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DETERMINANTAL CONSTRUCTION OF ORTHOGONAL POLYNOMIALS ASSOCIATED WITH ROOT SYSTEMS

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ABSTRACT. We consider semisimple triangular operators acting in the symmetric component of the group algebra over the weight lattice of a root system. We present a determinantal formula for the eigenbasis of such triangular operators. This determinantal formula gives rise to an explicit construction of the Macdonald polynomials and of the Heckman-Opdam generalized Jacobi polynomials.

CONTENTS

1. INTRODUCTION

The main objective of this work concerns the explicit computation of families of orthogonal polynomials associated with root systems. Key examples of the families under consideration are the Macdonald polynomials [M1, M2, M3] and the Heckman-Opdam generalized Jacobi polynomials [HS, O]. The origin of the Heckman-Opdam polynomials lies in the harmonic analysis of simple Lie groups, where they appear (for special parameter values) as zonal spherical functions on

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compact symmetric spaces [HS, He]. Other important applications of these poly-
nomials arise in mathematical physics, where they are used to express the eigen-
functions of the quantum Calogero-Sutherland one-dimensional many-body sys-
tems [S1, S2, OP, HS]. The Macdonald polynomials have similar applications:
they appear as zonal spherical functions on compact quantum symmetric spaces
[N, Su], and they are used to express the eigenfunctions of Ruijsenaars' (q-)difference
Calogero-Sutherland systems [R, D1]. Depending on the specific application of in-
terest, our work may thus be viewed as providing an explicit construction for the
zonal spherical functions on compact (quantum) symmetric spaces or for the eigen-
functions of the (difference) Calogero-Sutherland type quantum many-body models.

The usual definition of the Heckman-Opdam and Macdonald polynomials in-
volves a Gram-Schmidt type orthogonalization of the monomial basis with respect
to a generalized Haar measure [HS, O, M2, M3]. This definition, although most
appropriate from a theoretical point of view, is not very adequate for the explicit
computation of the polynomials in question. The main result of this paper is a
determinantal formula for the Heckman-Opdam and Macdonald polynomials that
gives rise to an efficient recursive procedure from which their expansion in the
monomial basis can be constructed explicitly. For the type A root systems the
Heckman-Opdam polynomials reduce (in essence) to Jack's polynomials [St, M1]
and the Macdonald polynomials reduce to Macdonald's symmetric functions [M1].
In this case the determinantal construction of the polynomials under consideration
was laid out in previous work by Lapointe, Lascoux, and Morse [LLM1, LLM2].
More specifically, the results of the present paper constitute a generalization of the
methods of Refs. [LLM1, LLM2] to the case of arbitrary root systems. For the
Heckman-Opdam families we consider general (not necessarily reduced) root sys-
tems and general values of the root multiplicity parameters. For the Macdonald
families, however, we restrict for technical reasons to those (reduced) root systems
for which the dual root system $R^\vee$ has a minuscule weight (thus including the types
A_N, B_N, C_N, D_N, E_6 and E_7 while excluding the types BC_n, E_8, F_4 and G_2).

The paper is organized as follows. Section 2 introduces the concept of a trian-
gular operator in the Weyl-group invariant component of the group algebra over
the weight lattice of a root system. In Section 3 we present a method for di-
agonalizing such triangular operators by means of a determinantal formula. The
Heckman-Opdam and Macdonald polynomials are defined in Section 4. We employ
the determinantal formula from Section 3 to build explicit expressions for the mono-
mial expansions of these polynomials in Section 5 (Heckman-Opdam) and Sections
6, 7 (Macdonald), respectively. For completeness, some technicalities concerning
the explicit evaluation of the determinant of a Hessenberg matrix are recalled in
Appendix A at the end of the paper. To facilitate explicit computations, we have
furthermore included a useful formula for the calculation of the orders of stabilizer
subgroups of the Weyl group in Appendix B.

2. Triangular Operators in the Symmetrized Group Algebra

In this section we define the concept of a triangular operator in the Weyl-group
invariant component of the group algebra over the weight lattice of a root system.
For preliminaries on root systems the reader is referred to e.g. Refs. [B, Hu].

Let E, $\langle \cdot, \cdot \rangle$ be a real Euclidean space spanned by an irreducible root system R
with Weyl group W. We write $\mathcal{Q}$ and $\mathcal{Q}^+$ for the root lattice and its nonnegative

semigroup generated by the positive roots R^+

$$\mathcal{Q} = \mathrm{Span}_{\mathbb{Z}}(R), \quad \mathcal{Q}^+ = \mathrm{Span}_{\mathbb{N}}(R^+). \tag{2.1}$$

The weight lattice $\mathcal{P}$ and the cone of dominant weights $\mathcal{P}^+$ are given by

$$\mathcal{P} = \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \forall \alpha \in R\} \tag{2.2}$$

and

$$\mathcal{P}^+ = \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N},\ \forall \alpha \in R^+\}, \tag{2.3}$$

where $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. The weight lattice is endowed with the natural partial order

$$\lambda \succeq \mu \iff \lambda - \mu \in \mathcal{Q}^+. \tag{2.4}$$

Let $\mathcal{Q}^\vee$ denote the dual root lattice generated by the dual root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$. The group algebra over the weight lattice $\mathbb{R}[\mathcal{P}]$ is the algebra generated by the formal exponentials e^λ, $\lambda \in \mathcal{P}$ subject to the multiplication relation $e^\lambda e^\mu = e^{\lambda+\mu}$. This algebra can be realized explicitly as the algebra $\mathcal{A}$ of (Fourier) polynomials on the torus $\mathbb{T} = E/(2\pi\mathcal{Q}^\vee)$ through the identification

$$e^\lambda = e^{i\langle \lambda, x \rangle}, \qquad \lambda \in \mathcal{P} \tag{2.5}$$

(with $x \in \mathbb{T}$). Symmetrization with respect to the action of the Weyl group produces the basis of monomial symmetric functions $\{m_\lambda\}_{\lambda \in \mathcal{P}^+}$ for the space $\mathcal{A}^W$ of Weyl-group invariant polynomials on $\mathbb{T}$, where

$$m_\lambda = \sum_{\mu \in W(\lambda)} e^\mu, \qquad \lambda \in \mathcal{P}, \tag{2.6}$$

with $W(\lambda)$ denoting the orbit of λ with respect to the action of the Weyl group.

We write $\mathcal{A}_\lambda^W$ for the finite-dimensional highest weight subspace of $\mathcal{A}^W$ with highest weight $\lambda \in \mathcal{P}^+$, i.e., $\mathcal{A}_\lambda^W = \mathrm{Span}\{m_\mu\}_{\mu \in \mathcal{P}^+,\, \mu \preceq \lambda}$.

Definition. A linear operator $D : \mathcal{A}^W \to \mathcal{A}^W$ is called *triangular* if $D(\mathcal{A}_\lambda^W) \subseteq \mathcal{A}_\lambda^W$ for all $\lambda \in \mathcal{P}^+$.

3. Determinantal Diagonalization

The triangularity of a linear operator D in $\mathcal{A}^W$ reduces its eigenvalue problem to a finite-dimensional one. In this section we diagonalize the triangular operators through a determinantal representation of the eigenfunctions.

Let D be a triangular operator and let $\{s_\lambda\}_{\lambda \in \mathcal{P}^+}$ be a second basis of $\mathcal{A}^W$ that is related to the monomial basis by a unitriangular transformation:

$$m_\lambda = \sum_{\mu \in \mathcal{P}^+,\, \mu \preceq \lambda} a_{\lambda\mu}\, s_\mu, \qquad a_{\lambda\lambda} = 1 \tag{3.1}$$

($\lambda \in \mathcal{P}^+$). The triangularity implies that the expansion of Dm_λ in the basis $\{s_\lambda\}_{\lambda \in \mathcal{P}^+}$ is of the form

$$D\, m_\lambda = \sum_{\mu \in \mathcal{P}^+,\, \mu \preceq \lambda} b_{\lambda\mu}\, s_\mu, \qquad b_{\lambda\lambda} = \epsilon_\lambda, \tag{3.2}$$

with the diagonal matrix elements ϵ_λ, $\lambda \in \mathcal{P}^+$ being precisely the eigenvalues of D.

Definition. The triangular operator D is called *regular* if $\epsilon_\mu \neq \epsilon_\lambda$ when $\mu \prec \lambda$.

For a regular triangular operator the eigenvalues ϵ_λ, $\lambda \in \mathcal{P}^+$ are semisimple. Let $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ be a corresponding basis of eigenfunctions diagonalizing D. Clearly, p_λ has a monomial expansion of the form

$$p_\lambda = \sum_{\mu \in \mathcal{P}^+,\, \mu \preceq \lambda} c_{\lambda\mu}\, m_\mu, \qquad c_{\lambda\lambda} = 1, \tag{3.3}$$

where we have normalized such that p_λ is monic. The following theorem provides an explicit determinantal formula for p_λ, given the action of D on m_λ expressed in the basis s_λ, i.e., given the expansion coefficients $a_{\lambda\mu}$ and $b_{\lambda\mu}$ in Eqs. (3.1) and (3.2).

Theorem 3.1 (Determinantal Formula)**.** *Let D be a regular triangular operator in $\mathcal{A}^W$ whose action on the monomial symmetric functions is given by Eqs. (3.1) and (3.2). Then the monic basis $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ of $\mathcal{A}^W$ diagonalizing D, in the sense that*

$$D p_\lambda = \epsilon_\lambda\, p_\lambda, \qquad \forall \lambda \in \mathcal{P}^+,$$

is given explicitly by the (lower) Hessenberg determinant

$$p_\lambda = \frac{1}{\mathcal{E}_\lambda} \begin{vmatrix} m_{\lambda^{(1)}} & \epsilon_{\lambda^{(1)}} - \epsilon_{\lambda^{(n)}} & 0 & \dots & \dots & 0 \\ m_{\lambda^{(2)}} & d_{\lambda^{(2)}\lambda^{(1)}} & \epsilon_{\lambda^{(2)}} - \epsilon_{\lambda^{(n)}} & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ m_{\lambda^{(n-1)}} & d_{\lambda^{(n-1)}\lambda^{(1)}} & d_{\lambda^{(n-1)}\lambda^{(2)}} & \cdots & & \epsilon_{\lambda^{(n-1)}} - \epsilon_{\lambda^{(n)}} \\ m_{\lambda^{(n)}} & d_{\lambda^{(n)}\lambda^{(1)}} & d_{\lambda^{(n)}\lambda^{(2)}} & \cdots & \cdots & d_{\lambda^{(n)}\lambda^{(n-1)}} \end{vmatrix}.$$

Here $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n-1)} < \lambda^{(n)} = \lambda$ denotes any linear ordering of the dominant weights $\{\mu \in \mathcal{P}^+ \mid \mu \preceq \lambda\}$ refining the natural order (2.4), the normalization is determined by

$$\mathcal{E}_\lambda = \prod_{\mu \in \mathcal{P}^+,\, \mu \prec \lambda} (\epsilon_\lambda - \epsilon_\mu),$$

and the matrix elements $d_{\lambda^{(j)}\lambda^{(k)}}$ $(n \geq j > k \geq 1)$ read

$$d_{\lambda^{(j)}\lambda^{(k)}} = b_{\lambda^{(j)}\lambda^{(k)}} - \epsilon_\lambda\, a_{\lambda^{(j)}\lambda^{(k)}}.$$

Proof. Expansion of the determinant with respect to the first column produces a linear combination of monomials in the highest weight space $\mathcal{A}_\lambda^W$. The coefficient of the leading monomial m_λ is given by $(-1)^{n-1}$ times the product of the elements on the super-diagonal, which are nonzero by the regularity condition on D. Division by $\mathcal{E}_\lambda$ thus gives rise to a monic polynomial. It remains to show that this polynomial is an eigenfunction of D with eigenvalue ϵ_λ. To this end one observes that the action of $(D - \epsilon_\lambda)$ on the determinant affects only its first column. Indeed, we get—upon

invoking the expansions (3.1) and (3.2)—that

$$
\left(D - \epsilon_\lambda\right)p_\lambda = \frac{1}{\mathcal{E}_\lambda}
\begin{vmatrix}
\cdots & \sum_{k=1}^{j-1} d_{\lambda^{(j)}\lambda^{(k)}} s_{\lambda^{(k)}} + \left(\epsilon_{\lambda^{(j)}} - \epsilon_\lambda\right)s_{\lambda^{(j)}} & \cdots \\
\cdots & d_{\lambda^{(j)}\lambda^{(1)}} & \cdots \\
& \vdots & \\
\cdots & d_{\lambda^{(j)}\lambda^{(j-1)}} & \cdots \\
\cdots & \epsilon_{\lambda^{(j)}} - \epsilon_\lambda & \cdots \\
\cdots & 0 & \cdots \\
& \vdots &
\end{vmatrix}
$$

(where, for typographical reasons, we have taken the transpose of our matrix). The latter determinant has a first row of the form $s_{\lambda^{(1)}}\mathrm{row}_2 + s_{\lambda^{(2)}}\mathrm{row}_3 + \cdots + s_{\lambda^{(n-1)}}\mathrm{row}_n$, and thus vanishes identically. $\qquad\square$

As a corollary of the determinantal formula for p_λ, one arrives at a linear recurrence relation encoding an efficient algorithm for the computation of the coefficients $c_{\lambda\mu}$ entering the monomial expansion (3.3).

Corollary 3.2 (Linear Recurrence Relation). *The monomial expansion of p_λ is of the form*

$$
p_\lambda = \sum_{\ell=1}^{n} c_{\lambda\lambda^{(\ell)}}\, m_{\lambda^{(\ell)}},
$$

with $c_{\lambda\lambda^{(n)}} = c_{\lambda\lambda} = 1$ and

$$
c_{\lambda\lambda^{(\ell-1)}} = \frac{1}{\epsilon_\lambda - \epsilon_{\lambda^{(\ell-1)}}} \sum_{k=\ell}^{n} c_{\lambda\lambda^{(k)}}\, d_{\lambda^{(k)}\lambda^{(\ell-1)}}
$$

$(1 < \ell \le n)$.

Proof. Immediate from Theorem 3.1 and the Hessenberg determinant evaluation given by the lemma in Appendix A at the end of the paper. $\qquad\square$

Moreover, by solving the recurrence relation we arrive at the following explicit expression for the coefficients $c_{\lambda\mu}$ of the monomial expansion (3.3).

Corollary 3.3 (Explicit Monomial Expansion). *The coefficients of the monomial expansion $p_\lambda = \sum_{\ell=1}^{n} c_{\lambda\lambda^{(\ell)}}\, m_{\lambda^{(\ell)}}$ are given explicitly by*

$$
c_{\lambda\lambda^{(\ell)}} = \sum_{\substack{\ell = j_r < j_{r-1} < \cdots < j_1 < j_0 = n \\ r = 1,\ldots,n-\ell}} \frac{d_{\lambda^{(j_0)}\lambda^{(j_1)}} d_{\lambda^{(j_1)}\lambda^{(j_2)}} \cdots d_{\lambda^{(j_{r-1})}\lambda^{(j_r)}}}{\left(\epsilon_\lambda - \epsilon_{\lambda^{(j_1)}}\right)\cdots\left(\epsilon_\lambda - \epsilon_{\lambda^{(j_r)}}\right)},
$$

with the convention that empty sums are equal to 1 (so $c_{\lambda\lambda^{(n)}} = c_{\lambda\lambda} = 1$).

Proof. In view of Corollary 3.2, it suffices to check that the stated expression for $c_{\lambda\lambda^{(\ell)}}$ represents the (unique) solution to the linear recurrence relation of Corollary 3.2, subject to the initial condition $c_{\lambda\lambda^{(n)}} = 1$. Firstly, the convention that empty sums are equal to 1 guarantees that the initial condition is satisfied. Secondly,

by isolating the last factor in the numerator and denominator, it is seen that for $1 < \ell \leq n$

$$
\begin{aligned}
c_{\lambda\lambda^{(\ell-1)}} &= \sum_{\substack{\ell-1=j_r<j_{r-1}<\cdots<j_1<j_0=n \\ r=1,\ldots,n-\ell+1}} \frac{d_{\lambda^{(j_0)}\lambda^{(j_1)}} d_{\lambda^{(j_1)}\lambda^{(j_2)}} \cdots d_{\lambda^{(j_{r-1})}\lambda^{(j_r)}}}{(\epsilon_\lambda - \epsilon_{\lambda^{(j_1)}})\cdots(\epsilon_\lambda - \epsilon_{\lambda^{(j_r)}})} \\
&= \frac{1}{\epsilon_\lambda - \epsilon_{\lambda^{(\ell-1)}}} \sum_{k=\ell}^n \left(d_{\lambda^{(k)}\lambda^{(\ell-1)}} \times \right. \\
&\qquad\qquad \left. \sum_{\substack{k=j_{r-1}<j_{r-2}<\cdots<j_1<j_0=n \\ r-1=1,\ldots,n-k}} \frac{d_{\lambda^{(j_0)}\lambda^{(j_1)}} d_{\lambda^{(j_1)}\lambda^{(j_2)}} \cdots d_{\lambda^{(j_{r-2})}\lambda^{(j_{r-1})}}}{(\epsilon_\lambda - \epsilon_{\lambda^{(j_1)}})\cdots(\epsilon_\lambda - \epsilon_{\lambda^{(j_{r-1})}})} \right) \\
&= \frac{1}{\epsilon_\lambda - \epsilon_{\lambda^{(\ell-1)}}} \sum_{k=\ell}^n d_{\lambda^{(k)}\lambda^{(\ell-1)}}\, c_{\lambda\lambda^{(k)}}.
\end{aligned}
$$

$\square$

4. Orthogonal Polynomials

In this section the Heckman-Opdam and Macdonald polynomials are defined.

Let $\Delta(x)$ be a positive continuous function on the torus $\mathbb{T}/(2\pi\mathcal{Q}^\vee)$ that is invariant with respect to the action of the Weyl group (i.e., $\Delta(w(x)) = \Delta(x)$ for all $w \in W$). We equip $\mathcal{A}^W$ with an inner product structure associated to the weight function Δ

$$
\langle f, g \rangle_\Delta = \frac{1}{|W|} \int_{\mathbb{T}} f\overline{g}\,\Delta\,\mathrm{d}x \qquad (f, g \in \mathcal{A}^W), \tag{4.1}
$$

where $\overline{g}$ denotes the complex conjugate of g and $|W|$ is the order of the Weyl group. Let $\{p_{\lambda,\Delta}\}_{\lambda\in\mathcal{P}^+}$ be the basis of $\mathcal{A}^W$ that is obtained from the monomial symmetric basis $\{m_\lambda\}_{\lambda\in\mathcal{P}^+}$ through application of the Gram-Schmidt process with respect to the partial order $\succeq$ (2.4). More specifically, by definition $p_{\lambda,\Delta}$ is the polynomial of the form

$$
p_{\lambda,\Delta} = \sum_{\mu\in\mathcal{P}^+,\,\mu\preceq\lambda} c_{\lambda,\mu}(\Delta)\,m_\mu, \qquad c_{\lambda,\lambda}(\Delta) = 1, \tag{4.2a}
$$

with coefficients $c_{\lambda,\mu}(\Delta)$ such that

$$
\langle p_{\lambda,\Delta}, m_\mu \rangle_\Delta = 0 \quad \text{for} \quad \mu \in \mathcal{P}^+ \text{ with } \mu \prec \lambda. \tag{4.2b}
$$

In the general, the basis $\{p_{\lambda,\Delta}\}_{\lambda\in\mathcal{P}^+}$ is *not* orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\Delta$ (4.1), as the natural order $\succeq$ (2.4) is not linear (unless $\dim(E) = 1$). However, for two important special choices of the weight function Δ it has been shown that the above partial Gram-Schmidt process indeed *does* produce an orthogonal basis [HS, O, M2, M3]:

$$
\Delta^{HO} = \prod_{\alpha\in R} (1 - e^\alpha)^{g_\alpha}, \tag{4.3a}
$$

with $g_\alpha \geq 0$ such that $g_{w(\alpha)} = g_\alpha$, $\forall w \in W$, and

$$
\Delta^M = \prod_{\alpha\in R} \frac{(e^\alpha; q)_\infty}{(t_\alpha e^\alpha; q)_\infty}, \tag{4.3b}
$$

with $(z; q)_\infty = \prod_{m=0}^\infty (1 - zq^m)$ and $0 < q, t_\alpha < 1$ such that $t_{w(\alpha)} = t_\alpha$, $\forall w \in W$.

Definition. The orthogonal polynomials $p_{\lambda, \Delta}$, $\lambda \in \mathcal{P}^+$ associated to the weight functions Δ^{HO} (4.3a) and Δ^M (4.3b) are called the *Heckman-Opdam polynomials* [HS, O] and the *Macdonald polynomials* [M2, M3], respectively.

Remark (i). In the above definition of the Heckman-Opdam polynomials we may allow for a root system R that is *nonreduced*. For the Macdonald polynomials, on the other hand, we always assume that R be *reduced*.

Remark (ii). The weight functions Δ^{HO} and Δ^M are invariant with respect to the action of the Weyl group on the variable (i.e. $\Delta(w(x)) = \Delta(x)$) because of the W-invariance of the orbit parameters g_α and t_α. The W-invariance of these parameters moreover implies that the values of g_α and t_α depend only on the length of the root α.

5. HECKMAN-OPDAM POLYNOMIALS

In this section we apply the formalism of Section 3 to arrive at a determinantal construction of the Heckman-Opdam polynomials for arbitrary (not necessarily reduced) root systems.

5.1. The hypergeometric differential operator.
To a vector $x \in E$ we associate the directional derivation ∂_x in $\mathcal{A}$, whose action on the exponential basis is given by

$$\partial_x e^\lambda = \langle \lambda, x \rangle \, e^\lambda \qquad (\lambda \in \mathcal{P}). \tag{5.1}$$

Definition ([HS, O]). Let $x_1, \ldots, x_N$ be an orthonormal basis of E. The second-order partial differential operator

$$D = \sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{\alpha \in R^+} g_\alpha \left(\frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \right) \partial_\alpha \tag{5.2}$$

is called the *hypergeometric differential operator* associated to the root system R.

Clearly the definition of D (5.2) does not depend on the particular choice for the orthonormal basis $x_1, \ldots, x_N$. It is known that the hypergeometric differential operator maps the space of invariants $\mathcal{A}^W$ into itself and, furthermore, that the Heckman-Opdam polynomials form a basis of eigenfunctions on which the operator acts diagonally [HS, O].

We will now compute the action of D (5.2) on the basis of monomial symmetric functions. To this end some notation is needed. We denote by $W_\lambda \subset W$ the stabilizer subgroup of $\lambda \in \mathcal{P}$

$$W_\lambda = \{ w \in W \mid w(\lambda) = \lambda \},$$

by $\rho_g \in E$ the weighted half-sum of the positive roots

$$\rho_g = \frac{1}{2} \sum_{\alpha \in R^+} g_\alpha \, \alpha,$$

by $r_\alpha : E \to E$ the orthogonal reflection in the hyperplane perpendicular to $\alpha \in R$ through the origin

$$r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \qquad (x \in E),$$

and by $[p]$ the function that extracts the integral part of a nonnegative real number p through truncation.

Lemma 5.1 (Action of the Hypergeometric Differential Operator)**.** *The action of* D *(5.2) on* m_λ*,* $\lambda \in \mathcal{P}^+$*, is given by*

$$
\begin{aligned}
D\, m_\lambda \;=\;& \big(\langle \lambda + \rho_g, \lambda + \rho_g\rangle - \langle \rho_g, \rho_g\rangle\big)\, m_\lambda \\
& + \frac{1}{|W_\lambda|} \sum_{\alpha \in R^+} \Big(g_\alpha \langle \lambda, \alpha\rangle \sum_{\ell=1}^{[\langle \lambda, \alpha^\vee\rangle/2]} |W_{\lambda-\ell\alpha}|\, |W^\alpha(\lambda - \ell\alpha)|\, m_{\lambda-\ell\alpha}\Big),
\end{aligned}
$$

where $W^\alpha \subset W$ *denotes the subgroup of order* 2 *generated by* r_α *(so* $|W^\alpha(\lambda - \ell\alpha)|$ *is equal to* 1 *if* $\ell = \langle \lambda, \alpha^\vee\rangle/2$ *and equal to* 2 *otherwise).*

Proof. The computation of the expansion of Dm_λ in the monomial basis hinges on the fundamental identity

$$
\begin{aligned}
\Big(&\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\Big)\partial_\alpha\big(e^\lambda + e^{r_\alpha(\lambda)}\big) \\
&= \langle \lambda, \alpha\rangle\, e^\lambda\, (1+e^{-\alpha})\Big(\frac{1-e^{-\langle\lambda,\alpha^\vee\rangle\,\alpha}}{1-e^{-\alpha}}\Big) \\
&= \langle \lambda, \alpha\rangle\big(e^\lambda + e^{\lambda - \langle\lambda,\alpha^\vee\rangle\,\alpha}\big) + 2\langle\lambda,\alpha\rangle\sum_{\ell=1}^{\langle\lambda,\alpha^\vee\rangle-1} e^{\lambda-\ell\alpha} \\
&= \langle \lambda, \alpha\rangle\big(e^\lambda + e^{r_\alpha(\lambda)}\big) \\
&\quad + \langle \lambda, \alpha\rangle \sum_{\ell=1}^{[\langle\lambda,\alpha^\vee\rangle/2]} |W^\alpha(\lambda - \ell\alpha)|\big(e^{\lambda-\ell\alpha} + e^{r_\alpha(\lambda-\ell\alpha)}\big). \qquad (5.3)
\end{aligned}
$$

Indeed, the following sequence of elementary manipulations reduces the computation of the action of the first-order component of D (5.2) on m_λ to an application of identity (5.3):

$$
\begin{aligned}
\sum_{\alpha \in R^+} &g_\alpha \Big(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\Big)\partial_\alpha m_\lambda \\
=\;& \frac{1}{|W_\lambda|}\sum_{\alpha \in R^+} g_\alpha\Big(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\Big)\partial_\alpha \sum_{w \in W} e^{w(\lambda)} \\
=\;& \frac{1}{|W_\lambda|}\sum_{w \in W} w\Big(\sum_{\alpha \in R^+} g_\alpha\Big(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\Big)\partial_\alpha e^\lambda\Big) \\
=\;& \frac{1}{2|W_\lambda|}\sum_{w \in W} w\Big(\sum_{\alpha \in R^+} g_\alpha\Big(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\Big)\partial_\alpha\big(e^\lambda + e^{r_\alpha(\lambda)}\big)\Big) \\
\overset{\text{Eq. }(5.3)}{=}\;& \frac{1}{|W_\lambda|}\sum_{\substack{w \in W \\ \alpha \in R^+}} w\Big(g_\alpha\langle\lambda,\alpha\rangle\, e^\lambda + g_\alpha\langle\lambda,\alpha\rangle\sum_{\ell=1}^{[\langle\lambda,\alpha^\vee\rangle/2]} |W^\alpha(\lambda-\ell\alpha)|\, e^{\lambda-\ell\alpha}\Big) \\
=\;& \sum_{\alpha \in R^+} g_\alpha\langle\lambda,\alpha\rangle\Big(m_\lambda + \frac{1}{|W_\lambda|}\sum_{\ell=1}^{[\langle\lambda,\alpha^\vee\rangle/2]} |W_{\lambda-\ell\alpha}|\, |W^\alpha(\lambda-\ell\alpha)|\, m_{\lambda-\ell\alpha}\Big).
\end{aligned}
$$

Combined with the action of the second-order component of D (5.2) on m_λ

$$\sum_{j=1}^{N} \partial_{x_j}^2 \, m_\lambda = \langle \lambda, \lambda \rangle \, m_\lambda,$$

this produces the formula of the lemma. $\qquad\square$

It is a standard property of root systems that for any $\lambda \in \mathcal{P}^+$ the integral convex hull $\mathcal{P}_\lambda = \{\mu \in \mathcal{P} \mid W(\mu) \preceq \lambda\}$ is saturated, i.e., if $\mu \in \mathcal{P}_\lambda$ then $\mu - \ell\alpha \in \mathcal{P}_\lambda$ for every integer ℓ between 0 and $\langle \mu, \alpha^\vee \rangle$ (extremal values included) [Hu]. Hence, it follows from Lemma 5.1 that the hypergeometric differential operator is triangular. To compute for μ dominant the coefficient of m_μ in $D \, m_\lambda$, it suffices to collect all terms in the lemma for which $\lambda - \ell\alpha \in W(\mu)$. Notice in this connection that for a given $\alpha \in R^+$ the α-string $\lambda - \alpha, \lambda - 2\alpha, \ldots, \lambda - [\langle \lambda, \alpha^\vee \rangle/2]\alpha$ may hit the Weyl orbit of μ at most once. Indeed, it is clear from expanding both sides of the equality $||\lambda - \ell'\alpha||^2 = ||\lambda - \ell\alpha||^2$ that $\lambda - \ell'\alpha \in W(\lambda - \ell\alpha)$—with $1 \le \ell', \ell \le [\langle \lambda, \alpha^\vee \rangle/2]$—implies $\ell' = \ell$.

We thus end up with the following explicit triangular matrix representation of the hypergeometric differential operator with respect to the monomial basis.

Proposition 5.2 (Triangular Expansion). *Let $\lambda \in \mathcal{P}^+$. We have that*

$$D \, m_\lambda = \epsilon_\lambda \, m_\lambda + \sum_{\mu \in \mathcal{P}^+, \, \mu \prec \lambda} b_{\lambda\mu} \, m_\mu,$$

with

$$
\begin{aligned}
\epsilon_\lambda &= \langle \lambda + \rho_g, \lambda + \rho_g \rangle - \langle \rho_g, \rho_g \rangle, \\
b_{\lambda\mu} &= \frac{|W_\mu|}{|W_\lambda|} \sum_{\alpha \in [\lambda, \mu]} g_\alpha \, \langle \lambda, \alpha \rangle \, n_{\lambda\mu}(\alpha).
\end{aligned}
$$

Here $[\lambda, \mu]$ denotes the subset of roots $\alpha \in R^+$ for which $\lambda - \ell\alpha \in W(\mu)$ for some (unique) $\ell \in \{1, 2, \ldots, [\langle \lambda, \alpha^\vee \rangle/2]\}$, and

$$n_{\lambda\mu}(\alpha) = \begin{cases} 1 & if \ \ ||\mu|| = ||P_\alpha(\lambda)|| \\ 2 & if \ \ ||\mu|| \ne ||P_\alpha(\lambda)|| \end{cases},$$

where $P_\alpha = (Id + r_\alpha)/2$ is the orthogonal projection onto the hyperplane perpendicular to α through the origin. (So for $\lambda - \ell\alpha \in W(\mu)$ we have that $n_{\lambda\mu}(\alpha) = 1$ if $\ell = \langle \lambda, \alpha^\vee \rangle/2$ and $n_{\lambda\mu}(\alpha) = 2$ otherwise.)

The regularity of the hypergeometric differential operator is immediate from the following proposition.

Proposition 5.3 (Monotonicity). *For nonnegative parameters g_α, the eigenvalues $\epsilon_\lambda = \langle \lambda + \rho_g, \lambda + \rho_g \rangle - \langle \rho_g, \rho_g \rangle$ are strictly monotonous in $\lambda \in \mathcal{P}^+$, i.e.,*

$$\forall \lambda, \mu \in \mathcal{P}^+ : \quad \mu \prec \lambda \Longrightarrow \epsilon_\mu < \epsilon_\lambda.$$

Proof. Assume λ, μ dominant with $\mu \prec \lambda$, and let $\nu = \lambda - \mu$ (so $\nu \in \mathcal{Q}^+$). Then

$$\epsilon_\lambda - \epsilon_\mu = \langle \nu, \nu \rangle + 2\langle \mu + \rho_g, \nu \rangle,$$

which is positive in view of the fact that $\langle \nu, \nu \rangle > 0$ and $\langle \mu + \rho_g, \nu \rangle \ge 0$ (since both μ and ρ_g lie in the closure of the dominant Weyl chamber, cf. remark below). $\qquad\square$

Remark. In the proof of Proposition 5.3, we used the fact that for nonnegative parameters g_α the weighted half-sum ρ_g lies in the closure of the dominant Weyl chamber $\{x \in E \mid \langle x, \alpha \rangle > 0, \ \forall \alpha \in R^+\}$. This follows from the fact that, for any root β, the partial half-sum $\rho(\beta)$ of positive roots with length $||\beta||$ lies in the dominant cone $\mathcal{P}^+$:

$$\rho(\beta) = \frac{1}{2} \sum_{\substack{\alpha \in R^+ \\ ||\alpha||=||\beta||}} \alpha = \sum_{\substack{j=1,\ldots,N \\ ||\alpha_j||=||\beta||}} \omega_j,$$

where $\{\alpha_j\}_{1 \le j \le N}$ denotes the basis of simple roots generating $\mathcal{Q}^+$ and $\{\omega_j\}_{1 \le j \le N}$ is the corresponding dual basis of fundamental weights generating $\mathcal{P}^+$, such that $\langle \omega_j, \alpha_k^\vee \rangle = \delta_{j,k}$. Indeed, since any simple reflection r_{α_j} permutes the positive roots other than α_j [Hu], one has that

$$r_{\alpha_j}(\rho(\beta)) = \begin{cases} \rho(\beta) - \alpha_j & \text{if } ||\alpha_j|| = ||\beta||, \\ \rho(\beta) & \text{otherwise.} \end{cases}$$

It thus follows, from working out both sides of the equality $\langle r_{\alpha_j}(\rho(\beta)), r_{\alpha_j}(\alpha_j^\vee) \rangle = \langle \rho(\beta), \alpha_j^\vee \rangle$, that $\langle \rho(\beta), \alpha_j^\vee \rangle$ is equal to 1 if $||\alpha_j|| = ||\beta||$ and is equal to 0 otherwise. This entails that $\rho(\beta) = \sum_{j=1}^N \langle \rho(\beta), \alpha_j^\vee \rangle \omega_j = \sum_{j=1, \, ||\alpha_j||=||\beta||}^N \omega_j \in \mathcal{P}^+$ as claimed.

5.2. **Determinantal construction.** It is known that the hypergeometric differential operator D (5.2) is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_\Delta$ (4.1), associated to the weight function Δ^{HO} (4.3a). Combined with the triangularity, this implies that the eigenbasis $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ diagonalizing D is given by the Heckman-Opdam polynomials $p_{\lambda \, \Delta}, \ \lambda \in \mathcal{P}^+$, defined in Section 4 through the (partial) Gram-Schmidt process [HS, O]. Moreover, since D is regular (Proposition 5.3), and its triangular action on the monomial basis is known explicitly (Proposition 5.2), we can in fact construct this eigenbasis in closed form by means of the determinantal construction in Section 3 (with $s_\lambda = m_\lambda$, so $a_{\lambda\mu} = 1$ if $\mu = \lambda$ and $a_{\lambda\mu} = 0$ otherwise). This gives rise to the following explicit representation of the Heckman-Opdam polynomials.

Theorem 5.4 (Determinantal Construction). *For $\lambda \in \mathcal{P}^+$, let*

$$p_\lambda = m_\lambda + \sum_{\mu \in \mathcal{P}^+, \, \mu \prec \lambda} c_{\lambda\mu} \, m_\mu$$

denote the (monic) Heckman-Opdam polynomial with parameters $g_\alpha \ge 0$. Then we have—upon setting for $\mu, \nu \in \mathcal{P}^+$

$$\begin{aligned} \epsilon_\mu &= \langle \mu + \rho_g, \mu + \rho_g \rangle - \langle \rho_g, \rho_g \rangle, \\ d_{\mu\nu} &= \frac{|W_\nu|}{|W_\mu|} \sum_{\alpha \in [\mu,\nu]} g_\alpha \langle \mu, \alpha \rangle \, n_{\mu\nu}(\alpha), \end{aligned}$$

with $[\mu, \nu] \subset R^+$ and $n_{\mu\nu}(\alpha)$ in accordance with the definition in Proposition 5.2—that:

i) *the polynomial p_λ is represented explicitly by the determinantal formula in Theorem 3.1,*

ii) *the coefficients $c_{\lambda\mu}$ of its monomial expansion are generated by the linear recurrence in Corollary 3.2,*

iii) *the expansion coefficients $c_{\lambda\mu}$ are given in closed form by the formula in Corollary 3.3.*

Given a concrete root system R, Theorem 5.4 turns into an efficient algorithm for the computation of the associated Heckman-Opdam polynomials. We will illustrate this below for the classical root systems.

Remark (i). The orders of the stabilizers in Theorem 5.4 can be computed by means of the formula

$$|W_\lambda| = \prod_{\substack{\alpha \in R^+ \\ \langle \lambda, \alpha^\vee \rangle = 0}} \frac{\langle \rho, \alpha^\vee \rangle + 1 + \frac{1}{2}\delta_{\frac{\alpha}{2}}}{\langle \rho, \alpha^\vee \rangle + \frac{1}{2}\delta_{\frac{\alpha}{2}}}, \tag{5.4}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, and $\delta_\alpha = 1$ if $\alpha \in R$ and zero otherwise. This expression can be found e.g. in Ref. [M3, Section 12], where it appears as a special case of the norm formulas for the Macdonald polynomials. For the reader's convenience, we have included a short proof of this formula in Appendix B.

Remark (ii). It is clear from Proposition 5.2 that the matrix $b_{\lambda\mu}$, representing the action of the hypergeometric differential operator with respect to the monomial basis, is quite sparse. The same is therefore true for the matrix $d_{\mu\nu}$ appearing in the determinantal formula for the Heckman-Opdam polynomials in Theorem 5.4. This means in practice that the algorithm for generating the Heckman-Opdam polynomials with the aid of the determinantal construction turns out to be much faster than one would expect based just on the size of the matrices involved.

Remark (iii). It is immediate from Theorem 5.4 that the coefficients in the monomial expansion of the monic Heckman-Opdam polynomial p_λ are of the form $c_{\lambda\mu} = p_{\lambda\mu}(g_\alpha)/q_{\lambda\mu}(g_\alpha)$, where $p_{\lambda\mu}(g_\alpha)$ and $q_{\lambda\mu}(g_\alpha)$ are polynomials in the parameters g_α that have nonnegative integral coefficients (and with the denominators $q_{\lambda\mu}(g_\alpha)$ dividing the normalization factor $\mathcal{E}_\lambda = \prod_{\mu \in \mathcal{P}^+, \mu \prec \lambda}(\epsilon_\lambda - \epsilon_\mu)$). Recently, a much stronger positive-integrality result for these expansion coefficients was found by Sahi [Sa] (see also [M1, KS] for the case $R = A_N$).

Remark (iv). It is clear from the proof of Proposition 5.3 that the hypergeometric differential operator D (5.2) is in fact regular for generic (complex) parameters g_α such that $\langle \nu, \nu \rangle + 2\langle \mu + \rho_g, \nu \rangle \neq 0$ for all $\nu \in \mathcal{Q}^+$ and $\mu \in \mathcal{P}^+$. Hence, the determinantal construction of the Heckman-Opdam polynomials in Theorem 5.4, as the eigenbasis for the hypergeometric differential operator, extends meromorphically to g_α in the complex plane.

Remark (v). It is known that the coefficients of the Heckman-Opdam polynomials can in principle be computed by means of (cumbersome) Freudenthal type recurrence relations [HS]. From this perspective, the determinantal construction of Theorem 5.4 thus provides the explicit solution to this Freudenthal type recurrence. The recurrence in Part *ii*) of Theorem 5.4—which arises as a particular case of the general recurrence scheme in Corollary 3.2 upon choosing for our triangular operator the hypergeometric differential operator—reads concretely

$$\left(\epsilon_\lambda - \epsilon_{\lambda^{(\ell-1)}}\right) c_{\lambda\lambda^{(\ell-1)}} =$$
$$\sum_{k=\ell}^n \frac{|W_{\lambda^{(\ell-1)}}|}{|W_{\lambda^{(k)}}|} \sum_{\alpha \in [\lambda^{(k)}, \lambda^{(\ell-1)}]} g_\alpha \langle \lambda^{(k)}, \alpha \rangle \, n_{\lambda^{(k)}\lambda^{(\ell-1)}}(\alpha) \, c_{\lambda\lambda^{(k)}}.$$

It may in fact be seen as a suitable symmetric reduction of the Freudenthal type recurrence relations, enabling their explicit solution in closed form via Corollary 3.3. When $g_\alpha = 1$, $\forall \alpha \in R$, our recurrence is closely related to the optimized Freudenthal recurrence scheme for the computation of weight multiplicities of characters of simple Lie groups due to Moody and Patera [MP].

5.3. **Tables for the classical root systems.** We will now provide tables of the matrix elements building the determinantal formulas for the Heckman-Opdam polynomials associated with the classical root systems. In each case, we will only list the minimum amount of information needed for constructing the matrix, viz., *i.* the cone of the dominant weights and its partial order, *ii.* the eigenvalues building the super-diagonal of the matrix, *iii.* and the values of the lower-triangular matrix elements. For further data on the root systems of interest we refer to the tables in Bourbaki [B].

It will be convenient to parameterize the dominant weights of the classical root systems in terms of N-tuples

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{5.5a}$$

of weakly decreasing (half-)integers (so $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$). Often we think of these N-tuples also as multi-sets of the form

$$\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \tag{5.5b}$$

where the parts λ_j are listed from largest to smallest. For $\lambda = (\lambda_1, \ldots, \lambda_N)$ $(= \{\lambda_1, \ldots, \lambda_N\})$, we define $\lambda^\varepsilon = (\lambda_1, \ldots, \lambda_{N-1}, \varepsilon|\lambda_N|)$ $(= \{\lambda_1, \ldots, \lambda_{N-1}, \varepsilon|\lambda_N|\})$, with $\varepsilon \in \{1, -1\}$. We need the following two operations on our weakly decreasing N-tuples:

$$\lambda \setminus \mu = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \setminus \{\mu_1, \mu_2, \ldots, \mu_N\}, \tag{5.6a}$$
$$\lambda \ominus \mu = (\lambda^+ \setminus \mu^+, (\mu^+ \setminus \lambda^+)^\varepsilon), \tag{5.6b}$$

where $\varepsilon = \text{sign}(\lambda_N) \times \text{sign}(\mu_N)$. The first operation takes the difference of λ and μ as multi-sets, i.e., taking into account the multiplicities of the parts. (By convention, we will list the parts of this difference again from large to small.) The second operation encodes—up to a possible sign—the symmetric difference of λ^+ and μ^+. For example: $(5, 3, 2\frac{1}{2}, 1, 1) \ominus (4, 3, 3, 1, -1) = (\{5, 2\frac{1}{2}\}, \{4, -3\})$. Finally, for future reference we furthermore introduce the operations

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N, \tag{5.7a}$$
$$\eta_\lambda(m) = |\{j = 1, \ldots, N \mid \lambda_j = m \vee \lambda_j = -m\}|, \tag{5.7b}$$
$$\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, -\lambda_N), \tag{5.7c}$$

producing, respectively, the sum of the parts, the number of parts with specified absolute value $|m|$, and the conjugate N-tuple with the sign of the last part flipped.

5.3.1. *The case $R = A_{N-1}$.* For the type A root system the Heckman-Opdam polynomials reduce (in essence) to Jack polynomials [St, M1]. The determinantal construction in Theorem 5.4 reproduces in this particular case the determinantal construction of the Jack polynomials found by Lapointe, Lascoux, and Morse [LLM2].

In dealing with the type A root system, it is more convenient to work with partitions rather than with the weights themselves. Let Λ_N be the set of partitions with at most N parts, i.e., the set of weakly decreasing N-tuples with components given

by nonnegative integers. For $\lambda, \mu \in \Lambda_N$ the dominance order on these partitions is defined as

$$\lambda \succeq \mu \iff |\lambda| = |\mu| \text{ and } \sum_{j=1}^{k}(\lambda_j - \mu_j) \geq 0 \text{ for } k = 1, \ldots, N - 1. \qquad (5.8)$$

We write $\hat{\lambda}$ for the orthogonal projection of $\lambda \in \Lambda_N$ onto the hyperplane $E \subset \mathbb{R}^N$ perpendicular to the vector $(1, 1, \ldots, 1)$:

$$\hat{\lambda} = (\lambda_1, \ldots, \lambda_N) - \frac{|\lambda|}{N}(1, \ldots, 1). \qquad (5.9)$$

The cone of dominant weights associated to the root system A_{N-1} is now given by the projection of Λ_N onto the hyperplane E, i.e. $\mathcal{P}_A^+ = \{\hat{\lambda} \mid \lambda \in \Lambda_N\}$, equipped with a partial order induced by the dominance ordering of the partitions in Eq. (5.8). Specifically, for given $\lambda \in \Lambda_N$ all dominant weights smaller or equal to $\hat{\lambda} \in \mathcal{P}_A^+$ are given by

$$\mathcal{P}_{\hat{\lambda},A}^+ = \{\hat{\mu} \mid \mu \in \Lambda_N \ \wedge \ \mu \preceq \lambda\}. \qquad (5.10)$$

The projection $\lambda \to \hat{\lambda}$ (5.9) has a nontrivial kernel of the form $(1, 1, \ldots, 1)\mathbb{N}$. The set in Eq. (5.10), however, is clearly independent of the particular choice for the partition λ projecting onto the dominant weight $\hat{\lambda}$.

The Weyl group acts transitively on the root system A_{N-1}, as all roots have the same length. Thus, the value of the root multiplicity parameter g_α does not depend on α, viz. $g_\alpha = g$ for all $\alpha \in R$. Given a partition $\lambda \in \Lambda_N$, let us define for $\mu \preceq \lambda$

$$\epsilon_\mu^A = \sum_{j=1}^{N} \mu_j \left(\mu_j + g\left(N + 1 - 2j\right)\right), \qquad (5.11a)$$

and for $\nu \prec \mu \preceq \lambda$

$$d_{\mu\nu}^A = \begin{cases} 2g\left(m_1 - m_2\right)\mathcal{N}_\nu(n_1, n_2) & \text{if } \mu \ominus \nu = (\{m_1, m_2\}, \{n_1, n_2\}) \\ & \text{with } m_1 - n_1 = n_2 - m_2 > 0, \qquad (5.11b) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{N}_\nu(n_1, n_2) = \begin{cases} \eta_\nu(n_1)\eta_\nu(n_2) & \text{if } |n_1| \neq |n_2|, \\ \binom{\eta_\nu(n_1)}{2} & \text{if } |n_1| = |n_2|, \end{cases} \qquad (5.12)$$

and $\eta_\nu(n)$ denotes the multiplicity counter defined in Eq. (5.7b). The super-diagonal $\epsilon_{\hat{\mu}}^A - \epsilon_{\hat{\lambda}}^A$ ($\hat{\mu} \preceq \hat{\lambda}$) and the lower triangular block $d_{\hat{\mu}\hat{\nu}}^A$ ($\hat{\nu} \prec \hat{\mu} \preceq \hat{\lambda}$) of the Hessenberg matrix in Theorem 5.4 become for the A_{N-1}-type Heckman-Opdam polynomial $p_{\hat{\lambda}}^A$:

$$\epsilon_{\hat{\mu}}^A - \epsilon_{\hat{\lambda}}^A = \epsilon_\mu^A - \epsilon_\lambda^A \quad \text{and} \quad d_{\hat{\mu}\hat{\nu}}^A = d_{\mu\nu}^A, \qquad (5.13)$$

respectively. (Notice in this connection that the expressions $\epsilon_\mu^A - \epsilon_\lambda^A$ and $d_{\mu\nu}^A$ on the r.h.s. are invariant with respect to the additive action of $(1, 1, \ldots, 1)\mathbb{N}$ on Λ_N.)

5.3.2. *The case D_N.* The cone of dominant weights $\mathcal{P}_D^+$ consists of the N-tuples $\lambda = (\lambda_1, \ldots, \lambda_N)$ with parts λ_j that are all integers or all half-integers subject to the ordering

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq |\lambda_N|. \tag{5.14}$$

The partial order on $\mathcal{P}_+^D$ is defined as

$$\lambda \succeq \mu \iff \begin{cases} \sum_{j=1}^k (\lambda_j - \mu_j) \in \mathbb{N} & \text{for } k = 1, \ldots, N-2, \\ \sum_{j=1}^{N-1} (\lambda_j - \mu_j) + \varepsilon(\lambda_N - \mu_N) \in 2\mathbb{N} & \text{for } \varepsilon = \pm 1. \end{cases} \tag{5.15}$$

The Weyl group again acts transitively on the root system D_N, so we have $g_\alpha = g$, $\forall \alpha \in R$. The super-diagonal $\epsilon_\mu^D - \epsilon_\lambda^D$ ($\mu \preceq \lambda$) and the lower triangular block $d_{\mu\nu}^D$ ($\nu \prec \mu \preceq \lambda$) of the Hessenberg matrix in Theorem 5.4 become for the D_N-type Heckman-Opdam polynomial p_λ^D of the form

$$\epsilon_\mu^D = \sum_{j=1}^N \mu_j \left(\mu_j + 2g \left(N - j \right) \right) \tag{5.16a}$$

and

$$d_{\mu\nu}^D = \begin{cases} \left(d_{m_1,m_2;n_1,n_2}^A + d_{m_1,\overline{m_2};n_1,n_2}^A + d_{m_1,m_2;n_1,\overline{n_2}}^A + d_{m_1,\overline{m_2};n_1,\overline{n_2}}^A \right) \mathcal{N}_\nu(n_1, n_2) \\ \qquad \text{if } \mu \ominus \nu = (\{m_1, m_2\}, \{n_1, n_2\}) \text{ with } n_\mu(0) \neq 0 \text{ and } m_2 \neq 0, \\ \\ \left(d_{m_1,m_2;n_1,n_2}^A + d_{m_1,\overline{m_2};n_1,\overline{n_2}}^A \right) \mathcal{N}_\nu(n_1, n_2) \\ \qquad \text{if } \mu \ominus \nu = (\{m_1, m_2\}, \{n_1, n_2\}) \text{ with } n_\mu(0) = 0 \text{ or } m_2 = 0, \\ \\ d_{m,\overline{\Delta^+};\Delta^+,\overline{n}}^A \, \mathcal{N}_\nu(\Delta^+, n) + d_{m,\overline{\Delta^-};\Delta^-,n}^A \, \mathcal{N}_\nu(\Delta^-, n) \\ \qquad \text{if } \mu \ominus \nu = (\{m\}, \{n\}) \text{ with } n_\mu(0) \neq 0, \\ \\ d_{m,\overline{\Delta^+};\Delta^+,\overline{n}}^A \, \mathcal{N}_\nu(\Delta^+, n) \\ \qquad \text{if } \mu \ominus \nu = (\{m\}, \{n\}) \text{ with } n_\mu(0) = 0, \\ \\ 0 \qquad \text{otherwise.} \end{cases} \tag{5.16b}$$

Here $d_{m_1,m_2;n_1,n_2}^A$ refers to the A_1-type matrix elements (cf. Eq. (5.11b)), viz.,

$$d_{m_1,m_2;n_1,n_2}^A = \begin{cases} 2\,g\,(m_1 - m_2) & \text{if } m_1 - n_1 = n_2 - m_2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.17}$$

and $\mathcal{N}_\nu(n_1, n_2)$ is the same as above (cf. Eq. (5.12)). Furthermore, $\overline{m}$ stands for $-m$ and $\Delta^\pm = (m \pm n)/2$.

Remark. In the first line of $d_{\mu\nu}^D$, at most two terms can be nonzero if $n_2 = 0$, and at most one term otherwise. Similarly, in the second line, at most one term can be nonzero.

5.3.3. *The case B_N.* The cone of dominant weights $\mathcal{P}_B^+$ consists of the N-tuples $\lambda = (\lambda_1, \ldots, \lambda_N)$ with parts λ_j that are all integers or all half-integers subject to the ordering

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N \geq 0. \tag{5.18}$$

The partial order on $\mathcal{P}_B^+$ is defined as

$$\lambda \succeq \mu \iff \sum_{j=1}^{k}(\lambda_j - \mu_j) \in \mathbb{N} \ \text{ for } k = 1, \ldots, N. \tag{5.19}$$

The B_N-type root system has two root lengths, so the action of the Weyl group splits up in two orbits. We will denote the root multiplicity parameters for the long and short roots by g and g_s, respectively. The super-diagonal $\epsilon_\mu^B - \epsilon_\lambda^B$ ($\mu \preceq \lambda$) and the lower triangular block $d_{\mu\nu}^B$ ($\nu \prec \mu \preceq \lambda$) of the Hessenberg matrix in Theorem 5.4 become for the B_N-type Heckman-Opdam polynomial p_λ^B of the form

$$\epsilon_\mu^B = \sum_{j=1}^{N} \mu_j \left(\mu_j + 2g\left(N - j\right) + g_s\right) \tag{5.20a}$$

and

$$d_{\mu\nu}^B = \begin{cases} d_{\mu\nu}^D + d_{\mu\nu}^{\text{short}} & \text{if } \mu = \bar{\mu}, \\ d_{\mu\nu}^D + d_{\bar{\mu}\nu}^D + d_{\mu\nu}^{\text{short}} & \text{if } \mu \neq \bar{\mu}, \end{cases} \tag{5.20b}$$

with $d_{\mu\nu}^D$ taken from Eq. (5.16b) and

$$d_{\mu\nu}^{\text{short}} = \begin{cases} 2g_s m \, \eta_\nu(n) & \text{if } \mu \ominus \nu = (\{m\}, \{n\}) \\ & \text{with } m - n > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

5.3.4. *The cases C_N and BC_N.* The cone of dominant weights $\mathcal{P}_{BC}^+$ consists of the partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ in Λ_N (cf. the A_{N-1}-type above). The partial order on $\mathcal{P}_{BC}^+$ is the same as in the B_N-case (cf. Eq. (5.19))

$$\lambda \succeq \mu \iff \sum_{j=1}^{k}(\lambda_j - \mu_j) \geq 0 \ \text{ for } k = 1, \ldots, N. \tag{5.22}$$

The BC_N-type root system has three root lengths, so the action of the Weyl group splits up into three orbits. We will denote the root multiplicity parameters for the long and short roots by g_l and g_s, respectively. The parameter for the remaining (i.e. middle) roots is g. The super-diagonal $\epsilon_\mu^{BC} - \epsilon_\lambda^{BC}$ ($\mu \preceq \lambda$) and the lower triangular block $d_{\mu\nu}^{BC}$ ($\nu \prec \mu \preceq \lambda$) of the Hessenberg matrix in Theorem 5.4 become for the BC_N-type Heckman-Opdam polynomial p_λ^{BC} of the form

$$\epsilon_\mu^{BC} = \sum_{j=1}^{N} \mu_j \left(\mu_j + 2g\left(N - j\right) + g_s + 2\,g_l\right) \tag{5.23a}$$

and

$$d_{\mu\nu}^{BC} = \begin{cases} d_{\mu\nu}^D + d_{\mu\nu}^{\text{short}} + d_{\mu\nu}^{\text{long}} & \text{if } \mu = \bar{\mu}, \\ d_{\mu\nu}^D + d_{\bar{\mu}\nu}^D + d_{\mu\nu}^{\text{short}} + d_{\mu\nu}^{\text{long}} & \text{if } \mu \neq \bar{\mu}, \end{cases} \tag{5.23b}$$

where $d_{\mu\nu}^D$ and $d_{\mu\nu}^{\text{short}}$ are taken from Eqs. (5.16b) and (5.21), respectively, and

$$d_{\mu\nu}^{\text{long}} = \begin{cases} 4g_l m\, \eta_\nu(n) & \text{if } \mu \ominus \nu = (\{m\},\{n\}) \\ & \text{with } m - n \in 2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

Remark (i). The C_N case is obtained from the BC_N case by setting $g_s = 0$. In this situation one generally can reduce the size of the Hessenberg matrix, as the partial order on the weights for the C_N root system, viz.

$$\lambda \succeq \mu \iff \begin{cases} \sum_{j=1}^k (\lambda_j - \mu_j) \in \mathbb{N} & \text{for } k = 1, \dots, N-1, \\ \sum_{j=1}^N (\lambda_j - \mu_j) \in 2\mathbb{N}, \end{cases} \tag{5.25}$$

is less refined than the partial order in Eq. (5.19) corresponding to the BC_N root system. More specifically, if the monomial on the lth row is not comparable to the leading monomial in the C_N ordering (5.25), then we may eliminate (for $g_s = 0$) the lth row together with the $(l+1)^{th}$ column from the Hessenberg matrix. (To keep the normalization monic, we should of course also delete the corresponding factors from the normalization constant $\mathcal{E}_\lambda$.)

Remark (ii). The BC_N Heckman-Opdam polynomials deserve special attention as they are universal in the sense that the polynomials associated with the other classical root systems can be obtained as special cases (the types B_N, C_N and D_N by specialization of the parameters g_l and g_s, and the type A_{N-1} by selecting the top-degree homogeneous component). For a systematic study of the properties of the BC_N-type Heckman-Opdam polynomials we refer to Refs. [BO, D2] and papers cited therein.

Remark (iii). Example: for $R = B_3$ and $\lambda = (2,1,0)$ the determinantal formula reads

$$p_{2,1,0} = \left((2+4\,g)(1+2\,g+g_s)(3+4\,g+g_s)(4+6\,g+2\,g_s)(5+10\,g+3\,g_s)\right)^{-1} \times$$

$$\begin{vmatrix} m_{0,0,0} & -5-10\,g-3\,g_s & 0 & 0 & 0 & 0 \\ m_{1,0,0} & 6\,g_s & -4-6\,g-2\,g_s & 0 & 0 & 0 \\ m_{1,1,0} & 24\,g & 4\,g_s & -3-4\,g-g_s & 0 & 0 \\ m_{2,0,0} & 12\,g_s & 4\,g_s & 4\,g & -1-2\,g-g_s & 0 \\ m_{1,1,1} & 0 & 8\,g & 2\,g_s & 0 & -2-4\,g \\ m_{2,1,0} & 0 & 24\,g+8\,g_s & 8\,g_s & 4\,g_s & 12\,g \end{vmatrix}.$$

This polynomial may also be interpreted as a special case of the BC_3-type Heckman-Opdam polynomial $p_{2,1,0}$ with $g_l = 0$. We observe in this connection that for $g_s = 0$, the 1st, the 3rd and the 4th row, together with the 2nd, the 4th and the 5th column, may be eliminated from the Hessenberg matrix (cf. Remark (i) above). Indeed, the weights $(0,0,0)$, $(1,1,0)$ and $(2,0,0)$ are not comparable to the highest weight $(2,1,0)$ with respect to the C_N-type partial order in Eq. (5.25). (To keep our normalization monic, we must also delete the 2nd, the 3rd, and the 5th factor from the normalization constant.)

6. Macdonald Polynomials: the case $t_\alpha = t$

In this section we apply the formalism of Section 3 to arrive at a determinantal construction of the Macdonald polynomials. Throughout this section it will be assumed that the root system R is reduced and that the dual root system $R^\vee$ has a minuscule weight (thus including the types A_N, B_N, C_N, D_N, E_6 and E_7 while

excluding the types BC_n, E_8, F_4 and G_2). We will furthermore restrict to the case that $t_\alpha = t$, $\forall \alpha \in R$ and—unless explicitly stated otherwise—we will consider the (q, t) parameters as indeterminates rather than real (or complex) numbers.

6.1. **The Macdonald operator.** For $x \in E$, we define the q-translation in $\mathcal{A}$ via its action on the exponential basis:

$$T_{x,q} e^\lambda = q^{\langle \lambda, x \rangle} e^\lambda \qquad (\lambda \in \mathcal{P}). \tag{6.1}$$

Definition ([M2, M3])**.** Let π be a minuscule weight for $R^\vee$, i.e., the vector $\pi \in E$ is such that $\langle \pi, \alpha \rangle \in \{0, 1\}$ for all $\alpha \in R^+$. The q-difference operator

$$D_\pi = \frac{1}{|W_\pi|} \sum_{w \in W} \Big(\prod_{\alpha \in R^+} \frac{1 - t^{\langle \pi, \alpha \rangle} e^{w(\alpha)}}{1 - e^{w(\alpha)}} \Big) T_{w(\pi),q} \tag{6.2}$$

is called the *Macdonald operator* associated to the minuscule weight π.

(The above definition of the Macdonald operator D_π is not precisely the same as the one employed by Macdonald [M2, M3]; both definitions do coincide upon restriction to the space of invariant polynomials $\mathcal{A}^W$ though.) In order to compute the action of D_π on the monomial basis we will make use of the Weyl characters χ_λ, $\lambda \in \mathcal{P}$:

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \det(w) e^{w(\lambda + \rho)}, \tag{6.3a}$$

where ρ and δ denote the half sum of the positive roots and the Weyl denominator respectively

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha, \tag{6.3b}$$

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} \det(w) e^{w(\rho)}. \tag{6.3c}$$

(Clearly the determinant $\det(w)$ is equal to $(-1)^{\ell(w)}$, where $\ell(w)$ represents the length of the (shortest) decomposition of w into a product of simple reflections.) It is well-known that for $\lambda \in \mathcal{P}^+$ one has that

$$\chi_\lambda = \sum_{\mu \in \mathcal{P}^+, \, \mu \preceq \lambda} K_{\lambda\mu} \, m_\mu, \qquad K_{\lambda\lambda} = 1, \tag{6.4}$$

with coefficients $K_{\lambda\mu} \in \mathbb{N}$. (In fact, the coefficients $K_{\lambda\mu}$, which are also known as Kostka numbers, count the multiplicity of the weight μ in the irreducible representation of the Lie algebra corresponding to the root system R with highest weight λ.) An efficient way to compute the coefficients $K_{\lambda\mu}$ is through the application of Theorem 5.4 with $g_\alpha = 1$, $\forall \alpha \in R$. However, for our purposes such a calculation is not necessary as we need the inverse of this basis transformation rather than Eq. (6.4) itself (cf. Corollary 6.3 below).

It is evident from the expansion in Eq. (6.4) that the Weyl characters $\{\chi_\lambda\}_{\lambda \in \mathcal{P}^+}$ form a basis of $\mathcal{A}^W$. The following lemma provides a formula for the action of the Macdonald operator D_π on the monomials m_λ in terms of Weyl characters. For the root system $R = A_N$ the formula in question is due to Macdonald [M1, M2].

Lemma 6.1 (Action of the Macdonald Operator). *Let $\lambda \in \mathcal{P}^+$. Then one has that*

$$D_\pi \, m_\lambda = t^{\langle \pi, \rho \rangle} \sum_{\nu \in W(\lambda)} \Big(\sum_{\tau \in W(\pi)} t^{\langle \tau, \rho \rangle} q^{\langle \tau, \nu \rangle} \Big) \chi_\nu.$$

Proof. Our starting point is the Weyl denominator formula in the form

$$e^{-\rho} \prod_{\alpha \in R^+} (e^\alpha - 1) = \sum_{w \in W} \det(w) \, e^{w(\rho)}.$$

By acting on both sides with the t-translator $T_{\pi,t}$ we obtain

$$t^{-\langle \pi, \rho \rangle} e^{-\rho} \prod_{\alpha \in R^+} (t^{\langle \pi, \alpha \rangle} e^\alpha - 1) = \sum_{w \in W} \det(w) \, t^{\langle \pi, w(\rho) \rangle} e^{w(\rho)}.$$

Division of the latter identity by the former gives rise to the following expansion for the coefficients of the Macdonald operator

$$\prod_{\alpha \in R^+} \frac{1 - t^{\langle \pi, \alpha \rangle} e^\alpha}{1 - e^\alpha} = \delta^{-1} t^{\langle \pi, \rho \rangle} \sum_{w \in W} \det(w) \, t^{\langle \pi, w(\rho) \rangle} e^{w(\rho)}.$$

Substitution of this expansion in the definition of D_π (taking into account the anti-symmetry of the Weyl denominator $w(\delta) = \det(w)\, \delta$), and acting on the exponential e^ν yields:

$$\begin{aligned}
D_\pi \, e^\nu \quad &= \quad \delta^{-1} \frac{t^{\langle \pi, \rho \rangle}}{|W_\pi|} \sum_{w_1, w_2 \in W} \det(w_1 w_2) \, t^{\langle \pi, w_2(\rho) \rangle} q^{\langle w_1(\pi), \nu \rangle} e^{\nu + w_1 w_2(\rho)} \\[2mm]
&= \quad \delta^{-1} \frac{t^{\langle \pi, \rho \rangle}}{|W_\pi|} \sum_{w_1, w_2 \in W} \det(w_1 w_2) \, t^{\langle w_1(\pi), w_1 w_2(\rho) \rangle} q^{\langle w_1(\pi), \nu \rangle} e^{\nu + w_1 w_2(\rho)} \\[2mm]
\overset{w_1 w_2 \to w}{=} \quad &\delta^{-1} t^{\langle \pi, \rho \rangle} \sum_{w \in W, \, \tau \in W(\pi)} \det(w) \, t^{\langle \tau, w(\rho) \rangle} q^{\langle \tau, \nu \rangle} e^{\nu + w(\rho)}.
\end{aligned}$$

Summation over ν in the orbit $W(\lambda)$ then entails:

$$\begin{aligned}
D_\pi \, m_\lambda \quad &= \quad \delta^{-1} t^{\langle \pi, \rho \rangle} \sum_{\substack{w \in W \\ \tau \in W(\pi), \, \nu \in W(\lambda)}} \det(w) \, t^{\langle \tau, w(\rho) \rangle} q^{\langle \tau, \nu \rangle} e^{\nu + w(\rho)} \\[2mm]
&= \quad \delta^{-1} t^{\langle \pi, \rho \rangle} \sum_{\substack{w \in W \\ \tau \in W(\pi), \, \nu \in W(\lambda)}} \det(w) \, t^{\langle \tau, w(\rho) \rangle} q^{\langle \tau, w(\nu) \rangle} e^{w(\nu + \rho)} \\[2mm]
&= \quad t^{\langle \pi, \rho \rangle} \sum_{\substack{\tau \in W(\pi) \\ \nu \in W(\lambda)}} t^{\langle \tau, \rho \rangle} q^{\langle \tau, \nu \rangle} \chi_\nu,
\end{aligned}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

For $\lambda \in \mathcal{P}$, let w_λ be the unique shortest Weyl group element such that $w_\lambda(\lambda) \in \mathcal{P}^+$. Then it follows from the definition of the Weyl characters that for $\nu \in \mathcal{P}$

$$\chi_\nu = \begin{cases} \det(w_{\nu+\rho}) \, \chi_{(w_{\nu+\rho}(\nu+\rho) - \rho)} & \text{if } |W_{\nu+\rho}| = 1, \\ 0 & \text{if } |W_{\nu+\rho}| > 1. \end{cases} \tag{6.5}$$

(Notice in this connection that—in view of Corollary B.3 in Appendix B—the stabilizer of a weight is nontrivial if and only if there exist a root $\alpha \in R^+$ perpendicular to it, i.e., if and only if there exists a reflection r_α, $\alpha \in R^+$ stabilizing the weight in

question.) Hence, to find for $\lambda, \mu \in \mathcal{P}^+$ the multiplicity of χ_μ in $D_\pi \, m_\lambda$, we have to collect all terms in the formula of Lemma 6.1 corresponding to weights $\nu \in W(\lambda)$ such that $w_{\nu+\rho}(\nu + \rho) - \rho = \mu$, or equivalently, $\nu = w_{\nu+\rho}^{-1}(\mu + \rho) - \rho$. Clearly the action of D_π is triangular [M2, M3], since

$$\mu = w_{\nu+\rho}(\nu + \rho) - \rho = w_{\nu+\rho}(\nu) - (\rho - w_{\nu+\rho}(\rho)) \preceq w_{\nu+\rho}(\nu) \preceq w_\nu(\nu) = \lambda$$

(where in the two last steps we used the fact that any dominant weight λ is maximal in its Weyl orbit, i.e., $w(\lambda) \preceq \lambda$ for all $w \in W$ [Hu]). We thus arrive at the following explicit triangular expansion of $D_\pi m_\lambda$ in terms of χ_μ.

Proposition 6.2 (Triangular Expansion). *Let $\lambda \in \mathcal{P}^+$. We have that*

$$D_\pi \, m_\lambda = \epsilon_\lambda \, \chi_\lambda + \sum_{\mu \in \mathcal{P}^+, \, \mu \prec \lambda} b_{\lambda\mu} \, \chi_\mu,$$

with

$$\epsilon_\lambda = t^{\langle \pi, \rho \rangle} \sum_{\tau \in W(\pi)} t^{\langle \tau, \rho \rangle} q^{\langle \tau, \lambda \rangle},$$

$$b_{\lambda\mu} = \sum_{\nu \in W(\lambda) \cap (W(\mu+\rho)-\rho)} \det(w_{\rho+\nu}) \, \epsilon_\nu.$$

For $t = 1$ the Macdonald operator D_π trivializes to $\sum_{\tau \in W(\pi)} T_{\tau, q}$, which acts diagonally on m_λ through multiplication by the eigenvalue $\sum_{\tau \in W(\pi)} q^{\langle \tau, \lambda \rangle}$. The formula of Lemma 6.1 reduces in this case (and upon division by the eigenvalue) to the following well-known relation between the symmetric monomials and the Weyl characters:

$$m_\lambda = \sum_{\mu \in W(\lambda)} \chi_\mu. \tag{6.6}$$

In the same way, one recovers from Proposition 6.2 the inverse of the expansion in Eq. (6.4).

Corollary 6.3 (Inverse Kostka Numbers). *Let $\lambda \in \mathcal{P}^+$. The expansion of the symmetric monomial m_λ in terms of Weyl characters is given by*

$$m_\lambda = \chi_\lambda + \sum_{\mu \in \mathcal{P}^+, \, \mu \prec \lambda} a_{\lambda\mu} \, \chi_\mu,$$

with

$$a_{\lambda\mu} = \sum_{\nu \in W(\lambda) \cap (W(\mu+\rho)-\rho)} \det(w_{\rho+\nu}).$$

The following proposition guarantees that the Macdonald operator D_π is regular.

Proposition 6.4 (Regularity). *The Macdonald operator D_π is regular in the sense that*

$$\forall \lambda, \mu \in \mathcal{P}^+ : \quad \mu \prec \lambda \Longrightarrow \epsilon_\mu(q, t) \neq \epsilon_\lambda(q, t)$$

(as (analytic) functions of the indeterminates q and t).

Proof. After setting $t = q^g$ and $q = \exp(z)$, we get

$$
\begin{aligned}
t^{-\langle \pi, \rho \rangle} \epsilon_\lambda &= \sum_{\tau \in W(\pi)} \exp(z \langle \tau, \lambda + g\rho \rangle) \\
&= |W(\pi)| + z \sum_{\tau \in W(\pi)} \langle \tau, \lambda + g\rho \rangle + \frac{z^2}{2} \sum_{\tau \in W(\pi)} \langle \tau, \lambda + g\rho \rangle^2 + O(z^3) \\
&= |W(\pi)| + c_\pi z^2 \langle \lambda + g\rho, \lambda + g\rho \rangle + O(z^3),
\end{aligned}
$$

with $c_\pi > 0$. (In the last step we employed the fact that the W-invariant linear form $\sum_{\tau \in W(\pi)} \langle \tau, x \rangle$ vanishes and that the W-invariant positive quadratic form $\sum_{\tau \in W(\pi)} \langle \tau, x \rangle^2$ must be proportional to $\langle x, x \rangle$, because the representation of the Weyl group on E is irreducible and unitary). When g is positive, one has that $\langle \mu + g\rho, \mu + g\rho \rangle < \langle \lambda + g\rho, \lambda + g\rho \rangle$ for all dominant weights μ, λ with $\mu \prec \lambda$ in view of Proposition 5.3. It thus follows that for comparable dominant weights $\mu \neq \lambda$ the corresponding eigenvalues $\epsilon_\mu(q,t)$ and $\epsilon_\lambda(q,t)$ cannot be equal as (analytic) functions of the indeterminates q and t.   □

6.2. **Determinantal construction.** We will now apply the determinantal formalism of Section 3 to construct the eigenbasis of D_π. To this end we pick for the second basis $\{s_\lambda\}_{\lambda \in \mathcal{P}^+}$ the basis of Weyl characters $\{\chi_\lambda\}_{\lambda \in \mathcal{P}^+}$. Specifically, by plugging in the eigenvalues ϵ_λ and off-diagonal matrix elements $b_{\lambda\mu}$ from Proposition 6.2, together with the inverse Kostka numbers $a_{\lambda\mu}$ from Corollary 6.3, the formulas of Theorem 3.1 and the Corollaries 3.2 and 3.3 give rise to the desired eigenbasis of the corresponding Macdonald operator D_π. For parameters such that $0 < q, t < 1$, this eigenbasis coincides with the Macdonald polynomials defined in Section 4 through the (partial) Gram-Schmidt process [M2, M3]. We thus end up with the following determinantal construction of the Macdonald polynomials (not necessarily with $0 < q, t < 1$).

Theorem 6.5 (Determinantal Construction). *For $\lambda \in \mathcal{P}^+$, let*

$$
p_\lambda = m_\lambda + \sum_{\mu \in \mathcal{P}^+,\, \mu \prec \lambda} c_{\lambda\mu}\, m_\mu
$$

denote the (monic) Macdonald polynomial with $t_\alpha = t$, $\forall \alpha \in R$. Then we have— upon setting for $\mu, \nu \in \mathcal{P}^+$

$$
\begin{aligned}
\epsilon_\mu &= t^{\langle \pi, \rho \rangle} \sum_{\tau \in W(\pi)} t^{\langle \tau, \rho \rangle} q^{\langle \tau, \mu \rangle}, \\
d_{\mu\nu} &= \sum_{\kappa \in W(\mu) \cap (W(\nu+\rho)-\rho)} \det(w_{\rho+\kappa})\, (\epsilon_\kappa - \epsilon_\lambda)
\end{aligned}
$$

(so $d_{\mu\mu} = \epsilon_\mu - \epsilon_\lambda$)—that:

   i) *the polynomial p_λ is represented explicitly by the determinantal formula in Theorem 3.1,*
   ii) *the coefficients $c_{\lambda\mu}$ of its monomial expansion are generated by the linear recurrence in Corollary 3.2,*
   iii) *the expansion coefficients $c_{\lambda\mu}$ are given in closed form by the formula in Corollary 3.3.*

Remark (i). To determine the matrix elements $d_{\mu\nu}$, it is not efficient to compute the intersection $W(\mu) \cap (W(\nu + \rho) - \rho)$ for each $\mu, \nu \in \mathcal{P}^+$ such that $\nu \prec \mu \preceq \lambda$. Indeed, because the matrices at issue are sparse, a better strategy is to construct the matrix row by row. For this purpose one first determines for each dominant weight $\mu \preceq \lambda$ the set $\Lambda_\mu = \{\tilde{\kappa} \in \mathcal{P} \mid \tilde{\kappa} \in \rho + W(\mu), \ |W_{\tilde{\kappa}}| = 1\}$ (i.e. all regular points of the translated Weyl orbit $\rho + W(\mu)$). Weyl-permuting the weights in Λ_μ to the dominant cone and translating over $-\rho$ produces all the nonzero contributions to the row μ. Specifically, the nonzero matrix elements on the row corresponding to μ occur in the columns corresponding to ν from the set $\{\nu \in \mathcal{P}^+ \mid \nu = w_{\tilde{\kappa}}(\tilde{\kappa}) - \rho, \ \tilde{\kappa} \in \Lambda_\mu\}$. The matrix elements in question are built of contributions of the form $\det(w_{\tilde{\kappa}})(\epsilon_{\tilde{\kappa}-\rho} - \epsilon_\lambda)$, $\tilde{\kappa} \in \Lambda_\mu$.

Remark (ii). It is well-known that for $t = q^g$ and $q \to 1$ the Macdonald polynomial p_λ tends to the corresponding Heckman-Opdam polynomial (with $g_\alpha = g, \forall \alpha \in R$) [M3]. To perform this limit at the level of the above determinantal construction, it suffices to determine the asymptotics of the eigenvalues ϵ_μ for $q \to 1$. The asymptotics in question is given by (cf. the proof of Proposition 6.4)

$$\epsilon_\mu t^{-\langle \pi, \rho \rangle} = |W(\pi)| + c_\pi \langle \mu + g\rho, \mu + g\rho \rangle (q-1)^2 + O((q-1)^3), \qquad (6.7)$$

where c_π is a positive constant that does not depend on μ and g. Since the formulas of the determinantal construction for the Macdonald polynomials are invariant with respect to an affine rescaling of the spectrum of the form $\epsilon_\mu \to a\epsilon_\mu + b$ (with $a \neq 0$), we only pick up the second-order term of the asymptotics in Eq. (6.7) when sending q to 1. The upshot is that by replacing ϵ_μ by $\langle \mu + g\rho, \mu + g\rho \rangle$ in Theorem 6.5, we wind up with an alternative determinantal formula for the Heckman-Opdam polynomials (with $g_\alpha = g, \forall \alpha \in R$, and with R such that $R^\vee$ has a minuscule weight). From a practical point of view the formulas coming from Theorem 6.5 are less efficient than those of Theorem 5.4, however, as the action of the Macdonald operator expanded in Weyl characters tends to be much less sparse than the action of the hypergeometric differential operator expanded in monomials. As a consequence, the matrices entering the determinantal formulas for the Heckman-Opdam polynomials coming from Theorem 6.5 are much less sparse than those of the type given by Theorem 5.4.

Remark (iii). The recurrence relation in Part *ii*) of Theorem 6.5 reads concretely

$$\left(\epsilon_\lambda - \epsilon_{\lambda^{(\ell-1)}}\right) c_{\lambda\lambda^{(\ell-1)}} =$$
$$\sum_{k=\ell}^{n} \sum_{\kappa \in W(\lambda^{(k)}) \cap (W(\lambda^{(\ell-1)}+\rho)-\rho)} \det(w_{\rho+\kappa}) \left(\epsilon_\kappa - \epsilon_\lambda\right) c_{\lambda\lambda^{(k)}}.$$

This relation should be regarded as a symmetrized Freudenthal type recurrence for the coefficients in the monomial expansion of the Macdonald polynomials. For $\epsilon_\mu = \langle \mu + g\rho, \mu + g\rho \rangle$, this recurrence degenerates to a recurrence for the coefficients of the Heckman-Opdam polynomials with $g_\alpha = g, \forall \alpha \in R$ (cf. Remark (ii) above). The recurrence in question is different from the previous recurrence for the Heckman-Opdam polynomials originating from the hypergeometric differential operator (cf. Remark (v) at the end of Section 5.2). In particular, for $g = 1$ this gives rise to an alternative system of symmetrized Freudenthal type recurrence relations for the weight multiplicities of characters of simple Lie groups.

6.3. **Tables for the classical root systems.** We now provide tables for the construction of the Macdonald polynomials associated with the classical root systems. The minimum information needed for invoking Theorem 6.5 consists of *i.* the cone of dominant weights and its partial order, *ii.* the half-sum of the positive roots ρ, *iii.* the action of the Weyl group, and *iv.* the eigenvalues ϵ_μ. Below we list items *ii.–iv* for $R = A_{N-1}$, B_N, C_N and D_N. For item *i* the reader is referred to Subsection 5.3.

6.3.1. *The case A_{N-1}.* The A_{N-1}-type Macdonald polynomials amount (in essence) to the Macdonald symmetric functions of Ref. [M1]. Theorem 6.5 reproduces in this case the determinantal construction of the Macdonald symmetric functions due to Lapointe, Lascoux, and Morse [LLM1].

We will again formulate the construction in terms of partitions with at most N parts, by adding a trivial center to the weight lattice (cf. Subsection 5.3). The Weyl group W is given by the permutation group of N letters Σ_N. A Weyl group element $w = \sigma \in \Sigma_N$ acts on a partition $\lambda \in \Lambda_N$ by rearranging its parts

$$\sigma(\lambda_1, \ldots, \lambda_N) = (\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_N}). \tag{6.8}$$

To construct the Macdonald symmetric function associated to a partition $\lambda \in \Lambda_N$, one employs Theorem 6.5 with $\mathcal{P}^+ = \Lambda_N$ endowed with the dominance order in Eq. (5.8), and

$$\epsilon_\mu^A \;=\; \sum_{j=1}^N t^{N-j} q^{\mu_j}, \tag{6.9a}$$

$$\rho_A \;=\; (N-1, N-2, \ldots, 1, 0). \tag{6.9b}$$

For a rearrangement κ of a partition $\mu \in \Lambda_N$, the sign $\det(w_{\rho+\kappa})$ is given by the signature of the shortest permutation $\sigma_{\rho+\kappa}$ rearranging $\rho + \kappa$ such that its parts become weakly decreasing (i.e. $\det(w_{\rho+\kappa}) = (-1)^{\ell(\sigma_{\rho+\kappa})}$, where $\ell(\sigma_{\rho+\kappa})$ denotes the number of transpositions of the permutation).

Projection of the resulting Macdonald symmetric function onto the space of homogeneous functions of degree zero (i.e., replacing m_μ by $m_{\hat\mu}$ in the monomial expansion) entails the A_{N-1}-type Macdonald polynomial $p_{\hat\lambda}^A$ associated to the weight $\hat\lambda$ (5.9).

6.3.2. *The case B_N.* The Weyl group is the semi-direct product of Σ_N and the N-fold product of $\mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z})$, i.e., $W = \Sigma_N \ltimes \mathbb{Z}_2^N$. A Weyl group element $w = (\sigma, \varepsilon)$ acts on a weight $\lambda \in \mathcal{P}_B^+$ as

$$w(\lambda_1, \ldots, \lambda_N) = (\varepsilon_1 \lambda_{\sigma_1}, \ldots, \varepsilon_N \lambda_{\sigma_N}), \tag{6.10}$$

with $\varepsilon_j \in \{1, -1\}$ for $j = 1, \ldots, N$.

To construct the B_N-type Macdonald polynomial p_λ^B associated to a weight $\lambda \in \mathcal{P}_+^B$, one employs Theorem 6.5 with

$$\epsilon_\mu^B \;=\; \sum_{j=1}^N \left(t^{2N-j} q^{\mu_j} + t^{j-1} q^{-\mu_j} \right), \tag{6.11a}$$

$$\rho_B \;=\; (N - \frac{1}{2}, N - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}). \tag{6.11b}$$

For a weight $\kappa \in W(\mu)$ with $\mu \in \mathcal{P}_B^+$, the sign $\det(w_{\rho+\kappa})$ is given by the signature of the shortest permutation rearranging $\rho + \kappa$ such that the absolute values of its parts become weakly decreasing, multiplied by $(-1)^{n_\varepsilon}$ where

$$n_\varepsilon = |\{j = 1, \ldots, N \mid \rho_j + \kappa_j < 0\}|. \tag{6.12}$$

6.3.3. *The case C_N.* The Weyl group and its action on a weight $\lambda \in \mathcal{P}_C^+$ are the same as in the B_N-case.

To construct the C_N-type Macdonald polynomial p_λ^C associated to a weight $\lambda \in \mathcal{P}_+^C$, one employs Theorem 6.5 with

$$\epsilon_\mu^C = \prod_{j=1}^N \left(t^{N+1-j} q^{\mu_j/2} + q^{-\mu_j/2} \right), \tag{6.13a}$$

$$\rho_C = (N, N-1, \ldots, 2, 1). \tag{6.13b}$$

The sign $\det(w_{\rho+\kappa})$, for $\kappa \in W(\mu)$ with $\mu \in \mathcal{P}_+^C$, is computed in the same way as in the B_N-case.

6.3.4. *The case D_N.* The Weyl group is given by $W = \Sigma_N \ltimes \mathbb{Z}_2^{N-1}$, and the action of $w = (\sigma, \varepsilon) \in W$ on a weight $\lambda \in \mathcal{P}_D^+$ is given by Eq. (6.10) with $\varepsilon_j \in \{1, -1\}$ for $j = 1, \ldots, N$ such that $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_N = 1$.

To construct the D_N-type Macdonald polynomial p_λ^D associated to a weight $\lambda \in \mathcal{P}_+^D$, one employs Theorem 6.5 with

$$\epsilon_\mu^D = \sum_{j=1}^N \left(t^{2N-j-1} q^{\mu_j} + t^{j-1} q^{-\mu_j} \right), \tag{6.14a}$$

$$\rho_D = (N-1, N-2, \ldots, 1, 0). \tag{6.14b}$$

For a weight $\kappa \in W(\mu)$ with $\mu \in \mathcal{P}_D^+$, the sign $\det(w_{\rho+\kappa})$ is given by the signature of the shortest permutation rearranging $\rho + \kappa$ such that the absolute values of its parts become weakly decreasing.

Remark (i). Example: for $R = D_3$ and $\lambda = (2, 1, 0)$ the determinantal formula reads

$$p_{2,1,0} = \left((\epsilon_{2,1,0}^D - \epsilon_{1,1,1}^D)(\epsilon_{2,1,0}^D - \epsilon_{1,1,-1}^D)(\epsilon_{2,1,0}^D - \epsilon_{1,0,0}^D) \right)^{-1} \times$$

$$\begin{vmatrix} m_{1,0,0} & \epsilon_{2,1,0}^D - \epsilon_{1,0,0}^D & 0 & 0 \\ m_{1,1,-1} & -\epsilon_{2,1,0}^D + \epsilon_{1,-1,1}^D & \epsilon_{2,1,0}^D - \epsilon_{1,1,-1}^D & 0 \\ m_{1,1,1} & -\epsilon_{2,1,0}^D + \epsilon_{1,-1,-1}^D & 0 & \epsilon_{2,1,0}^D - \epsilon_{1,1,1}^D \\ m_{2,1,0} & -\epsilon_{1,-2,0}^D + \epsilon_{-1,2,0}^D & -2\,\epsilon_{2,1,0}^D + \epsilon_{1,0,-2}^D + \epsilon_{0,2,-1}^D & -2\,\epsilon_{2,1,0}^D + \epsilon_{1,0,2}^D + \epsilon_{0,2,1}^D \end{vmatrix},$$

with $\epsilon_{m_1,m_2,m_3}^D = (t^4 q^{m_1} + q^{-m_1}) + (t^3 q^{m_2} + t\, q^{-m_2}) + (t^2 q^{m_3} + t^2 q^{-m_3})$.

Remark (ii). For the root systems B_N and C_N the minuscule weights π of the dual root systems are unique. Specifically, for $R = B_N$ the minuscule weight is given by lowest fundamental weight ω_1 of $R^\vee$ $(= C_N)$, and for $R = C_N$ it is given by the highest fundamental weight ω_N of $R^\vee$ $(= B_N)$. In the cases of the root systems A_{N-1} and D_N the above formulas for the eigenvalues ϵ_μ correspond to picking for the minuscule weight π the lowest fundamental weight ω_1 of $R^\vee$ $(= R)$. However,

in these cases there exist actually several alternative possibilities for the choice of the minuscule weight π. For $R = A_{N-1}$ we could work with each of the eigenvalues

$$\epsilon_{\mu,r}^A = \sum_{\substack{J \subset \{1,\ldots,N\} \\ |J|=r}} \prod_{j \in J} t^{N-j} q^{\lambda_j}, \quad r = 1, \ldots, N-1,$$

corresponding to the fundamental weights $\omega_1, \ldots, \omega_{N-1}$, respectively. For $R = D_N$ we could alternatively work with the eigenvalues

$$\epsilon_{\mu,N-1}^D = (\epsilon_{\mu,+}^D - \epsilon_{\mu,-}^D)/2 \quad \text{or} \quad \epsilon_{\mu,N}^D = (\epsilon_{\mu,+}^D + \epsilon_{\mu,-}^D)/2,$$

where

$$\epsilon_{\mu,+}^D = q^{-\frac{1}{2}\sum_{j=1}^N \mu_j} \prod_{j=1}^N \left(t^{N-j} q^{\mu_j} + 1 \right),$$

$$\epsilon_{\mu,-}^D = q^{-\frac{1}{2}\sum_{j=1}^N \mu_j} \prod_{j=1}^N \left(t^{N-j} q^{\mu_j} - 1 \right),$$

corresponding to the fundamental (spin) weights ω_{N-1} and ω_N, respectively. From a computational point of view, however, in this last case it is more efficient to work simply with the eigenvalues $\epsilon_{\mu,+}^D$, corresponding to the linear combination of Macdonald operators $D = D_{\omega_{N-1}} + D_{\omega_N}$. For instance, in the example of Remark (i) above, this amounts to replacing the eigenvalues ϵ_{m_1,m_2,m_3}^D by $\epsilon_{m_1,m_2,m_3,+}^D = q^{-(m_1+m_2+m_3)/2}(t^2 q^{m_1} + 1)(t q^{m_2} + 1)(q^{m_3} + 1)$.

7. MACDONALD POLYNOMIALS: THE CASE OF GENERAL t_α

In this section we will briefly indicate how to generalize the results of the previous section to the case of Macdonald polynomials with general parameters t_α such that $t_{w(\alpha)} = t_\alpha$ for all $w \in W$. We will keep the restriction that our root system R is reduced and that the dual root system $R^\vee$ has a minuscule weight π.

For general W-invariant t_α-parameters the Macdonald operator becomes [M3]

$$D_\pi = \frac{1}{|W_\pi|} \sum_{w \in W} \left(\prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \pi, \alpha \rangle} e^{w(\alpha)}}{1 - e^{w(\alpha)}} \right) T_{w(\pi),q}. \tag{7.1}$$

It is convenient to reparameterize the t_α as

$$t_\alpha = q^{g_\alpha}$$

(with $g_{w(\alpha)} = g_\alpha$, $\forall w \in W$). The action of D_π on the monomial basis can be written as [M3, Section 5]

$$D_\pi m_\lambda = q^{\langle \pi, \rho_g \rangle} \sum_{X \subset R^+} (-1)^{|X|} \sum_{\nu \in W(\lambda)} q^{\langle \pi, \nu + \rho_g(X^c) - \rho_g(X) \rangle} \chi_{\nu - 2\rho(X)}, \tag{7.2}$$

where $X^c = R^+ \setminus X$ and

$$\rho(X) = \frac{1}{2} \sum_{\alpha \in X} \alpha, \qquad \rho_g(X) = \frac{1}{2} \sum_{\alpha \in X} g_\alpha \, \alpha$$

(so $\rho = \rho(R^+)$ and $\rho_g = \rho_g(R^+)$). Bringing the action in Eq. (7.2) to triangular form gives

$$D_\pi m_\lambda = \epsilon_\lambda \chi_\lambda + \sum_{\mu \in \mathcal{P}^+, \, \mu \prec \lambda} b_{\lambda\mu} \chi_\mu, \tag{7.3}$$

with

$$\epsilon_\lambda \;\;=\;\; q^{\langle \pi, \rho_g \rangle} \sum_{\tau \in W(\pi)} q^{\langle \tau, \lambda + \rho_g \rangle},$$

$$b_{\lambda\mu} \;\;=\;\;$$
$$\sum_{\substack{\nu \in W(\lambda) \cap (W(\mu+\rho)+\rho(X)-\rho(X^c)) \\ X \subset R^+}} (-1)^{|X|} \det(w_{\nu+\rho(X^c)-\rho(X)}) q^{\langle \pi, \nu+\rho_g(X^c)-\rho_g(X) \rangle}.$$

We thus wind up with the following determinantal construction of the Macdonald polynomials for general W-invariant parameters.

Theorem 7.1 (Determinantal Construction). *For $\lambda \in \mathcal{P}^+$, let*

$$p_\lambda = m_\lambda + \sum_{\mu \in \mathcal{P}^+,\, \mu \prec \lambda} c_{\lambda\mu}\, m_\mu$$

denote the (monic) Macdonald polynomial with $t_\alpha = q^{g_\alpha}$. Then we have—upon setting for $\mu, \nu \in \mathcal{P}^+$

$$\epsilon_\mu \;\;=\;\; q^{\langle \pi, \rho_g \rangle} \sum_{\tau \in W(\pi)} q^{\langle \tau, \mu + \rho_g \rangle},$$

$$d_{\mu\nu} \;\;=\;\; b_{\mu\nu} - \epsilon_\lambda a_{\mu\nu},$$

with

$$a_{\mu\nu} \;\;=\;\; \sum_{\kappa \in W(\mu) \cap (W(\nu+\rho)-\rho)} \det(w_{\rho+\kappa}),$$

$$b_{\mu\nu} \;\;=\;\;$$
$$\sum_{\substack{\kappa \in W(\mu) \cap (W(\nu+\rho)+\rho(X)-\rho(X^c)) \\ X \subset R^+}} (-1)^{|X|} \det(w_{\kappa+\rho(X^c)-\rho(X)}) q^{\langle \pi, \kappa+\rho_g(X^c)-\rho_g(X) \rangle}$$

(so $d_{\mu\mu} = \epsilon_\mu - \epsilon_\lambda$)—that:

   i) *the polynomial p_λ is represented explicitly by the determinantal formula in Theorem 3.1,*

   ii) *the coefficients $c_{\lambda\mu}$ of its monomial expansion are generated by the linear recurrence in Corollary 3.2,*

   iii) *the expansion coefficients $c_{\lambda\mu}$ are given in closed form by the formula in Corollary 3.3.*

Remark (i). From a computational standpoint the formulas of Theorem 7.1 are much less effective than the determinantal constructions for the (q, t) Macdonald polynomials (Theorem 6.5) and (especially) for the Heckman-Opdam polynomials (Theorem 5.4). This is because the action of the general Macdonald operator on the monomial basis (cf. Eq. (7.2)) is much less sparse and the matrix elements are moreover much more complex than in these two previous cases. This renders Theorem 7.1 presumably only of limited practical value.

Remark (ii). The most general class of Macdonald polynomials admits a richer parameter structure connected with admissible pairs of root systems (R, S) [M3]. (From this perspective the polynomials studied here are of the type (R, R).) Since Macdonald in fact gives the action of the Macdonald operator on the monomial basis for general admissible pairs, it is not difficult to generalize Theorem 7.1 also

to this context (at the expense of having to introduce a more elaborate notational apparatus).

APPENDIX A. DETERMINANTS OF HESSENBERG MATRICES

In this appendix we recall a classic recursive method for the efficient evaluation of the determinant of a Hessenberg matrix [W]. This recursive method was used in Section 3 for the explicit evaluation of our determinantal formula for the eigenbasis of the regular triangular operators in $\mathcal{A}^W$.

Lemma ([W]). *Let $|D|$ be the Hessenberg determinant*

$$|D| = \begin{vmatrix} m_1 & -d_{1,2} & 0 & \cdots & 0 \\ m_2 & d_{2,2} & -d_{2,3} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ m_{n-1} & d_{n-1,2} & d_{n-1,3} & \cdots & -d_{n-1,n} \\ m_n & d_{n,2} & d_{n,3} & \cdots & d_{n,n} \end{vmatrix},$$

with nonzero elements on the super-diagonal: $d_{j-1,j} \neq 0$ $(1 < j \leq n)$. Then the expansion of $|D|$ with respect to the first column is of the form

$$|D| = \sum_{\ell=1}^{n} c_\ell \, m_\ell,$$

with $c_n = d_{1,2} d_{2,3} \cdots d_{n-1,n}$ and

$$c_{\ell-1} = \frac{1}{d_{\ell-1,\ell}} \sum_{j=\ell}^{n} c_j \, d_{j,\ell}$$

$(1 < \ell \leq n)$.

Proof. ([LLM2]) We denote the columns of our Hessenberg matrix by the n-dimensional vectors $\mathbf{m}$ and $\mathbf{d}^{(k)}$ $(1 < k \leq n)$, respectively. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ be a nonzero vector perpendicular to the hyperplane spanned by the $(n-1)$ linear independent columns $\mathbf{d}^{(2)}, \ldots, \mathbf{d}^{(n)}$. Because the value of the determinant is equal to the (hyper)volume of the polygon determined by $\mathbf{d}^{(2)}, \ldots, \mathbf{d}^{(n)}$, multiplied by the height of $\mathbf{m}$ in the perpendicular direction $\mathbf{c}$ (possibly up to a sign), we conclude that $|D|$ must be proportional to the scalar product of $\mathbf{c}$ and the first column $\mathbf{m}$, i.e. $|D| \sim (\mathbf{c}, \mathbf{m}) = \sum_{\ell=1}^{n} c_\ell \, m_\ell$. Furthermore, since the cofactor of m_n in $|D|$ equals the product of the elements on the super-diagonal, one sees that in fact $|D| = \sum_{\ell=1}^{n} c_\ell \, m_\ell$ provided the normalization of $\mathbf{c}$ is fixed such that $c_n = d_{1,2} d_{2,3} \cdots d_{n-1,n}$. The lemma now follows from the observation that the requirement that $\mathbf{c}$ be orthogonal to the columns $\mathbf{d}^{(2)}, \ldots, \mathbf{d}^{(n)}$ translates itself directly into the stated recurrence relations for the components c_j, $j = 1, \ldots, n$. $\square$

APPENDIX B. COUNTING THE ORDERS OF STABILIZER SUBGROUPS

To build the determinantal formula for the Heckman-Opdam polynomials in Theorem 5.4, one frequently needs to compute the orders of stabilizer subgroups of the Weyl group. In this appendix we include a short proof of a useful formula for the orders of these stabilizers that can be found e.g. in Ref. [M3, Section 12].

Proposition B.1 (Orbit Size). *Let* $\lambda \in \mathcal{P}$. *Then the size of the Weyl orbit through* λ *is given by*

$$|W(\lambda)| = \prod_{\substack{\alpha \in R^+ \\ \langle \lambda, \alpha^\vee \rangle \neq 0}} \frac{\langle \rho, \alpha^\vee \rangle + 1 + \frac{1}{2}\delta_{\frac{\alpha}{2}}}{\langle \rho, \alpha^\vee \rangle + \frac{1}{2}\delta_{\frac{\alpha}{2}}},$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, *and*

$$\delta_\alpha = \begin{cases} 1 & \text{if } \alpha \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us first assume that λ is dominant. The orbit size can be obtained in this case from the following evaluation formula for the Heckman-Opdam polynomial p_λ at $x = 0$ (i.e. at the identity element of the torus $\mathbb{T} = E/(2\pi\mathcal{Q}^\vee)$) [HS]

$$p_\lambda(0) = \prod_{\alpha \in R^+} \frac{[\langle \rho_g, \alpha^\vee \rangle + g_\alpha + \frac{1}{2}g_{\frac{\alpha}{2}}]_{\langle \lambda, \alpha^\vee \rangle}}{[\langle \rho_g, \alpha^\vee \rangle + \frac{1}{2}g_{\frac{\alpha}{2}}]_{\langle \lambda, \alpha^\vee \rangle}},$$

where $[a]_m = a(a+1)\cdots(a+m-1)$ (with the convention that $[a]_0 = 1$), and by definition $g_\alpha = 0$ if $\alpha \notin R$. Indeed, setting $g_\alpha = g$ for all $\alpha \in R$, and performing the limit $g \to 0$, readily entails the formula of the proposition. (Here one uses that $\lim_{g_\alpha \to 0} p_\lambda = m_\lambda$ and that $m_\lambda(0) = |W(\lambda)|$.) The extension to non-dominant weights λ is immediate (cf. also the two corollaries below). $\square$

By picking λ regular (for instance strongly dominant), one gets a formula for the order of the Weyl group.

Corollary B.2 (Order of the Weyl Group). *The order of the Weyl group is given by*

$$|W| = \prod_{\alpha \in R^+} \frac{\langle \rho, \alpha^\vee \rangle + 1 + \frac{1}{2}\delta_{\frac{\alpha}{2}}}{\langle \rho, \alpha^\vee \rangle + \frac{1}{2}\delta_{\frac{\alpha}{2}}}.$$

Dividing the order of the Weyl group by the size of the orbit produces the order of the stabilizer subgroup.

Corollary B.3 (Order of the Stabilizer). *Let* $\lambda \in \mathcal{P}$. *The order of the stabilizer of* λ *is given by*

$$|W_\lambda| = \prod_{\substack{\alpha \in R^+ \\ \langle \lambda, \alpha^\vee \rangle = 0}} \frac{\langle \rho, \alpha^\vee \rangle + 1 + \frac{1}{2}\delta_{\frac{\alpha}{2}}}{\langle \rho, \alpha^\vee \rangle + \frac{1}{2}\delta_{\frac{\alpha}{2}}}.$$

Remark (i). It is clear that the formulas of Proposition B.1 and Corollary B.3 in fact serve to compute the sizes of the orbit $W(x)$ and the stabilizer W_x for *any* vector $x \in E$ (i.e. not just weight vectors).

Remark (ii). For a reduced root system R the above formulas simplify somewhat as $\delta_{\frac{\alpha}{2}} = 0$ in this case.

References

BO.   R.J. Beerends and E.M. Opdam, Certain hypergeometric series related to the root system *BC*, Trans. Amer. Math. Soc. **339** (1993), 581–609.

B.   N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4–6, Hermann, Paris, 1968.

D1.   J.F. van Diejen, On the diagonalization of difference Calogero-Sutherland systems, in: Symmetries and Integrability of Difference Equations (D. Levi, L. Vinet and P. Winternitz, Eds.), CRM Proc. Lecture Notes, **9**, Amer. Math. Soc., Providence, RI, 1996, pp. 79–89.

D2.   J.F. van Diejen, Properties of some families of hypergeometric orthogonal polynomials in several variables, Trans. Amer. Math. Soc. **351** (1999), 233–270.

HS.   G. Heckman and H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces, Perspect. Math. **16**, Academic Press, San Diego, 1994.

He.   S. Helgason, Geometric Analysis on Symmetric Spaces, Mathematical Surveys and Monographs **39**, American Mathematical Society, Providence, RI, 1994.

Hu.   J. Humphreys, Introduction to Lie Algebras and Representation Theory, Second Printing, Springer-Verlag, New York, 1972.

KS.   F. Knop and S. Sahi, A recursion and a combinatorial formula for Jack polynomials, Invent. Math. **128** (1997), 9–22.

LLM1.   L. Lapointe, A. Lascoux, and J. Morse, Determinantal expressions for Macdonald polynomials, Internat. Math. Res. Notices **1998**, 957–978.

LLM2.   L. Lapointe, A. Lascoux, and J. Morse, Determinantal expression and recursion for Jack polynomials, Electron. J. Combin. **7** (2000), No. 1, Note 1, 7 pp. (electronic).

M1.   I.G. Macdonald, Symmetric Functions and Hall Polynomials, Second Edition, Oxford University Press, New York, 1995.

M2.   I.G. Macdonald, Symmetric Functions and Orthogonal Polynomials, University Lecture Series **12**, Amer. Math. Soc., Providence, RI, 1998.

M3.   I.G. Macdonald, Orthogonal polynomials associated with root systems, Sém. Lothar. Combin. **45** (2000/01), Art. B45a, 40 pp. (electronic).

MP.   R.V. Moody and J. Patera, Fast recursion formula for weight multiplicities, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 237–242.

N.   M. Noumi, Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, Adv. Math. **123** (1996), 16–77.

OP.   M.A. Olshanetsky and A.M. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rep. **94** (1983), 313–404.

O.   E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, Acta Math. **175** (1995), 75–121.

R.   S.N.M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Comm. Math. Phys. **110** (1987), 191–213.

Sa.   S. Sahi, A new formula for weight multiplicities and characters, Duke Math. J. **101** (2000), 77–84.

St.   R.P. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math. **77** (1989), 76–115.

Su.   T. Sugitani, Zonal spherical functions on quantum Grassmann manifolds, J. Math. Sci. Univ. Tokyo **6** (1999), 335–369.

S1.   B. Sutherland, Exact results for a quantum many-body system in one dimension, Phys. Rev. **A4** (1971), 2019–2021.

S2.   B. Sutherland, Exact results for a quantum many-body system in one dimension II, Phys. Rev. **A5** (1972), 1372–1376.

W.   J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, New York, 1988.

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