Advanced Optimization and Statistical Methods in Portfolio Optimization and Supply Chain Management

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ABSTRACT

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This dissertation is on advanced mathematical programming with applications in portfolio optimization and supply chain management. Specifically, this research started with modeling and solving large and complex optimization problems with cone constraints and discrete variables, and then expanded to include problems with multiple decision perspectives and nonlinear behavior. The original work and its extensions are motivated by real world business problems.

The first contribution of this dissertation, is to algorithmic work for mixed-integer second-order cone programming problems (MISOCPs), which is of new interest to the research community. This dissertation is among the first ones in the field and seeks to develop a robust and effective approach to solving these problems. There is a variety of important application areas of this class of problems ranging from network reliability to data mining, and from finance to operations management.

This dissertation also contributes to three applications that require the solution of complex optimization problems. The first two applications arise in portfolio optimization, and the third application is from supply chain management. In our first study, we consider both single-period and multi-period portfolio optimization problems based on the Markowitz (1952) mean/variance framework. We have also included transaction costs, conditional value-at-risk (CVaR) constraints, and diversification constraints to approach more realistic scenarios that an investor should take into account when he is constructing his portfolio. Our second work proposes the empirical validation of posing the portfolio selection problem as a Bayesian decision problem dependent on mean, variance and skewness of future returns by comparing it with traditional mean/variance efficient portfolios. The last work seeks supply chain coordination under multi-product batch production and truck shipment scheduling under different shipping policies. These works present a thorough study of the following research foci: modeling and solution of large and complex optimization problems, and their applications in supply chain management and portfolio optimization.

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to my family

Chapter 1

Introduction

This dissertation is on advanced mathematical programming with applications in portfolio optimization and supply chain management. Specifically, we focus on three types of problems arising as follows:

1. Portfolio optimization models with discrete decisions and risk constraints modeled as cone constraints,

2. Inclusion of skewness in portfolio optimization frameworks as a Bayesian decision problem, which can be modeled as a bilevel optimization problem.

3. Economic lot scheduling problem with discrete choices modeled as a mixed-integer nonlinear programming problem (MINLP).

Within the scope of our study, we also observed a need for robust and efficient methods for mixed-integer second-order cone programming problems (MISOCP)

to address application (1). Therefore, we conducted algorithmic development, implementation and numerical studies to fill this gap. Since such methods for bilevel optimization and MINLP already exist, along with efficient software, it sufficed for us to be users rather than developers to address applications (2) and (3).

1.1. Algorithmic Studies: MISOCP

In our algorithmic work, we study mixed-integer second-order cone programming problems (MISOCPs) of the form

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & c^T x \\ \text{s.t.} \quad & \|A_i x + b_i\| \leq a_{0i}^T x + b_{0i}, \qquad i = 1, \ldots, m \end{aligned} \tag{1.1}$$

where x is the n-vector of decision variables, $\mathcal{X} = \{(y, z) : y \in \mathcal{Z}^p, z \in \mathcal{R}^k, p+k = n\}$, and the data are $c \in \mathcal{R}^n$, $A_i \in \mathcal{R}^{m_i \times n}$, $b_i \in \mathcal{R}^{m_i}$, $a_{0i} \in \mathcal{R}^n$, and $b_{0i} \in \mathcal{R}$ for i = 1,...,m. The notation $\|\cdot\|$ denotes the Euclidean norm, and the constraints are said to define the *second-order cone*, also referred to as the *Lorentz cone*.

MISOCP is a category of mixed-integer nonlinear optimization (MINLP) problems, where there is a special structure. For this category of problems, we want to minimize a linear objective function, subject to second-order constraint(s). Note that this form also accommodates, and generally includes, linear constraints: if $m_i = 0$, the ith constraint is linear. The decision variables can be continuous or

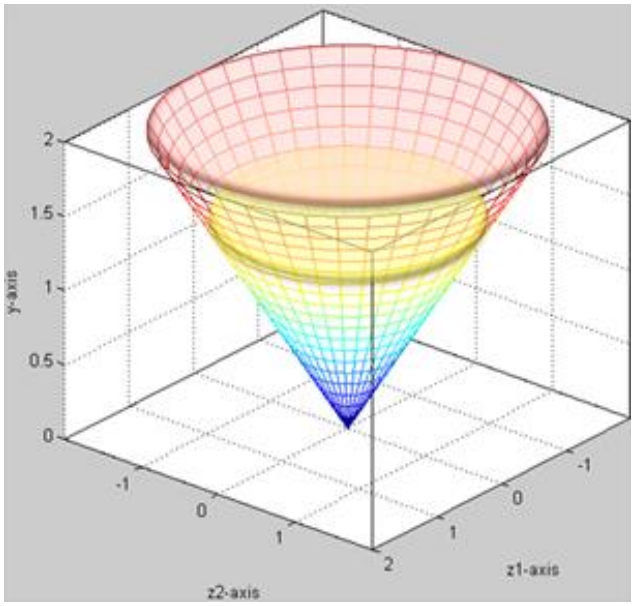

Figure 1.1: Feasible Region of an MISOCP are horizontal slices of the Lorentz Cone

discrete, and while the form of 1.1 allows cases where $p = 0$ or $k = 0$, we are only interested in those cases where $p > 0$. The feasible region of an MISOCP with $p = 1$, $k = 2$ consists of the slices of the ice-cream cone, as shown in Figure 1.1.

When $p = 0$, 1.1 reduces to the continuous problem referred to a second-order cone programming problem (SOCP). SOCPs have been well studied in literature, and computationally efficient implementations of solution algorithms exist. We will provide a thorough survey of these algorithms in Section 3.4 as characteristics of these algorithms will greatly impact the methodologies developed for this dissertation. Of relatively new interest to the research community is the extension to MISOCP fueled by interest in portfolio optimization problems in business

and network reliability models in engineering. Comparatively, MISOCP is a less mature field than SOCP, and this dissertation is among the first ones to branch into this exiting area.

Although this field is only 5-6 years old, this class of problems arise in a variety of important application areas ranging from finance to electrical engineering, from operations management to statistics. In Section 2.2.3, we will discuss these important applications areas in detail, but we will list several examples here as well to show their importance:

- In [121], Pinar studies a multiperiod pricing problem for an American option under uncertainty where the objective function is to maximize at the end of period expected wealth subject to the second-order cone constraints that arise as risk constraints providing a lower bound for the Sharpe ratio of the final wealth position of the buyer. The binary variables are introduced to denote the decision whether to exercise the option at each node of the scenario tree, and additional constraints enforce that the option is exercised at no more than 1 node in each sample path. (Please see Section 2.2.3.1 for more detail about the application.)

- In [42], Cheng et.al. present multi-point transmission problem for cellular networks. The binary variables represent the assignment of mobile units to base stations, where multiple base stations can coordinate the transmission.

The second-order cone constraints are formulated for each base station and serve to limit the total power transmitted from the base station to all of the mobile units that it serves. (Please see Section 2.2.3.4 for more detail about the application.)

- In [132], Taylor and Hover consider several problems from power distribution system reconfiguration. The second-order cone constraints arise in the approximation of flow distribution equations. The binary variables appear as switching variables. (Please see Section 2.2.3.5 for more detail about the application.)

- In [31], Brandenberg and Roth propose a new algorithm for the Euclidean k-center problem. The binary variables denote the assignment of the points to the balls, and second-order cone constraints are used to denote that if a point is assigned to a ball, then the Euclidean distance between the point and the center of the ball must be no more than the radius of the ball. (Please see Section 2.2.3.7 for more detail about the application.)

- In [55], Du et.al. present an MISOCP as a relaxation of the MINLP that arises in the problem of determining the berthing positions and order for a group of vessels waiting at a container terminal in order to minimize the total waiting time of the vessels. Binary variables are used to denote the relative positions of pairs of vessels (whether one vessel is to the left of

another and whether one vessel is earlier than another.) The second-order cone constraints arise in a reformulation of a nonlinear fuel consumption constraint. (Please see Section 2.2.3.9 for more detail about the application.)

As shown above, MISOCPs are very common in a variety of application areas, because of two reasons: (1) risk (volatility) constraints can easily be formulated as second-order cone constraints, and (2) binary choices and discrete decisions are quiet common in the real world. Then the question arises: Despite the many important application areas of this class of problems, why has there been so little development this field until the last decade? There are several natural reasons for this:

- About two decades ago, SOCPs became very popular, as this class of problems arises in important application areas ranging from financial engineering to electrical engineering, as well. However due to their special structure, SOCPs were viewed as extensions of linear programming problems over the Lorenz cone, instead of as nonlinear programming problems. While this feature was essential in the development of algorithms for SOCP, the nonlinear nature of the cone constraints complicates the successful extension of similar concepts from mixed-integer linear programming, such as column generation and cutting plane methods, to MISOCP. Moreover, viewing MISOCPs as mixed-integer nonlinear problems (MINLP) has had its share of

challenges as well, due to MINLP algorithms requiring twice continuously differentiable constraint functions, a property violated by the constraints of 1.1. On a more basic level, due to their long treatment as LP extensions, SOCPs were largely unknown by the NLP community, and by extension, MISOCPs were largely unknown by the MINLP community.

- Compared to two decades ago, today we have more powerful computers that allow us to solve large-scale complex optimization problems. Here, the propose a branch-and-bound algorithm to handle the discrete variables in the single-portfolio optimization problem, which is formulated as an MISOCP. We may need to solve up to $2^n - 1$ subproblems, where n is the number of discrete variables in the model, for this algorithm. Due to the previously discussed application areas leading to large-scale problems, the computational studies conducted for this dissertation may not have been possible before.

In the MISOCP framework, we need to handle two different types of constraints, second-order cone and discrete constraints that bring extra difficulty to the problem. In this dissertation, we use interior-point methods, which require twice continuous differentiability,so we propose the ratio reformulation to rewrite the cone constraint to obtain a smooth convex formulation, for solving portfolio optimization problems. We propose two MINLP methods, branch-and-bound and outer approximation to handle discrete variables. The primal-dual penalty

method is applied to the interior-point algorithm to enable warmstarts and infeasibility detection. We investigate the application of our proposed techniques to portfolio optimization problems that can be formulated as MISOCPs.

1.2. Application 1: Portfolio Selection Models as MISOCPs

Classical portfolio selection models are based on the Markowitz mean/variance framework (1.2), where there is a trade-off between expected return and the risk that the investor may be willing to take on, in a single-period time horizon.

$$\begin{aligned} \max_{w} \quad & r^T w \\ \text{s.t.} \quad & w^T Q w \leq \sigma^2 \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & g(w) \leq 0 \\ & w \geq 0 \end{aligned} \tag{1.2}$$

Although there have been substantial developments in portfolio selection models since the publication of [103], there is a huge gap between the theoretical work and real world application. Therefore, there is still work to be done to incorporate more complex components into these models and solve them efficiently and reliably. In this dissertation, our aim is to model more realistic scenarios when an investor experience when he is constructing his portfolio. Therefore, we have cho-

sen to incorporate transaction costs, conditional value-at-risk (CVaR) constraints, diversification requirements by sectors, and buy-in-thresholds into our framework. These model components/features have been adapted from [1], [29], [67], [73], and [99], and we come up with a very comprehensive portfolio selection model in the portfolio optimization literature. The first two components of the model require the use of second-order cone constraints while the latter two are implemented using binary variables, resulting in an MISOCP. In Chapter 3, Section 3.2 we attempt to solve these comprehensive problems. We propose two algorithms for MISOCP: one with a branch-and-bound framework and the other with an outer approximation framework, both using a primal-dual penalty interior-point method to solve the underlying SOCPs. Both algorithms can accommodate the various cuts appearing in MISOCP literature for further improvement, and they take into account issues such as the non-differentiability of the underlying SOCP, warm-starting, and infeasibility detection. We have implemented both branch-and-bound and outer approximation frameworks that use this method, and use them to solve single-period portfolio optimization problems that can be formulated as MISOCPs. In addition, in Section 3.3 we further extend this model to the multi-period case that is obtained using a binary scenario tree that is constructed with monthly returns of the closing price of the stocks from the S&P 500. We solve these models with the MATLAB-based Mixed Integer Linear and Nonlinear Optimizer (MILANO) solver that implements a variety of methods for handling integer variables, cone

constraints, linear and nonlinear subproblems. The real-world data are used for the numerical examples. Numerical results show that we can solve instances with up to 400 stocks successfully. The infeasibility detection capability provided by the primal-dual penalty approach allows us to either solve or declare infeasibility at each node, thereby leading to a robust method. The warm-start capability is shown to significantly improve algorithm efficiency.

1.3. Application 2: Skewness in Portfolio Selection Models

In Chapter 4, we consider the paradigm of mean/variance efficient portfolios where the investor's objective function is to choose the portfolio weights to maximize expected return subject to predetermined level of risk. Posing the portfolio selection problem as a Bayesian decision problem we investigate which reasonable assumptions on agent's utility and the underlying probability model lead to asset allocation rules that depend on mean, variance and skewness of future returns. The main contribution of this article is the empirical validation of this argument by comparing it with traditional mean/variance efficient portfolios. In this study, we consider two competing descriptions of portfolio selection, the traditional mean/variance efficient portfolio versus a generalization allowing for decision makers to consider

skewness in their asset allocation. We develop a framework to attempt explaining observed investor preferences by the two alternative utility functions. Minimizing the discrepancy between the optimal decision under the considered utility functions and the observed data formalizes the comparison. The discrepancy between the market weight and optimal portfolio weight is formulated as an SOCP and the model becomes bilevel second-order cone programming (BLSOCP) problem where the constraint of an upper level optimization problem, is also an optimization problem. In our framework, the outer problem is to maximize the investor's objective function which is penalized by risk for the first model and it is constrained by both variance/covariance and skewness matrices for the second model. For the inner problem, we solve an optimization problem, minimizing the discrepancy between market and the optimal portfolio. The described algorithm is highly computation intensive. Our numerical experiments are conducted on portfolio drawn from 30 different stocks from the Dow Jones. Numerical results show that investor's preferences are better explained when skewness is taken into account.

1.4. Application 3: Supply Chain Management

The advanced optimization modeling techniques and algorithms we used for portfolio optimization extended naturally to cover some models in supply chain management. Today's supply chains are impacted by increased complexity, unpre-

dictable economic conditions, operational risks, environmental regulations, globalization and rising fuel costs. Historically, optimization projects within the supply chain have been cumbersome, time-consuming undertakings. Many companies find themselves in a constant struggle to maintain efficiency at every stage along the supply chain, attempting to reduce costs and increase productivity within their procurement-production-distribution networks, in the face of intense competitive pressures. In this context, holistic integration of decisions involving serial stages of activities has received attention from researchers in recent years. We attempt to extend the classical and well-known economic lot scheduling problem (ELSP) by incorporating the transportation decision, accounting for finished goods inventories in discrete, sizeable lots. In Chapter 5 we formulate mathematical models that attempt to integrate the production lot scheduling problem with outbound shipment decisions. The optimization objective is to minimize the total relevant costs of a manufacturer, which distributes a set of products to multiple retailers. In making the production/distribution decisions, the common cycle approach is employed to solve the ELSP, for simplicity. Two different shipping scenarios, i.e. periodic full truckload (TL) peddling shipments and less than truckload (LTL) direct shipping, are integrated with and linked to the multi-product batching decisions. The resulting mixed-integer, non-linear programming models (MINLPs) are solved by the BONMIN solver. We illustrate and evaluate a set of numerical examples to find the relative efficiencies of these policies.

1.5. Contributions to Literature

In summary, this thesis provides:

- a thorough survey of the literature on applications of and algorithms for MISOCP
- proposed branch-and-bound and outer approximation algorithms to handle discrete variables in this class of problems
- comprehensive portfolio selection models which have included transaction cost with risk constraint and more realistic diversification requirements
- better explanations of investors' preferences when skewness is taken into account for the portfolio selection models
- an insight to supply chain practitioners about the importance of integrating the production schedule with transportation planning and selecting an appropriate method of distribution.

1.6. Organization of the Thesis

After providing a survey of all related literature, we have divided the dissertation into two parts. The first part covers the portfolio selection model as a MISOCP and all the accompanying algorithmic and computational studied for this class of

problems. To continue with the theme of portfolio optimization, we also include our work evaluation of the inclusion of skewness in these models. The second part covers models from supply chain management.

As such, the balance of this thesis is organized as follows:

- Chapter 2 provides a thorough literature review of three different streams of existing literature: mixed-integer second-order cone programming (2.2), optimization methods arising in portfolio optimization (2.3) and supply chain management (2.4), that provide an overall view of the concepts that will be utilized in the subsequent chapters of this dissertation.

- As mentioned, there are two chapters in the first part of the dissertation. In Chapter 3, we consider single- and multiperiod portfolio optimization with second-order cone constraint and discrete decisions. In Chapter 4, we study the effect of skewness of the future returns for the portfolio selection models.

- In Part II, Chapter 5, we formulate mathematical models that attempt to integrate the production scheduling problem with the outbound decision.

- Finally Chapter 6 provides summary and some concluding remarks for each application as well as some future research directions regarding to the algorithmic work that we will present.

Chapter 2

Literature Review

2.1. Introduction

This chapter first presents a review of three different streams of the existing literature: mixed-integer second-order cone programming (2.2), and optimization methods arising in portfolio optimization (2.3) and supply chain management (2.4), that provide an overall view of the concepts that will be utilized in the subsequent chapters of this dissertation.

In the next section, we provide a brief overview of relevant solution methods for SOCP, since the choice of method for the underlying continuous relaxation greatly influences the design and performance of the overall solution method for MISOCP. These solution methods include interior-point methods designed specifically for SOCP, adaptations of interior-point methods for nonlinear programming

to the case of SOCP, and lifted polyhedral relaxations, and they are incorporated into the MISOCP algorithms presented in Section 2.2.2. In general, MISOCP algorithms fall into two groups: extensions of MILP approaches (since the second-order cone can be viewed as an extension of the linear cone) or special-purpose MINLP approaches (since SOCPs can also be viewed as nonlinear programming problems with special structure). As such, we will present an overview of cuts, including extensions of Gomory and rounding cuts, and relaxations for MISOCP, while discussing the adaptation of branch-and-cut, branch-and-bound, and outer approximation methods to the case of this class of problems. In Section 2.2.3, we start the literature review on applications ranging from scheduling to electrical engineering, from finance to operations management. Several of the examples arise as reformulations or even relaxations of MINLPs, as MISOCPs can sometimes present advantages over MILP in this regard.

In Section 2.3, we start the literature review on portfolio optimization. In Chapters 3 and 4, we consider a single-period portfolio optimization problem which is based on the Markowitz mean-variance framework [104], where there is a trade-off between expected return and risk (market volatility) that the investor may be willing to take on. Therefore, we provide thorough literature review on mean-variance framework in 2.3.1. In Sections 2.3.2 and 2.3.3, we provide the additional literature directly related to each of our models in Chapter 3 and Chapter 4, respectively.

In Section 2.4, we provide a review of the relevant research literature on multi-product batch production and truck shipment scheduling under different shipping policies, which is presented in Chapter 5 in detail.

2.2. Mixed-Integer Second-Order Cone Programming

In this section, we provide a thorough overview of the existing literature on MISOCPs. While the field may not be as mature as SOCP, there have been a wide variety of algorithms studied in the last decade, and the application areas range from supply chain management to electrical engineering to asset pricing. we will focus on portfolio optimization models in Chapter 3 and will provide a literature review for this particular application area as well.

2.2.1. Algorithms for Second Order Cone Programming

In this section, we give a brief overview of several algorithms for solving the underlying SOCPs. These algorithms will have a significant impact on the design and efficiency of the overall MISOCP methods that will be discussed in the next section. For a thorough overview of SOCP, including theory, applications, and solution algorithms, we refer the reader to [2].

2.2.1.1. Interior-Point Methods for SOCP

The continuous relaxation of (1.1) is given by a problem of the same form as (1.1), but with $x \in \mathcal{R}^n$. In order to write the dual problem, let us first introduce auxiliary variables $(t_{0i}, t_i) \in \mathcal{R}^{m_i+1}$ for $i = 1, \ldots, m$ and rewrite the continuous relaxation of (1.1) as

$$
\begin{array}{llcll}
\min\limits_{x,t_0,t} & c^T x & & & \\
\text{s.t.} & t_{0i} & = & a_{0i}^T x + b_{0i}, & i = 1, \ldots, m \\
 & t_i & = & A_i x + b_i, & i = 1, \ldots, m \\
 & \|t_i\| & \leq & t_{0i}, & i = 1, \ldots, m.
\end{array}
\tag{2.1}
$$

The dual problem can now be written as

$$
\begin{array}{llcll}
\max\limits_{\lambda_0,\lambda} & \sum\limits_{i=1}^{m} (b_i^T \lambda_i + b_{0i}^T \lambda_{0i}) & & & \\
\text{s.t.} & \sum\limits_{i=1}^{m} (A_i^T \lambda_i + a_{0i} \lambda_{0i}) & = & c & \\
 & \|\lambda_i\| & \leq & \lambda_{0i}, & i = 1, \ldots, m,
\end{array}
\tag{2.2}
$$

where $(\lambda_{0i}, \lambda_i) \in \mathcal{R}^{m_i+1}$ for $i = 1, \ldots, m$ are the dual variables. Assuming that we have strict interiors for (2.1) and (2.2), strong duality holds and the optimality

conditions for (2.1) are

$$
\begin{aligned}
t_{0i} &= a_{0i}^T x + b_{0i}, && i = 1, \ldots, m \\
t_i &= A_i x + b_i, && i = 1, \ldots, m \\
\sum_{i=1}^{m} (A_i^T \lambda_i + a_{0i}\lambda_{0i}) &= c && \\
\|t_i\| &\leq t_{0i}, && i = 1, \ldots, m \\
\|\lambda_i\| &\leq \lambda_{0i}, && i = 1, \ldots, m \\
(t_{0i}, t_i) \circ (\lambda_{0i}, \lambda_i) &= 0, && i = 1, \ldots, m,
\end{aligned}
\tag{2.3}
$$

where

$$(t_{0i}, t_i) \circ (\lambda_{0i}, \lambda_i) = (t_i^T \lambda_i, t_{0i}\lambda_i + \lambda_{0i} t_i)^T.$$

As with linear programming, an interior-point method starts by introducing a barrier parameter $\mu > 0$, perturbing the last (complementarity) condition in (2.3) as

$$(t_{0i}, t_i) \circ (\lambda_{0i}, \lambda_i) = 2\mu e_i, \qquad i = 1, \ldots, m,$$

with $e_i = \begin{pmatrix} 1 \\ 0^{m_i} \end{pmatrix}$, and initializing t and λ on the strict interior of the second-order cone. The Newton system associated with the perturbed conditions

$$
\begin{aligned}
t_{0i} &= a_{0i}^T x + b_{0i}, && i = 1, \ldots, m \\
t_i &= A_i x + b_i, && i = 1, \ldots, m \\
\sum_{i=1}^{m} (A_i^T \lambda_i + a_{0i}\lambda_{0i}) &= c && \\
(t_{0i}, t_i) \circ (\lambda_{0i}, \lambda_i) &= 2\mu e_i, && i = 1, \ldots, m,
\end{aligned}
$$

is solved at each iteration, but a scaling, such as HRVW/KSH/M ([82],[88],[110]), AHO [3], or NT ([114],[115]), may be needed to do obtain iterates on the interior of the second-order cone. The barrier parameter is also reduced at each iteration. Optimality is declared when (2.3) are satisfied to within a small tolerance.

Interior-point methods for SOCP are theoretically robust and computationally efficient. However, there are drawbacks when they are used within an MISOCP framework, including the need to start from a strictly feasible primal-dual pair of solutions and the accuracy level of the optimal solution obtained. The former makes it hard to warmstart the algorithm from a previously obtained solution, while the latter may create issues with declaring feasibility with respect to the integer variables and adding cuts to the underlying SOCP.

2.2.1.2. Extensions of Interior-Point Methods for NLP to SOCP

While SOCP can be seen as an extension of linear programming, it can also be seen as a special case of nonlinear programming (NLP). The formulation (2.1) is already in the form of a structured, convex NLP. Therefore, another possibility for a solution algorithm is to use interior-point methods that have been developed for NLP. However, an interior-point method for NLP requires that all objective and constraint functions be twice continuously differentiable, so the main challenge in using such a method is the nondifferentiability of the second-order cone constraint functions due to the use of the Euclidean norm.

In [23], Benson and Vanderbei investigated the nondifferentiability of an SOCP and proposed several reformulations of the second-order cone constraint to overcome this issue. Note that the nondifferentiability is only an issue if it occurs at the optimal solution. Since an initial solution can be randomized, especially when using an infeasible interior-point method to solve the SOCP, the probability of encountering a point of nondifferentiability is 0.

For a constraint of the form

$$\|t_i\| \leq t_{0i}, \tag{2.4}$$

Benson and Vanderbei proposed the following:

- Exponential reformulation: Replacing (3.4) with $e^{(t_i^T t_i - t_{0i}^2)/2} \leq 1$ and $t_{0i} \geq 0$ gives a smooth and convex reformulation of the problem, but numerical issues frequently arise due to the exponential.

- Smoothing by perturbation: Introducing a scalar variable v into the norm gives a constraint of the form $\sqrt{v^2 + t_i^T t_i} \leq t_{0i}$, but in order for the formulation to be smooth, we need $v > 0$. This is ensured by setting $v \geq \epsilon$ for a small constant ϵ, usually taken around $10^{-6} - 10^{-4}$.

- Ratio reformulation: Replacing (3.4) with $\frac{t_i^T t_i}{t_{0i}} \leq t_{0i}$ and $t_{0i} \geq 0$ yields a convex reformulation of the problem, but the constraint function may still not be smooth. Nevertheless, in many applications, such as the portfolio

optimization problems to be studied in the next section, the right-hand side of the second-order cone constraint in (1.1) is either a scalar or bounded away from zero at the optimal solution.

Once the reformulation is complete, the problem can be solved using any variant of an interior-point method. In [23], Benson and Vanderbei used the infeasible primal-dual interior-point method that was implemented in LOQO [135]. Introducing a barrier parameter $\mu > 0$ and slack variables $w, s \geq 0$, the perturbed optimality conditions for an SOCP that has undergone the ratio reformulation can be expressed as

$$
\begin{aligned}
t_{0i} &= a_{0i}^T x + b_{0i}, && i = 1, \ldots, m \\
t_i &= A_i x + b_i, && i = 1, \ldots, m \\
\sum_{i=1}^{m} (A_i^T \lambda_i + a_{0i}\lambda_{0i}) &= c && \\
\frac{t_i^T t_i}{t_{0i}} + w &= t_{0i}, && i = 1, \ldots, m \\
\frac{\lambda_i^T \lambda_i}{\lambda_{0i}} + s &= \lambda_{0i}, && i = 1, \ldots, m \\
w_i s_i &= \mu, && i = 1, \ldots, m.
\end{aligned} \tag{2.5}
$$

Starting at an initial solution with $w, s > 0$, we solve the Newton system associated with (2.5) at each iteration, find an appropriate steplength using a merit function or a filter, and update the barrier parameter as needed. The algorithm stops when it satisfies (2.5), with μ sufficiently close to 0, to a desired level of accuracy.

Using this approach, the accuracy of the solution to the SOCP still remains a

concern. However, due to recent work in the area, the warmstart issue is starting to get resolved. I refer the reader to [15] for details on a primal-dual penalty method that enables warmstarts when solving SOCPs.

2.2.1.3. Lifted Polyhedral Relaxation

A very different perspective on (approximately) solving SOCPs is to employ a polyhedral relaxation of the convex feasible region and to solve a related linear programming problem instead. However, in doing so, it is important to ensure that the size of the linear programming problem remains tractable. Ben-Tal and Nemirovski [12] have presented a lifted polyhedral relaxation that uses a polynomial number of constraints and auxiliary variables, and this relaxation method has been further refined by Glineur [70].

Starting with an SOCP of the form (2.1), let us focus on the constraint $\|t_i\| \leq t_{0i}$ for some i. The goal is to construct a polyhedron that is ϵ-tight, i.e. satisfies $\|t_i\| \leq (1+\epsilon)t_{0i}$ for a small $\epsilon > 0$. For ease of presentation, we assume that m_i is an integer power of 2. (We refer the reader to the details provided in [12] and [70] for the case when m_i is not an integer power of 2.) If the variables are grouped into $r/2$ pairs and an auxiliary variable ρ_j is associated with the jth pair, then the set of points satisfying the original cone constraint canbe rewritten as

$$\{(t_{0i}, t_i) \in \mathcal{R}^{m_i+1} : \exists \rho \in \mathcal{R}^{m_i/2} \text{ s.t. } \rho^T \rho \leq t_{0i}^2, t_{i(2j-1)}^2 + t_{i(2j)}^2 \leq \rho_j^2, j = 1, \ldots, m_i\}.$$

This new definition uses one cone of dimension $m_i/2+1$ and $m_i/2$ cones of dimension 3. This process is recursively applied to cone of dimension $m_i/2+1$, until there are only 3-dimensional cones left. Then, each 3-dimensional cone can be replaced with a polyhedral relaxation of the form

$$\{(r_0, r_1, r_2) \in \mathcal{R}^3 : r_0 \geq 0 \text{ and } \exists(\alpha, \beta) \in \mathcal{R}^{2s} \text{ s.t.} \tag{2.6}$$

$$\begin{aligned}
r_0 &= \alpha_s \cos\left(\tfrac{\pi}{2^s}\right) + \beta_s \sin\left(\tfrac{\pi}{2^s}\right) \\
\alpha_1 &= r_1 \cos(\pi) + r_2 \sin(\pi) \\
\beta_1 &\geq |r_2 \cos(\pi) - r_1 \sin(\pi)| \\
\alpha_{i+1} &= \alpha_i \cos\left(\tfrac{\pi}{2^i}\right) + \beta_i \sin\left(\tfrac{\pi}{2^i}\right), \quad i = 1, \ldots, s-1 \\
\beta_{i+1} &\geq \left|\beta_i \cos\left(\tfrac{\pi}{2^i}\right) - \alpha_i \sin\left(\tfrac{\pi}{2^i}\right)\right|, \quad i = 1, \ldots, s-1,
\end{aligned}$$

for some $s \in \mathcal{Z}$.

Given that the resulting problem is a linear program, it can be solved using any number of suitable methods, including the simplex method or a crossover approach, both of which would yield very efficient warmstarts.

2.2.2. Algorithms for Mixed-Integer Second-Order Cone Programming

One very straightforward way to devise a method for solving MISOCPs is to use a branch-and-bound algorithm that calls an interior-point method designed specifically for SOCPs at each node. However, if such a method is to be competitive on

large-scale MISOCPs, it is important to reduce the number of nodes in the tree using cuts and relaxations designed specifically for MISOCP and to reduce the runtime at each node using an SOCP solver that is capable of warmstarting and infeasibility detection.

There are other approaches for MINLP besides branch-and-bound which can similarly be adopted for the case of MISOCP, using the fact that the underlying SOCPs are essentially convex NLPs. These approaches include outer approximation [57], extended cutting-plane methods [139], and generalized Benders decomposition [69]. However, the nondifferentiability of the constraint functions in (1.1) is of particular concern when generating the gradient-based cuts required by these methods, and their application to MISOCP should be done carefully and by considering this special case.

Additionally, any method that can convert the underlying SOCPs into linear programming problems can take advantage of the efficient algorithms designed for MILP.

A number of studies appear in literature dealing with algorithms specifically for MISOCP, and we will now present them here.

2.2.2.1. Gomory Cuts and Tight Relaxations

In [39], Cezik and Iyengar study mixed-integer conic programming problems (MICPs), of which both mixed-integer linear programming problems and MISOCPs are sub-

sets. Their approach is to extend some well-known techniques for mixed-integer linear programming to mixed-integer programs involving second-order cone and/or semidefinite constraints. Since the problem setup in [39] includes a more general cone than ours, we have adapted their discussion to the case of the second-order cone.

Their first extension is that of Gomory cuts to integer conic programs. For the case of integer SOCPs, they note that

$$\left\{x \in \mathcal{R}^n : \|A_i x + b_i\| \leq a_{0i}^T x + b_{0i}, i = 1, \ldots, m\right\} \Leftrightarrow \tag{2.7}$$
$$\left\{x \in \mathcal{R}^n : \left(\sum_{i=1}^{m}(a_{0i}u_{0i} + \sum_{j=1}^{n} a_{ij}^T u_i)\right)^T x \geq \sum_{i=1}^{m}(b_{0i}u_{0i} + b_i^T u_i), (u_{0i}, u_i)^T \in K_i^*, i = 1, \ldots, m\right\},$$

where $A_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and K_i^* is the dual cone of the ith second order cone. This equivalence leads to the following natural extension of the Chvatal-Gomory procedure for integer SOCPs:

1. Choose $(u_{0i}, u_i)^T \in K_i^*, i = 1, \ldots, m$. Then,
$$\left(\sum_{i=1}^{m}(a_{0i}u_{0i} + \sum_{j=1}^{n} a_{ij}^T u_i)\right)^T x \geq \sum_{i=1}^{m}(b_{0i}u_{0i} + b_i^T u_i).$$
2. Without loss of generality, $x \geq 0$, so
$$\left(\sum_{i=1}^{m}(\lceil a_{0i}\rceil u_{0i} + \sum_{j=1}^{n} \lceil a_{ij}\rceil^T u_i)\right)^T x \geq \sum_{i=1}^{m}(b_{0i}u_{0i} + b_i^T u_i).$$
3. By the integrality of x, it holds that
$$\left(\sum_{i=1}^{m}(\lceil a_{0i}\rceil u_{0i} + \sum_{j=1}^{n} \lceil a_{ij}\rceil^T u_i)\right)^T x \geq \sum_{i=1}^{m}(\lceil b_{0i}\rceil u_{0i} + \lceil b_i\rceil^T u_i)$$

is a valid linear inequality that can be added to the cone constraints.

The authors also prove that every valid inequality for the convex hull of the feasible region of an integer SOCP can be obtained by repeating the above procedure a finite number of times.

The second extension is that of sequential convexification to the case of integer conic programs. This approach, which was studied in [7], [126], [127], [100], [92], for pure and mixed-integer linear programming problems can provide tighter relaxations than the continuous relaxation of the integer SOCP. To extend the Lovasz-Schrijver and Balas-Ceria-Cornuejols hierarchies, the authors start by picking a subset of size l of the variables and introduce $Y^0 = [y_1^0 \dots y_l^0]$ and $Y^1 = [y_1^1 \dots y_l^1]$ with

$$y_k^0 = (1 - x_{j_k}) \begin{pmatrix} 1 \\ \\ x \end{pmatrix}, \qquad y_k^1 = x_{j_k} \begin{pmatrix} 1 \\ \\ x \end{pmatrix}, \qquad k = 1, \dots, l.$$

Then, the following is a relaxation for (1.1) with all binary variables:

$$
\begin{array}{llcll}
\min & c^T x & & & \\
\text{s.t.} & y_k^0 + y_k^1 & = & \begin{pmatrix} 1 \\ \\ x \end{pmatrix}, & k = 1, \ldots, l \\
& \|\sum_{j=1}^{n} y_{jk}^0 a_{ij} + y_{0k}^0 b_i\| & \leq & \sum_{j=1}^{n} y_{jk}^0 a_{0j} + y_{0k}^0 b_0, & k = 1, \ldots, l \\
& \|\sum_{j=1}^{n} y_{jk}^1 a_{ij} + y_{0k}^1 b_i\| & \leq & \sum_{j=1}^{n} y_{jk}^1 a_{0j} + y_{0k}^1 b_0, & k = 1, \ldots, l \\
& y_{kk}^0 & = & 0, & k = 1, \ldots, l \\
& y_{kk}^1 & = & y_{0k}^1, & k = 1, \ldots, l \\
& Y^1 & = & (Y^1)^T. &
\end{array}
$$

To extend the Sherali-Adams and Laserre hierarchies, the authors also start by picking a subset of size l of the variables and call this subset B. Let y be a vector that is indexed by the empty set, subsets $H \subseteq B$, and sets of the form $H \cup \{j\}$ for j not picked for B, and define y as follows:

$$
y_I = \begin{cases} 1, & I = \emptyset \\ \\ \prod_{j \in I} x_j, & \text{otherwise.} \end{cases}
$$

Then, define $z_0^I \in \mathcal{R}$ and $z^I \in \mathcal{R}^n$ for $I \subseteq B$:

$$
\begin{array}{lcccl}
z_0^I & = & \prod_{j \in I} x_j \prod_{j \in B \setminus I} (1 - x_j) & = & \sum_{I \subseteq H \subseteq B} (-1)^{|B \setminus H|} y_H \geq 0, \\
z_k^I & = & x_k \prod_{j \in I} x_j \prod_{j \in B \setminus I} (1 - x_j) & = & \sum_{I \subseteq H \subseteq B} (-1)^{|B \setminus H|} y_{H \cup \{k\}}, \qquad k = 1, \ldots, n.
\end{array}
$$

Thus, the following problem is a relaxation of the binary SOCP:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x_j = y_{\{j\}}, && k = 1, \ldots, n \\ & \|A_i z^I + b_i z_0^I\| \leq a_{0i}^T z^I + b_{0i} z_0^I, && I \subseteq B. \end{aligned}$$

Additional hierarchies based on these principles are also discussed in the paper.

The authors propose a cut algorithm can use the Chvatal-Gomory procedure and the tighter relaxations. However, the success of this algorithm is rather limited since the authors consider only interior-point methods for SOCP as the solution algorithm for the underlying SOCPs and implement it using SeDuMi. As they note, the use of interior-point methods results in a solution that is feasible subject to a tolerance and may need rounding prior to applying the cut generation procedure. In addition, warmstarts from feasible dual solutions are not available within SeDuMi, as is the case for most other codes for mixed-integer conic programs. Noting these limitations, the authors present preliminary numerical results and pointers for future improvement.

2.2.2.2. Rounding Cuts

In [5], Atamturk and Narayanan focus on MISOCPs and their solution using a branch-and-bound framework. They introduce rounding cuts obtained by first decomposing each second-order cone constraint into polyhedral sets. In order to introduce this approach, we first note that according to the definition of (1.1), the

variable x can be decomposed into $(y, z) : y \in \mathcal{Z}^p, z \in \mathcal{R}^k, p + k = n$. For each second-order cone $i = 1, \ldots, m$, and partitioning the columns of A_i into A_i^y and A_i^z and the vector a_{0i} into a_{0i}^y and a_{0i}^z, the constraints of (2.1) can be rewritten as follows:

$$\begin{aligned} t_{0i} &\leq (a_{0i}^y)^T y + (a_{0i}^z)^T z + b_{0i} \\ t_i &\geq |A_i^y y + A_i^z z - b_i| \\ \|t_i\| &\leq t_{0i}. \end{aligned}$$

We assume that the absolute value in the second constraint is elementwise and focus on one such constraint which we will write as

$$|a^y y + a^z z + b| \leq t \tag{2.8}$$

where a^y is a row of A_i^y for some i, a^z is the corresponding row in A_i^z, and b and t are the corresponding elements of b_i and t_i, respectively. This form is both for ease of notation and to better match the exposition in [5]. The set S is defined as $\{y \in \mathcal{Z}^p, z \in \mathcal{R}^k, t \in \mathcal{R} : |a^y y + a^z z + b| \leq t, y \geq 0, z \geq 0\}$, where the nonnegativities of y and z can be imposed without loss of generality (i.e., if a variable is free, we can always split it into two nonnegative ones). Grouping the terms of $a^z z$ with positive and negative coefficients into z^+ and z^-, respectively, (2.8) is rewritten as

$$|a^y y + z^+ - z^- + b| \leq t. \tag{2.9}$$

The authors first define a rounding function φ_f for $0 \leq f < 1$ as

$$\varphi_f(v) = \begin{cases} (1-2f)n - (v-n), & \text{if } n \leq v < n+f \\ (1-2f)n + (v-n) - 2f, & n+f \leq v < n+1, \end{cases}$$

where $n \in \mathcal{Z}$. Then, they show that the following is a valid inequality for S

$$\sum_{j=1}^{n} \varphi_f(a_j/\alpha)y_j - \varphi_f(b/\alpha) \leq (t + z^+ + z^-)/|\alpha|$$

for any $\alpha \neq 0$ and $f = b/\alpha - \lfloor b/\alpha \rfloor$. In addition, if $b/a_i > 0$ for some i and $\alpha = a_i$, then the above inequality is shown to be facet-defining for the convex hull of S.

These rounding cuts are added at the root node of the branch-and-bound tree, and the preliminary results in [5] show that the cuts can significantly reduce the number of nodes in the tree. The authors provide a more thorough analysis and further examples showing the success of their approach in [6].

2.2.2.3. MILP methods applied to lifted polyhedral relaxation

In [137], Vielma et.al. propose using a lifted polyhedral relaxation ([12], [70], and described in Section 3.3) of the underlying SOCPs, thereby solving the MISOCPs using a linear programming based branch-and-bound framework. Their approach can be generalized to any convex MINLP, does not use gradients to generate the cuts, and benefits from the linear programming structure that can use a simplex-based method with warmstarting capabilities within the solution process.

The *Lifted LP Branch-and-Bound Algorithm* presented in [137] is too detailed to present in its entirety here, so we will give a brief outline and the interested

reader is referred to [137], particularly Figure 1 of that paper. In general, the algorithm proceeds as the usual branch-and-bound method for MILPs by branching on discrete variables with non-integer values, except solving the lifted polyhedral relaxation of the associated SOCP at each node. If a feasible solution is found at any node, the continuous relaxation of the MISOCP is solved at that node to see if the exact solution (rather than an ϵ-tight relaxation), still yields a feasible solution. If so, the node is fathomed by integrality. Otherwise, we continue branching on a discrete variable with a non-integer value. This approach ensures that only linear programming problems are solved at most nodes of the tree and limits the solution of the underlying SOCPs to a much smaller number of nodes.

Numerical studies on portfolio optimization problems show that the method outperforms CPLEX and Bonmin. A similar method is used in [128] by Soberanis to solve the MISOCP reformulations of risk optimization problems with p-order conic constraints.

2.2.2.4. Extensions of Convex MINLP methods to MISOCP

In [54], Drewes proposes both a branch-and-cut method and a hybrid branch-and-bound/outer approximation method for solving MISOCPs. The branch-and-cut method uses techniques similar to [39] and those developed in [130] for mixed-integer convex optimization problems with binary variables. Therefore, given its similarity to the method presented in Section 4.1, we will not present this

approach, but instead provide details on the hybrid branch-and-bound/outer approximation method. Numerical results for both methods are provided in [54] for a number of test problems.

The hybrid approach extends outer approximation methods, which use gradient-based techniques to generate cuts, to the case of MISOCPs using subgradients. As in outer approximation, constraints of the form $\|t_i\| \leq t_{0i}$ are replaced by

$$(\|\bar{t}_i\| - \bar{t}_{0i}\|) + \xi_i^T(t_i - \bar{t}_i) + \xi_{0i}(t_{0i} - \bar{t}_{0i}) \leq 0,$$

where $(\bar{t}_{0i}, \bar{t}_i)$ is part of the solution of a continuous relaxation of the MISOCP and (ξ_0, ξ) is a subgradient of the second-order cone constraint function $\|t_i\| - t_{0i}$ at $(\bar{t}_{0i}, \bar{t}_i)$. If $\bar{t}_i \neq 0$, the gradient can be used and set

$$\xi_{0i} = -1 \text{ and } \xi_i = \frac{\bar{t}_i}{\|\bar{t}_i\|}.$$

Otherwise, the dual variables $(\bar{\lambda}_{0i}, \bar{\lambda}_i)$ can be used to get an appropriate subgradient. In [54], Drewes proposes that

$$\xi_{0i} = -1 \text{ and } \xi_i = \begin{cases} -\frac{\bar{\lambda}_i}{\bar{\lambda}_{0i}}, & \text{if } \bar{\lambda}_{0i} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Additional cuts are generated from infeasible instances and are described in [54].

In [14] and [15], Benson and Sağlam propose two MINLP methods, branch-and-bound and outer approximation, for solving MISOCPs. Since the underlying problems are smoothed using the ratio reformulation, as described in Section

3.2, and a primal-dual penalty method is applied to the interior-point algorithm to enable warmstarts and infeasibility detection, both MINLP methods can be applied directly and efficiently to solve an MISOCP. Preliminary numerical results on problems arising in portfolio optimization are encouraging.

2.2.3. MISOCPs Arising in Applications

In this section, we give an overview of MISOCPs arising in a variety of application areas in business, engineering, and statistics. It should be noted that this is only a representative list and not an exhaustive one by any means. One of the challenges in gathering a literature review on MISOCPs is that, many times, authors do not recognize the special structure of the problem and simply identify the model as a MINLP, solved using traditional MINLP methods. Therefore, we have included in this section only those models that have been recognized by the authors as MISOCPs.

We should also note that due to the wide variety of models, each subsection below will have a self-contained list of notation. We will return to the formulation (1.1) and related notation in the next section.

2.2.3.1. Options Pricing

In [121], Pinar describes a pricing problem for an American option in a financial market under uncertainty. The multiperiod, discrete time, and discrete state space

structure is modeled using a scenario tree, and, therefore, the resulting problem is large-scale. The set of nodes is denoted by N, and the nodes corresponding to time period t are denoted by N_t. The planning horizon is at time T. $\pi(n)$ denotes the parent node of n, and $A(n)$ denotes the ascendant nodes of n, including itself. The probability of each $n \in N_T$ is denoted by p_n.

We assume that there is a market consisting of $J+1$ securities, with prices at node n given by $z_n = (z_n^0, z_n^1, \ldots, z_n^J)^T$. The security with index 0 is assumed to be risk-free. The decision variables $\theta_n \in \mathcal{R}^{J+1}$ denote the portfolio allocations at node n, and thus, $z_n^T \theta_n$ denotes the value of the portfolio at the node. The binary decision variables e_n indicate whether an American option is exercised at node n, and, if exercised, the holder would have a payoff of h_n. Auxiliary variables, x_n (free) and v_n (nonnegative) are also introduced to denote that the final wealth position can be unrestricted in sign. An additional auxiliary variable,

v, is introduced as the initial wealth of the portfolio.

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & \sum_{n \in N_T} p_n x_n - \lambda \sqrt{\sum_{n \in N_T} p_n \left(x_n - \sum_{k \in N_T} p_k x_k \right)^2} \geq 0 \\
& \sum_{m \in A(n)} e_m \leq 1, && n \in N_T \\
& z_0^T \theta_0 = h_0 e_0 - v \\
& z_n^T (\theta_n - \theta_{\pi(n)}) = h_n e_n, && n \in N_t, t = 1, \ldots, T \\
& z_n^T \theta_n - x_n - v_n = 0, && n \in N_T \\
& v_n \geq 0, && n \in N_T \\
& e_n \in \{0, 1\}, && n \in N.
\end{aligned} \tag{2.10}
$$

The second-order cone constraints arise as risk constraints that provide a lower bound for the Sharpe ratio of the final wealth position of the buyer. The term

$$\sum_{n \in N_T} p_n x_n$$

is the expected value of the final wealth position,

$$\sum_{n \in N_T} p_n \left(x_n - \sum_{k \in N_T} p_k x_k \right)^2$$

is its variance, and λ is the lower bound on the Sharpe ratio.

The second set of constraints enforce that the option is exercised at no more than 1 node in each sample path, and the remaining linear constraints ensure flow balance through the scenario tree. The numerical results show that these problems, with over 20,000 continuous variables, 5,000 discrete variables, and 30,000

constraints to accommodate large enough scenario trees, are quite challenging for existing MISOCP software.

2.2.3.2. Network Design and Operations

We now present a group of problems which we have loosely termed under the heading of Network Design and Operations. They arise in telecommunications networks that model the flow of commodities, cellular networks which must assign base stations to mobile units, power systems, and highway networks with vehicular traffic. Despite the similarities in the structures of the systems, the applications all have different objectives and concerns, so there is a variety of different uses for the binary variables and the second-order cone constraints in the following four applications.

2.2.3.3. Delays in Telecommunication Networks

In [83], Hijazi et.al. investigate a telecommunications network problem that seeks to minimize the network response time to a user request. The network is represented by vertices V and edges E, and vectors of capacities c and routing costs w for the edges are given. We assume that the network can handle multiple commodities grouped by the set K, there is an amount $\bar{v}_k$ of commodity k, and that each commodity k has a set of candidate paths $P(k)$, leading from its source to its destination. The decision variables in the problem are continuous variables x_e

representing the flow along edge e and ϕ_{ik} representing the fraction of commodity k routed along path P_{ik}, as well as binary variables z_{ik} which indicate whether the path P_{ik} is open.

The initial model has the following form:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} w_e x_e & \\
\text{s.t.} \quad & \sum_{e \in P_{ik}} \frac{1}{c_e - x_e} \leq \alpha_k, & k \in K, P_{ik} \in P(k) \text{ if } z_{ik} = 1 \\
& \sum_{i: P_{ik} \in P(k)} \phi_{ik} = 1, & k \in K \\
& \sum_{k \in K} (\bar{v}_k \sum_{P_{ik}: e \in P_{ik}} \phi_{ik}) = x_e, & e \in E \\
& x_e \leq c_e, & e \in E \\
& \sum_{P_{ik} \in P(k)} z_{ik} \leq N, & k \in K \\
& \phi_{ik} \leq z_{ik}, & k \in K, P_{ik} \in P(k) \\
& z_{ik} \in \{0, 1\}, & k \in K, P_{ik} \in P(k) \\
& \phi_{ik} \geq 0, & k \in K, P_{ik} \in P(k).
\end{aligned}
\tag{2.11}
$$

The objective function minimizes the total cost of all the flows along the edges of the network. With c_e denoting the capacity along edge e, the average queueing plus transmission delay using an M/M/1 model is computed to be $\frac{1}{c_e - x_e}$. Thus, the first constraint ensures that the total end-to-end delay on any active path through which a commodity k must travel is no greater than some parameter α_k, and it is this constraint that will require further examination. The second, third, and fourth constraints ensure that all parts of a commodity are routed, that the

flow along each edge is the total flow over all the commodities that use the edge as a part of one or more associated paths, and that the total flow along an edge does not exceed the capacity of the edge. The fifth constraint states that the commodity cannot be partitioned to more than N paths, and the sixth constraint ensures that only the paths that will be opened for the commodity are allowed to have flow of that commodity along them.

The authors re-examine the first constraint, and note that since the delay along each open path is uncertain, they can also model it using a robust constraint. These constraints are also disjunctive since they are only used if the path is open. To handle both the uncertainty and the disjunction, the authors propose an extended formulation and a perspective function approach. The additional details and notation required to introduce these MISOCPs is beyond the scope of this paper, and the interested reader is referred to [83]. The numerical testing shows that CPLEX has trouble solving large MISOCP instances, while related MINLPs are solved in within reasonable time requirements by Bonmin.

2.2.3.4. Coordinated Multi-point Transmission in Cellular Networks

In [42], Cheng et.al. model and solve a coordinated multi-point transmission problem for cellular networks. For a network with L multiple-antenna base stations and K single-antenna mobile stations, the problem is to find w_{kl} as the beamforming vector used at base station l to transmit to mobile station k using the

following model

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{K}\sum_{l=1}^{L}(\|w_{kl}\|^2 + \lambda_{kl}U(\|w_{kl}\|)) \\
\text{s.t.} \quad & \sum_{l=1}^{L} U(\|w_{kl}\|) \le c_k, && k = 1,\dots,K \\
& \mathrm{SINR}_k \ge \gamma_k, && k = 1,\dots,K \\
& \sum_{k=1}^{K} \|w_{kl}\|^2 \le P_l, && l = 1,\dots,L,
\end{aligned} \tag{2.12}
$$

where λ_{kl} denotes the penalty of serving mobile station l by base station k, the function $U(x) = 0$ if $x = 0$ and 1 otherwise, c_k is the maximum number of base stations that can be assigned to mobile station k, SINR_k is the receive signal-to-interference-plus-noise ratio (SINR) at mobile station k, γ_k is the minimum SINR level required to provide sufficient quality of service at k, and P_l is the maximum available transmit power at base station l.

The SINR constraints can be reformulated as second-order cone constraints of the form

$$
\|(h_k^H W, \sigma_k)^T\| \le \sqrt{1 + 1/\gamma_k}\mathrm{Re}\{h_k^H w_k\}, \qquad \mathrm{Im}\{h_k^H w_k\} = 0, \qquad k = 1,\dots,K,
$$

where h_k represent the matrix of frequency-flat vectors to mobile station k, W is the matrix whose columns are w_k, $k = 1,\dots,K$, and σ_k is the standard deviation of the white noise at mobile station k. In addition, binary variables a_{kl} and auxiliary continuous variables t_{kl} are introduced to convert (2.12) to the following

MISOCP:

(2.13)

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{K}\sum_{l=1}^{L}(t_{kl}+\lambda_{kl}a_{kl}) & \\
\text{s.t.} \quad & \|(2w_{kl}^T, a_{kl}-t_{kl})^T\| \le a_{kl}+t_{kl}, & k=1,\dots,K, l=1,\dots,L \\
& \sum_{k=1}^{K} t_{kl} \le P_l, & l=1,\dots,L \\
& \|(h_k^H W, \sigma_k)^T\| \le \sqrt{1+1/\gamma_k}\mathrm{Re}\{h_k^H w_k\}, & k=1,\dots,K \\
& \mathrm{Im}\{h_k^H w_k\} = 0, & k=1,\dots,K \\
& \sum_{l=1}^{L} a_{kl} \le c_k, & k=1,\dots,K \\
& a_{kl} \in \{0,1\}, & k=1,\dots,K, l=1,\dots,L \\
& t_{kl} \ge 0, & k=1,\dots,K, l=1,\dots,L,
\end{aligned}
$$

where the first set of second-order cone constraints are reformulations of the quadratic constraints

$$\|w_{kl}\|^2 \le a_{kl}t_{kl}, \qquad k=1,\dots,K, l=1,\dots,L$$

as described in Subsection 2.1.1 for the portfolio optimization problem.

Due to the large size of the problem instances, the authors propose a heuristic, which is able to obtain slightly worse solutions in significantly less CPU times than CPLEX.

2.2.3.5. Power Distribution Systems

In [132], Taylor and Hover present several problems from power distribution system reconfiguration, one of which is formulated as an MISOCP. Given a set of lines W and a set of buses B, along with subsets $W^S \subseteq W$ with switches and $B^F \subseteq B$ with substations, the goal is to minimize the loss by choosing the right combination of open and closed switches along the system. The problem data include the real and reactive powers from each substation i, p_i^F and q_i^F; the real and reactive loads at a bus i without a substation, p_i^L and q_i^L; and resistance of the line from bus i to j, r_{ij}. The model using the *DistFlow* equations of [9] has

the following form:

(2.14)

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in W} r_{ij}(p_{ij}^2 + q_{ij}^2) \\
\text{s.t.} \quad & \sum_{k:(i,k)\in W} p_{ik} = p_{ji} - r_{ij}\frac{p_{ji}^2 + q_{ji}^2}{v_j^2} - p_i^L, && i \in B\backslash B^F \\
& \sum_{k:(i,k)\in W} q_{ik} = q_{ji} - x_{ij}\frac{p_{ji}^2 + q_{ji}^2}{v_j^2} - q_i^L, && i \in B\backslash B^F \\
& v_i^2 = v_j^2 - 2(r_{ij}p_{ji} + x_{ij}q_{ji}) + (r_{ij}^2 + x_{ij}^2)\tfrac{p_{ji}^2+q_{ji}^2}{v_j^2}, && (i,j) \in W \\
& \sum_{j:(i,j)\in W} p_{ij} = p_i^F, && i \in B^F \\
& \sum_{j:(i,j)\in W} q_{ij} = q_i^F, && i \in B^F \\
& 0 \le p_{ij} \le M z_{ij}, && (i,j) \in W \\
& 0 \le q_{ij} \le M z_{ij}, && (i,j) \in W \\
& z_{ij} \ge 0, && (i,j) \in W \\
& z_{if} = 0, && (i,f) \in W : f \in B^F \\
& z_{ij} + z_{ji} = 1, && (i,j) \in W\backslash W^S \\
& z_{ij} + z_{ji} = y_{ij}, && (i,j) \in W^S \\
& \sum_{j:(i,j)\in W} z_{ij} = 1, && i \in B^F \\
& y_{ij} \in \{0,1\}, && (i,j) \in W^S
\end{aligned}
$$

The decision variables are the continuous p_{ij} and q_{ij} denoting the real power flow from bus i to j, continuous z_{ij} denoting the orientation of the line (i,j), and the discrete y_{ij} denoting whether the switch on the line (i,j) will be open or

closed. In addition, the squared-variables v_i^2 are the voltage magnitude. The first three constraints represent the *DistFlow* equations, followed by two flow balance constraints. The sixth and seventh constraints ensure that the power flow only occurs along edges with open switches, and the remaining constraints seek to define that power will flow only in one direction and in a manner consistent with the network configuration.

When converting the problem into an MISOCP, the authors drop the last term in the third constraint and replace the first three constraints with the following system which includes auxiliary variables $\tilde{p}$, $\tilde{q}$, and $\tilde{v}^2$:

$$\begin{aligned}
&\sum_{j:(i,j)\in W} (p_{ij} - p_{ji}) - p_i^L = \tilde{p}_i, && i \in B \backslash B^F \\
&\sum_{j:(i,j)\in W} (q_{ij} - q_{ji}) - q_i^L = \tilde{q}_i, && i \in B \backslash B^F \\
&\tilde{v}_i^2 \leq v_j^2 + M(1 - z_{ji}), && (i,j) \in W \\
&\tilde{v}_i^2 \geq v_j^2 - M(1 - z_{ji}), && (i,j) \in W \\
&r_{ij}(p_{ji}^2 + q_{ji}^2) \leq \tilde{v}_i^2 \tilde{p}_i, && (i,j) \in W \\
&x_{ij}(p_{ji}^2 + q_{ji}^2) \leq \tilde{v}_i^2 \tilde{q}_i, && (i,j) \in W \\
&v_i^2 \leq v_j^2 - 2(r_{ij}p_{ji} + x_{ij}q_{ji}) + M(1 - z_{ij}), && (i,j) \in W \\
&v_i^2 \geq v_j^2 - 2(r_{ij}p_{ji} + x_{ij}q_{ji}) - M(1 - z_{ij}), && (i,j) \in W
\end{aligned}$$

While the authors do not mention doing so, we would also need to introduce an auxiliary variable to move the quadratic objective function into a constraint and then replace the constraint with an equivalent second-order cone constraint.

Numerical results are presented on 32 to 880 bus systems using CPLEX.

2.2.3.6. Battery Swapping Stations on a Freeway Network

In [101], Mak et.al. consider the problem of creating a network infrastructure and providing coverage for battery swapping stations to service electric vehicles. Given an existing freeway network, they consider candidate locations, J, and use a binary variable, x_j, for each candidate j to denote whether or not a swapping station is located there. Additional binary variables, y_{jp} and z_{jq} denote whether vehicles traveling along a path $p \in P$ or a portion $q \in Q$ of a path along the network will visit swapping station j. The number of electric vehicles that travel along each portion of a path is random, so demand at each swapping station is uncertain. The model seeks to minimize the total cost, which consists of the fixed costs associated with opening and operating the swapping stations and the

expected holding costs at each station.

$$
\begin{aligned}
\min \quad & \sum_{j \in J} (f_j x_j + h G_j(y)) & \\
\text{s.t.} \quad & \sum_{j \in J} a_{jq} z_{jq} \geq 1, & q \in Q \\
& y_{jp} \geq b_{pq} z_{jq}, & j \in J, p \in P, q \in Q \\
& y_{jp} \leq x_j, & j \in J, p \in P \\
& H_j(y) \geq 1 - \epsilon, & j \in J \\
& x_j \in \{0, 1\}, & j \in J \\
& y_{jp} \in \{0, 1\}, & j \in J, p \in P \\
& z_{jq} \in \{0, 1\}, & j \in J, q \in Q.
\end{aligned} \tag{2.15}
$$

In the objective function, f_j is the annualized fixed cost incurred if a station is located at $j \in J$, and h is the annualized holding cost per battery. $G_j(y)$ denotes the expected largest total demand at swapping station j given the assignments of stations to paths. If Q only contains those portions that are longer than a maximum length dictated by battery life and denote by a_{jq} a binary parameter that indicates whether station j is on portion q, then the first constraint states that there needs to be at least one swapping station along the portion q. In addition, the second constraint states that, if portion q, with a station, is a part of multiple paths as indicated by the binary parameter b_{pq} with $p \in P$, then each of those paths inherit the swapping station at q. The third constraint ensures that vehicles are assigned only to stations that are open. In the fourth constraint,

$H_j(y)$ is the worst-case probability of the demand at station j being less than the number of simultaneous recharges permitted by the grid, and a worst-case service level of at least 1 - ϵ is guaranteed, where $\epsilon > 0$ is a small constant.

There are two parts of the problem, the nonlinear term in the objective function and the chance constraint, that have to be dealt with before obtaining an MISOCP. To handle the objective function term, auxiliary variables $v_j \geq 0$ are introduced for each station j, modify the objective function to

$$\sum_{j \in J} (f_j x_j + h v_j),$$

and let

$$v_j \geq G_j(y), \qquad j \in J.$$

It is shown in [101] that the worst-case scenario demand at j, $G_j(y)$, has an upper bound that consists of the sum of a Euclidean norm of a linear vector involving y and another linear term also involving y. Due to the multitude of additional notation in the calculation of this upper bound, we have not included the exact formulation here and invite the interested reader to read the details in [101]. We simply note that such a construct leads to a second-order cone constraint. The numerical studies conducted by the authors indicate that the upper bound is accurate, and they also show that it is asymptotically tight if the underlying uncertainties share the same descriptive statistics.

To handle the chance constraint indicating a robust service level requirement,

the authors introduce a Conditional Value-at-Risk constraint was used in our earlier discussion on portfolio optimization problems. The resulting problem is an MISOCP, and they solve it with data from the San Francisco freeway network using CPLEX.

2.2.3.7. Euclidean k-center

In [31], Brandenberg and Roth introduce a new algorithm for the Euclidean k-center problem, which deals with the clustering of a group of points among k balls and arises in facility location and data classification applications. Without loss of generality, assume that sets $S_1, \ldots, S_k$ exist of points that are to be clustered together and that there are still remaining points in S_0 that have not yet been assigned a cluster. There are a total of m points in $\mathcal{R}^n$. The clusters, as stated, will be enclosed in balls, and the continuous variables in the problem are the coordinates of the centers, $c \in \mathcal{R}^n$, for each ball. The binary variable, λ_{ij}, denotes the assignment of the points $p_j \in S_0$ to ball i. The model can be formulated as follows:

$$
\begin{aligned}
\min \quad & \rho \\
\text{s.t.} \quad & \|p_j - c_i\| \leq \rho, && p_j \in S_i, i = 1, \ldots, k \\
& \|\lambda_{ij} p_j - c_i + (1 - \lambda_{ij}) q_{ij}\| \leq \rho, && p_j \in S_0, i = 1, \ldots, k \\
& \sum_{i=1}^{k} \lambda_{ij} = 1, && p_j \in S_0 \\
& \lambda_{ij} \in \{0, 1\}, && p_j \in S_0, i = 1, \ldots, k.
\end{aligned} \tag{2.16}
$$

The first set of second-order cone constraints is a reformulation of the requirement to minimize the maximum Euclidean distance between a point and the center of a cluster, and it is obtained by introducing an auxiliary variable ρ to denote the maximum distance. The second set of second-order cone constraints are used to denote that if a point is assigned to a ball, then the Euclidean distance between the point and the center of the ball must be no more than the radius of the ball. If the assignment is not made, then the constraint reduces to a given reference point q_{ij}, already in the ball, being within the radius. This reference point is usually chosen as the point in S_i closest to p_j.

We should note here that the authors consider norms other than the Euclidean norm in the paper, and the MISOCP is a special case of their basic model. Numerical results are obtained using a branch-and-bound method calling SeDuMi.

2.2.3.8. Operations Management

In [4], Atamturk et.al. explore a joint facility location and inventory management model under stochastic retailer demand. The binary variables arise in the choice of candidate locations at which to open distribution centers and the assignment of retailers to the distribution centers. The second-order cone constraints appear in the reformulation of the uncapacitated problem to move the nonlinear objective function terms denoting the fixed costs of placing and shipping orders and the expected safety stock cost into the constraints. The complete model has the

following form:

$$
\begin{aligned}
\min \quad & \sum_{j \in J} \left(f_j x_j + \sum_{i \in I} d_{ij} y_{ij} + K_j s_j + q_j t_j \right) \\
\text{s.t.} \quad & \sqrt{\sum_{i \in I} \mu_i y_{ij}^2} \leq s_j, && j \in J \\
& \sqrt{\sum_{i \in I} \sigma_i^2 y_{ij}^2} \leq t_j, && j \in J \\
& \sum_{j \in J} y_{ij} = 1, && i \in I \\
& y_{ij} \leq x_j, && i \in I, j \in J \\
& x_j \in \{0, 1\}, s_j, t_j \geq 0, && j \in J \\
& y_{ij} \in \{0, 1\}, && i \in I, j \in J,
\end{aligned}
\tag{2.17}
$$

where I is the set of existing retailers, J is the set of candidate locations for opening distribution centers, and the variables $x \in \mathcal{R}^{|J|}$ represent choices among the candidates, with $y \in \mathcal{R}^{|I| \times |J|}$ assigning existing retailers to the new distribution centers. Auxiliary variables are introduced to denote cost terms that are computed nonlinearly

$$
s_j = \sqrt{\sum_{i \in I} \mu_i y_{ij}}, \qquad t_j = \sqrt{\sum_{i \in I} \sigma_i^2 y_{ij}},
$$

and the fact that $y_{ij} = y_{ij}^2$ is used to obtain the first two constraints in (2.17), which are second-order cone constraints. In the otherwise linear problem, f is the vector of fixed costs for opening a distribution center at each candidate location, d is the matrix of unit shipping costs between retailers and distribution centers, K and q aid in the calculation of costs for shipping, safety stocks, and any related inventory costs for the assignments made to each distribution center, and μ and

σ denote the mean and standard deviation, respectively, of the daily demand at each retailer. The third and fourth constraints in (2.17) ensure that each retailer is assigned to only one distribution center and that assignment is only made to those distribution centers which are open.

The model with capacities additionally has a second-order cone constraint arising from moving an objective function term for the average inventory holding cost into a constraint, and another one arising from the reformulation of a capacity constraint. Other related models with similar features are provided in the paper. Numerical results are conducted using algorithms studied in [124], [118], and CPLEX.

2.2.3.9. Scheduling and Logistics

In [55], Du et.al. present an MISOCP as a relaxation of the MINLP that arises in the problem of determining the berthing positions and order for a group of vessels, V, waiting at a container terminal in order to minimize the total fuel cost

and waiting time of the vessels. The MINLP is formulated as follows:

$$
\begin{aligned}
\min \quad & \sum_{i \in V} (c_i^0 a_i + c_i^1 m_i^{\mu_i} a_i^{1-\mu_i}) + \lambda \sum_{i \in V} (y_i + h_i - d_i)^+ \\
\text{s.t.} \quad & x_i + l_i \leq L, && i \in V \\
& x_i + l_i \leq x_j + L(1 - \sigma_{ij}), && i, j \in V, i \neq j \\
& y_i + h_i \leq y_j + M(1 - \delta_{ij}), && i, j \in V, i \neq j \\
& 1 \leq \sigma_{ij} + \sigma_{ji} + \delta_{ij} + \delta_{ji} \leq 2, && i, j \in V, i < j \\
& \underline{\mathrm{a}}_i \leq a_i \leq \bar{a}_i, && i \in V \\
& a_i \leq y_i, && i \in V \\
& a_i, x_i \geq 0, && i \in V \\
& \sigma_{ij}, \delta_{ij} && i, j \in V, i \neq j
\end{aligned}
\tag{2.18}
$$

In (2.18), binary variables, σ and δ, are used to denote the relative positions of pairs of vessels (whether one vessel is to the left of another and whether one vessel is earlier than another). Additional continuous variables, x, a, and y, denote the leftmost berthing positions, the terminal arrivals, and the start of the berthing times for each vessel, respectively. In the problem data, L denotes the wharf length at the terminal and l, h, and d denote the length, handling time, and requested departure time of each vessel, respectively. As is customary, M denotes an arbitrary large constant. The first four constraints of the problem are linear and serve to set the rules on wharf length and the positions and handling time of each vessel. Figure 1 from [55] depicts an example which clarify these relationships. The fifth and sixth constraints allow the vehicle to adjust its sailing speed in order

to save fuel—the actual arrival time at the terminal is allowed to be in an interval $[\underline{a}_i, \bar{a}_i]$, while still remaining before the berthing time y_i.

The MINLP model (2.18) incorporates two objective functions, fuel consumption and total departure delay, both of which are minimized. We have introduced a weight of λ in order to combine these two objective functions and simplify the problem. Let us first discuss the fuel consumption objective function. This function is obtained using regression analysis for each vessel, and c^0 and c^1 denote the regression coefficients, m denotes the distance of the vessels from the terminal, and $\mu_i \in \{3.5, 4, 4.5\}$ for each $i \in V$. Introducing auxiliary variables $q \in \mathcal{R}^{|V|}$, this function can be rewritten as

$$\sum_{i \in V} (c_i^0 a_i + c_i^1 m_i^{\mu_i} q_i)$$

if the constraints

$$a_i^{1-\mu_i} \leq q_i, \qquad q_i \geq 0 \qquad i \in V$$

are introduced. These constraints can then be transformed into hyperbolic inequalities and then rewritten as second-order cone constraints. When $\mu_i = 3.5$, for example, additional variables $u_{i1}, u_{i2} \geq 0$ and the additional constraints

$$\|(2u_{i1}, a_i - 1)\| \leq a_i + 1, \qquad \|(2u_{i2}, u_{i1} - q_i)\| \leq u_{i1} + q_i, \qquad \|(2, a_i - u_{i2})\| \leq a_i + u_{i2}$$

can be introduced. Similar transformations for $\mu_i = 4$ and 4.5 are given in [55].

The second objective function is handled by introducing auxiliary variables

$t \in \mathcal{R}^{|V|}$, rewriting it as

$$\sum_{i \in V} t_i,$$

and introducing the linear constraints

$$y_i + h_i - d_i \leq t_i, \qquad t_i \geq 0, \qquad i \in V.$$

With these transformations, the resulting problem is an MISOCP. Numerical results are conducted for instances up to 28 vessels using CPLEX, which has runtime and memory problems as the problem size grows.

2.3. Optimization Problems Arising in Portfolio Selection

2.3.1. Markowitz Mean-Variance Framework

In the next two chapters, we extend the classical portfolio selection model developed by [103]. There are multiple reformulations of this model, in Chapter 3 we will focus on the case 1.2 where the investor's objective function is to choose the trading strategy to maximize expected return subject to constraints on the maximum risk that the investor may be willing to take on, and in Chapter 4 we will focus on the formulation that the investor's objective function is to choose the trading strategy to maximize expected return which is penalized variance-

covariance (market volatility) matrix:

$$\begin{aligned} \max_{w} \quad & r^T w - \lambda w^T Q w \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & x \geq 0 \end{aligned} \tag{2.19}$$

where $Q_{ij} = Cov(R_i, R_j)$ is variance-covariance matrix of the vector of returns, R, λ is risk aversion parameter. w is a vector of portfolio weights, and short-sale is restricted with nonnegative weights in this model. We will consider the extended version of (2.19) in both single and multi-period frameworks in Chapter 3.

In Chapter 4, we will consider two competing descriptions of portfolio selection, the traditional mean variance efficient portfolio which is modeled as (1.2) versus a generalization allowing for decision makers to consider skewness in their asset allocation.

The classical mean-variance framework has been quite popular for portfolio optimization problems since the 1960s. In [120], Phelps discusses an individual's optimal consumption policy using dynamic programming that maximizes wealth under capital risk. In [112], Mossin provides optimal policies for both single- and multi-period portfolio selection problems using dynamic programming tools. Samuelson uses dynamic stochastic programming to find an optimal lifetime consumption and investment policy in [123]. In [62], Fama also shows optimal consumption policies for both single- and multi-period portfolio selection problems. Hakansson has significant contributions to multi-period mean-variance portfolio

selection literature with [74], [75], [76], [77], [78]. Multi-period mean-variance portfolio selection has also been studied in [60], [61], [64], [71], [94], [95], [96], [129], [140], and [143].

2.3.2. Single- and Multi-Period Portfolio Optimization with Cone Constraints and Discrete Decisions

Although there have been substantial developments in portfolio selection models during last six decades, there is still work to be done to incorporate more complex components into these models and solve them efficiently. In this study, we have chosen to incorporate transaction costs, conditional value-at-risk (CVar) constraints, diversification requirements by sectors, and buy-in-thresholds into our framework. The first two require the use of second-order cone constraints while the latter two are implemented using binary variables, resulting in an MISOCP. In addition, we further extend this model to the multiperiod case. These model components/features have been adapted from [1], [29], [67], [73], and [99] as follows:

- According to [109], transaction costs include a number of factors, such as price impacts of transactions, brokerage commissions, bid-ask spreads, and taxes. About two decades ago, transaction costs started to be taken into account by [48], [68], and [113] in portfolio optimization problems. In [141],

Yoshimoto models V-shaped transaction cost function in the mean-variance portfolio optimization framework. However there are a number of different ways to model transaction costs, including linear, piecewise linear and convex or concave nonlinear cost functions. These transaction costs have also been studied by [49], [56], [87], and [89] in a mean-variance framework.

We will use a quadratic cost function for the single-period model as proposed in [67]:

$$\frac{1}{2}u_t \Lambda u_t,$$

where $u_t = x_t - x_{t-1}$, and $\Lambda \in \mathcal{R}^{(n+1)\times(n+1)}$ is the trading cost matrix and is obtained as a positive multiple of the covariance matrix of the expected returns. Because of this connection to a covariance matrix, Λ is symmetric and positive definite. Note that both buy and sell transactions receive the same transaction cost.

The multi-period portfolio optimization problem is obtained using a binary scenario tree, and for this model we have to modify our transaction cost function because of the fact that this quadratic convex formulation cause non-convex algorithm, and for the multi-period model, we will use proportional transaction cost in our model as Gurpinar et.al. do in [73] to preserve convexity.

- We adopt our CVaR constraint from [99], where Lobo et.al. consider a

single-period portfolio selection problem with linear and fixed transaction costs. They introduce a shortfall risk constraint in order to ensure that the terminal wealth is greater than a predetermined threshold level. They obtain second-order cone constraint from this formulation. We allow short-selling in our model as they did. They consider specific portfolio optimization model that is formulated as follows:

(2.20)

$$
\begin{aligned}
\max \quad & \bar{\alpha}^T(w + x^+ - x^-) \\
\text{s.t.} \quad & \mathbf{1}^T(x^+ - x^-) + \textstyle\sum_{i=1}^n (a_i^+ x_i^+ + a_i^- x_i^-) \leq 0 \\
& x_i^+ \geq 0, \quad x_i^- \geq 0, \quad i = 1, \ldots, n \\
& w_i + x_i^+ - x_i^+ \geq s_i, \quad i = 1, \ldots, n \\
& \Phi^{-1}(\eta_j) \|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\| \leq \bar{\alpha}^T(w_j + x_j^+ - x_j^-) - W_j^{low}, \quad j = 1, \ldots, M
\end{aligned}
$$

where $\bar{\alpha}^T \in \mathbf{R}^n$ is the vector of expected returns on each asset, $w \in \mathbf{R}^n$ is the vector of current holdings in each asset, and $x \in \mathbf{R}^n$ is the vector of amounts transacted in each asset and Φ is the transaction cost function. By using linear transaction costs, they obtain a convex optimization problem which can be solved by using the general purpose software *SOCP*. When they introduce fixed transaction costs into their framework, their model is no longer convex. They relax their transaction cost constraint in order to obtain a convex problem and solve the relaxed problem by using branch and bound method. Their second approach is to provide an iterative heuristic. They

obtain a suboptimal solution with this method by solving a small number of convex optimization problems. They show that there is a small gap between the suboptimal heuristic solution and the guaranteed upper bound with computational experiments. They suggest that these two methods can be incorporated for further accuracy levels.

- In [29], Bonami and Lejeune study a single-period portfolio optimization problem under stochastic and integer constraints as an extension of the classical mean-variance portfolio optimization framework. First they introduce a probabilistic portfolio optimization model where expected asset returns are stochastic and then they obtain their deterministic equivalents of these models to test different probability distributions that can/can't provide an exact or approximate closed-form solution. They focus on different types of constraints that traders should take into account when they are constructing their portfolio, such as diversification by sectors, buy-in-threshold and round lot constraints. The probabilistic Markowitz model with diversification-by-

sectors constraint is formulated as follows:

$$
\begin{aligned}
\min \quad & w^T \Sigma w \\
\text{s.t.} \quad & \mu^T w + F^{-1}(1-p)\sqrt{w^T \Sigma w} \geq R \\
& w_0 + \textstyle\sum_{j=1}^{r} w_j = 1 \\
& s_{min}\zeta_k \leq \textstyle\sum_{j \in S_k} w_k \leq s_{min} + (1 - s_{min})\zeta_k, \quad k = 1, \ldots, L \\
& \textstyle\sum_{k=1}^{L} \zeta_k \geq L_{min} \\
& \zeta \in \{0,1\}^L \\
& w \in \mathcal{Z}_+^{r+1}
\end{aligned}
\tag{2.21}
$$

where F^{-1} is the inverse cumulative probability distribution of the portfolio returns, L_{min} is different economic sectors, and s_{min} is the pre-determined minimum value of investment level.

The probabilistic Markowitz model with buy-in-threshold constraint is formulated as follows:

$$
\begin{aligned}
\min \quad & w^T \Sigma w \\
\text{s.t.} \quad & \mu^T w + F^{-1}(1-p)\sqrt{w^T \Sigma w} \geq R \\
& w_0 + \textstyle\sum_{j=1}^{r} w_j = 1 \\
& w_{min} \leq \delta_j, \quad j = 1, \ldots, r \\
& w_{min}\delta_j \leq w_j, \quad j = 1, \ldots, r \\
& \delta \in \{0,1\}^r \\
& w \in \mathcal{R}_+^{r+1}
\end{aligned}
\tag{2.22}
$$

These constraints are studied by [85] in absence of uncertainty about a decade ago. These sets of constraints provide binary and integer variables. They use branch and bound algorithm with two new proposed branching rules: *Idiosyncratic risk* and *portfolio risk* branching. Numerical results are presented up to 200 assets with comparing standard MINLP solvers and [28]. They suggest that the portfolio risk branching rule performs best in terms of robustness and speed. We adopt these formulations into our framework.

- In [73], Gurpinar et.al. introduce multi-period stochastic mean-variance portfolio optimization problem. They include proportional transaction costs in their model. The stochastic data is obtained by a scenario tree. They obtain multistage stochastic quadratic programming model which is solved

by *foliage*, is a financial software package coded in $C++$.

$$
\begin{aligned}
\max_{w,b,s} \quad & \textstyle\sum_{t=1}^{T} \alpha_t \sum_{e\in\mathcal{N}_t} P_e[(w_{a(e)} - \bar{w}_{a(e)})'(\Lambda_e + \hat{r}_e\hat{r}_e^{'})(w_{a(e)} - \bar{w}_{a(e)})] \\
\text{s.t.} \quad & p + (1-c_b)b_0 - (1+c_s)s_0 = w_0 \\
& 1'b_0 - 1's_0 = 1 - 1'p \\
& \hat{r}_e \circ w_{a(e)} + (1-c_b)\circ b_e - (1+c_s)\circ s_e = w_e, \quad e \in \mathcal{N}_I \\
& 1'b_e - 1's_e = 0, \quad e \in \mathcal{N}_I \\
& \textstyle\sum_{e\in\mathcal{N}_T} P_e[\hat{r}_e^{'}(w_{a(e)} - \bar{w}_{a(e)})] \geq \mathcal{W} \\
& w_e^L \leq w_e \leq w_e^U, \quad e \in \mathcal{N} \\
& 0 \leq b_e \leq b_e^U, \quad e \in \mathcal{N}_I \cup 0 \\
& 0 \leq s_e \leq s_e^U, \quad e \in \mathcal{N}_I \cup 0
\end{aligned}
\tag{2.23}
$$

They test their model with WATSON dataset. Besides that they provide computational backtesting experiments by using historical stock prices.

Although we are inspired from [67], [99], [29], and [73] when we build model, we take forward all these studies in terms of comprehensiveness and complexity. We consider real world portfolio constraints such as diversification by sectors, buy-in thresholds constraints and total transaction costs. Although these constraints provide meaningful financial interpretations to the portfolio, they result in a much more complicated model. The overall model is a mixed-integer second-order cone programming problem, a relatively new area of research. We consider this model in both single and multi-period frameworks. We solve these model with a MATLAB based Mixed Integer Linear and Nonlinear Optimization ([16]) solver that

implements a variety of methods for handling integer variables, cone constraints, linear and nonlinear sub-problems. We have devised and implemented a solution method for such problems and demonstrate its efficiency on large-scale portfolio optimization models. We provide substantial improvement with warm-starting in both branch-and-bound and outer approximation algorithms in terms of number of iterations. I will discuss this study in Chapter 3.

2.3.3. Revealed Preferences for Portfolio Selection - Does Skewness Matter?

Bilevel programming problem (BLPP) is a hierarchical optimization problem where the constraint of an upper level optimization problem, is also an optimization problem. In this framework, there are two independent decision makers, leader and follower, who want to optimize their objectives. The leader moves first and optimize her objective function with solving upper level optimization problem, and the follower observes the leader's action and she moves sequentially to optimize her objective function with solving lower level optimization problem with given parameters of the upper level optimization problem. Therefore, this framework is very similar with the Stackelberg leadership model which was proposed in [138] by Von Stackelberg. In this model, leader firm, moves first and chooses quantity where follower firm observes the leader's action and chooses her quantity

which maximizes her profit. When both of the firms choose their quantities, then market clearing price is set.

The general BLPP is formulated as follows:

$$
\begin{aligned}
\min_{x \in X} \quad & F(x, y) \\
\text{s.t.} \quad & G(x, y) \leq 0 \\
& H(x, y) = 0 \\
\min_{y \in Y} \quad & f(x, y) \\
\text{s.t.} \quad & g(x, y) \leq 0 \\
& h(x, y) = 0 \\
& x, y \geq 0
\end{aligned}
$$

In this framework the leader moves first, to choose the optimal $\mathbf{x}$ vector to optimize her objective function $F(x, y)$. The follower observes this action and moves sequentially to choose the optimal $\mathbf{y}$ vector to optimize her objective function $f(x, y)$.

Although the BLPP model first proposed by Bracken and McGill in [30], in [36] Chandler and Norton first mentioned the term of "multilevel". Last five decades, the BLPP models have been used to describe varies application problems in the literature:

Agriculture: In [35], Chandler et al. study the potential role of multilevel pro-

gramming in agricultural economics. In [116], Onal et al. show that availability of the agricultural subsidy provide an increase in both aggregate agricultural output and rural income by using bilevel programming model. In [107], Miljkovic uses BLPP formulation to show the the effects of privatization in YugoslaviaâŁ™s agricultural sector.

Economics: As we said before, bilevel programming problems subsume the Stackelberg duopoly model as discusses in [63]. In [125], Sherali et al. study the existence and uniqueness of a Stackelberg-Nash-Cournot equilibrium in an oligopoly model by using bilevel programming model. In addition bilevel programming problems also study the principal-agent problem. In [134], Ackere studies these problems in this context by analyzing a batch-size problem.

Engineering: In [44], Clark and Westerberg study BLPP for steady-state chemical process design with thermodynamic equilibrium. BLP is also studied in bioengineering. In [33], Burgard et al. use BLP framework to identify gene knockout strategies for microbial strain optimization.

Government Policy: In [37], Cassidy et al. study the distribution of government resources in this framework.

Management: In [10], Bard discusses coordination of decentralized organization

by using BLPP model. In [108], Miller et al. study facility location problem under delivered pricing strategy. In [47], CÃ´tÃ© study airline revenue management problem that solves the capacity allocation and pricing sub-problems.

Transportation: BLPs are heavily used by [11], [43], [93], and [102] for network design problems.

Please see [45], [51], [106], and [136] for more comprehensive literature review and varies applications of BLPP models.

Although BLPP models are widely used in varies application as seen above, there is only a couple of literature that related portfolio optimization. In [46], Conn and Vicente study BLPP when both upper and lower level objective function don't have available derivative. They apply their derivative-free bilevel method (Algorithm 5.2) to the robust optimization of the Omega function that is the ratio of the weighted gains over the weighted losses. In the second study [97], Liou and Yao introduce BLPP model for the mean-variance portfolio optimization problem. They obtain an unique results for its MPEC (Mathematical Programming with Equilibrium Constraints) problem formulation.

2.4. Optimization Problems Arising in Supply Chain Management

Over the past several decades, there have been a variety of papers published on the ELSP. Earliest works in this area include [58], [79], [122] and [105]. In these studies, the lower bound (LB) for the ELSP solution was calculated using an independent solution methodology, which ignored the sharing constraint and the machine capacity issues. An improved LB approach was developed by [27], in which the Karush-Kuhn-Tucker conditions were applied to the ELSP to account for only the capacity constraint. Several researchers have utilized this LB for comparative purposes ([111]). The bulk of the ELSP research has focused on cyclic schedules which satisfy the Zero-Switch-Rule (ZSR), meaning an item is produced only if its inventory depletes to a zero level. Nevertheless here are some cases, such as [50] and, [105], where the ZSR was not considered.

As noted earlier, there are three approaches to solve the ELSP. The CC approach provides an upper bound to the optimal solution and yields very good results under certain conditions ([65] and, [86]). The various heuristic methods developed using the BP approach first selects a frequency for each item (i.e., the number of times an item is produced in a production cycle). After the frequency is determined, a basic time period to satisfy this frequency is then determined. Earlier efforts along these lines include [27], [53]. In [59], Elmaghraby provided a

comprehensive review of this research. In [84], Hsu showed that using the basic period approach to solve the ELSP is NP-hard and the NP-hardness increased with an increase in the facility utilization ratio. Unlike the basic period approach, the time-varying lot size approach does not require equal production runs. This lot sizing approach was first examined by [105]. Subsequently, in [50], Delporte and Thomas, and in [52], Dobson developed efficient heuristic techniques to show that given enough time for production and set-up, any production sequence could be converted into a feasible production schedule, although the timings and lot sizes may not be equal. In [66], Gallego and Shaw provided support that the time-varying lot size approach to the ELSP was generally NP-hard. More recent explorations in this area has shown that Dobson's heuristic in [52] can be integrated with Zipkin's optimum-seeking algorithm in [144], in order to generate near optimal schedules in an efficient manner in [111].

In today's competitive business environment, customers require dependable on-time delivery at minimum cost from their suppliers and the ELSP can play a key role in coordinating all the necessary activities of the various participants at each stage of a supply network. However, as pointed out earlier, one characteristic of much of the research on ELSP is that the finished products are consumed at continuous rates. This implies that retail market demands for these products are satisfied directly from the manufacturing facility. In today's supply chains, however, employing complex distribution networks, involving production plants,

vehicle terminals, airline hubs, warehouses, distribution centers, retail outlets, etc., finished goods inventories from manufacturing plants are usually shipped in bulk to succeeding stages along the distribution process. Moreover, existing transport economies often tend to favor full truckload, rather that partial or less than truckload shipments, in discrete, sizeable lots, for efficient movement of such goods. Thus, it becomes necessary to re-examine the ELSP, with a focus on coordination and integration of the production schedule of a manufacturing process with the transportation function, towards achieving greater supply chain efficiency.

In a review paper about the integrated analysis of production, distribution and inventory planning, in [25], Bhatnagar et al. address the issue of coordination of activities in organizations. Two levels of coordination are discussed, i.e. coordination of inter-organizational functions and coordination within the same function at different echelons of an organization. In [40], Chandra and Fisher align the production scheduling with the vehicle routing problem for examining the value of coordination between these functions, employing a simulation study of a two-echelon supply chain and a with one manufacturing plant and several retailers.

In [26], Blumenfeld et al. and in [13], Benjamin consider multiple locations within an echelon for the integrated analysis of production, inventory and transportation decisions, under deterministic conditions. Blumenfeld et al. investigate the trade-offs between transportation, inventory holding and production setup

costs in a supply chain. These authors analyze the cases of direct shipping between nodes, shipping through a terminal and a combination of both, and obtained shipment sizes that consider the trade-offs between these costs. They are not concerned with the capacity and the number of vehicles, but focus on obtaining the value of the shipment size that trades off the respective costs. In [13], Benjamin considers the simultaneous optimization of the production lot size, the transportation decision and the economic order quantity. He accounts for supply constraints and explicitly considers inventory costs; his emphasis is on finding optimal production batch sizes for supply points and order quantities for demand points. Our analysis assumes an unconstrained transportation system and direct shipments between nodes. Therefore, no routing issues were considered. Although this work considers multiple products, no product to truck allocation decisions are made.

Generally speaking, coordination in production, inventory, and delivery has been well addressed in the recent literature. There are a number of models on integrated production, inventory and delivery decisions. In [142], Jonrinaldi and Zhang proposed an integrated production and inventory model in an entire supply chain system, which consists of several raw materials and parts suppliers, a manufacturer, multiple distributors and retailers. They propose a methodology for determining integrated production and inventory cycles for multiple raw materials, parts, and products in a supply chain involving reverse logistics concepts. A significant amount of recent research has focused on the area of linking produc-

tion scheduling and delivery activities, under various assumptions and objectives. In [131], Lee and Yoon propose a coordinated production scheduling and delivery batching model where different inventory holding costs are considered between production and delivery stage. In [90], Georgios et al. formulate a mixed integer programming (MIP) model pertaining to the simultaneous food processing and logistics planning problem for multiple products at various sites. In addition to finding a feasible and optimal schedule, his proposed model help all the participants in a supply chain collaborative process for obtaining the best balance between production, inventory level, and distribution efficiency. Also, in a recent study, in [98] Liu et al develop a multi-objective mixed-integer linear programming (MILP) model with the minimization of total cost, total flow time, and total lost sales as the objectives towards making optimal decisions with respect to production, distribution, and capacity expansion.

Part I

Optimization Problems Arising in Portfolio Optimization Models

Chapter 3

Single- and Multi-Period Portfolio Optimization with Cone Constraints and Discrete Decisions

3.1. Introduction

In this study, we extend the classical portfolio selection model developed by [103]. Although there are multiple reformulations of this model, we will focus only on the case 1.2 where the investor's objective function is to choose the trading strategy to maximize expected return subject to constraints on the maximum risk that the investor may be willing to take on.

In this chapter we focus on comprehensive MISOCP models that contain all

components/features proposed in [1], [29], [67], [73], and [99] that fit into our framework. Our goal is to study how such models can be solved efficiently by exploiting existing methods for MINLP and specialized approaches to solve the underlying SOCPs.

The remainder of this chapter is organized as follows: The next section of this chapter presents the single period portfolio optimization model. We discuss multi-period portfolio optimization model and its formulation in depth in Section 3.3. In Section 3.4, we propose two new algorithms for MISOCP, based on popular algorithms for mixed-integer nonlinear programming (MINLP): a branch-and-bound method and an outer approximation method. Both algorithms use a version of the primal-dual penalty interior-point method proposed in [20] and [21] for solving the underlying SOCPs, which allows us to perform warmstarts and detect infeasibilities in an efficient manner. In addition, we reformulate the second-order cone constraint as discussed in [23] in order to convert the underlying SOCPs into smooth convex nonlinear programming problems (NLP). Numerical testing for these portfolio optimization problems have been conducted using the Matlab-based solver MILANO [16] and are documented in Section 3.5. We will conclude and discuss some future directions of this study in Chapter 6.

3.2. Single Period Portfolio Optimization Model

The single-period portfolio optimization model considered in this chapter can be formulated as

(3.1)

$$
\begin{aligned}
\max_{x^+,x^-,\zeta} \quad & r^T(w + x^+ - x^-) - \frac{1}{2}(x^+ + x^-)^T \Lambda (x^+ + x^-) \\
\text{s.t.} \quad & \Phi^{-1}(\eta_k)\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\| \leq r^T(w + x^+ - x^-) - W_k^{low},\ k = 1, \ldots, M \\
& s_{min}\zeta_k \leq \sum_{j \in S_k} (w_j + x_j^+ - x_j^-) \leq s_{min} + (1 - s_{min})\zeta_k,\ k = 1, \ldots, L \\
& \sum_{k=1}^{L} \zeta_k \geq L_{min} \\
& \sum_{j=0}^{n} (w_j + x_j^+ - x_j^-) = 1 \\
& w_j + x_j^+ - x_j^- \geq -s_j, \qquad j = 1, \ldots, n \\
& x^+, x^- \geq 0 \\
& \zeta \in \{0, 1\}^L,
\end{aligned}
$$

where we consider cash (index 0) and n risky assets from L different sectors for inclusion in our portfolio. The decision variables are $x^+ \in \mathcal{R}^{n+1}$ and $x^- \in \mathcal{R}^{n+1}$, which denote the buy and sell transactions, respectively, and $\zeta \in \{0, 1\}^L$, the elements of which denote whether there are sufficient investments in each sector. We describe the remaining model components in detail below.

3.2.1. Objective Function

The investor's objective is to choose the optimal trading strategies to maximize the end-of-period expected total return. Denoting the expected rates of return by $r \in \mathcal{R}^{n+1}$ and the current portfolio holdings by $w \in \mathcal{R}^{n+1}$, the expected total portfolio value at the end of the period is given by

$$r^T(w + x^+ - x^-).$$

However, both the buy and sell transactions are penalized by transaction costs. According to recent dynamic portfolio choice literature ([67] and [32], for example), transaction costs include a number of factors, such as price impacts of transactions, brokerage commissions, bid-ask spreads, and taxes. As such, there are a number of different ways to model transaction costs, including linear and convex or concave nonlinear cost functions. In this work, we have decided to use the quadratic convex transaction cost formulation of [67] as it provides best fit to our framework. Therefore, the total transaction costs appear as a penalty term in the objective function:

$$\frac{1}{2}(x^+ + x^-)^T \Lambda (x^+ + x^-),$$

where $\Lambda \in \mathcal{R}^{(n+1)\times(n+1)}$ is the trading cost matrix and is obtained as a positive multiple of the covariance matrix of the expected returns. Because of this connection to a covariance matrix, Λ is symmetric and positive definite. Note that both buy and sell transactions receive the same transaction cost, but it would be

straightforward to instead include two quadratic terms in the objective function with different trading cost matrices for each type of transaction.

As we will see in the following discussion, the continuous relaxation of (3.1) includes only linear and second-order cone constraints. However, the quadratic term in the objective function prevents the overall problem from being formulated as an MISOCP. While we could simply classify the problem as a MINLP, we choose to instead reformulate it as an MISOCP so that the efficient algorithm we will describe in the next section can be applied, allowing us to take advantage of the special structure in the problem. We introduce a new variable $\rho \in \mathcal{R}$ and rewrite the objective function of (3.1) as

$$\sum_{j=0}^{n} r_j(w_j + x_j^+ - x_j^-) - \rho,$$

with

$$\frac{1}{2}(x^+ + x^-)^T \Lambda (x^+ + x^-) \leq 2\rho.$$

Note that this constraint is equivalent to

$$(x^+ + x^-)^T \Lambda (x^+ + x^-) \leq (1+\rho)^2 - (1-\rho)^2,$$

and moving the last term to the left-hand side and taking the square root of both sides gives the following second-order cone constraint:

$$\left\| \begin{pmatrix} \Lambda^{\frac{1}{2}}(x^+ + x^-) \\ 1-\rho \end{pmatrix} \right\| \leq 1+\rho.$$

$\Lambda^{\frac{1}{2}}$ exists since Λ is positive-definite. Additionally, this conversion does not increase the difficulty of solving the problem significantly—we add only one auxiliary variable, so the Newton system does not become significantly larger. Also, worsening the sparsity of the problem is not a concern here, since the original problem (3.1) has a quite dense matrix in the Newton system due to the covariance matrix and the related trading cost matrix both being dense.

3.2.2. Shortfall Risk Constraint

As stated above, we are considering both return and risk in this model. In the objective function, we focus on maximizing the expected total return less transaction costs, so we will seek to limit our risk using constraints. To that end, we will use Conditional Value-at-Risk (CVaR) constraints, as was done by Lobo et.al. in [99].

For each CVaR constraint k, $k = 1, \ldots, M$, we will require that our expected wealth at the end of the period be above some threshold level W_k^{low} with a probability of at least η_k. Thus, letting

$$W = \hat{r}^T(w + x^+ - x^-),$$

where $\hat{r}$ is the random vector of returns, we require that

$$\mathcal{P}(W \geq W_k^{low}) \geq \eta_k, \qquad k = 1, \ldots, M.$$

We assume that the elements of r have jointly Gaussian distribution so that W is normally distributed with a mean of

$$r^T(w + x^+ - x^-)$$

and a standard deviation of

$$\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|,$$

where Σ is the covariance matrix of the returns.

Therefore, the CVaR constraints can be formulated as

$$\mathcal{P}\left(\frac{W - r^T(w + x^+ - x^-)}{\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|} \geq \frac{W_k^{low} - r^T(w + x^+ - x^-)}{\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|}\right) \geq \eta_k,$$

for each $k = 1, \ldots, M$. This implies that

$$1 - \Phi\left(\frac{W_k^{low} - r^T(w + x^+ - x^-)}{\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|}\right) \geq \eta_k, \qquad k = 1, \ldots, M,$$

where Φ is the cumulative distribution function for a standard normal random variable, or

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

Rearranging the terms and taking the inverse gives us

$$\frac{W_k^{low} - r^T(w + x^+ - x^-)}{\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|} \leq \Phi^{-1}(1 - \eta_k), \qquad k = 1, \ldots, M.$$

Using the symmetry of the standard normal distribution function, we can rewrite the constraint again as

$$-\frac{W_k^{low} - r^T(w + x^+ - x^-)}{\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\|} \geq \Phi^{-1}(\eta_k), \qquad k = 1, \ldots, M.$$

Finally, rearranging the terms gives us the second-order cone constraint in (3.1)

$$\Phi^{-1}(\eta_k)\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\| \leq r^T(w + x^+ - x^-) - W_k^{low}, \qquad k = 1, \ldots, M.$$

3.2.3. Diversification By Sectors

Diversification is another important instrument used to reduce the level of risk in the portfolio. In this part, we impose a diversification requirement to the investor to allocate sufficiently large amounts in at least L_{min} of the L different economic sectors. This type of constraint was considered by [29].

To express this diversification requirement, we start by defining binary variables $\zeta_k \in \{0, 1\}$, $k = 1, \ldots, L$ for each economic sector k. If $\zeta_k = 1$, our total portfolio allocation in assets from sector k will be at least s_{min} (and, of course, no more than 1). Otherwise, it will mean that our total portfolio allocation in those assets fell short of the threshold level s_{min}. We can express these requirements with a constraint in the following form:

$$s_{min}\zeta_k \leq \sum_{j \in S_k} (w_j + x_j^+ - x_j^-) \leq s_{min} + (1 - s_{min})\zeta_k, \quad k = 1, \ldots, L,$$

where S_k is the set of assets that belong to economic sector k, $k = 1, \ldots, L$.

In order to express the diversification requirement, we also need to introduce a cardinality constraint:

$$\sum_{k=1}^{L} \zeta_k \geq L_{min}. \tag{3.2}$$

3.2.4. Portfolio Constraints

The remaining constraints in our problem are grouped into the general category of portfolio constraints. The first of these,

$$\sum_{j=0}^{n}(w_j + x_j^+ - x_j^-) = 1,$$

requires that we allocate 100% of our portfolio at the end of the investment period. Since we start with

$$\sum_{j=0}^{n} w_j = 1,$$

this constraint can also be written as

$$\sum_{j=0}^{n} x_j^+ = \sum_{j=0}^{n} x_j^-,$$

which provides a balance between the buy and sell transactions.

Additionally, we have another constraint that allows for shortsales of the nonliquid assets by stating that we can take a limited short position for each one:

$$w_j + x_j^+ - x_j^- \geq -s_j, \qquad j = 1, \ldots, n,$$

where s represents the short position limit for each nonliquid asset.

Finally, we require that x^+ and x^-, the variables associated with the buy and sell transactions must be nonnegative.

With the modifications to the model due to the transaction costs, the MISOCP

we will be solving in our numerical testing will have the form

(3.3)

$$
\begin{aligned}
\max_{x^+,x^-,\zeta,\rho} \quad & r^T(w + x^+ - x^-) - \rho \\
\text{s.t.} \quad & \left\| \begin{pmatrix} \Lambda^{\frac{1}{2}}(x^+ + x^-) \\ \\ 1 - \rho \end{pmatrix} \right\| \leq 1 + \rho \\
& \Phi^{-1}(\eta_k)\|\Sigma^{\frac{1}{2}}(w + x^+ - x^-)\| \leq r^T(w + x^+ - x^-) - W_k^{low}, \ k = 1, \ldots, M \\
& s_{min}\zeta_k \leq \sum_{j \in S_k} (w_j + x_j^+ - x_j^-) \leq s_{min} + (1 - s_{min})\zeta_k, \ k = 1, \ldots, L \\
& \sum_{k=1}^{L} \zeta_k \geq L_{min} \\
& \sum_{j=0}^{n} (w_j + x_j^+ - x_j^-) = 1 \\
& w_j + x_j^+ - x_j^- \geq -s_j, \qquad j = 1, \ldots, n \\
& x^+, x^- \geq 0 \\
& \zeta \in \{0, 1\}^L.
\end{aligned}
$$

As mentioned above, there are two additional types of constraints appearing in literature that we would like to include in future testing. The first of these, buy-in-threshold constraints, require additional binary variables and linear constraints, and therefore keep the problem as an MISOCP. We did not include them in this study since we already have an MISOCP and the particular data set we chose led to either infeasible or trivially solved problems when the buy-in-thresholds were added. The second type of constraint, round-lot constraints, could introduce nonlinear functions into our constraints, and we wanted to focus on MISOCPs

in this chapter and leave MINLPs with second-order cone constraints for future work. For completeness, however, we include a brief description of both of these types of constraints.

3.2.5. Buy-in-Threshold Constraints

Since we have included transaction costs in our model (3.3), we will be mindful of the number of transactions, as well. Therefore, we can impose a requirement that the investors do not hold very small active positions (see [29]). Introducing new binary variables $\delta \in \{0,1\}^n$, we can write this requirement using constraints of the following form:

$$w_{min}\delta_j \leq w_j + x_j^+ - x_j^- \leq \delta_j, \quad j = 1, \ldots, n,$$

where w_{min} is a predetermined proportion of the available capital.

3.2.6. Round-Lot Constraints

For certain types of investments, such as real estate, we might be required to hold an integer number of assets. Therefore, we could consider adding a constraint of the form

$$w_j + x_j^+ - x_j^- = \frac{p_j \gamma_j M_j}{\sum_{k=0}^{n} p_k (w_k + x_k^+ - x_k^-)} \qquad j = 1, \ldots, n,$$

where p_j is the face value of one unit of asset j, $\gamma_j \in \mathcal{Z}$ is the (nonnegative integer) decision variable denoting the number of assets held of that type, and M_j is the

batch size for the asset.

3.3. Multi-Period Portfolio Optimization Model

In this section we consider the multi-period portfolio optimization problem. There are multiple ways to formulate and solve multi-period portfolio optimization problems in literature, such as dynamic programming [38] and robust optimization [24]. In this study, we obtain the multi-period model by constructing a scenario tree. These model features were adapted from [73].

3.3.1. Scenario Tree

The use of scenario trees is not a new concept for multi-period financial/portfolio optimization problems. In multi-period portfolio allocation literature, the generation of scenario trees is discussed in [72] and [119], and scenario trees studied in [34] and [73]. We construct a binary scenario tree in Figure 1 to illustrate some of the important concepts and notation. There are discrete decision periods $t = 1, \ldots, T$ at which to reallocate the volumes of n risky assets and a riskless asset in the portfolio. $\mathcal{N}$ represents the set of all nodes in the scenario tree, $e \in \mathcal{N}$ represents the index of the event (s, t), the ordered pair of scenario s and time period t. The parent node of e in the scenario tree is denoted by $a(e)$. The branching probability is denoted by $\mathcal{P}_e = \Pi_{i=1,\ldots,t} p_{(s,i)}$.

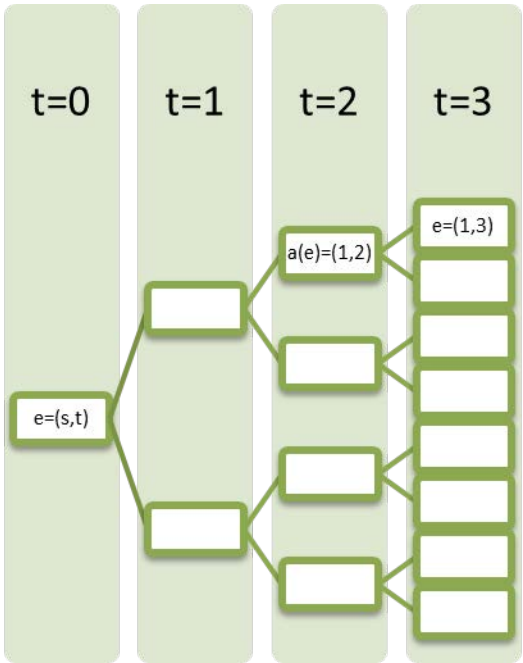

Figure 3.1: Scenario Tree

3.3.2. Objective Function

The investor's objective function is to choose the optimal trading strategies ($x_e = x_e^+ - x_e^-$), to maximize the-end-of period expected return. The expected rate of return is denoted by $r_e \in \mathbb{R}^{n+1}$ and the current portfolio holdings are denoted by $w_e \in \mathbb{R}^{n+1}$ for event e. The end-of-period expected return is formulated as:

$$
\begin{aligned}
\mathcal{W}_T &= \mathbb{E}[r_T(\xi^T)w_{T-1}] \\
&= \mathbb{E}[r_T(\xi_T|\xi^{T-1})w_{T-1}] \\
&= \mathbb{E}\left[\sum_{e\in\mathcal{N}_T} \mathcal{P}_e r_T^\top w_{a(e)}\right] \\
&= \sum_{e\in\mathcal{N}_T} \mathcal{P}_e \widehat{r}_e w_{a(e)}
\end{aligned}
$$

where ξ_t is the stochastic data at time t, ξ^t represents historic data up to time t and $\widehat{r}_e$ is the stochastic realization of $r_T(\xi_T|\xi^{T-1})$.

3.3.3. Transaction Costs Constraints

In a multi-period framework, we assume that transaction costs are paid on a period-by-period basis. As such, the payment of transaction costs needs to be incorporated into the flow balance constraints, which are modeled an equalities. If we were to keep the quadratic transaction costs of the single-period framework, the nonlinearities in the flow balance constraints would lead to the MINLP having nonconvex nonlinear relaxations. Therefore, we have decided to use linear transaction costs for the multi-period model and leave the quadratic case for future work on MINLPs. Such linear transaction costs accurately model brokerage commissions on transacted assets.

We impose transaction cost for both buying (c_b) and selling (c_s) strategies.

Therefore, we obtain following balance constraint:

$$w_e = \widehat{r}_e \circ w_{a(e)} + x_e^+ \circ (1 - c_b) - x_e^- \circ (1 + c_s), \quad \forall e \in \mathcal{N}_I$$

We also require that

$$1^\top x_e^+ = 1^\top x_e^- \quad \forall e \in \mathcal{N}_I$$

where $\mathcal{N}_I$ represents the set of all interior nodes of the scenario tree.

3.3.4. Shortfall Risk Constraints

To model shortfall risk, we follow the same procedure as in the single period model in Section 3.2.2. This constraint provides a requirement that the end-of-period wealth W stay above of some undesired level W^{low} with a probability greater than η. Therefore, we can formulate the end-of-period shortfall risk constraint using the following steps:

Letting

$$W = \sum_{e \in \mathcal{N}_T} \mathcal{P}_e \hat{r}_e w_{a(e)},$$

where $\hat{r}_e$ is the stochastic realization of $r_T(\xi_T|\xi^{T-1})$, we require that

$$W \geq W_k^{low} \geq \eta_k, \qquad k = 1, \ldots, M.$$

We assume that the elements of r have jointly Gaussian distribution so that W is normally distributed with a mean of

$$\sum_{e \in \mathcal{N}_T} \mathcal{P}_e \hat{r}_e w_{a(e)}$$

and a standard deviation of

$$\|\Sigma^{\frac{1}{2}} w_e\|,$$

where Σ is the covariance matrix of the returns. As shown in [29], we can expand this assumption to a more general class of probability distributions, including symmetric probability distributions and positively skewed probability distributions. In fact, [29] also shows that our proposed model can approximate an even greater class of distributions, which encompasses any distribution that can be characterized by its first two moments.

Therefore, the CVaR constraints can be formulated as

$$\mathcal{P}\left(\frac{W - \hat{r}_e w_{a(e)}}{\|\Sigma^{\frac{1}{2}} w_e\|} \geq \frac{W_k^{low} - \hat{r}_e w_{a(e)}}{\|\Sigma^{\frac{1}{2}} w_e\|}\right) \geq \eta_k,$$

for each $k = 1, \ldots, M$. This implies that

$$1 - \Phi\left(\frac{W_k^{low} - \hat{r}_e w_{a(e)}}{\|\Sigma^{\frac{1}{2}} w_e\|}\right) \geq \eta_k, \qquad k = 1, \ldots, M.$$

Rearranging the terms and taking the inverse gives us

$$\frac{W_k^{low} - \hat{r}_e w_{a(e)}}{\|\Sigma^{\frac{1}{2}} w_e\|} \leq \Phi^{-1}(1 - \eta_k), \qquad k = 1, \ldots, M.$$

Using the symmetry of the standard normal distribution function, we can rewrite the constraint again as

$$-\frac{W_k^{low} - \hat{r}_e w_{a(e)}}{\|\Sigma^{\frac{1}{2}} w_e\|} \geq \Phi^{-1}(\eta_k), \qquad k = 1, \ldots, M.$$

Finally, rearranging the terms gives us the second-order cone constraint in (3.1)

$$\Phi^{-1}(\eta_k)\|\Sigma^{\frac{1}{2}}w_e\| \leq \hat{r}_e w_{a(e)} - W_k^{low}, \qquad k = 1, \ldots, M.$$

In addition, we include the following to impose risk constraints for the interior branches of the tree:

$$w_e \geq 0.85 w_{a(e)} \qquad e = 1, \ldots, T-1$$

Using different constraints on wealth allows us to be less conservative in the interior branches, where we might take a slight loss in one period to realize bigger gains later.

3.3.5. Diversification By Sectors

We already discussed this constraint in Section 3.2.3 in the single period model. In this part, we just manipulate the constraints to satisfy the multi-period setting. Therefore we obtain the following set of constraints.

$$s_{min}\zeta_{ke} \leq \sum_{i \in S_k} w_e \leq s_{min} + (1 - s_{min})\zeta_{ke}, \qquad k = 1, ..., L$$
$$\sum_{k=1}^{L} \zeta_{ke} \geq L_{min}, \qquad \zeta \in \{0,1\}^{L \times \mathcal{N}}$$

3.3.6. Portfolio Constraints

We manipulate the portfolio constraints that are discussed in Section 3.2.4 to apply them in multi-period framework:

$$x_e^+ \geq 0, \qquad x_e^- \geq 0, \qquad w_e \geq -s, \qquad \forall e \in \mathcal{N}.$$

3.3.7. Buy-in-Threshold Constraints

In this section, we adopt the constraints introduced in Section 3.2.5 to the multi-period setting:

$$w_{min}\delta_e \leq w_e \leq \delta_e, \qquad \forall e \in \mathcal{N} \quad \text{and} \quad \delta \in \{0,1\}^{n\times\mathcal{N}}.$$

We obtain the following MISOCP prbolem for the multi-period portfolio optimization problem:

$$
\begin{aligned}
\max_{w_e, x_e^+, x_e^-, \zeta_e, \delta_e} \quad & \sum_{e \in \mathcal{N}_T} \mathcal{P}_e \widehat{r}_e w_{a(e)} \\
\text{s.t.} \quad & w_e = \widehat{r}_e \circ w_{a(e)} + x_e^+ \circ (1 - c_b) - x_e^- \circ (1 + c_s), \quad \forall e \in N_I \\
& 1^\top x_e^+ = 1^\top x_e^- \quad \forall e \in N_I \\
& \Phi^{-1}(\eta_j) \|\Sigma^{\frac{1}{2}} w_e\| \leq \widehat{r}_e w_{a(e)} - W_j^{low}, \quad j = 1, \ldots, m, \quad \forall e \in N_T \\
& w_e \geq 0.85 w_{a(e)} \qquad \forall e \in N_I \\
& s_{min} \zeta_{ke} \leq \sum_{i \in S_k} w_e \leq s_{min} + (1 - s_{min}) \zeta_{ke}, \quad k = 1, ..., L, \quad \forall e \in \mathcal{N} \\
& \sum_{k=1}^{L} \zeta_{ke} \geq L_{min}, \qquad \forall e \in \mathcal{N} \\
& \zeta \in \{0, 1\}^{L \times \mathcal{N}} \\
& w_e \leq \delta_e, \quad \forall e \in \mathcal{N} \\
& w_{min} \delta_e \leq w_e, \qquad \forall e \in \mathcal{N} \\
& \delta \in \{0, 1\}^{n \times \mathcal{N}} \\
& x_e^+ \geq 0, \qquad \forall e \in \mathcal{N} \\
& x_e^- \geq 0, \qquad \forall e \in \mathcal{N} \\
& w_e \geq -s, \quad \forall e \in \mathcal{N}
\end{aligned}
$$

3.4. Solving the MISOCP

In this section, we will describe two MINLP approaches that we have adapted for MISOCP. As stated, there are three important issues to consider: nondifferentiability of the underlying SOCP, warmstarting when solving a sequence of SOCPs, and infeasibility detection. We will address the first using a smooth convex reformulation of the SOCP and the latter two using a primal-dual penalty interior-point method.

3.4.1. The Ratio Reformulation

In [23], Benson and Vanderbei investigated the nondifferentiability of an SOCP and proposed several reformulations of the second-order cone constraint to overcome this issue. Note that the nondifferentiability is only an issue if it occurs at the optimal solution. Since an initial solution can be randomized, especially when using an infeasible interior-point method to solve the SOCP, the probability of encountering a point of nondifferentiability is 0.

For a constraint of the form

$$\|u\| \leq t \tag{3.4}$$

where u is a vector and t is a scalar, Benson and Vanderbei proposed the following:

- Exponential reformulation: Replacing (3.4) with $e^{(u^T u - t^2)/2} \leq 1$ and $t \geq 0$

gives a smooth and convex reformulation of the problem, but numerical issues frequently arise due to the exponential.

- Smoothing by perturbation: Introducing a scalar variable v into the norm gives a constraint of the form $\sqrt{v^2 + u^T u} \leq t$, but in order for the formulation to be smooth, we need $v > 0$. This is ensured by setting $v \geq \epsilon$ for a small constant ϵ, usually taken around $10^{-6} - 10^{-4}$.

- Ratio reformulation: Replacing (3.4) with $\frac{u^T u}{t} \leq t$ and $t \geq 0$ yields a convex reformulation of the problem, but the constraint function may still not be smooth. Nevertheless, in many applications, such as the portfolio optimization problems to be studied in the next section, the right-hand side of the second-order cone constraint in (1.1) is either a scalar or bounded away from zero at the optimal solution.

While the exponential reformulation and smoothing by perturbation resolve the nondifferentiability issue for the general SOCP, the ratio reformulation will be our pick for this study since we will focus only on portfolio optimization problems. The numerical issues due to the exponential function were causing failures during our numerical studies, and the smallest lower bounds that would avoid numerical nondifferentiability were still too big for the scale of the numbers in the second-order cone constraints in our problems. Applying the ratio reformulation, we will

be solving the following MISOCP instead of (1.1)

$$
\begin{array}{llll}
\min\limits_{x\in\mathcal{X}} & c^T x & & \\
\text{s.t.} & \dfrac{(A_i x + b_i)^T (A_i x + b_i)}{a_{0i}^T x + b_{0i}} & \leq & a_{0i}^T x + b_{0i}, \qquad i = 1, \ldots, m \\
 & a_{0i}^T x + b_{0i} & \geq & 0, \qquad i = 1, \ldots, m.
\end{array} \tag{3.5}
$$

We picked the ratio reformulation in order to guarantee that the underlying SOCPs would be smooth. We will now examine this choice for the second-order cone constraints included in (3.3).

For the transaction cost constraints, the right-hand side term is $1 + \rho$. Since the total transaction cost paid will be 2ρ, we have that $\rho \geq 0$. Therefore, $1+\rho \geq 1$, and the right-hand side is bounded away from 0.

For the shortfall constraints, note that we start with $\sum_{j=0}^{n} w_j = 1$ and that, since we are focusing on shortfalls, $W_k^{low} < 1$. Also note that if we assume that our initial asset allocation satisfies the diversification by sector constraints, we can define a feasible solution that does not require us to buy or sell any assets. Our objective is to maximize our end-of-period expected total return, which means that we expect our optimal allocation to do at least as well as this feasible solution. Thus, we can guarantee that

$$\sum_{j=0}^{n} r_j (w_j + x_j^+ - x_j^-) \geq 1 > W_k^{low}, \qquad k = 1, \ldots, M,$$

which means that the right-hand side is bounded away from 0.

With these reformulations, the first two constraints in (3.3) can be rewritten

as

$$\begin{array}{rcl} \dfrac{(x^+ + x^-)^T \Lambda (x^+ + x^-) + (1-\rho)^2}{1+\rho} & \leq & 1+\rho \\ \dfrac{(\Phi^{-1}(\eta_k))^2 (w + x^+ - x^-)^T \Sigma (w + x^+ - x^-)}{r^T(w + x^+ - x^-) - W_k^{low}} \| & \leq & r^T(w + x^+ - x^-) - W_k^{low}, \quad k = 1, \ldots, M. \end{array}$$

3.4.2. The Primal and Dual Penalty Problems

In order to solve the SOCPs that will arise during the course of the branch-and-bound and the outer approximation methods, we will use the primal-dual penalty interior-point method that was introduced in [20] for linear programming and in [21] for nonlinear programming. This approach includes relaxation/penalty terms in both the primal and the dual problems, which imbues the algorithm with the ability to perform warmstarts and detect infeasibilities. The new terms do not change the structure of the problem, that is, we will still solve an SOCP and can continue to use a highly efficient interior-point method to do so. In addition, the relaxation scheme creates strict interiors for the feasible regions of both the primal and the dual problems, thereby providing a regularization and allowing for the solution of SOCPs that may not otherwise satisfy standard assumptions for the interior-point method to work.

Even though our approach is to solve the SOCP as a nonlinear programming problem, the particular relaxation/penalty scheme differs slightly from the one presented in [21]. If we were to follow the outline of the approach presented in

that paper, the relaxed SOCP constraint would have the form

$$\begin{aligned} \frac{(A_i x + b_i)^T (A_i x + b_i)}{a_{0i}^T x + b_{0i}} &\leq a_{0i}^T x + b_{0i} + \xi_i \\ a_{0i}^T x + b_{0i} + \rho_i &\geq 0 \\ \xi_i, \rho_i &\geq 0, \end{aligned}$$

where $\xi, \rho \in \mathcal{R}^m$ are the relaxation variables that would get penalized in the objective function. While this would provide a sufficient relaxation for our purposes, we have decided to use the following relaxation instead:

$$\begin{aligned} \frac{(A_i x + b_i)^T (A_i x + b_i)}{a_{0i}^T x + b_{0i} + \xi_i} &\leq a_{0i}^T x + b_{0i} + \xi_i \\ a_{0i}^T x + b_{0i} + \xi_i &\geq 0 \\ \xi_i &\geq 0, \end{aligned} \tag{3.6}$$

This form of the relaxation can be obtained in two different ways:

- If we apply the relaxation scheme from [21] to the second-order cone constraint in (1.1), we obtain

  $$\|A_i x + b_i\| \leq a_{0i}^T x + b_{0i} + \xi_i, \qquad \xi_i \geq 0.$$

  Note that we have a second-order cone and a linear constraint after the relaxation. If we apply the ratio reformulation now, we obtain (3.6).

- The ratio reformulation constraint in (3.5) can also be written as a semidefinite constraint of the form

  $$\begin{bmatrix} (a_{0i}^T x + b_{0i}) I & A_i x + b_i \\ (A_i x + b_i)^T & a_{0i}^T x + b_{0i} \end{bmatrix} \succeq 0,$$

where I is the $m_i \times m_i$ identity matrix. As outlined in [23], the semidefinite constraint is equivalent to the entries of the diagonal matrix D in the LDL^T factorization of the above matrix being nonnegative. Without permutation, we have that

$$D_{jj} = \begin{cases} a_{0i}^T x + b_{0i}, & j = 1 \dots m_i \\ a_{0i}^T x + b_{0i} - \dfrac{(A_i x + b_i)^T (A_i x + b_i)}{a_{0i}^T x + b_{0i}}, & j = m_i + 1, \end{cases}$$

so $D_{jj} \geq 0$ for $j = 1, \dots m_i + 1$ matches the constraints in (3.5). The semidefinite constraint can be relaxed by adding a positive definite diagonal matrix to the left-hand side:

$$\begin{bmatrix} (a_{0i}^T x + b_{0i}) I & A_i x + b_i \\ (A_i x + b_i)^T & a_{0i}^T x + b_{0i} \end{bmatrix} + \xi_i \hat{I} \succeq 0, \qquad \xi_i \geq 0,$$

where $\hat{I}$ is the $(m_i+1) \times (m_i+1)$ identity matrix. The first two inequalities in (3.6) correspond to nonnegativity requirements on the entries of the diagonal matrix in the LDL^T factorization of this matrix. The third inequality in (3.6) exactly matches the nonnegativity of ξ to ensure that this is indeed a relaxation.

One advantage of this relaxation formulation over the one presented in [20] and [21] is that we only use m relaxation variables instead of $2m$. Doing so means that we will have not only fewer variables but also fewer penalty parameters to control in the resulting primal-dual penalty problem.

Thus, the *primal penalty problem* can be formulated as

$$
\begin{array}{llll}
\min\limits_{x,\xi} & c^T x + d^T \xi & & \\
\text{s.t.} & \dfrac{(A_i x + b_i)^T (A_i x + b_i)}{a_{0i}^T x + b_{0i} + \xi_i} & \leq & a_{0i}^T x + b_{0i} + \xi_i, \qquad i = 1, \ldots, m \\
 & a_{0i}^T x + b_{0i} + \xi_i & \geq & 0, \qquad i = 1, \ldots, m \\
 & a_{0i}^T x + b_{0i} & \leq & u_i, \qquad i = 1, \ldots, m \\
 & \xi_i & \geq & 0, \qquad i = 1, \ldots, m,
\end{array}
\tag{3.7}
$$

where d and u are the strictly positive primal and dual penalty parameters, respectively. As discussed in [20] and [21], relaxing a constraint in the primal problem leads to the primal penalty parameter of the relaxation acting as an upper bound on the dual variables. In order to establish a similar relaxation on the dual side, we introduce an upper bound on the primal side, and, again, this upper bound ends up serving as the dual penalty parameter of the dual relaxation. In fact, the dual problem has the following form:

$$
\begin{array}{llll}
\max\limits_{y_0, y, \psi} & -\sum\limits_{i=1}^{m} (b_i^T y_i + b_{0i} y_{0i} + u_i \psi_i) & & \\
\text{s.t.} & \sum\limits_{i=1}^{m} (A_i^T y_i + a_{0i} y_{0i}) & = & c \\
 & y_0 + \psi & \leq & d \\
 & y_0 + \psi & \geq & 0 \\
 & \dfrac{y_i^T y_i}{y_{0i} + \psi_i} & \leq & y_{0i} + \psi_i, i = 1, \ldots, m \\
 & \psi & \geq & 0,
\end{array}
\tag{3.8}
$$

where $y_i \in \mathcal{R}^{m_i}$, $i = 1, \ldots, m$ and $y_0 \in \mathcal{R}^m$ are the dual variables and $\psi \in \mathcal{R}^m$

are the dual relaxation variables.

Note that for sufficiently large d and u, both (3.7) and (3.8) have strictly feasible interiors. For the primal problem, we can pick any x, set u to satisfy $a_{0i}^T x + b_{0i} < u_i$ for $i = 1, \ldots, m$, and we can let

$$\xi_i > \max\{0, -(a_{0i}^T x + b_{0i}), \|A_i x + b_i\| - (a_{0i}^T x + b_{0i})\}.$$

Similarly, for the dual problem, pick any y and set y_0 in order to satisfy the first constraint of (3.8). (Since we no longer require $y_0 \geq 0$, it is possible to do so.) Then, we can pick any

$$\psi_i > \max\{0, -y_{0i}, \|y_i\| - y_{0i}\}$$

and set $d_i > y_{0i} + \psi_i$ for $i = 1, \ldots, m$.

Having strictly feasible interiors for both the primal and the dual problems means that both (3.7) and (3.8) have optimal solutions, and there is no duality gap. Thus, the pair (3.7) and (3.8) satisfy the regularity assumptions of standard interior-point algorithms for both SOCP and general NLP ([2], [19]).

Nevertheless, even though (3.7) and (3.8) exhibit regularity, the original SOCP may not. In fact, as it quite often happens within a branch-and-bound framework, the original SOCP may not even be feasible. It is shown in [19] that a solution with a duality gap, if it exists, can be recovered as the penalty parameters (either the primal or the dual, while keeping the other fixed) tend to infinity. Similarly, it is well-known that the original objective function can be dropped and a *feasibility*

problem can be solved as needed. One advantage of having a relaxation/penalty scheme for both the primal and the dual problems is that a feasibility problem can be designed for either one, in order to detect primal or dual infeasibility for the original SOCP.

3.4.3. A Primal-Dual Penalty Interior-Point Method

Since we will solve the pair (3.7)-(3.8) as NLPs, we will now describe the application of a standard interior-point method to these problems. This method, along with approaches to manage the penalty parameters, has been discussed extensively in [21] for a general NLP, so we will only provide a brief outline here, adapted to the case of a reformulated SOCP. Since the relaxed constraint (3.6) looks slightly different than the relaxed constraint in [21], we will need to introduce the appropriate first-order conditions, but the general outline of the overall solution method will be the same.

We start by introducing some auxiliary variables that will help simplify our

formulation:

$$
\begin{array}{llcll}
\min\limits_{x,\xi,f,g} & c^T x + d^T \xi & & & \\
\text{s.t.} & f_i & = & A_i x + b_i, & i = 1, \ldots, m \\
 & g_i & = & a_{0i}^T x + b_{0i} + \xi_i, & i = 1, \ldots, m \\
 & g_i - \dfrac{f_i^T f_i}{g_i} & \geq & 0, & i = 1, \ldots, m \\
 & g_i & \geq & 0, & i = 1, \ldots, m \\
 & u_i - g_i + \xi_i & \geq & 0, & i = 1, \ldots, m \\
 & \xi_i & \geq & 0, & i = 1, \ldots, m,
\end{array}
\tag{3.9}
$$

where $f_i \in \mathcal{R}^{m_i}$ and $g_i \in \mathcal{R}$, $i = 1, \ldots, m$ are the auxiliary variables. Since the first two constraints that serve to introduce these variables are affine equality constraints, (3.9) remains a convex nonlinear programming problem.

Formulating the log-barrier problem for (3.9):

(3.10)

$$
\begin{array}{lll}
\min\limits_{x,\xi,f,g} & c^T x + d^T \xi - \mu \sum\limits_{i=1}^{m} \left(\log \left(g_i - \dfrac{f_i^T f_i}{g_i} \right) + \log g_i + \log(u_i - g_i + \xi_i) + \log \xi_i \right) & \\
\text{s.t.} & f_i = A_i x + b_i, & i = 1, \ldots, m \\
 & g_i = a_{0i}^T x + b_{0i} + \xi_i, & i = 1, \ldots, m,
\end{array}
$$

where $\mu > 0$ is the barrier parameter.

The first-order conditions for this problem are:

$$
\begin{aligned}
A_i x - f_i + b_i &= 0, \qquad i = 1, \ldots, m \\
a_{0i}^T x + \xi_i - g_i + b_{0i} &= 0, \qquad i = 1, \ldots, m \\
c - \sum_{i=1}^m A_i^T y_i - \sum_{i=1}^m a_{0i} y_{0i} &= 0 \\
\psi_i(u_i - g_i + \xi_i) &= \mu, \qquad i = 1, \ldots, m \\
\xi_i(d_i - y_{0i} - \psi_i) &= \mu, \qquad i = 1, \ldots, m \\
(y_{0i} + \psi_i)\left(g_i - \frac{f_i^T f_i}{g_i}\right) &= \mu, \qquad i = 1, \ldots, m \\
\frac{y_{0i} + \psi_i}{g_i} f_i + y_i &= 0, \qquad i = 1, \ldots, m.
\end{aligned}
\tag{3.11}
$$

Note that the last condition implies the second-order cone constraint in (3.8) since we would have that

$$y_i^T y_i = (y_{0i} + \psi_i)^2 \frac{f_i^T f_i}{g_i^2}$$

and $\frac{f_i^T f_i}{g_i^2} \leq 1$ in each iteration.

The first-order conditions are solved using Newton's Method while performing a linesearch to guarantee progress toward optimality and modifying the value of μ at each iteration (see [21] or [22] for details). Of course, we need to also control the penalty parameters to guarantee that we have found a solution for the original SOCP or provide a certificate of infeasibility. In [21], Benson and Shanno discuss two approaches, static and dynamic updating, to resolve this issue.

- For static updating, the values of d and u are kept constant, and the problem is solved to optimality. Then, if $\xi > 0$ (or $\psi > 0$) at the optimal solution, the

primal (or the dual) penalty parameters are increased and the new problem is solved. After a fixed number of updates are performed, the problem is declared a candidate for infeasibility. If another update is necessary, c is set to 0 before solving the system again to detect primal infeasibility (or b is set to 0 to detect dual infeasibility). If a feasible solution (for the original SOCP) is obtained at the end of this process, we return to solving (3.11) with higher values of the penalty parameters. Otherwise, we declare the problem to be infeasible.

- For dynamic updating, the progress of $g_i + \xi_i$ and $y_{0i} + \psi_i$ for $i = 1, \ldots, m$ toward their upper bounds of u_i and d_i, respectively, are monitored at each iteration. If any of them are too close to their upper bounds, those bounds are increased. If any single bound is increased more than a fixed number of times, we modify the corresponding problem as described in static updating to enter the infeasibility detection phase. Similarly, if a feasible solution is found, we return to solving the original problem. Otherwise, we declare the problem to be infeasible.

While the static update is rather straightforward, it may require the complete solution of multiple problems. Therefore, as was the case in [21], the dynamic updating approach is preferred here as well.

In addition to its warmstarting capabilities, the primal-dual penalty approach

also allows us to (approximately) solve SOCPs that have duality gaps at the optimal pair of primal-dual solutions. This asymptotic behavior of the relaxed problem is analyzed in [19].

3.4.4. Warmstarting

Most successful implementations for mixed-integer linear programming either use a simplex-type method to solve the underlying linear programming problems, or they use a crossover approach which starts simplex iterations and crosses over to an interior-point method as needed. This is due to the fact that a simplex-type method (or an active-set approach in nonlinear programming) is quite easy to restart from a previous solution. In contrast, starting an interior-point method from the optimal solution of another problem causes issues due to at least one of a complementary pair of primal-dual variables already being at its bound. A thorough analysis of the numerical difficulties is presented in [20] and [21] for general linear and nonlinear programming warmstarts, respectively, and in [17] and [18] for warmstarts within branch-and-bound and outer approximation frameworks, respectively, for mixed-integer nonlinear programming. In all instances, it is shown that a standard interior-point method, applied directly to the original problem, will not only fail to warmstart but fatally stall if initialized from the optimal solution of a previously solved problem.

As pointed out in these papers, the primal-dual penalty approach serves as a remedy to the stalling issue by un-stalling the iterates and even improves on the iteration count over a coldstart. This is attained by keeping the optimal values for the primal-dual variables x, g, y, and y_0, but slightly perturbing the primal-dual relaxation variables ξ and ψ away from 0 (and recomputing f. This perturbation can be quite small (10^{-4} usually suffices), since both ξ and ψ are variables and their values can increase as needed. This framework avoids stalling by moving all the terms of the complementarity conditions in (3.11) away from 0, but still close to the central path for a small value of μ.

3.4.5. Handling the discrete variables

For our numerical experiments, we have implemented both a branch-and-bound method [91] and an outer approximation method [57] for a generic MINLP. Branch-and-bound conducts a search through a tree where each node is obtained by adding a bound to its parent to eliminate a noninteger solution and where each node requires the solution of a continuous NLP. Outer approximation alternates between the solution of an NLP obtained by fixing the integer variables and of an MILP obtained using linearizations of the objective function and the constraints at the solutions of the NLP. These methods and their use in conjunction with the primal-dual penalty interior-point method were analyzed in [17] and [18]. We refer the

reader to these papers for further details.

3.5. Numerical Results

3.5.1. Numerical Results for the Single Period Model

In our numerical testing, we consider one riskless and 20-400 risky assets for trading. The risky assets are chosen from the S&P500 list of companies in alphabetical order, and each stock is matched with its real world economic sector. The geometric mean and the covariance of the risky assets were calculated from the closing prices of the stocks in 2010. The riskless asset which refers to investment in the money market has a 1% return.

As we discussed before, we follow both [99] and [29] formulation in our framework. Therefore, we generally use the same constraint parameters with these two studies for consistency.

Initial weights for the stocks $w_j = 1/(n+1)$, $j = 0, \ldots, n$

Shortfall risk constraint parameters $\eta_1 = 95\%$, $W_1^{low} = 0.90$, $\eta_2 = 99.7\%$, $W_2^{low} = 0.95$

Diversification by sectors parameters $L_{min} = \lceil 0.5 \times L \rceil$, $s_{min} = 0.01$

Shortsale portfolio constraints $s_j = 0.5/n$, $j = 1, \ldots, n$, $s_0 = 0.5$

The problem instances are modeled using Matlab and solved using the Matlab-based solver MILANO ([16]) Version 1.4 which implements both branch-and-bound and outer approximation algorithms and uses the primal-dual penalty interior-point approach that allows warmstarting, as described in Section 4. The mixed-integer LPs arising in the outer approximation algorithm are solved using Gurobi [117]. Table 1 illustrates the result of the branch and bound algorithm while Table 2 presents the results of the outer approximation algorithm. The first column is the number of assets considered for the instance, the second column is number of different economic sectors, and the third column gives the number of CVaR constraints included in the model. The next four columns show the numbers of nodes and iterations that are required to solve the problem after either a coldstart or a warmstart. The last column represents the percentage improvement in the average number of iterations per node, as attained by warmstarting, and the numbers show that we obtain substantial improvements by using warmstarting for both the branch-and-bound and outer approximation algorithms.

3.5.2. Numerical Results for the Multi-Period Model

In our numerical testing, we consider 4-10 risky assets for trading. Each asset is randomly selected from the different economic sectors of S&P500 list of com-

Table 3.1: Results of the Branch-and-Bound Algorithm

			Coldstart		**Warmstart**		
n	L	M	Nodes	Iters	Nodes	Iters	% Impr
20	6	2	7	111	7	63	43.2
50	10	2	25	424	27	282	33.5
100	10	2	33	705	33	446	36.7
200	10	2	11	261	11	184	29.5
400	10	2	19	527	11	238	22.9

Table 3.2: Results of the Outer Approximation Algorithm for Single-Period

			Coldstart		**Warmstart**		
n	L	M	Nodes	Iters	Nodes	Iters	% Impr
20	6	2	2	36	2	31	13.9
50	10	2	3	65	3	52	20.0
100	10	2	3	89	3	60	32.3
200	10	2	2	57	3	69	27.7
400	10	2	2	69	2	53	23.2

panies. Therefore, each asset represents a different, real-world economic sector. The scenario tree is constructed using monthly returns of the closing price of the stocks from September 2005 to December 2010.

We use the same constraint parameters as the single period model for consistency.

Initial weights for the stocks $w_j = 1/(n+1)$, $j = 0, \ldots, n$

Shortfall risk constraint parameters $\eta_1 = 95\%$, $W_1^{low} = 0.90$, $\eta_2 = 99.7\%$, $W_2^{low} = 0.95$

Diversification by sectors parameters $L_{min} = \lceil 0.5 \times L \rceil$, $s_{min} = 0.01$

Shortsale portfolio constraints $s_j = 0.5/n$, $j = 1, \ldots, n$, $s_0 = 0.5$

The problem instances are modeled and solved as in the single-period case. Table 3 illustrates the results of the outer approximation algorithm for the multi-period model. The first column presents the different data sets which are denoted by TPNS where T represents the number of time period where N represents the number of stocks in the portfolio. The second column is the number of assets considered for the instance, the third column is number of different economic sectors, and the fourth column gives the number of CVaR constraints included in the model. The next three columns show the number of discrete variables (DV), the number of continuous variables (CV) and the number of second-order

cone constraint blocks respectively. The next four columns show the numbers of nodes and iterations that are required to solve the problem after either a coldstart or a warmstart. The last column represents the percentage improvement in the average number of iterations per node, as attained by warmstarting, and the numbers show that we obtain substantial improvements by using warmstarting for the multi-period model with outer approximation algorithm.

Table 3.3: Results of the Outer Approximation Algorithm for Multi-Period Model

							Coldstart		**Warmstart**		
Data	N	L	M	DV	CV	SOCC	Nodes	Iters	Nodes	Iters	% Impr
3P4S	4	4	2	112	168	16	2	45	2	41	8.9
3P6S	6	6	2	168	252	16	2	51	2	46	9.8
3P8S	8	8	2	224	336	16	2	59	2	53	10.2
3P10S	10	10	2	280	420	16	2	60	2	56	6.7
4P4S	4	4	2	240	360	32	2	70	2	66	5.7
4P6S	6	6	2	360	540	32	2	78	2	73	6.4
4P8S	8	8	2	480	720	32	2	82	2	69	15.9
4P10S	10	10	2	600	900	32	2	79	2	69	12.7
5P4S	4	4	2	496	744	64	2	72	2	64	11.1
5P6S	6	6	2	744	1116	64	2	84	2	72	14.3
5P8S	8	8	2	992	1488	64	2	97	2	79	18.6
5P10S	10	10	2	1240	1860	64	5	243	2	89	8.4

Chapter 4

Revealed Preferences for Portfolio Selection - Does Skewness Matter?

4.1. Introduction

We take a critical look at the paradigm of mean/variance efficient portfolios. Posing the portfolio selection problem as a decision problem we show how reasonable assumptions on utility and the probability model lead to asset allocation rules that depend on mean, variance and skewness of future returns. The main contribution of this article is the empirical validation of this argument by comparing it with traditional mean variance efficient portfolios. As with any decision problem the main challenge in setting up a fair comparison of alternative loss functions is the choice of an appropriate benchmark. Using any of the two competing loss

functions would unfairly bias the comparison in favor of the chosen loss, and the comparison is meaningless if the other loss function is a better representation of investor preferences. To enable a meaningful comparison we set up a revealed preference study. We develop a framework to explain observed investor preferences by the two alternative utility functions. Minimizing the discrepancy between the optimal decision under the considered utility functions and the observed data formalizes the comparison. In other words, we implement the inverse problem of expected utility maximization. Given observed decisions we back out inference about the underlying probability model and utility function, revealing the implied risk preference profile of the investor(s).

We cast the portfolio selection problem as a Bayesian decision problem. The elements of a decision problem are a probability model for the unknown future asset returns, a decision variable representing the portfolio choice as a vector of weights across a given set of assets, and a utility function that models preferences over consequences. In this context, it can be argued that a rational decision maker selects a portfolio by maximizing expected utility. The expectation is with respect to the probability model on the unknown future returns, conditional on all presently available information, i.e., the posterior predictive distribution. Assuming that given future returns an investor's utility is a quadratic function of the realized returns, it follows that the optimal asset allocation is determined by the first two moments only. Using a second order expansion, the argument remains

approximately valid for an arbitrary utility function. The apprxomation remains valid up to a third order expansion if the distribution of future returns follows a multivariate normal model.

The remainder of this chapter is organized as follows: In Section 4.2. we describe a probabilty model and a class of utility functions that lead to mean, variance, skewness efficient portfolios. Section 4.3 develops a framework for a comparison of alternative utility functions and probability models. In Section 4.4 we report the implementation and results of the proposed comparison with monthly data from the *Dow Jones Industrial Average* from August 2008 to January 2013. We will conclude and discuss some future directions of this study in Chapter 6.

4.2. Model

4.2.1. Mean Variance Efficient Portfolios

Markowitz (1952) proposed the idea of selecting portfolio weights based on the certainty equivalent framework using the mean and variance of historical returns. He stated that parameter uncertainty should be considered in the allocation problem, but did not actually address it. In this paper study we follow the implementation of Harvey *et al.* (2010) [80] to include parameter uncertainty in the two moment

portfolio problem. This is done by using Bayesian methods complete with drawing from posterior predictive distributions for the asset returns, and using summaries of these for estimates for the mean and variance. Specifically we implement the following setup.

We define the sampling distribution as

$$p_m : \quad x_t \overset{\text{iid}}{\sim} N(\mu, \Sigma), \tag{4.1}$$

for $t = 1 \ldots T$. The posterior predictive distribution as

$$p_m(x_s \mid x_1, \ldots, x_t), \text{for any future time } s > t.$$

The posterior predictive mean as

$$\bar{m}_m = E(x \mid x_1, \ldots, x_t),$$

where the expectation is with respect to p_m, and $x = x_{t+1}$ generically denotes a future observation. The utility function is

$$u_m(w, x) = w'x - \lambda_m[w'(x - \bar{m}_m)]^2.$$

Finally the expected utility to be maximized is

$$U_m(w) = w'\bar{m}_m - \lambda_m w'\bar{V}_m w,$$

where $\bar{V}_m$ is the predictive variance.

4.2.2. Asset Allocation with Higher Moments

Harvey *et al.* (2010) [80] propose a decision problem, i.e., a probability model and a utility function, to describe portfolio selection with higher order moments. The probability model is independent sampling from a multivariate skewed normal distribution. Utility is a third order polynomial of future returns. Details are described below.

Skew normal distributions provide a technically convenient generalization of normal models. Several multivariate versions of skew normals have been proposed in the literature, differing mainly in the number of parameters that are used to define skewness. Azzalini and Dalla Valle (1996) define a multivariate skew normal distribution by multiplying a multivariate normal density with a univariate normal c.d.f. This is generalized by Branco and Dey (2001) and Sahu *et al.* (2002) by replacing the univariate normal c.d.f. by a more flexible multivariate normal c.d.f. We choose the latter to define a probability model for asset returns.

A constructive definition of the multivariate skew normal is as a convolution of a multivariate normal and a linear function of a truncated multivariate normal. We say $X \sim SN(\mu, \Sigma, \Delta)$ if

$$X = \mu + \Delta Z + \epsilon \text{ with } \epsilon \sim N(0, \Sigma) \text{ and } Z \sim N(0, I),\ Z_i \geq 0. \tag{4.2}$$

The distribution is indexed by a location parameter μ, a scale matrix Σ and a (co-)skewness parameter Δ. Sahu *et al.* (2002) restrict Δ to a diagonal matrix. In

Harvey *et al.* (2010) [80] the model is generalized to unrestricted Δ to facilitate inference on co-skewness. Harvey *et al.* (2010) [80] discuss properties, convenient prior choices and details of posterior inference for model (4.2).

Let vec(A) denote a representation of an $(m \times n)$ matrix A as a $(mn \times 1)$ vector of the stacked columns. We assume a multivariate normal prior for μ and vec(Δ), and a Wishart prior for Σ^{-1}.

We now use the multivariate skew normal distribution to set up a description of portfolio selection as a decision problem. Let x_t denote the returns of the assets under consideration at time t. We assume

$$p_h : \quad x_t \overset{\text{iid}}{\sim} SN(\mu, \Sigma, \Delta), \tag{4.3}$$

$t = 1, \ldots, T$. To simplify notation in the following discussion we generically use $x = x_{t+1}$ for future returns in the posterior predictive distribution $p(x_{t+1} \mid x_1, \ldots, x_t)$. Let $\bar{m}_h = E(x \mid x_1, \ldots, x_t)$ denote the posterior predictive mean. Let $w = (w_1, \ldots, w_p)$ denote an investor's portfolio choice, with w_i being the relative weight of the i-th asset. We hypothesize that an investor's preferences can be described in terms of future reward $w'x$ and second and third moments:

$$u_h(w, x) = w'x - \lambda_h[w'(x - \bar{m})]^2 + \gamma_h[w'(x - \bar{m})]^3 \tag{4.4}$$

The function $u_h(w, x)$ is the reward for portfolio choice w under assumed future returns x. The scalars λ_h and γ_h are relative weights describing the investor's

risk averseness. Of course, at the time of the asset allocation decision future returns are unknown. It can be argued that a rational decision maker proceeds by maximizin expected utility, marginalizing x with respect to the posterior predictive distribituion $p(x \mid x_1, \ldots, x_t)$. Let $\bar{V}_h$ and $\bar{S}_h$ denote the predictive moments. Then

$$U_h(w) = \int u_h \, dp(x \mid x_1, \ldots, x_t) = w'\bar{m} - \lambda_h w'\bar{V}w + \gamma_h w'\bar{S}w \otimes w. \tag{4.5}$$

Optimal portfolio selection under the probability model (4.3) and utility (4.4) proceeds by maximizing $U(w)$ with respect to w.

4.3. Revealed Preferences

We have described two competing descriptions of portfolio selection, the traditional mean variance efficient portfolio and a generalization allowing for decision makers to consider skewness in their asset allocation. Both setups are formally coherent and justifyable as decision theoretically optimal actions. A critical comparison of the competing approaches is only possible by validating the models with observed investor behavior.

We consider a broad based panel of assets, chosen to allow a wide variety of portfolio choices. At each time t we record total shares outstanding of the stocks represented in the panel. The relative size $\widehat{w}_{ti}$ of the shares outstanding for the i-th asset in period t quantifies a typical investor's portfolio weight for asset i. We

refer to $\widehat{w}_t = (\widehat{w}_1, \ldots, \widehat{w}_p)$ as the observed portfolio weights. Using $\widehat{w}_t$ as observed data we proceed by finding in each period for both utility functions under consideration the optimal portfolio under the respective utility function that can best approximate the observed weights $\widehat{w}_t$. We denote with w_t^{h*} and w_t^{m*} the optimal portfolio under the higher order moment framework and under the mean variance efficient framework, respectively. The distances $d(w_t^{h*}, \widehat{w}_t)$ and $d(w_t^{m*}, \widehat{w}_t)$ evaluate the fit of the two utility functions to the observed data. Finally, summarizing the comparison over time provides the desired criterion to evaluate the relative merit of the two utility functions in the light of the market data. Details are given in the following algorithm.

In the following description we will use M and H to refer to the two decision models. Model M refers to the independent normal sampling model (4.1), together with utility function $u_m(w, x)$, and Model H refers to skew normal sampling (4.3), together with utility function $u_h(w, x)$.

Algorithm: Revealed Preferences in Asset Allocation

Repeat the follwoing steps 1. through 3. for $t = 1, \ldots, T$

1. *Posterior predictive inference.*

   Find the posterior predictive distributions under both models, $p_m(x \mid x_1, \ldots, x_t)$ and $p_h(x \mid x_1, \ldots, x_t)$, and evaluate the posterior predictive moments $(\bar{m}_m, \bar{V}_m)$

and $(\bar{m}_h, \bar{V}_h, \bar{S}_h)$.

2. Find w_t^{m*} and w_t^{h*}.

   2.1. *Optimal portfolio for given utility parameters.*

   Let $w_m(\lambda_m)$ denote the optimal portfolio under model M, using coefficient λ_m.

   Let $w_h(\lambda_h, \gamma_h)$ denote the optimal portfolio under model H, using coefficients (λ_h, γ_h).

   2.2. *Approximate the observed weights.*

   Find the utility parameters λ_m and (λ_h, γ_h) that best approximate the observed data: Let

$$\lambda_{mt} = \arg\min_{\lambda_m} d\left[\widehat{w}_t, w_m(\lambda_m)\right].$$

   and similarly for model H:

$$(\lambda_{ht}, \gamma_{ht}) = \arg\min_{\lambda_h, \gamma_h} d\left[\widehat{w}_t, w_h(\lambda, \gamma)\right].$$

   2.3. *Optimal portfolios to approximate data.*

   We define $w_t^{m*} = w_m(\lambda_{mt})$ and $w_t^{h*} = w_h(\lambda_{ht}, \gamma_{ht})$ as the optimal approximations to $\widehat{w}$. Using $d(v, w) = \sum(v_i - w_i)^2$ the portfolios w_t^{h*} and w_t^{m*} are least squares approximations to w_t^{h*}, under decision models H and M, respectively.

3. *Summarize the approximation residuals.*

   Plot $d(\widehat{w}_t, w_t^{m*})$ and $d(\widehat{w}_t, w_t^{h*})$ against t. The relative position of the two curves formalizes the comparison of the two decision models. If desired, a suitable summary statistic can serve as a single number comparison. For example, we could use $\sum_t d(\widehat{w}_t, w_t^{m*}) - d(\widehat{w}_t, w_t^{h*})$.

The described algorithm is highly computation intensive. At each time t we solve an optimization problem, minimizing the discrepancy between market and optimal portfolio. The minimization is with respect to the utility parameters λ_m and (λ_h, γ_h), respectively. Nested within this optimization is a second optimization problem. For each utility parameter λ_m (or (λ_h, γ_h)) under consideration we solve another minimization problem to find the optimal portfolio.

4.4. Results

We used daily returns on stock prices for the *Dow Jones Industrial Average* from August 2008 to January 2013, a total of 1075 data points as our historical data. Based on that data we sampled from the posterior predictive distribution for $T = 25$ additional steps in to the future.

We were able to show that the three moment decision model uniformly beat the two moment model in matching the observed portfolio, see Figure 4.1, and Table 4.1 for comparisons.

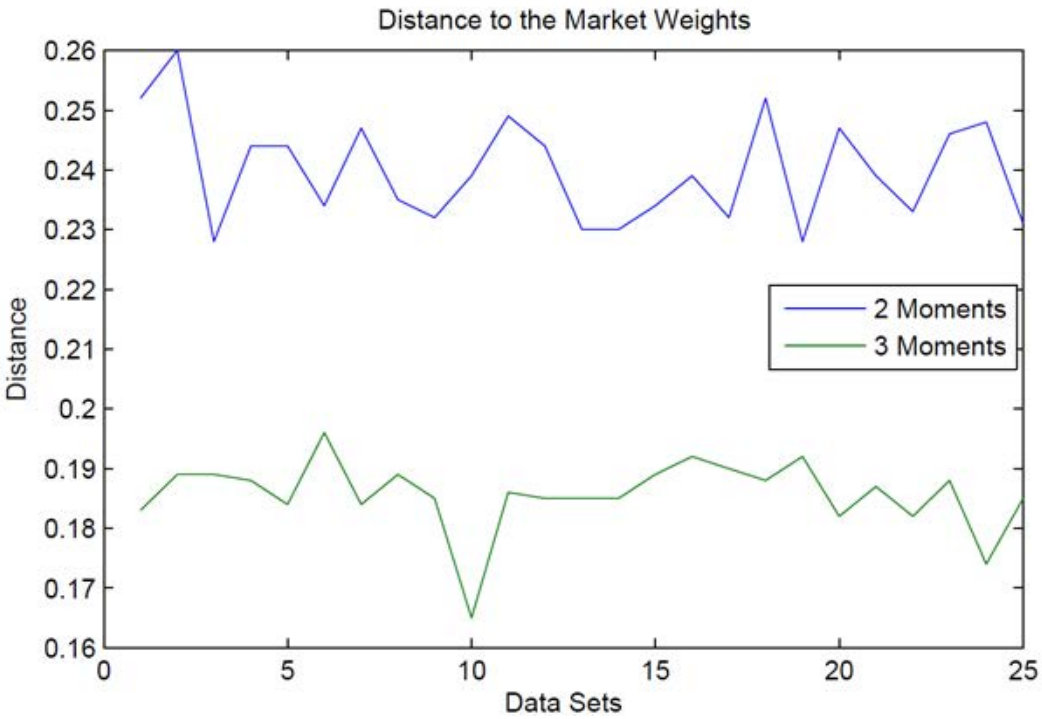

Figure 4.1: Distance to the Market Weights: Distance from observed weight to two and three moment weights. Three moment weights are always closer to the observed weights.

Table 4.1: Distance to the Market Weights: Distance from observed weight to two and three moment weights. Three moment weights are always closer to the observed weights.

Distance to the Market Weights		
Data Sets	2 Moments	3 Moments
1076	0.252	0.183
1077	0.260	0.189
1078	0.228	0.189
1079	0.244	0.188
1080	0.244	0.184
1081	0.234	0.196
1082	0.247	0.184
1083	0.235	0.189
1084	0.232	0.185
1085	0.239	0.165
1086	0.249	0.186
1087	0.244	0.185
1088	0.230	0.185
1089	0.230	0.185
1090	0.234	0.189
1091	0.239	0.192
1092	0.232	0.190
1093	0.252	0.188
1094	0.228	0.192
1095	0.247	0.182
1096	0.239	0.187
1097	0.233	0.182
1098	0.246	0.188
1099	0.248	0.174
1100	0.231	0.185

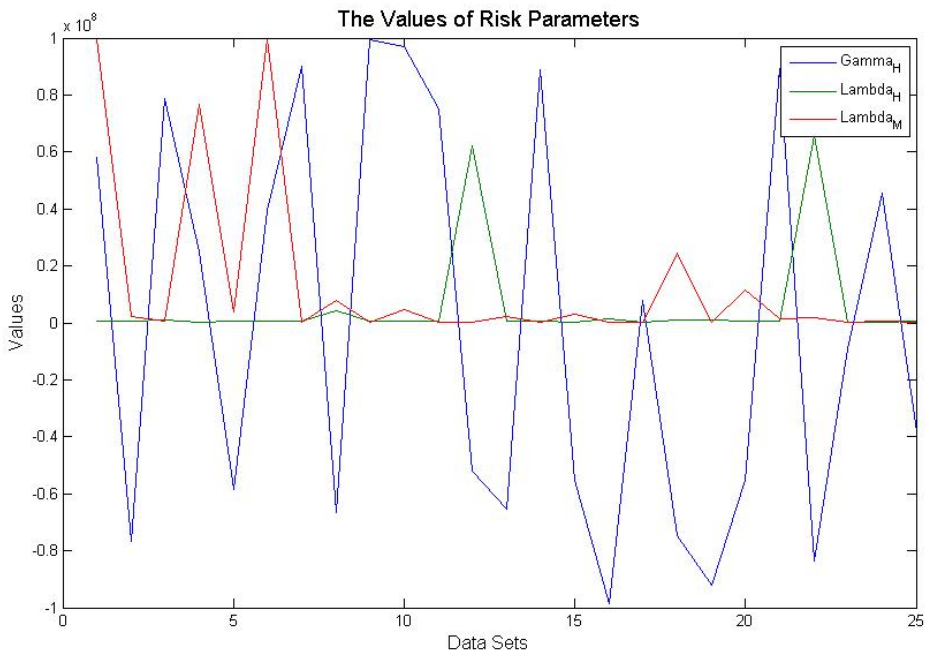

Figure 4.2: The Values of Risk Parameters: The large values for γ_h suggest that the typical investor has a strong preference for positive skewness.

Also of interest is the implied risk preferences of the market investor. We can see from Figure (4.2) and Table (4.2) that λ_m and (λ_h, γ_h) are quite substantial in magnitude, and quite unstable over time. The rather large values for γ_h suggest that the typical investor has a strong preference for positive skewness, which is consistent with economic theory (see Harvey & Siddique 2000 [81] who argue that investors are typically willing to trade expected return for positive skewness).

Table 4.2: Risk Parameters

	Risk Parameters		
	2 Moments	3 Moments	
Data Sets	Lambda	Lambda	Gamma
1076	1.00E+08	4.80E+05	5.79E+07
1077	2.09E+06	7.54E+05	-7.68E+07
1078	6.58E+05	9.37E+05	7.88E+07
1079	7.68E+07	1.91E+05	2.51E+07
1080	4.03E+06	5.56E+05	-5.85E+07
1081	1.00E+08	4.80E+05	3.96E+07
1082	1.27E+04	4.29E+05	9.01E+07
1083	7.83E+06	4.32E+06	-6.67E+07
1084	6.86E+04	5.77E+05	9.94E+07
1085	4.62E+06	6.02E+05	9.69E+07
1086	7.62E+03	6.02E+05	7.50E+07
1087	8.47E+02	6.23E+07	-5.19E+07
1088	2.12E+06	4.34E+05	-6.53E+07
1089	2.82E+02	4.80E+05	8.87E+07
1090	3.06E+06	2.36E+05	-5.53E+07
1091	3.14E+01	1.44E+06	-9.85E+07
1092	3.14E+01	2.06E+05	7.96E+06
1093	2.46E+07	1.03E+06	-7.46E+07
1094	3.14E+01	1.17E+06	-9.20E+07
1095	1.16E+07	6.02E+05	-5.52E+07
1096	1.23E+06	4.34E+05	8.93E+07
1097	1.84E+06	6.58E+07	-8.37E+07
1098	8.47E+02	2.06E+05	-9.14E+06
1099	7.41E+05	2.04E+05	4.57E+07
1100	8.47E+02	4.80E+05	-3.82E+07

Table 4.3: Current *Dow Jones 30* Stocks

Symbol	Name
AA	Alcoa Inc.
AXP	American Express Company
BA	The Boeing Company
BAC	Bank of America Corporation
CAT	Caterpillar Inc.
CSCO	Cisco Systems, Inc.
CVX	Chevron Corporation
DD	E. I. du Pont de Nemours and Company
DIS	The Walt Disney Company
GE	General Electric Company
HD	The Home Depot, Inc.
HPQ	Hewlett-Packard Company
IBM	International Business Machines Corporation
INTC	Intel Corporation
JNJ	Johnson & Johnson
JPM	JPMorgan Chase & Co.
KO	The Coca-Cola Company
MCD	McDonald's Corp.
MMM	3M Company
MRK	Merck & Co. Inc.
MSFT	Microsoft Corporation
PFE	Pfizer Inc.
PG	Procter & Gamble Co.
T	AT&T, Inc.
TRV	The Travelers Companies, Inc.
UNH	UnitedHealth Group Incorporated
UTX	United Technologies Corp.
VZ	Verizon Communications Inc.
WMT	Wal-Mart Stores Inc.
XOM	Exxon Mobil Corporation

Part II

Optimization Problems Arising in Supply Chain Management

Chapter 5

Multi-Product Batch Production and Truck Shipment Scheduling under Different Shipping Policies

5.1. Introduction

Over the past several years, with advances in the notions concerning efficient supply chains, relationships between customers, manufacturers and suppliers have undergone numerous notable changes by removing non-value added activities in procurement, production and distribution. These progressive paradigm changes tend to view individual decisions as parts of an integrated series of business activities that span across the entire supply chain. Today's supply chains are impacted

by increased complexity, unpredictable economic conditions, operational risks, environmental regulations, globalization and rising fuel costs. Historically, optimization projects within the supply chain have been cumbersome, time-consuming undertakings. Many companies find themselves in a constant struggle to maintain efficiency at every stage along the supply chain, attempting to reduce costs and increase productivity within their procurement-production-distribution networks, in the face of intense competitive pressures. In this context, holistic integration of decisions involving serial stages of activities has received attention from researchers in recent years.

This study focuses on a specific supply chain scenario, where a single manufacturing plant produces multiple products for satisfying customer demands that occur at several retail outlets. The production facility can produce only one product at a time, but shipments can be made either directly to each individual retailer via relatively small, less than truckload (LTL) quantities or via larger full truckload (TL) quantities, where deliveries are made to all the retailers according to a peddling arrangement. In the TL transportation mode, a full truckload represents the aggregate retail demand during a common delivery cycle. The required lot sizes are then dropped off at the respective retail locations from the same transport vehicle, which incurs a fixed shipping charge. In the case of LTL shipping, the delivery cycle times for the various products may be different, but any given item has the same inventory cycle time at all retail locations. The shipments

are made directly from the supplier to the various retailers individually and the respective shipping costs depend on the amount of load delivered, based on a variable transportation charge.

For either shipment policy, the transportation schedule is directly linked to the batch production schedule for the multiple items at the manufacturing facility. It is to be noted that the production batch sizing issue here is represented by the well-known economic lot scheduling problem (ELSP). Our analysis differs from existing work in this area in two important ways. First, the inventories of the different products are depleted at uniform market demand rates in the traditional treatment of the ELSP, whereas in this paper, we allow such depletions to occur in discrete lot sizes, depending on the transportation policy in effect. Secondly, we make an attempt to integrate the production plan with either the TL or LTL shipment schedule, as the case may be. It is well known that the ELSP addresses the lot sizing issue for several items with static and deterministic demands over an infinite planning horizon at a single facility. In this paper, we recast this problem in a way that ties the production and shipping schedules together with the objective of minimizing the sum of all the relevant costs, including setup and other fixed costs, as well as inventory holding and other variable costs, while satisfying the market demands for all products at the various retail locations. The solution involves determining a consistent and repetitive production schedule for all products to meet the necessary demands ([41]). Since the ELSP has been

shown to be NP-hard, the focus of most research efforts has been to generate near optimal cyclic schedules with three well known policies, viz. the common cycle, basic period (or multiple cycle) and time varying lot size approaches ([133]).

The common cycle (CC) approach always produces a feasible schedule and is the simplest to implement. However, in some cases, the CC solution, when compared to the lower bound (LB) solution, turns out to be of poor quality. Unlike the common cycle approach, the basic period (BP) approach allows different cycle times for different products, where the individual item cycle times are integer multiples of a basic period. Although this approach generally tends to yield better solutions to the ELSP than the CC methodology, obtaining a feasible schedule is NP-hard ([27]). Moreover, the computational effort required for implementing the BP solution is considerable greater compared to the CC solution. Finally, the time-varying lot size approach, being more ï¬,exible than the aforementioned procedures, allows for different lot sizes for the different products in a cycle. In [52], Dobson showed that the time-varying lot size technique always produces feasible schedules, while generating better quality solutions. Nevertheless, the computational burden associated with this procedure is significantly higher than those for the other two approaches. Thus, in order to keep the computational complexity to a minimum, as well for the sake of simplicity of implementation, we adopt the common cycle approach in our integrated analysis for addressing the production batching issue.

This chapter attempts to extend the classical ELSP model by incorporating the transportation decision, accounting for finished goods inventories in discrete, sizeable lots. It may be beneficial to deliver quantities of the various products, using either the full truckload (TL) or less than truckload (LTL) shipment policies. In the case of the TL policy, each truckload consists of a mix of all the items. As mentioned earlier, we also adopt, for simplicity, the common production cycle (CC) approach (see, for example, [105]), with a delivery cycle that is common to all the individual items. For coordination purposes, this delivery cycle is a multiple integer of the overall production cycle, also common to all items. Under the LTL shipping policy, the different items may have different delivery cycles, where individual products are shipped directly to the retailers. Nevertheless, for each product, each item's delivery cycle is an integer multiple of the overall production cycle, which is common to all the products.

This work extends the multiproduct model presented by Banerjee [8]. He formulated an analytical model to align the production schedule of multiple products with a full truckload delivery plan and develops a heuristic solution methodology. We propose a generalized mixed integer, non-linear mathematical programming model (MINLP) for developing a multi-product batch production schedule, which coordinates finished goods availabilities with their outbound TL or LTL shipment plans. The transportation cost for a TL shipment is a fixed cost, whereas LTL shipment costs are based on a variable shipping charge. Finally, the models and

the concepts regarding coordinated production and shipment decisions developed in this study are illustrated through numerical examples.

The remainder of this chapter is organized as follows: The next section of this chapter outlines the assumptions made and the notation used in our models and in Section 3, the proposed models are described in detail. A numerical example and some selected sensitivity analyses are presented in Section 4. Finally, Section 5 provides a summary and some concluding remarks.

5.2. Assumptions and notation

In this Section we present the assumptions that we need and the important notation that we use throughout this paper.

5.2.1. Assumptions

The following assumptions are made in describing the manufacturing-distribution scenario adopted in this paper and for formulating our models that follow:

1. Market demands for the various products are deterministic and stationary.

2. A set of products are manufactured in a single capacitated batch production facility, with different production rates for the various items.

3. Only a single product may be produced at any given time.

4. Stockouts are not permitted.

5. The common cycle (CC) approach is deployed to solve the ELSP, where each product is produced exactly once in every production cycle.

6. Each of the products is transported via truck and is delivered to one or more given demand locations, depending on one of two shipping policies in effect.

7. Under a full truckload (TL) shipping policy, a mix of all products, constituting a full load, is delivered to all retail locations on the basis of a peddling arrangement. The LTL transportation mode, on the other hand, implies direct shipment of each product to each retailer.

8. These two scenarios impose different transportaion costs. The TL mode involves a capacitated vehicle, incurring only a fixed cost for all the peddling shipments made in a single delivery, while for LTL shipments, each direct shipment cost is based on a load-based variable cost

9. Under the TL policy, an integer number, K, of deliveries are made at equal intervals of time over a production cycle.

10. For the LTL case, the number of deliveries made per common production cycle may vary for the different products, but are still integer multiples, K_1, K_2, etc., of the production cycle.

11. At each of the various demand locations, stocks are replenished via a periodic review, order-up-to level inventory control system, when TL shipping is in effect. For coordination purposes, all the items at all the demand locations share a common fixed review period.

12. Under the LTL shipment policy, although the review periods for the various items may be different, for any given product, all retail locations share a common review period, for coordination purposes.

5.2.2. Notation

The notational scheme is adopted in the formulation of our models is given in Table 5.1.

5.3. Model Development

In this section, we present the details of the two shipment policies adopted in this paper, based on direct shipment and peddling shipment modes. These distribution policies are depicted in Figure 5.1.

Table 5.1: Notation

i	An index used to denote a specific product, $i = 1, 2, \ldots, n$
D_i	The demand rate for product i (units/time unit)
P_i	The production rate for product i (units/time unit)
A_i	Manufacturing setup cost per production batch for product i (\$/batch)
h_i	Inventory holding (carrying) cost for product i (\$/unit/time unit)
Q_i	Amount of product i contained in each TL shipment (units)
K	A positive integer, representing the number of shipments per production cycle
T	The shipment interval in time units (common to all products and locations)
KQ_i	The production lot size (in units) for product i
KT	Production cycle length in time units
C	The FTL capacity, i.e. maximum total load (or volume) allowable per truckload
w_i	Weight (or volume) of each unit of product i
$\bar{I}_t$	Average inventory level (units) of product i
TRC	Total relevant cost (\$) per time unit
γ	Fixed cost of initiating one truck dispatch (\$/dispatch) for TL policy
v	Unit shipment cost of products for less than truckload (LTL) amounts (\$/pound).

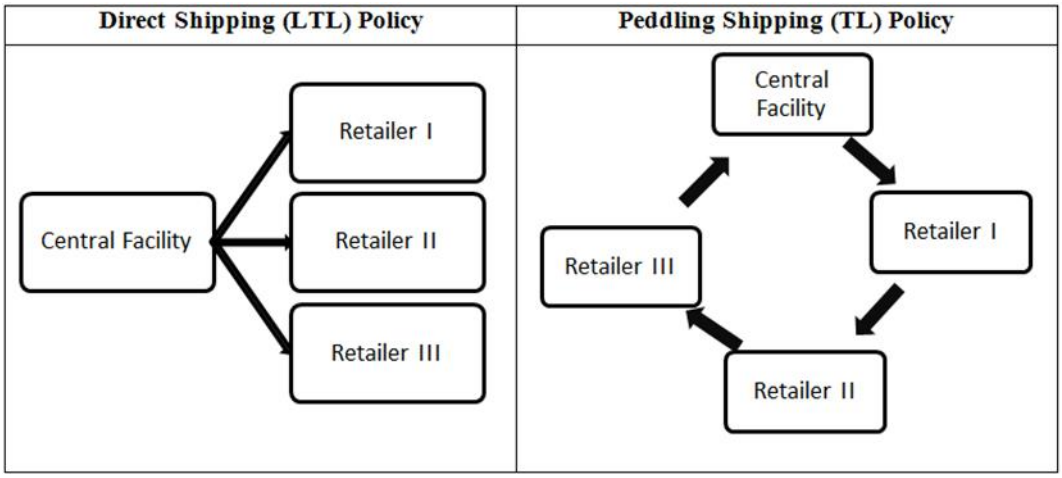

Figure 5.1: Direct Shipping (LTL) vs. Peddling Shipping (TL) Policies

5.3.1. Shipment Policies

Our analyses are based on extensions of the multiproduct model presented in Banerjee (2009). We develop an analytical model and methods to minimize total inventory and transportation related costs when a supplier distributes a set of different items to several retailers or customers. This paper evaluates and compares two different distribution policies: direct shipping and peddling.

The direct shipping distribution policy involves shipping separate loads from the supplier directly to each customer, whereas peddling shipping dispatches a fully loaded truck in each distribution cycle, that deliver items to all of the customers, based on each locations demand during this cycle. The latter is depicted in Figure 5.2 and a LTL distribution situation is shown in Figure 5.3.

5.3.2. TL policy model formulation

For illustrative purposes, the inventory-time plots for a TL distribution scenario are shown in Figure 5.2. This plot illustrates a situation that involves three products ($n = 3$) with negligibly small set up times and transit times and three full TL shipments for each production cycle. Figure 5.2 shows that there is a common delivery cycle time of T. Each truck with a limited capacity, C, contains Q_i units of product i, $(i = 1, 2, 3)$. The products should be sequenced to minimize the total set up cost, inventory holding cost and transportation cost (see [8], for an explanation of this).

We obtain the following the average inventory values for the three items:

$$\bar{I}_1 = \frac{[\frac{1}{2}KQ_1(\frac{KQ_1}{P_1}) + KQ_1(\frac{KQ_2}{P_2} + \frac{Q_3}{P_3}) + (K-1)(\frac{Q_1}{D_1})Q_1 + (K-2)(\frac{Q_1}{D_1})Q_1 + \cdots + (\frac{Q_1}{D_1})Q_1]}{KQ_1/D_1}$$

$$= KD_1(\frac{Q_1}{2P_1} + \frac{Q_2}{P_2}) + D_1(\frac{Q_2}{P_2}) + (K-1)(\frac{Q_1}{2})$$

$$\bar{I}_2 = \frac{[\frac{1}{2}KQ_2(\frac{KQ_2}{P_2}) + KQ_2(\frac{Q_3}{P_3}) + (K-1)(\frac{Q_2}{D_2})Q_2 + (K-2)(\frac{Q_2}{D_2})Q_2 + \cdots + (\frac{Q_2}{D_2})Q_2]}{KQ_2/D_2}$$

$$= KD_2(\frac{Q_2}{2P_2}) + D_2(\frac{Q_3}{P_3}) + (K-1)(\frac{Q_2}{2})$$

$$\bar{I}_3 = \frac{Q_2}{2}[\frac{D_3}{P_3}(2-K) + (K-1)]$$

5.3.2.1. Objective Function

In consideration of these results, we formulate the minimization objective function (the total relevant cost per time unit), as shown below. This expression includes the inventory holding, setup, and the transportation costs per time unit. Note that the cost function below is non-linear, with an integrality requirement. The

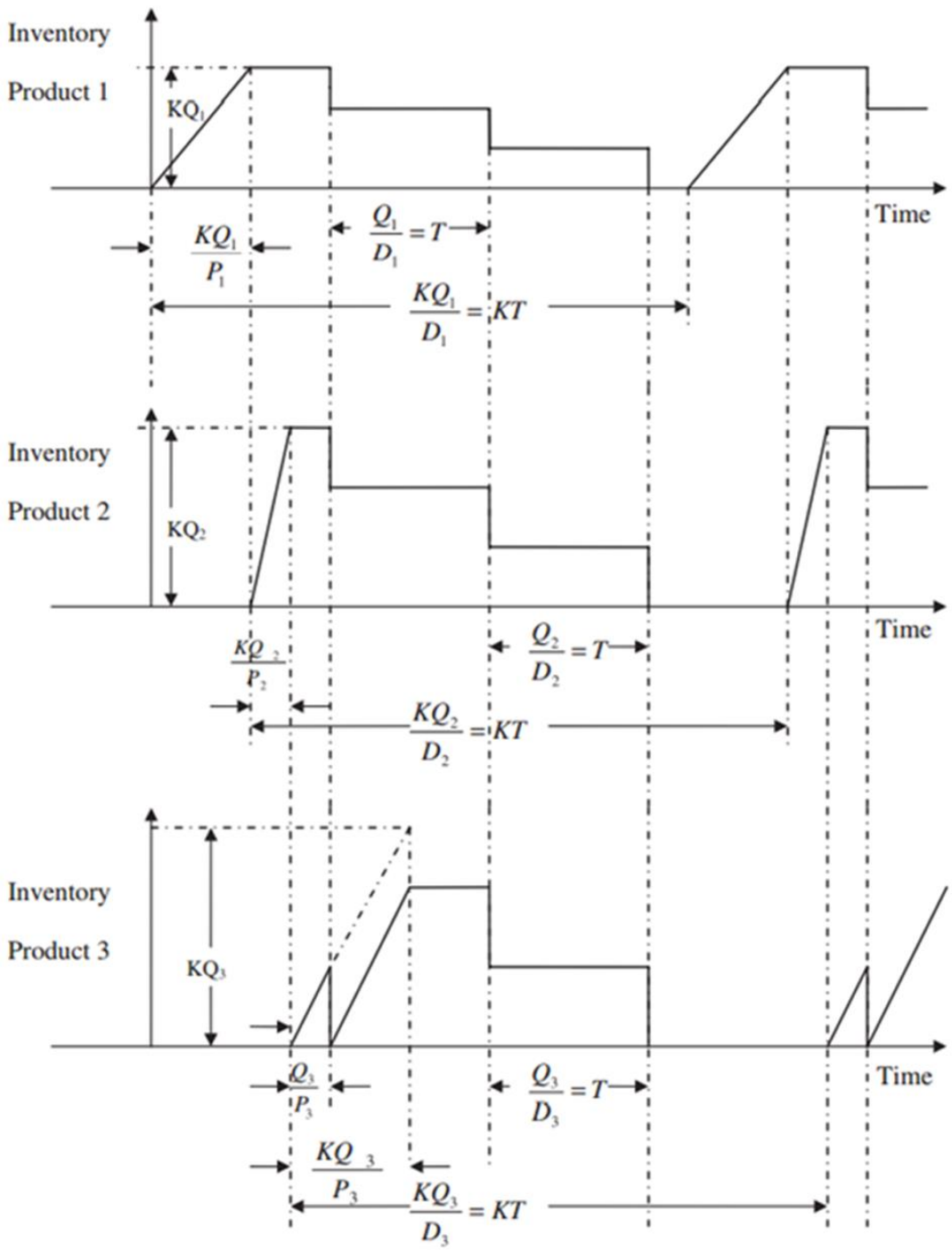

Figure 5.2: Inventory-Time Plots for a Peddling Shipment Policy ($n = 3, K = 3$)

(As shown in [8])

decision variables are the amounts of all the products, Q_i (for all i), contained in each TL shipment and the number of shipments per production cycle which is denoted by K, an integer.

$$Minimize \quad TRC(Q,K) = \sum_{i=1}^{n} \frac{D_i A_i}{KQi} + K \sum_{i=1}^{n-1} D_i h_i [\frac{Q_i}{2P_i} + \sum_{j=i+1}^{n-1} \frac{Q_j}{P_j}] + \frac{Q_n}{P_n} \sum_{i=1}^{n-1} D_i h_i +$$
$$(K-1)\frac{Q_n}{P_n} \sum_{i=1}^{n-1} \frac{Q_i h_i}{2} + \frac{Q_n h_n}{2} [\frac{D_n}{P_n}(2-K) + (K-1)] + \gamma \frac{1}{T}$$

The first term above represents the total manufacturing setup cost per production batch for n products. The last part of the objective function represents the total transportation cost that is obtained by the multiplication of the fixed cost of initiating one truck dispatch and the total number of TL shipment per unit of time. The remaining terms capture the inventory holding cost per time unit for items $1, 2, \ldots, n$, respectively. Finally, we obtained convex objective function which is shown in Appendix.

5.3.2.2. Constraints

1. The delivery cycle is common to all products:

$$\frac{Q_1}{D_1} = \frac{Q_2}{D_2} = \cdots = \frac{Q_n}{D_n} = T \quad \text{or} \quad Q_i = TD_i \quad \text{where} \quad i = 1, 2, \ldots, n$$

2. Production schedule should be feasible, i.e. total production time should be less than the manufacturing cycle time (without loss of generality, we

assume that the manufacturing setup times are negligibly small):

$$K\sum_{i=1}^{n-1}\frac{Q_i}{P_i}+\frac{Q_n}{P_n}\leq KT \quad \text{or} \quad \sum_{i=1}^{n-1}\frac{Q_i}{P_i}+\frac{Q_n}{KP_n}\leq T$$

3. The load capacity of a truck is limited by the total weight (or volume) of the products, i.e.

$$\sum_{i=1}^{n} w_i Q_i = C$$

4. At least one TL shipment must be made over a production cycle:

$$K \geq 1 \quad \text{where} \quad K \in \mathcal{Z}_+$$

The TL policy, i.e. a mixed integer non-linear programming (MINLP), model formulated above may be solved using one of several computer based solvers available. We employ the BONMIN solver for this purpose and obtain the optimal solution.

5.3.3. LTL policy model formulation

For illustrative purposes, the inventory-time plots for a direct shipment based LTL distribution policy are shown in Figure 5.3, which represemts a scenario involving three products ($n = 3$) with negligible set up and transit times. Note that LTL shipment for each production cycle.

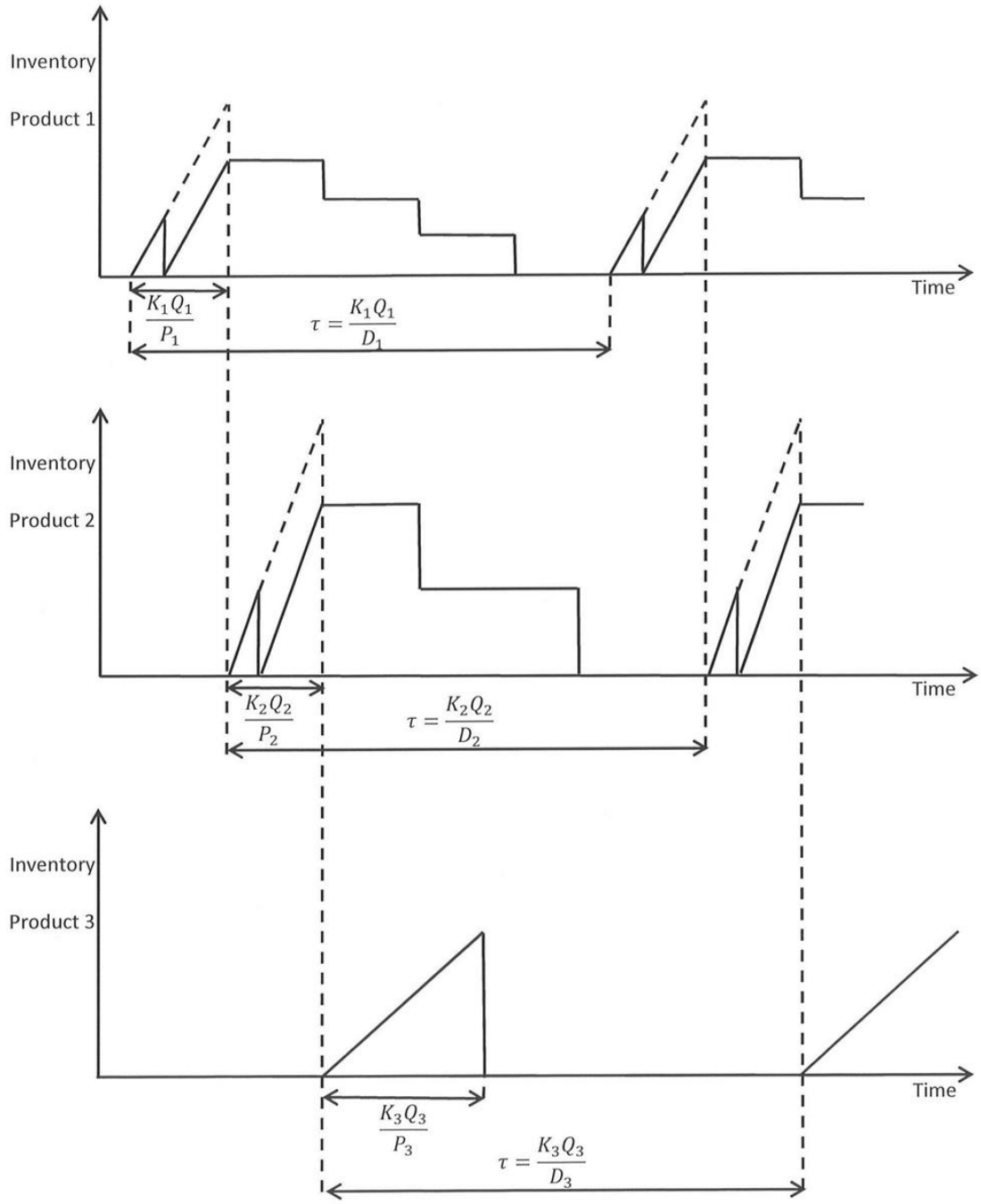

Figure 5.3: Inventory-Time Plots for a Direct Shipment Policy ($n = 3, K_1 = 4, K_2 = 3, K_3 = 1$)

5.3.3.1. Objective Function

As before, the objective is to minimize the total relevant cost per time unit, which includes the inventory holding, setup and the transportation costs. Also, the decision variables consist of the amount of product i contained in each LTL shipment (denoted by Q_i) and the number of shipments per production cycle for product i, K_i, which are restricted to positive integers. The objective function of the LTL model then can be expressed as:

$$Minimize \quad TRC(Q,K) = \frac{1}{\tau}\sum_{i=1}^{n} A_i + \tau \sum_{i=1}^{n} \frac{D_i h_i}{2}[\frac{D_i}{P_i}(2 - \frac{1}{K_i})] + \sum_{i=1}^{n} w_i D_i v_i$$

where $\tau = K_i T_i$.

The first term above represents the total manufacturing setup cost per production batch for n products, second term captures the total inventory holding cost for all items and the last term denotes the total transportation cost per unit of time. Finally, we obtained convex objective function which is shown in Appendix.

5.3.3.2. Constraints

1. The delivery cycle time is common to all products:

$$\frac{Q_1}{D_1} = \frac{Q_2}{D_2} = \cdots = \frac{Q_n}{D_n} = \tau; \quad \tau = K_i T_i \quad \text{so} \quad Q_i = D_i K_i T_i, \quad i = 1, 2, \ldots, n$$

2. At least one shipment per truck should be made within a production cycle:

$$K \geq 1 \forall i, \quad \text{where} \quad K \in \mathcal{Z}_+$$

Table 5.2: Example Problem Parameters

Product (i)	D_i (units/year)	P_i (units/year)	A_i (\$/setup)	h_i (\$/unit/year)	w_i (lbs./unit)
1	8,000	30,000	1,500	40	20
2	12,000	50,000	3,000	72	50
3	15,000	40,000	2,400	60	40

Once again, the BONMIN solver is utilized so find the optimal solution to the MINLP LTL policy model formulated above.

5.4. Numerical Example

This section presents an illustrative example involving three products. The relevant data pertaining to the problem are shown in Table 5.2.

Truck capacity is varied between 10000 lbs. and 70000 lbs. at 5000 lbs. increments, for full truckload shipments. In addition, the unit variable shipment cost of products for the less than truckload mode is varied from \$0.24 to \$0.18 per lb. in increments of \$0.01. As mentioned before, we obtain the optimal solutions to the mixed integer nonlinear optimization problems (MINLPs) for both TL and

LTL shipment policies using the BONMIN solver, which provides global optimal solutions for the MINLPs. Table 5.3 presents the summary of the computational results for TL shipments with varying truck capacities and Table 5.4 shows the summary results for the LTL shipping policy, incorporating different unit variable shipping costs.

Table 5.3 indicates the TL delivery cycle time varies from 0.00735 year to 0.05147 year. These increase as the truck capacity goes up. The number of TL deliveries per production cycle (K) tends to decrease with increasing truck capacity. These results are not unexpected, since the fixed cost per shipment tends to increase with larger vehicle capacities. To compensate for this phenomenon, the production cycle time is increased, together with fewer deliveries per manufacturing cycle. Needless to say that delivery lot sizes also increase with higher truck capacities. Interestingly, the total relevant cost function value tends to exhibit a convex behavior with respect to vehicle capacity. Clearly, due to the effects of economies of scale, the TRC decreases with a larger and larger vehicle size. Nevertheless, after a certain truck size, the initial cost advantage of scale seems to be more than offset by the higher annual truck dispatching costs, as well as higher inventory holding costs, resulting from the need to hold more output in stock

Table 5.3: Numerical Results for TL Shipment Policy with Varying Truck Capacities

Capacity	*Gamma*	T	K	Q_1	Q_2	Q_3	*TRC*
10000	3000	0.00735294	10	58.82	88.24	110.29	595789
15000	3300	0.01102940	7	88.24	132.35	165.44	486356
20000	3630	0.01470590	5	117.65	176.47	220.59	432714
25000	3993	0.01838240	4	147.06	220.59	275.74	402135
30000	4392	0.02205880	3	176.47	264.71	330.88	383902
35000	4832	0.02573530	3	205.88	308.82	386.03	371083
40000	5315	0.02941180	3	235.29	352.94	441.18	366286
45000	5846	0.03308820	2	264.71	397.06	496.32	358604
50000	6431	0.03676470	2	294.12	441.18	551.47	355050
55000	7074	0.04044120	2	323.53	485.29	606.62	355146
60000	7781	0.04411760	2	352.94	529.41	661.77	358114
65000	8559	0.04779410	2	382.35	573.53	716.91	363439
70000	9415	0.05147060	1	411.77	617.65	772.06	370674

Table 5.4: Numerical Results for LTL Shipment Policy

T	K_i	Q_1	Q_2	Q_3	TRC	v
0.14798	1,1,1	1183.84	1775.76	2219.7	419656	0.24
0.14798	1,1,1	1183.84	1775.76	2219.7	406056	0.23
0.14798	1,1,1	1183.84	1775.76	2219.7	392456	0.22
0.14798	1,1,1	1183.84	1775.76	2219.7	378856	0.21
0.14798	1,1,1	1183.84	1775.76	2219.7	365256	0.20
0.14798	1,1,1	1183.84	1775.76	2219.7	351656	0.19
0.14798	1,1,1	1183.84	1775.76	2219.7	338056	0.18

before a larger vehicle can be fully loaded. For the given problem parameters, it appears that under a peddling distribution policy, TL shipments with a 50,000 lbs. truck capacity yields the lowest total relevant cost per year of \$355,050.

The results for the LTL direct shipment policy, as shown in Table 5.4, lead to some interesting observations. First, in the absence of a fixed shipping cost, a common production cycle leads to a lot-for-lot (with respect to aggregate market demand) delivery policy for each of the products concerned, i.e. $K_i = 0, \forall i$. From the minimization objective function of the LTL policy model, it is clear that for any given production cycle time, τ, the second term, representing the total holding cost, is minimal when all K_i values are set to 0. Thus, if a feasible solution (i.e.

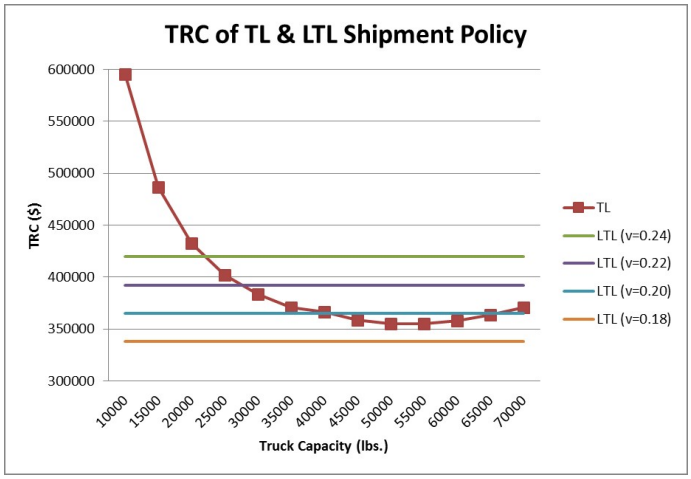

Figure 5.4: Comparison of TL and LTL Shipment Policies

sufficient production capacity) exists for this model, the optimization task in this case is to determine the appropriate value of τ. Once this is accomplished, the problem is essentially solved. Hence, we observe that regardless of the value of the unit variable shipping charge, the production and delivery cycles remain the same, although, the annual total relevant cost increases with increasing variable transportation cost. For the example chosen, the optimal cycle time remains fixed at 0.14798 year, with the same lot-for-lot product deliveries for differing variable shipping charges. Figure 5.4, comparing the TRC values for the TL policies with varying truck capacities and the

LTL policy with changing variable shipping costs, further indicates that when the unit shipping cost is sufficiently low, the latter policy is always superior from a cost perspective. Otherwise, there is clearly a beak-even point between these two policies, with respect to vehicle capacity. For relatively small-sized trucks, the

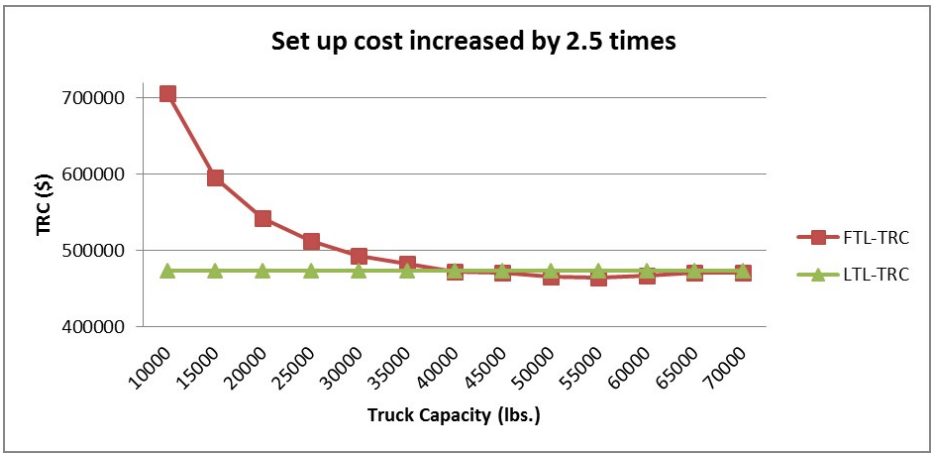

Figure 5.5: Policy Comparison with 250% Increase in Setup Cost

LTL policy is likely to be more desirable, whereas the TL shipping policy tends to yield lower TRC values, beyond the break-even truck capacity level, before the effect of diseconomies of scale take effect. This is not surprising, since with larger trucks the fixed charge structure is based on a decreasing prorated cost per unit shipped.

For the purpose of sensitivity analysis, we vary the manufacturing setup cost, A_i, and the fixed TL shipping cost, γ, values. The results of these analysis are summarized in Figures 5.5, 5.6, 5.7 and 5.8, which

indicate that with increasing fixed TL shipping charges, the LTL policy tends to become more dominant. By the same token, if this cost decreases, the TL policy tends to be superior to the LTL shipping mode. Additionally, increasing the production setup cost appears to have a similar effect with respect to the two distribution policies examined here. In other words, all else being equal, higher

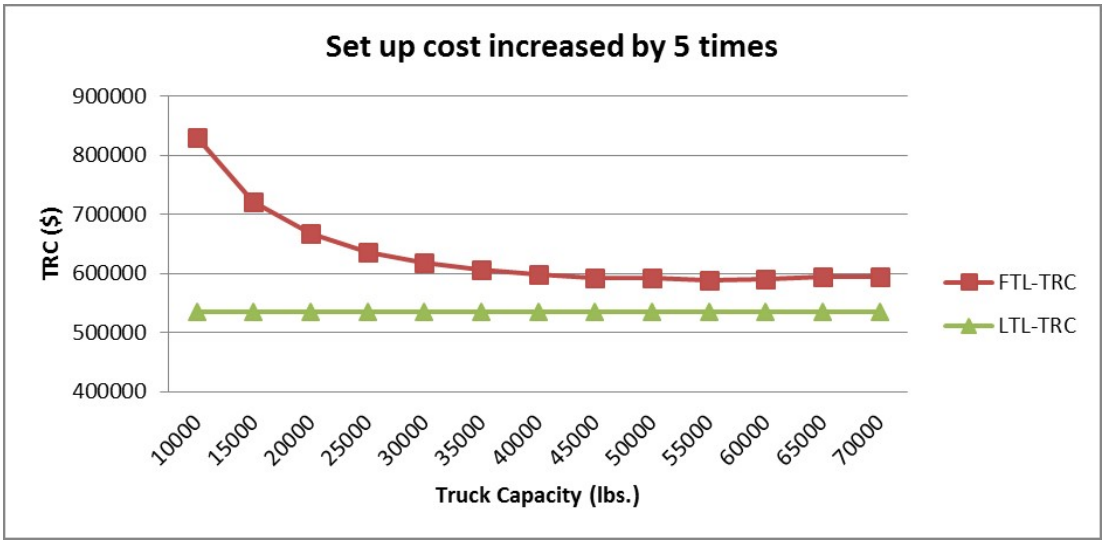

Figure 5.6: Policy Comparison with 500% Increase in Setup Cost

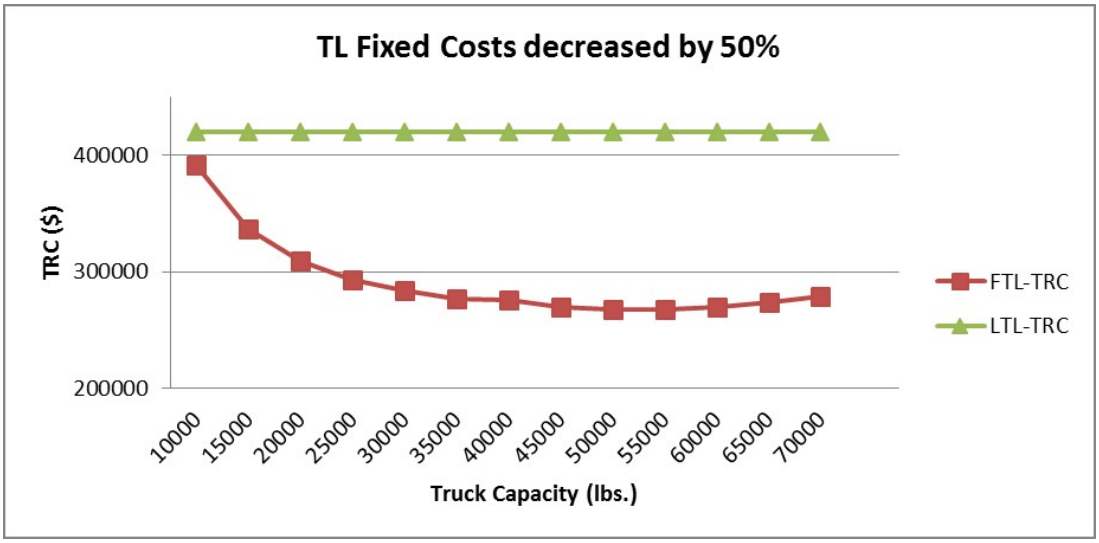

Figure 5.7: Policy Comparison with 50% Reductions in TL Shipping Costs

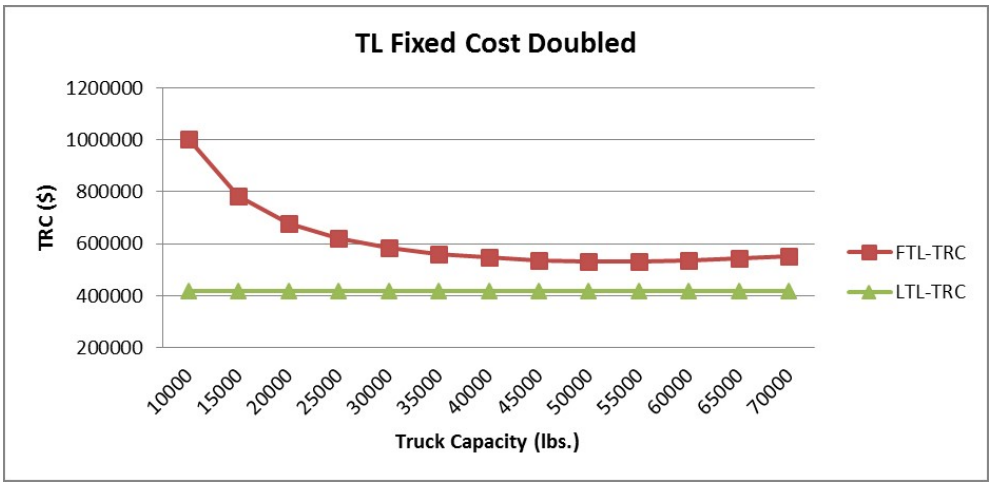

Figure 5.8: Policy Comparison with 100% Increase TL Shipping Costs

setup costs tend to render the LTL policy a better alternative to TL distribution.

Chapter 6

Conclusions and Future Research Directions

6.1. Conclusions

In this chapter, we will provide concluding remarks for each of the applications covered in Chapters 3, 4 and 5. The overall contribution of this dissertation is the use of these applications to motivate the development and use of advanced statistical and optimization techniques to solve business problems. Given the success of our solution methods and/or existing approaches on these applications, we believe that we have provided sufficient motivation for future researchers.

6.1.1. Portfolio Selection Models as MISOCPs

In Chapter 2, we gave an overview of the state-of-the-art in mixed-integer second-order cone programming problems. We described numerous applications and a handful of solution algorithms. Given the wide range of fields from which the applications arise, we anticipate that this problem class will continue to flourish. The solution methods for MISOCP are still at their infancies, however, so for the growth of this problem class, it is important to continue to address issues of warmstarts and levels of accuracy in methods for solving the continuous relaxations and to add to the types of cuts available to improve the efficiency of overall solution approaches. The lifted LP branch-and-bound algorithm presents another opportunity for algorithmic improvement, and it may be useful to investigate other approaches for solving SOCPs using an LP-based approach within the MISOCP framework.

In Chapter 3, we presented a set of techniques for solving MISOCPs as MINLPs whose underlying NLPs are smooth, regularized, and convex. A ratio reformulation was used to smooth the underlying SOCPs. The primal-dual penalty interior-point method, modified from that presented in [17] and [18], was then used to provide warmstarts, regularization, and infeasibility detection capabilities, and the modification also exploited the structure of the MISOCP. We have implemented both branch-and-bound and outer approximation frameworks that use

this method, and use them to solve portfolio optimization problems. Numerical results show that we can solve small to medium-sized instances successfully. The infeasibility detection capability provided by the primal-dual penalty approach allows us to either solve or declare infeasibility at each node, thereby leading to a robust method. The warmstart capability is shown to significantly improve algorithm efficiency.

In future work, we hope to extend our approach to general MISOCPs by having a dynamic choice of constraint reformulations to resolve nonsmoothness issues. For handling the integer variables, our proposed frameworks can accommodate the various cuts appearing in MISOCP literature, and we will investigate such algorithmic improvements as well. Additionally, we will continue our work on portfolio optimization models by working to include round-lot constraints in our models for both single and multi-period portfolio optimization model.

6.1.2. Skewness in Portfolio Selection Models

We have proposed and implemented a competition between traditional mean variance efficient portfolio selection and an alternative portfolio selection paradigm based on higher order moments. We have shown that the higher order moment model does a better job of describing the "typical investor's" portfolio and allows us to estimate the revealed preferences of the market.

The comparison is fair in the sense that it is based on market data and is not unfairly hinged upon one or the other decision criterion. However, several limitations remain. Perhaps the most important limitations are related to the appropriate interpretation of the market data. We used total shares outstanding of stocks in a broad-based set of assets to define maket weights that reflect a "typical investor" and proceeded to approximate these weights under the two models of interest. But of course the sum of the optimal solutions of all investors does not necessarily take the form of the optimal solution of an average investor.

Another limitation is the constraint to a fixed set of assets. In reality, investors have choices beyond the limited number of assets considered. We mitigate this problem by considering a widely diversified mix of assets.

6.1.3. Supply Chain Management

In Chapter 5, we have made an attempt to integrate the lot scheduling decisions for multiple products produced in a single facility, with their shipment schedules under two different types of transportation cost structures under deterministic conditions. One common type of shipping rate regime found in the real world involves full truckload (TL) or carload movement of goods, where only a fixed cost is incurred depending on the points of origin and destination, as well as the type of commodity moved. An alternative transportation mode is the less than

truckload (LTL), or carload shipping, where there is no fixed cost. The cost of a specific shipment is based on a variable cost per unit moved from an origin to a destination. In our analysis, we incorporate both of these transportation scenarios for a single manufacturer and several retailers. Furthermore, under a TL shipping policy, we employ a peddling type of distribution arrangement, where a fully loaded vehicle containing a mix of all the products is dispatched to all the retail locations for simultaneous delivery. In the case of LTL shipments, each product has its own delivery cycle and shipments are made directly and individually to each of the retailers when a batch of the item is completed.

We construct constrained (MINLP) models for linking the production and distribution decisions under both of the distribution policies described above and employ widely available solver software for finding globally optimal solutions. Through a set of numerical experiments we show that the respective magnitudes of the various cost parameters play a crucial role in selecting either a TL or LTL distribution method. An important finding of this work is that when transportation involves no fixed cost, but only a variable charge per unit shipped, the optimal shipment schedule is essentially lot-for-lot with respect to aggregate retail demand. We have observed that under TL distribution, the production, as well as the delivery cycle lengths tend to go as vehicles of larger capacities are employed. Also, we have attempted to outline the parametric conditions under which either of the two transportation modes will dominate the other from a cost perspective.

It is hoped that this study will provide a helpful tool for supply chain practitioners, in terms of integrating the production schedule with transportation planning and selecting an appropriate method of distribution. We also hope that future research endeavors in this area will find some value in this work and will extend our findings under more complex and realistic supply chain environments.

6.2. Future Research Directions

As we mentioned before, mixed-integer second-order cone programming problems arise a variety of important application areas ranging from finance to electrical engineering, from operations management to statistics because of two types of constraints: (1) risk (volatility) constraints can easily be formulated as second-order cone constraints, and (2) binary choices and discrete decisions are quite common in the real world. Therefore, exploring the use of the improved solution approaches proposed here to a variety of different application areas will be a significant part of my future research agenda. Given the current advanced state of my research, I believe that I am poised to make contributions to the various fields in a timely manner.

In fact, we have already started working on the solution of a humanitarian logistics problem that arises in a real-world application. ABC is a utility company that delivers natural gas to its customers via underground pipelines. This utility

company conducts the installation and maintenance of the pipelines as well as responds to gas leak emergencies. It wants to improve the operational performance of its emergency crews which one subject to response to the constraints by state law. We are using real-world historical data which was gathered the dispatching office of the company from the gas leakage calls. The company's objective is to find the optimal number of centers that minimize the total travel distance from employees' homes to centers and from centers to leakage areas subject to the state-law constraint. Second-order cone constraint arise in the calculation of the total travel distance and binary variables arise in assigning each employee to centers and centers to the leakage areas. Therefore, the model is formulated as a k-centers problem that fits the MISOCP framework. Another version of the problem uses nonlinear constraints to formulate distances using latitude-longitude information, resulting in the overall k-centers problem to be formulated as a MINLP.

The techniques and applications discussed in this dissertation readily extend to problems such as that of ABC. I look forward to making further contributions to the field by pursuing research in such directions.

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