and} \quad \Lambda^{(n)} = \Lambda/\langle m_\lambda : \lambda_1 \geq n \rangle \,.$$

These spaces are naturally paired under the Hall-inner product on Λ.

The Schur function basis for Λ is self-dual with respect to the Hall-inner product ([Sag00]). Recall that this basis is a fundamental combinatorial tool to study tensor products of irreducible representations and intersections in the geomety of the Grassmannian variety.

Refinements of the Schur basis for Λ to bases for $\Lambda^{(n)}$ and $\Lambda_{(n)}$ give a combinatorial framework that can be applied to the cohomology and homology of Gr. Let $k = n - 1$ throughout. The basis of k-Schur functions for $\Lambda_{(n)}$ was introduced in [LM05], inspired by the Macdonald polynomial study of [LLM03] summarized in the introduction. The basis for $\Lambda^{(n)}$ that is dual to the k-Schur basis with respect to the Hall-inner product arose in the context of the quantum cohomology of Grassmannians in [LM08]. Appealing to the algebraic nil-Hecke ring construction of Kostant and Kumar [KK86] and the work of Peterson [Pet97], Lam [Lam08] proved that the Schubert classes ξ_w and ξ^w can be represented explicitly by the k-Schur functions in $\Lambda_{(n)}$ and $\Lambda^{(n)}$, respectively.

For our purposes, we define the k-Schur functions of $H^*(\mathrm{Gr})$ as the weight generating functions of a combinatorial object called *affine factorizations* and then introduce the homology k-Schur functions by duality.

Definition 140. *For any composition $\alpha \in \mathbb{N}^\ell$ with parts smaller than n and $w \in \tilde{S}_n$ of length $|\alpha|$, an* **affine factorization** *for w of weight α is a decomposition*

$$w = v^\ell \cdots v^1 \,,$$

where v^i is a cyclically decreasing permutation of length α_i.

The representatives for the Schubert classes of $H^*(\mathrm{Gr})$ are then defined, for $\lambda \in C^n$, by

$$\mathfrak{S}^{(n)}_\lambda = \sum_{w_\lambda = v^r \cdots v^1} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)} \,, \tag{9.1}$$

over all affine factorizations $v^r \cdots v^1$ of w (In [Lam06], by dropping the condition that w_λ is affine Grassmannian, these are extended to a more general family of functions that relate to the stable limits of Schubert polynomials [LS82, Sta84]). The set $\{\mathfrak{S}^{(n)}_\lambda\}_{\lambda \in C^n}$ is a basis for $\Lambda^{(n)}$ and we take the k-Schur representatives for Schubert classes of $H_*(\mathrm{Gr})$ to be the dual basis $\{s^{(n)}_\nu\}_{\nu \in C^n}$ with respect to the Hall-inner product. That is, the k-Schur functions are defined by the relation

$$\langle \mathfrak{S}^{(n)}_\lambda, s^{(n)}_\nu \rangle = \delta_{\lambda\nu} \,. \tag{9.2}$$

9.2 Affine Bruhat countertableaux

Here we derive a new combinatorial object with which to study $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ by considering the association between cyclically decreasing permutations and horizontal strong strips that was made in Section 7.2. Recall that the sequence (3.1) can be represented by its *countertableau* filling, derived by placing an $r+1-i$ in $\lambda^{(i)}/\lambda^{(i-1)}$.

Definition 141. *Fix composition $\alpha = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_i < n$ and n-core $\lambda^{(r)}$ of degree $|\alpha|$. An* **affine Bruhat countertableau** *of shape $\lambda^{(r)}$ and weight α is a skew tableau $\lambda^{(r)} = \mu^{(0)} \subset \cdots \subset \mu^{(r)}$ such that*

$$\mu^{(x)} = (\mu_1^{(1)}, \ldots, \mu_{x-1}^{(x-1)}, \lambda_1^{(r-x)} + n - 1, \lambda^{(r-x)}), \tag{9.3}$$

where $(\lambda^{(x-1)}, \lambda^{(x)})$ is a horizontal strong $(n-1-\alpha_x)$-strip for $1 \leq x \leq r$ and $\lambda^{(0)} = \emptyset$.

An affine Bruhat countertableau (or *ABC*) is represented by its skew countertableau filling where $r - x + 1$ is placed in the cells of $\mu^{(x)}/\mu^{(x-1)}$. We denote the set of *ABC*'s of shape λ and weight α by $ABC(\lambda, \alpha)$ and let $ABC(\lambda)$ be their union over all weights α.

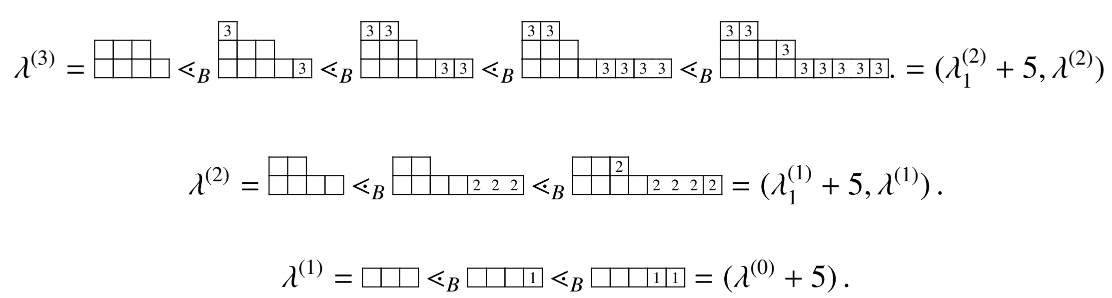
Example 142. *For $n = 6$, $\mu^{(0)} = (4,3,0) \subset (9,4,2) \subset (9,8,3) \subset (9,8,5) = \mu^{(3)}$ is an ABC of shape $(4,3)$ and weight $(3,3,1)$, represented by its countertableau filling*

$A =$

3	3	2	1	1				
			3	2	2	2	2	
				3	3	3	3	3

.

The corresponding strong strips are



Lemma 143. *[DM13] Relation* (9.3) *uniquely identifies the element* $\mu^{(0)} \subset \cdots \subset \mu^{(r)}$ *in* $ABC(\mu^{(0)}, \alpha)$ *with the sequence* $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \mu^{(0)}$ *where* $(\lambda^{(p-1)}, \lambda^{(p)})$ *is a horizontal strong* $(n-1-\alpha_p)$*-strip,*

Proof. The forward direction is immediate from the definition of ABC. On the other hand, consider a sequence $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$ where $(\lambda^{(p-1)}, \lambda^{(p)})$ is a horizontal strong $(n-1-\alpha_p)$-strip for $1 \leq p \leq r$. Let $\mu^{(0)} = \lambda^{(r)}$, and for $1 \leq p \leq r$, define $\mu^{(p)}$ by Relation (9.3). From this, we find that

$$\mu^{(p)} = (\lambda_1^{(r-1)} + n - 1, \ldots, \lambda_1^{(r-p+1)} + n - 1, R(n-1, \lambda^{(r-p)}))\,,$$

and need only to confirm that $\mu^{(p)}/\mu^{(p-1)}$ is a horizontal strip for $1 \leq p \leq r$. Note that

$$\mu^{(p-1)} = (\lambda_1^{(r-1)} + n - 1, \ldots, \lambda_1^{(r-p+1)} + n - 1, \lambda^{(r-p+1)})\,.$$

The claim follows by recalling that when $(\lambda^{(r-p)}, \lambda^{(r-p+1)})$ is a horizontal strong strip, $R(n-1, \lambda^{(r-p)})/\lambda^{(r-p+1)}$ is a horizontal strip. □

Theorem 144. *[DM13] For any n-core* λ*,*

$$\mathfrak{S}_\lambda^{(n)} = \sum_{A \in ABC(\lambda)} x^{weight(A)}\,. \tag{9.4}$$

Proof. Fix a composition α of length r with parts smaller than n and an n-core λ such that $deg(\lambda) = |\alpha|$. Define a map on domain $ABC(\lambda, \alpha)$ by sending

$$\Theta : A \longmapsto v^r \cdots v^1 \quad \text{where } v^i = w_{\lambda^{(i)}} w_{\lambda^{(i-1)}}^{-1}\,,$$

for the unique sequence $\lambda^{(0)} \subset \cdots \subset \lambda^{(r)}$ associated to A via Lemma 143. We claim that Θ is a bijection whose image is the set of affine factorizations of w_λ of weight α. For this, we must prove that v^i is a cyclically decreasing permutation of length α_i, $\ell(w_\lambda) = |\alpha|$ and that Θ is a bijection.

When $\alpha = (\alpha_1)$ has only one part, the unique element of $ABC(\lambda, \alpha)$ corresponds to the sequence $\emptyset \subset \lambda$ where $\lambda = (\alpha_1)$. Its image under Θ is $v^1 = w_{(\alpha_1)} = s_{\alpha_1 - 1} \cdots s_0$, the only decomposition of w_λ into one cyclically decreasing permutation v^1 of length α_1. By induction, assume that the sequence $\emptyset = \lambda^{(0)} \subset \cdots \subset \lambda^{(r-1)}$, where $(\lambda^{(j-1)}, \lambda^{(j)})$ is a horizontal strong $(n-1-\alpha_j)$-strip, corresponds uniquely to a decomposition of $w_{\lambda^{(r-1)}} = v^{r-1} \cdots v^1$ into cyclically decreasing permutations v^j of length α_j, for $j < r$. Since $v^r = w_{\lambda^{(r)}} w^{-1}_{\lambda^{(r-1)}}$ is cyclically decreasing of length α_r if and only if $(\lambda^{(r-1)}, \lambda^{(r)})$ is a horizontal strong $(n-1-\alpha_r)$-strip by Theorem 128, the result follows by induction.

□

Because $\mathfrak{S}^{(n)}_\lambda$ is a symmetric function, the coefficient of x^α in (9.4) equals the coefficient of m_μ, where μ is the non-increasing rearrangement of the parts of α. The set of monomial symmetric functions indexed by elements in $\mathcal{P}^n = \{\lambda \in \mathcal{P} : \lambda_1 < n\}$ is a basis for $\Lambda^{(n)}$ and in fact, the transition matrix from $\{\mathfrak{S}^{(n)}_\lambda\}_{\lambda \in C^n}$ to $\{m_\mu\}_{\mu \in \mathcal{P}^n}$ is unitriangular. The unitriangularity relation is described by an identification from [LM05] of n-cores with partitions of $\mathcal{P}^n$, given by the map

$$\mathfrak{c} : \mathcal{P}^n \longrightarrow C^n ,$$

where $\mathfrak{c}^{-1}(\gamma) = (\lambda_1, \ldots, \lambda_\ell)$ and λ_i is the number of cells in row i of γ with hook-length smaller than n. The unitriangularity relation is taken with respect to the dominance order on partitions of the same degree, where $\lambda \lhd \mu$ when $\lambda_1 + \cdots + \lambda_s \leq \mu_1 + \cdots + \mu_s$ for all s. It was proven in [LM05, Lam06] that for any $\lambda \in \mathcal{P}^n$,

$$\mathfrak{S}^{(n)}_{\mathfrak{c}(\lambda)} = m_\lambda + \sum_{\substack{\mu \in \mathcal{P}^n \\ \mu \not\rhd \lambda}} K^n_{\lambda\mu} m_\mu \,. \tag{9.5}$$

Corollary 145. *[DM13] For $\lambda, \mu \in \mathcal{P}^n$, $K^n_{\lambda\mu}$ is the number of ABC's of shape $\mathfrak{c}(\lambda)$ and weight μ. In particular, there is a unique ABC of shape $\mathfrak{c}(\lambda)$ and weight λ and $ABC(\mathfrak{c}(\lambda), \mu) = \emptyset$ when $\mu \not\lhd \lambda$.*

Proposition 146. *[DM13] [LM07] For an n-core λ where deg(λ) < n, we have $\mathfrak{S}^{(n)}_\lambda = s_\lambda$.*

Proof. Given an n-core λ where $deg(\lambda) < n$ and a partition μ of length r where $|\mu| = deg(\lambda)$, we shall prove that there is a bijection between $ABC(\lambda, \mu)$ and $SSYT(\lambda, \mu)$.

By Lemma 143, $A \in ABC(\lambda, \mu)$ is defined by a sequence of n-cores

$$\emptyset \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda \tag{9.6}$$

where $(\lambda^{(i-1)}, \lambda^{(i)})$ is a horizontal strong $(n-1-\mu_i)$-strip. Note in particular that $n-1-\mu_i = n-1+deg(\lambda^{(i-1)}) - deg(\lambda^{(i)})$. Since Theorem 128 implies that $w_{\lambda^{(i)}} w^{-1}_{\lambda^{(i-1)}}$ is cyclically decreasing of length $\ell(w_{\lambda^{(i)}}) - \ell(w_{\lambda^{(i-1)}}) = \mu_i$, there are μ_i distinct residues labelling the cells of the horizontal strip $\lambda^{(i)}/\lambda^{(i-1)}$ by Lemma 117. If $\lambda \in C^n$ with $deg(\lambda) < n$, then no two cells that lie at the top of their column in $\lambda^{(i)} \subset \lambda$ can have the same n-residue. Therefore, $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal μ_i-strip, implying that (9.6) is an element of $SSYT(\lambda, \mu)$.

On the other hand, given a semi-standard tableau $T \in SSYT(\lambda, \mu)$ defined by (9.6), it suffices to show that $(\lambda^{(i-1)}, \lambda^{(i)})$ is a horizontal strong $n-1-\mu_i$-strip for all i. By definition of semi-standard tableau, $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal μ_i-strip for $1 \leq i \leq r$. Since $deg(\lambda) < n$, μ_i distinct residues label the cells of $\lambda^{(i)}/\lambda^{(i-1)}$ and $deg(\lambda^{(i)}) - deg(\lambda^{(i-1)}) = \mu_i$. Therefore, by Lemma 117, $w_{\lambda^{(i)}} w^{-1}_{\lambda^{(i-1)}}$ is cyclically decreasing with length μ_i. Theorem 128 then implies the result. □

An *ABC* countertableau A comes equipped with a ribbon tiling specified by its defining strong strips. Let the **column residue** of every cell in column c be $c-1 \pmod n$. Recall that the bijection ϕ identifies a horizontal strong $(n-1-m)$-strip $(\lambda^{(i)}, \lambda^{(i+1)})$ with a reduced word $j_1 \cdots j_m$ for $w_{\lambda^{(i+1)}} w^{-1}_{\lambda^{(i)}}$ by a tiling of $R(n-1, \lambda^{(i)})/\lambda^{(i+1)}$ with ribbons of height one that are determined by placing a tail in the rightmost cell of the bottom row with residue $t \in \{x-1, \ldots, x+1\} \backslash \{j_1, \ldots, j_m\}$ for $x = \lambda^{(i)}_1 - 1 \pmod n$. Therefore, since A is given by $\mu^{(0)} \subset \cdots \subset \mu^{(r)}$ where $\mu^{(r+1-i)} = (\mu^{(r-i)}_1, \ldots, \mu^{(r-i)}_{r-i}, R(n-1, \lambda^{(i)}))$, these ribbons tile row $r+1-i$ of A where the residues of their tails are now column residues and ribbons containing letter $j > i$ in row i are copies of the ribbons specified in row j.

A diagram derived from an *ABC* A of weight $\mu \in \mathcal{P}^n$ called the *extension* $ext(A)$ is a useful tool to convert between affine factorizations and ABC's.

Definition 147. *For a given ABC A of weight $\mu \in \mathcal{P}^n$, the* **extension** *$ext(A)$ is formed by appending a ribbon of length $\lambda_1^{(x)} - \lambda_1^{(x-1)} + 1$ to the end of row x, and then deleting any letter larger than x in row x and the tail of every ribbon containing x.*

Example 148. *For $n = 6$ and an ABC A of weight $(3, 3, 3, 1)$, we have*

	4	2	1	1						
		4	3	2	2	2	2			
			4	4	4	3	3	3		
						4	4	4	4	4

$\longmapsto$

						1	1	1			
					2	2			2		
							3			3	3
									4		

Lemma 149. *[DM13] For $A \in ABC(\lambda)$, $j_1 \cdots j_\ell$ is a reduced word for v^i in the affine factorization $\Theta(A) = v^r \cdots v^1$ if and only if the cells containing i in $ext(A)$ have column residues $\{j_1, \ldots, j_\ell\}$.*

Proof. Consider the ABC given by $\mu^{(0)} \subset \cdots \subset \mu^{(r)}$ where

$$\mu^{(r-i+1)} = (\lambda_1^{(r-1)} + n - 1, \ldots, \lambda_1^{(i)} + n - 1, R(n-1, \lambda^{(i-1)}))$$

and

$$\mu^{(r-i)} = (\lambda_1^{(r-1)} + n - 1, \ldots, \lambda_1^{(i)} + n - 1, \lambda^{(i)}) \,.$$

The parts of $\mu^{(r-i+1)}$ and $\mu^{(r-i)}$ differ only in the top i rows where we find the cells of A containing an i forming the horizontal strip $D = R(n-1, \lambda^{(i-1)})/\lambda^{(i)}$. In particular, the bottom row of D in A has an i in columns $\lambda_1^{(i)} + 1, \ldots, \lambda_1^{(i-1)} + n - 1$. To determine which cells of $ext(A)$ contain i, a ribbon of length $\lambda^{(i)} - \lambda^{(i-1)} + 1$ is appended to the end of this row and we must appeal to the strong strips to delete the tails.

Theorem 144 implies that $\Theta(A) = v^r \cdots v^1$ is an affine factorization for w_λ where $v^i = w_{\lambda^{(i)}} w^{-1}_{\lambda^{(i-1)}}$. For each i, the proof of Theorem 9.4 uniquely identifies a reduced word $j_1 \cdots j_{n-1-m}$ for v^i with the strong chain

$$\lambda^{(i)} = \nu^{(0)} \lessdot_B \cdots \lessdot_B \nu^{(m)} = R(n-1, \lambda^{(i-1)}) \,,$$

where $\nu^{(i-1)}$ is obtained from $\nu^{(i)}$ by deleting all removable copies of the ribbon whose tail has residue a_{m-i+1} and lies in the bottom row, for the elements $x+1 \leq a_m < \cdots < a_1 \leq x-1$ of $\{x-1,\ldots,x+1\} \backslash \{j_1,\ldots,j_{n-1-m}\}$ and $x = \lambda^{(i-1)} - 1 \pmod{n}$. Appending a ribbon of length $\lambda_1^{(i)} - \lambda_1^{(i-1)} + 1$ to the end of the bottom row of $R(n-1, \lambda^{(i-1)})/\lambda^{(i)}$ gives a skew shape where $\{j_1,\ldots,j_{n-1-m}\}$ is the set of column residues labeling the cells in the bottom row excluding ribbon tails. □

Chapter 10

The t-generalized affine Schubert polynomials

Macdonald's basis of P-functions (see [Mac95]) are defined by

$$P_\lambda(x;t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w(x^\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}) \tag{10.1}$$

where $v_\lambda(t) = \prod_{j \geq 0} \prod_{i=1}^{m_j} \frac{1-t^i}{1-t}$ for m_j the multiplicity of j in λ. For convenience, we work with the deformation $\tilde{P}_\lambda(x;t) = t^{-n(\lambda)} P_\lambda(x;t^{-1})$ where $n(\lambda) = \sum_i (i-1)\lambda_i$. The set of $\tilde{P}$-functions forms a basis for Λ that generalizes the monomial basis; when $t = 1$, $\tilde{P}_\mu(x;1) = m_\mu$. One of the most important features of this basis is that the Kostka-Foulkes polynomials are inscribed in the Schur to $\tilde{P}$-function transition matrix:

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t)\, \tilde{P}_\mu(x;t)\,. \tag{10.2}$$

Moreover, the $q = 0$ case of the Macdonald polynomials $\{H_\mu(x;0,t)\}$ arise as the the dual basis to $\{\tilde{P}_\mu\}$ with respect to the Hall-inner product.

10.1 Cocharge of an ABC

The n-cocharge of an ABC depends on computing an index vector in a similar spirit. However, the role of n brings forth an additional concept.

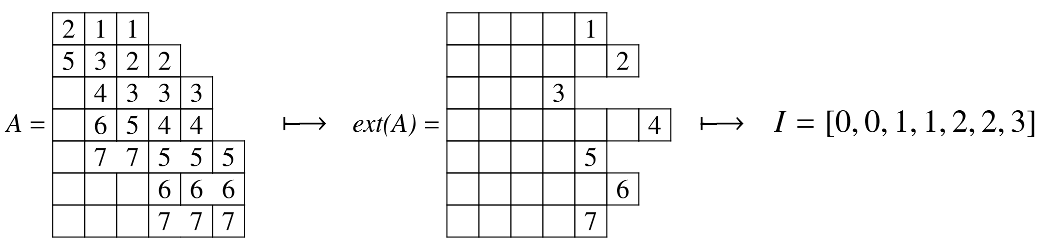
Definition 150. *Any ribbon in an ABC that is filled with letter i but does not lie in row i is called an* **offset**. *Set the number of cells that are not tails in all the offsets,*

$$off(A) = \sum_{R:\ \textit{offset in } A} (size(R) - 1)\,,$$

The $off(A)$ is one of the two components needed when computing n-cocharge. The index is the other, computed on the extension of A. Our construction of the index considers only the cells in $ext(A)$. For A of weight 1^m, $ext(A)$ is standard; there is exactly one cell in each row i (coming from the single ribbon head with an i in row i of A). In this case, the k-cocharge is defined by computing an index vector $I = [0, I_2, \ldots, I_m]$ defined by

$$I_{r+1} = \begin{cases} I_r & \text{when } r+1 \text{ is east of } r \\ I_r + 1 & \text{when } r+1 \text{ is west of } r\,. \end{cases}$$

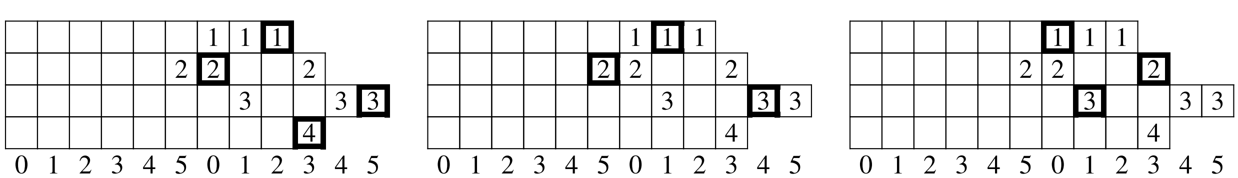
Example 151. *From the ABC of weight* (1^7)*:*



Equipped with a method to obtain the index when $ext(A)$ has a single i in row i, we describe a method for extracting standard fillings from an ABC of arbitrary weight $\mu \in \mathcal{P}^n$.

Algorithm 152. *Given an ABC A of weight* $\mu \in \mathcal{P}^n$*, consider its labelling by column residues. Iteratively earmark a standard sequence starting with the rightmost 1. From an* x *(of column residue i) the appropriate choice of* $x+1$ *will be determined by choosing its column residue from the set* $\mathcal{B}$ *of all column residues labelling the* $x+1$*'s. Reading counter-clockwise from i, this choice is the closest* $j \in \mathcal{B}$ *on a circle labelled clockwise with* $0, 1, \ldots, n-1$.

Example 153.



$I = [0, 1, 1, 2]$ $\qquad I = [0, 1, 1]$ $\qquad I = [0, 0, 1]$

Definition 154. *For an ABC A of weight $\mu \in \mathcal{P}^n$, the n-**cocharge** of A is defined by*

$$n\text{-}cocharge(A) = \sum_r I_r(A) + off(A)\,.$$

We use the Schur expansion in $\tilde{P}$-functions (10.2) as a guide to introduce a new family of symmetric functions involving the parameter t that play a role in affine Schubert calculus and the theory of Macdonald polynomials.

Definition 155. *For $\lambda \in \mathcal{P}^n$, set*

$$\mathfrak{S}^{(n)}_{\mathfrak{c}(\lambda)}(x;t) = \sum_{\mu\in\mathcal{P}^n} K^n_{\lambda\mu}(t)\,\tilde{P}_\mu(x;t)\,, \tag{10.3}$$

where the coefficients are taken to be the n-cocharge generating functions of ABC's (or weak Koskta-Foulkes polynomials),

$$K^n_{\lambda\mu}(t) = \sum_{A\in ABC(\mathfrak{c}(\lambda),\mu)} t^{n\text{-}cocharge(A)}\,. \tag{10.4}$$

For each $n > 1$, consider the restricted linear span of Macdonald polynomials $H_\lambda(x;0,t)$ and the $\tilde{P}$-functions defined by

$$\Lambda^t_{(n)} = \mathcal{L}\{H_\lambda(x;0,t) : \lambda_1 < n\} \quad \text{and} \quad \Lambda^{(n)^t} = \mathcal{L}\{\tilde{P}_\lambda(x;t) : \lambda_1 < n\}\,.$$

When $t = 1$, these reduce to $\Lambda_{(n)}$ and $\Lambda^{(n)}$, respectively.

Proposition 156. *[DM13] For $n > 1$, the set $\{\mathfrak{S}^{(n)}_\lambda(x;t)\}_{\lambda\in C^n}$ forms a basis for $\Lambda^{(n)^t}$ that reduces to a set of representatives for the Schubert cohomology classes of Gr when $t = 1$.*

Proof. Since $\tilde{P}_\mu(x;1) = m_\mu$, we have that $\{\mathfrak{S}^{(n)}_\lambda(x;1)\} = \{\mathfrak{S}^{(n)}_\lambda\}$ which gives a set of Schubert representatives $\{\xi^{w_\lambda}\}$ in $H^*(\mathrm{Gr})$ by Theorem 144. The set $\{\tilde{P}_\lambda(x;t) : \lambda \in \mathcal{P}^n\}$ is linearly independent and therefore a basis for $\Lambda^{(n)^t}$. Therefore, $\mathfrak{S}^{(n)}_\lambda(x;t) \in \Lambda^{(n)^t}$ implies that it suffices to show the transition matrix between $\tilde{P}$-functions and $\{\mathfrak{S}^{(n)}_\lambda(x;t)\}_{\lambda\in C^n}$ is invertible. The matrix is square since there is a bijection between n-cores and elements of $\mathcal{P}^n$ and in fact invertible since Corollary 145 implies

$$K^n_{\eta\mu}(t) = t^{n\text{-cocharge}(B)} + \sum_{\substack{A\in ABC(\mathfrak{c}(\eta),\mu)\\ \eta \ntrianglelefteq \mu}} t^{n\text{-cocharge}(A)}$$

where B is the unique ABC of wieght η and shape $\mathfrak{c}(\eta)$. □

Let the set of functions $\{s^{(n)}_\gamma(x;t)\}_{\gamma\in C^n}$ be the basis for $\Lambda^t_{(n)}$ defined by the duality relation, with respect to the Hall-inner product,

$$\langle \mathfrak{S}^{(n)}_\mu(x;t), s^{(n)}_\gamma(x;t)\rangle = \delta_{\mu\gamma}\,. \tag{10.5}$$

Since the Macdonald polynomial $H_\lambda(x;q,t)$ reduces to the Hall-Littlewood polynomial $H_\lambda(x;t)$ when $q = 0$, we are now able to prove there is a natural tie between affine Schubert calculus and Macdonald polynomials.

Corollary 157. *[DM13] For each n-core λ, $s^{(n)}_\lambda(x;1)$ represents the Schubert class ξ_{w_λ} in $H_*(Gr)$ and for every $\mu \in \mathcal{P}^n$, the Macdonald polynomial at $q = 0$ satisfies the non-negative expansion*

$$H_\mu(x;0,t) = \sum_\lambda K^n_{\lambda\mu}(t)\, s^{(n)}_{\mathfrak{c}(\lambda)}(x;t)\,.$$

Proof. The result follows by noting that (10.5) reduces to (9.2) when $t = 1$ by Proposition 156 and that (9.2) defines the k-Schur functions representing the Schubert class ξ_{w_λ}. □

Combinatorial results on ABC's can be used to prove that when $n > |\lambda|$, both $s^{(n)}_\lambda(x;t)$ and $\mathfrak{S}^{(n)}_\lambda(x;t)$ reduce to the Schur function s_λ.

Lemma 158. *[DM13] Given a partition λ where $|\lambda| < n$, if T is a semi-standard tableau of shape λ then Definition 198 reduces to Definition 64.*

Proof. Consider x of n-residue i in T. Note that $|\lambda| < n$ implies that there is a unique cell c of residue i that contains x. Let j be the first entry on the circle reading counter-clockwise from i that is a residue of a cell containing $x+1$. If there is an $x+1$ above c, then the south-easternmost cell containing an $x+1$ that is above c has residue j since there are no $x+1$'s of a residue counter-clockwise between i and j. If there are none above c, then for the same reason, the south-easternmost cell containing an $x+1$ has residue j. □

Proposition 159. *[DM13] For $\lambda \in C^n$ with $deg(\lambda) < n$, $\mathfrak{S}^{(n)}_\lambda(x;t) = s^{(n)}_\lambda(x;t) = s_\lambda$.*

Proof. Let $\lambda \in C^n$ with $deg(\lambda) < n$. By Lemma 143, $A \in ABC(\lambda, \mu)$ is defined by a sequence of n-cores

$$\emptyset \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda \tag{10.6}$$

where $(\lambda^{(i-1)}, \lambda^{(i)})$ is a horizontal strong $(n-1-\mu_i)$-strip. This sequence is in one to one correspondence with a unique element $T \in SSYT(\lambda, \mu)$ by Proposition 146. Lemmas 117 and 149 imply that the set of residues labelling cells of $\lambda^{(i)}/\lambda^{(i-1)}$ is the same as the set of column residues of cells containing the letter i in $ext(A)$. Since Definition 198 depends only on residues, it remains to show that the index vector on a standard sequence matches.

Note that the cocharge index of a standard sequence of T can be defined by ordering the residues of T with respect to

$$x+1 < x+2 < \cdots < 0 < 1 < \cdots < x-1 < x, \quad \text{where } x = \lambda^{(i)}_1 - 1 \pmod{n}, \tag{10.7}$$

and setting $I_i = I_{i-1}$ when the residue of the cell containing i is larger than the residue of $i-1$ and $I_i = I_{i-1} + 1$ otherwise. Recall also that the index for an ABC is computed by geographically comparing a cell containing $i-1$ in $ext(A)$ to a cell containing i; if i is east of $i-1$ then $I_i = I_{i-1}$

and $I_i = I_{i-1} + 1$ otherwise. We claim that the colum residue of the cell containing i is larger than that containing $i-1$ only when i is east of $i-1$.

First we claim that if $\lambda_1^{(i-1)} < \lambda_1^{(i)}$, then there is no ribbon S containing $i-1$ that has a non-tail cell in columns $[\lambda_1^{(i-1)} + 1, \lambda_1^{(i)}]$ of A. By way of contradiction suppose such a ribbon S containing $i-1$ with a non-tail cell c of column residue r exists in A. By Lemma 149, $\lambda^{(i-1)}$ has a cell of residue r. Since c is in columns $[\lambda_1^{(i-1)} + 1, \lambda_1^{(i)}]$, where $\lambda_1^{(i-1)} < \lambda_1^{(i)}$, then $\lambda^{(i)}/\lambda^{(i-1)}$ has a cell in the bottom row of residue r. Thus T has two distinct cells, not on the same diagonal and of residue r. This contradicts the assumption that $deg(\lambda) < n$.

Observe that the $(i-1)^{st}$ row of A has its rightmost cell in column $\lambda_1^{(i-2)} + n - 1$. Since $ext(A)$ is constructed by first appending $\lambda_1^{(i-1)} + \lambda_1^{(i-2)} + 1$ cells to the $(i-1)^{st}$ row of A, then the cells containing $i-1$ in $ext(A)$ are within columns

$$[\lambda_1^{(i-1)} + 1, (\lambda_1^{(i-2)} + n - 1) + (\lambda_1^{(i-1)} - \lambda_1^{(i-2)} + 1)] = [\lambda_1^{(i-1)} + 1, \lambda_1^{(i-1)} + n].$$

Similarly, the cells containing i in $ext(A)$ are within columns $[\lambda_1^{(i)} + 1, \lambda_1^{(i)} + n]$. Since there are no ribbons containing $i-1$ that have a non-tail cell in columns $[\lambda_1^{(i-1)} + 1, \lambda_1^{(i)}]$ of A, then the cells containing $i-1$ in $ext(A)$ are actually within columns $[\lambda_1^{(i)} + 1, \lambda_1^{(i-1)} + n]$. Since $[\lambda_1^{(i)} + 1, \lambda_1^{(i-1)} + n] \subseteq [\lambda_1^{(i)} + 1, \lambda_1^{(i)} + n]$, then when i is east of $i-1$ in $ext(A)$, its column residue is larger with respect the same ordering (10.7).

It remains to prove that $off(A) = 0$. By contradiction, if $off(A) > 0$ then A has a ribbon O of length greater than 1 and filled with the letter i that is not in the i^{th} row, for some $1 \leq i \leq \ell(\mu)$. On the one hand, note that O has cells which are at the top of their columns in $R(n-1, \lambda^{(i-1)})$, and not in the bottom row. There is a horizontal ribbon in the bottom row of $R(n-1, \lambda^{(i-1)})$ whose cells are at the top of their columns and of the same residue as those of O. By Lemmas 149, the set of residues labeling cells of $\lambda^{(i)}/\lambda^{(i-1)}$ is the same as the set of column residues of cells containing letter i in $ext(A)$. Thus, in $R(n-1, \lambda^{(i-1)})$, the residues of non-tail cells of O are the same as the residues labeling a horizontal strip S in $\lambda^{(i)}/\lambda^{(i-1)}$.

On the other hand, if $\lambda \in C^n$ with $deg(\lambda) < n$, then no two cells that lie at the top of their columns in $\lambda^{(i)} \subset \lambda$ can have the same n-residue. Thus the residues labeling cells of $\lambda^{(i)}/\lambda^{(i-1)}$ are unique. Since O is not in the bottom row of $R(n-1, \lambda^{(i-1)})$, and it is of length greater than 1, then it must be the case that there is a non-tail cell of it which is above a cell of S. This contradicts the fact that $R(n-1, \lambda^{(i-1)})/\lambda^{(i-1)}$ is a horizontal strip; constructed by adding a cell to the top of every column of $\lambda^{(i-1)}$ and $n-1$ cells to its bottom row. □

10.2 Charge of an *ABC*

In this section we are concerned with defining a *n-charge* statistic over *ABC*'s which is in similar spirits to that of the cocharge statistic. To do this, we set the bottom row of *ABC A* is row one, and the left-most column is column one. With this in mind, a cell of an *ABC* which is in row r and column c will be sometimes referred to as (r, c).

Definition 160. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$. For each $1 \leq i \leq \ell(\mu)$, let $A_{\leq i}$ be the ABC A restricted to the top i rows of A and those cells with filled with letters $t \leq i$.*

Example 161. *If the ABC*

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline & 4 & 2 & 1 & 1 \\ \hline & & 4 & 3 & 2 & 2 & 2 & 2 \\ \hline & & & 4 & 4 & 4 & 3 & 3 & 3 \\ \hline & & & & & & 4 & 4 & 4 & 4 & 4 \\ \hline \end{array} \quad \Longrightarrow \quad A_{\leq 3} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & 2 & 1 & 1 \\ \hline & & & 3 & 2 & 2 & 2 & 2 \\ \hline & & & & & & 3 & 3 & 3 \\ \hline \end{array}$$

Definition 162. *Given $\alpha \in \mathcal{P}^n$, an n-**tableau** of weight α is a sequence of n-cores $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(\ell(\alpha))}$, where $(\lambda^{(i-1)}, \lambda^{(i)})$ is a horizontal strong $(n-1-\alpha_i)$-strip. A n-tableau is represented by its tableau filling where i is placed in the cells of $\lambda^{(i)}/\lambda^{(i-1)}$. If T is an n-tableau of weight α, then for each $1 \leq i \leq \ell(\alpha)$, let $T_{\leq i}$ be the tableau restricted to those cells with letters $t \leq i$. Observe that $T_{\leq i}$ is an n-tableau for each $1 \leq i \leq \ell\alpha$.*

Example 163. *If the* 5*-tableau*

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 3 \\ \cline{1-2} 2 & 2 & 3 \\ \cline{1-3} 1 & 1 & 1 & 2 & 3 & 3 \\ \hline \end{array} \qquad \Longrightarrow \qquad T_{\leq 3} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 \\ \cline{1-2} 2 & 2 & 3 \\ \cline{1-3} 1 & 1 & 1 & 2 & 3 & 3 \\ \hline \end{array}$$

In this section we will consider properties of certain types of *ABC*'s. Namely, those *ABC*'s whose shape and weight are related.

One of our goals is to characterize the unique *ABC* with $shape(A) = \mathfrak{c}(weight(A)) \in C^n$. To reach this goal we slightly modify the definition of the extension of an *ABC* A given in [DM13] (pg. 16) for ease of notation.

Definition 164. *The* **extension** $ext(A)$ *is formed from* A *by appending a ribbon of length* $\lambda_1^{(x)} - \lambda_1^{(x-1)} + 1$ *to the end of row* x *and then deleting any letter from a ribbon of length one. An* **extended ribbon** *is the appended ribbon of length* $\lambda_1^{(x)} - \lambda_1^{(x-1)} + 1$ *to the end of row* x*.* **Consecutive extended ribbons** *are two extended ribbons such that there is no extended ribbon in any row between the rows with them.*

With this definition, we show an example of an *ABC* A with $shape(A) = \mathfrak{c}(weight(A))$, and make some observations.

Example 165. *If we have the ABC of weight* $(3, 2, 2, 2, 1)$ *and shape* $\mathfrak{c}(3, 2, 2, 2, 1) = (7, 4, 3, 2, 1)$*,*

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \, & 5 & 2 \\ \cline{1-6} \, & \, & 3 & 3 & 3 & 2 \\ \cline{1-8} \, & \, & \, & 5 & 4 & 3 & 3 & 3 \\ \hline \, & \, & \, & \, & 5 & 5 & 5 & 4 \\ \hline \, & \, & \, & \, & \, & \, & \, & 5 & 5 & 5 \\ \hline \end{array} \Longrightarrow ext(A) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \, & \, & \, & 1 & 1 & 1 & 1 \\ \cline{1-9} \, & \, & 3 & 3 & 3 & \, & 2 & 2 & 2 \\ \cline{1-9} \, & \, & \, & \, & \, & 3 & 3 & 3 \\ \hline \, & \, & \, & \, & 5 & 5 & \, & \, & 4 & 4 & 4 \\ \hline \, & \, & \, & \, & \, & \, & \, & 5 & 5 & \, \\ \cline{1-10} \end{array}$$

The extended ribbons of $ext(A)$ *are the 4-ribbon filled with one's, the 3-ribbon filled with two's, and the 3-ribbon filled with four's. The extended ribbon filled with two's and the extended ribbon filled with four's are two consecutive extended ribbons.*

Remark 166. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and shape$(A) = \mathfrak{c}(\mu)$. Observe that for each $1 \leq i \leq \ell(\mu)$, ext(A) has only one ribbon filled with i and in row i. The length of this ribbon is $\mu_i + 1$. Furthermore, the ribbons from a staircase pattern. Namely, any row between two consecutive extended ribbons has a ribbon whose tail is exactly one column to the left of the tail of the ribbon in the row above.*

These observations on the *ABC*'s A with $\mathfrak{c}(weight(A)) = shape(A)$ help to shape our key Lemma's.

Lemma 167. *Let A be an ABC of weight μ and T the corresponding n-tableau. A has an i-offset, of length x, with a tail in (r, c) if and only if T has no ribbon filled with the letter i, of length $x - 1$, whose tail is in (r, c).*

Proof. Suppose T is defined by the sequence of n-cores $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(\ell(\mu))}$. A has an i-offset of length x with a tail in (r, c) if and only if $R(n-1, \lambda^{(i-1)})/\lambda^{(i)}$ has non-empty cells in row r and in columns $c, c+1, \ldots, c+x-1$. This is true if and only if $\lambda^{(i)}/\lambda^{(i-1)}$ has no cells in row r and columns $c, c+1, \ldots, c+x-1$ and the r^{th} row from the top. □

For an *ABC* with weight $\mu \in \mathcal{P}^n$ and shape $\mathfrak{c}(\mu)$, Lemma 167 reduces very nicely.

Corollary 168. *Let A is an ABC with weight $\mu \in \mathcal{P}^n$ and shape$(A) = \mathfrak{c}(\mu)$. Then A has an i-offset, of length $\mu_i + 1$, with tail in row r if and only if T has no μ_i-ribbon filled with i and in row r.*

Lemma 169. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and T the corresponding n-tableau. The ext(A) has a extended ribbon in the i^{th} row if and only if T has a ribbon of length μ_i whose cells contain i in the bottom row.*

Proof. Using Γ, let T be the n-tableau, corresponding to A, defined by the sequence of n-cores $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(\ell(\mu))}$. The *ext*$(A)$ has a extended ribbon in the i^{th} row, of column residue x, if and only if the southeastern-most cell of $R(n-1, \lambda^{(i-1)})$ has residue $(x-1)$ mod n. Using the definition of $R(n-1, \lambda^{(i-1)})$, we then have that the southeastern-most cell of $\lambda^{(i-1)}$ has residue $x-1$, which implies $\lambda^{(i)}/\lambda^{(i-1)}$ has a μ_i-ribbon in the bottom row. Reversing this argument completes the proof. □

Lemma 170. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and shape$(A) = \mathfrak{c}(\mu)$. If ext(A) has two consecutive extended ribbons in rows i and j, with $i < j$, then $\mu_i + (j-i) = n$. Furthermore, the head of the extended ribbon in the i^{th} row is in the same column as the tail of the extended ribbon in the j^{th} row.*

Proof. Using Γ, let T be the n-tableau, corresponding to A, defined by the sequence of n-cores $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(\ell(\mu))}$. By Lemma 169, *ext*$(A)$ has extended ribbons in rows i and j if and only if $\lambda^{(i)}/\lambda^{(i-1)}$ has a μ_i-ribbon in the bottom row and $\lambda^{(j)}/\lambda^{(j-1)}$ has a μ_j-ribbon also in the bottom row. Since $i < j$, then the μ_i-ribbon is to the left of the μ_j-ribbon in the bottom row of $\lambda^{(j)}/\lambda^{(i-1)}$. Furthermore, since *ext*(A) has no extended ribbons in any row between i and j, then the μ_i-ribbon is directly to the left of the μ_j-ribbon in the bottom row of $\lambda^{(j)}/\lambda^{(i-1)}$. This tells us that in $\lambda^{\ell(\mu)}$, the cell in the j^{th} row and first column has residue $x+1$ if and only if the cell in the i^{th} row, μ_i^{th} column has residue x. Finally since the length of the i^{th} row in $\lambda^{(i)}$ is μ_i, it follows that $\mu_i + (j-i) = n$.

To see the next implication, let R be the extended ribbon in the j^{th} row and Q be the extended ribbon in the i^{th} row of *ext*(A). Let S be the ribbon in the $(j-1)^{th}$ row that is not an offset in *ext*(A). Since R is a extended ribbon in a row directly below to the one S is in, then the tail of R is n columns to the right of the tail of S in *ext*(A). On the other hand, the head of Q is $\mu_i + (j-i)$ columns to the right of the tail of S. The implication follows since $n = \mu_i + (j-i)$. □

Definition 171. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and shape$(A) = \mathfrak{c}(\mu)$. For each $1 \leq i \leq \ell(\mu)$, $\mathcal{R}_i$ is the set of left-most μ_i cells of every extended ribbon in ext$(A)_{\leq i}$, except for their tails.*

Note that every *ABC* has at least one extended ribbon; the one in the top row. This tells us that $\mathcal{R}_i \neq \emptyset$.

Example 172. *For the ABC A in Example 165, since*

$$ext(A) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & \ & 3 & 3 & 3 & \ & 2 & 2 & 2 \\ \hline \ & \ & \ & \ & \ & 3 & 3 & 3 \\ \hline \ & \ & \ & \ & 5 & 5 & \ & \ & 4 & 4 & 4 \\ \hline \ & \ & \ & \ & \ & \ & \ & 5 & 5 & \ \\ \hline \end{array} \implies ext(A)_{\leq 3} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & 1 & \boxed{1} & \boxed{1} & 1 \\ \hline \ & \ & 3 & 3 & 3 & \ & 2 & \boxed{2} & \boxed{2} \\ \hline \ & \ & \ & \ & \ & 3 & 3 & 3 \\ \hline \end{array}$$

Since the weight μ of A is such that $\mu_3 = 2$, then $\mathcal{R}_3$ is the set of bold cells in $ext(A)_{\leq 3}$. Namely, This tells us that

$$\mathcal{R}_3 = \{(2,8),(2,9),(3,5),(3,6)\}\,.$$

Definition 173. *For any cell of an ABC, we define the diagonal through that cell to be the line going through its bottom left corner and the top right corner.*

One of our objectives is to show that for a given *ABC A*, where $\mathfrak{c}(weight(A)) = shape(A)$, the ribbon in the bottom row of $ext(A)_{\leq i}$, filled with i has a diagonal through its tail which also goes through the tail of a unique extended ribbon. The example below illustrates this for an *ABC A* with $\mathfrak{c}(weight(A)) = shape(A)$.

Example 174. *For the ABC A in Example 165, we know*

$$ext(A)_{\leq 3} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & \ & 3 & 3 & 3 & \ & \boxed{2} & 2 & 2 \\ \hline \ & \ & \ & \ & \ & \boxed{3} & 3 & 3 \\ \hline \end{array}$$

Observe that the bottom row of $ext(A_{\leq 3})$ has a ribbon of filled with 3. Furthermore, the diagonal through the tail of this ribbon goes through the tail of the extended ribbon filled with 2 as indicated by the bold cells.

Proposition 175. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$. Let I be the ribbon filled with i and in the bottom row of $ext(A)_{\leq i}$. The diagonal through the tail of I goes through the tail of some extended ribbon R in $ext(A)_{\leq i}$. Furthermore, there is no extended ribbon in any row below the row with R.*

Proof. If I is a extended ribbon of $ext(A)_{\leq i}$, then we are done. If I is not a extended ribbon, then let t be the cell of $ext(A)_{\leq i}$, located in some row j and t is on the same diagonal as the one through the tail of I. Furthermore, suppose that there is no row $r < j$ which has a cell on the same diagonal as the one through t. Since $shape(A) = \mathfrak{c}(weight(A))$, and t is on the same diagonal as the one through the tail of I, then the ribbon filled with the letter j in the j^{th} row of $ext(A)_{\leq i}$, must have a tail of column residue the same as that of t. Remark 166 tells us that t must be the tail of the unique ribbon filled with the letter j in the j^{th} row of $ext(A)_{\leq i}$. Let R be this unique ribbon filled with j in the j^{th} row of $ext(A)_{\leq i}$.

Since there is no cell in $ext(A)_{\leq i}$ in any rows above j that is on the diagonal throught t, then the ribbon, S, in the $(j-1)^{th}$ row filled with $j-1$ must have a tail in some column that is west of the column with t. Since the $shape(A) = \mathfrak{c}(weight(A))$, then the column residue of the tail of S must be $r+1$, where r is the column residue of t. So if the tail of S is in a column west of the column with t, then the tail of S must be exactly $n-1$ columns west of the column t is in. This implies that R must be a extended ribbon as it is not in $A_{\leq i}$.

Finally observe that if there is a row $r > j$ which has a extended ribbon, then since the $shape(A) = \mathfrak{c}(weight(A))$, we have that the bottom row of $ext(A)_{\leq i}$ must have at least two ribbons filled with i. This contradicts the facts of Remark 166. □

Suppose A is an ABC with weight $\mu \in \mathcal{P}^n$ and $shape(A) = \mathfrak{c}(\mu)$. If $ext(A)_{\leq i}$ has any ribbon I filled with the letter i and in the bottom row, then Proposition 175 tells us that there are μ_i cells in the bottom row of $ext(A)_{\leq i}$ such that the diagonal through them goes through certain cells of the lowest extended ribbon. More precisely, if I is the ribbon filled with i, located in the bottom row of $ext(A)_{\leq i}$, then define $\mathcal{B}_i$ to be all the cells of I excluding its tail. This definition gives us the following Corollary to Proposition 175.

Corollary 176. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$. Fix an i such that $1 \leq i \leq \ell(\mu)$. The diagonal through each $(r, c) \in \mathcal{B}_i$ goes through a unique cell of $\mathcal{R}_i$.*

We now show that for a given *ABC* A, such that $\mathfrak{c}(weight(A)) = shape(A)$, the lowest extended ribbon of $ext(A)_{\leq i}$ has a diagonal through its tail which also goes through the tail of the ribbon in the bottom row filled i.

Example 177. *For the ABC A in Example 165 with weight* $\mu = (3, 2, 2, 2, 1)$*, we know by Example 174 that*

$ext(A)_{\leq 3} =$

			1	1	1	1		
		3	3	3		2	**2**	**2**
					3	**3**	**3**	

and

$ext(A)_{\leq 5} =$

			1	1	1	1				
		3	3	3		2	2	2		
					3	3	3			
				5	5			4	**4**	4
							5	**5**		

For $i = 3$*,* $\mathcal{B}_3 = \{(1, 7), (1, 8)\}$ *and Example 172 gives us* $\mathcal{R}_3 = \{(2, 8), (2, 9), (3, 5), (3, 6)\}$*. Observe that the diagonal through the cell* $(1, 7) \in \mathcal{B}_3$ *goes through the cell* $(2, 8) \in \mathcal{R}_3$ *and the diagonal through the cell* $(1, 8) \in \mathcal{B}_3$ *goes through the cell* $(2, 9) \in \mathcal{R}_3$*, as indicated by the bold cells in* $ext(A)_{\leq 3}$*.*

For $i = 5$*,* $\mathcal{B}_5 = \{(1, 9)\}$ *and* $\mathcal{R}_5 = \{(2, 10), (4, 8), (5, 5)\}$*. Observe that the diagonal through the cell* $(1, 9) \in \mathcal{B}_5$ *goes through the cell* $(2, 10) \in \mathcal{R}_5$*, as indicated by the bold cells in* $ext(A)_{\leq 5}$*.*

Proposition 178. *Let A be an ABC of weight* $\mu \in \mathcal{P}^n$*, and* $shape(A) = \mathfrak{c}(\mu)$*. Let I be the ribbon in the* i^{th} *row of* $ext(A)_{\leq i}$ *and filled with the letter i. Let R is a extended ribbon such that there is no extended ribbon in any row below the row with R. Then the diagonal through the tail of R goes through the tail of I.*

Proof. If the column residue of the tail of R is x and if the $shape(A) = \mathfrak{c}(weight(A))$, then the column residue of the tail of I is $(x - i + b) \bmod n$, where b is the row that R is in. Since R is the lowest extended ribbon in $ext(A)_{\leq i}$, then the claim follows by considering the column residues of the tail of non-offset ribbons in rows $b, b + 1, \ldots, i$. □

Our next objective is to show that for a given *ABC* A, such that $\mathfrak{c}(weight(A)) = shape(A)$, the diagonal through the tail of an *i-offset* goes through the tail of a unique extended ribbon.

Definition 179. *Given an ABC A, for any i in* $1 \leq i \leq weight(A)$*, an* ***i*-offset** *of* $ext(A)$ *is an offset filled with i.*

The example below illustrates this for an *ABC A* with $\mathfrak{c}(weight(A)) = shape(A)$.

Example 180. *For the ABC A in Example 165, we know that*

$ext(A)_{\leq 5} =$

			1	1	1	1				
		3	3	3		**2**	2	2		
					3	3	3			
				5	5			4	4	4
							5	5		

Observe that the diagonal through the tail of the 5-offset goes through the tail of the extended ribbon filled with 2, as indicated by the bold cells in $ext(A)_{\leq 5}$*.*

Proposition 181. *Let A be an ABC of weight* $\mu \in \mathcal{P}^n$*, and* $shape(A) = \mathfrak{c}(\mu)$*. Fix i such that* $1 \leq i \leq \ell(\mu)$*. If* $ext(A)_{\leq i}$ *has an i-offset, then the diagonal through the tail of it goes through the tail of a unique extended ribbon.*

Proof. The proof will be given by induction on the i-offsets of $ext(A)_{\leq i}$. For the base case, consider the lowest i-offset, say S, in $ext(A)_{\leq i}$. From the definition of A, there is a ribbon F filled with i in the bottom row of $ext(A)_{\leq i}$. The tail of S is q columns west of the column with the tail of F and S is $n - q$ rows above F.

Let R be a extended ribbon in some row b such that there are no extended ribbons in any row $r > b$ of $ext(A)_{\leq i}$. Note that such a extended ribbon must exist in $ext(A)_{\leq i}$ since the top row has a extended ribbon. Propositions 175 and 178 give us that F has a tail which is d columns west of the column with the tail of R and F is d rows below the row with R.

We now show that there is a extended ribbon in some row above the row with R. By way of contradiction, suppose R is in the top row of $ext(A)_{\leq i}$. This implies that its tail is in column n, so the number of columns between the tail of S and the tail of R must be less than n. Since the tail of S is $q + d - 1$ columns west of the column with the tail of R, then $d \leq n - q + 1$. Recall that F is

d rows below R and that S is $n - q$ rows above F, so $d = n - q + 1$. Now S is in the top row of $ext(A)_{\leq i}$, which contradicts Corollary 168. Thus there is a extended ribbon in some row above the row with R. Let Q and R be consecutive ribbons, where Q is in some row $a < b$. Lemma 170 tells us that the tail of R is in the same column as the head of Q and that $\mu_a + (b - a) = n$.

We now show that the tail of S is in a column that is west of the column with the tail of Q. By way of contradiction, suppose that the tail of S is in a column that is weakly east of the column with the tail of Q. Since the column with the tail of S is $q + d$ west of the tail of R and the column with the tail of Q is μ_a columns west of the tail of R, then $q + d \leq \mu_a$. Since $\mu_a + (b - a) = n$, then $d + b - a \leq n - q$. Since F is $d + b - a$ rows south of the row with Q and F is also $n - q$ rows south of row with S, then we get that the row with S is north of the row with Q. Thus the tail of S is north-west of the tail of Q in $ext(A)_{\leq i}$. This is a contradiction as S is an i-offset implying that any cell below it in $ext(A)_{\leq i}$ is empty. A similar argument shows that the tail of S is in a row that is below the row that the tail of Q is in. Hence the tail of Q is north-east of the tail of S in $ext(A)_{\leq i}$.

Figure 10.1 shows us the relative placement of S, F, R, Q and the distances between them. If Q is x rows north of the row with S, then it is a check to see that the tail of Q is x columns east of the column with the tail of S.

For the induction step, we have almost the same setup as Figure 10.1, except now F is an i-offset. The argument then follows similar to the base case. □

Suppose A is an ABC with weight $\mu \in \mathcal{P}^n$ and $shape(A) = \mathfrak{c}(\mu)$. If $ext(A)_{\leq i}$ has any i-offsets for each $1 \leq i \leq \ell(\mu)$, then Proposition 181 tells us that there are μ_i cells of each i-offset such that the diagonal through each one goes through certain cells of a extended ribbon. More precisely, let O_i to be the set of all cells in $ext(A)_{\leq i}$ which belong to any i-offset, excluding its tail. Note that if $ext(A)_{\leq i}$ has no i-offsets, then $O_i = \emptyset$. However, when $ext(A)_{\leq i}$ has at least one i-offset, then we have the following Corollary to Proposition 181.

Corollary 182. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$. Fix an i such that*

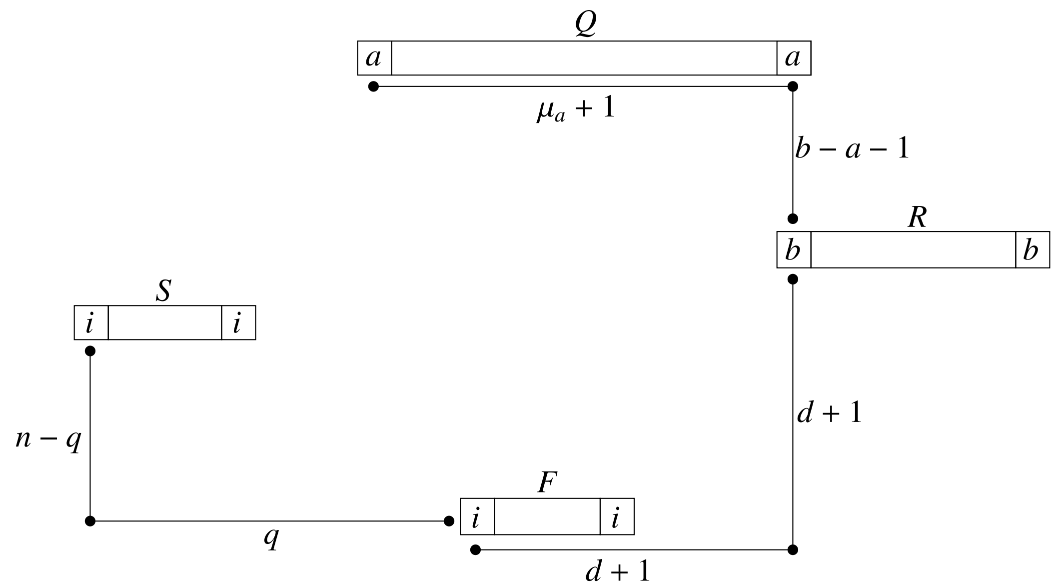


Figure 10.1: Ribbons S, F, R, Q of $ext(A)_{\leq i}$

$1 \leq i \leq \ell(\mu)$. *Suppose that* $ext(A_{\leq i})$ *has at least one i-offset in it. The diagonal through any cell* $(r, c) \in O_i$ *goes through a unique cell of* $\mathcal{R}_i$.

Example 183. *For the ABC A in Example 165, we know by Example 180 that*

$ext(A)_{\leq 5} =$

			1	1	1	1				
		3	3	3		2	**2**	2		
					3	3	3			
				5	**5**			4	4	4
							5	5		

Thus for $i = 5$, $O_5 = \{(2, 6)\}$ *and* $\mathcal{R}_5 = \{(2, 10), (4, 8), (5, 5)\}$. *Observe that the diagonal through the cell* $(2, 6) \in O_5$ *goes through the cell* $(4, 8) \in \mathcal{R}_5$, *as indicated by the bold cells.*

Next we consider certain types of ribbons in an n-tableau and show how they are connected to cells of extended ribbons in an ABC. To see this connection, we first need a definition.

Definition 184. *For a given* $\mu \in \mathcal{P}^n$, *Let* T *be the* n*-tableau defined by the sequence* $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(\ell(\mu))} = \mathfrak{c}(\mu)$. *For each* $1 \leq i \leq \ell(\mu)$, *we say that a* μ_i*-ribbon of* T *is an* n*-connected*

μ_i-ribbon of i if and only if $\lambda^{(i)}/\lambda^{(i-1)}$ has that μ_i-ribbon and it is not the top most μ_i-ribbon.

When T is an n-tableau whose corresponding ABC A has weight $\mu \in \mathcal{P}^n$ and shape $\mathfrak{c}(\mu)$, then $T_{\leq i}$ has a μ_i-ribbon in its top row whose tail is in the first column.

Example 185. *For the ABC A in Example 165, since the corresponding* 4*-tableau is*

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 5 \\ \cline{1-2} 4 & 4 \\ \cline{1-3} 3 & 3 & 5 \\ \cline{1-4} 2 & 2 & 4 & 4 \\ \hline 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ \hline \end{array} \qquad \Longrightarrow \qquad T_{\leq 4} = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-2} 4 & 4 \\ \cline{1-2} 3 & 3 \\ \cline{1-4} 2 & 2 & 4 & 4 \\ \hline 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ \hline \end{array}$$

Observe that $T_{\leq 4}$ has a 4*-connected* 2*-ribbon of $i = 4$ in rows one and two.*

We are almost ready to show the correspondence between certain ribbons in n-tableau and extended ribbons of ABC's. We first show this correspondence through an example.

Example 186. *The ABC A in Example 165 has the corresponding 4-tableau T of Example 185. Thus*

$$T_{\leq 4} = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-2} 4 & 4 \\ \cline{1-2} 3 & 3 \\ \cline{1-4} 2 & 2 & \boxed{4} & 4 \\ \hline 1 & 1 & 1 & 2 & 2 & \boxed{4} & 4 \\ \hline \end{array} \qquad \Longleftrightarrow \qquad ext(A)_{\leq 4} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \cline{1-7} & & & \boxed{1} & 1 & 1 & 1 \\ \cline{1-9} & & \boxed{3} & 3 & 3 & & \boxed{2} & 2 & 2 \\ \cline{1-9} & & & & & \boxed{3} & 3 & 3 & \\ \hline & & & & & & & & 4 & 4 & 4 \\ \hline \end{array}$$

Example 185 tells us that in $T_{\leq 4}$, the tails of the 4*-connected* 2*-ribbons filled with* 4 *are in cells* $(2, 3)$ *and* $(1, 6)$*. Observe that each of the diagonals through the cells* $(2 + 1, 3) = (3, 3)$ *and* $(1 + 1, 6) = (2, 6)$ *of $ext(A)_{\leq 4}$ go through the tails of the extended ribbon in rows four and two, respectively.*

Proposition 187. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$. Let T be the corresponding n-tableau of A. Fix i such that $1 \leq i \leq \ell(\mu)$. Suppose that $T_{\leq i}$ has at least one n-connected μ_i-ribbon of i with tail in some position (r, c). Then the diagonal through the cell in position $(r+1, c)$ of $ext(A)_{\leq i}$ goes through the tail of a unique extended ribbon.*

Proof. The proof is by induction on the n-connected μ_i-ribbons. For the base case, consider the top-most n-connected μ_i-ribbon of i, say S, of $T_{\leq i}$. Suppose that S has a tail in position (r, c). From the definition of A, we see that $ext(A)_{\leq i}$ has a cell e in position $(r+1, c)$. We show that the diagonal through the cell e also goes through the tail of the unique extended ribbon R in the top row of $ext(A)_{\leq i}$. Since $shape(A) = \mathfrak{c}(weight(A))$, then $shape(A_{\leq i}) = \mathfrak{c}(weight(A_{\leq i}))$, which says that the top row of $T_{\leq i}$ has μ_i cells filled with the letter i. If S is the top-most n-connected μ_i-ribbon filled with the letter i, then $r = n - c$. This tells us that e is $n - c - 1$ rows bellow the row R is in. On the other hand, since the tail of R is in the n^{th} column of $ext(A)_{\leq i}$, then the number of columns between e and the tail of R is $n - 1 - c$. Thus the diagonal through e goes through the tail of R.

Next suppose that the $(p-1)^{th}$ n-connected μ_i-ribbon of i in $T_{\leq i}$ has a tail in position (r, c), for some $p > 1$. Let e be the cell in position $(r+1, c)$ of $ext(A)_{\leq i}$. Furthermore suppose that there is a diagonal through e which also goes through the tail of a unique extended ribbon R. This tells us that e is d columns west and d rows south of the tail of R.

If the p^{th} n-connected μ_i-ribbon in $T_{\leq i}$ has a tail in column $c+q$ then that tail is in row $r+n-q$. Thus, there is a cell f in $ext(A)_{\leq i}$ which is q columns east and $n-q$ rows south of e.

Next, suppose there is no extended ribbon in any rows below the row with R. Lemma 169 tells us then that the p^{th} n-connected μ_i-ribbon in $T_{\leq i}$ can't be in the bottom row. So there is a ribbon in the bottom row of $A_{\leq i}$ filled with the letter i, such that the diagonal through the tail, t, goes through the tail of R. This implies that the cell t is the cell e, giving us a contradiction as t is in the lowest row of $A_{\leq i}$ and f is in some row below it. Thus there is a extended ribbon, Q, in some row below the row with R. Let Q and R be consecutive ribbons, where Q is in some row a and R is in some row $b < a$. Lemma 170 tells us that the head of R is in the same column as the tail of Q and that $\mu_b + (a-b) = n$.

Now if the tail of Q is in a row weakly south of the row with f, then we have that $n+d \leq a-b+q$. Since $\alpha_b + (a-b) = n$, then we get that $\alpha_b + d \leq q$. This says that the tail of Q is in a column weakly west of the column with f. Thus the tail of Q is weakly south-west of f. This is a contradiction as there are only empty cells weakly south-west of f in $ext(A)_{\leq i}$. A similar argument shows us that the tail of Q is in a column east of the column with f. Hence the tail of Q is north-east from the cell f.

Figure 10.2 shows us the relative placements of R, Q, e, f and the distances between them. If f is x rows south of the row with Q, then it is a check to see that f is x columns west of the column with the tail of Q. Finally, Lemma 170 gives us the uniqueness of Q. □

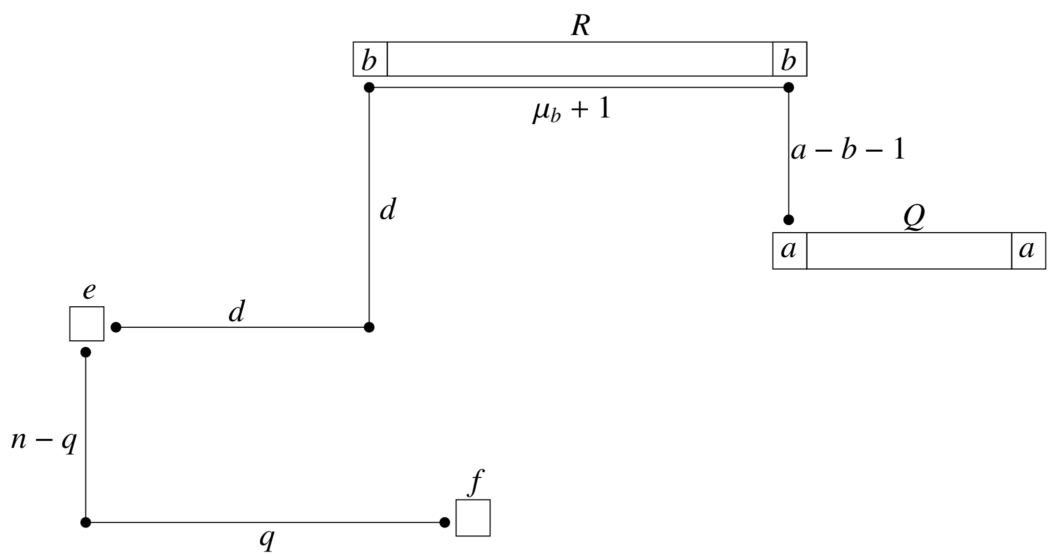


Figure 10.2: Cells e, f and ribbons R, Q in $ext(A)_{\leq i}$

Suppose A is an ABC with weight $\mu \in \mathcal{P}^n$ and $\mathfrak{c}(\mu) = shape(A)$. Let T be the n-tableau corresponding to A. Fix i in $1 \leq i \leq \ell(\mu)$. If $T_{\leq i}$ has any n-connected μ_i-ribbbons of i, then Proposition 187 tells us that the diagonal through each cell of this ribbon goes through certain cells of a unique extended ribbon. More precisely, let $\mathcal{I}_i$ be the set of all cells in $T_{\leq i}$ which belong to any n-connected μ_i-ribbon of i except for its tail. Note that if $T_{\leq i}$ has no n-connected μ_i-ribbon of i, then $\mathcal{I}_i = \emptyset$. However, when $T_{\leq i}$ has at least one n-connected μ_i-ribbon of i, then we have the following Corollary to Proposition 187.

Corollary 188. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$. Let T be the corre-*

sponding n-tableau. Fix an i such that $1 \leq i \leq \ell(\mu)$*. Suppose that* $T_{\leq i}$ *has at least one n-connected* μ_i*-ribbon of i. If* $(r, c) \in \mathcal{I}_i$*, then the diagonal through the cell* $(r+1, c+1)$ *of* $A_{\leq i}$ *goes through a unique cell of* $\mathcal{R}_i$*.*

Example 189. *The ABC A in Example 165 has the corresponding 4-tableau T of Example 185. Thus*

$$T_{\leq 5} = \begin{array}{|c|c|c|c|c|c|c|}\hline 5 \\ \hline 4 & 4 \\ \hline 3 & 3 & \boxed{5} \\ \hline 2 & 2 & 4 & 4 \\ \hline 1 & 1 & 1 & 2 & 2 & 4 & 4 \\ \hline \end{array} \qquad \Longleftrightarrow \qquad ext(A)_{\leq 5} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & 1 & \boxed{1} & 1 & 1 \\ \hline & & 3 & \boxed{3} & 3 & & 2 & 2 & 2 \\ \hline & & & & & 3 & 3 & 3 \\ \hline & & & & 5 & 5 & & & 4 & 4 & 4 \\ \hline & & & & & & & 5 & 5 & \\ \hline \end{array}$$

For $i = 5$*,* $\mathcal{I}_5 = \{(3, 3)\}$ *and* $\mathcal{R}_5 = \{(2, 10), (4, 8), (5, 5)\}$*. Observe that the diagonal through the cell* $(3+1, 3+1) = (4, 4)$ *goes through* $(5, 5) \in \mathcal{R}_5$*, as indicated by the bold cells.*

The sets $\mathcal{I}_i, \mathcal{O}_i, \mathcal{B}_i$ from Corollaries 176, 182, 188 have an interesting property about their size.

Example 190. *For the ABC A in Example 165 and its corresponding* 4*-tableau from Example 185, we know by Examples 177, 183, and 189 that* $\mathcal{B}_5 = \{(1, 9)\}, \mathcal{O}_5 = \{(2, 6)\}$ *and* $\mathcal{I}_5 = \{(3, 3)\}$*. Since*

$$\mathcal{B}_5 \cap \mathcal{O}_5 = \mathcal{B}_5 \cap \mathcal{I}_5 = \mathcal{O}_5 \cap \mathcal{I}_5 = \emptyset \quad \Longrightarrow \quad |\mathcal{B}_5 \cup \mathcal{O}_5 \cup \mathcal{I}_5| = |\mathcal{B}_5| + |\mathcal{O}_5| + |\mathcal{I}_5| = 3$$

Proposition 191. *Let A be an ABC of weight* $\mu \in \mathcal{P}^n$*, and shape*$(A) = \mathfrak{c}(\mu)$*. For each i with* $1 \leq i \leq \ell(\mu)$*,*

$$|\mathcal{I}_i \cup \mathcal{O}_i \cup \mathcal{B}_i| = |\mathcal{I}_i| + |\mathcal{O}_i| + |\mathcal{B}_i|.$$

Proof. Suppose the corresponding n-tableau of A is defined by the sequence $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(\ell(\mu))}$. If $\mathcal{I}_i$. Observe that if $\lambda^{(i)}/\lambda^{(i-1)}$ has any non-empty cells in the bottom row, then it has exactly μ_i non-empty cells in the bottom row. Lemma 169 then tells us that $ext(A)_{\leq i}$ has a extended ribbon in its bottom row. Thus, if $(i, c) \in \mathcal{I}_i$ then $(i, c) \notin \mathcal{B}_i$. Hence, $\mathcal{I}_i \cap \mathcal{B}_i = \emptyset$. Lemma 167 tells us that $\mathcal{I}_i \cap \mathcal{O}_i = \emptyset$. The definition of an offset gives us that $\mathcal{O}_i \cap \mathcal{B}_i = \emptyset$, and the claim follows. □

We now show how the tails of each extended ribbon are mapped to either tails of an i-offset or tails of an n-connected μ_i-ribbon.

Proposition 192. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and shape$(A) = \mathfrak{c}(\mu)$. Let T be the corresponding n-tableau of A. Fix i such that $1 \leq i \leq \ell(\mu)$. Suppose ext$(A)_{\leq i}$ has a extended ribbon R such that there is a extended ribbon in some row below the row with R. Then the diagonal through the tail of R goes through a cell t, in some position (r, c), such that either t is the tail of an i-offset, or the cell in position $(r-1, c)$ of $T_{\leq i}$ is the tail of an n-connected μ_i-ribbon, but not both.*

Proof. We proceed by way of induction on the extended ribbons. For the base case, let R be the extended ribbon in the top row of $ext(A)_{\leq i}$. Let t be the cell in position (r, c) such that t is on the diagonal going through the tail of R and there is no cell in row $r-1$ which is also on the diagonal. If t is on the diagonal, and t is in position (r, c), then t is r rows south and r columns west of from the tail of R. Furthermore, since the tail of R is in position $(1, n)$, then we must have that $r + c = n + 1$.

If there is an extended ribbon in some row below the row with R, then Proposition 178 tells us that t is not in the bottom row of $ext(A)_{\leq i}$. Since the tail of R is in position $(1, n)$, then we see that $r + c = n + 1$.

Suppose that t is not the tail of an i-offset. We show that the cell $(r-1, c)$ of $T_{\leq i}$ is the tail of a n-connected μ_i-ribbon. Suppose that T is defined by the sequence $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(i)} \subset \cdots \subset \lambda^{(\ell(\mu))}$. Since there is no cell in any row below r which is also on the diagonal through t, then the cell $\tilde{t}$ in $(r-1, c)$ of $T_{\leq i}$ is in $\lambda^{(i)}/\lambda^{(i-1)}$. Furthermore, if $r + c = n + 1$, then $\tilde{t}$ has the same residue, say x, as that of the cell in $(1, \ell(\lambda^{(i)}))$ of T. Since $shape(A) = \mathfrak{c}(\mu)$, then the top row of $T_{\leq i}$ has μ_i cells of residue $x, x + 1, \ldots, x + \mu_i - 1$, each taken $\mod n$. This implies that row $r - 1$ of $\lambda^{(i)}/\lambda^{(i-1)}$ has μ_i cells of residue $x, x + 1, \ldots, x + \mu_i - 1$, each taken $\mod n$. Furthermore, the cell $\tilde{t}$ is the cell of residue x.

If the cell directly west of $\tilde{t}$ is in $\lambda^{(i)}/\lambda^{(i-1)}$, then there would be an i-offset in $ext(A)_{\leq i}$ with tail in $(r-1, c-1)$. This would contradict the assumption that there is no cell in any row below r which is also on the diagonal through t. Hence there is a n-connected μ_i-ribbon in $T_{\leq i}$ with tail in $(r + 1, c)$.

For the induction step, suppose there is a n-connected μ_i-ribbon of $T_{\leq i}$ with tail in $(\tilde{r}, \tilde{c})$ where the cell $\tilde{t}$ in position $(\tilde{r}+1, \tilde{c})$ is on the same diagonal as the one through the tail of a unique extended ribbon R of $ext(A)_{\leq i}$. Let Q be a consecutive extended ribbon in some row below the row with R. Suppose R is in some row a and Q is in some row $b > a$. Lemma 170 tells us that the tail of R is in the same column as the head of Q and that $\mu_a + (b - a) = n$.

Let t be the cell in position (r, c) of $ext(A)_{\leq i}$, which is on the same diagonal as the tail of Q. Furthermore suppose there is no cell in any row below r which is also on the diagonal through t. Figure 10.3 shows us the relative placement of $\tilde{t}, t, R, Q$ and the distances between them. Using the fact that $\mu_a + (b - a) = n$, it is a check to see that if t is q columns east of $\tilde{t}$, then t is $n - q$ rows south of $\tilde{t}$. Thus, if we assume that t is not the tail of an i-offset, then the cell at $(r - 1, c)$ of $T_{\leq i}$ is the tail of an n-connected μ_i-ribbon. A similar argument as the induction step shows that if $\tilde{t}$ is the tail of an i-offset in Figure 10.3, then t is also the tail of an i-offset. Finally Lemma 167 tells us that either t is the tail of an i-offset or the cell $(r - 1, c)$ is the tail of an n-connected μ_i-ribbon, but not both.

□

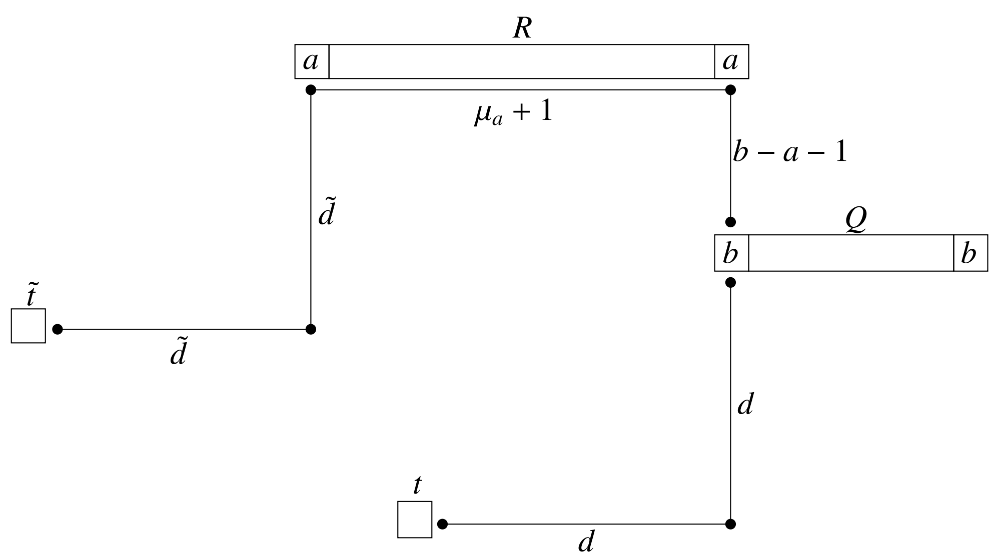


Figure 10.3: Cells e, f and ribbons R, Q in $ext(A)_{\leq i}$

Example 193. *The ABC A in Example 165 which has the corresponding 4-tableau T from Example 185 gives us*

$ext(A)_{\le 5} =$

			1	1	1	1				
		3	3	3		**2**	2	2		
					3	3	3			
				5	5			**4**	4	4
							5	5		

$\iff$ $T_{\le 5} =$

5						
4	4					
3	3	**5**				
2	2	4	4			
1	1	1	2	2	4	4

We observe that the diagonal through the tail at $(2, 9)$ *goes through the tail at* $(1, 8)$ *of a ribbon filled with five. The diagonal through the tail at* $(4, 7)$ *goes through the tail at* $(2, 5)$ *of a* 5*-offset. The diagonal through the tail at* $(5, 4)$ *goes through the cell at* $(4, 3)$*, and the cell* $(4 - 1, 3) = (3, 3)$ *of* $T_{\le 5}$ *s the tail of a* 4*-connected* 1*-ribbon with* $i = 5$.

We are now ready to state our main theorem of this section which relates the set $\mathcal{R}_i$ and $\mathcal{I}_i \cup \mathcal{O}_i \cup \mathcal{B}_i$.

Theorem 194. *Let A be an ABC of weight* $\mu \in \mathcal{P}^n$*, and* $shape(A) = \mathfrak{c}(\mu)$*. Let T be the corresponding n-tableau of A. Fix i such that* $1 \le i \le \ell(\mu)$*. There is a bijection between the set* $\mathcal{R}_i$ *and the set* $\mathcal{I}_i \cup \mathcal{O}_i \cup \mathcal{B}_i$*.*

Proof. Corollaries 176, 182, 188 tells us the diagonal through each cell $(r, c) \in \mathcal{I}_i \cup \mathcal{O}_i \cup \mathcal{B}_i$ also goes through a unique element $(x, y) \in \mathcal{R}_i$. Furthermore, Propositions 178 and 192 tells us that the diagonal through $(x, y) \in \mathcal{R}_i$ goes through a unique cell $(a, b) \in \mathcal{I}_i \cup \mathcal{O}_i \cup \mathcal{B}_i$. □

Example 195. *The ABC A in Example 165 and its corresponding 4-tableau T from Example 185 gives us*

$ext(A)_{\le 5} =$

			1	**1**	1	1				
		3	3	3		2	**2**	2		
					3	3	3			
				5	**5**			4	**4**	4
							5	**5**		

$\iff$ $T_{\le 5} =$

5						
4	4					
3	3	**5**				
2	2	4	4			
1	1	1	2	2	4	4

We know by Examples 177 and 190 that

$$\mathcal{B}_5 \cup \mathcal{O}_5 \cup \mathcal{I}_5 = \{(3,3),(2,6),(1,9)\} \quad \textit{and} \quad \mathcal{R}_5 = \{(2,10),(4,8),(5,5)\}\,.$$

Theorem 194 tells us that $(3,3)$ *corresponds with* $(5,5)$*,* $(2,6)$ *corresponds with* $(4,8)$ *and* $(1,9)$ *corresponds with* $(2,10)$*.*

The n-cocharge of an *ABC* A was defined in [DM13]. The n-cocharge of an *ABC* was used to introduce a new family of symmetric functions, $\mathfrak{S}^{(n)}_{\mathfrak{c}(\lambda)}(x;t)$, which play a role in affine Schubert calculus and the theory of Macdonald polynomials. When the functions $\mathfrak{S}^{(n)}_{\mathfrak{c}(\lambda)}(x;t)$ are expanded in terms of a deformation of Macdonald's basis of P-functions (See [Mac95]), the coefficients are the n-cocharge generating functions of *ABC*'s (or *weak Kostka-Foulkes polynomials*),

$$K^n_{\lambda\mu}(t) = \sum_{A \in ABC(\mathfrak{c}(\lambda),\mu)} t^{n\text{-}cocharge(A)}\,.$$

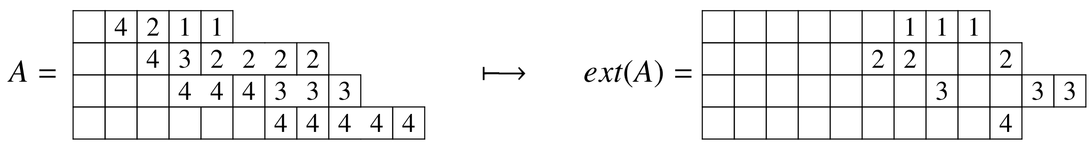
In [DM13], the weak Kostka-Foulkes polynomials are shown to generalize the Kostka-Foulkes polynomials. These Kostka-Foulkes polynomials, $K_{\lambda\mu}(t)$, were beautifully characterized by Lascoux and Schützenberger [LS78] by computing an index vector on certain sub-words of a semi-standard tableau of shape λ and weight μ. The weak Kostka-Foulkes polynomials were computed using a similar index vector on sub-words of an *ABC* along with another component involving offsets of that *ABC*.

The n-charge statistic of an *ABC* depends on computing an index vector in a similar spirit to the n-cocharge statistic. To see the definition of n-charge of A, we first define some of its components. Recall from [DM13] that the number of cells that are not tails in all the offsets is

$$off(A) = \sum_{\text{R: offset in } A} (\text{size}(R) - 1)\,.$$

We recall the definition for the extension of an *ABC* A from [DM13]. The extension of A is formed by appending a ribbon of length $\lambda_1^{(x)} - \lambda_1^{(x-1)} + 1$ to the end of row x, and then deleting any letter larger than x in row x and the tail of every ribbon containing x.

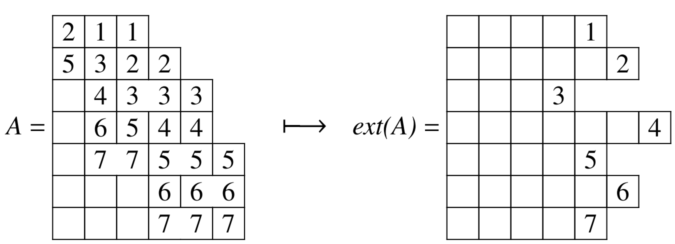
Example 196. *For* $n = 6$ *and an ABC A of weight* $(3, 3, 3, 1)$*, we have*



Our construction of the index considers only the cells in $ext(A)$. For A of weight 1^m, $ext(A)$ is standard; there is exactly one cell in each row i (coming from the single ribbon head with an i in row i of A). In this case, the n-charge is defined by computing an index vector $\tilde{I} = [0, \tilde{I}_2, \ldots, \tilde{I}_m]$ defined by

$$\tilde{I}_{r+1} = \begin{cases} \tilde{I}_r & \text{when } r+1 \text{ is west of } r \\ \tilde{I}_r + 1 & \text{when } r+1 \text{ is east of } r\,. \end{cases}$$

Example 197. *From the ABC of weight* (1^7)*:*

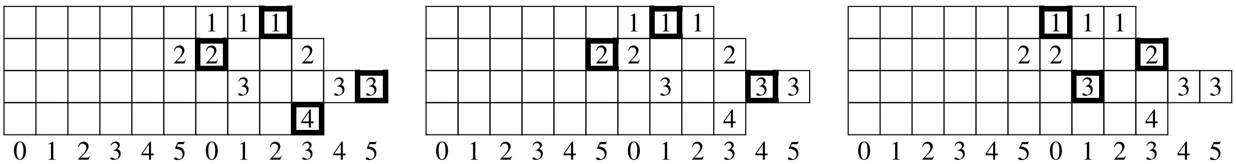


$$\longmapsto \quad \tilde{I} = [0, 1, 1, 2, 2, 3, 3]$$

Equipped with a method to obtain the index when $ext(A)$ has a single i in row i, we describe a method for extracting standard fillings from an *ABC* of arbitrary weight $\mu \in \mathcal{P}^n$.

Algorithm 198. *Given an ABC A of weight* $\mu \in \mathcal{P}^n$*, consider its labelling by column residues. Iteratively earmark a standard sequence starting with the rightmost 1. From an x (of column residue i) the appropriate choice of* $x + 1$ *will be determined by choosing its column residue from the set* $\mathcal{B}$ *of all column residues labelling the* $x + 1$*'s. Reading counter-clockwise from i, this choice* *is the closest* $j \in \mathcal{B}$ *on a circle labelled clockwise with* $0, 1, \ldots, n-1$.

Example 199. *For the ABC A from Example 196, we have the following standard sequences and their respective index vectors.*



$\tilde{I} = [0, 0, 1, 1]$ $\qquad$ $\tilde{I} = [0, 0, 1]$ $\qquad$ $\tilde{I} = [0, 1, 1]$

Definition 200. *For an ABC A of weight* $\mu \in \mathcal{P}^n$*, the n-charge of A is defined by*

$$n\text{-}charge(A) = \sum_r \tilde{I}_r(A) - off(A) - \beta(A)\,,$$

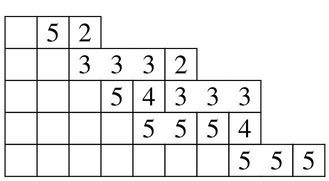
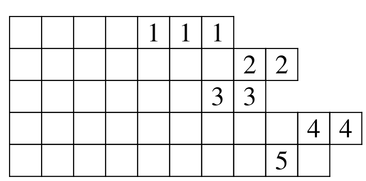
where $\beta(A)$ *is the number of cells in the shape*(A) *whose hook-length exceeds n.*

Example 201. *For the ABC A from Example 196, we have that* $off(A) = 1$*. Since the shape*$(A) = (6, 3, 2, 1) \in C^6$*, then* $\beta(A) = 2$*. Example 199 tells us that the sum of all the charge indices is 5. Thus we have that the*

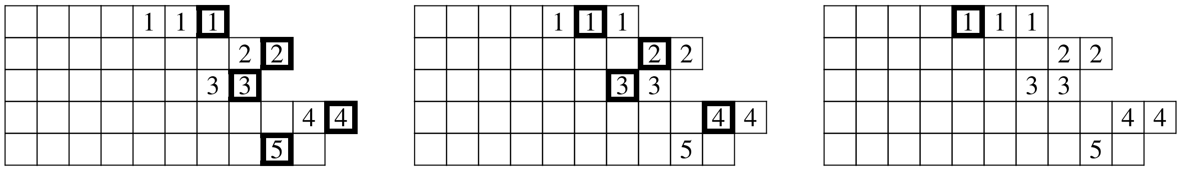
$$n\text{-}charge(A) = 5 - 1 - 2 = 2\,.$$

We now consider the n-charge for ABC's of $shape = \mathfrak{c}(weight)$. For such ABC's it will be shown that their n-charge is exactly zero.

Example 202. *For* $n = 4$*, since the ABC of weight* $(3, 2, 2, 2, 1)$ *and shape* $\mathfrak{c}(3, 2, 2, 2, 1) = (7, 4, 3, 2, 1)$,

$A =$  $\Longrightarrow ext(A) =$ 

Observe that since the shape$(A) = (7, 4, 3, 2, 1)$ *then* $\beta(A) = 7$*, and A tells us that* $off(A) = 3$*. We* *have the standard sequence given by the bold cells below.*



$I = [0, 1, 1, 2, 2]$ $\qquad$ $I = [0, 1, 1, 2]$ $\qquad$ $I = [0]$

This gives us the

$$n\text{-}charge(A) = 10 - 3 - 7 = 0\,.$$

Theorem 203. *Let A be an ABC of weight $\mu \in \mathcal{P}^n$, and $shape(A) = \mathfrak{c}(\mu)$, then*

$$n\text{-}charge(A) = 0.$$

Proof. It suffices to show that $\sum_r I_r(A) = off(A) + \beta(A)$. Fix any i in $1 \le i \le \mu_i$. If A has x extended ribbons, then we see that the sum of the charge index of the letter i in A is $\mu_i\, x - \mu_i$. Since $|\mathcal{R}_i| = \mu_i\, x$, then $\mu_i\, x - \mu_i = |\mathcal{R}_i| - \mu_i$. Theorem 194 and Proposition 191 tells us that $|\mathcal{R}_i| = |\mathcal{I}_i| + |\mathcal{O}_i| + |\mathcal{B}_i|$. Thus, the sum of the charge index of the letter i in A is

$$\begin{aligned} \mu_i\, x - \mu_i &= |\mathcal{I}_i| + |\mathcal{O}_i| + |\mathcal{B}_i| - \mu_i \\ &= |\mathcal{I}_i| + |\mathcal{O}_i| + \mu_i - \mu_i \\ &= |\mathcal{I}_i| + |\mathcal{O}_i|. \end{aligned}$$

Next, observe that

$$off(A) = \sum_{i=1}^{\ell(\mu)} |\mathcal{O}_i| \qquad \& \qquad \beta(A) = \sum_{i=1}^{\ell(\mu)} |\mathcal{B}_i|.$$

A double sum over $1 \le i \le \ell(\mu)$, and for each i the charge index of the letter i is the same as

$\sum_r I_r(A)$. This gives us that the

$$\begin{aligned}\sum_r I_r(A) &= \sum_{i=1}^{\ell(\mu)} (\mu_i\, x - \mu_i) \\ &= \sum_{i=1}^{\ell(\mu)} (|\mathcal{I}_i| + |\mathcal{O}_i|) \\ &= off(A) + \beta(A).\end{aligned}$$

□

10.3 Weak Kostka-Foulkes polynomials

Recall from [DM13] the definition of the n-cocharge of an ABC A is

$$n\text{-cocharge}(A) = \sum_r I_r(A) + off(A)\,,$$

where the method of extracting a standard filling is the same as that of Definition 198, and the index vector is given by

$$I_{r+1} = \begin{cases} I_r & \text{when } r+1 \text{ is east of } r \\ I_r + 1 & \text{when } r+1 \text{ is west of } r\,. \end{cases}$$

In [DM13] a new family of symmetric functions, $\mathfrak{S}^{(n)}_{\mathfrak{c}\lambda}(x;t)$, were defined in terms of a deformation of the Macdonald's P-functions (see [Mac95]), where the coefficients are taken to be the n-cocharge generating functions of ABC's. For $\lambda, \mu \in \mathcal{P}^n$, the *weak Kostka-Foulkes polynomials* are

$$K^n_{\lambda,\mu}(t) = \sum_{A \in ABC(\mathfrak{c}(\lambda),\mu)} t^{n\text{-cocharge}(A)}\,.$$

The symmetric functions, $s^{(n)}_{c(\lambda)}(x;t)$, which are dual to $(S)^{(n)}_{c(\mu)}$ under the Hall-inner product, are connected to the $q = 0$ case of the Macdonald polynomials

Theorem 204. *[DM13] For $\mu \in \mathcal{P}^n$, the Macdonald polynomials at $q = 0$ satisfies the non-negative expansion*

$$H_\mu(x;0,t) = \sum_{\lambda \in \mathcal{P}^n} K^n_{\lambda,\mu}(t) s^{(n)}_{c(\lambda)}(x;t)\,.$$

The functions $s^{(n)}_{c(\lambda)}(x;t)$ and $\mathfrak{S}^{(n)}_{c(\mu)}(x;t)$ are shown to connect to the cohomology of the affine Grassmannian in [DM13], but here we focus on the weak Kostka-Foulkes polynomials, $K^n_{\lambda,\mu}(t)$, and show that they form a change of basis matrix between $H_\mu(x;0,t)$ and $s^{(n)}_{c(\lambda)}(x;t)$.

Definition 205. *Given $2 \leq n \leq m$, we set $S^n_m = \{\nu \in \mathcal{P}^n \mid |\nu| = m\}$. The weak Kostka-Foulkes matrix is*

$$[K^n_{\lambda,\mu}(t)]_{\lambda,\mu \in S^n_m}\,,$$

where the columns and the rows are arranged by elements of S^n_m in lexicographic order.

Example 206. *For $m = 5$ and $n = 4$, we have $S^4_5 = \{(1^5), (2,1,1), (2,2,1), (3,1,1), (3,2)\}$. By Theorem 204 we have the system*

$$\begin{bmatrix} H_{(1^5)}(x;0,t) \\ H_{(2,1,1)}(x;0,t) \\ H_{(2,2,1)}(x;0,t) \\ H_{(3,1,1)}(x;0,t) \\ H_{(3,2)}(x;0,t) \end{bmatrix} = \begin{bmatrix} t^8 & t^7+t^6+t^5 & t^6+t^5+t^4+t^3 & t^4+t^3+t^2 & 1 \\ 0 & t^5 & t^4+t^3 & t^3+t^2 & 1 \\ 0 & 0 & t^3 & t^2 & 1 \\ 0 & 0 & 0 & t^2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^{(4)}_{(1^5)}(x;t) \\ s^{(4)}_{(2,1,1)}(x;t) \\ s^{(4)}_{(2,2,1)}(x;t) \\ s^{(4)}_{(3,1,1)}(x;t) \\ s^{(4)}_{(3,2)}(x;t) \end{bmatrix}$$

where the coefficient matrix is $[K^n_{\lambda,\mu}(t)]_{\lambda,\mu \in S^n_m}$.

The weak Kostka-Foulkes matrix of Example 206 is invertible for all $t \neq 0$. In fact, this is true for any $2 \leq n \leq m$. To show the invertibility of the weak Kostka-Foulkes matrix, we need to establish a relationship between the n-charge and n-cocharge of a given ABC.

Theorem 207. *For any ABC A of weight* $\mu \in \mathcal{P}^n$*,*

$$n\text{-}cocharge(A) = n(\mu) - \beta(A) - n\text{-}charge(A)\,.$$

Proof. The proof follows immediately from showing that

$$\sum_{r=1}^{m} \tilde{I}_r + \sum_{r=1}^{m} I_r = n((1^m))\,,$$

where (1^m) is a partition of length m whose parts are all ones. This follows by induction and the observation that $I_{m+1} + \tilde{I}_{m+1} = m$. □

Example 208. *Example 201 shows us that the ABC A from Example 196 has n-charge*$(A) = 2$*. The n-cocharge*$(A) = 8$*. Since the weight of A is* $\mu = (3,3,3,1)$ *and the shape of A is* $(6,3,2,1)$*, then* $n(\mu) = 12$ *and* $\beta(A) = 2$*. Applying Theorem 207 gives us* $8 = 12 - 2 - 2$*.*

Corollary 21 of [DM13] tells us that there is a unique ABC of weight $\mu \in \mathcal{P}^n$ and shape $\mathfrak{c}(\mu)$. Thus each diagonal entry of the weak Kostka-Foulkes matrix corresponds to a unique ABC whose weight is μ and shape is $\mathfrak{c}(\mu)$. For these types of ABC's, Theorem's 203 and 207 show us that the n-cocharge is non-negative. This gives rise to the following Corollary of the weak Kostka-Foulkes matrix.

Corollary 209. *For* $2 \leq n \leq m$*, the weak Kostka-Foulkes matrix*

$$\left[K^n_{\lambda,\mu}(t)\right]_{\lambda,\mu \in S^n_m}$$

is invertible for all values of $t \neq 0$*.*

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Publications

- A. Dalal. **Unitriangularity of the weak Kostka-Foulkes matrix**. *Preprint*.
- A. Dalal and J. Morse. **Quantum and affine Schubert calculus and Macdonald polynomials**. *Submitted to American Journal of Mathematics*.
- A. Dalal and J. Morse. **A t-generalization for Shubert representatives of the affine Grassmannian**. *DMTCS Proceedings*, 2013.
- A. Dalal and J. Morse. **The ABC's of affine Grassmannians and Hall-Littlewood polynomials**. *DMTCS Proceedings*, 2012.
- A. Dalal and E. Schumtz. **Compositions of random functions on a finite set**. *Electronic Journal of Combinatorics, 9*, R26 2002.

Teaching experience

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- Conduct research in Algebraic Combinatorics and provide weekly updates to my advisor
- Teach freshmen level math courses under a given curriculum, write and grade quizzes and exams
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