Supply Chain Coordination Contracts with Free Replacement Warranty

A Thesis

Submitted to the Faculty

of

Drexel University

by

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in partial fulfillment of the

requirements for the degree

of

PhD in Decision Sciences

2008

Acknowledgements

I would like to express my gratitude to many people who helped me through this work. I wan to thank Dr. Avijit Banerjee and Dr. Seung-Lae Kim, my advisors, for your tremendous time and energy devoted in my research. Thank you for careful reading and editing this manuscript, and helping me in scientific writing with great patience.

I am very grateful to Dr. Hande Benson for working with me through the analysis and giving me insightful comments. I enjoy every discussion in our independent study, and this work will not be completed without your inspiring and guiding. I admire you for your thinking and personality, your patience and assertiveness. You have influenced me in many ways.

I want to express my gratitude to my exceptional doctoral committee. Thank Dr. Darrell Radson for giving me directions when I needed them most. The book you suggested initiated my thoughts in quality improvement. Thank Dr. Thomas Hindelang for your continual support and encouragement.

I would like to thank all the faculty members, staffs and colleagues in the Decision Sciences Department at Drexel University. I would also like to extend my thanks to my friends. Thank you all for making my life joyful and memorable.

Finally, I want to thank my parents for their unconditional love and support. The weekly calls to my parents have been a constant source of encouragement and strength. My deep gratitude goes to my husband, whose optimism and humorousness always inspire me to greater challenges. To them I dedicate this thesis.

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Abstract

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This thesis investigates a coordination mechanism for a supply chain with one manufacturer and one retailer in a single period, single product newsvendor model. It looks beyond the conventional supply chain coordination problem by incorporating a specific form of warranties. The manufacturer provides a free replacement warranty in case of product failure within a specified after-sale interval. We assume that the expected value of stochastic market demand is an increasing function of this warranty period length. The supply chain is coordinated if its optimal actions (production quantity and warranty length) are realized while each party maximizes its own respective profit. Any deviation by either party from the terms of a coordinated contract cannot improve its performance.

We consider different types of contracts between the two parties: a wholesale price only or a revenue sharing contract with shared warranty costs or such costs borne by the manufacturer alone. The manufacturer decides the warranty period, K, and other contract parameters, such as the wholesale price, shares of revenue, and warranty cost sharing arrangements. The retailer accepts the contract and determines the order amount, as long as it is able to make positive profit. The manufacturer then produces and delivers the order quantity for the selling season. Each party makes its decisions to maximize its own profit, hence the realized decisions may differ from the supply chain's optimal solutions, if the contract is not coordinated. Thus, we examine whether the supply chain can be coordinated under each type of contract outlined above. For coordinated contracts we focus on the issue of profit allocation. If a contract type is non-coordinating, we attempt to highlight the factors that affect its efficiency, where the efficiency of a non-coordinating contract is defined as the ratio of realized supply chain profit over its optimal profit.

The results obtained from this research leads to some interesting managerial insights. Under the wholesale price only contract types, we find that even if the retailer is willing to share the warranty fulfillment costs with the manufacturer, the resulting supply chain profit is less than the optimal value, leading to suboptimal performance. Under a revenue sharing contract, however, the production/ order quantity and the warranty length are coordinated, if the warranty costs are shared by the two parties in the same proportion as the profits. The profit allocation of each party under coordination is flexible from 0 to 100% of chain profit. This concept is illustrated by a numerical example of additive demand case followed by an extensive sensitivity analysis, which leads to some important insight.

The major contribution of this thesis is its novel aspect of considering warranty period optimization towards supply chain coordination. We provide the guidelines for designing a contract between a manufacturer and a retailer so that the supply chain's performance is optimized in terms of the production/ order quantity and the warranty period, while each party in the chain achieves its maximal profit. Without the presence of a third party, the contract coordinates the supply chain with less cost. A non-coordinating contract may still be desirable if it entails relatively low administrative cost while achieving a high degree of efficiency, as defined before. The concepts developed here are easy to implement in real world supply chain, and can provide valuable insights into more complex types of supply chain contracts.

0.1 List of Notations

D	demand over the considered selling season (1)
p	product price ($ /unit) (1)
K	warranty interval in years (1)
Q	retailer's order amount in units (1)
W	wholesale price ($ /unit) (1)
$f(\cdot\|K)$	probability density function of demand over selling season for any given K (1)
$F(\cdot\|K)$	cumulative probability function of demand over selling season for any given K (1)
c_s	manufacturer's production cost ($ /unit) (1)
c_r	retailer's procuring cost ($ /unit) (1)
c	supply chain's product cost ($ /unit); $c = c_s + c_r$ (1)
v	salvage value ($ /unit) (1)
g_s	manufacturer's lost sales cost ($ /unit) (1)
g_r	retailer's lost sale cost ($ /unit) (1)
g	supply chain's lost sale cost ($ /unit); $g = g_s + g_r$ (1)
r	replacement cost for each warranty claim ($ /unit) (1)
Y	years until a product fails in time units (1)
$\frac{1}{\beta}$	mean failure time of products in years (1)
N	number of products that fail during the warranty period (1)
ϕ	% of revenue, taken by the retailer in the revenue sharing contract (1)
θ	% of warranty cost, borne by the retailer in the revenue and warranty sharing contract (1)

1. Introduction

The thesis considers some types of contracts as coordination mechanisms for a manufacturer's warranty of free product replacement, if a product fails within a specific after-sale interval. Research interest in supply chain coordination via contracts has grown rapidly and a number of different possibilities have been explored over the past ten years or so. We consider in this research a new scenario where the warranty decision needs to be coordinated with the supply chain's production level. We first introduce the concept of supply chain coordination, review current research in supply chain coordination and warranty related matters. The behaviors of different types of contracts towards coordinating the supply chain and their coordination flexibility are investigated. Finally, we examine the efficiency of a class of non-coordinating contracts.

A typical supply chain consists of suppliers, distributors, retailers and end customers. Raw materials are procured, and various products are produced at one or more factories, shipped to warehouses for intermediate storage, and then transported to retailers to reach end customers. Consequently, to reduce associated costs and improve customer service, effective strategies, policies and operational decisions must take into account the interactions between different stages, requiring advanced supply chain management techniques. Supply chain management is a set of approaches used to effectively integrate the parties involved in the various stages of a chain so that the systemwide costs are minimized while satisfying each party's profit requirements. It is an integrated holistic approach for systems, often involving considerable real world complexities.

Research in Supply Chain Management has grown dramatically in the past decade as firms have intensified efforts to streamline the operations for competing in a global economic environment. Market's pressure for lower prices, better quality and higher levels of services present significant challenges to today's supply chains, forcing retailers, transporters and manufacturers to achieve greater cost efficiencies and improve after-sales service. Supply chain efficiency is, thus, a key factor in gaining competitive advantage. At the same time, with the proliferation of the internet and the development of robust business-to-business (B2B) technologies, supply chain solutions are scalable, in conjunction with automated information exchanges between trading partners. These make the collaboration and coordination among supply chain players possible. Such opportunities, however, introduce new challenges and complexities. Different stages in supply chains frequently have different, conflicting objectives. Each party is primarily concerned with optimizing its own objectives which may sacrifice the entire chain's overall performance. For instance, manufacturers typically want retailers to commit themselves to purchasing large quantities of a product in stable volumes with flexible delivery dates. Unfortunately, most retailers need to be flexible to their customers' needs and changing demands. They are also conservative in the order amount, as they usually undertake the risk of excess inventories. The actual order quantity may not coincide with the supply chain optimum. There exist opportunities, nevertheless, that if they make decisions as if they were part of an integrated, seamless pipeline, the entire supply chain optimum is achieved, and each party may be better off as they now share a bigger pie.

Companies see significant leverage in coordinating the supply chain. How to align incentives with the total chain objectives remains a question. One way is to introduce an independent third party authority that proposes global optimal decisions for all the parties to follow and allocates profits after the selling season. If the manufacturer and retailer are parts of a larger supply chain owned by a single corporate entity, this entity may act as a third party and force coordination of the supply chain. However, the cost of coordination incurred by a third party and spent during the process may make such an arrangement impractical. Besides, keeping the third party fair and neutral presents new challenges. With more supply chain members involved in a transaction, the complexity increases significantly, which may incur transaction and service delays. There may exist a way to ensure the proper alignment of incentives without presence of a third party authority. It has been shown that a properly designed contract can coordinate the chain automatically through appropriate risk sharing and profit allocation. Over the last decade, the issue of supply chain coordination through a variety of contract structures has attracted the attention of many researchers, resulting in substantial contributions to the literature.

A contract is said to coordinate the supply chain if the adoption of a set of supply chain optimal decisions by the respective constituents results in the optimization of the objective of each of them. The most studied issue in this respect is the coordination of the retailer's order quantity (which in the single buyer, single vendor, single period environment is the same as the manufacturer's production quantity) with the supply chain's optimum quantity. A number of different supply chain coordinating contract types are identified. It is shown that in a simple newsvendor problem, buyback contracts (Pasternack, 1985), revenue sharing contracts (Cachon & Lariviere, 2005), quantity flexibility contracts (Tsay, 1999), sales rebate contracts (Taylor, 2002; Krishnan, Kapuscinski, & Butz, 2004) and quantity discount contracts (Moorthy, 1987) can coordinate the production quantity of a supply chain. The related compliance regime (Chen, 1999; Cachon & Lariviere, 2001; Porteus, 2000) and administration cost related matters are also discussed by these authors.

Within the literature pertaining to supply chain coordination with contracts, however, no existing research sufficiently addresses the incorporation of manufacture's warranty as a coordination mechanism.

As a common practice in industry, warranties have received the attention of researchers from many diverse disciplines. Menke (1969) and Lowerre (1968) conducted probabilistic analyses of warranty costs under free replacement warranties with a rebate. Subsequently, warranty related issues have been examined in legal, economic, behavioral, marketing and management science contexts from implementation and operational impact perspectives. A comprehensive review of this research area in different disciplines can be found in (Blischke & Murthy, 1996).

A great deal of work has been done by operations researchers in exploring various aspects of manufacturer's warranties, such as warranty types (Blischke & Murthy, 1992, 1994; Djamaludin, Murthy, & Blischke, 1996; Murthy & Jack, 2003), product failures during warranty period (Gerner & Bryant, 1980), warranty claims (Barlow & Hunter, 1960; Wasserman, 1992), warranty costs (Menke, 1969; Lowerre, 1968), and warranty logistics (Murthy, Solem, & Roren, 2004; Alfredsson, 1997; Cassady, Murdock, & Pohl, 2001; Cohen & Lee, 1990; Jack & Van der Duyn Schouten, 2000). Studies examining the impact of warranties in a supply chain context are rare. Consequently, questions regarding optimal warranty strategy for a supply chain and incentives necessary for adopting specific warranty policies, etc. have remained largely unanswered to date.

Manufacturer's warranties may involve free replacement, and/ or coverage of parts and/ or labor work, and/ or parts repair in case of product failure within a specified interval. It is known that a longer warranty period signals the manufacturer's confidence in its products quality and tends to boost sales. Automobiles produced by Hyundai and Kia are well known for their extensive warranty coverage, whereas GM and Ford have recently extended the powertrain warranty for their 2007 vehicles from 3 years/36,000 miles to 5 years/100,000 miles and 5 years/60,000 miles respectively (Scherer, 2006). Such extensions have helped GM, particularly, in instilling a higher level of confidence regarding product quality in the minds of consumers. This has resulted in higher sales, bringing greater profit (Connelly, 2006). After the GM upgrades, several imported brands have offered broader powertrain coverage on their 2007 vehicles. Mitsubishi now offers warranty coverage over 10 years or 100,000 miles. Similarly Suzuki vehicles have powertrain warranty coverage for 7 years or 100,000 miles (Scherer, 2006).

Longer warranty periods, however, imply higher warranty service costs. These costs include the costs of spare parts, costs associated with operating a service center, shipping costs and those involving handling of claims. For a product having a certain level of quality, a longer warranty implies more warranty claims, and consequently, the manufacturer needs to hold larger spare parts inventories and have larger service center capacity. This raises the interesting question concerning the optimal warranty period, which will result in maximum supply chain profit. We explore a number of such contracts that align the manufacturer's warranty incentive with the chain, and investigate the impact of exogenous parameters on the coordinated decisions. If a contract does not coordinate the supply chain, We analyze its efficiency, as defined above.

The purpose of this thesis is to investigate some of the questions raised above. We consider a newsvendor environment for a two echelon supply chain with one manufacturer and one retailer dealing with a single product. The manufacturer offers the end customers a free replacement warranty (FRW) if the product fails within a stipulated interval after sale or delivery. Both the manufacturer and the retailer make their decisions to maximize their respective individual profit. The following events occur during the decision process:

- The manufacturer decides the warranty period K in time units;
- In addition, the manufacturer determines the contract parameters, i.e. wholesale price W and/or each party's share of revenue ϕ, as well as that pertaining warranty servicing costs θ, and offers the contract to the retailer;
- the retailer decides whether to accept or reject the contract offer; it will accept it if its expected profit is positive;
- if the retailer accepts the contract, it decides the order quantity Q based on the demand forecast given K;
- the manufacturer produces and delivers Q to the retailer before the selling season

begins;

- demand is realized;
- as some of the sold products start failing within the warranty period, the manufacturer handles the claims in honoring the terms of the free replacement warranty;
- all transfer payments are made between the two supply chain members after the sales season based on the terms of the contract.

During this process, the manufacturer attempts to optimize its expected profit in deciding the warranty period and other contract parameters while the retailer attempts to optimize its own expected profit by determining the order quantity. The manufacturer understands that a longer warranty period results in a demand forecast with a higher expected value, and the retailer will order a larger quantity, albeit with higher expected warranty service costs. A higher unit price can bring in more unit profit, but also discourages the retailer from ordering a large amount. There is a tradeoff between better service and costs, as well as price and quantity. The tradeoff for the retailer lies in the inventory risks. Holding large inventories for the season reduces potential shortage loss but increases the risk of leftover stock. For both parties, the manufacturer and the retailer, under each contract, there exists a set of optimal actions (for example, optimal warranty period and wholesale price for the manufacturer and the optimal order quantity for the retailer) that maximizes their respective profits. This set may be different across different contracts. For instance, Cachon and Lariviere (Cachon & Lariviere, 2005) have shown that under a revenue sharing contract the optimal order quantity for the retailer is more than that under a wholesale only contract. These optimal actions will be realized as both parties behave rationally and selfishly to optimize their own decisions. We assume each party enjoys full knowledge of market information and relevant data covering the other party, such as demand distribution, the retailer's and manufacture's cost parameters. The optimization process can be described as follows: The manufacturer knows that for any warranty length, K, there will be a demand distribution with respect to K: $f(D|K)$. Based on this demand forecast, the retailer will determine an optimal order quantity Q_r^* in response to the contract offer from the manufacturer. That is, given K, the final realized actions will be among the set of pairs $(contract_i, Q_r^*(contract_i))$. The manufacturer thus maximizes its profit by choosing the optimal contract $contract_s^*$ and expect the retailer's response $Q_r^*(contract_s^*)$. This optimal contract differs with different warranty length K. That is, for K_j, there exist at least one optimal contract $contract_s^*(K_j)$ such that $(contract_s^*(K_j), Q_r^*(contract_s^*))$ maximizes its profit given K_j. It then optimizes over K to find the optimal K^* so that $(K^*, contract_s^*(K^*), Q_r^*(contract_s^*))$ maximizes its profit among all the decisions it can make. It offers this contract to the retailer and informs about it the warranty decision. The retailer then orders the quantity as expected.

The supply chain assumes to take the warranty risk and inventory risk. If there exists a neutral third party on behalf of the supply chain, it will optimize the chain through the production and warranty decisions (Q and K). The chain is coordinated if the manufacturer and retailer's decisions fall into the optimal decision sets of the supply chain.

We investigate this newsvendor model under four different contract types: a wholesale price only contract with manufacturer taking warranty cost; a wholesale price only contract with warranty cost sharing; a revenue sharing contract with manufacturer bearing warranty costs; a revenue sharing contract with warranty cost sharing. We examine the supply chain's optimal decision set (Q^*, K^*). Under each contract, we first check with any determined K as to what contract offer aligns the retailer's order quantity with the supply chain's optimum and whether the manufacturer is willing to offer such a contract. If yes, we examine the flexibility of the profit allocation under coordination. High flexibility is preferred because this indicates the contract adoption is not limited by the relative power positions of the parties. If a coordinating contract can allocate profits arbitrarily, then there always exists a contract that Pareto dominates a non-coordinating contract, i.e., each firm's profit is no worse off and at least one firm is strictlybetter off with the coordinating contract. (Cachon, 2003) We then examine whether the warranty length K can be coordinated by a subset of contract offers in the above coordinating set. Our results show that the wholesale only contracts never coordinate the order quantity, no matter the warranty costs are shared or not. The revenue sharing contract without warranty cost sharing coordinates the order quantity with full flexibility of profit allocation, but coordinates the warranty with only one allocation. The revenue and warranty cost sharing contract, as the best among four, coordinates both decisions, if the share of the warranty cost equal to the share of revenue for each party. Theory is then tested in a deterministic additive demand case. The coordinating contract parameters and profit allocations are obtained. We then considered a fair and equitable profit allocation as sharing profit according to each party's relevant cost. We prove this can be realized by defining the warranty share as same as the proportion of retailer's unit wholesale and procuring cost to manufacturer's unit production cost. Theory is then tested in a deterministic additive demand case.

Although coordination and flexible profit allocation are desirable features, contracts with those properties might be costly to administer if they require extra information collection and coordination actions. As a result, a simple non-coordinating contract may be worth adopting if it is "efficient". The efficiency of a non-coordinating contract is defined as the ratio of supply chain profit with the contract to the supply chain's optimal profit. We investigate the efficiency of wholesale price only contract and theoretically prove that higher wholesale price lowers retailer's profit and makes the supply chain less efficient; improvement in production cost, warranty cost and quality helps to build a more efficient supply chain.

The paper is organized as follows: in Chapter 2 we review the current research in supply chain coordination and warranty; in Chapter 3 we explore the optimal solutions of order quantity and warranty policy for the supply chain; in Chapter 4 and 5 the wholesale contract and revenue sharing contract are analyzed for supply chain coordination; in Chapter 6 we provide a guideline to allocate supply chain profit at coordination in a fair manner; in Chapter 7 we investigate how quality improvement will affect our optimal solutions and benefit the supply chain; in Chapter 8 we test our theory in additive demand; in section 9 we derive the efficiency of wholesale only contract and explore how different parameters affect supply chain efficiency; in Chapter 10 we provide numerical examples and conduct some sensitivity analysis. The proof of lemmas and theorems is available in Appendix 1, and concavity check in Appendix 2. The tables and figures are provided in Appendix 3 and 4.

2. Literature Survey

2.1 Warranty Literature

2.1.1 Background

Modern industrialized societies are characterized by (Murthy, Solem, & Roren, 2004)

- new products (consumer durable, industrial and commercial products) appearing at an ever-increasing rate on the market,
- products getting more complex (due to technology advances),
- more demanding customers and,
- more stringent government regulations regarding product liability.

Buyers of product want assurance that the product will perform satisfactorily over a reasonable lifetime. To satisfy that need, manufacturers provide post-sale support that includes instruction and training, repair and maintenance, follow-on sales among others. As an important component of post-sale support service, warranty offered by a manufacturer or a dealer establishes liability among the two parties (manufacturer and buyer) in the event that an item fails. It is a contractual obligation in connection with the sale of a product (Murthy & Blischke, 1992). The warranty specifies that the manufacturer/ dealer agrees to rectify certain defects or failures in the product for a specified period of time (sometimes within specified amount of cost) after the sale of the product.

According to Murthy et al. (2004), warranties serve different purposes for buyer and seller. From the buyer's point of view, the main role of a warranty is protectional; it provides a means of redress if the item, when properly used, fails to perform as intended or as specified by the seller. Specifically, the warranty assures the buyer that a faulty item will either be repaired or replaced at no cost or at reduced cost. A second role is informational. Many buyers infer that a product with a relatively long warranty period is a more reliable and long-lasting product than one with a shorter warranty period.

Warranty is becoming an increasingly important dimension of competitive strategy for manufacturer. It is an integral part of the bundle of satisfaction which the buyer receives when he purchases a product. The three major objectives of warranty are promotional, protective and profitable.

Promotional Warranty The promotional warranty is designed to encourage purchases by reducing risks for the consumer. It usually promises extensive service or complete satisfaction, and its duration often covers a significant proportion of the life span of the product (Udell & Anderson, 1968). Since buyers believe long warranty often signals higher quality, warranty has been used as an effective advertising tool. It is often adopted when marketing new and innovative products, which may be viewed with a degree of uncertainty by many potential consumers; or with significant quality improvement and customers awareness being rebuilt. GM and Ford has just extended the powertrain warranty for their 2007 vehicles from 3 years/36,000 miles to 5 years/100,000 miles and 5 years/60,000 miles respectively (Scherer, 2006). It helped to establish a higher confidence level of product quality among customers, which resulted in higher volume of contracts, bringing greater profit (Connelly, 2006). In addition, warranty is also an instrument, similar to product performance and price, used in competition with other manufacturers in the marketplace.

Protective Warranty The purpose of protective warranty is to guard the manufacturer/ dealer from unreasonable claims of purchasers. It Warranty terms often specify the use and conditions of use for which the product is intended, and limits the manufacturer's responsibility to defects in materials and manufacturing which develop in the normal or improper use of the product during the warranty time. (Udell & Anderson, 1968)

Profitable Warranty Priced extended warranty also contributes significantly to the profits of the manufacturer. Typically, the profit margin for post-sale service is roughly 30% as opposed to 10% on the initial sale. Due to the fierce competition, most electronic and automobile manufacturers are surviving due to the profits made on offering extended warranties (Murthy, Solem, & Roren, 2004).

To the public policy makers the warranty plays a statutory role to enact laws to see that warranty terms are fair and there are mechanisms to resolve conflicts arising from warranty claims.

Murthy and Blischke (1992) give a comprehensive review of the mathematical models of warranty by classifying them into 3 categories based on consumer, manufacturer, and public policy decision maker perspectives. In current literature a number of models have been built to study the warranty process, such as cost accounting of warranty, warranty servicing, and dispute resolution. In addition, a variety of mathematical models are developed to study the effect of warranties on market behavior and the resulting social welfare implications are examined. These fall into 4 categories based on the role assigned to warranties:

- warranties as insurance,
- warranties as signals,
- warranties as incentives,
- warranties as marketing devices.

2.1.2 Disciplines Related to Warranty

Because of this diversity of purpose, product warranty has received the attention of researchers from many diverse disciplines. Blischke and Murthy (Blischke & Murthy, 1996) review the papers written by researchers from different disciplines. These issues are interlinked and hence proper study of warranty requires interdisciplinary approach. The study of warranties deals with different issues as illustrated by the following list: (Murthy & Djamaludin, 2002)

Legal: Court action, dispute resolution, product liability.

Economic: Market equilibrium, social welfare.

Behavioral: Buyer reaction, influence on purchase decision, perceived role of warranty, claims behavior.

Consumerist: Product information, consumer protection.

Engineering: Design, manufacturing, quality control, testing.

Statistics: Data acquisition and analysis, databased reliability analysis.

Operations Research: Cost modeling, optimization.

Accounting: Tracking of costs, time of accrual.

Marketing: Assessment of consumer attitudes, assessment of the marketplace, use of warranty as a marketing tool, warranty and sales.

Management: Integration of many of the previous items, determination of warranty policy, warranty servicing decisions.

Societal: public policy issues.

2.1.3 Express versus Implied Warranty

Warranty is a contractual theory of recovery governed by principles of sales (Kimble & Lesher, 1979). Researchers consider both express warranties and implied warranties In marketing literature. However, the vast operations management literature in warranty focuses most on the express warranty. The difference between express and implied warranties is explained in (Morgan, 1982)

A detailed description of express warranties can be found in Uniform Commercial Code (Register, 1979). The key in determining the existence of an express warranty is the extent of allowable seller puffing. General statements or affirmations that are nothing more than the seller's opinions about the product do not create an express warranty. Advertising, catalog statements, besides the explicitly stated warranty contracts with purchase can be the basis for an express warranty.

An implied warranty may exist as a matter of law even when no express warranty is stated. An implied warranty of merchantability is part of a sales contract, unless explicitly modified or negated, whenever the seller regularly offers the product in question for sale (Register, 1979). The implied warranty means that the item is of average quality and can be used for the purpose for which such a product typically is used. Advertising and labels can created certain reasonable expectations in consumer's minds which, if not fulfilled, could lead to breach of implied warranty.

2.1.4 Product Failure Analysis

A warranty contract specifies the expected performance and the redress available to the user if a failure occurs. A product fails if it is unable to perform satisfactorily its intended function when properly used. Failures are a function of several variables and these include product reliability (influenced by design and manufacturing decisions of the manufacturer) and the usage mode and environment (influenced by the consumer). A complete product failure analysis over warranty period can be found in (Blischke & Murthy, 1994). The product failures are modeled in literature either at component level or at product level in various distributions including exponential, Weibull, gamma, lognormal and mixed exponential.

At component level, failure of each component is modeled separately according to the component's physical, operational characteristics and reliability. The subsequent failures of a component may need to be modeled different from first failure of the same component depending on the type of components (reparable/ non-repairable) and the type of rectification actions (minor repair, major repair or replacement with a new or used one). According to that, the renewal process is classified into ordinary renewal process and delayed renewal process:

Ordinary Renewal Process If the product is non-repairable and replaced by a new one, the subsequent failures are modeled as the first failure with same distribution. Or, if the product is composed of many components and the failure is due to failure of one (or very few) component(s) and the failed item is restored to operational state by either repairing or replacing the failed components. (Gerner & Bryant, 1980)

Delayed Renewal Process If the product is repaired, the subsequent failures are modeled with different distributions, which incorporate the factor that the repaired used product is usually more vulnerable to failures than a new one.

Interactive Renewal Process If the failure of one component induces the failure of the other component, the subsequent failures of both components are modeled with different distributions. Murthy and Nguyen (1985b) developed a model for a two-component product with interactive renewal process. They then extended it to a system with n components (Murthy & Nguyen, 1985a).

2.1.5 Warranty Claims

The topic of warranty claims was first discussed by Gerner and Bryant (1980). They proposed a model to examine the function of expected warranty claims with an ordinary renewal process over product's life cycle. Let L denote the product life cycle. This is the period from the time the product is first introduced into the market to the instant when it is withdrawn from the market due to the appearance of a new and better product that replaces it. The sales rate during L increases initially and then starts decreasing over time. Let K denote the warranty period. In the case where items are sold with non-renewing FRW policy, the warranty claims occur over the period $[0, L + K)$. These occur as random points along the time axis and are a function of the reliability. For example, the first sale of a product occurs on day 0 and its one year warranty starts immediately. We expect its claim can possibly come in any time between day 0 and the end of year 1. If the product's life cycle is 5 years, which means the last product on the market is sold out on the last day of year 5, this product's warranty does not expire until the end of year 6. So the interim during which the warranty service center expects to receive claims is from day 0 to the end of year 6, which is $[0, L+K)$. The last Let $F(t)$ denote distribution function for the time to first failure and the failed item is made operational by minimal repair (Barlow & Hunter, 1960) with the failure rate just after repair being the same as that just before failure. If the time to repair is small relative to the time between failures, then warranty claims over the warranty period for a single item occur according to a point process with intensity function $\lambda(t)$ that is the same as the failure rate $r(t) = f(t)/[1-F(t)]$. Then with sales rate given by $s(t)$, the warranty claims occur according to a point process with intensity function given by (Blischke & Murthy, 1996)

$$v(t) = \int_{\phi}^{t} s(x)r(t-x)dx$$

where

$$\phi = max0, t-K$$

for $0 < t < L+K$. Note that $v(t)$ is also the expected claims rate (expected claims per unit time).The total expected warranty claims (EWC) over the life cycle is given by

$$EWC = \int_{0}^{K+L} v(t)dt.$$

Wasserman (Wasserman, 1992) uses a time-series approach to predict warranty claims.

2.1.6 Warranty Cost Analysis

The servicing of warranty requires service channels, repair facilities, spares, equipment to carry our repair and replacement, which involves various service cost. A lot of research has been done to predict failure, modeling warranty cost and analysis with time or usage as variable. Menke (1969) and Lowerre (1968) conducted the first probabilistic analysis of warranty cost for free replacement warranty under rebate policy. Blischke and Murthy (1992) explore the warranty cost from an engineering point of view. Blischke and Murthy

(1994) define several costs of interest to manufacturers and buyers:

1. Warranty cost per unit sale;

2. warranty cost over the lifetime of an item (life cycle cost LCC-I): This is buyer oriented and includes elements such as purchase cost, maintenance and repair costs following expiration of the warranty coverage, operating costs and disposal costs;

3. warranty cost over the product lifecycle (life cycle cost LCC-II): This is dependent on the interval over which buyers purchase the product. This life cycle begins with the launch of the product onto the marketplace and ends when it is withdrawn;

4. cost per unit time: This is useful for managing warranty servicing resources such as parts inventories, labor and costs over time with dynamic sales.

2.1.7 Warranty Types and Policies

Various warranty types have been practiced and studied from the perspectives of warranty design, warranty marketing, warranty servicing and warranty process management. Blischke and Murthy (1992) proposed a taxonomy for classification of warranties. The first criterion for classification of a warranty is whether or not the warranty requires development after sale of the product. Policies which do not involve product development can be further divided into two groups Group A, consisting of policies applicable for single item sales, and Group B, policies used in the sale of groups of items (called lot or batch sales). Group C policies involve development subsequent to the sale.

Policies in Group A can be subdivided into two sub-groups, based on whether the policy is renewing or non-renewing. In the case of one-dimensional renewing warranty policy, the warranty gets renewed with each failure occurring within the warranty interval so that the warranty ceases only when an item operates satisfactorily with no failures over the warranty interval. In the case of two-dimensional warranties, the warranty is characterized by a region over a two-dimensional plane with the two axes representing the age and usage of item. A further subdivision comes about in that warranties may be classified as simple or combination. The free replacement (FRW) and pro-rata (PRW) are two simple policies. A combination policy is a simple policy combined with some additional features or a policy, which combines the terms of two or more simple policies. Group C policies are used principally in industry and government acquisition of large, complex items for example locomotives, power plants, aircraft or military equipment. Such warranties involve several different characteristics (some of which are reliability oriented) that change over time. One such class of policies is the reliability improvement warranty (RIW) policies. The basic idea of a RIW is to extend the notion of a basic consumer warranty (usually the FRW) to include guarantees on the reliability of the item and not just on its immediate or short-term performance. This is particularly appropriate in the purchase of complex, repairable equipment that is intended for relatively long use. The intent of reliability improvement warranties is to negotiate warranty terms that will motivate a manufacturer to continue improvements in reliability after a product is delivered.Consumer durables are sold with warranty policies from Group A as they are bought as single items. Industrial and commercial products are sold with policies from Group A if bought individually or with policies from Group B if bought in lots. Finally, specialised industrial and defence products built to customer requirements (and often involving new and cutting edge technologies) are sold with warranty policies belonging to Group C.

Blischke and Murthy (1992) give details of several different warranty policies belonging to each of the three groups. A thorough review of papers on warranty policies can be found in (Djamaludin, Murthy, & Blischke, 1996). Warranty policies that receive much research interest are base warranty, extended warranty, lifetime warranty, warranty for used products and service contract.

Base and Extended Warranty

A base warranty is an integral part of a product sale and its cost is factored into the sales price (Murthy & Jack, 2003). It is usually free of charge to consumers as assumed in this paper. Consumers who prefer extra protection purchase additional coverage in the form of extended warranty which is an obligation of responsibility assumed by the manufacturer or dealer for further service to buyers beyond base warranty for a certain premium. Limited research has been done on extended warranty. Because consumers differ in their choices of the extended warranty some papers analyze the impact of consumer's behavior and financial status to their warranty preference, and then explore the best strategy to attract consumers that value more to the manufacturer/ dealer (Padmanabhan, 1995; Lutz & Padmanabhan, 1998). Padmanabhan (1995) used utility functions to model the buyers and manufacturers risk attitudes and discovered that heavy users of the product prefer extended warranty more than the light users. Lutz and Padmanabhan (1998) proposed that high-income consumers are more attracted to the extended warranty because they require lower marginal utility of their wealth. Because of that the manufacturer is attempted to the moral hazard to offer different versions of the products to different types of consumers. Higher quality and better warranty products with higher price target high valuation high income consumers; lower quality, lower warranty products with lower price target low income consumers. Other research focuses on the duration and cost of extended warranty (Hollis, 1999; Mitra & Patnakar, 1997; Rinsaka & Sandoh, 2001). Mitra and Patanakar (1997) analyzed the warranty policy under which the buyer has the option to extend the warranty till the product failure has not occurred during the base warranty period. If failure occurs during base warranty, the consumer will receive total price refund; if it occurs during extended warranty, he will receive a refund which is proportional to the time left in warranty. Rinsaka and Sandoh (2001) analyzed the extension of the contract period. Yeh and Peggo (2001) proposed optimal cost models for extended warranty according to two cost criteria, namely total expected discounted cost and long-run average cost per unit time.

Warranty for used products

Warranty is also offered with sale of second-hand products from the dealer, such as used cars. It is estimated that in US the used cars sales was 40% of that for new cars in numbers and 22% in values.(Genesove, 1993) Decisions related to second-hand products are more complex compared to new products due to the fact that each second-hand product is statistically different due to variation of age, usage and previous maintenance history. The paper by Chattopadhyay and Murthy (1996) is the first one to set up a cost model to analyze for a proper warranty policy for second-hand products. In another paper they proposed stochastic models to estimate the expected warranty cost for second-hand products (Chattopadhyay & Murthy, 2000). They then developed and analyzed stochastic models for three new cost sharing warranty policies for second-hand products, i.e., specific parts exclusion (SPE), limit on individual cost (LIC) and limit on individual and total cost (LITC) (Chattopadhyay & Murthy, 2001). Under the SPE policy the components of the product are grouped into two disjoint sets: I for inclusion and E for exclusion. Failures due to components in set I are rectified at no cost while failures due to components in E are rectified with costs borne by the buyer. Under LIC policy the dealer takes the cost of a rectification is it is below a certain limit c_i, otherwise the buyer pays the extra. If the warranty policy is an LITC, similar to LIC, the cost to the dealer has an upper limit for each rectification, however, it also has an upper limit for the total warranty servicing cost; when the total cost exceeds the limit, the warranty terminates, and the buyer pays the exceeded cost of that rectification.

Lifetime Warranty

With enhanced customer demand on aftermarket service and advanced technology that ensures longer-life of durable products, manufactures start to offer lifetime warranties. Lifetime warranty means a commitment to provide repair or replacement service in case of failure throughout the useful life of the product or the buyer's ownership of the product (Rahman & Chattopadhyay, 2004). Usually the warranty provider defines the termination of such warranty for reasons of technological obsolescence, design modifications, or even with the change of the ownership (Rahman & Chattopadhyay, 2004). Rahman and Chattopadhyay (2004) provided a framework of four types of product lifetime.

- Technical life/ physical life: the period over which the product lasts physically, until replacement or major rehabilitation.
- Technological life: The period until technological obsolescence dictating replacement due to the development of a superior alternative technology.
- Commercial life:The period over which the demand for the product exists on the market.
- Ownership life/ social and legal life: The period until human desire or legal requirement dictates replacement or change of ownership occurs.

Service Contract

The maintenance of some complex industry equipment, such as a production line or a power generation turbine, requires expertise and specialized facility. Service contract is a maintenance contract between the user and the service provider, who will carry out maintenance actions on the user's demand. The provider could be the manufacturer or an independent third party. The difference between a warranty and a service contract is that the latter is entered into voluntarily and is purchased separately the buyer may even have a choice of terms, whereas a warranty is part of product purchase and integral to the sale. A few research work on service contract can be found in (Ashgarizadeh & Murthy, 2000), (Blischke & Murthy, 2000) and (Murthy & Ashgarizadeh, 1995).

2.1.8 Warranty Servicing Actions

The warranty servicing cost for a claim includes repair and/ or replacement, shipping and handling and other administrative costs. The manufacture/ dealer chooses the most cost efficient action, repairing or replacing with a new or used item, with considerations of long term costs and benefits, such as customer goodwill and future failures over the warranty period. It is shown that costs can be reduced by conducting an inspection and cost assessment before the service (Murthy & Nguyen, 1988). If the repair cost exceeds a certain limit, replacement is suggested, otherwise the repair is carried out. Murthy and Jack(2003) analyzed the cost of repairing failed item over the warranty period and proposed that it can be minimized through optimal corrective maintenance decisions. Chukova et al.(2004) developed a model to analyze warranty cost when imperfect repair is undertaken. Iskander et al. (2005) suggested a new repair replacement strategy for products sold with two-dimensional warranties.

2.1.9 Warranty Logistics

Warranty logistics deals with all the issues relating to warranty servicing. It has a significant impact on both warranty servicing cost and customer satisfaction. Murthy et al.(2004) give a thorough review of the literature on warranty logistics and discuss potential future research for operations researchers.

Among all the costs of interest reviewed above, the warranty servicing cost per unit time is the most relevant from the warranty logistics perspective. This includes the costs associated with the operation of service centers (facilities and equipment) and warehouses (spare parts inventories), the servicing of claims (material and labor for shipping and handling), and spare part inventories. It comprises of fixed costs that are independent of claims rate and variable costs that are dependent on the claims rate. As claims occur in an uncertain manner and the cost of each repair is also a random variable.

As customers demand greater assurance and most countries have either enacted or are in the process of enacting stricter legislation to protect consumer interests, proper management of warranty logistics is becoming critical for business survival and success. To achieve this, the manufacturer can choose to perform warranty service inhouse or outsource it to a third party. The ability to provide satisfactory warranty service depends on the manufacturer's / service delivery network that involves service facilities for spare parts storage and provide a base for field service and a delivery system that transport returned failed items and materials needed for warranty servicing and rectified items. The realized service level of the network is also affected by the geographical distribution of customers and by their demand for prompt response.

The process of design and building a service network, and providing ongoing service via the network involve several strategic and operational issues. The strategic issues are

- the number of service centers and their locations,
- the capacity and staffing level for each service center to ensure desired response time for customer satisfaction, and
- whether to own these centers or outsource them so that the service is carried out by independent agents.

The tactical and operational issues are

- transportation of the materials needed for warranty servicing,
- spare parts inventory management,
- scheduling of jobs, and
- optimal repair or replace decisions.

Warranty logistics: Strategic issues

As mentioned above, the main strategic issues are the location and capacity of warehouses for stocking spares needed, service centers for carrying out product rectification, and the channels for warranty servicing. The locations of both service centers and warehouses depend on the geographical distribution of customers who have bought the product and the type of product and its reliability characteristics. Another strategic decision with regard to warranties is whether to undertake warranty service inhouse or outsource it to an external agent. When the channel involves an independent agent, several problems may arise for the manufacturer, such as the monitoring of service level and contracts between parties. These issues need to be understood and resolved properly.

Location of warehouses Warehouses are needed to stock spare parts for components that are non-repairable and need to be replaced on failure. The number, location, capacity and size of demand served by the warehouses determine the paths by which products are directed to the marketplace. As a result, the distribution network comprises of warehouses, manufacturing plants and service centers as the nodes, and a transportation system as the routes. Decisions of warehouse location need to consider all product movements and associated costs. The objective function can be the average travel time, maximum travel time, territory covered, total system cost or utilization of the facilities etc. Mirchandani and Francis (1990), Dresner (1992), Daskin (1995), and Beckmann (1999) are examples of works dealing with the location problem.

In the logistics literature, components that are non-repairable and need to be replaced on failure are called “discardables”. Depending on the geographical area of the market, the manufacturer might need to have a network of warehouses with a multi-echelon structure involving one or more levels. For example, a multi-national manufacturer might have a regional warehouse (level 3) receiving parts from different component manufacturers and feeding to national warehouses (level 2) which feed local distributed warehouses (level 1)which, in turn, feed parts to service centers. The location of the warehouses and the capacity of each warehouse are determined through an analysis of the quantities of different components that need to be stocked. This problem has received some attention in the logistic literature. Handler and Mirchandani (1979) classify location problems based on the objective function, the points of demand, the potential facility sites and the number of facilities to be located. A small illustrative sample of related journal papers is as follows: Daskin and Stern (1981) deal with location of emergency medical services, Alfredsson (1995) studies location of repair facilities, and Dasci and Verter (2001) explore location in the context of production distribution systems. Schilling et al. (1993) explore covering models to decide on facility location and Owen and Daskin (1998) discuss strategic facility location.

However, most models do not take into account the location of service centers and the reliability characteristics of the product. Future research may modify the current models to incorporate the following issues:

- Transportation time and cost for moving parts between warehouses (in the case of multi-echelon warehouses) and from warehouses to service centers;
- the cost of operating the warehouses;
- the capacity of each warehouse based on the demand for spares from the various service centers that are serviced by the warehouse.

Location of service centers Many products are complex systems that can be decomposed into many different levels. When an item fails, the first task is to determine and identify the most likely cause of failure. Certain products (such as an elevator in a multistorey building) require on-site evaluation of the failed item. For others, the failed item is brought to either the retailer (in the case of most consumer durables) or to some designated service center. For most products, the failed item is made operational through appropriate actions at this level. However, in some instances, it is not possible to rectify all failures at this level due to a lack of resources such as special equipment and/or an appropriately trained workforce. In this case, the failed component needs to be removed and shipped to a higher level service center for rectification. Often, there can be more than two levels depending on the complexity of the product and the type of resources needed for rectification. For example, in the case of a jet engine, it might involve a service facility at major airports (level 1) followed by a national (or regional) service center (level 2) and a service center at the manufacturing plant (level 3). This problem has received some attention in the logistic literature and is referred to as level-of-repair analysis (LORA). This basically deals with the task of determining whether an item is to be treated as discardable (also called consumable) or as repairable. If the item is to be treated as repairable, the objective is to determine where it should be repaired in a multi-echelon repair facility. Alfredsson (1997) deals with decisions with regard to LORA and the spare parts and test equipment needed to support a system. Barros and Riley (2001) explores the optimization of the LORA. A related issue is the choice of the best from a set of desirable maintenance actions, which is discussed in Cassady et al. (2001). Fortuin and Martin (1999) have thoroughly examined this topic. Models to determine the number of levels, the location of the service centers and their capacities must take into account the following:

- Transportation time and cost for moving failed and repaired items between service centers;
- the cost of operating the service centers (equipment and skilled persons needed at each center);
- the capacity needed at each service center, depending on the demand for services at the center;
- this in turn depends on the geographical distribution of sales, as well as product reliability.

The location problem needs to address the following issues:

- Customers coverage (so that all customers are covered);
- distance that a failed item needs to travel;
- distance that a repairman has to travel in case of a field visit or that a customer has to travel to bring a failed item to a service center or collection point.

Spare parts demand Besides the location and capacity decisions, the management of inventories and the ordering policies of the spare parts warehouse also impact the service level and total system costs. As other inventory decisions in a supply chain setting, there is a trade-off between the cost associated with, and the benefits derived from holding inventories. To derive an optimal inventory policy, we need to model the spare parts demand patterns.

At the highest level of warehouse, the total demand for spares of a component over the product lifecycle can be modeled as follows. If the replacement time of a failed component is small relative to the mean life of the component, then the demand for replacements over the warranty period occur according to a point process with intensity function $\rho(t)$ given by Blischke and Murthy (1996), i.e.,

$$\rho(t) = \int_{\phi}^{t} s(x)m(t-x)dx,\ \ for\ \ 0 < t < L + W,$$

where $m(t)$ is the renewal density function associated with the component failure density function $g(t)$ and is given by

$$m(t) = g(t) + \int_{0}^{t} m(t-x)g(x)dx.$$

The expected replacement (or demand) rate is also given by $m(t)$. The expected total spares (ETS) required over the product life cycle is given by

$$ETS = \int_{0}^{L+W} \rho(t)dt.$$

For further discussion on spares in the context of post-sale support, see Cohen and Lee (1990).

Service Channels In practice products are distributed through one of four main distribution channels to reach customers:

1. manufacturer direct,
2. company-owned dealerships,

3. independent retailer and

4. some combination of the first three.

The co-ordination between the different entities has received considerable attention in the marketing literature in the context of both supply chain and service response logistics. Works on management of marketing channels, for example Lewis (1968) and Rosenbloom (1995), deal with this topic in more detail. The linkage between product distribution and service support channels is discussed in Loomba (1996).

Similar to distribution channels, a manufacturer can choose between the two following options to serve their warranty (Murthy, Solem, & Roren, 2004):

1. Service provided by the manufacturer (through retail or service centers owned and operated by the manufacturer);

2. service provided by an independent agent.

Independent service agents Many manufacturers employ independent service agents to carry out the warranty servicing under a properly drafted contract. Two contracts are commonly presented in the literature:

Contract A: The manufacturer pays a lump sum to the agent and in return the agent has to service all claims during the warranty period at no additional cost to the manufacturer.

Contract B: The service agent charges the manufacturer for each warranty service.

Several new issues arise as the interests of the manufacturer and the agent are different. Manufacturers seek to optimize the warranty service quality to protect their reputation with minimum payment to the agent; while the objective of the agent is to maximize its profit, which may put customer service level under risk. Related issues that are considered in the literature include (Murthy, Solem, & Roren, 2004):

Informational asymmetry: The manufacturer has better knowledge of product reliability compared to the agent and similarly the agent has better information regarding field failures than the manufacturer. This asymmetry can lead to each party deciding on actions that are optimal from their own individual perspective but overall sub-optimal.

Moral hazard: This situation arises when the agent shirks in the effort expended (under Contract A) or carries out over-servicing (under Contract B) and the manufacturer is unable to observe the service agent's effort. In the former case, it can lead to customer dissatisfaction and thus affecting the manufacturer's reputation and sales.

Monitoring: The manufacturer can obtain new information by monitoring the agent's actions. Such information will allow the manufacturer to assess the warranty servicing carried out by the agent. However, this results in additional effort and cost to the manufacturer.

Adverse selection: This issue arises when the manufacturer has to choose one or more service agents to carry out warranty servicing through a pool of service agents. The service agents can misrepresent their ability and competencies and the manufacturer is unable to assess them prior to the signing of the contract. This can lead to the selection of inappropriate agents for warranty servicing.

Incentives: The manufacturer can provide proper incentives to a service agent so that the actions of the agent are in the best interests of the manufacturer and avoid the need for monitoring. A proper contract provides the right incentives for the service agent to provide the optimal amount of effort.

Agency cost: The structuring, administering, and enforcing of contracts causes result in a cost which is referred to as the agency cost.

Risks: In general, the manufacturer and the service agents have partly differing goals and risk preferences and these impact on their individual actions.

The "agency theory" is applied to deal with the above issues. The agency theory has been studied extensively and the literature is vast,(see for example, Eisenhardt, 1989). The study of warranty servicing logistics based on agency theory is a topic that has not received much attention and offers scope for considerable new research.

Warranty logistics: Tactical and operational issues

The tactical and operational issues deal with short term activities and decisions at the service center level. These issues include spare part inventory levels, transportation of spares from warehouses to service centers, jobs scheduling and repair versus replacement decisions.

Spare part inventory The key issues in spare parts inventories are the following (Murthy, Solem, & Roren, 2004):

1. Which components should be carried as spare parts?
2. What should be the inventory levels?
3. When should the spares be reordered?
4. What quantities of spares must be ordered?

These need to be linked to failures of components over time and these in turn are related to the sales over time and component reliability.

The existent literature on inventory management is considerable. Most models dealing with spare parts inventory assume very simple forms for the depletion of inventory. In the warranty-servicing context, the depletion of a particular component occurs according to a point process with the intensity function related to the dynamic sales over the region serviced by the servicing center and the reliability of the product. Optimal decisions of inventory levels and ordering policies need to take into account this stochastic nature of depletion. Besides, most products are complex systems involving many components. They differ in their usage and reliability, thus, have different failure rates. These add extra complexity to the modeling of overall inventory policies.

Material transportation In supply chain raw materials and components flow from suppliers to manufacturing plants and finished products flow from plants to retail markets via a hierarchy of warehouses and retail outlets. Similarly, in warranty service logistics failed items flow from end users to service center, spare parts flow from high level warehouse to service centers via multi-echelon holding points, and rectified products are transported from service centers to users.

Transportation of materials has received significant research attention in materials management and operations management. Related more specifically to warranty servicing logistics, one can define three kinds of material transportation as indicated below:

1. Transportation of failed units from lower level to higher level in the case of multi-echelon service structure;
2. transportation of repaired items from service centers to customers or pick up points where customers can collect them;
3. transportation of spares to and from warehouses.

The quantities transported are random variables related to sales, product reliability and location's distribution. Transportation can be carried out either by the manufacturer internally or outsourced to an independent agent. In the latter case, a contract between the manufacturer and the independent agent needs to take into account the time and cost of transportation, and the agreement between the parties, that specify the service quality metrics and payment forms.

Scheduling of jobs, repairs and traveling repairman problem Warranty service involves repairing or replacing failed items. The scheduling and execution of the transportation impacts both manufacturer's cost efficiency and customer satisfaction. The failed item may be brought to a service center or retail outlet by the customer while the cost is covered by the manufacturer/ dealer. Items are processed usually based on the first come first served rule. If we consider some form of penalty resulting from delay in our model, such as lower customer satisfaction or higher shipping cost through expedited shipping, the objective function is to minimize the penalty, or some form of overall delay. If a repairman needs to go onsite to examine and repair the failed item, such as an elevator in a building, the scheduling of jobs is a traveling repairman's problem. This has been discussed by Afrati, Cosmadakis, Papadimitriou, Papageorgiou and Papakostantinou (1986), Yang (1989) and Agnihothri (1998). The objective of job scheduling is to reduce the time spent in traveling between jobs.

Replace versus repair strategies Whenever a repairable item fails under warranty, the manufacturer has the option of either repairing the failed item or replacing it by a new item. In the case of repair, the manufacturer needs to choose between different repair actions, which impact on customer satisfaction, as well as the warranty servicing cost. Murthy et al. (2004) reviewed some of the strategic issues of replace versus repair decisions.

Strategies based on age and usage at failure Here the decision to repair or replace is based on the age of the item at failure (in the case of one dimensional warranties) and on the age and/or usage (in the case of two-dimensional warranties). The optimal strategy is selected to minimize the expected cost of servicing the warranty over the warranty period. Blischke and Murthy (1994, 1996) discuss two sub-optimal strategies for one-dimensional warranties and Jack and Van der Duyn Schouten (2000) deal with the optimal strategy. Jack and Murthy (2001) examine a suboptimal policy that is very close to the optimal strategy involving at most one replacement over the warranty period.

Cost repair limit strategy In general, the cost to repair a failed item is a random variable, which can be characterized by a distribution function $H(t)$. Analogous to the notion of a failure rate, one can define a repair cost rate given by $h(z)/[1 - H(z)]$ where $h(z)$ is the derivative of $H(z)$ Depending on the form of $H(z)$, the repair cost rate can increase, decrease or remain constant with z. A decreasing repair cost rate is usually an appropriate characterization of the repair cost distribution (Mahon & Bailey, 1975). Optimal repair limit strategies are discussed in Blischke and Murthy (1994) and Zuo et al.(2000).

Warranty logistics: Other issues

Customer satisfaction Customer dissatisfaction can arise due to poor performance of the purchased item and/or the quality of warranty service provided by the manufacturer. In either case, there is a negative impact on overall business performance. This could be either due to dissatisfied customers switching to a competitor or losing potential new customers due to negative word-of-mouth effect. The consequence of poor warranty servicing is more difficult and costly to rectify and hence it is very important that the manufacturer avoids this occurring in the first instance. A proper contract between the manufacturer and service agents and the monitoring of the agent's actions are critical for ensuring high level customer satisfaction.

Service quality has received a lot of attention in the literature; see for example Haugen and Hill (1999). There are several dimensions of service quality and many of these are intangible and can vary significantly from customer to customer. Often customers can have undue expectations regarding product performance for a variety of reasons (exaggerated statements made during promotion, customer being not fully informed etc.). However, other dimensions are more tangible and can be objectively assessed. These include response time to attend to a warranty claim, the time for rectifying a failed item, delays resulting from lack of spares, workshop resources etc. Through effective warranty logistics the negative impact, resulting from these can be minimized.

Dispute resolution Disputes in the context of warranties arise when the manufacturer (or service agent) refuses to admit a warranty claim as a valid (or legitimate) claim for a variety of reasons (for example, misuse of the product) or the customer is unhappy with the warranty service provided. In either case, the problem needs to be resolved. The former is a legal issue and has been discussed extensively in the legal literature. The latter is influenced by the warranty logistics, i.e, poor warranty logistics can lead to greater number of disputes that need to be resolved. The resolution can involve a third party (small claims tribunal for relatively inexpensive claims or higher legal institution in the case of more costly claims). This topic has received some attention, see, for example Steele (1975), Palfrey and Romer (1983) and Cooter and Rubenfeld (1989).

Data collection and analysis During the servicing of warranties a lot of data is generated. such data can be classified into different categories as indicated below.

Product related: Modes of failures, time between failures, operating environment etc.

Customer related: Satisfaction with regards to product, warranty service etc.

Servicing: Spare parts inventories, utilization of service centers, transportation of material etc.

Economic: Costs associated with different aspects of warranty servicing.

These data need to be collected properly and analyzed to extract useful information that can be used for improving service activities. Technical data are relevant for design changes, servicing data are important for improving warranty logistics and, customer and financial data are useful for improving overall business performance.

Product recall Occasionally, a manufacturer finds it necessary to recall either a fraction or all of the items sold, for some rectification action as a way of reducing the overall warranty servicing costs. The recall of only a fraction of the total production arises when items are produced in batches and some of the batches are defective due to inferior components or materials having been used that are not detected during quality control. A total recall situation usually arises because of poor design specifications that can lead to malfunction under certain conditions and is discovered only after the items have been produced and sold. In such cases, the manufacturer can be held responsible for damages caused under the terms of warranty for fitness and the recall is to replace one or more old components by newly designed ones.

2.1.10 Warranty and Quality

Conventional wisdom supports the notion that better product quality means lower warranty costs for the manufacturer, and lower maintenance costs for the users of a manufactured product. This section reviews the literature pertaining to quality measurement, quality improvement strategies, and their impact on warranty costs.

Measuring Quality

One of the traditional measures of quality is the mean time between failures (MTBF). However, because the mean alone provides incomplete information of the failure distribution, most reliability measures in the literature require high-order distributional measures or the entire failure time distribution. One of them is to take the variance into account, which makes it possible to define meaningful means of quality improvement. One way is to increase the MTBF while keeping the variance constant. This is largely achieved by better production inputs or operational procedures. Adoption of new manufacturing technology upgrades, on the other hand, may impact on higher moments of the failure time distribution, such as variance reduction.

Stochastic ordering and mixture models are used for measuring quality improvement (Sahin & Polatoglu, 1998). The theoretical base of this work is a time-varying failure-rectification process (which includes replacement, minimal repair, and imperfect repair) as alternative rectification modes that may be available to the manufacturer or the user in warranty-servicing. The use of this process enables one to investigate jointly optimal repair-effort/warranty-policy and repair-effort/maintenance-strategy configurations for repairable units.

Quality Improvement and Warranty

Sahin and Polatoglu (1998) investigate the impact of product quality on warranty and maintenance costs and strategies, from the perspectives of both manufacturers and users. On general grounds, there are three ways to improve product quality: upgrading the manufacturing process, performing inspection on final products before release, or a burn-in program to eliminate infant mortality.

The manufacturing process can be improved through higher quality inputs, superior production equipment, and more effective process control or inspection during production. Murthy and Nguyen (1998) link manufacturing quality with product reliability by modeling the failure time distribution of the product by $F(t; \omega)$, where ω indicates the manufacturing quality. They develop a model to optimally select the warranty period together with product reliability and price. Mamer (1987) examines the relationship between warranty and quality control using the same representation. He determines a warranty cycle length that allows the expected discounted cost of ownership of a sequence of products under warranty with a random parameter to be the same as a target level of ownership cost.

Another way to improve quality is to perform outgoing inspections to weed out nonconforming items. These can be on the samples in the form of life testing or on every item with nondestructive testing. Djamaludin et al. (1994) investigate life testing as a means to prevent defective items from being released and examine the optimal testing period under different warranty policies. Unit manufacturing cost is represented as the sum of the production cost per item, testing cost per item (which is an increasing function of the testing period) and the cost of scrapping a defective item. This cost is then added to the unit warranty cost per item, the optimal testing period is derived, and its cost is compared with the total unit cost under no inspection to determine the best policy.

The reliability of a durable product can also be improved through a burn-in program, which has its origins in the fact that in some manufacturing processes the majority of failures occur either right at the beginning of a product's life or close to its end. A burn-in test is designed to catch some of those infant stage failures and remove the faulty parts before failure. Most maintenance strategies are based on the use of planned replacements made before failure and service rectifications made after failure. Planned replacements are generally less expensive than service replacements because they are performed regularly within predetermined time windows and involve a large number of units which result in economies of scale. The service replacements, on the other hand, involve interruption cost for repair performance and failure cost for user.

2.2 Literature on Supply Chain Coordination Contracts

Parties in a supply chain have full or partial access to the information needed to determine the optimal actions for the supply chain, which may maximize their own objectives (profit for risk neutral parties, for example). However, if they do not, they may lack the incentive to implement those actions. To create such incentives the firms can adjust their terms of trade via a contract that establishes a transfer payment scheme. Motivated by these challenges, a substantial amount of research has been done to explore supply chain coordination via contracts. This section reviews the supply chain literature pertaining to the management of incentive coordination with contracts.

As mentioned earlier, a contract is said to coordinate the supply chain if the realized decisions made by the parties in the supply chain actually optimizes the supply chain's performance and no party has the incentive to move away from this optimum solution under the same circumstance. In the newsvendor model the action to coordinate is the retailer's order quantity, which in our settings, the manufacturer's production quantity as well.

Flexibility is another issue that is worth of study. A coordinating contract is flexible if it allows for multiple divisions of the supply chain's profit among the parties through adjusting the parameters. If it is able to allocate profits arbitrarily, then there always exists a contract that Pareto dominates a noncoordinating contract, i.e., each party's profit is no worse off and at least one firm is strictly better off with the coordinating contract. With any given profit allocation, for instance, equally between manufacturer and retailer, each party obtains more from sharing a “bigger pie”.

Cachon(2003) provides a thorough review of supply chain contract models in the literature. A number of different contract types are identified and their benefits and drawbacks are illustrated. The same analysis recipe is generally followed: identify the type of contracts that can coordinate the supply chain, propose the supply chain optimal actions, determine for each contract type the set of parameters that achieves coordination, and evaluate for each coordinating contract type the possible range of profit allocations, i.e., what fraction of the supply chain's profit can be earned by each member in the supply chain with a coordinating contract. Implementation issues are then explored: e.g., is a contract type compliant with legal restrictions; what are the consequences for failing to comply with the contractual terms; and what is a contract's administrative burden (e.g., what types of data need to be collected and how often must data be collected).

2.2.1 Simple Newsvendor Coordination

Under a newsvendor environment, our model considers a single manufacturer selling a product to a single retailer. The following sequence of events occurs in this game: the manufacturer offers the retailer a contract; the retailer accepts or rejects the contract; assuming the retailer accepts the contract, it specifies an order quantity, q_r, to the manufacturer; the manufacturer produces and delivers the order to the retailer before the selling season; season demand occurs; and finally transfer payments are made between the firms based on the agreed upon contract. If the retailer rejects the contract, the game ends and each firm earns a default payoff.

With a standard wholesale price only contract, it has been shown that the retailer does not order enough inventory to maximize the supply chain's total profit due to “double marginalization”. Hence, coordination requires that the retailer is given an incentive to increase its order size. Several different contract types are shown to coordinate the supply chain and arbitrarily divide its profit; namely buyback contracts, revenue sharing contracts, quantity flexibility contracts, sales rebate contracts and quantity discount contracts.

2.2.2 Coordinating Contracts

Different contract models are reviewed in the light of their coordination properties in the following sections. In general, a contract coordinates the retailer's and the supplier's actions whenever each firm's profit is an affine function of the supply chains profit, which resembles a profit sharing arrangement. Jeuland and Shugan (1983) note profit sharing can coordinate a supply chain, but they do not offer an influential contract for achieving profit sharing. Caldentey and Wein (2003) show profit sharing occurs when each firm receives a fixed fraction of every other firm's utility.

The Wholesale Price Only Contract With a wholesale price only contract the manufacturer charges the retailer W per unit purchased. Bresnahan and Reiss (1985) study the wholesale price contract with deterministic demand. Lariviere and Porteus (2001) conducted a complete analysis of this contract in the context of the newsvendor problem. They show that only if the manufacturer charges a price less than the production cost, the wholesale price contract coordinates the channel. This means, the manufacturer has to earn a non-positive profit. As a result, the wholesale price contract is generally not considered a coordinating contract. Spengler (1950) was the first to identify the problem of "double marginalization. The supply chain's optimal decision is to hold inventory until the marginal profit equals the marginal cost. The same holds for the retailer. Since the retailer's marginal cost is higher as long as the manufacturer charges a wholesale price higher than the production cost, the retailer always order an amount less than the chain optimum.

The wholesale price only contract is commonly observed in practice because it is easy to implement. The contract designer (the manufacturer or the retailer depending on their respective power positions) may prefer the wholesale price contract over a coordinating contract if the additional administrative burden associated with coordinating the contract exceeds its potential profit increase.

The contract model has been extended to multiple periods or multiple newsvendors. Cachon (2004) studies a two period version of the model which has excess inventory and demand updating. Dong and Rudi (2004) study the wholesale price contract with two newsvendors and transshipment of inventory between them. They find that the manufacturer is generally able to capture most of the benefits of transshipment and the retailers are worse off with transshipment. This is consistent with Lariviere and Porteus (2001) finding that the manufacturer is better off and the retailer worse off with less variable demand. Chod and Rudi (2005) study a manufacturer selling a single resource to a downstream firm that can use that resource to produce multiple products. Gilbert and Cvsa (2000) study the wholesale price contract with demand uncertainty and costly investment to reduce production costs. They demonstrate that a trade exists between the beneficial flexibility of allowing the wholesale price to adjust to market demand and the need to provide incentives to reduce production costs.

The Buyback Contract With a buy back contract the manufacturer charges the retailer w per unit purchased, but buys back the unsold inventory from the retiler at b per unit at the end of the season. Several motivations exist for the manufacturer to implement this return policy (Padmanabhan & Png, 1995):

- A manufacturer may offer a return policy to prevent the retailer from discounting left over items, thereby weakening the suppliers brand image. For instance, suppliers of fashion apparel have large marketing budgets to enhance the popularity of their clothes. It is difficult to convince consumers that your clothes are popular if they can be found in the discount rack at the end of the season.

- A manufacturer may accept returns to rebalance inventory among retailers. A number of papers consider stock rebalancing in a centralized system (see Lee, 1987, Tagaras and Cohen 1992, and in decentralized systems (Rudi & Pyke, 2001; Anupindi, Bassok,

& Zemel, 2001).

Pasternack (1985) conducted a detailed analysis of buy back contracts in the context of the newsvendor problem and demonstrates that the buy back contract coordinates with voluntary compliance and the coordination of the supply chain requires the simultaneous adjustment of both the wholesale price and the buy-back price. This implies that in the bargaining process, negotiations should always allow simultaneous changes in both the wholesale price and the buy-back price to reach a pareto optimal decision.

There is a substantial amount of existing work on buyback contracts. Padmanabhan and Png(1997) examines the situation where a manufacturer uses a buyback contract to manipulate competition between retailers. Emmons and Gilbert (Emmons & Gilbert, 1998) study buy back contracts with a retail price setting newsvendor. Taylor (Taylor, 2002) incorporates a buy back contract with a sales rebate contract to coordinate the newsvendor with effort dependent demand. Donohue (Donohue, 2000) provides analysis of buy back contracts in a model with multiple production opportunities and improving demand forecasts. Anupindi and Bassok(1999) demonstrate buy back contracts can coordinate a two-retailer supply chain in which consumers search among the retailers to identify available inventory. Lee, Padmanabhan, Taylor and Wang(2000) model price protection policies in a way that closely resembles a buyback policy.

The Revenue Sharing Contract With a revenue sharing contract the manufacturer charges W_r per unit purchased plus a percentage of retailer's revenue. Let ϕ be the fraction of supply chain revenue kept by the retailer, i.e..,$(1-\phi)$ is the fraction of revenue that the manufacturer earns. Revenue sharing contracts have been applied in the video cassette rental industry with fully success. Cachon and Lariviere(2005) provide an analysis of these contracts in a more general setting and demonstrate that revenue sharing contracts coordinate the supply chain and arbitrarily allocate its profit to the two parties. Mortimer(2006) provides a detailed econometric study of the impact of revenue sharing contracts in the video rental industry. He found that the adoption of these contracts increased total supply chain profits by about seven percent. Dana and Spier(2001) study these contracts in the context of a perfectly competitive retail market. Consistent with Mortimer(2006), they find that revenue sharing achieves the first best outcome by softening retail price competition without distorting retailers' inventory decisions. Gerchak, Cho and Ray (2006) consider a video retailer that determines the number of tapes to purchase and time to keep them. Revenue sharing coordinates their supply chain, but provides a unique division of profit. They, thus, propose a licensing fee as an additional lever.

The Quantity Flexibility Contract With a quantity flexibility contract the manufacturer charges W_q per unit purchased but then compensates the retailer for its losses on unsold units up to a threshold. To be specific, the retailer receives a credit from the manufacturer at the end of the season equal to $(W_q+c_r-v)\min(I, \delta q)$ where c_r is the retailer's per unit procurement cost, v is the salvage value, I is the amount of left over inventory, q is the number of units purchased and $0 \leq \delta \leq 1$ is a contract parameter. Thus, quantity flexibility contract fully protects the retailer on a portion of its order, whereas the buyback contract provides only partial protection on the retailer's entire order. If the manufacturer does not compensate the retailer for the procurement cost then the retailer would receive only partial compensation on a limited number of units, which is called a backup agreement. Those contracts are studied by Pasternack (1985), Eppen and Iyer (1997) and Barnes-Schuster, Bassok and Anupindi (2002). It has been found that quantity flexibility contracts coordinate the supply chain only with forced compliance, whereas the manufacturer has to deliver the order amount; otherwise it has incentives to deliver less than the order.

Tsay (1999) studies supply chain coordination with quantity flexibility contracts where the retailer receives an imperfect demand signal before submitting its order. Nevertheless, since production is completed before actual demand information is available, the centralized solution in Tsay(1999) is the same as the above newsvendor problem. The demand signal does not change the analysis or the outcome if the retailer returns units only at the end of the season: by then the demand signal is no longer relevant. However, if the retailer is able to return units after observing the demand signal and before the selling season starts, then this signal does matter. Because the inventory that is produced is sunk, the supply chain optimal solution is to keep all inventory at the retailer regardless of the signal received. Allowing the retailer to return inventory (alternatively, allowing the retailer to cancel a portion of the initial order) creates a stranded inventory problem. In other words, inventory can be stranded at the manufacturer's, and is not available for satisfying demand. In such a situation, as shown in Tsay (1999), a quantity flexibility contract may actually prevent supply chain coordination.

Other papers that study the quantity flexibility contract, or a closely related contract include Tsay and Lovejoy(1999), Bassok and Anupindi(1997), Cachon and Lariviere(2001). Tsay and Lovejoy (1999) study quantity flexibility contracts for multiple locations, multiple demand periods with lead time and demand forecast updates. Bassok and Anupindi(1997) propose a rolling horizon flexibility contract and provide an in-depth analysis for a single stage system with more general assumptions than Tsay and Lovejoy(1999). In multiple period models it is observed that these contracts dampen supply chain order variability, which is a potentially beneficial feature that the single period model does not capture.

The Sales Rebate Contracts With a sales rebate contract the manufacturer charges W_s per unit purchased but then gives the retailer a rebate for r per unit (called markdown allowance) sold above a threshold t. With these three parameters, a sales rebate contract has more than sufficient parameters to coordinate the supply chain and allocate profit arbitrarily between the parties. It is observed that there is generally a set of contracts that generate any profit allocation.

There are several papers that study this type of contracts or those closely related to it, such as Taylor(2002) and Krishnan et al.(2004), where the latter refers to the rebate as a "markdown allowance. Both papers allow the retailer to exert effort for increasing demand. In Taylor's paper(2002), the effort is chosen simultaneously with the order quantity, whereas in Krishnan, Kapuscinski and Butz (2004) the retailer chooses an order quantity, a signal of demand is observed and then the effort is exerted. In this case, if the demand signal is strong relative to the order quantity, the retailer chooses not to exert much effort.

The Quantity Discount Contract There are many types of quantity discounts. This section considers an "all unit quantity discount, i.e., the transfer payment is $T_d(q) = W_d(q)q$. $W_d(q)$ is the per unit wholesale price that is decreasing in order amount q. As with other coordinating contracts, the quantity discount contract achieves coordination by manipulating the retailer's marginal cost curve, while leaving the retailers marginal revenue curve untouched. Coordination is achieved if the marginal revenue and marginal cost curves intersect at the optimal quantity. Hence, there is an infinite number of marginal cost curves that intersect the marginal revenue curve at a single point (Cachon, 2003) and, thus, many coordinating quantity discount schedules exist (Moorthy, 1987). Kolay et al.(2004) provide a discussion on different types of quantity discounts. Wilson (Wilson, 1993) gives a much broader discussion on non-linear pricing.

2.2.3 Compliance Regime

It is assumed that the manufacturer cannot force the retailer to accept more product than it orders, but it is debatable whether the manufacturer is required to deliver the retailer's entire order. This influences the types of contracts that coordinate the supply chain, i.e., some contracts coordinate with one compliance regime, but not with another.

Under voluntary compliance, the manufacturer delivers the amount (not to exceed the retailers order) that maximizes its profit given the terms of the contract. Voluntary compliance increases the robustness of the supply chain. For example, if the manufacturer offers generous buyback terms to the retailer, the manufacturer does not want the retailer to order too much product. In case the retailer is not rational and orders more than the chain optimal quantity ($q > q^o$), under voluntary compliance the manufacturer can avoid this excessive ordering error by shipping only q^o. See Chen(1999) and Porteus(2000) for further discussion on the robustness of a coordination scheme to irrational ordering.

With voluntary compliance, however, the manufacturer may ship less than the retailer's order even if all parties are quite rational. For a fixed wholesale price, which is no less than the production cost, the supplier's profit is non-decreasing in the order amount, so that the manufacturer produces and delivers whatever quantity the retailer orders. The buyback contract coordinates with voluntary compliance. Revenue sharing and quantity discounts always coordinate the manufacturer's action with voluntary compliance, quantity flexibility contracts generally coordinate the supplier's action, whereas sales rebate contracts never do.

Alternatively, the retailer may believe the manufacturer will never deliver less than the order, because the consequences for doing so are sufficiently deleterious, e.g., court action or a loss of reputation. This is called a forced compliance regime. With forced compliance the manufacturer bears the full risk of an irrational retailer, a risk that even a risk neutral manufacturer may choose to avoid. Cachon and Lariviere(2001) outline an additional discussion on compliance regimes, where they check under each contract type if the manufacturer has any incentive to deliver less than the entire order.

2.2.4 Administration Cost

The various coordinating contracts discussed above may not be equally costly to administer. A wholesale price only contract is easy to describe and requires a single transaction between the firms. The quantity discount also requires only a single transaction, but it is more complex to describe. The other coordinating contracts are more costly to administer: the manufacturer must monitor the number of units the retailer has left at the end of the season, or the remaining units must be transported back to the manufacturer, depending on where the units are salvaged. A contract designer may prefer a non-coordinating contract if it is "highly efficient" (explained below) and easy to administer.

2.2.5 Efficiency of Non-coordinating Contracts

The three performance measures to a non-coordinating contract are the efficiency of the contract defined as the realized chain profit over optimal profit $\Pi(q_{real})/\Pi(q^*)$, the suppliers profit share over realized profit and the retailer's profit share over realized profit. From the supplier's perspective, the wholesale price contract is an attractive option if both of those measures are high: the product of these ratios is the supplier's share of the supply chain's optimal profit.

2.2.6 Independence of Contract Parameters

The independence of a contract to some parameter is also advantageous if the manufacturer lacks information regarding this parameter. For example, a manufacturer does not need to know a retailer's demand distribution to coordinate the supply chain with a revenue sharing contract, but would need to know the retailer's demand distribution with a quantity flexibility, sales rebate or quantity-discount contract.

3. Supply Chain Optimization

3.1 Introduction

Research in Supply Chain Management has grown dramatically in the past decade as firms have intensified efforts to streamline operations for higher efficiency. Central to this theme is the need to coordinate with the upstream suppliers and downstream retailers to ensure that the supply chain is both efficient and responsive to dynamic market needs. These coordination opportunities introduce new challenges and complexities, because different players have potentially conflicting incentives. Each party is primarily concerned with optimizing its own objectives which may sacrifice the chain's overall performance. However, optimal performance is achievable if the parties coordinate on a contract which aligns their objectives with the chain's through a proper profit allocation. Over the last decade the issue of supply chain coordination through contracting has attracted many researchers and the literature is vast.

A contract is said to coordinate the supply chain if the set of supply chain optimal actions is adopted while each party optimizes its own objective. The most studied issue is to coordinate the retailer's order quantity (which in turn determines the manufacturer's production quantity) with the supply chain's optimum. A number of different supply chain coordinating contract types are identified. It is shown that in a simple newsvendor problem, buy back contracts (Pasternack, 1985), revenue sharing contracts (Cachon & Lariviere, 2005), quantity flexibility contracts (Tsay, 1999), sales rebate contracts (Taylor, 2002; Krishnan, Kapuscinski, & Butz, 2004) and quantity discount contracts (Moorthy, 1987) can coordinate the production quantity of a supply chain. The related compliance regime (Chen, 1999; Porteus, 2000; Cachon & Lariviere, 2001) and administration cost is also discussed.

Despite the vast literature, none of the existing research considers the problem with the presence of manufacture's warranty decision. How does a manufacture's warranty affect the supply chain's optimal inventory and its retailer's order quantity?

Motivated by this challenge, this thesis focuses on the supply chain coordination by contracts with consideration of free replacement manufacture warranty (FRW hereafter).

As a common practice in industry, warranty has received the attention of researchers from many diverse disciplines. Menke (Menke, 1969) and Lowerre (Lowerre, 1968) conducted the first probabilistic analysis of warranty cost for free replacement warranty under rebate policy. After that, warranty has been researched in legal, economic, behavioral, marketing and management science to examine its operations and impacts in various aspects. A comprehensive review of warranty research in different disciplines can be found in "Product Warranty Handbook" (Blischke & Murthy, 1996).

A lot of research has been done to discuss various aspects of warranty. However, it is rare to see discussions in supply chain coordination. What is the optimal warranty strategy for a supply chain? Does the warranty designer (the manufacturer in this paper) have sufficient incentive to adopt the supply chain optimal warranty strategy?

The purpose of this thesis is to answer the above questions. We consider a single period newsvendor model in a two echelon supply chain with one manufacturer and one retailer. The manufacturer offers FRW to the end customers, who react positively to warranty offering. The manufacturer decides the warranty length to maximize its profit while retailer decides order quantity, which determines the manufacturer's production level, to maximize its own. The chain is coordinated if their decisions are supply chain's optimal production level and warranty length. It is found that simple wholesale price only contract between them does not coordinate production or warranty. Only if both revenue and warranty costs are shared, with contract parameters satisfying some conditions, the supply chain is coordinated in both aspects. At coordination, the supply chain enjoys a full freedom profit allocation. That is, we are able to allocate each party a profit share from 0% to 100% by changing contract parameters.

3.2 The Free Replacement Warranty

We consider FRW for inexpensive small electronic appliances such as printers and washers. With its relatively low production cost, it is more cost efficient to replace the product instead of repairing it. If a product fails during warranty K, the consumer files a warranty claim to the manufacturer/ manufacturer. Here we adopt the common assumption that time before product failure follows an exponential distribution with mean $\frac{1}{\beta}$. Thus for any purchase the probability of a replacement during warranty is $1 - e^{-\beta K}$. We assume the product enjoys good quality and warranty time is reasonably short, so the probability of two or more failures during warranty is extremely low and can be ignored. When a warranty claim arrives, it incurs warranty servicing cost to the manufacturer. A detailed warranty cost analysis from manufacturer's and consumer's point can be found in "Warranty Cost Analysis" (Blischke & Murthy, 1994). Here we consider the common cost structure that includes unit production cost for replacement unit, shipping and handling cost, customer service cost, administrative cost and goodwill lost. If the warranty policy requires the defective product to be returned, the returned units provide components for future production. The final service cost per claim varies upon products and warranty policies. In this research we assume a constant average warranty servicing cost per claim r, which in practice can be estimated from the total cost of warranty and the number of claims per year.

The expected per unit warranty cost for each unit sale is therefore the replacement cost r times the probability that the sold unit fails during warranty time K. Demand D is considered to be stochastic with its cumulative distribution as $F(\cdot)$. We incorporate the warranty's positive influence on demand by assuming $\frac{\partial F(\cdot|K)}{\partial K} < 0$. Specifically, we assume the mean of demand distribution becomes larger with larger K, while its variance remains constant.

3.3 Total Supply Chain Function

We follow the standard newsvendor model assumptions that the production and sales are solely for one season; the leftover products are salvaged at the end of the season. Let Q be the retailer's order quantity, which is also the manufacturer's production quantity as the manufacturer serves only one retailer. The supply chain obtains expected revenue from expected regular sales $S(Q, K)$ with product unit price p and season end salvage sales $I(Q, K)$ with salvage price at v. Its profit is revenue minus total cost, which includes per unit cost c that consists of manufacturer's production cost c_s and retailer's procurement cost c_r; per unit lost sales cost g that consists of manufacturer's lost sales cost g_s and retailer's lost sales cost g_r; and per unit warranty service cost r. We assume $r > p - c$ as the supply chain does not make a profit from a product that fails during warranty; and $c = c_s + c_r > v$ as it does not profit from units salvaged. Neither the retailer nor the manufacturer is able to profit from defective products that fail in warranty period, or salvaged inventories. Therefore it is assumed that under whole sale only contract $r > p - c_r - W$ for the retailer and $r > W - c_s$ for the manufacturer; under revenue sharing contract $r > \phi p - c_r - W$ for the retailer and $r > (1-\phi)p - c_s + W$ for the manufacturer, where ϕ is the retailer's revenue share stipulated in the contract.

Let $\mu(K)$ denote the expected demand given warranty length K. It is shown that
$\mu(K) = \int_0^\infty (1 - F(D|K))dD$
Let $S(Q, K)$ represent expected sales for the given Q and K. It can be expressed as

$$\begin{aligned} S(Q, K) &= E\left[\min(Q, D)\right] \\ &= Q - \int_0^Q F(D|K)dD. \end{aligned}$$

Let $I(Q,K)$ represent expected leftover inventory, then

$$\begin{aligned} I(Q,K) &= \int_0^Q F(D|K)dD \\ &= Q - S(Q,K). \end{aligned}$$

Let $L(Q,K)$ represent expected lost sales. It is expressed as

$$\begin{aligned} L(Q,K) &= \mu - Q + \int_0^Q F(D|K)dD \\ &= \mu(K) - S(Q,K). \end{aligned}$$

The probability that a product fails during warranty is

$$P(Y \leq K) = 1 - e^{-\beta K}.$$

The expected number of warranty claims for the sale amount $S(Q,K)$ is

$$E[N] = P(Y \leq K)S(Q,K).$$

Let Π be the supply chain's expected profit. No matter what contract is adopted the chain's profit is

$$\Pi(Q,K) = [p - v + g - rP(Y \leq K)]\, S(Q,K) - (c - v)Q - g\mu(K). \tag{3.3.1}$$

The following events occur in the decision process:

- The manufacturer decides the warranty period K.
- We assume the manufacturer dominates the relationship and determines the contract parameters. He determines the wholesale price W and/or revenue share ϕ and/or warranty cost share θ, and offers the contract to the retailer. At the same time, he informs the retailer the warranty policy.

- The retailer decides whether to accept or reject the contract. He will accept it as long as his expected profit is positive.
- If the retailer accepts the offer, he decides the order amount Q based on the demand forecast given K.
- The manufacturer delivers Q units of products to the retailer before the selling season.
- The selling season starts and the demand is realized.
- Warranty claims are filed and warranty service is fulfilled by the manufacturer.
- The warranty period expires, and full transfer payments are made between the parties according to the contract.

For any set of cost parameters and warranty time K the supply chain profit function is concave in order quantity Q,

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial Q^2} &= (p - v + g - rP(Y \leq K))\frac{\partial^2 S(Q,K)}{\partial Q^2} \\ &= (p - v + g - rP(Y \leq K))(-f(Q|K)) \\ &< 0. \end{aligned} \tag{3.3.2}$$

Therefore, the necessary and sufficient condition to optimize the supply chain's profit with respect to Q is

$$\frac{\partial \Pi}{\partial Q} = (p - v + g - rP(Y \leq K))\frac{\partial S(Q,K)}{\partial Q} - (c - v) = 0.$$

The optimal order quantity for the supply chain is then always

$$\frac{\partial S(Q,K)}{\partial Q} = 1 - F(Q|K) = \frac{c - v}{p - v + g - rP(Y \leq K)}. \tag{3.3.3}$$

We need the expected warranty servicing cost per unit sale relatively small that

$$rP(Y \leq K) < p + g - v$$

for the supply chain to earn a positive profit. This is realized either through low servicing cost per claim r or high product quality (small $P(Y \leq K)$).

4. Wholesale Contract

In the absence of warranty costs considered in the supply chain it is generally known that due to the double marginalization the wholesale contract coordinates the supply chain if and only if the manufacturer earns a non-positive profit.(Lariviere & Porteus, 2001) Our analysis shows that with the existence of warranty the result is also not improved with or without warranty cost sharing.

4.1 Wholesale without Warranty Cost Sharing

For a wholesale contract without warranty cost sharing, the manufacturer takes all the warranty cost. The manufacturer determines the warranty length and quotes a wholesale price, the retailer then determines the order quantity. The decision making process can be described as:

- Given K and W the retailer chooses order quantity Q to maximize its profit knowing that the demand follows a distribution with cdf as $F(D|K)$.
- Knowing that the retailer will choose its optimal Q^* corresponding to the wholesale price and warranty length, the manufacturer chooses the wholesale price W.
- The manufacturer decides warranty length K to maximize its profit.

The manufacturer's expected profit Π_s is

$$\begin{aligned} \Pi_s(W, K; Q^*) &= WQ - c_sQ - g_sL(Q, K) - rE[n] \\ &= (W - c_s)Q + (g_s - rP(Y \leq K))S(Q, K) - g_s\mu(K). \end{aligned} \tag{4.1.1}$$

The retailer's expected profit Π_r is

$$\begin{aligned}\Pi_r(q;W,K) &= pS(Q,K) + vI(Q,K) - WQ - g_r L(Q,K) - c_r q \\ &= pS(Q,K) + v\left[Q - S(Q,K)\right] - WQ - g_r\left[\mu(K) - S(Q,K)\right] - c_r q \\ &= (p - v + g_r)S(Q,K) - (W + c_r - v)Q - g_r\mu(K).\end{aligned} \tag{4.1.2}$$

The objective of the supply chain is to maximize its expected profit:

$$\max_{Q,K} \Pi(Q,K)$$

The retailer chooses order quantity Q to maximize its expected profit:

$$\max_{Q} \Pi_r(Q;W,K).$$

The manufacturer's optimization problem is

$$\max_{w,K} \Pi_s(W,K;Q^*).$$

Whether there exists one or more (W,K,q) to coordinate the supply chain depends on whether the above profit functions can be maximized simultaneously.

Lemma 4.1.1. *For any given warranty time K, when the manufacturer incurs all of the warranty cost the supply chain cannot be coordinated by choosing an appropriate wholesale price W.*

The proof is shown in the appendix. It shows that the manufacturer should choose

$$W = \frac{(p - v + g_r)(c - v)}{p - v + g - rP(Y \leq K)} - (c_r - v)$$

to coordinate the supply chain. This is impossible because the manufacturer then earns a negative profit. Practically, the wholesale contract with the manufacturer taking all the warranty cost does not coordinate the supply chain.

4.2 Wholesale with Warranty Cost Sharing

As demand is increasing with longer warranty and retailer also benefits from greater demand, for the contract to better reflect this benefit sharing the retailer should take part of the warranty cost. In this section we consider the retailer takes a percentage of θ of warranty cost. The rest is on the manufacturer. The manufacturer's expected profit is thus

$$\Pi_s(W, K; Q^*) = (g_s - (1-\theta)rP(Y \leq K))S(Q, K) - (c_s - W)Q - g_s\mu(K). \tag{4.2.1}$$

The retailer's expected profit is

$$\Pi_r(W, q, K) = (p - v + g_r - \theta rP(Y \leq K))S(Q, K) - (W + c_r - v)Q - g_r\mu(K). \tag{4.2.2}$$

Lemma 4.2.1. *For given warranty time K the whole sale contract with warranty cost sharing does not coordinate the order quantity by choosing an appropriate wholesale price W and warranty cost share θ.*

The proof is shown in the appendix.

Because the marginal profit of the retailer is less than that of the chain at the same Q,the retailer stops ordering before reaching the channel's optimal order quantity. This is called double marginalization, which is first identified by Spengler (Spengler, 1950). The consideration of warranty cost, whether shared or not, does not help to synchronize the marginal profits of both parties and the supply chain,and thus, does not coordinate the order quantity.

5. Revenue Sharing

Revenue sharing has been known as a coordinating contract in the ordinary news vendor model. It has been applied in the video rental industry with much success. (Cachon & Lariviere, 2005) Blockbuster's market share has increased from 24% in 1997 to 40% in 2002 (Warren & Peers, 2002). A detailed econometric study of the impact of revenue sharing contracts in the video rental industry is conducted by Mortimer (2002). it finds that the adoption of these contracts increased the supply chain profits by seven percent.

With full freedom to choose the revenue share ϕ and wholesale price W revenue sharing contract coordinates the supply chain and arbitrarily allocates the profit. Cachon and Lariviere (2005) provide an analysis of these contracts in a general setting. Pasternack (2005) presents a single retailer newsvendor model in which the retailer can purchase some units with revenue sharing and other units with a wholesale price contract. Cachon (2003) points out that with price dependent demand revenue sharing is able to coordinate the retailer's quantity and retail price automatically with the same set of contracts used when the retailer price is fixed.

In this section we present two models: revenue sharing contract with warranty cost on manufacturer and revenue sharing contract with warranty cost sharing. The process to determine the contract parameters is similar to the wholesale but the manufacturer also chooses the optimal revenue share ϕ at the same time of the wholesale price W. We find that supply chain coordination can be achieved only when the warranty cost is shared.

5.1 Revenue Sharing with Manufacturer Taking All Warranty Cost

Assume the contract dictates revenue sharing in both regular price sales and salvage revenues. The retailer takes the share of ϕ of the total revenue including regular sale and salvage and the manufacturer takes the rest. As salvage in reality usually implies a loss we assume again $c > v$ and $w + c_r > \phi$. That is, the retailer does not gain any profit from any unit salvaged. The transfer payment to the manufacturer in a revenue sharing contract is

$$T_r(W, q, \phi) = (W + (1-\phi)v)q + (1-\phi)(p-v)S(Q). \tag{5.1.1}$$

The manufacturer provides the replacement service for defect products and bear all the costs from warranty. The fact that it is not able to profit from a defect product motivates its quality control and improvement. We model this by assuming $r > p(1-\phi) - c_s + W$. His expected profit function is

$$\begin{aligned} \Pi_s(W, q, K) &= T_r(W, q, \phi) - c_s Q - g_s L(Q, K) - rE[n] \\ &= ((p-v)(1-\phi) + g_s - rP(Y \leq K))S(Q, K) - ((c_s - W) - (1-\phi)v)q - g_s\mu(K). \end{aligned} \tag{5.1.2}$$

The retailer's expected profit is

$$\begin{aligned} \Pi_r(W, q, K) &= p\phi S(Q, K) + v\phi I(Q, K) - WQ - g_r L(Q, K) - c_r q \\ &= ((p-v)\phi + g_r)S(Q, K) - (W + c_r - v\phi)Q - g_r\mu(K) \end{aligned} \tag{5.1.3}$$

.

Lemma 5.1.1. *Given any specific K and the manufacturer pays all the warranty cost, revenue sharing contract coordinates the supply chain order quantity if the contract parameters satisfy*

$$\frac{\phi(p-v) + g_r}{p + g - v - rP(Y \leq K)} = \frac{W + c_r - v\phi}{c - v}. \tag{5.1.4}$$

The coordinating order quantity Q^* satisfies

$$1 - F(Q^*|K) = \frac{c - v}{p - v + g - rP(Y \leq K)}.$$

The proof is shown in appendix.

If we define this ratio $\frac{\phi(p-v)+g_r}{p+g-v-rP(Y \leq K)} = \lambda$, at coordination the retailer's profit is $(\Pi +$

$g\mu(K))\lambda - g_r\mu(K)$; the manufacturer's profit is $(\Pi + g\mu(K))(1-\lambda) - g_s\mu(K)$. This ratio λ is the profit allocation parameter and varying it results in different profit share. When $\lambda = \frac{g_r\mu(K)}{\Pi+g\mu(K)}$, the manufacturer obtains whole chain's profit and the retailer earns zero dollar; when $\lambda = \frac{\Pi+g_r\mu(K)}{\Pi+g\mu(K)}$ the retailer obtains whole chain's profit. For a feasible profit allocation that both parties get non-negative profit, λ needs to be in the range $(\frac{g_r\mu(K)}{\Pi+g\mu(K)}, \frac{\Pi+g_r\mu(K)}{\Pi+g\mu(K)})$, i.e., the contract parameters need to satisfy $\frac{g_r\mu(K)}{\Pi+g\mu(K)} < \frac{\phi(p-v)+g_r}{p+g-v-rP(Y\leq K)} = \frac{W+c_r-v\phi}{c-v} < \frac{\Pi+g_r\mu(K)}{\Pi+g\mu(K)}$. within the above range the ratio, and hence the profit allocation depends on the negotiation between the parties.In the special case when there is no lost sales cost, i.e., $g_r = g_s = 0$, λ exactly equals the retailer's final share of profit, and can be any value between zero and one.

This conclusion is consistent with the simple revenue sharing model without warranty. Our model shows that when warranty is considered and taken all by the manufacturer, after the warranty time K has been determined as long as the manufacturer chooses the contract parameters from the above coordinating parameters set the retailer will automatically choose the supply chain first best order quantity. With the freedom of choosing two contract parameters the manufacturer can arbitrarily allocate the supply chain's profit between two parties. Each party's profit can be allocated at any value between zero to the whole chain's profit.

Lemma 5.1.2. *The coordinating Q^* also maximizes the manufacturer's profit given K and other contract parameters, so the manufacturer will voluntarily comply with the retailer's order quantity.*

Since the retailer's order quantity will also maximizes the manufacturer's profit, the manufacturer will comply and deliver the exact amount as ordered. If in case the retailer orders the wrong amount, the manufacturer has the incentive to correct it to the chain optimal quantity. This is what we call "double optimization" system.

As another endogenous parameter in the contract, the warranty time K should then be determined by the manufacturer to optimize the supply chain. There exists an optimal K^* to coordinate the supply chain if and only if it is optimal for both the manufacturer and the supply chain, and it coordinates the supply chain with 5.1.4 satisfied.

Note from 3.3.2 that $\Pi(Q,K)$ is concave in Q for any given K. From 5.1.4 we see there is a unique optimal Q for each K, while the integrated channel profit function may not be concave or unimodal in K. We employ the method in (Petruzzi & Dada, 1999). Assume there exists a finite but may not unique optimal quantity-warranty pair, Q^*, K^*. Let $K^*(q)$ be the supply chain optimal warranty time for a given Q. There might be multiple $K^*(q)$ for any given Q. But all of them must satisfy the necessary but not sufficient first order condition of the supply chain

$$\begin{aligned} \tfrac{\partial \Pi}{\partial K} &= -rS(Q,K)\tfrac{\partial P(Y\leq K)}{\partial K} + (p-v+g-rP(y\leq K))\tfrac{\partial S(Q,K)}{\partial K} - g\tfrac{\partial \mu(K)}{\partial K} \\ &= 0. \end{aligned} \tag{5.1.5}$$

Because the manufacturer determines the warranty period, a coordinating K should optimize the manufacturer's and the channel integrated profit simultaneously. A contract fails to coordinate if it is unable to satisfy the supply chain's and the manufacturer's first order conditions at the same $K^*(q)$ or it is able to satisfy the first order conditions at $K^*(q)$ only with parameters that fail to coordinate the quantity decision. From this we derive lemma 5.1.3.

Lemma 5.1.3. *If the profit functions are concave in (Q,K) and*

$$\frac{g_r\mu(K^*)}{\Pi + g\mu(K^*)} < \frac{g_r}{p-v+g-rP(Y\leq K)} \cdot \frac{\frac{\partial \mu(K)}{\partial K}\mid_{K^*}}{\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}} < \frac{\Pi + g_r\mu(K^*)}{\Pi + g\mu(K^*)},$$

revenue sharing contract without warranty cost sharing coordinate the supply chain with the allocation parameter

$$\lambda^* = \frac{g_r}{p-v+g-rP(Y\leq K)} \cdot \frac{\frac{\partial \mu(K)}{\partial K}\mid_{K^*}}{\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}}.$$

.

The proof is shown in the appendix. In the revenue sharing without warranty cost sharing contract the retailer pays no marginal cost of warranty while enjoys a share of λ of the marginal revenue that longer warranty brings.The manufacturer, on the other hand, pays the whole marginal cost of providing longer warranty as the supply chain while enjoys only a share $(1-\lambda)$ of the marginal revenue that the warranty brings. Unless the manufacturer gets some marginal compensation somewhere else, it usually earns less marginal profit than the supply chain and chooses a shorter warranty period than the chain optimal. Lemma 5.1.3 shows that there is only one instance that the manufacturer chooses the supply chain optimal warranty period, and the realized profit is allocated in a specific way. This result may not be of much help in practice, as the profit allocation in a chain depends on the negotiation power of the members, and may not coincide with above allocation.

5.2 Revenue and Warranty Sharing Contract

Let θ be the warranty cost share that the retailer pays and the rest of $1-\theta$ is paid by the manufacturer.
The retailer's profit function is

$$\Pi_r = \left[(p-v)\phi - \theta r P(Y \leq K) + g_r\right] S(Q,K) - (W + c_r - v\phi)Q - g_r\mu(K). \tag{5.2.1}$$

The manufacturer provides the replacement service for defective products and bear a proportion of $1-\theta$ of the total costs from warranty. His earning from a defective product is not able to cover its share of the product's replacement cost, which gives him strong incentive to control defect rate. This is modeled by defining $(1-\theta)r > p(1-\phi) - c_s + W$. The manufacturer's profit function is

$$\Pi_s = \left[(p-v)(1-\phi) - (1-\theta)rP(Y \leq K) + g_s\right] S(Q,K) - (c_s - W_s - v(1-\phi))Q - g_s\mu(K). \tag{5.2.2}$$

We first consider whether the quantity decision can be coordinated for a given K. By taking the first orders of the profit functions we derive the following lemma:

Lemma 5.2.1. *Given any given K, the revenue and warranty sharing contract coordinates the supply chain if the contract parameters satisfy*

$$\frac{\phi(p-v)+g_r-\theta rP(Y\le K)}{p+g-v-rP(Y\le K)}=\frac{W+c_r-v\phi}{c-v}. \tag{5.2.3}$$

The coordinating order quantity Q satisfies

$$1-F(Q|K)=\frac{c-v}{p-v+g-rP(Y\le K)}.$$

As the conclusion of lemma 5.1.1, if we define the ratio $\frac{\phi(p-v)+g_r-\theta rP(Y\le K)}{p+g-v-rP(Y\le K)}=\lambda$, λ is the profit allocation parameter. With $\frac{g_r\mu(K)}{\Pi+g\mu(K)}<\lambda<\frac{\Pi+g_r\mu(K)}{\Pi+g\mu(K)}$, the retailer's profit ranges from 0% to 100% of the supply chain's. Thus there always exists a contract that Pareto dominates a non-coordinating contract, i.e., each firm's profit is no worse off and at least one firm is strictly better off with the coordinating contract.

Because the retailer also shares the burden of warranty cost intuitively the warranty time is possibly determined by the joint decision of both parties. Hereby we first consider the simultaneous optimization of the manufacturer and the integrated chain and then we check whether the retailer will comply the contract voluntarily or object to it.

If it is a coordinating contract that can allocate revenues arbitrarily then there always exists a contract that Pareto dominates a non-coordinating contract, i.e., each firm's profit is no worse off and at least one firm is strictly better off with the coordinating contract.

We derive the following lemma:

Lemma 5.2.2. *If the integrated channel's profit function is concave in K, the revenue and warranty sharing contract coordinates the supply chain with K^* and $\theta^*=\lambda$, where K^**

satisfies

$$(p+g-v-rP(Y\leq K))\frac{\partial S(Q,K)}{\partial K}-rS(Q,K)\frac{\partial P(Y\leq K)}{\partial K}-g\frac{\partial \mu(K)}{\partial K}=0; \qquad (5.2.4)$$

and λ *satisfies 5.2.3.*

The proof is shown in appendix.

Lemma 5.2.2 shows that as long as the chain profit function is concave the revenue and warranty sharing contract coordinates the chain and arbitrarily allocates the profit. The allocation is explicitly defined by their warranty cost shares. This freedom of profit allocation is made possible by the ample numbers of contract parameters: W, ϕ and θ. Basically with three parameters the revenue and warranty sharing contract is able to make the manufacturer's (and thus the retailer's) marginal profit with respect to K an affine function of the integrated chain's marginal profit, which induce the manufacturer's warranty decision coincides with the integrated channel's. One special case is when $\phi=\theta=\frac{g_r}{g}$. That is, if the manufacturer and retailer agrees on the same percentage share of revenue and warranty expense, this percentage should coincide with the respective percentage of the unit shortage cost.

Theorem 5.2.3. *The revenue and warranty sharing contract coordinates the supply chain as long as*

$$w+c_r-\phi p-g_r=\theta(c-p-g) \qquad (5.2.5)$$

A special attribute of the contract is that the coordination does not depend on the quality level (reflected by failure rate) or demand distribution. That means, if the manufacturing process or inspection procedure is improved, or new technology is adopted, so that the failure rate is decreased, the equilibrium still holds. There is no need to redesign the parameters of the contract to coordinate two parties. However, the optimal warranty length might be different and the retailer may be willing to take more inventory risk because he expects to pay less warranty cost. These result in larger order quantities, and thus higher overall sales and profits for both parties.

6. Fair Profit Share at Coordination

Our theorem shows that the revenue and warranty sharing contract coordinates the supply chain as long as $\theta = \lambda$. At coordination the retailer and manufacture's profit shares are respectively

$$\Pi_r = \theta\Pi + \theta g\mu(K) - g_r\mu(K),$$

$$\Pi_s = (1-\theta)\Pi - \theta g\mu(K) + g_r\mu(K).$$

θ is the retailer's share of warranty cost.

By determining θ, we can arbitrarily allocate profits among two parties and the allocation is explicitly defined by their warranty cost share. This is a unique property of revenue and warranty sharing contract; it allocates profit with full freedom. This raises another interesting question: can we recommend a fair profit allocation?

As suggested by Banerjee and Kim (1995), we might consider a fair and equitable manner of sharing benefit from coordination as sharing profit by the relevant cost / capital invested by each party. In our model the relevant cost of retailer RC_r and cost of manufacturer RC_s are respectively

$$RC_r = \theta r P(Y \le K)S(Q,K) + c_rQ + WQ + g_r[\mu(K) - S(Q,K)],$$

$$RC_s = (1-\theta) r P(Y \le K)S(Q,K) + c_sQ + g_s[\mu(K) - S(Q,K)].$$

We seek a fair manner to share the supply chain profit by the relevant costs: $\Pi_r/\Pi_s = RC_r/RC_s$.

6.1 Special Case

If the supply chain lost sales cost g is so small that it can be ignored in the total cost, the cost functions of the two parties are

$$RC_r = \theta r P(Y \leq K) S(Q, K) + c_r Q + WQ,$$

$$RC_s = (1 - \theta) r P(Y \leq K) S(Q, K) + c_s Q.$$

and their profits at coordination are

$$\Pi_r = \theta \Pi,$$

$$\Pi_s = (1 - \theta) \Pi.$$

θ exactly allocates the profit.

If we define θ as

$$\theta / (1 - \theta) = (c_r + W) / c_s,$$

the profit is then allocated in the same manner as cost

$$\frac{\Pi_r}{\Pi_S} = \frac{\theta}{1 - \theta} = \frac{c_r + W}{c_s} = \frac{RC_r}{RC_s}.$$

This provides an easy and explicit way to implement the fair and equitable profit allocation. As long as the contract states the allocation of warranty cost in the same manner as retailer's total procuring cost to manufacture's production cost, the final profit will automatically be allocated according to each party's relevant cost.

6.2 General Case

Generally we consider lost sales cost is not ignorable. It mainly consists of lost profit. If we assume the unit lost sale cost is the average unit profit,

$$g_r = \Pi_r / Q,$$

$$g_s = \Pi_s/Q,$$

and the supply chain lost sale cost is

$$g = \Pi/Q;$$

the retailer's profit is therefore

$$\Pi_r = \theta(\Pi + \Pi\mu(K)/Q) - \Pi_r\mu(K)/Q.$$

That is,

$$\Pi_r(1 + \mu(K)/Q) = \theta\Pi(1 + \mu(K)/Q).$$

So

$$\Pi_r = \theta\Pi,$$

and through the same analysis

$$\Pi_s = (1 - \theta)\Pi.$$

The above analysis shows, whatever percentage θ is, it exactly allocates total profits, and

$$g_r = \theta g$$

$$g_s = (1 - \theta)g.$$

If we define θ as the the proportion of retailer's procuring cost versus manufacturer's production cost

$$\theta/(1 - \theta) = (c_r + W)/c_s,$$

the relevant cost of retailer is

$$\begin{aligned} RC_r &= \theta r P(Y \le K)S(Q, K) + c_r Q + WQ + \theta g[\mu(K) - S(Q, K)] \\ &= \theta(RC_s + RC_r), \end{aligned}$$

and that of manufacturer is

$$\begin{aligned} RC_s &= (1-\theta)rP(Y \leq K)S(Q,K) + c_sQ + (1-\theta)g[\mu(K) - S(Q,K)] \\ &= (1-\theta)(RC_s + RC_r). \end{aligned}$$

It is allocated in the same manner as profit. This shows, even in the general case when we consider lost sale cost as lost profit, we are able to implement the fair and equitable profit allocation by previous solution, that is, sharing the warranty cost by the rate of procuring cost to production cost.

7. Quality Improvement at Coordination

There is an increasing focus on "quality" throughout United States. Many books have been written describing the philosophy and methods used in the quality movement. This section is to investigate the economic effects of quality improvement on supply chain coordination, which further stresses the importance of quality and its longrun benefit to each party in the supply chain.

Sahin and Polatoglu (1998) investigate the impact of product quality on warranty costs and strategies. On general grounds, there are three ways to improve product quality: upgrading the manufacturing process, performing inspection on final products before release, or a burn-in program to eliminate infant mortality. In this paper, we model quality improvement as reduction in probability of aftersale failure. Quality improvement brings multiple benefits. For current selling season, when the warranty policy is published and production is already decided, the supply chain will pay less warranty service cost, as fewer failures will occur. If the revenue and warranty sharing contract is adopted, when the retailer expects to pay less warranty cost, he is willing to order more in the future selling seasons. The manufacturer will also redesign the warranty policy to extract maximum benefit from quality improvement program. We thus expect to observe a recoordination of the supply chain decision variables Q and K. This raises interesting questions:

- How to recoordinate the supply chain?
- What is the new equilibrium? Which direction each decision variable will move from old equilibrium?
- How much is the ivestment to achieve profitable quality improvement? Is manufacturer motivated to initiate a program?

How are the benefits shared between two parties?

This Chapter attempts to answer above questions through detailed analysis.

In the long run, better quality improves brand image and boost sales. The market demand function moves to new function $F(\tilde{D}|K)$, which incorporates positive response of demand to continuous quality improvement practice. This long run economic effects have huge potential. Detail analysis of this situation is left for future research.

7.1 Quality Improvement to Benefit Current Season

A general way to measure quality improvement is to measure the change of failure rate. In literature failure rate is understood as the rate parameter β in the exponential distribution, if the time to failure is assume to follow an exponential distribution. As we know, quality improvement activities usually involve investments in upgrading equipments, design, inspection process or production. Sahin and Polatoglu (1998) investigate the investments and propose to model them as an overhead cost and a variable cost linear to the proportional improvement. For example, the variable cost of lowering failure rate (denoted from 20% to 10% should be comparable to that from 10% to 5%, because both programs lower the rate by half. It matches our intuition that 5% decrease from 10% should be considerably harder than same 5% decrease from 50%, which has a bigger room to improve. Mathematically, this can be modeled by $a_0 + a_1(ln\beta_1 - ln\beta_2)$, where failure rate is improved from β_1 to β_2 ($\beta_2 < \beta_1$). a_0 is the fix overhead, and a_1 is the variable cost. The expected product lifetime is extended from $\frac{1}{\beta_1}$ to $\frac{1}{\beta_2}$.

If the retailer has accepted the contract and ordered the amount for the current selling season, quality improvement during this time will not change any decision variables (Q, K). It will benefit the supply chain through warranty cost savings only. The savings are represented by

$$r(e^{-\beta_2 K^*} - e^{-\beta_1 K^*})S(Q^*, K^*),$$

Q^* and K^* are optimal order quantity and warranty length coordinated by the revenue and warranty sharing contract, and assumed constant in this case as the quality improvement occurs after warranty length and production amount are determined.

The cost of improving β_1 to β_2 is

$$a_0 + a_1(ln\beta_1 - ln\beta_2).$$

The supply chain is motivated to take improvement actions to benefit current season, if there exists β_2 that

$$r(e^{-\beta_2 K^*} - e^{-\beta_1 K^*})S(Q^*, K^*) \geq a_0 + a_1(ln\beta_1 - ln\beta_2).$$

The marginal warranty cost saving with respect to β_2 is

$$rS(Q^*, K^*)K^* e^{-\beta_2 K^*}$$

while the marginal quality improvement cost is

$$a_1 \frac{1}{\beta_2},$$

it is possible that there exists no β_2 that makes quality improvement "worth the cost" for current season.

The manufacturer takes a share of $(1 - \theta)$ of warranty cost savings, as dictated in revenue and warranty sharing contract. Assume the quality improvement cost is solely on manufacturer, as it usually is, the manufacturer has motivation to launch an improvement program if

$$(1-\theta)r(e^{-\beta_2 K^*} - e^{-\beta_1 K^*})S(Q^*, K^*) \geq a_0 + a_1(ln\beta_1 - ln\beta_2).$$

The marginal saving is

$$(1-\theta)rS(Q^*, K^*)K^* e^{-\beta_2 K^*},$$

which is smaller than supply chain. The pool of feasible β_2 that motivates manufacturer is smaller than that of supply chain. It is possible that the manufacturer is reluctant to take actions even if it benefits the chain immediately.

Of course considering current benefit is a little bit "short-sighted" as in the long run, superior quality boosts sales and supply chain will recoordinate its production and warranty to maximize the benefit. Quality improvement has larger profit potential than warranty cost savings alone. This is investigated in the next section.

7.2 Quality Improvement to Benefit Future Seasons

7.2.1 Impact on Optimal Q and K

If β is lowered from β_1 to β_2, the supply chain profit is still concave in Q

$$\frac{\partial^2 \Pi}{\partial Q^2} = (p - v + g - r(1 - e^{-\beta_2 K}))(-f(Q|K)) < 0.$$

The optimal production is larger than previous Q_1, however, as it satisfies

$$\frac{\partial S(Q,K)}{\partial Q}|_{Q_2} = 1 - F(Q_2|K) = \frac{c - v}{p - v + g - r(1 - e^{-\beta_2 K})}.$$

$\beta_2 < \beta_1$, thus $Q_2 > Q_1$. Quality improvement boosts production for the supply chain.

If the manufacturer informs the retailer of quality improvement before ordering, the retailer will adjust the order to maximize his profit. Its profit is still concave in Q with new

β_2

$$\frac{\partial^2 \Pi_r}{\partial Q^2} = ((p-v)\phi + g_r - \theta r(1-e^{-\beta_2 K}))(-f(Q|K)) < 0.$$

The optimal order quantity is larger than previous order quantity/ production level, as it satisfies

$$\frac{\partial S(Q,K)}{\partial Q}|_{Q_2} = 1 - F(Q_2|K) = \frac{W + c_r - v\phi}{(p-v)\phi + g_r - \theta r(1-e^{-\beta_2 K})}.$$

$\beta_2 < \beta_1$, thus $Q_2 > Q_1$. Besides, from Lemma (5.2.1), we know that the coordinating contract has the property of

$$\begin{aligned} \phi(p-v) + g_r - \theta r P(Y \leq K) &= \lambda(p + g - v - rP(Y \leq K)) \\ W + c_r - v\phi &= \lambda(c - v); \end{aligned} \quad (7.2.1)$$

so the above optimal retailer order condition is same as supply chain optimal production condition. After quality improvement the supply chain is automatically recoordinated from Q_1 to Q_2. For the same reason, warranty length is also recoordinated automatically from K_1 to K_2. There is not need to change or redesign the contract to achieve new maximum profit for the chain. This can be seen in Theorem (5.2.3), which shows the coordination requirement does not involve quality parameters.

Intuitively, with lower failure rate, the optimal warranty length for the supply chain should be longer than before. This is because the tradeoff between benefit and cost of warranty moves to make longer warranty more attractive. However, this cannot be completely verified theoretically. Shown in Lemma (5.2.2), the supply chain always choose the optimal K that satisfies

$$(p + g - v - rP(Y \leq K))\frac{\partial S(Q,K)}{\partial K} - rS(Q,K)\frac{\partial P(Y \leq K)}{\partial K} - g\frac{\partial \mu(K)}{\partial K} = 0.$$

When β becomes smaller, how will the solution of K for above equation cannot be explicitly determined. It depends on how demand distribution changes over K; whether the sales and demand distribution is concave or convex to K. We will show that in the numerical examples.

7.2.2 Motivation

We show in the last section that the warranty cost savings from quality improvement will immediately benefit the supply chain in current season. However, sometimes the cost of quality improvement sets off the warranty savings, and for the current season, the supply chain expects a loss from that initiative. Does that imply the supply chain is not motivated to improve quality?

It is still profitable for supply chain to take actions on quality if the benefit after recoordinating Q and K is enough to set of the cost. The benefit will be realized in the next selling season, when both retailer and manufacturer have chance to adjust their ordering and warranty decisions. With original β_1, the supply chain profit is

$$\Pi_1(Q_1^*, K_1^*) = \left[p - v + g - r(1 - e^{-\beta_1 K_1^*})\right] S(Q_1^*, K_1^*) - (c - v)Q_1^* - g\mu(K_1^*);$$

after decreasing β_1 to β_2,

$$\Pi_2(Q_2^*, K_2^*) = \left[p - v + g - r(1 - e^{-\beta_2 K_2^*})\right] S(Q_2^*, K_2^*) - (c - v)Q_2^* - g\mu(K_2^*).$$

As long as the extra profit $\Pi_2(Q_2^*, K_2^*) - \Pi_1(Q_1^*, K_1^*)$ is bigger than cost $a_0 + a_1(ln\beta_1 - ln\beta_2)$, the supply chain will gain a positive return of the investment immediately in the next season, when the manufacturer redesigns warranty length and the retailer adjusts order quantity.

The manufacturer, however, takes only a share of $1 - \theta$ of the extra profit. It will take action to improve quality only if this share is able to cover the cost. It is reluctant to improve quality sometimes, even it will benefit the supply chain.

This paper investigates the motivation in single selling season. Quality improvement has long term impact to boost sales, establish brand image and save warranty service cost. If the quality improvement is stable and recognized by the customers, the demand distribution will move to the right (expected demand increases) even for the same warranty policy. If the supply chain and manufacturer are more tolerant to the time needed to breakeven, they will be more motivated to initiate an improvement program. Taking the above factors into account, future research can be conducted to consider the benefits of quality improvement in multiple period. We expect the extra profit potential is huge.

8. Additive Demand Case

Consider a supply chain that faces the random warranty-dependent demand function as in previous sections. The randomness in demand is warranty independent and can be modeled either in an additive or a multiplicative fashion. Specifically, demand is defined as $D(K,\varepsilon) = y(K) + \varepsilon$ in the additive case (Mills, 1959) and $D(K,\varepsilon) = y(K)e^{\varepsilon}$ in the multiplicative case (Arrow, S.Karlin, & Scarf, 1962), where $y(K)$ is an increasing function that captures the dependency between demand and warranty length, and ε is a random variable defined on the range [-A, A]. We let $y(K) = a + bK(a > 0), (b > 0)$ in the additive case, and $y(K) = aK^b(a > 0), (b > 0)$ in the multiplicative case. Both representations of $y(K)$ are similar to the common representations of price-dependent-demand in the economics literature, with the former representing a linear demand curve and the latter representing an iso-elastic demand curve (Petruzzi & Dada, 1999). To ensure the demand is nonnegative in both cases, we require $A \leq a$.

In the additive demand case, $D(K,\varepsilon) = y(K) + \varepsilon$, where $y(K) = a + bK(a > 0), (b > 0)$, $\varepsilon \in [-A, A]$ and $D \in [y(K) - A, y(K) + A]$. Let $g(\cdot)$ be the probability density function of ε, and $G(\cdot)$ be the corresponding cumulative function, we have

$$f(\alpha|K) = g(\alpha - y(K)); F(\alpha|K) = G(\alpha - y(K)).$$

The expected sales, leftover inventories and lost sales with (Q, K) are respectively

$$\begin{aligned} S(Q,K) &= Q - \textstyle\int_{y(K)-A}^{Q} F(\alpha|K)d\alpha \\ &= Q - \textstyle\int_{-A}^{q-y(K)} G(u)d\alpha; \end{aligned}$$

$$\begin{aligned} \tfrac{\partial F(D|K)}{\partial K} &= \tfrac{\partial G(D-a-bK)}{\partial K} \\ &= g(D-a-bK)\tfrac{\partial(D-a-bK)}{\partial K} \\ &= g(D-a-bK)(-b) \end{aligned}$$

$$\frac{\partial^2 F(D|K)}{\partial K^2} = b^2 \frac{dg(\varepsilon)}{d\varepsilon};$$

$$I(Q,K) = \int_{-A}^{Q-y(K)} G(u)du$$

and

$$L(Q,K) = \mu(K) - Q + \int_{-A}^{Q-y(K)} G(u)du.$$

With the revenue and warranty cost sharing contract the retailer's expected profit is

$$\Pi_r = (p\phi - \theta r P(Y \leq K) + g_r - W - c_r)Q - [(p-v)\phi - \theta r P(Y \leq K) + g_r] \int_{-A}^{Q-y(K)} G(u)du - g_r\mu(K).$$

The manufacturer's expected profit is

$$\Pi_s = [p(1-\phi) - (1-\theta)rP(Y \leq K) + g_s - c_s + W_s]\, Q - [(p-v)(1-\phi) - (1-\theta)rP(Y \leq K) + g_s] \int_{-A}^{Q-y(K)} G(u)du - g\mu(K).$$

The integrated channel's profit is

$$\Pi = [p + g - c - rP(Y \leq K)]\, Q - [p - v + g - rP(Y \leq K)] \int_{-A}^{Q-y(K} G(u)du - g\mu(K).$$

Again, according to lemma 5.2.3, for any given K, revenue and warranty sharing contract coordinates the order quantity if the contract parameters satisfy

$$\begin{aligned} \phi(p-v) + g_r - \theta r P(Y \leq K) &= \lambda(p + g - v - rP(Y \leq K)) \\ W + c_r - v\phi &= \lambda(c - v) \end{aligned}$$

with $\frac{g_r\mu(K)}{\Pi+g\mu(K)} < \lambda < \frac{\Pi+g_r\mu(K)}{\Pi+g\mu(K)}$. The coordinating order quantity Q satisfies

$$1 - G(Q - y(K)) = \frac{c - v}{p - v + g - rP(Y \leq K)}.$$

8.1 Deterministic Additive Demand

8.1.1 Chain Optimization

Deterministic increasing demand is a simple case of additive demand. We assume $\varepsilon = 0$, that the demand is linear increasing with K

$$D = a + bK.$$

For any given K, the supply chain optimal Q^* is just to order same amount as the demand. This can be proved by comparing the profit functions when $Q < D$, $Q = D$ and $Q > D$. When $Q < D$,

$$\Pi(Q, K) = (p - v + g - rP(Y \leq K))Q - (c - v)Q - gD.$$

When $Q > D$,

$$\Pi(Q, K) = (p - v + g - rP(Y \leq K))D - (c - v)Q - gD.$$

Both of them are less than the profit when $Q = D$,

$$\begin{aligned} \Pi(Q, K) &= (p - v + g - rP(Y \leq K))D - (c - v)D - gD \\ &= (p - c - rP(Y \leq K))D \\ &= (p - c - rP(Y \leq K))(a + bK). \end{aligned}$$

When $Q^* = D$,

$$\frac{d\Pi(K; Q = D)}{dK} = b(p - c - r + re^{-\beta K}) - (a + bK)r\beta e^{-\beta K}.$$

$$\frac{d^2\Pi(K;Q=D)}{dK^2} = (a+bK)r\beta^2 e^{-\beta K} - 2br\beta e^{-\beta K}.$$

When $K < \frac{2}{\beta} - \frac{a}{b}$, the chain profit function is concave in K; when $K \geq \frac{2}{\beta} - \frac{a}{b}$,

$$\begin{aligned} \tfrac{d\Pi(K;Q=D)}{dK} &= b(p-c-r+re^{-\beta K}) - (a+bK)r\beta e^{-\beta K} \\ &\leq b(p-c-r-re^{-\beta K}). \end{aligned}$$

We know that $r > p - c$, so $b(p-c-r-re^{-\beta K}) < 0$, and thus $\frac{d\Pi(K;Q=D)}{dK} < 0$. The chain profit decreases with K when $(a+bK)\beta \geq 2b$. That means, given the demand elasticity to warranty (given a and b), increasing warranty length K beyond $\frac{2}{\beta} - \frac{a}{b}$ will diminish the profit.

So the chain optimal K^* satisfies first order condition

$$\frac{d\Pi(K;Q=D)}{dK} = b(p-c-r+re^{-\beta K}) - (a+bK)r\beta e^{-\beta K} = 0.$$

8.1.2 Coordination of Revenue Sharing Contract with Manufacturer Taking All the Warranty Cost

The retailer with this contract will automatically choose the chain optimal order amount $Q_r^* = Q^* = D$. This can be shown by comparing its profit functions when $Q < D$, $Q = D$ and $Q > D$. When $Q < D$,

$$\begin{aligned} \Pi_r(Q,K) &= p\phi Q - (W+c_r)Q - g_r(D-Q) \\ &= (p\phi - W - c_r)Q - g_r(D-Q). \end{aligned}$$

When $Q > D$,

$$\begin{aligned} \Pi_r(Q,K) &= p\phi D - (W+c_r)Q + \phi v(Q-D) \\ &= ((p-v)\phi)D - (W+c_r-v\phi)Q. \end{aligned}$$

Both of them are less than the profit when $Q = D$,

$$\begin{aligned}\Pi_r(Q,K) &= ((p-v)\phi + g_r)D - (W + c_r - v\phi)D - g_r D \\ &= ((p-v)\phi)D - (W + c_r - v\phi)D \\ &= (p\phi - W - c_r)D \\ &= (p\phi - W - c_r)(a + bK).\end{aligned}$$

So the order quantity is coordinated at the supply chain optimum $Q^* = D$.

When $Q^* = D$,

$$\Pi_s(Q,K) = (p(1-\phi) - c_s + W - r(1 - e^{-\beta K}))(a + bK).$$

Therefore,

$$\frac{d^2\Pi_s(K; Q = D)}{dK^2} = (a + bK)r\beta^2 e^{-\beta K} - 2br\beta e^{-\beta K} = \frac{d^2\Pi(K; Q = D)}{dK^2}.$$

The manufacturer's profit has the same concavity condition with the chain. When $K < \frac{2}{\beta} - \frac{a}{b}$, the profit function is concave in K; when $K \geq \frac{2}{\beta} - \frac{a}{b}$,

$$\begin{aligned}\tfrac{d\Pi_s(K;Q=D)}{dK} &= b(p(1-\phi) - c_s + W - r + re^{-\beta K}) - (a + bK)r\beta e^{-\beta K} \\ &\leq b(p(1-\phi) - c_s + W - r - re^{-\beta K}).\end{aligned}$$

We know that $r > p(1-\phi) - c_s + W$, so $b(p(1-\phi) - c_s + W - r - re^{-\beta K}) < 0$, and thus $\frac{d\Pi_s(K;Q=D)}{dK} < 0$. Same as the chain profit, the manufacturer's profit decreases with K when $(a + bK)\beta \geq 2b$.

When $(a + bK)\beta < 2b$, the profit function is concave in K. The manufacturer's optimal K^* satisfies first order condition

$$\frac{d\Pi_s(K; Q = D)}{dK} = b(p(1-\phi) - c_s + W - r + re^{-\beta K}) - (a + bK)r\beta e^{-\beta K} = 0.$$

Compared with chain's optimal condition,

$$\frac{d\Pi(K;Q=D)}{dK} = b(p-c-r+re^{-\beta K}) - (a+bK)r\beta e^{-\beta K} = 0,$$

coordination requires,

$$p(1-\phi) - c_s + W = p - c.$$

But that means,

$$\Pi_s(K^*;Q=D) = \Pi(K^*;Q=D).$$

The manufacturer takes all the chain profit and leaves the retailer nonprofitable. If the retailer makes a positive profit, it requires

$$p(1-\phi) - c_s + W < p - c.$$

Then,

$$\frac{d\Pi_s(K;Q=D)}{dK} < \frac{d\Pi(K;Q=D)}{dK},$$

manufacturer will always choose a warranty length that is shorter than the chain optimum. ($K_s^* > K^*$).

It is interesting to observe that both wholesale price only contract and revenue sharing without warranty cost sharing contract distorts the warranty length decision by making the manufacturer provide a shorter warranty.

8.1.3 Coordination of Revenue and Warranty Sharing Contract

The retailer will automatically choose the chain optimal order amount $Q_r^* = Q^* = D$. This can be shown by comparing its profit functions when $Q < D$, $Q = D$ and $Q > D$. When $Q < D$,

$$\Pi_r(Q,K) = ((p-v)\phi + g_r - \theta r P(Y \le K))q - (W + c_r - v\phi)Q - g_r D.$$

When $Q > D$,

$$\Pi_r(Q,K) = ((p-v)\phi + g_r - \theta r P(Y \le K))D - (W + c_r - v\phi)Q - g_r D.$$

Both of them are less than the profit when $Q = D$,

$$\begin{aligned}\Pi_r(Q,K) &= ((p-v)\phi + g_r - \theta r P(Y \le K))D - (W + c_r - v\phi)D - g_r D \\ &= (p\phi - W - c_r - \theta r P(Y \le K))D \\ &= (p\phi - W - c_r - \theta r P(Y \le K))(a + bK).\end{aligned}$$

So the order quantity is coordinated at the supply chain optimum $Q^* = D$.

When $Q^* = D$,

$$\Pi_s(Q,K) = (p(1-\phi) - c_s + W - (1-\theta)r(1 - e^{-\beta K}))(a + bK).$$

Therefore,

$$\frac{d^2\Pi_s(K; Q = D)}{dK^2} = (a + bK)r(1-\theta)\beta^2 e^{-\beta K} - 2b(1-\theta)r\beta e^{-\beta K} = (1-\theta)\frac{d^2\Pi(K; Q = D)}{dK^2}.$$

The manufacturer's profit has the same concavity condition with the chain. When $(a + bK)\beta < 2b$, the profit function is concave in K; when $(a + bK)\beta \ge 2b$,

$$\begin{aligned}\frac{d\Pi_s(K;Q=D)}{dK} &= b(p(1-\phi) - c_s + W - (1-\theta)rP(Y \le K) - (1-\theta)(a + bK)r\beta e^{-\beta K} \\ &\le b[p(1-\phi) - c_s + W - (1-\theta)r] - b(1-\theta)re^{-\beta K}).\end{aligned}$$

We know that $(1-\theta)r > p(1-\phi) - c_s + W$, so $b[p(1-\phi) - c_s + W - (1-\theta)r] - b(1-\theta)re^{-\beta K}) < 0$, and thus $\frac{d\Pi_s(K;Q=D)}{dK} < 0$. Same as the chain profit, when K is greater than $\frac{2}{\beta} - \frac{a}{b}$, the profit will decrease with K. More interesting, if we have parameter b smaller than $\frac{a\beta}{2}$, the optimal solution for the manufacturer is to provide no warranty to maximize profit.

When $(a + bK)\beta < 2b$, the profit function is concave in K. The manufacturer's optimal K^*

satisfies first order condition

$$\frac{d\Pi_s(K;Q=D)}{dK} = b(p(1-\phi)-c_s+W-(1-\theta)rP(Y\leq K)-(1-\theta)(a+bK)r\beta e^{-\beta K} = 0.$$

Compared with chain's optimal condition,

$$\frac{d\Pi(K;Q=D)}{dK} = b(p-c-r+re^{-\beta K})-(a+bK)r\beta e^{-\beta K} = 0,$$

coordination requires,

$$p(1-\phi)-c_s+W = (1-\theta)(p-c).$$

That means,

$$\Pi_s(K^*;Q=D) = (1-\theta)\Pi(K^*;Q=D).$$

9. Numerical Example and Sensitivity Analysis

In this chapter, we demonstrate supply chain coordination with revenue and warranty cost sharing contract via a numerical example of a single period case. We assume that the period's demand has a normal distribution with a standard deviation of 100. The mean of the demand distribution is a linear function of the warranty period length, i.e., as $\mu(K) = a + bK$, and we let $a = 500$, $b = 30$. The remaining parameters in the data set are as follows:

The exogenous parameters are listed in Table 9.1:

Table 9.1: Exogenous Parameters of the Numerical Example

p	c_r	c_s	c	g_r
100	10	40	50	8

g_s	g	v	r	β
5	13	20	50	1/20

According to the coordination condition (5.2.3), we construct the contract parameters as shown in Table 9.2:

Table 9.2: Contract Parameters of the Numerical Example

W	ϕ	θ
22.8	0.5	0.35

We assume $\beta = 1/20$, i.e., the expected lifetime of the product is 20 years. Practically any warranty period length less than 10 years sounds realistic. If the warranty period is longer than 10 years, the probability of two failures is significant and cannot be ignored. For example, for a warranty period of 10 years, the probability of having two or more failures is only 0.09, whereas under a 15 years warranty, this probability increases to 0.17 and under a 20 years warranty, this probability is 0.26. Two or more claims during the warranty period of one purchase is beyond the scope of this research.

9.1 Coordination

We iterate the warranty period length from 0 to 10 years, with 1 year increment. The supply chain profits for various Q and K are shown in Figure 15.1. The profits are concave in both Q and K, and a longer warranty generally requires larger Q to achieve the supply chain maximum profit. For each warranty period length, we derive the supply chain's optimal order quantity Q^* according to (3.3.3). We then calculate the supply chain's profit for each K and find the largest one as the chain optimal solution of K^* and Q^*. The results are shown in Table 14.1, which shows that the optimal warranty length for the supply chain and the manufacturer are both 4 years. The retailer's optimal order quantity and the supply chain's production level is 657 units. Both production level and the warranty period length are coordinated. The supply chain now has a total profit of m$\$22,245$. The manufacturer and the retailer both achieve their maximum profits of $\$6,775$ and $\$15,470$, respectively.

9.2 Robustness of The Revenue and Warranty Sharing Contract

What if one party makes an error and does not make its optimal decision? For example, what if the manufacturer does not offer the optimal and coordinating warranty of 4 years, but instead makes a conservative offer of 3 or 2 years; or the warranty policy is 4 years, but the retailer chooses an order amount other than the optimal one? In this section, we analyze the robustness of the supply chain performance to deviations from optimal decisions.

We expect the supply chain will perform suboptimally as a result of such deviation. From

Table 14.1, we see that if the manufacturer provides a 3 year warranty, the retailer will respond to this by ordering 629 units. This amount is optimal for the adopted warranty length, but fewer than the theoretical amount of 656 units under a 4 year warranty period. The supply chain now achieves a total profit of the $22,205, which is 99.8% of optimum. If the manufacturer provides a 5 year warranty, the retailer will respond to this policy by ordering 684 units, which is larger than the optimum 656 warranty of under a 4 year warranty. The supply chain, thus, achieves a total profit of $22,150, which is 99.8% of the optimum. The results show that even if the manufacturer's warranty decision deviates significantly from optimum, the final profit does not deviate much from the maximum, because the retailer's ordering decision will be the same as the supply chain's optimum. The supply chain's performance is quite robust with respect to deviations from the manufacturer's optimal warranty policy. This robustness results from the "Double Optimization" scheme of the revenue and warranty sharing contract. If the manufacturer's warranty decision deviates from the optimum, the retailer will respond to this. Instead of responding to the theoretical optimal order amount associated with the optimal K^*, the retailer maximizes its profit under the actual warranty policy. Under coordination its decision also maximizes the supply chain's total profit under actual warranty period length. Thus, the retailer's action partially recovers the loss from the "erroneous" warranty policy.

Under the optimal warranty policy of 4 years, if the retailer makes a conservative decision of ordering 20% less, as shown in Table 14.2, the supply chain will achieve 87.6% of maximum, and the manufacturer and the retailer are able to achieve 91.5% and 86.0% of their respective maximum profits. This shows that the supply chain's performance is fairly insensitive to the retailer's error. This is because the manufacturer has already made the right decision of the warranty length and has partially ensured good performance of the supply chain. It is another example of the advantage of the "Double Optimization" aspect of the revenue and warranty sharing contract.

9.3 Sensitivity Analysis of Market Demand Responsiveness

We assume that the market demand responds positively to the warranty period, as represented by the function $\mu(K) = a + bK$ $(b > 0)$. The value of b indicates to which the warranty period length impacts expected market demand. For every one year increase in the warranty period, the expected market demand will increase by b units. It represents the market demand responsiveness to a change in the warranty length. The parameter a is the expected market demand if no warranty is offered. It is set to 500 units in this example. We vary the parameter from 25 to 35 and examine how the optimal solutions of Q^* and K^* move with b, resulting in changes in the supply chain maximum profit and those of each chain member. The results are shown in Table 14.6.

Figure 15.2 shows that the optimal warranty period K^* increases with increasing b. Longer warranty period convinces consumers that the product quality is superior, or consumers value the warranty as an important part of the product and service package, and hence their intent to buy can be positively influenced by a longer warranty length. However, a longer warranty period incurs more expected warranty service costs. The tradeoff between higher demand and larger costs makes the optimal warranty period K^* concave in b, as shown in Table 15.2.

Figure 15.2 also shows that the optimal order quantity increases with b. This matches our intuition that if the market demand is more sensitive to the warranty period, the manufacturer is likely to offer a longer warranty period, hence the demand forecast increases, which encourages the retailer to order more.

Figure 15.4 illustrates the optimal profits of the supply chain and each of its members for different values of b. As expected, with larger b, the market demand is more sensitive to the warranty period. A longer warranty is thus effective in stimulating demand. This is reflected by larger expected revenue and profits for the supply chain and its members. One interesting observation is that the manufacturer seems to benefit more from this than the retailer, due to the profit allocation shown below

$$\Pi_r = \theta\Pi + \theta g\mu(K) - g_r\mu(K),$$

$$\Pi_s = (1-\theta)\Pi - \theta g\mu(K) + g_r\mu(K).$$

In this example we let $\theta = 0.35$ and is held constant. The parameters $g_r = 8 > \theta g = 4.55$, so the retailer, compared with the manufacturer loses more for unit of lost sales. With a larger demand forecast, the retailer will order more, but the percentile of its order on the demand distribution is lower (as shown in Figure 15.3) so that the optimal Q^* approaches expected demand with larger b. When the demand distribution shifts to the right, the retailer experiences larger lost sales. The tradeoff between the expected revenue increase and more lost sales makes the retailer's benefit less from a better warranty.

9.4 Sensitivity Analysis of Quality Improvement

We define β as the failure rate, which represents the expected number of failures within one year for each unit purchased. We follow the conventional model that the quality improvement cost is loglinear in the failure rate (Sahin & Polatoglu, 1998), and as expressed as $a_0 + a_1(ln\beta_1 - ln\beta_2)$.

9.4.1 Quality Improvement to Benefit a Single Period

We demonstrated in Chapter 7 that quality improvement will benefit the supply chain, as well as each party in the chain through warranty cost savings, a larger retailer order amount, or a larger demand through a better warranty. We proved that the contract does not need to be redesigned to coordinate the chain with improved quality. However, the optimal order quantity does increase. We cannot show explicitly whether the optimal warranty length will increase, because it depends on how the demand distribution responds quantitatively to change in warranty, and whether the relationship is concave or convex. We show in Table

14.5 that in our numerical example, the optimal warranty period length does increase.

We set $a_0 = 800$, $a_1 = 1200$, and the original failure rate $\beta_1 = 0.05$. As shown in Figure 15.5, the quality improvement cost increases as the failure rate decreases, and is a convex function of the new failure rate. Reducing β from 0.032 to 0.03 requires more investment than lowering it from 0.048 to 0.046.

In a scenario involving a single period and a single product, improved quality lowers the warranty service cost, and encourages the manufacturer to offer a longer warranty period and hence boosts expected demand. We assume $a_0 = 800$ and $a_1 = 1200$ to represent respectively the fixed overhead and variable costs of quality improvement. In Figure 15.6 it appears that the supply chain optimal profit is convex in quality cost and increases with each additional $\$1,000$ invested in quality. Without considering the quality improvement cost, the supply chain profit is higher. The net profit (profit without the quality improvement cost), however, is first lower than original, due to the overhead cost, but then goes up when the improvement in β is significant.

Figure 15.7 shows that the expected demand is convex and increases with the investment in quality. Also the retailer's optimal order quantity increases with demand. However, the difference between them becomes smaller. With a constant demand standard deviation, the retailer is more conservative in increasing the order quantity than the supply chain in case of higher demand forecast. This is a result of the tradeoff of inventory risk, lost sales risk and sales increase from ordering more, and is another reason that the retailer does not benefit as much from a demand increase as the manufacturer, as shown in Figure 15.4.

As shown in Figure 15.8, the optimal warranty period is convex and increases with the quality investment, consistent with expected demand.

9.5 Fair and Equitable Profit Allocation

9.5.1 Special Case

In Chapter 6, we considered a fair and equitable manner of sharing the benefits resulting from coordination, by allocating profits proportionally to the relevant costs of the two parties involved. We showed in the special case, when lost sales cost g_r and g_s are negligible, that this fair allocation can be implemented by simply defining the warranty cost share θ using $\theta/(1-\theta) = (c_r + W)/c_s$, and by letting θ dictate the retailer's profit share. In our numerical example, if $g_r = g_s = 0$, we show the fair profit allocations in Table 14.3. The proportion of the retailer's unit procurement cost to manufacturer's production cost is $(c_r + W)/c_s = \theta/(1-\theta) = 0.724$, which means that $\theta = 0.42$. The coordinating contract parameters are, thus, $W = 18.99$, $\phi = 0.5$ and $\theta = 0.42$. The optimal solution shows that the optimal warranty length as 4 years and optimal order quantity as 639 units. The retailer takes 41% of the supply chain profit, while the manufacturer takes 59%. This is roughly the same as that dictated by $\theta = 0.42$. The cost sharing follows exactly the same allocation: retailer bears 42% of the total supply chain cost, while manufacturer bears 58%.

9.5.2 General Case

We now analyze the case where the lost sales cost is not negligible. Since the unit lost sales cost consists mainly of lost profit, we show in the following example that a fair profit allocation is achieved the same way as setting $\theta/(1-\theta) = (c_r + W)/c_s$.

By setting $g_r = \Pi_r/Q$ and $g_s = \Pi_s/Q$, we assume that the lost sales cost is equal to lost profit opportunity. As before, from the exogenous parameters c_r, W and c_s, we derive the fair θ as 0.42. The corresponding coordinating contract has the wholes sale price W equal to 18.67 and the revenue sharing parameter ϕ equal to 0.5. The optimal warranty policy is still 4 years and the optimal order quantity is 679 units. The final profit and cost are shown in Table 14.4. As expected, both the profit share and the cost share matches the warranty cost share $\theta = 0.42$. Besides, the unit lost sales cost for the retailer and the manufacturer are respectively $g_r = \Pi_r/Q^* = 15$ and $g_s = \Pi_s/Q^* = 22$.

10. Efficiency of Non-coordinating Contract

Although coordination and flexible profit allocation are desirable features, contracts with those properties might be costly to administer if they require extra information collection and coordination actions. As a result, a simple non-coordinating contract may be worth adopting if it is "efficient". The efficiency of a non-coordinating contract E is defined as the ratio of supply chain profit with the contract to the supply chain's optimal profit $\frac{\Pi}{\Pi^o}$. In this Chapter we theoretically prove that higher wholesale price lowers retailer's profit and makes the supply chain less efficient; improvement in production cost, warranty cost and quality helps to build a more efficient supply chain.

10.1 Efficiency of Wholesale Contract without Warranty Cost Sharing

10.1.1 Efficiency of Order quantity

To obtain insight of efficiency with respect to order quantity, we first assume the manufacturer chooses the chain optimal warranty length $K_s = K^o$. Define the efficiency in this case E_Q given K_s.

For any wholesale price offer W, the retailer optimizes its profit by choosing $Q_r^*(W)$. Based on our previous analysis, $Q_r^*(W)$ is unique for any given W, but varies with different W. The set of $(Q_r^*(W), W)$ contains all possible wholesale price offers and their corresponding order responses. Nevertheless, all $Q_r^*(W)$ are less than the supply chain optimal production level as long as W is larger than chain optimal price for manufacturer to make a positive profit. The realized supply chain is sub-optimal.

The manufacturer, on the other hand, has full knowledge of retailer's optimization problem and expects retailer's response $Q_r^*(W)$ for any W it offers. The manufacturer thus optimizes its own profit from choosing an optimal pair of $(Q_r^*(W_s^*), W_s^*)$ among the set of all wholesale offers and corresponding retailer's responses. For manufacturer, the optimal solution $(Q_s^* = Q_r^*(W_s^*), W_s^*)$ is unique and does not depend on any other endogenous parameters. In this Chapter we first investigate how the wholesale price affects retailer's order decision and the supply chain expected sales, and thus the realized retailer's profit, supply chain's profit and the contract efficiency; then explore the manufacturer's optimization problem. This problem is not fully examined yet in the previous analysis where we only confirm that the supply chain's optimal wholesale price and production is never optimal to manufacturer. Here we introduce an interesting property of most demand distributions: the Increasing Generalized Failure Rate (IGFR) and discover that if the demand is IGFR, the supplier's profit function is unimodal. Our analysis shows that improvement in production, warranty serving cost and production quality increases supply chain's realized production and profit. The supplier and retailer are both better off through lower wholesale price. The results are intuitive, however, our analysis provides the quantitative marginal change of retailer's order, expected sales, retailer's and supply chain's profit with respect to manufacturer's wholesale price; and the marginal change of that wholesale price with respect to exogenous parameters.

Sensitivity of Order Quantity to Wholesale Price

We know the retailer's expected profit Π_r is

$$\Pi_r(Q; W, K) = (p - v + g_r)S(Q) - (W + c_r - v)Q - g_r\mu(K).$$

Based on our previous analysis, the retailer chooses optimal Q_r^* that satisfies

$$\frac{\partial S(Q)}{\partial Q} = 1 - F(Q_r^*) = \frac{W + c_r - v}{p - v + g_r}$$

to maximize its profit.

Thus

$$\frac{dQ_r^*}{dW} = -\frac{1}{f(Q_r^*)(p - v + g_r)}.$$

It shows the marginal change of order quantity with wholesale price. It is not surprising that higher wholesale price discourages the retailer to order more. Besides, the amount of change depends on the demand density of order amount. It is consistent with our intuition that if the probability of demand as Q_r^* is high, the retailer is more willing to stick to current order amount and less sensitive to wholesale price change. A salvage item brings retailer a revenue of v, one unit stock out causes a shortage loss of g_r; thus the revenue of one unit regular sale can be standardized as $p - v + g_r$. Our analysis also shows that with bigger standardized regular revenue, the retailer is less sensitive to the order price.

The sensitivity of sales to wholesale price is

$$\begin{aligned} \frac{dS(Q_r^*)}{dW} &= \frac{dS(Q_r^*)}{dQ_r^*} \cdot \frac{dQ_r^*}{dW} \\ &= -\frac{W + c_r - v}{f(Q_r^*)(p - v + g_r)} \\ &= -\frac{1 - F(Q_r^*)}{f(Q_r^*)(p - v + g_r)}. \end{aligned}$$

It shows sales also move negatively with wholesale price. The marginal sales increase with lower whole sale price is related to the classical hazard rate $h(Q) = \frac{f(Q)}{\bar{F}(Q)}$, which gives roughly the percentage decrease in the probability of a stock out from increasing the stocking quantity by one unit. Our analysis shows the marginal increase in sales is proportional to the reverse of hazard rate.

Define $\upsilon(Q_r^*) = -\frac{W}{Q_r^*}\frac{dQ_r^*}{dW}$ the order quantity elasticity with respect to W is (Kreps, 1990),

$$\upsilon(Q_r^*) = \frac{\bar{F}(Q_r^*) - (c_r - v)/(p - v + g_r)}{Q_r^* f(Q_r^*)}.$$

Define $g(Q) = \frac{Qf(Q)}{\bar{F}(Q)}$ the generalized failure rate. Different from hazard rate, it gives the percentage decrease in the probability of a stock out from increasing the stocking quantity by 1% roughly. A distribution has an increasing generalized failure rate (IGFR) if $g(Q)$ is weakly increasing for all Q such that $F(Q) < 1$ (Lariviere & Porteus, 2001). If $c_r = v$,

$\upsilon(Q_r^*) = \frac{1}{g(Q_r^*)}$. Then if demand is IGFR, the order amount Q has a decreasing elasticity.

Let Π^D be the realized decentralized supply chain profit given in comparison with optimal chain profit Π^o. Define $E_Q = \frac{\Pi^D}{\Pi^o}$ the efficiency of the contract with respect to Q, we derive the theorem

Theorem 10.1.1. *Higher wholesale price makes the supply chain less efficient.*

10.1.2 Retailer Profit with Respect to W

At optimum Q_r^*

$$\frac{dS(Q_r^*)}{dW} = -\frac{1 - F(Q_r^*)}{f(Q_r^*)(p - v + g_r)},$$

$$\frac{dQ_r^*}{dW} = -\frac{1}{f(Q_r^*)(p - v + g_r)},$$

and

$$1 - F(Q_r^*) = \frac{W + c_r - v}{p - v + g_r}.$$

Thus, the derivative of Π_r with respect to W is

$$\begin{aligned} \frac{\partial \Pi_r}{\partial W} &= (p - v + g_r)\frac{\partial S(Q_r)}{\partial W} - Q_r - (W + c_r - v)\frac{\partial Q_r}{\partial W} \\ &= -Q_r^*. \end{aligned}$$

It is not surprising that higher wholesale price lowers retailer profit. Our analysis provides shows that this marginal profit is equal to the order quantity and is more significant when W is small (lower W, bigger Q_r^*).

10.1.3 Manufacturer Profit

With the knowledge that the retailer will order $Q_r^*(W)$ corresponding to each W, the manufacturer will maximize its profit

$$\Pi_s(W; Q_r^*(W), K) = (W - c_s)Q_r^*(W) + (g_s - rP(Y \leq K))S(Q_r^*(W)) - g_s\mu(K);$$

Check whether Π_s is unimodal on W.

The first order derivative of Π_s to W is

$$\begin{aligned}\frac{\partial \Pi_s}{\partial W} &= Q_r^* - \frac{\bar{F}(Q_r^*)(p-v+g-rP(y\le K))-(c-v)}{f(Q_r^*)(p-v+g_r)} \\ &= Q_r^* - \frac{W-c_s+(g_s-rP(y\le K))(W+C_r-v)/(p-v+g_r)}{(f(Q_r^*))(p-v+g_r)}.\end{aligned}$$

From supply chain's profit $\Pi \geq 0$ we have

$$p - v + g - rP(y \leq K) \geq 0.$$

Therefore,

$$(p - v + g - rP(y \leq K)) - (p - v + g_r) = g_s - rP(y \leq K) \geq -(p - v + g_r).$$

That is,

$$\frac{g_s - rP(y \leq K)}{p - v + g_r} \geq -1.$$

So if the demand follows a uniform distribution, the second order derivative

$$\frac{\partial^2 \Pi_s}{\partial W^2} = \frac{\partial Q_r^*}{\partial W} - \frac{1 + (g_s - rP(y \leq K))/(p - v + g_r)}{f(Q_r^*)(p - v + g_r)} \leq 0.$$

and thus $\Pi_s(W; Q_r^*(W), K)$ is concave on W.

Generally,

$$\begin{aligned}\frac{\partial^2 \Pi_s}{\partial W^2} &= \frac{\partial Q_r^*}{\partial W} - \frac{1 + (g_s - rP(y \leq K))/(p - v + g_r)}{f(Q_r^*)(p - v + g_r)} \\ &\quad + \frac{W + (g_s - rP(y \leq K))(W + C_r - v)/(p - v + g_r)}{(f(Q_r^*))^2(p - v + g_r)^2} \cdot \frac{\partial f(Q_r^*)}{\partial W}.\end{aligned}$$

Since $\frac{\partial f(Q_r^*)}{\partial W} = \frac{df(Q)}{dQ} \cdot \frac{\partial Q_r^*(W)}{\partial W}$, the above equation is non-positive if the demand distribution is increasing ($df(Q)/dQ \geq 0$).

If Π_s is concave in W and W^* satisfies $\frac{\partial \Pi_s}{\partial W} = 0$, W^* is the optimal price to the manufacturer.

So the manufacturer will choose (W^*, Q_r^*) that follow

$$Q_r^* = \frac{\bar{F}(Q_r^*)(p - v + g - rP(y \le K)) - (c - v)}{f(Q_r^*)(p - v + g_r)}$$

and

$$\bar{F}(Q_r^*) = \frac{W + c_r - v}{p - v + g_r}.$$

If we assume $c_r, v, g_r, g_s = 0$ to standardize the problem, Q_r^* of the decentralized supply chain will settle to the solution of the equation with one unknown

$$Q_r^* = \frac{\bar{F}(Q_r^*)(p - rP(y \le K)) - c_s}{f(Q_r^*)p},$$

and the wholesale price determined by the manufacturer to induce this Q_r^* is

$$W^* = p\bar{F}(Q_r^*).$$

Unfortunately most conventional demand distributions, such as normal, gamma and Poisson do not necessarily satisfy these conditions to ensure concavity of Π_s.

10.1.4 Unimodality of Manufacturer Profit Function

We know that

$$\begin{aligned}
\frac{\Pi_s(Q_r^*; Q_r^*(W), K)}{Q_r^*} &= (W - c_s) + Q_r^* \frac{dW}{dQ_r^*} + (g_s - rP(y \le K)) \frac{\partial S(Q_r^*)}{\partial Q_r^*} \\
&= \bar{F}(Q_r^*)[(p - v + g - rP(y \le K)) - \frac{Q_r^* f(Q_r^*)}{\bar{F}(Q_r^*)}(p - v + g_r)] - (c - v).
\end{aligned}$$

We show that

Theorem 10.1.2. *If the demand distribution is IGFR, the manufacturer's expected profit function is unimodal in Q_r^* and the unique optimal Q_s^* satisfies*

$$Q_s^* = \frac{\bar{F}(Q_s^*)(p - v + g - rP(y \le K)) - (c - v)}{f(Q_s^*)(p - v + g_r)}. \quad (10.1.1)$$

10.1.5 Sensitivity of Manufacturer Profit Function

The optimal Q_s^* is obtained from (10.1.1) and thus the maximum Π_s can be solved for any set of exogenous parameters p, v, g, c etc. How does it react to those parameters?

Sensitivity to Net Production Cost

Suppose the manufacturer improves the production process by, for example, implementing advanced technology or achieving more bargain power in procurement. His unit production cost c_s gets smaller. Equation (10.1.1) shows optimal quantity Q_s^* gets larger. The manufacturer is better off by charging a lower W to encourage the retailer to order more. This is proved as follows.

Let c_{s1} be the original production cost, c_{s2} be the current production cost ($c_{s1} > c_{s2}$), then from (10.1.1), the manufacturer now asks for a smaller wholesale price $(W_{w1}) > (W_{w2})$. Let Q_{s1}^*, Q_{s2}^* be the original and current order quantity, respectively, then $Q_{s1}^* < Q_{s2}^*$ and $S(Q_{s1}^*) < S(Q_{s2}^*)$ because $(\frac{\partial S(Q)}{Q} > 0)$. Then the original manufacturer's profit

$$\begin{aligned}
\Pi_{s1}^* &= (W_{w1} - c_{s1})Q_{s1}^* + (g_s - rP(Y \leq K))S(Q_{s1}^*) - g_s\mu(K) \\
&< (W_{w1} - c_{s2})Q_{s1}^* + (g_s - rP(Y \leq K))S(Q_{s1}^*) - g_s\mu(K)
\end{aligned}$$

because $c_{s1} > c_{s2}$.
$Q_{s1}^* = Q_r^*(W_{w1})$ is a function of W and unrelated to the change in c_s, so (W_{w1}, Q_{s1}^*) is also a feasible solution of Π_{s2} with c_{s2}. But its profit is less than the maximum profit Π_{s2}^*, because (W_{w2}, Q_{s2}^*) is the unique optimal solution.

$$(W_{w1} - c_{s2})Q_{s1}^* + (g_s - rP(Y \leq K))S(Q_{s1}^*) - g_s\mu(K) < \Pi_{s2}^*.$$

This shows improved production increases manufacturer's profit. The retailer's profit is also better off. The whole supply chain achieves a higher profit. The proof is analogous to above.

Sensitivity to Unit Warranty Cost

How does the order amount and manufacturer's profit change with unit warranty cost r?

From (10.1.1), we have

$$\frac{Q_s^* f(Q_r^*)}{\bar{F}(Q_r^*)} = \frac{(p - v + g - rP(y \leq K))}{p - v + g_r} - \frac{(c - v)}{(p - v + g_r)(\bar{F}(Q_r^*))}$$

If the manufacturer is able to improve the warranty process to reduce r, the optimal order amount Q_s^* for the manufacturer will increase if the demand is IGFR. That means, the manufacturer is able to charge a lower whole sale price to encourage retailer to order more. The manufacturer's's total profit is bigger, so is the retailer's. The total supply chain's profit increases. The proof is analogous to the sensitivity to production cost.

Let r_1 be the original warranty cost, r_2 be the current warranty cost $r_1 > r_2$. Let Q_{s1}^*, Q_{s2}^* be the original and current order quantity, respectively, then $Q_{s1}^* < Q_{s2}^*$ and $S(Q_{s1}^*) < S(Q_{s2}^*)$. Then from the elasticity of Q_r^* to W, $(W_{w1}) > (W_{w2})$. Then original manufacturer's maximum profit

$$\begin{aligned}
\Pi_{s1}^* &= (W_{w1} - c_s)Q_{s1}^* + (g_s - r_1 P(Y \leq K))S(Q_{s1}^*) - g_s \mu(K) \\
&< (W_{w1} - c_s)Q_{s1}^* + (g_s - r_2 P(Y \leq K))S(Q_{s1}^*) - g_s \mu(K)
\end{aligned}$$

because $r_1 > r_2$.

$Q_{s1}^* = Q_r^*(W_{w1})$ is a function of W and unrelated to the change in r, so (W_{w1}, Q_{s1}^*) is also a feasible solution of Π_{s2} with r_2. But its profit is less than the maximum Π_{s2}^*, because (W_{w2}, Q_{s2}^*) is the unique optimal solution. The manufacturer's maximum profit is higher. So is the retailer's profit because it increases with lower whole sale price. Thus the total supply chain's profit increases.

Sensitivity to Product Quality

From

$$\frac{Q_s^* f(Q_r^*)}{\bar{F}(Q_r^*)} = \frac{(p - v + g - rP(y \leq K))}{p - v + g_r} - \frac{(c - v)}{(p - v + g_r)(\bar{F}(Q_r^*))}$$

we see that if the product quality improves, i.e., for same warranty length K, the failure rate $P(y \leq K)$ is reduced, the optimal order amount Q_s^* for the manufacturer will increase if the demand is IGFR. That means, the manufacturer is able to charge a lower whole sale price to encourage retailer to order more. The manufacturer's total profit is bigger, so is the retailer's. The total supply chain's profit increases. The proof is analogous to above.

11. Conclusion and Future Research

In today's highly competitive global marketplace, customers demand lower prices, better quality and higher levels of service. To compete in such business environments, enterprises need to exert substantial efforts to streamline their operations, and improve communications and collaboration with every other entity in the supply chains. With the development of the Internet and Business-to-Business technologies, supply chain solutions are more scalable and information exchanges are faster and easier. These make close collaboration among supply chain players possible. However, without an effective coordination mechanism each member in the chain is primarily concerned with its own individual profit, and the total supply chain performance may not optimally be achieved unless incentives of the supply chain are not aligned. A supply chain is coordinated if the optimal decisions of the individual parties are the same as the supply chain's optimal solution. This is desired because with the supply chain's maximum profit achieved, both entities in the chain have a "bigger pie" to share.

This thesis proposed a contract mechanism to coordinate the supply chain's decision of production quantity in conjunction with a free replacement warranty. We examined four forms of contracts, namely wholesale price only contracts and revenue sharing contracts with or without warranty cost sharing, for a supply chain with one manufacturer and one retailer in the context of a single period, single product newsvendor model. This work contributes to the existing literature by considering warranty period optimization towards a supply chain coordination. We demonstrated that the wholesale price only contract does not coordinate the order quantity or the warranty length decision due to "double marginalization". The manufacturer, in order to make a positive profit, offers a contract with a unit wholesale price higher than its unit production cost. Under such a wholesale contract price, the retailer always orders a quantity less than the supply chain's optimum, as its marginal cost of ordering one more unit is the unit wholesale price plus the procurement cost, which is higher than the marginal cost of the supply chain. The profit realized is, thus, less than the supply chain's optimum. The revenue sharing contract with the manufacturer bearing all the warranty costs coordinates the order quantity. However, this type of a contract coordinates the warranty length with only one manner of profit allocation. In practice, the profit allocation between the parties depends on their respective negotiation power, and a contract needs to be flexible in terms of allocating profit in order to be adopted under different situations. The revenue and warranty cost sharing contract, on the other hand, coordinates the production/order quantity and the warranty period length, with the contract parameters satisfying the quantitative conditions in Theorem 5.2.5. With coordination, the profit of each party ranges from 0 to 100% of chain profit depending on the parameter, which is derived explicitly for contract implementation.

The results obtained from this research lead to some interesting managerial insights. They suggest that revenue and warranty cost should be shared between the members in a supply chain, so that the inventory risk, demand risk and warranty service risk are shared. In this way, the incentives of the different members are aligned with the supply chain goals through the alignment of their marginal revenues and costs with those of the supply chain.

After developing a general framework for coordinated revenue and warranty sharing contracts, the coordination equilibrium was examined in details. A fair and equitable manner of sharing benefit from coordination was considered as sharing profit by the relevant cost or capital invested by each party. We showed that this is achieved by dictating the warranty share same as the variable cost per unit, i.e., the manufacturer's unit production cost to retailer's unit purchase and procurement cost.

The impact of quality improvement was also investigated. The results supported the insight from Theorem 5.2.5 that, at coordination if product quality changes, the contract with its original parameters (the wholesale price, and revenue and warranty cost share) still coordinates the supply chain and does not need to be redesigned to maximize the supply chain's profit. However, the optimal decisions order quantity and warranty length decisions are different in the new equilibrium. With better quality both of them are larger. Furthermore, the manufacturer was assumed to bear all the quality improvement costs, as commonly observed in practice, while it takes only a portion of the extra revenue from the stimulated demand. Its marginal profit from quality improvement is less than the supply chain's and it is, thus, less motivated to initiate a quality improvement program.

Numerical testing of the case of additive demand under revenue and warranty sharing contract was conducted, followed by an extensive sensitivity analysis, in order to understand the impact of change in contract parameters on the optimal solutions and profits realized at the coordination. It was illustrated that the coordination is robust, that is, in the aspect that if either party deviates from the optimal decisions, the final profit achieved by both members and the supply chain does not deviate much from optimum. Additionally, we showed that the optimal decisions at coordination change with different levels of market responsiveness to warranty length. Serving a market more responsive to warranty length, the manufacturer has a longer optimal warranty period. The maximum profits achieved by the supply chain and both members in the chain also increase with responsiveness. This demonstrated that if a supply chain faces a market where the customers' incentive to purchase can be largely influenced by warranty length, a longer warranty period should be offered to stimulate the demand and achieve maximum profit. Also, our numerical results showed the supply chain's net profit (profit minus quality improvement cost) can increase in a single period due to quality improvement. The one time improvement has to be significant above a threshold (represented by the failure rate lower than a specific value) to cancel out the fixed overhead cost and increase net profit.

A simple non-coordinating contract, such as wholesale price only contract, requires less information exchange and administration and is less costly to implement. Compared with a coordinated contract, it might be worth adopting in practice if it is "efficient", where "efficiency" is defined as the ratio of realized supply chain profit under the contract over the theoretical optimal supply chain's profit. It was demonstrated that with a wholesale price only contract, a higher wholesale price lowers retailer's profit and makes the supply chain less efficient; reduction in production cost, warranty cost and improvement in quality helps to build a more efficient supply chain.

This dissertation represented a preliminary study in supply chain coordination via contracts with warranty. It was developed under some simplifying assumptions, which may be extended in future research to account for more complicated situations. To date we developed a coordinated contract by primarily focusing on wholesale contracts and revenue sharing contracts. In practice various other contract forms, such as buyback contracts and sales rebate contracts, are frequently adopted. Those contract types offer great potential for future research. Additionally, this research was limited to the single product, single period model. Future research can explore multiple period models, where inventory holding cost throughout multiple periods may play a role in the profit function. Quality improvement can also be analyzed in the long term for its impact on customer satisfaction, brand image and reputation besides the short term benefit of saving warranty service costs.

It was assumed that the manufacturer dominates the relationship in the supply chain and is the contract designer. The retailer can only decide whether to accept the contracts. This can be extended to the situation where both parties have some decision power on the contract design and modeled by setting positive thresholds of each party's expected profit. The retailer will accept a contract only if it offers an expected profit higher than the threshold.

This research considered a one-to-one relationship between the members in a supply chain. In practice it is common, though, that the manufacturer serves multiple retailers, or the retailer procures from multiple competing manufacturers. Addressing coordination in a competitive many-to-many relationship presents another challenge for future research.

This thesis examined the supply chain coordination with a focus on manufacturer free replacement warranty. Future research may explore coordination under different warranty forms, such as priced warranty and renewing warranty, among others. Finally, the manufacturer and the retailer were assumed to be risk neutral in this research. Future research should explore those situations where parties in the supply chain are risk averse or risk seeking, and consider supply chain coordination when each member maximizes his individual utility.

12. Appendix 1 Proof of Lemmas and Theorems

12.1 Proof of a Preliminary Lemma

Lemma 12.1.1. *For any function* $f(x, y)$*, if*

$$\begin{cases} x^*(y) = \arg\max_x f(x; y) \\ y^* = \arg\max_y f(x^*(y), y) \end{cases},$$

then $(x^* = x^*(y^*), y^*) = \arg\max_{(x,y)} f(x, y)$.

Proof. Lemma 12.1.1

if

$$\frac{df(x; y)}{dx} \mid_{x^*(y)} = 0;$$

$$\frac{d^2 f(x; y)}{dx^2} \leq 0,$$

then $x^*(y) = \arg\max_x f(x; y)$.

If

$$\frac{df(x^*(y); y)}{dy} \mid_{y^*} = 0;$$

and

$$\frac{d^2 f(x^*(y); y)}{dy^2} \leq 0.$$

then $y^* = \arg\max_y f(x^*(y), y)$.

We know that

$$\begin{aligned} \frac{df(x^*(y), y)}{dy} &= \frac{\partial f(x,y)}{\partial x}\Big|_{x^*(y)} \frac{dx^*(y)}{dy} + \frac{\partial f(x,y)}{\partial y} \\ &= 0 \cdot \frac{dx^*(y)}{dy} + \frac{\partial f(x,y)}{\partial y} \\ &= \frac{\partial f(x,y)}{\partial y} \end{aligned}$$

thus

$$\frac{\partial f(x, y)}{\partial y} \mid_{y^*} = 0.$$

Plus

$$\frac{d^2 f(x^*(y),y)}{dy^2} = \frac{\partial^2 f(x,y)}{\partial x^2}\left(\frac{\partial x^*(y)}{\partial y}\right)^2 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{dx^*(y)}{dy} + \frac{\partial f(x,y)}{\partial x}\frac{d^2 x^*(y)}{dy^2} + \frac{\partial^2 f(x,y)}{\partial y^2}.$$

Because $x^*(y)$ satisfies the first order condition

$$\frac{\partial f(x,y)}{\partial x} = \frac{df(x;y)}{dx} = 0,$$

the above second order condition to y is

$$\frac{d^2 f(x^*(y),y)}{dy^2} = \frac{\partial^2 f(x,y)}{\partial x^2}(\frac{dx^*(y)}{dy})^2 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{dx^*(y)}{dy} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

so

$$\begin{aligned}\frac{\partial^2 f(x,y)}{\partial x^2}\frac{d^2 f(x^*(y),y)}{dy^2} &= (\frac{\partial^2 f(x,y)}{\partial x^2})^2(\frac{dx^*(y)}{dy})^2 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{dx^*(y)}{dy}\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial^2 f(x,y)}{\partial x^2} \\ &= \frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial^2 f(x,y)}{\partial x^2} + (\frac{\partial^2 f(x,y)}{\partial x^2}\frac{dx^*(y)}{\partial y} + \frac{\partial^2 f(x,y)}{\partial x \partial y})^2 - (\frac{\partial^2 f(x,y)}{\partial x \partial y})^2.\end{aligned}$$

From $\frac{df(x;y)}{dx}\mid_{x^*(y)}= 0$, we have

$$\begin{aligned}\frac{\partial^2 f(x,y)}{\partial x^2}\frac{dx^*(y)}{\partial y} + \frac{\partial^2 f(x,y)}{\partial x \partial y} &= \frac{\partial \frac{\partial f(x,y)}{\partial x}}{\partial y} \\ &= 0,\end{aligned}$$

so the second derivative with respective to y is

$$\frac{\partial^2 f(x,y)}{\partial x^2}\frac{d^2 f(x^*(y),y)}{dy^2} = \frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial^2 f(x,y)}{\partial x^2} - (\frac{\partial^2 f(x,y)}{\partial x \partial y})^2 \leq 0.$$

Thus, as long as

$$\begin{cases} x^*(y) = \arg\max_x f(x;y) \\ y^* = \arg\max_y f(x^*(y),y) \end{cases},$$

we have

$$\begin{cases} \frac{d^2 f(x;y)}{dx^2} \leq 0 \\ \frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial^2 f(x,y)}{\partial x^2} - (\frac{\partial^2 f(x,y)}{\partial x \partial y})^2 \leq 0 \end{cases}.$$

This means the Hessian matrix of $f(x,y)$ is negative semidefinite.

What is more, it also satisfies

$$\frac{\partial f(x,y)}{\partial x} |_{x^*(y^*)} = 0;$$
$$\frac{\partial f(x,y)}{\partial y} |_{y^*} = 0.$$

This proves that if

$$\begin{cases} x^*(y) = \arg\max_x f(x;y) \\ y^* = \arg\max_y f(x^*(y),y) \end{cases},$$

we have

$$(x^* = x^*(y^*), y^*) = \arg\max_{(x,y)} f(x,y).$$

□

12.2 Lemmas and Theorems of Wholesale Only Contracts

Proof. Lemma 4.1.1

As long as the retailer earns a positive profit we have

$$p - v + g_r > 0.$$

Its profit function is then concave in Q for any fixed K because

$$\begin{aligned} \frac{\partial^2 \Pi_r}{\partial Q^2} &= (p - v + g_r)\frac{\partial^2 S(Q,K)}{\partial Q^2} \\ &= (p - v + g_r)(-f(Q|K)) \\ &< 0. \end{aligned}$$

The necessary and sufficient condition to maximize the retailer's profit for given K is

$$\frac{\partial \Pi_r}{\partial Q} = (p - v + g_r)\frac{\partial S(Q,K)}{\partial Q} + (v - W - c_r) = 0.$$

That is,

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{w + c_r - v}{p - v + g_r}.$$

From (3.3.3) we know that the supply chain optimal condition is

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{c - v}{p - v + g - rP(Y \leq K)}.$$

Since

$$\frac{\partial S(Q)}{\partial Q} = 1 - F(Q|k)$$

is decreasing in Q for given K, the sufficient condition for coordination is to make the retailer's profit an affine function of the supply chain's profit:

$$\frac{w + c_r - v}{p - v + g_r} = \frac{c - v}{p - v + g - rP(y \leq K)}.$$

In other words the manufacturer should choose

$$W = \frac{(p - v + g_r)(c - v)}{p - v + g - rP(Y \leq K)} - (c_r - v).$$

Let

$$\lambda = \frac{p - v + g_r}{p - v + g - rP(Y \leq K)} = \frac{w + c_r - v}{c - v},$$

the retailer's profit is then

$$\Pi_r = \lambda(\Pi + g\mu(K)) - g_r\mu(K).$$

If $\lambda < 1$, the manufacturer has to charge a wholesale price less than the production cost to coordinate the chain. If $\lambda \geq 1$, the manufacturer charges a wholesale price higher than production cost but still earns a negative profit to coordinate the supply chain because the retailer's profit is greater than the whole supply chain's profit. In that case the manufacturer pays more warranty cost and lost sales cost than its revenue from the wholesale. Practically the wholesale contract without warranty cost sharing does not coordinate the supply chain.

□

Proof. Lemma 4.2.1

As long as the retailer earns a positive profit we have its profit function concave in Q for any fixed K because

$$\begin{aligned}\frac{\partial^2 \Pi_r}{\partial Q^2} &= (p - v + g_r - \theta r P(Y \le K))\frac{\partial^2 S(Q,K)}{\partial Q^2} \\ &= (p - v + g_r - \theta r P(Y \le K))(-f(Q|K)) \\ &< 0.\end{aligned}$$

The necessary and sufficient condition to maximize the retailer's profit for given K is

$$\frac{\partial \Pi_r}{\partial Q} = (p - v + g_r - \theta r P(Y \le K))\frac{\partial S(Q,K)}{\partial Q} - (W + c_r - v) = 0.$$

That is,

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{W + c_r - v}{p - v + g_r - \theta r P(Y \le K)}.$$

From (3.3.3) we know that the supply chain optimal condition is

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{c - v}{p - v + g - r P(Y \le K)}.$$

Since

$$\frac{\partial S(Q)}{\partial Q} = 1 - F(Q|k)$$

is decreasing in Q for given K, the sufficient condition for coordination is to make the retailer's profit an affine function of the supply chain's profit:

$$W + c_r - v = \lambda(c - v); p - v + g_r - \theta r P(Y \le K) = \lambda(p - v + g - r P(y \le K)).$$

The retailer's profit is then

$$\Pi_r = \lambda(\Pi + g\mu(K)) - g_r\mu(K).$$

For the manufacturer to earn a positive profit, W has to be greater than c_s. Therefore, $\lambda > 1$. The retailer's profit in this case is larger than the whole channel's profit. Practically the whole sale contract with warranty cost sharing does not coordinate the order quantity. In fact, when both parties earn positive profit, we need parameters that

$$p - v + g_r - \theta r P(Y \leq K) = \lambda(p - v + g - rP(y \leq K)).$$

Therefore the retailer stops ordering when the marginal sales $\frac{\partial S(Q,K)}{\partial Q}$ reaches $\frac{W+c_r-v}{p-v+g_r-\theta r P(Y \leq K)}$, which is greater than the channel marginal sales at the optimum $\partial S(Q,K)\partial Q = \frac{c-v}{p-v+g-rP(Y \leq K)}$. Because the marginal sales $S(Q,K)$ decreases with Q, the retailer orders less than the channel optimum. □

12.3 Lemmas and Theorems of Revenue Sharing Contracts

Proof. Lemma 5.1.1

As long as the retailer earns a positive profit we have

$$(p - v)\phi + g_r > 0.$$

Its profit function is then concave because

$$\begin{aligned} \frac{\partial^2 \Pi_r}{\partial Q^2} &= ((p - v)\phi + g_r)\frac{\partial^2 S(Q,K)}{\partial Q^2} \\ &= ((p - v)\phi + g_r)(-f(Q|K)) \\ &< 0. \end{aligned}$$

Thus for fixed K the sufficient condition to optimize the retailer's profit with respect to

Q is

$$\frac{\partial \Pi_r}{\partial Q} = [(p-v)\phi + g_r]\,\frac{\partial S(Q,K)}{\partial Q} - (W + c_r - v\phi) = 0.$$

That is,

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{W + c_r - v\phi}{(p-v)\phi + g_r)}.$$

From 3.3.3 we know the supply chain optimal Q satisfies

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{c-v}{p-v+g-rP(y \le K)}.$$

Since

$$\frac{\partial S(Q)}{\partial Q} = 1 - F(Q|k)$$

is decreasing in Q for given K, the necessary and sufficient condition for coordination is

$$\frac{c-v}{p-v+g-rP(Y \le K)} = \frac{W + c_r - v\phi}{(p-v)\phi + g_r}.$$

That is the retailer's profit function is an affine function of the supply chain's

$$\Pi_r = \lambda(\Pi + g\mu(K)) - g_r\mu(K).$$

and

$$\phi(p-v) + g_r = \lambda(p + g - v - rP(Y \le K));$$

$$W + c_r - v\phi = \lambda(c - v).$$

For both party to obtain positive profits we require $0 < \lambda < 1$. This is made possible if $W < c_s - (1-\phi)v$. This is made possible by revenue sharing. The manufacturer first charges a small premium for each unit ordered. Though this amount is not enough to cover its production cost, its revenue share through the retail sale will cover the production cost and earns a positive profit. It is shown that under the above condition the manufacturer earns a positive profit of

$$\Pi_s = (1-\lambda)\Pi + \mu(K)(g_r - \lambda g).$$

From this we derive the range of λ for both party to earn a positive profit is

$$\frac{g_r\mu(K)}{\Pi + g\mu(K)} < \lambda < \frac{\Pi + g_r\mu(K)}{\Pi + g\mu(K)}.$$

□

Proof. Lemma 5.1.2 As long as the manufacturer earns a positive profit we have

$$(p-v)(1-\phi) + g_s - rP(Y \leq K) > 0.$$

The manufacturer's profit is concave in Q, as

$$\frac{\partial^2 \Pi_s}{\partial Q^2} = ((p-v)(1-\phi) + g_s - rP(y \leq K))(-f(Q|K)) < 0.$$

Thus given K and other contract parameters the necessary and sufficient condition to maximize the manufacturer's profit is

$$\frac{\partial \Pi_s}{\partial Q} = ((p-v)(1-\phi) + g_s - rP(y \leq K))\frac{\partial S(q;K)}{\partial Q} - (c_s - W - (1-\phi)v) = 0,$$

which means,

$$\frac{\partial S(q;K)}{\partial Q} = \frac{c_s - W - (1-\phi)v}{(p-v)(1-\phi) + g_s - rP(y \leq K))}$$

We know that under coordination the optimal Q^* satisfies

$$\frac{\partial S(q;K)}{\partial Q}|_{Q^*} = \frac{c-v}{p+g-v-rP(Y \leq K)}$$

and the condition to reach the coordination is specified in equations (5.1.4) as

$$\phi(p-v) + g_r = \lambda(p+g-v-rP(Y \leq K))$$

$$W + c_r - v\phi = \lambda(c - v),$$

which basically means,

$$(p - v)(1 - \phi) + g_s - rP(Y \leq K) = (1 - \lambda)(p + g - v - rP(Y \leq K))$$

$$c_s - W - (1 - \phi)v = (1 - \lambda)(c - v)$$

. So

$$\frac{\partial S(q; K)}{\partial Q}|_{Q^*} = \frac{c - v}{p + g - v - rP(y \leq K)} = \frac{c_s - W - (1 - \phi)v}{(p - v)(1 - \phi) + g_s - rP(y \leq K))}.$$

At the supply chain optimal order quantity Q^*, the manufacturer's maximum profit is also achieved. □

Proof. Lemma 5.1.3

The first order condition with respect to K of the manufacturer profit is,

$$\begin{aligned} \frac{\partial \Pi_s}{\partial K} &= -rS(Q, K)\frac{\partial P(Y \leq K)}{\partial K} + ((p - v)(1 - \phi) + g_s - rP(y \leq K))\frac{\partial S(Q, K)}{\partial K} - g_s\frac{\partial \mu(K)}{\partial K} \\ &= 0. \end{aligned} \tag{12.3.1}$$

We know that the first order condition of the supply chain is,

$$\begin{aligned} \frac{\partial \Pi}{\partial K} &= -rS(Q, K)\frac{\partial P(Y \leq K)}{\partial K} + (p - v + g - rP(y \leq K))\frac{\partial S(Q, K)}{\partial K} - g\frac{\mu(K)}{\partial K} \\ &= 0. \end{aligned}$$

Because

$$\frac{\partial P(Y \leq K)}{\partial K} = \beta e^{-\beta K} > 0,$$

$$\frac{\partial S(Q, K)}{\partial K} = -\int_0^Q \frac{\partial F(D|K)}{\partial K} dD > 0,$$

and

$$\frac{\partial \mu(K)}{\partial K} = -\int_0^\infty \frac{\partial F(D|K)}{\partial K} dD > \frac{\partial S(Q,K)}{\partial K} > 0,$$

it is possible for some probability function $F(D|K)$ to satisfy above first order conditions simultaneously. If the contract coordinates with K^*, it must satisfy

$$[(p-v)(1-\phi)+g_s-rP(Y\le K)]\frac{\partial S(Q,K)}{\partial K}-g_s\frac{\partial \mu(K)}{\partial K} = [p-v+g-rP(Y\le K)]\frac{\partial S(Q,K)}{\partial K}-g\frac{\mu(K)}{\partial K}. \tag{12.3.2}$$

From 5.1.4 we derive that

$$(p-v)(1-\phi)+g_s-rP(Y\le K) = (1-\lambda)\left[p-v+g-rP(y\le K)\right].$$

Let $p-v+g-rP(Y\le K)=x$, the above coordination condition is

$$(1-\lambda)x\frac{\partial S(Q,K)}{\partial K} - g_s\frac{\partial \mu(K)}{\partial K} = x\frac{\partial S(Q,K)}{\partial K} - g\frac{\mu(K)}{\partial K}.$$

That is, the coordinating λ^* must satisfy

$$\lambda^* = \frac{g-g_s}{p-v+g-rP(Y\le K)}\cdot\frac{\frac{\partial \mu(K)}{\partial K}\mid_{K^*}}{\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}}.$$

Because $\frac{\partial \mu(K)}{\partial K}\mid_{K^*} > \frac{\partial S(Q,K)}{\partial K}\mid_{K^*}$, $g-g_s=g_r<p-v+g-rP(Y\le K)$ usually, it is possible that $\frac{g_r\mu(K^*)}{\Pi+g\mu(K^*)}<\lambda^*<\frac{\Pi+g_r\mu(K^*)}{\Pi+g\mu(K^*)}$.

Thus if the profit functions are concave in (Q,K), the contract is able to coordinate the supply chain in Q and K if

$$\frac{g_r\mu(K^*)}{\Pi+g\mu(K^*)} < \frac{g_r}{p-v+g-rP(Y\le K)}\cdot\frac{\frac{\partial \mu(K)}{\partial K}\mid_{K^*}}{\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}} < \frac{\Pi+g_r\mu(K^*)}{\Pi+g\mu(K^*)}.$$

The coordination condition is

$$\lambda^* = \frac{g_r}{p-v+g-rP(Y\le K)}\cdot\frac{\frac{\partial \mu(K)}{\partial K}\mid_{K^*}}{\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}}.$$

The optimal Q^* and K^* satisfy

$$\frac{\partial \Pi}{\partial K}\mid_{K^*}= -rS(Q^*,K^*)\frac{\partial P(Y\leq K)}{\partial K}\mid_{K^*}+(p-v+g-rP(y\leq K^*))\frac{\partial S(Q,K)}{\partial K}\mid_{K^*}-g\frac{\partial \mu(K)}{\partial K}=0$$

$$1-F(Q^*|K^*)=\frac{c-v}{p-v+g-rP(y\leq K^*)}.$$

The profit allocation at coordination is

$$\Pi_r=\lambda^*(\Pi+g\mu(K^*))-g_r\mu(K^*)$$

$$\Pi_s=(1-\lambda^*)(\Pi+g\mu(K^*))-g_s\mu(K^*).$$

□

Proof. Lemma 5.2.1 As long as the retailer earns a positive profit we have

$$(p-v)\phi+g_r-\theta rP(Y\leq K)>0.$$

Its profit function is then concave because

$$\begin{aligned}\frac{\partial^2\Pi_r}{\partial Q^2} &= ((p-v)\phi+g_r-\theta rP(Y\leq K))\frac{\partial^2 S(Q,K)}{\partial Q^2}\\ &= ((p-v)\phi+g_r)(-f(Q|K))\\ &< 0.\end{aligned}$$

Thus for fixed K the sufficient condition to optimize the retailer's profit with respect to Q is

$$\frac{\partial \Pi_r}{\partial Q}=[(p-v)\phi+g_r-\theta rP(Y\leq K)]\frac{\partial S(Q,K)}{\partial Q}-(W+c_r-v\phi)=0.$$

That is,

$$\frac{\partial S(Q,K)}{\partial Q}=\frac{W+c_r-v\phi}{(p-v)\phi+g_r-\theta rP(Y\leq K))}.$$

From 3.3.3 we know the supply chain optimal Q satisfies

$$\frac{\partial S(Q,K)}{\partial Q} = \frac{c-v}{p-v+g-rP(y \leq K)}.$$

Since

$$\frac{\partial S(Q)}{\partial Q} = 1 - F(Q|k)$$

is decreasing in Q for given K, the necessary and sufficient condition for coordination is

$$\frac{c-v}{p-v+g-rP(Y \leq K)} = \frac{W+c_r-v\phi}{(p-v)\phi+g_r-\theta rP(Y \leq K)}.$$

That is the retailer's profit function is an affine function of the supply chain's

$$\Pi_r = \lambda(\Pi + g\mu(K)) - g_r\mu(K).$$

and

$$\phi(p-v) + g_r - \theta rP(Y \leq K) = \lambda(p+g-v-rP(Y \leq K))W + c_r - v\phi = \lambda(c-v).$$

The manufacturer earns a positive profit of

$$\Pi_s = (1-\lambda)\Pi + \mu(K)(g_r - \lambda g).$$

From this we derive the range of λ for both party to earn a positive profit is

$$\frac{g_r\mu(K)}{\Pi + g\mu(K)} < \lambda < \frac{\Pi + g_r\mu(K)}{\Pi + g\mu(K)}.$$

□

Proof. Lemma 5.2.2 For any given Q, the first order condition of the manufacturer is,

$$\begin{aligned}\frac{\partial \Pi_s}{\partial K} &= -(1-\theta)rS(Q,K)\frac{\partial P(Y\leq K)}{\partial K} + ((p-v)(1-\phi)+g_s-(1-\theta)rP(y\leq K))\frac{\partial S(Q,K)}{\partial K}\\ &= 0.\end{aligned} \tag{12.3.3}$$

We know that the first order condition of the supply chain is,

$$\begin{aligned}\frac{\partial \Pi}{\partial K} &= -rS(Q,K)\frac{\partial P(Y\leq K)}{\partial K} + (p-v+g-rP(y\leq K))\frac{\partial S(Q,K)}{\partial K}\\ &= 0.\end{aligned}$$

Because

$$\frac{\partial P(Y\leq K)}{\partial K} = \beta e^{-\beta K} > 0,$$

and

$$\frac{\partial S(Q,K)}{\partial K} = -\int_0^Q \frac{\partial F(D|K)}{\partial K} dD > 0,$$

it is possible for some probability function $F(D|K)$ to satisfy above first order conditions simultaneously. If the contract coordinates with K^*, it must satisfy the necessary but not sufficient condition of

$$(p-v)(1-\phi)+g_s-(1-\theta)rP(Y\leq K) = (1-\theta)\left[p-v+g-rP(Y\leq K)\right]. \tag{12.3.4}$$

If the integrated chain's profit function is concave,

$$\begin{aligned}\frac{\partial^2 \Pi}{\partial K^2} &= -rS(Q,K)\frac{\partial^2 P(Y\leq K)}{\partial K^2} - 2r\frac{\partial P(Y\leq K)}{\partial K}\frac{\partial S(Q,K)}{\partial K} + \left[p+g-v-rP(Y\leq K)\right]\frac{\partial S^2(Q,K)}{\partial K^2}\\ &< 0.\end{aligned}$$

The manufacturer's profit function is then concave, too, because

$$
\begin{aligned}
\frac{\partial^2 \Pi_s}{\partial K^2} =& -(1-\theta) r S(Q,K) \frac{\partial^2 P(Y \le K)}{\partial K^2} - 2(1-\theta) r \frac{\partial P(Y \le K)}{\partial K} \frac{\partial S(Q,K)}{\partial K} + \\
& \left[(p-v)(1-\phi) + g_s - (1-\theta) r P(Y \le K)\right] \frac{\partial S^2(Q,K)}{\partial K^2} \\
=& (1-\theta) \frac{\partial^2 \Pi}{\partial K^2} \\
<& 0.
\end{aligned}
$$

The above first order conditions are then sufficient for optimization. From 5.2.3 we know that

$$
(p-v)(1-\phi) + g_s - (1-\theta) r P(Y \le K) = (1-\lambda)\left[p - v + g - r P(y \le K)\right].
$$

Thus the warranty and quantity decisions are automatically coordinated if and only if $\theta = \lambda$.
□

12.4 Lemmas and Theorems of Efficiency of Non-coordinating Contracts

Proof. Theorem 10.1.1

We know the retailer's expected profit

$$
\Pi_r(Q; W, K) = (W + c_r - v)\left[\frac{p - v + g_r}{W + c_r - v} S(Q) - Q\right] - g_r \mu(K),
$$

the supply chain's expected profit

$$
\Pi^D(Q, K) = (c - v)\left[\frac{p - v + g - r P(y \le K)}{c - v} S(Q) - Q\right] - g_r \mu(K).
$$

Because $W + c_r - v > c - v$ and the retailer retains profit less than supply chain, $\frac{p-v+g_r}{W+c_r-v} < \frac{p-v+g-rP(y \le K)}{c-v}$. That is,

$$
c - v < \frac{(p - v + g - r P(y \le K))(W + c_r - v)}{p - v + g_r}
$$

At optimal Q_r^*, $\bar{F}(Q_r^*) = \frac{W+c_r-v}{p-v+g_r}$. Therefore,

$$\begin{aligned}\frac{\partial \Pi^D(Q,K)}{\partial W} &= -(p-v+g-rP(y\leq K))\bar{F}(Q_r^*)\frac{1}{f(Q_r^*)(p-v+g_r)} + (c-v)\frac{1}{f(Q_r^*)(p-v+g_r)} \\ &= (-\frac{1}{f(Q_r^*)(p-v+g_r)})[\frac{(p-v+g-rP(y\leq K))(W+c_r-v)}{p-v+g_r} - (c-v)] \\ &\leq 0.\end{aligned}$$

The decentralized supply chain's profit decreases with wholesale price. As efficiency $E_Q = \frac{\Pi^D}{\Pi^o}$, we have E_Q also decrease with W. Higher wholesale price makes the supply chain less efficient. □

Proof. Theorem 10.1.2

Demand D is distributed on $[0, +\infty)$. When $D = 0$, its IGFR $\frac{Q_r^* f(Q_r^*)}{\bar{F}(Q_r^*)} = 0$; when $D = Max(D)$, its IGFR $\frac{Q_r^* f(Q_r^*)}{\bar{F}(Q_r^*)} = +\infty$. So the domain of IGFR is $[0, +\infty)$.

Let ε be the least upper bound on the set of points such that $g(\varepsilon) = \frac{Q_r^* f(Q_r^*)}{\bar{F}(Q_r^*)} \leq \frac{p-v+g-rP(y\leq K)}{p-v+g_r}$. Thus on $[0, \varepsilon]$,

$$(p-v+g-rP(y\leq K)) - g(Q_r^*)(p-v+g_r)] \geq 0.$$

Because $\bar{F}(Q)$ is decreasing on Q, and $g(Q)$ is increasing on Q, we have

$$\frac{\Pi_s}{Q_r^*} = \bar{F}(Q_r^*)[(p-v+g-rP(y\leq K)) - g(Q_r^*)(p-v+g_r)] - (c-v)$$

decreasing on $[0, \varepsilon]$. Π_s is concave on this domain and the optimal Q_s^* is obtained from $\frac{\Pi_s}{Q_r^*} = 0$. That is,

$$Q_s^* = Q_r^* = \frac{\bar{F}(Q_r^*)(p-v+g-rP(y\leq K)) - (c-v)}{f(Q_r^*)(p-v+g_r)}$$

If $c_r, v, g_r, g_s = 0$

$$Q_s^* = Q_r^* = \frac{\bar{F}(Q_r^*)(p-rP(y\leq K)) - c_s}{f(Q_r^*)p},$$

For any $Q_r^* \in (\varepsilon, +\infty)$, because $g(Q_r^*) > \frac{p-v+g-rP(y\leq K)}{p-v+g_r}$, the first order derivative of Π_s to

Q_r^*

$$\frac{\Pi_s}{Q_r^*} = \bar{F}(Q_r^*)[(p - v + g - rP(y \leq K)) - g(Q_r^*)(p - v + g_r)] - (c - v) < -(c - v) < 0.$$

Π_s is decreasing and the maximum Π_s on $(\varepsilon, +\infty)$ is smaller than the Π_s of Q_r^* on $[0, \varepsilon]$.

To summarize, if demand distribution is IGFR, the manufacturer profit function is unimodal and the optimal Q_s^* is obtained from $\frac{\Pi_s(Q_r^*; Q_r^*(W), K)}{Q_r^*} = 0$ as

$$Q_s^* = \frac{\bar{F}(Q_s^*)(p - v + g - rP(y \leq K)) - (c - v)}{f(Q_s^*)(p - v + g_r)}.$$

This solution is unique. The above equation can be written as

$$\frac{Q_s^* f(Q_s^*)}{\bar{F}(Q_s^*)} = \frac{(p - v + g - rP(y \leq K))}{p - v + g_r} - \frac{(c - v)}{(p - v + g_r)(\bar{F}(Q_s^*))}.$$

If demand is IGFR, there is only one Q_s^* that satisfies above. Here is the proof.

Assume $Q_s' \neq Q_s^*$ also satisfies above.
If $Q_s' > Q_s^*$, because $f(Q)$ is IGFR, the LHS of the equation $\frac{Q_s' f(Q_s')}{\bar{F}(Q_s')} > \frac{Q_s^* f(Q_s^*)}{\bar{F}(Q_s^*)}$; but because $\bar{F}(Q_s') < \bar{F}(Q_s^*)$, the RHS is less than that of Q_s^*. So Q_s' cannot be greater than Q_s^*.

Same as above, if $Q_s' < Q_s^*$, because $f(Q)$ is IGFR, the LHS of the equation $\frac{Q_s' f(Q_s')}{\bar{F}(Q_s')} < \frac{Q_s^* f(Q_s^*)}{\bar{F}(Q_s^*)}$; but because $\bar{F}(Q_s') > \bar{F}(Q_s^*)$, the RHS is greater than that of Q_s^*. So Q_s' cannot be smaller than Q_s^*. Q_s' can only be same as Q_s^* to satisfy the first order condition. Thus Q_s^* is the unique optimal solution of Π_s given demand distribution is IGFR.

□

13. Appendix 2 Concavity of the profit functions

13.1 Concavity of Chain Profit Function

Some useful results

$$\frac{\partial P(Y \leq K)}{\partial K} = \beta e^{-\beta K} > 0,$$

$$\frac{\partial S(Q,K)}{\partial K} = -\int_0^Q \frac{\partial F(D|K)}{\partial K} dD > 0,$$

$$\frac{\partial \mu(K)}{\partial K} = -\int_0^\infty \frac{\partial F(D|K)}{\partial K} dD > 0,$$

$$\frac{\partial S(Q,K)}{\partial Q} = 1 - F(Q|K) > 0,$$

$$\frac{\partial^2 S(Q,K)}{\partial Q^2} = -f(Q|K) < 0,$$

The chain profit function is concave in (Q, K) if and only if

$$\frac{\partial^2 \Pi}{\partial Q^2} < 0,$$

$$\frac{\partial^2 \Pi}{\partial K^2} < 0,$$

$$\frac{\partial^2 \Pi}{\partial Q^2} \cdot \frac{\partial^2 \Pi}{\partial K^2} - (\frac{\partial^2 \Pi}{\partial Q \partial K})^2 > 0.$$

The profit function

$$\Pi(Q,K) = (p - v + g - rP(Y \leq K))S(Q,K) - (c - v)q - g\mu(K)$$

is concave in Q, as

$$\frac{\partial \Pi}{\partial Q} = (p - v + g - rP(Y \leq K))\frac{\partial S(Q,K)}{\partial Q} - (c - v)$$

$$\frac{\partial^2 \Pi}{\partial Q^2} = (p - v + g - rP(y \leq K))(-f(Q|K)) < 0.$$

The profit function may or may not be concave in K, as

$$\begin{aligned}\frac{\partial \Pi}{\partial K} &= (p - v + g - rP(Y \leq K))\frac{\partial S(Q,K)}{\partial K} - rS(Q,K)\frac{\partial P(Y \leq K)}{\partial K} \\ &= (p - v - rP(Y \leq K))(-\int_0^Q \frac{\partial F(D|K)}{\partial K} dD) - rS(Q,K)\beta e^{-\beta K} + g\int_Q^\infty \frac{\partial F(D|K)}{\partial K} dD,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \Pi}{\partial K^2} =& -(p - v - rP(Y \leq K))\int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + r\frac{\partial P(Y \leq K)}{\partial K}\int_0^Q \frac{\partial F(D|K)}{\partial K} dD \\ & - r\beta e^{-\beta K}\frac{\partial S(Q,K)}{\partial K} + rS(Q,K)\beta^2 e^{-\beta K} + g\int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD \\ =& -(p - v - rP(Y \leq K))\int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + 2r\beta e^{-\beta K}\int_0^Q \frac{\partial F(D|K)}{\partial K} dD \\ & + r\beta^2 e^{-\beta K} S(Q,K) + g\int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD.\end{aligned}$$

The cross derivative of Π to (Q, K) is

$$\frac{\partial^2 \Pi}{\partial K \partial Q} = -(p - v + g - rP(Y \leq K))\frac{\partial F(Q|K)}{\partial K} - r\beta e^{-\beta K}(1 - F(Q|K)).$$

The Hessian determinant is

$$\begin{aligned}& \frac{\partial^2 \Pi}{\partial Q^2} \cdot \frac{\partial^2 \Pi}{\partial K^2} - (\frac{\partial^2 \Pi}{\partial Q \partial K})^2 \\ =&(p - v + g - rP(y \leq K))(-f(Q|K))[-(p - v - rP(Y \leq K))\int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + \\ & 2r\beta e^{-\beta K}\int_0^Q \frac{\partial F(D|K)}{\partial K} dD + r\beta^2 e^{-\beta K} S(Q,K) + g\int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD] - \\ & [-(p - v + g - rP(Y \leq K))\frac{\partial F(Q|K)}{\partial K} - r\beta e^{-\beta K}(1 - F(Q|K))]^2.\end{aligned}$$

13.2 Concavity of Revenue Sharing Contract with Manufacturer Taking All Warranty Cost

13.2.1 Concavity of Retailer's Profit Function

The retailer's profit function is

$$\Pi_r(W, q, K) = ((p - v)\phi + g_r)S(Q, K) - (W + c_r - v\phi)Q - g_r\mu(K).$$

It is concave in (Q, K) if and only if

$$\frac{\partial^2 \Pi_r}{\partial Q^2} < 0,$$

$$\frac{\partial^2 \Pi_r}{\partial K^2} < 0,$$

$$\frac{\partial^2 \Pi_r}{\partial Q^2} \cdot \frac{\partial^2 \Pi_r}{\partial K^2} - (\frac{\partial^2 \Pi_r}{\partial Q \partial K})^2 > 0.$$

We know that

$$\frac{\partial^2 \Pi_r}{\partial Q^2} = ((p - v)\phi + g_r)(-f(Q|K)) < 0,$$

$$\frac{\partial^2 \Pi_r}{\partial K^2} = -(p - v)\phi \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + g_r \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD,$$

and

$$\frac{\partial^2 \Pi_r}{\partial Q \partial K} = -(\phi(p - v) + g_r)\frac{\partial F(Q|K)}{\partial K},$$

so,

$$\begin{aligned}
&\frac{\partial^2 \Pi_r}{\partial Q^2} \cdot \frac{\partial^2 \Pi_r}{\partial K^2} - (\frac{\partial^2 \Pi_r}{\partial Q \partial K})^2 \\
=&((p - v)\phi + g_r)(-f(Q|K)) \cdot [-(p - v)\phi \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + \\
&g_r \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD] - [(\phi(p - v) + g_r)\frac{\partial F(Q|K)}{\partial K}]^2.
\end{aligned}$$

13.2.2 Concavity of Manufacturer's Profit Function

The manufacturer's profit function is

$$\Pi_s(W,q,K) = ((p-v)(1-\phi)+g_s-rP(Y\leq K))S(Q,K)-((c_s-W)-v(1-\phi))Q-g_s\mu(K).$$

It is concave in (Q,K) if and only if

$$\frac{\partial^2\Pi_s}{\partial Q^2} < 0,$$

$$\frac{\partial^2\Pi_s}{\partial K^2} < 0,$$

$$\frac{\partial^2\Pi_s}{\partial Q^2}\cdot\frac{\partial^2\Pi_s}{\partial K^2} - (\frac{\partial^2\Pi_s}{\partial Q\partial K})^2 > 0.$$

We know that

$$\frac{\partial^2\Pi_s}{\partial Q^2} = ((p-v)(1-\phi)+g_s-rP(Y\leq K))(-f(Q|K)) < 0,$$

$$\begin{aligned}\frac{\partial^2\Pi_s}{\partial K^2} =& -((p-v)(1-\phi)-rP(Y\leq K))\int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2}dD + 2r\beta e^{(-\beta K)}\int_0^Q \frac{\partial F(D|K)}{\partial K}dD \\ &+ r\beta^2 e^{-\beta K}S(Q,K) + g_s\int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2}dD,\end{aligned}$$

and

$$\frac{\partial^2\Pi_s}{\partial Q\partial K} = -((1-\phi)(p-v)+g_s-rP(Y\leq K))\frac{\partial F(Q|K)}{\partial K} - r\beta e^{-\beta K}(1-F(Q|K)),$$

so,

$$\frac{\partial^2 \Pi_s}{\partial Q^2} \cdot \frac{\partial^2 \Pi_s}{\partial K^2} - (\frac{\partial^2 \Pi_s}{\partial Q \partial K})^2$$
$$=((p-v)(1-\phi) + g_s - rP(Y \leq K))(-f(Q|K))$$
$$[-((p-v)(1-\phi) - rP(Y \leq K)) \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + 2r\beta e^{(-\beta K)}$$
$$\int_0^Q \frac{\partial F(D|K)}{\partial K} dD + r\beta^2 e^{-\beta K} S(Q,K) + g_s \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD] -$$
$$[((1-\phi)(p-v) + g_s - rP(Y \leq K)) \frac{\partial F(Q|K)}{\partial K} + r\beta e^{-\beta K}(1 - F(Q|K))]^2.$$

13.3 Concavity of Revenue Sharing Contract with Warranty Cost Sharing

13.3.1 Concavity of Retailer's Profit Function

The retailer's profit function is

$$\Pi_r = [(p-v)\phi - \theta r P(Y \leq K) + g_r] \, S(Q,K) - (W + c_r - v\phi)Q - g_r \mu(K).$$

It is concave in (Q, K) if and only if

$$\frac{\partial^2 \Pi_r}{\partial Q^2} < 0,$$
$$\frac{\partial^2 \Pi_r}{\partial K^2} < 0,$$
$$\frac{\partial^2 \Pi_r}{\partial Q^2} \cdot \frac{\partial^2 \Pi_r}{\partial K^2} - (\frac{\partial^2 \Pi_r}{\partial Q \partial K})^2 > 0.$$

We know that

$$\frac{\partial^2 \Pi_r}{\partial Q^2} = ((p-v)\phi - \theta r P(Y \leq K) + g_r)(-f(Q|K)) < 0,$$

$$\frac{\partial^2 \Pi_r}{\partial K^2} = -\left((p-v)\phi - \theta r P(Y \leq K)\right) \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD +$$
$$2\theta r \beta e^{-\beta K} \int_0^\infty \frac{\partial F(D|K)}{\partial K} dD + g_r \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD + \theta r \beta^2 e^{-\beta K} S(Q,K),$$

and

$$\frac{\partial^2 \Pi_r}{\partial Q \partial K} = -(\phi(p-v) - \theta r P(Y \le K) + g_r)\frac{\partial F(Q|K)}{\partial K} - \theta r \beta e^{-\beta K}(1 - F(Q|K)),$$

so,

$$\begin{aligned}
&\frac{\partial^2 \Pi_r}{\partial Q^2} \cdot \frac{\partial^2 \Pi_r}{\partial K^2} - (\frac{\partial^2 \Pi_r}{\partial Q \partial K})^2 \\
=&((p-v)\phi - \theta r P(Y \le K) + g_r)(-f(Q|K))[-((p-v)\phi - \theta r P(Y \le K)) \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD \\
&+ 2\theta r \beta e^{-\beta K} \int_0^\infty \frac{\partial F(D|K)}{\partial K} dD + g_r \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD + \theta r \beta^2 e^{-\beta K} S(Q,K)] \\
&- [(\phi(p-v) - \theta r P(Y \le K) + g_r)\frac{\partial F(Q|K)}{\partial K} + \theta r \beta e^{-\beta K}(1 - F(Q|K))]^2
\end{aligned}$$

13.3.2 Concavity of Manufacturer's Profit Function

The manufacturer's profit function is

$$\Pi_s = [(p-v)(1-\phi) - (1-\theta) r P(Y \le K) + g_s]\, S(Q,K) - (c_s - W_s - v(1-\phi))Q - g_s \mu(K).$$

It is concave in (Q, K) if and only if

$$\frac{\partial^2 \Pi_s}{\partial Q^2} < 0,$$

$$\frac{\partial^2 \Pi_s}{\partial K^2} < 0,$$

$$\frac{\partial^2 \Pi_s}{\partial Q^2} \cdot \frac{\partial^2 \Pi_s}{\partial K^2} - (\frac{\partial^2 \Pi_s}{\partial Q \partial K})^2 > 0.$$

We know that

$$\frac{\partial^2 \Pi_s}{\partial Q^2} = ((p-v)(1-\phi) + g_s - (1-\theta) r P(Y \le K))(-f(Q|K)) < 0,$$

$$\frac{\partial^2 \Pi_s}{\partial K^2} = -\left((p-v)(1-\phi) - (1-\theta) r P(Y \le K)\right) \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD$$
$$+ 2(1-\theta) r \beta e^{(-\beta K)} \int_0^Q \frac{\partial F(D|K)}{\partial K} dD + (1-\theta) r \beta^2 e^{-\beta K} S(Q,K)$$
$$+ g_s \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD,$$

and

$$\frac{\partial^2 \Pi_r}{\partial Q \partial K} = -((1-\phi)(p-v) + g_s - (1-\theta) r P(Y \le K)) \frac{\partial F(Q|K)}{\partial K} - (1-\theta) r \beta e^{-\beta K} (1 - F(Q|K)),$$

so,

$$\frac{\partial^2 \Pi_r}{\partial Q^2} \cdot \frac{\partial^2 \Pi_r}{\partial K^2} - \left(\frac{\partial^2 \Pi_r}{\partial Q \partial K}\right)^2$$
$$=((p-v)(1-\phi) + g_s - (1-\theta) r P(Y \le K))(-f(Q|K))$$
$$[-((p-v)(1-\phi) - (1-\theta) r P(Y \le K)) \int_0^Q \frac{\partial^2 F(D|K)}{\partial K^2} dD + 2(1-\theta) r \beta e^{(-\beta K)}$$
$$\int_0^Q \frac{\partial F(D|K)}{\partial K} dD + (1-\theta) r \beta^2 e^{-\beta K} S(Q,K) + g_s \int_Q^\infty \frac{\partial^2 F(D|K)}{\partial K^2} dD] -$$
$$[((1-\phi)(p-v) + g_s - (1-\theta) r P(Y \le K)) \frac{\partial F(Q|K)}{\partial K} + (1-\theta) r \beta e^{-\beta K} (1 - F(Q|K))]^2$$

14. Appendix 3 Tables

Table 14.1: Results of Revenue and Warranty Sharing Contract

K	$Q*$	Π	Π_r	Π_s	$\% of Maximum$
0	546	21660	6539	15121	97.4%
1	574	21919	6638	15281	98.5%
2	601	22099	6709	15390	99.3%
3	629	22205	6754	15451	99.8%
4	657	22245	6775	15470	100.0%
5	684	22224	6774	15450	99.9%
6	712	22150	6754	15395	99.6%
7	740	22026	6717	15309	99.0%
8	767	21858	6663	15195	98.3%
9	795	21651	6595	15056	97.3%
10	823	21409	6515	14894	96.2%

Table 14.2: Sensitivity to Order Amount

	K	Q	Π	Π_r	Π_s
Optimal	4	657	22244	6774	15469
Actual	4	525	19496	6197	13299
% to Optimal	100%	80%	87.6%	91.5%	86.0%

Table 14.3: Fair Profit Allocation: Zero Lost Sale cost

	Chain	Retailer	Manufacturer
Profit	22601	9175	13426
Percentage		41%	59%
Cost	49454	20781	28673
Percentage		42%	58%

Table 14.4: Fair Profit Allocation: Lost Sale cost as Lost Profit

	Chain	Retailer	Manufacturer
Profit	21755	8894	12861
Percentage		41%	59%
Cost	52086	21748	30338
Percentage		42%	58%
Lost Sale Cost	37	15	22
Percentage		41%	59%

Table 14.5: Impact of Quality Improvement on Coordination

β	K^*	Q^*	Π	$\%Increase$	Π_r	$\%Increase$	Π_s	$\%Increase$
20	4	657	22245	0.0%	6775	0.0%	15470	0.0%
21	5	685	22516	1.2%	6873	1.4%	15643	1.1%
22	6	713	22820	2.6%	6980	3.0%	15839	2.4%
23	7	742	23151	4.1%	7097	4.8%	16054	3.8%
24	8	770	23505	5.7%	7220	6.6%	16285	5.3%
25	9	799	23879	7.3%	7350	8.5%	16529	6.8%

Table 14.6: Sensitivity Analysis of Demand Responsiveness

b	Chain \$ (1.0e+004)	Retailer \$ (1.0e+004)	%	Manufacturer \$ (1.0e+004)	%	Q	K	Demand Mean	Sales
25	2.17	0.478	22.0%	1.691	78.0%	566	0.9	522	500
26	2.17	0.474	21.8%	1.700	78.2%	584	1.6	542	520
27	2.18	0.472	21.6%	1.711	78.4%	603	2.3	563	540
28	2.20	0.471	21.4%	1.724	78.6%	622	3.0	583	560
29	2.21	0.470	21.3%	1.739	78.7%	641	3.6	604	580
30	2.22	0.470	21.1%	1.755	78.9%	661	4.1	624	600
31	2.24	0.470	21.0%	1.773	79.0%	680	4.7	645	620
32	2.26	0.471	20.8%	1.791	79.2%	699	5.2	665	640
33	2.28	0.472	20.7%	1.810	79.3%	718	5.6	686	660
34	2.30	0.474	20.6%	1.831	79.4%	738	6.1	706	680
35	2.33	0.476	20.4%	1.852	79.6%	757	6.5	726	700

15. Appendix 4 Figures and Graphics

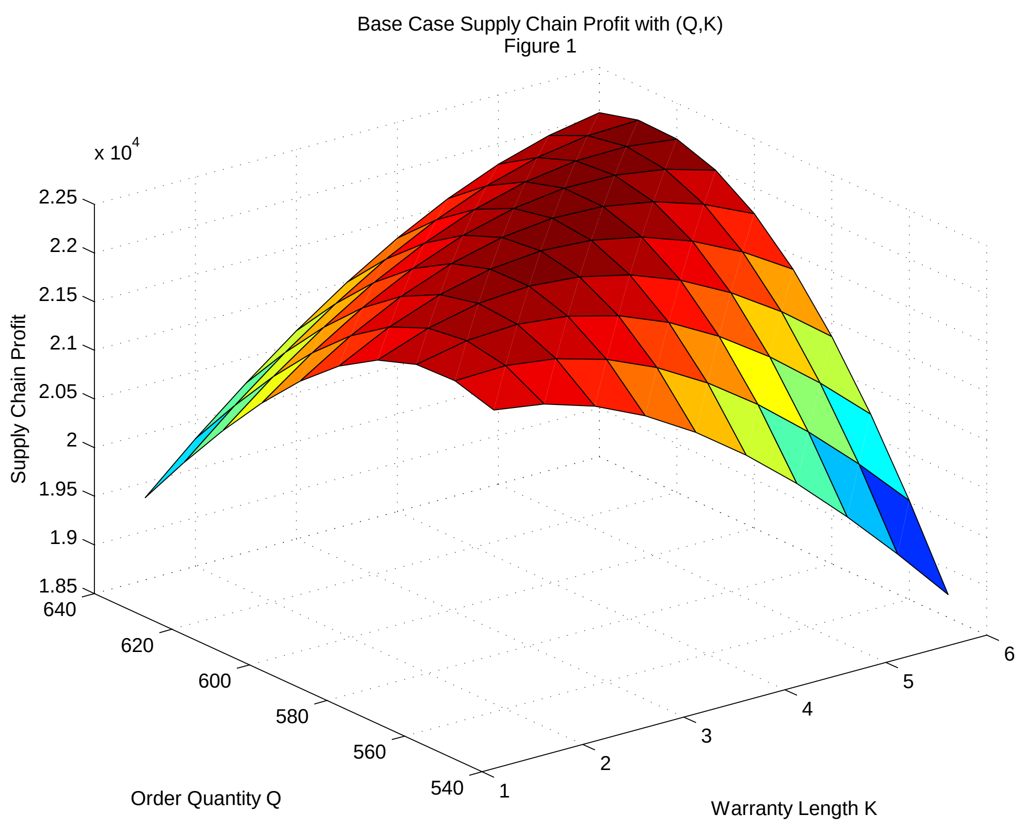

Figure 15.1: Supply Chain Profit to Q and K

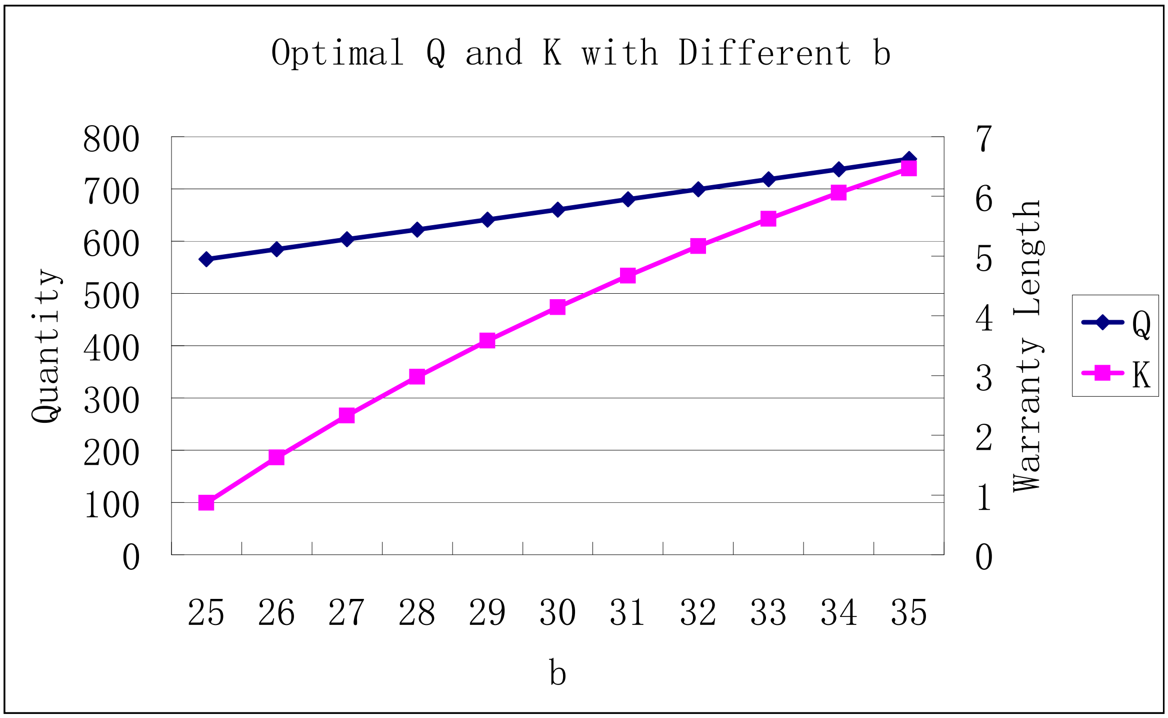

Figure 15.2: Optimal Q and K with Different b

Figure 15.3: Optimal Q and Expected Demand with Different b

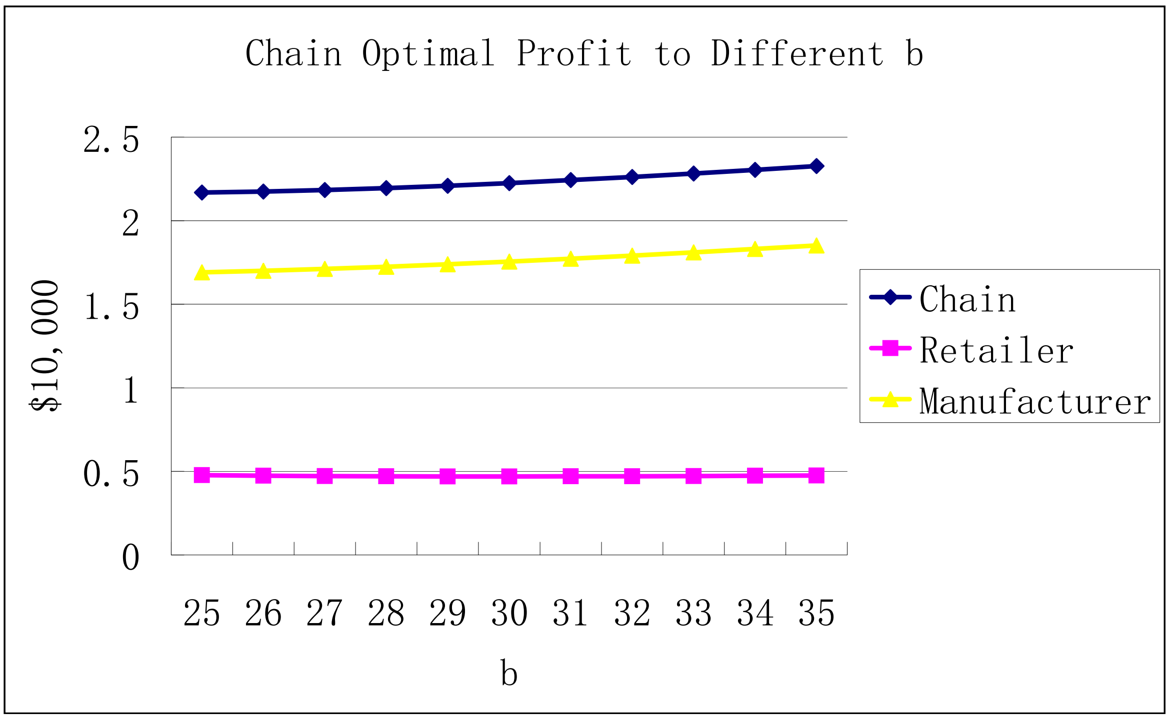

Figure 15.4: Optimal Profits with Different b

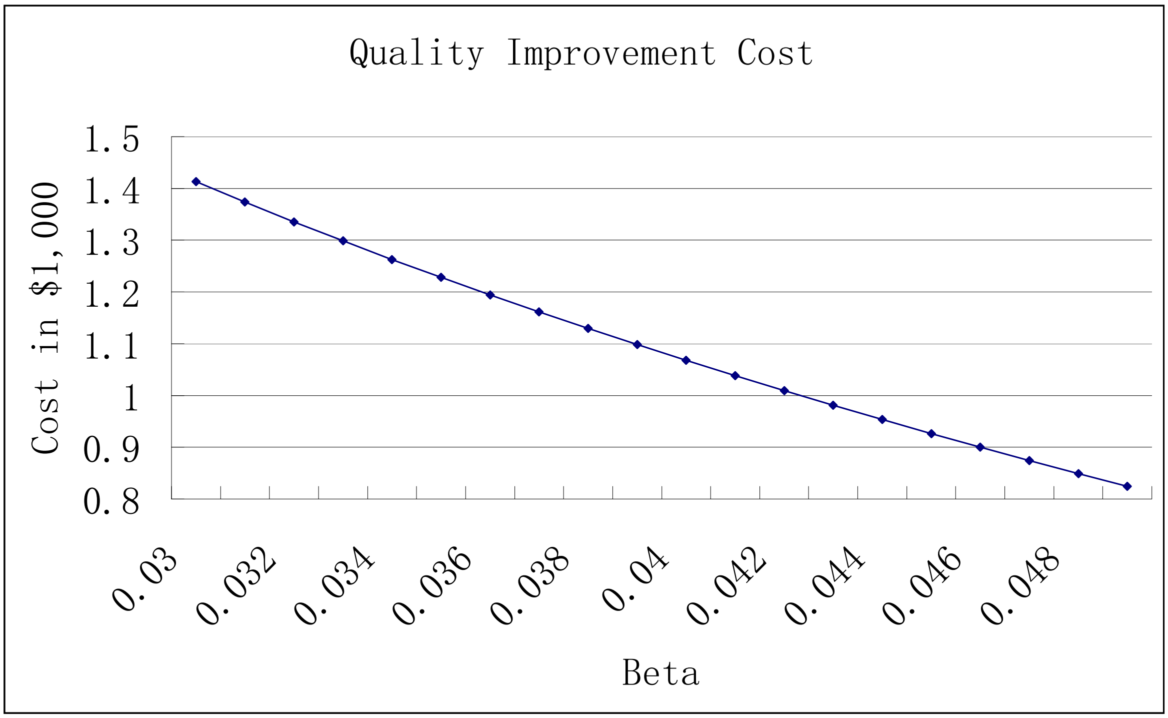

Figure 15.5: Quality Investment with Different Failure Rate

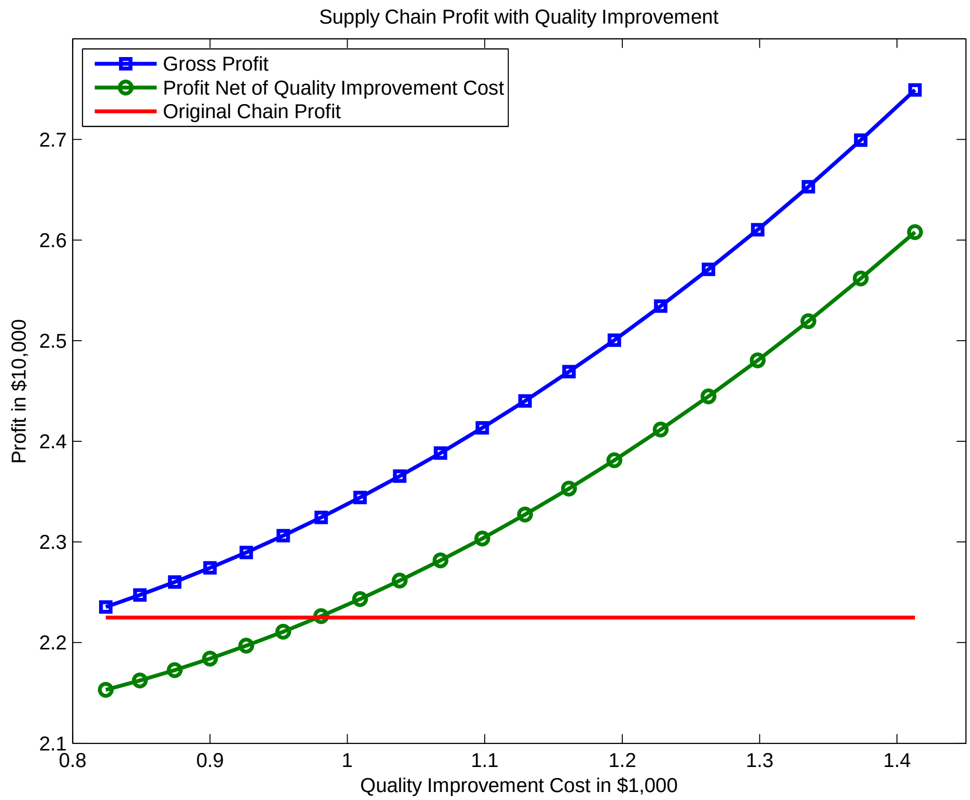

Figure 15.6: Chain Profits with Different Quality Investment

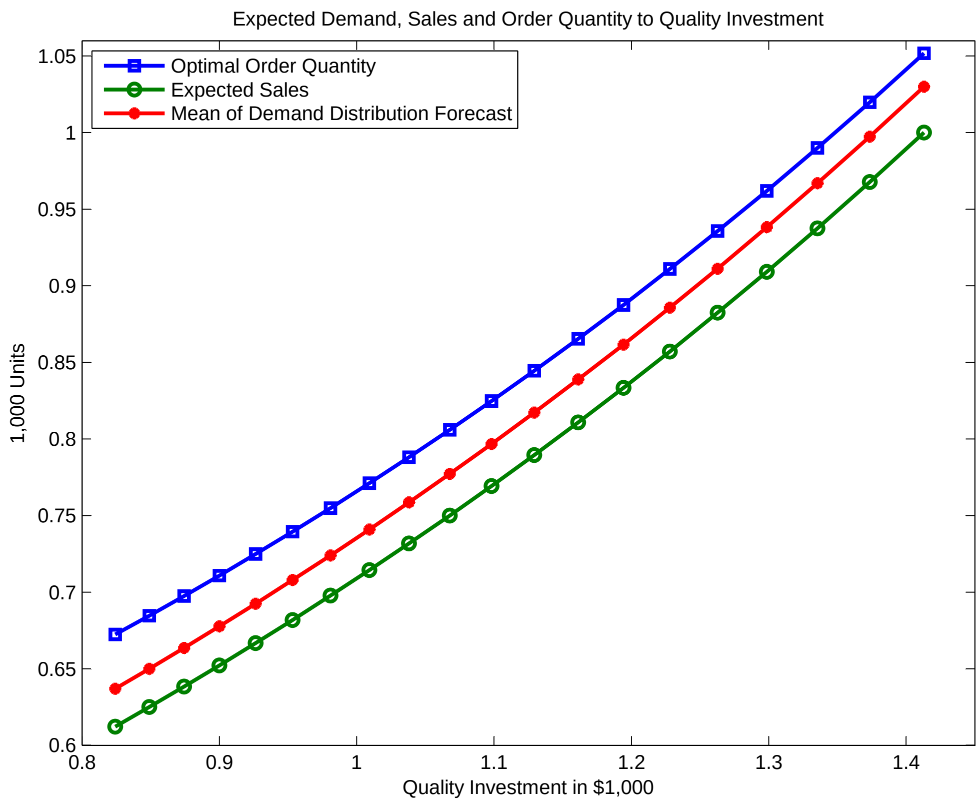

Figure 15.7: Expected Demand, Sales and Optimal Order Quantity with Different Quality Investment

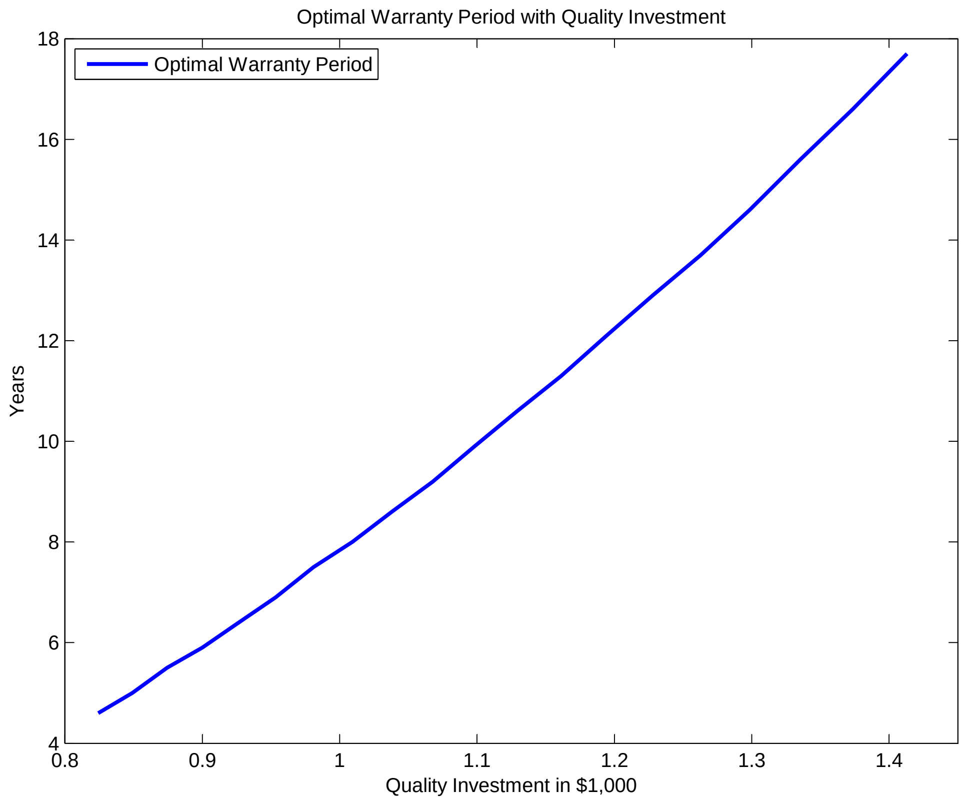

Figure 15.8: Optimal Warranty Period with Different Quality Investment

Bibliography

Alfredsson, P. (1997). Optimization of multi-echelon repairable item inventory systems with simultaneous location of repair facilities. *European Journal of Operational Research*, *99*, 584–595.

Anupindi, R. & Bassok, Y. (1999). Centralization of stocks: retailers vs. manufacturer. *Management Science*, *45*, 178–191.

Anupindi, R., Bassok, Y., & Zemel, E. (2001). A general framework for the study of decentralized distribution systems. *Manufacturing and Service Operations Management*, *3*, 349–368.

Arrow, K., S.Karlin, & Scarf, H. (Eds.) (1962). *Studies in Applied Probability and Management Science*. Stanford University Press.

Ashgarizadeh, E. & Murthy, D. (2000). Service contract: a stochastic model. *Mathematical and Computer Modelling*, *31*, 11–20.

Banerjee, A. & Kim, S.-L. (1995). An integrated jit inventory model. *International Journal of Operations and Production Management*, *15*, 237.

Barlow, R. & Hunter, L. (1960). Optimum preventive maintenance policies. *Operations Research*, *8*, 90–100.

Barnes-Schuster, Y. B., D. & Anupindi, R. (2002). Coordination and flexibility in supply contracts with options. *Manufacturing and Service Operations Management*, *4*, 171–207.

Bassok, Y. & Anupindi, R. (1997). Analysis of supply contracts with total minimum commitment. *IIE Transactions*, *29*.

Blischke, W. & Murthy, D. (1992). Product warranty management-i: Taxonomy for warranty policies. *European Journal of Operational Research*, *62*, 127–148.

Blischke, W. & Murthy, D. (Eds.) (1994). *Warranty Cost Analysis*. New York: Marcel Dekker.

Blischke, W. & Murthy, D. (1996). *Product Warranty Handbook*. New York:Marcel Dekker.

Blischke, W. & Murthy, D. (2000). *Reliability Modelling, Prediction and Optimisation*. New York: John Willey and sons, Inc.

Bresnahan, T. & Reiss, P. (1985). Dealer and manufacturer margins. *Rand Journal of Economics*, *16*, 253–268.

Cachon, G. (2004). The allocation of inventory risk in a supply chain: push, pull and advanced purchase discount contracts. *Management Science*, *50*, 222–238.

Cachon, G. & Lariviere, M. (2001). Contracting to assure supply: how to share demand forecasts in a supply chain. *Management Science*, *47*, 629–646.

Cachon, G. & Lariviere, M. (2005). Supply chain coordination with revenue sharing: strengths and limitations. *Management Science*, *51*, 30.

Cachon, G. P. (2003). Handbooks in operations research and management science: Supply chain management.

Caldentey, R. & Wein, L. (2003). Analysis of a decentralized production-inventory system. *Manufacturing and Service Operations Management*, *5*, 1–17.

Cassady, C., Murdock, W., & Pohl, E. (2001). Selective maintenance of support equipment involving multiple maintenance actions. *European Journal of Operational Research*, *129*, 252–258.

Chattopadhyay, G. & Murthy, D. (1996). Modeling warranties for second-hand products. *Australia-Japan Workshop on Stochastic Models in Engineering, Technology and Management*, *2*, 91–99.

Chattopadhyay, G. & Murthy, D. (2000). Warranty cost analysis for second-hand products. *Journal of Mathematical and Computer Modeling*, *31*, 81–88.

Chattopadhyay, G. & Murthy, D. (2001). Cost sharing warranty policies for second-hand products. *International Transactions in Operational Research*, *8(1)*, 47–60.

Chen, F. (1999). Decentralized supply chains subject to information delays. *Management Science*, *45*, 1076–1090.

Chod, J. & Rudi, N. (2005). Resource flexibility with responsive pricing. *Operations Research*, *53*, 532–548.

Chukova, H., Arnold, R., & Wang, D. (2004). Warranty analysis: an approach to modeling imperfect repair. *International Journal of Production Economics*, *89*, 57–68.

Cohen, M. & Lee, H. (1990). Out of touch with customer needs? spare parts and after sales service. *Sloan Management Review*, *31*, 55–66.

Connelly, M. (2006). Dealers: Extended factory warranties boost profits. *Automotive News.Detroit*, *81*, 46.

Dana, J. D. J. & Spier, K. E. (2001). Revenue sharing and vertical control in the video rental industry. *The Journal of Industrial Economics*, *49*, 223–246.

Djamaludin, I., Murthy, D., & Blischke, W. (1996). An annotated bibliography on warranties. *Product Warranty Handbook*.

Dong, L. & Rudi, N. (2004). Who benefits from transshipment? exogenous vs. endogenous wholesale prices. *Management Science*, *50*, 645–657.

Donohue, K. (2000). Efficient supply contracts for fashion goods with forecast updating and two production modes. *Management Science*, *46*, 1397–1411.

Emmons, H. & Gilbert, S. (1998). Returns policies in pricing and inventory decisions for catalogue goods. *Management Science*, *44*, 276–283.

Eppen, G. & Iyer, A. (1997). Backup agreements in fashion buying - the value of upstream flexibility. *Management Science*, *43*, 1469–1484.

Genesove, D. (1993). Adverse selection in the wholesale used car market. *Journal of Political Economy*, *101*, 644–665.

Gerchak, Y., Cho, R., & Ray, S. (2006). Coordination of quantity and shelf-retention timing in the video movie rental industry. *IIE Transactions*, *38*, 525 – 536.

Gerner, J. L. & Bryant, W. K. (1980). The demand for repair service during warranty. *Journal of Business*, *53*, 397–414.

Gilbert, S. & Cvsa, V. (2000). Strategic supply chain contracting to stimulate downstream process innovation. *Manufacturing and Service Operations Conference Proceedings*.

Hollis, A. (1999). Extended warranties, adverse selection and after markets. *The Journal of Risk and Insurance*, *66(3)*, 321–343.

Iskander, B., Murthy, D., & Jack, N. (2005). A new repair-replacement strategy for items sold with a two-dimensional warranty. *Computers and Operations Research*, *32(3)*, 669–682.

Jack, N. & Van der Duyn Schouten, F. (2000). Optimal repairreplace strategies for a warranted product. *International Journal of Production Economics*, *67*, 95–100.

Jeuland, A. & Shugan, S. (1983). Managing channel profits. *Marketing Science*, *2*, 239–272.

Kimble, W. & Lesher, R. O. (1979). *Products Liability*. St. Paul, MN: West Publishing Company.

Kolay, S., Shaffer, G., & Ordover, J. (2004). All-units discounts in retail contracts. *Journal of Economics and Management Strategy*, *13*, 429–459.

Kreps, D. (1990). *A Course in Microeconomic Theory*. Princeton University Press, Priceton, NJ.

Krishnan, H., Kapuscinski, R., & Butz, D. (2004). Coordinating contracts for decentralized supply chains with retailer promotional effort. *Management Science*, *50*, 48 – 63.

Kumar, U. & Chattopadhyay, G. (2004). Mathematical models for analysis of extended warranty. *Proc. the Fifth Asia-Pacific Industrial Engineering and Management Systems Conference*, 249.

Lariviere, M. A. & Porteus, E. L. (2001). Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing and Service Operations Management*, *3*, 293–305.

Lee, H. (1987). A multi-echelon inventory model for repairable items with emergency lateral transshipments. *Management Science*, *33*, 1302–1316.

Lee, T. T., H.and V. Padmanabhan & Whang, S. (2000). Price protection in the personal computer industry. *Management Science*, *46*, 467–482.

Lowerre, J. (1968). On warranties. *Journal of Industrial Engineering*, *19*, 359–360.

Lutz, N. & Padmanabhan, V. (1998). Warranties, extended warranties and product quality. *International Journal of Industrial Organization*, *16*, 463–493.

Mahon, B. & Bailey, R. (1975). A proposed improved replacement policy for army vehicles. *Operational Research Quarterly*, *26*, 477–494.

Menke, W. (1969). Determination of warranty reserves. *Management Science*, *15*, 542–549.

Mills, E. S. (1959). Uncertainty and price theory. *Quart. J. Econom.*, *73*, 116–130.

Mitra, A. & Patnakar, J. (1997). Market share and warranty cost for renewable warranty programs. *International Journal of Production Economics*, *50*, 155–168.

Moorthy, K. S. (1987). Managing channel profits: comment. *Marketing Science*, *6*, 375–379.

Morgan, F. W. (1982). Marketing and product liability: A review and update. *Journal of Marketing*, *46*, 69 – 78.

Mortimer, J. (2002). The effects of revenue-sharing contracts on welfare in vertically separated markets: evidence from video rental industry. *Harvard Institute of Economic Research Discussion Paper*.

Mortimer, J. H. (2006). Vertical contracts in the video rental industry. *Working Paper*.

Murthy, D. & Ashgarizadeh, E. (1995). Modelling service contracts. *Presented at the INFORMS Meeting*.

Murthy, D. & Blischke, W. (1992). Product warranty and management -ii: An integrated framework for study. *European Journal of Operational Research*, *62*, 261–281.

Murthy, D. & Djamaludin, I. (2002). New product warranty:a literature review. *Int.J.Production Economics*, *79*, 231–260.

Murthy, D. & Jack, N. (2003). Warranty and maintenance. *Handbook of Reliability Engineering*, 305–314.

Murthy, D. & Nguyen, D. (1985a). Study of multi-component system with failure interaction. *European Journal of Operational Research*, *21*, 330–338.

Murthy, D. & Nguyen, D. (1985b). Study of two component system with failure interaction. *Naval Research Logistics Quarterly*, *32*, 239–247.

Murthy, D. & Nguyen, D. (1988). An optimal repair cost limit policy for servicing warranty. *Mathematical and Computer Modeling*, *11*, 595–599.

Murthy, D., Solem, O., & Roren, T. (2004). Product warranty logistics: Issues and challenges. *European Journal of Operational Research*, *156*, 110.

Padmanabhan, V. (1995). Usage hetrogeneity and extended service contracts. *Journal of Economics and Management Strategy*, *12*, 33–54.

Padmanabhan, V. & Png, I. P. L. (1995). Returns policies: make money by making good. *Sloan Management Review*, *Fall*, 65–72.

Padmanabhan, V. & Png, I. P. L. (1997). Manufacturers returns policy and retail competition. *Marketing Science*, *16*, 81–94.

Pasternack, B. (1985). Optimal pricing and returns policies for perishable commodities. *Marketing Science*, *4*, 166–76.

Pasternack, B. (2005). Using revenue sharing to achieve channel coordination for a newsboy type inventory model. *Supply Chain Management: Models, Applications, and Research Directions*, *62*, 117–136.

Petruzzi, N. & Dada, M. (1999). Pricing and the newsvendor problem: a review with extensions. *Operations Research*, *47*, 183–194.

Porteus, E. (2000). Responsibility tokens in supply chain management. *Manufacturing and Service Operations Management*, *2*, 203–219.

Rahman, A. & Chattopadhyay, G. (2004). Lifetime warranty policies: complexities in modelling and industrial application. *Proc. 5th Asia-Pacific Industrial Engineering and Management Systems Conference*, *Gold Coast, Australia, 12-15 December*, 249.

Register, F. (1979). Uniform commercial code (1972). *44(January 12)*, 2996–3019.

Rinsaka, K. & Sandoh, H. (2001). study on extension contract of warranty period. *Transaction of the Institute of Electronics, Inofrmation and Communication Engineers*, *84*, 528–542.

Rudi, S. K., N. & Pyke, D. (2001). A two-location inventory model with transhipment and local decision making. *Management Science*, *47*, 1668–1680.

Sahin, I. & Polatoglu, H. (1998). *Quality, Warranty and Preventive Maintenance*. Springer.

Scherer, K. (2006). Gm warranties take big leap. *Automotive News.Detroit*, *81*, 30.

Spengler, J. (1950). Vertical integration and antitrust policy. *Journal of Political Economy*, 347–352.

Tagaras, G. & Cohen, M. A. (1992). Pooling in two-location inventory systems with non-negligible replenishment lead times. *Management Science*, *38*, 1067–1083.

Taylor, T. (2002). Coordination under channel rebates with sales effort effect. *Management Science*, *48*, 992–1007.

Tsay, A. (1999). Quantity-flexibility contract and supplier-customer incentives. *Management Science*, *45*, 1339–1358.

Tsay, A. & Lovejoy, W. S. (1999). Quantity-flexibility contracts and supply chain performance. *Manufacturing and Service Operations Management*, *1*, 89–111.

Udell, J. G. & Anderson, E. E. (1968). The product warranty as an element of competitive strategy. *Journal of Marketing*, *32*, 1–8.

Warren, A. & Peers, M. (2002). Video retailers have day in court plaintiffs say supply deals between blockbuster inc. and studios violate laws. *Wall Street Journal*, *13*, B10.

Wasserman, G. (1992). An application of dynamic linear models for predicting warranty claims. *Computers and Industrial Engineering*, *22*, 37–47.

Wilson, R. (1993). *Nonlinear Pricing*. New York: Oxford University Press.

Yeh, L. & Peggo, K. (2001). An extended warranty policy with options open to consumers. *European Journal of Operational Research*, *131*, 514–529.