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# TWO VARIABLE ORTHOGONAL POLYNOMIALS ON THE BICIRCLE AND STRUCTURED MATRICES*

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**Abstract.** We consider bivariate polynomials orthogonal on the bicircle with respect to a positive linear functional. The lexicographical and reverse lexicographical orderings are used to order the monomials. Recurrence formulas are derived between the polynomials of different degrees. These formulas link the orthogonal polynomials constructed using the lexicographical ordering with those constructed using the reverse lexicographical ordering. Relations between the coefficients in the recurrence formulas are derived and used to give necessary and sufficient conditions for the existence of a positive linear functional. These results are then used to construct a class of two variable measures supported on the bicircle that are given by one over the magnitude squared of a stable polynomial. Applications to Fejér–Riesz factorization are also given.

**Key words.** bivariate orthogonal polynomials, positive definite linear functionals, moment problem, doubly Toeplitz matrices, recurrence coefficients

**AMS subject classifications.** 42C05, 30E05, 47A57, 15A48, 47B35

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**1. Introduction.** Bivariate polynomials orthogonal on the bicircle have been investigated mostly in the electrical engineering community in relation to the design of stable recursive filters for two-dimensional filtering. In particular we note the work of Genin and Kamp [7] who were interested in the following problem. Given any two variable polynomials $q(z,w)$, with $q(0,0) \neq 0$, let $a_{k,l}(z,w)$ be its planar least squares inverse polynomial of degree $(k,l)$; i.e., $a_{k,l}$ minimizes the mean quadratic value of $1 - a_{k,l}q$ on the bicircle. What properties does $a_{k,l}$ have? At the time it was conjectured the minimizing polynomials were stable, i.e., $a_{k,l}(z,w) \neq 0,\ |z| \leq 1,\ |w| \leq 1$, which they showed was false. Their investigation was carried further by Delsarte, Genin, and Kamp [4] who developed the connection between these polynomials and matrix polynomials orthogonal on the unit circle [3]. In the development of this connection these authors were led to examine moment matrices that were block Toeplitz matrices where each block entry is itself a Toeplitz matrix. Such structured matrices are called doubly Toeplitz matrices and arise naturally in the bivariate trigonometric moment problem. These types of matrices arose more recently in the work of Geronimo and Woerdeman [8] in their investigation of the bivariate Fejér–Riesz factorization theorem. These authors were able to resolve the question when a strictly positive bivariate trigonometric polynomial of a certain degree can be written as the magnitude squared of a stable polynomial of the same degree. In this work the authors used the fact that the theory of orthogonal polynomials on the unit circle provides a proof of the one variable Fejér–Riesz theorem which does not use the fundamental theorem of algebra. We intend here to continue to investigate the properties of bivariate polynomials orthogonal on the bicircle and clarify their role in the Fejér–Riesz theorem.

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A major difficulty encountered in the theory of orthogonal polynomials of more than one variable is which monomial ordering to use. For bivariate real orthogonal polynomials the preferred ordering is the total degree ordering which is the one set by Jackson [14]. For polynomials with the same total degree the ordering is lexicographical. As noted in Delgado et al. [2] in their study of orthogonal polynomials associated with doubly Hankel matrices, there is a good reason for choosing this ordering which is that if new orthogonal polynomials of higher degree are to be constructed, then their orthogonality relations will not affect the relations governing the lower degree polynomials. However, in order for the moment matrix to be doubly Toeplitz the monomial orderings that need to be used are lexicographical and reverse lexicographical.

We begin in section 2 by considering finite-dimensional subspaces spanned by the monomials $z^i w^j, |i| \le n$, $|j| \le m$, and exhibiting the connection between positive linear functionals defined on this space and positive definite doubly Toeplitz matrices. We then introduce certain matrix orthogonal polynomials and show how they give the Cholesky factors for the inverse of the doubly Toeplitz matrices considered above. The results in [8] show that these polynomials play a role in the parametric moment problem. In section 3 we construct two variable orthogonal polynomials, where the monomials are ordered according to the lexicographical ordering. When these polynomials are organized into vector orthogonal polynomials, they can be related to the matrix orthogonal polynomials constructed previously. From this relation it is shown that these vector polynomials are the minimizers of a certain quadratic functional. Using the orthogonality relation, recurrence relations satisfied by the vector polynomials and their counterparts in the reverse lexicographical ordering are derived, and relations between these recurrence coefficients are exhibited. In section 4 a number of Christoffel–Darboux-like formulas are derived. In section 5 we use the relations between the coefficients derived in section 3 to develop an algorithm to construct the coefficients in the recurrence formulas at a particular level $(n, m)$, say, in terms of the coefficients at the previous levels plus a certain number of unknowns. The collection of these unknowns is in one to one correspondence with the number of moments needed to construct the vector polynomials up to level $(n, m)$. This is used in section 6 to construct a positive linear functional from the recurrence coefficients. The construction allows us to find necessary and sufficient conditions on the recurrence coefficients for the existence of a positive linear functional which is in one to one correspondence with the set of positive definite doubly Toeplitz matrices. In section 7 we examine conditions under which the linear functional can be represented as a positive measure supported on the bicircle having the form of one over the magnitude squared of a stable polynomial. This gives a new proof of the Fejér–Riesz result of [8]. Finally in section 8 examples are given that illustrate various aspects of the theory developed.

**2. Positive linear functionals and doubly Toeplitz matrices.** In this section we consider moment matrices associated with the lexicographical ordering, which is defined by

$$(k,\ell) <_{\text{lex}} (k_1,\ell_1) \Leftrightarrow k < k_1 \text{ or } (k = k_1 \text{ and } \ell < \ell_1),$$

and the reverse lexicographical ordering, defined by

$$(k,\ell) <_{\text{revlex}} (k_1,\ell_1) \Leftrightarrow (\ell,k) <_{\text{lex}} (\ell_1,k_1).$$

Both of these orderings are linear orders, and in addition they satisfy

$$(k,\ell) < (m,n) \Rightarrow (k+p,\ell+q) < (m+p,n+q).$$

In such a case, one may associate a half-space with the ordering which is defined by $\{(k,l) \ : \ (0,0) < (k,l)\}$. In the case of the lexicographical ordering we shall denote the associated half-space by $H$ and refer to it as *the standard half-space*. In the case of the reverse lexicographical ordering we shall denote the associated half-space by $\tilde{H}$. Instead of starting with the ordering, one may also start with a half-space $\hat{H}$ of $\mathbb{Z}^2$ (i.e., a set $\hat{H}$ satisfying $\hat{H}+\hat{H} \subset \hat{H}$, $\hat{H} \cap (-\hat{H}) = \emptyset$, $\hat{H} \cup (-\hat{H}) \cup \{(0,0)\} = \mathbb{Z}^2$) and define an ordering via

$$(k,l) <_{\hat{H}} (k_1,l_1) \Longleftrightarrow (k_1-k, l_1-l) \in \hat{H}.$$

We shall refer to the order $<_{\hat{H}}$ as *the order associated with* $\hat{H}$. Note that the lexicographical and reverse lexicographical orderings do not respect total degree.

Let $\prod^{n,m}$ denote the bivariate Laurent linear subspace span$\{z^i w^j, \ -n \le i \le n, \ -m \le j \le m\}$. Let $\mathcal{L}_{n,m}$ be a linear functional defined on $\prod^{n,m}$ by

$$\mathcal{L}_{n,m}(z^{-i}w^{-j}) = c_{i,j} = \overline{\mathcal{L}(z^i w^j)}.$$

We will call $c_{i,j}$ the $(i,j)$ moment of $\mathcal{L}_{n,m}$ and $\mathcal{L}_{n,m}$ a moment functional. If we form the $(n+1)(m+1) \times (n+1)(m+1)$ matrix $C_{n,m}$ for $\mathcal{L}_{n,m}$ in the lexicographical ordering, then, as noted in the introduction, it has the special block Toeplitz form

$$C_{n,m} = \begin{bmatrix} C_0 & C_{-1} & \cdots & C_{-n} \\ C_1 & C_0 & \cdots & C_{-n+1} \\ \vdots & & \ddots & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \end{bmatrix}, \tag{2.1}$$

where each $C_i$ is an $(m+1)\times(m+1)$ Toeplitz matrix as follows:

$$C_i = \begin{bmatrix} c_{i,0} & c_{i,-1} & \cdots & c_{i,-m} \\ \vdots & & \ddots & \vdots \\ c_{i,m} & & \cdots & c_{i,0} \end{bmatrix}, \qquad i=-n,\dots,n. \tag{2.2}$$

Thus $C_{n,m}$ has a doubly Toeplitz structure. If the reverse lexicographical ordering is used in place of the lexicographical ordering, we obtain another moment matrix $\tilde{C}_{n,m}$ where the roles of $n$ and $m$ are interchanged.

Let us introduce the notion of centrotranspose symmetry. We denote the transpose of a matrix $A$ by $A^T$. A square matrix $A$ is said to be *centrotranspose symmetric* if $JAJ = A^T$, where $J$ is the matrix with ones on the antidiagonal and zeros elsewhere. Note that a Toeplitz matrix is centrotranspose symmetric. We have the following useful lemmas which characterize Toeplitz and doubly Toeplitz matrices in terms of centrotranspose symmetry.

LEMMA 2.1. *An* $(n+1)\times(n+1)$ *matrix* $A = (a_{i,j})_{i,j=0}^n$ *is Toeplitz if and only if both* $A$ *and* $\hat{A} := (a_{i,j})_{i,j=0}^{n-1}$ *are centrotranspose symmetric.*

*Proof.* Notice that $JAJ = A^T$ is equivalent to $a_{n-i,n-j} = a_{j,i}$, $0 \le i,j \le n$. Similarly, the centrotranspose symmetry of $\hat{A}$ is equivalent to $a_{n-1-i,n-1-j} = a_{j,i}$, $0 \le i,j \le n-1$. But then

$$a_{i+1,j+1} = a_{n-j-1,n-i-1} = a_{i,j}, \quad 0 \le i,j \le n-1,$$

and thus it follows that $A$ is Toeplitz.

As $A$ and $\hat{A}$ are Toeplitz, the converse is immediate. ∎

LEMMA 2.2. *Let $A = (A_{i,j})$, $i, j = 1, \ldots, k$, where each $A_{i,j}$ is a complex $m \times m$ matrix. Then $A$ is a doubly Toeplitz matrix if and only if $A^T = JAJ$, $A_1^T = J_1 A_1 J_1$, and $A_2^T = J_1 A_2 J_1$. Here $A_1$ is obtained from $A$ by deleting the last block row and column, and $A_2$ is obtained from $A$ by removing the last row and column of each $A_{i,j}$. The matrices $J$ and $J_1$ are square matrices of appropriate size with ones on the antidiagonal and zeros everywhere else.*

*Proof.* Again the necessary conditions follow from the structure of $A$. To see the converse note that $A^T = JAJ$ implies that $A_{j,i}^T = J_2 A_{k-i,k-j} J_2$, where $J_2$ is the $m \times m$ matrix with ones on the reverse diagonal and zeros everywhere else. This coupled with the condition on $A_1$ implies that $A$ is a block Toeplitz matrix from Lemma 2.1 and $J_2 A_{i,j} J_2 = A_{i,j}^T$. These relations plus the condition on $A_2$ and Lemma 2.1 give the result. ∎

*Remark* 2.3. The conclusions of the above lemmas hold if we replace deleting the last (last block) row and column by deleting the first (first block) row and column.

We say that the moment functional $\mathcal{L}_{n,m} : \prod^{n,m} \to \mathbb{C}$ is positive definite or positive semidefinite if

$$\mathcal{L}_{n,m}(|p|^2) > 0 \quad \text{or} \quad \mathcal{L}_{n,m}(|p|^2) \geq 0 \tag{2.3}$$

for every nonzero polynomial $p \in \prod^{n,m}$. It follows from a simple quadratic form argument that $\mathcal{L}_{n,m}$ is positive definite or positive semidefinite if and only if its moment matrix $C_{n,m}$ is positive definite or positive semidefinite, respectively.

We will say that $\mathcal{L}$ is positive definite or positive semidefinite if

$$\mathcal{L}(|p|^2) > 0 \qquad \text{or} \qquad \mathcal{L}(|p|^2) \geq 0$$

for all nonzero polynomials, respectively. Again these conditions are equivalent to the moment matrices $C_{n,m}$ being positive definite or positive semidefinite for all positive integers $n$ and $m$. The above discussion leads to the following.

LEMMA 2.4. *Let $C_{n,m}$ be a positive (positive semi-) definite $(n+1)(m+1) \times (n+1)(m+1)$ matrix given by* (2.1) *and* (2.2). *Then there is a positive (positive semi-) definite moment functional $\mathcal{L}_{n,m} : \prod^{n,m} \to \mathbb{C}$ associated with $C_{n,m}$ given by*

$$c_{i,j} = \mathcal{L}_{n,m}(z^{-i} w^{-j}) = \overline{\mathcal{L}_{n,m}(z^i w^j)}, \qquad -n \leq i \leq n, \qquad -m \leq j \leq m.$$

*The converse also holds.*

Let $\prod_{m+1}^n$ be the set of all $(m+1) \times (m+1)$ complex-valued matrix polynomials of degree $n$ or less, $\prod_{m+1}$ the set of all $(m+1) \times (m+1)$ complex-valued matrix polynomials, and $M^{m,n}$ the space of $m \times n$ matrices. For a matrix $M$ we let $M^\dagger$ denote the conjugate transpose (or the adjoint) of $M$. For a polynomial $Q(z,w)$ we let $Q^\dagger(z,w)$ denote the polynomial in $z^{-1}$ and $w^{-1}$ defined by $Q(z,w)^\dagger = Q^\dagger(\frac{1}{\bar{z}}, \frac{1}{\bar{w}})^\dagger$. If the positive moment functional $\mathcal{L}_{n,m} : \prod^{n,m} \to \mathbb{C}$ is extended to two variable polynomials with matrix coefficients in the obvious way, we can associate it with a positive matrix function $\mathcal{L}_m : \prod_{m+1}^n \times \prod_{m+1}^n \to M^{m+1,m+1}$ defined by

$$[\mathcal{L}_m(P(z), Q(z))]_{i,j} = \mathcal{L}_{n,m}(\left[P(z,w)\ Q^\dagger(z,w)\right]_{i,j}), \quad 1 \leq i,j \leq m+1 \tag{2.4}$$

where

$$P(z,w) = P(z) \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} \text{ and } Q(z,w) = Q(z) \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix}.$$

Equation (2.4) shows that if $\mathcal{L}_{n,m}$ can be represented in terms of a positive measure $\mu$ supported on the bicircle, then for $f$ an $(m+1)\times(m+1)$ matrix function continuous on the unit circle,

$$\mathcal{L}_m(f) = \int_{-\pi}^{\pi} f(\theta) dM_m(\theta),$$

where $M_m$ is the $(m+1)\times(m+1)$ matrix measure given by

$$dM_m(\theta) = \int_{\phi=-\pi}^{\pi} \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} d\mu(\theta,\phi) \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix}^{\dagger},$$

which shows that $M_m$ is Toeplitz.

Because of the structure of $C_{n,m}$ we can associate with $\mathcal{L}_m$ matrix valued orthogonal polynomials in the following manner [3], [4], [8]. Let $\{R_i^m(z)\}_{i=0}^n$ and $\{L_i^m(z)\}_{i=0}^n$ be $(m+1)\times(m+1)$ complex-valued matrix polynomials given by

$$R_i^m(z) = R_{i,i}^m z^i + R_{i,i-1}^m z^{i-1} + \cdots, \qquad i = 0,\ldots,n, \tag{2.5}$$

and

$$L_i^m(z) = L_{i,i}^m z^i + L_{i,i-1}^m z^{i-1} + \cdots, \qquad i = 0,\ldots,n, \tag{2.6}$$

satisfying

$$\mathcal{L}_m(R_i^{m\dagger}, R_j^{m\dagger}) = \delta_{ij} I_{m+1} \tag{2.7}$$

and

$$\mathcal{L}_m(L_i^m, L_j^m) = \delta_{ij} I_{m+1}, \tag{2.8}$$

respectively, where $I_{m+1}$ denotes the $(m+1)\times(m+1)$ identity matrix. The above relations uniquely determine the sequences $\{R_i^m\}_{i=0}^n$ and $\{L_i^m\}_{i=0}^n$ up to a unitary factor, and this factor will be fixed by requiring $R_{i,i}^m$ and $L_{i,i}^m$ to be upper triangular matrices with positive diagonal entries. We write

$$L_i^m(z) = [0 \cdots 0 \ L_{i,i}^m \ L_{i,i-1}^m \ \cdots \ L_{i,0}^m] \begin{bmatrix} z^n I_{m+1} \\ z^{n-1} I_{m+1} \\ \vdots \\ I_{m+1} \end{bmatrix}, \tag{2.9}$$

and

$$\hat{L}_n^m(z) = \begin{bmatrix} L_n^m(z) \\ L_{n-1}^m(z) \\ \vdots \\ L_0^m(z) \end{bmatrix} = L \begin{bmatrix} z^n I_{m+1} \\ z^{n-1} I_{m+1} \\ \vdots \\ I_{m+1} \end{bmatrix}, \tag{2.10}$$

where

$$L = \begin{bmatrix} L_{n,n}^m & L_{n,n-1}^m & \cdots & L_{n,0}^m \\ 0 & L_{n-1,n-1}^m & \cdots & L_{n-1,0}^m \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & L_{0,0}^m \end{bmatrix}. \tag{2.11}$$

In an analogous fashion write

$$\hat{R}_n^m(z) = \begin{bmatrix} R_0^m(z) \\ R_1^m(z) \\ \vdots \\ R_n^m(z) \end{bmatrix} = \begin{bmatrix} I_{m+1} & \dots & z^n I_{m+1} \end{bmatrix} R, \tag{2.12}$$

where

$$R = \begin{bmatrix} R_{0,0}^m & R_{1,0}^m & \cdots & R_{n,0}^m \\ 0 & R_{1,1}^m & \cdots & R_{n,1}^m \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & R_{n,n}^m \end{bmatrix}. \tag{2.13}$$

By lower (respectively, upper) Cholesky factor $A$ (respectively, $B$) of a positive definite matrix $M$, we mean

$$M = AA^\dagger = BB^\dagger, \tag{2.14}$$

where $A$ is a lower triangular matrix with positive diagonal elements, and $B$ is an upper triangular matrix with positive diagonal elements. With the above we have the following well known lemma [15].

LEMMA 2.5. *Let $C_{n,m}$ be a positive definite block Toeplitz matrix given by* (2.1)*; then $L^\dagger$ is the lower Cholesky factor, and $R$ is the upper Cholesky factor of $C_{n,m}^{-1}$.*

*Proof.* To obtain (2.14) note that (2.8) implies that

$$I = \mathcal{L}_m(\hat{L}_n^m, \hat{L}_n^m) = L\mathcal{L}_m \left( \begin{bmatrix} z^n I_{m+1} \\ z^{n-1} I_{m+1} \\ \vdots \\ I_{m+1} \end{bmatrix}, \begin{bmatrix} z^n I_{m+1} \\ z^{n-1} I_{m+1} \\ \vdots \\ I_{m+1} \end{bmatrix} \right) L^\dagger = LC_{n,m}L^\dagger,$$

where $I$ is the $(n+1)(m+1) \times (n+1)(m+1)$ identity matrix. Since $C_{n,m}$ is invertible we find

$$C_{n,m}^{-1} = L^\dagger L.$$

The result for $R$ follows in an analogous manner. $\square$

From this formula and (2.11) we find

$$L_n^m(z) = \left[ ({L_{n,n}^m}^\dagger)^{-1}, 0, 0, \dots 0 \right] C_{n,m}^{-1} [z^n I_{m+1}, z^{n-1} I_{m+1}, \dots, I_{m+1}]^T, \tag{2.15}$$

and

$$R_n^m(z) = [I_{m+1}, zI_{m+1}, \dots, z^n I_{m+1}]\, C_{n,m}^{-1} \left[ 0, 0, \dots, 0, (\bar{R}_{n,n}^m)^{-1} \right]^T. \tag{2.16}$$

Note that ${L_{n,n}^m}^\dagger$ is the lower Cholesky factor of $[I_{m+1}, 0, \cdots, 0] C_{n,m}^{-1}\, [I_{m+1}, 0, \cdots, 0]^T$, while $R_{n,n}^m$ is the upper Cholesky factor of $[0, \cdots, I_{m+1}] C_{n,m}^{-1}\, [0, \cdots, I_{m+1}]^T$.

The theory of matrix orthogonal polynomials (see [3], [15], [17], [19]) can be applied to obtain the recurrence formulas

$$\begin{aligned} A_{i+1,m} L_{i+1}^m(z) &= zL_i^m(z) - E_{i+1,m} \overleftarrow{R}_i^m(z), \quad i = 0, \dots, n-1, \\ R_{i+1}^m(z) \hat{A}_{i+1,m} &= zR_i^m(z) - \overleftarrow{L}_i^m(z) E_{i+1,m}, \quad i = 0, \dots, n-1, \end{aligned} \tag{2.17}$$

where

$$E_{i+1,m} = \mathcal{L}_m(zL_i^m, \overleftarrow{R}_i^m) = \mathcal{L}_m(\overleftarrow{L}_i^{m^\dagger}, (zR_i^m)^\dagger) \tag{2.18}$$

and

$$\begin{aligned} A_{i+1,m} &= \mathcal{L}_m(zL_i^m,\ L_{i+1}^m) = L_{i,i}^m(L_{i+1,i+1}^m)^{-1}, \\ \hat{A}_{i+1,m} &= \mathcal{L}_m(R_{i+1}^{m\dagger}, (zR_i^m)^\dagger) = (R_{i+1,i+1}^m)^{-1}R_{i,i}^m. \end{aligned} \tag{2.19}$$

For a matrix polynomial $B$ of degree $n$ in $z$, $\overleftarrow{B}(z) = z^n \sum_{i=0}^n B_i^\dagger z^{-i}$. By multiplying the first equation in (2.17) on the left by $\bar{z}L_i^m(z)^\dagger$ and the second equation on the right by $\bar{z}R_i^m(z)^\dagger$ and then integrating, we see that

$$\begin{aligned} A_{i+1,m}A_{i+1,m}^\dagger &= I_{m+1} - E_{i+1,m}E_{i+1,m}^\dagger, \\ \hat{A}_{i+1,m}^\dagger \hat{A}_{i+1,m} &= I_{m+1} - E_{i+1,m}^\dagger E_{i+1,m}. \end{aligned} \tag{2.20}$$

The above equations and the properties of $A_{i+1,m}$ and $\hat{A}_{i+1,m}$ show that $E_{i+1,m}$ is a strictly contractive matrix and that $A_{i+1,m}$ is the upper Cholesky factor of $I_{m+1} - E_{i+1,m}E_{i+1,m}^\dagger$. Similarly $\hat{A}_{i,m}^\dagger$ is the lower Cholesky factor of $I_{m+1} - E_{i+1,m}^\dagger E_{i+1,m}$. Furthermore (2.19) and (2.20) show that

$$\det((L_{i+1,i+1}^m)^\dagger L_{i+1,i+1}^m)^{-1} = \det(C_0)\prod_{j=1}^{i+1}\det(I_{m+1} - E_{j,m}E_{j,m}^\dagger). \tag{2.21}$$

The recurrence formulas (2.17) can be inverted in the following manner. Multiply the reverse of the second equation in (2.17) on the right by $E_{i+1,m}$ to obtain

$$E_{i+1,m}\hat{A}_{i+1,m}^\dagger \overleftarrow{R}_{i+1}^m(z) = E_{i+1,m}\overleftarrow{R}_i^m(z) - zE_{i+1,m}E_{i+1,m}^\dagger L_i^m(z).$$

Add this equation to the first equation in (2.17) and then use (2.20) to eliminate $A_{i+1,m}$ and $\hat{A}_{i+1,m}^\dagger$ to find

$$(A_{i+1,m}^\dagger)^{-1}L_{i+1}^m(z) + E_{i+1,m}(\hat{A}_{i+1,m})^{-1}\overleftarrow{R}_{i+1}^m(z) = zL_i^m(z). \tag{2.22}$$

In a similar manner we find

$$R_{i+1}^m(z)(\hat{A}_{i+1,m}^\dagger)^{-1} + \overleftarrow{L}_{i+1}^m(z)(A_{i+1,m})^{-1}E_{i+1,m} = zR_i^m(z). \tag{2.23}$$

From the recurrence formulas it is not difficult to derive the Christoffel–Darboux formulas [3]:

$$\begin{aligned} \overleftarrow{R}_k^m(z)^\dagger \overleftarrow{R}_k^m(z_1) - \bar{z}z_1 L_k^m(z)^\dagger L_k^m(z_1) &= (1-\bar{z}z_1)\sum_{i=0}^k L_i^m(z)^\dagger L_i^m(z_1), \\ \overleftarrow{L}_k^m(z_1)\overleftarrow{L}_k^m(z)^\dagger - \bar{z}z_1 R_k^m(z_1)R_k^m(z)^\dagger &= (1-\bar{z}z_1)\sum_{i=0}^k R_i^m(z_1)R_i^m(z)^\dagger. \end{aligned} \tag{2.24}$$

These formulas give rise to the matrix Gohberg–Semencul formulas [11], [15] when the linear equations obtained by equating like powers of $\bar{z}^i z_1^j$ are put in matrix form.

Some properties that follow from the above formulas [3, Theorems 9, 14, and 15] are that $\overleftarrow{R}_i^m(z)$ and $\overleftarrow{L}_k^m(z)$ have empty kernels for $|z| \leq 1$; i.e.,

$$\det(\overleftarrow{R}_i^m(z)) \neq 0 \neq \det(\overleftarrow{L}_k^m(z)), \ |z| \leq 1. \tag{2.25}$$

Such polynomials are called stable matrix polynomials, and if we write

$$W_k(z) = \left[\overleftarrow{L}_k^m(z)\overleftarrow{L}_k^m(z)^\dagger\right]^{-1} \tag{2.26}$$

and

$$C_j^k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta} W_k(e^{i\theta}) d\theta,$$

then

$$C_j^k = C_j, \quad |j| \leq k. \tag{2.27}$$

Furthermore

$$W_k = \left[\overleftarrow{R}_k^m(z)^\dagger \overleftarrow{R}_k^m(z)\right]^{-1}. \tag{2.28}$$

If $\overleftarrow{L}_k^m(z)$ ($\overleftarrow{R}_k^m(z)$) satisfies (2.25) and (2.27), we will say it is stable and has spectral matching (up to level $k$). Another useful result shown in [3] is

$$\log\det((L_{i+1,i+1}^m)^\dagger L_{i+1,i+1}^m)^{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det W_k(\theta) d\theta. \tag{2.29}$$

From the stability of $\overleftarrow{R}_{i+1}^m$ and $\overleftarrow{L}_{i+1}^m$, (2.22) and (2.23) give the following formulas for the recurrence coefficients $E_{i+1,m}$:

$$\begin{aligned} E_{i+1,m} &= -(A_{i+1,m}^\dagger)^{-1} L_{i+1}^m(0) \overleftarrow{R}_{i+1}^m(0)^{-1} \hat{A}_{i+1,m} \\ &= -A_{i+1,m} \overleftarrow{L}_{i+1}^m(0)^{-1} R_{i+1}^m(0) (\hat{A}_{i+1,m}^\dagger)^{-1}. \end{aligned} \tag{2.30}$$

We also note that $\overleftarrow{L}_k^m(z)$ and $\overleftarrow{R}_k^m(z)$ are minimizers of certain quadratic functions. To see this denote the set of $(m+1)\times(m+1)$ hermitian matrices as $\mathrm{Herm}(m+1)$, and let $\mathcal{M} : \prod_{m+1} \to \mathrm{Herm}(m+1)$ be given by

$$\mathcal{M}[X(z)] = \mathcal{L}_m(X, X) - (X(0) + X(0)^\dagger), \tag{2.31}$$

then Delsarte, Genin, and Kamp have shown [3] that for a given degree $k$, $\mathcal{M}$ is minimized by $\overleftarrow{L}_k^m(z) L_{k,m}^m$ with the value $(L_{k,m}^m)^\dagger L_{k,m}^m$. Likewise $\hat{\mathcal{M}} : \prod_{m+1} \to \mathrm{Herm}(m+1)$ given by

$$\hat{\mathcal{M}}[X(z)] = \mathcal{L}_m(X^\dagger, X^\dagger) - (X(0) + X(0)^\dagger) \tag{2.32}$$

is minimized by $R_{k,m}^m \overleftarrow{R}_k^m(z)$ and takes the value $R_{k,m}^m (R_{k,m}^m)^\dagger$. Thus we find

$$(L_{k,m}^m)^\dagger L_{k,m}^m \geq (L_{k+1,m}^m)^\dagger L_{k+1,m}^m \tag{2.33}$$

and

$$R^m_{k,m}(R^m_{k,m})^\dagger \geq R^m_{k+1,m}(R^m_{k+1,m})^\dagger. \tag{2.34}$$

Here $A \geq B$ for two $(m+1)\times(m+1)$ matrices means that $A - B$ is positive semidefinite. The above discussion leads to Burg's entropy theorem. Consider the class of $M^m$ of $(m+1)\times(m+1)$ matrix Borel measures on the unit circle, and for each such measure $\mu$ write the Lebesgue decomposition of $\mu = \mu_{ac} + \mu_s$, where $d\mu_{ac}/d\theta = W(\theta)$. Let $S^m_n$ be the subset of $M^m$ such that each $\mu \in S^m_n$ has the same Fourier coefficients $C_i$, $|i| \leq n$, and $\mathcal{E}(\mu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det(W)d\theta > -\infty$. Then there is a unique measure which maximizes the above entropy function $\mathcal{E}(\mu)$, and this measure is given by $d\mu = W(\theta)d\theta$, with $W(\theta) = Q^m_n(\theta)^{-1}$, where $Q^m_n(\theta)$ is a positive $(m+1)\times(m+1)$ matrix trigonometric polynomial of degree $n$.

This leads to a simple proof of the matrix Fejér–Reisz factorization theorem (Helson [13], Dritschel [5], McLean and Woerdeman [16], Geronimo and Lai [10]) which will be useful later.

Lemma 2.6. *Let $Q^m_n(\theta)$ be a strictly positive $(m+1)\times(m+1)$ matrix trigonometric polynomial; then $Q^m_n(\theta) = \overleftarrow{L}^m_n(z)(\overleftarrow{L}^m_n(z))^\dagger$, $z = e^{i\theta}$, where $\overleftarrow{L}^m_n$ is a stable $(m+1)\times(m+1)$ matrix polynomial of degree $n$. Furthermore $L^m_n$ is given by* (2.15).

*Proof.* Since $Q^m_n(\theta)$ is strictly positive we can compute the moments $C_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}Q^m_n(\theta)^{-1}d\theta$. If we compute the matrix orthogonal polynomials associated with these Fourier coefficients, we find that $W_n$ has spectral matching up to $n$. That is, its Fourier coefficients match $C_i$ for $|i| \leq n$. The maximum entropy theorem implies that $Q^m_n(\theta) = W_n^{-1}$, which gives the result. □

The matrix Fejér–Riesz theorem now follows.

Theorem 2.7. *Let $Q^m_n(\theta) \geq 0$ be a positive $(m+1)\times(m+1)$ matrix trigonometric polynomial; then $Q^m_n(\theta) = P^m_n(z)(P^m_n(z))^\dagger$, $z = e^{i\theta}$, where $P^m_n$ is an outer (nonzero for $|z| < 1$) $(m+1)\times(m+1)$ matrix polynomial.*

*Proof.* Let $Q^m_{n,\epsilon} = \epsilon I + Q^m_n$ , $\epsilon > 0$; then $Q^m_{n,\epsilon}$ satisfies the hypotheses of the above lemma. Thus $Q^m_{n,\epsilon} = P^m_{n,\epsilon}(P^m_{n,\epsilon})^\dagger$. The proof now follows by taking the limit as $\epsilon$ tends to zero. □

It was observed by Delsarte et al. [4] that if the $C_k$ in $C_{n,m}$ are centrotranspose symmetric, then

$$(L^{m\,\dagger}_{i,i} L^m_i(z))^T = J_m R^m_i(z) R^{m\,\dagger}_{i_i} J_m, \quad i = 0, \ldots, n, \tag{2.35}$$

where $J_m$ is the $(m+1)\times(m+1)$ matrix with ones on the reverse diagonal and zeros everywhere else. This can easily be seen from (2.15) and (2.16) since in this case from Lemma 2.2 $C^T_{n,m} = JC_{n,m}J$, with $J$ the $(n+1)(m+1)\times(m+1)(n+1)$ matrix with ones down the antidiagonal and zeros everywhere else. This leads to the following characterization of positive definite doubly Toeplitz matrices in terms of certain recurrence coefficients. We will denote by $C^m_0$ the $m\times m$ matrix obtained from $C_0$ by eliminating the first row and first column of $C_0$.

Theorem 2.8. *Suppose $C_{n,m}$ is positive definite. Then the Fourier coefficients $C_i, |i| \leq n$, are centrotranspose symmetric if and only if $E_{k,m}, k = 1, \ldots, n$, and $C_0$ are centrotranspose symmetric. Consequently, $C_{n,m}$ is doubly Toeplitz if and only if $E_{k,i}, k = 1, \ldots, n, i = m-1, m$, $C_0$, and $C^m_0$ are centrotranspose symmetric.*

*Proof.* Examining the leading coefficients in (2.35) and using the fact that $L^m_{i,i}$ and $R^m_{i,i}$ are upper triangular, we find that (see also [4]) $(L^m_{i,i})^T = J_m R^m_{i,i} J_m$ for $i = 0, \ldots, n$. Thus

$$L^m_i(z)^T = J_m R^m_i(z) J_m, \ i = 0, \ldots, n. \tag{2.36}$$

The above equation and (2.19) imply that

$$J_m A_{i+1,m} J_m = A^T_{i+1,m}. \tag{2.37}$$

Using this coupled with (2.36) and its reverse in (2.30) yields

$$\begin{aligned} J_m E_{i+1,m} J_m &= -J_m (A^\dagger_{i+1,m})^{-1} L^m_{i+1}(0) \overleftarrow{R}^m_{i+1}(0)^{-1} \hat{A}_{i+1,m} J_m \\ &= -(A_{i+1,m} \overleftarrow{L}^m_{i+1}(0)^{-1} R^m_{i+1}(0)(\hat{A}^\dagger_{i+1,m})^{-1})^T = E^T_{i+1,m}. \end{aligned} \tag{2.38}$$

To show the converse note that if $E_{i,m}$ is centrotranspose symmetric, then from (2.20) we obtain

$$J_m (A_{i,m} A^\dagger_{i,m})^T J_m = J_m (I_m - E_{i,m} E^\dagger_{i,m}) J_m = \overline{(I - E^\dagger_{i,m} E_{i,m})} = \hat{A}^T_{i,m} \overline{\hat{A}_{i,m}},$$

which gives (2.37). Since $C_0$ is centrotranspose symmetric and $L^\dagger_{0,m}(z)$ is the lower Cholesky factor of $C_0$, we see that $J_m L^m_0 J_m = R^{m\,T}_0$. Thus by induction using (2.17) we find that $J_m L^m_n(z) J_m = R^m_n(z)^\top$. The first part of the result now follows from the spectral matching of $W_n$, (2.26), and (2.28). The second part of the theorem follows by applying the above argument to $C_{n,m-1}$ and $C^m_0$ and then using Lemma 2.1. □

In the next two sections we present recurrence formulas and an algorithm that computes recurrence coefficients for a positive definite doubly Toeplitz matrix.

## 3. Bivariate orthogonal polynomials.

In this section we examine the properties of two variable orthogonal polynomials where the monomial ordering is either lexicographical or reverse lexicographical. The study of orthogonal polynomials on the bicircle with this ordering was begun by Delsarte et al. [4] and extended in [8]. Given a positive definite linear functional $\mathcal{L}_{N,M} : \prod^{N,M} \to \mathbb{C}$ we perform the Gram–Schmidt procedure using the lexicographical ordering and define the orthonormal polynomials $\phi^l_{n,m}(z,w)$, $0 \le n \le N$, $0 \le m \le M$, $0 \le l \le m$, by the equations

$$\begin{aligned} &\mathcal{L}_{N,M}(\phi^l_{n,m} z^{-i} w^{-j}) = 0, \quad 0 \le i < n \text{ and } 0 \le j \le m \quad \text{ or } i = n \text{ and } 0 \le j < l, \\ &\mathcal{L}_{N,M}(\phi^l_{n,m} (\phi^l_{n,m})^\dagger) = 1, \end{aligned} \tag{3.1}$$

and

$$\phi^l_{n,m}(z,w) = k^{n,l}_{n,m,l} z^n w^l + \sum_{(i,j) <_{\mathrm{lex}} (n,l)} k^{i,j}_{n,m,l} z^i w^j. \tag{3.2}$$

With the convention $k^{n,l}_{n,m,l} > 0$, the above equations uniquely specify $\phi^l_{n,m}$. Polynomials orthonormal with respect to $\mathcal{L}_{N,M}$ but using the reverse lexicographical ordering will be denoted by $\tilde{\phi}^l_{n,m}$. They are uniquely determined by the above relations with the roles of $n$ and $m$ interchanged.

Set

$$\Phi_{n,m} = \begin{bmatrix} \phi^m_{n,m} \\ \phi^{m-1}_{n,m} \\ \vdots \\ \phi^0_{n,m} \end{bmatrix} = K_{n,m} \begin{bmatrix} z^n w^m \\ z^n w^{m-1} \\ \vdots \\ 1 \end{bmatrix}, \tag{3.3}$$

where the $(m+1) \times (n+1)(m+1)$ matrix $K_{n,m}$ is given by

$$K_{n,m} = \begin{bmatrix} k^{n,m}_{n,m,m} & k^{n,m-1}_{n,m,m} & \cdots & \cdots & \cdots & k^{0,0}_{n,m,m} \\ 0 & k^{n,m-1}_{n,m,m-1} & \cdots & \cdots & \cdots & k^{0,0}_{n,m,m-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & k^{n,0}_{n,m,0} & k^{n-1,m}_{n,m,0} & \cdots & k^{0,0}_{n,m,0} \end{bmatrix}. \tag{3.4}$$

As indicated above denote

$$\tilde{\Phi}_{n,m} = \begin{bmatrix} \tilde{\phi}^n_{n,m} \\ \tilde{\phi}^{n-1}_{n,m} \\ \vdots \\ \tilde{\phi}^0_{n,m} \end{bmatrix} = \tilde{K}_{n,m} \begin{bmatrix} w^m z^n \\ w^m z^{n-1} \\ \vdots \\ 1 \end{bmatrix}, \tag{3.5}$$

where the $(n+1) \times (n+1)(m+1)$ matrix $\tilde{K}_{n,m}$ is given similarly to (3.4) with the roles of $n$ and $m$ interchanged. For the bivariate polynomials $\phi^l_{n,m}(z,w)$ above we define the reverse polynomials $\overleftarrow{\phi}^l_{n,m}(z,w)$ by the relation

$$\overleftarrow{\phi}^l_{n,m}(z,w) = z^n w^m \bar{\phi}^l_{n,m}(1/z, 1/w). \tag{3.6}$$

With this definition $\overleftarrow{\phi}^l_{n,m}(z,w)$ is again a polynomial in $z$ and $w$, and furthermore

$$\overleftarrow{\Phi}_{n,m}(z,w) := \begin{bmatrix} \overleftarrow{\phi}^m_{n,m} \\ \overleftarrow{\phi}^{m-1}_{n,m} \\ \vdots \\ \overleftarrow{\phi}^0_{n,m} \end{bmatrix}^T. \tag{3.7}$$

An analogous procedure is used to define $\overleftarrow{\tilde{\phi}}^l_{n,m}$.

In order to ease the notation to find recurrence formulas for the vector polynomials $\Phi_{n,m}$, we introduce the inner product

$$\langle X, Y \rangle = \mathcal{L}_{N,M}(XY^\dagger). \tag{3.8}$$

Let $\hat{\prod}^{n,m}$ be the linear span of $z^i w^j$, $0 \le i \le n, 0 \le j \le m$, $\hat{\prod}^{n,m}_k$ be the vector space of $k$ dimensional vectors with entries in $\hat{\prod}^{n,m}$, and $\hat{\prod}^m_{m+1} = \hat{\prod}^{\infty,m}_{m+1}$.

Utilizing the orthogonality relations (3.1) we obtain the following auxiliary results.

Lemma 3.1. *Suppose $\Phi \in \hat{\prod}^{n,m}_k$. If $\Phi$ satisfies the orthogonality relations*

$$\langle \Phi, z^i w^j \rangle = 0, \quad 0 \le i < n, \quad 0 \le j \le m, \tag{3.9}$$

*then $\Phi = T\Phi_{n,m}$, where $T$ is a $k \times (m+1)$ matrix. If $k = m+1$, $T$ is upper triangular with positive diagonal entries, and if $\langle \Phi, \Phi \rangle = I_{m+1}$, then $T = I_{m+1}$.*

Lemma 3.2. *Suppose $\tilde{\Phi} \in \hat{\prod}^{n,m}_k$. If $\tilde{\Phi}$ satisfies the orthogonality relations*

$$\langle \tilde{\Phi}, z^i w^j \rangle = 0, \quad 0 \le i \le n, \quad 0 \le j < m, \tag{3.10}$$

*then $\tilde{\Phi} = T\tilde{\Phi}_{n,m}$, where $T$ is a $k \times (n+1)$ matrix. If $k = n+1$, $T$ is upper triangular with positive diagonal entries, and if $\langle \tilde{\Phi}, \tilde{\Phi} \rangle = I_{n+1}$, then $T = I_{n+1}$.*

With the above we can make contact with the matrix orthogonal polynomials introduced in section 2. This was observed by Delsarte et al. [4].

Lemma 3.3. *Let $\Phi_{n,m}$ be given by* (3.3). *Then*

$$\Phi_{n,m} = L^m_n(z)[w^m, w^{m-1}, \dots, 1]^T, \tag{3.11}$$

$$\overleftarrow{\Phi}_{n,m} = [1, w, \dots, w^m] J_m \overleftarrow{R}^m_n(z)^T J_m, \tag{3.12}$$

*and*

$$\begin{bmatrix} \Phi_{n,m}(z,w) \\ \Phi_{n-1,m}(z,w) \\ \vdots \\ \Phi_{0,m}(z,w) \end{bmatrix} = \begin{bmatrix} L_n^m(z) \\ L_{n-1}^m(z) \\ \vdots \\ L_0^m(z) \end{bmatrix} [w^m, w^{m-1}, \ldots, 1]^T$$

$$= L \begin{bmatrix} z^n I_{m+1} \\ z^{n-1} I_{m+1} \\ \vdots \\ I_{m+1} \end{bmatrix} [w^m, w^{m-1}, \ldots, 1]^T. \tag{3.13}$$

*Proof.* If we substitute the equation

$$\Phi_{n,m} = \hat{L}_n(z)[w^m \ \cdots \ 1]^T = \sum_i \hat{L}_{n,i} z^i [w^m \ \cdots \ 1]^T$$

into (3.9), where $\hat{L}_n(z)$ is an $(m+1) \times (m+1)$ matrix polynomial of degree $n$, we find, for $j = 0, \ldots, n-1$,

$$\begin{aligned} 0 &= \left\langle \Phi_{n,m},\, z^j \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} \right\rangle = \sum_{i=0}^n \hat{L}_{n,i} \left\langle z^i \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix},\, z^j \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} \right\rangle \\ &= \sum_{i=1}^n \hat{L}_{n,i} \begin{bmatrix} \mathcal{L}_{NM}(z^{i-j}) & \cdots & \mathcal{L}_{NM}(z^{i-j} w^{-m}) \\ \vdots & & \vdots \\ \mathcal{L}_{NM}(z^{i-j} w^m) & \cdots & \mathcal{L}_{NM}(z^{i-j}) \end{bmatrix} \\ &= \sum_{i=1}^n \hat{L}_{n,i} \mathcal{L}_m(z^i,\ z^j) = \mathcal{L}_m(\hat{L}_n(z),\ z^j). \end{aligned}$$

Similarly,

$$\langle \Phi_{n,m},\, \Phi_{n,m} \rangle = I_{m+1} = \mathcal{L}_m \langle \hat{L}_n(z),\, \hat{L}_n(z) \rangle.$$

This, coupled with (2.8) and the fact that (3.3) implies that $\hat{L}_{n,m}$ is upper triangular with positive diagonal entries, gives (3.11). Equation (3.12) follows from (3.11) and (2.36), while (3.13) follows from (3.11) and the definition of $L$. ∎

Analogous formulas for bivariate orthogonal polynomials in the reverse lexicographical ordering are obtained by interchanging the roles on $n$ and $m$.

The function $\mathcal{M}$ given by (2.31) can be used to show that $\overleftarrow{\Phi}_{n,m}$ satisfies a minimization condition. Define $\bar{\mathcal{M}} : \hat{\prod}_{m+1}^m \to \mathrm{Herm}(m+1)$ by

$$\bar{\mathcal{M}}(\Phi) = \langle \Phi^\dagger,\, \Phi^\dagger \rangle - (\Phi_0 + \Phi_0^\dagger).$$

We find the following.

LEMMA 3.4. *The polynomial $\overleftarrow{\Phi}_{n,m}$ is the unique minimizer on $\hat{\prod}_{m+1}^{n,m}$.*

*Proof.* Since $\Phi \in \hat{\prod}_{m+1}^{n,m}$ can be represented as

$$\Phi(z,w) = [1, w, \ldots, w^m] \hat{\Phi}(z) = [1, w, \ldots, w^m] \sum_{i=0}^n \Phi_i z^i,$$

and from (2.4)

$$\left\langle z^i \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix}, z^j \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \mathcal{L}_{NM}(z^{i-j}) & \cdots & \mathcal{L}_{NM}(z^{i-j}w^{-m}) \\ \vdots & & \vdots \\ \mathcal{L}_{NM}(z^{i-j}w^m) & \cdots & \mathcal{L}_{NM}(z^{i-j}) \end{bmatrix} = \mathcal{L}_m(z^i, z^j),$$

we find $\bar{\mathcal{M}}(\Phi) = \hat{\mathcal{M}}(\hat{\Phi})$. The result now follows from (3.12) and the fact that $R^m_{n,m} \overleftarrow{R}^m_n(z)$ minimizes $\hat{\mathcal{M}}$ on $\prod^n_{m+1}$. ∎

We can now derive recurrence relations between the various polynomials.

THEOREM 3.5. *Given $\{\Phi_{n,m}\}$ and $\{\tilde{\Phi}_{n,m}\}$, $0 \le n \le N$, $0 \le m \le M$, the following recurrence formulas hold:*

$$A_{n,m}\Phi_{n,m} = z\Phi_{n-1,m} - \hat{E}_{n,m}\overleftarrow{\Phi}^T_{n-1,m}, \tag{3.14}$$

$$\Phi_{n,m} + A^\dagger_{n,m}\hat{E}_{n,m}(A^T_{n,m})^{-1}\overleftarrow{\Phi}^T_{n,m} = A^\dagger_{n,m} z\Phi_{n-1,m}, \tag{3.15}$$

$$\Gamma_{n,m}\Phi_{n,m} = \Phi_{n,m-1} - \mathcal{K}_{n,m}\tilde{\Phi}_{n-1,m}, \tag{3.16}$$

$$\Gamma^1_{n,m}\Phi_{n,m} = w\Phi_{n,m-1} - \mathcal{K}^1_{n,m}\overleftarrow{\tilde{\Phi}}^T_{n-1,m}, \tag{3.17}$$

$$\Phi_{n,m} = I_{n,m}\tilde{\Phi}_{n,m} + \Gamma^\dagger_{n,m}\Phi_{n,m-1}, \tag{3.18}$$

$$\overleftarrow{\Phi}^T_{n,m} = I^1_{n,m}\tilde{\Phi}_{n,m} + (\Gamma^1_{n,m})^T\overleftarrow{\Phi}^T_{n,m-1}, \tag{3.19}$$

*where*

$$\hat{E}_{n,m} = \langle z\Phi_{n-1,m}, \overleftarrow{\Phi}^T_{n-1,m}\rangle = E_{n,m}J_m = \hat{E}^T_{n,m} \in M^{m+1,m+1}, \tag{3.20}$$

$$A_{n,m} = \langle z\Phi_{n-1,m}, \Phi_{n,m}\rangle \in M^{m+1,m+1}, \tag{3.21}$$

$$\mathcal{K}_{n,m} = \langle \Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle \in M^{m,n}, \tag{3.22}$$

$$\Gamma_{n,m} = \langle \Phi_{n,m-1}, \Phi_{n,m}\rangle \in M^{m,m+1}, \tag{3.23}$$

$$\mathcal{K}^1_{n,m} = \langle w\Phi_{n,m-1}, \overleftarrow{\tilde{\Phi}}^T_{n-1,m}\rangle \in M^{m,n}, \tag{3.24}$$

$$\Gamma^1_{n,m} = \langle w\Phi_{n,m-1}, \Phi_{n,m}\rangle \in M^{m,m+1}, \tag{3.25}$$

$$I_{n,m} = \langle \Phi_{n,m}, \tilde{\Phi}_{n,m}\rangle \in M^{m+1,n+1}, \tag{3.26}$$

$$I^1_{n,m} = \langle \overleftarrow{\Phi}^T_{n,m}, \tilde{\Phi}_{n,m}\rangle \in M^{m+1,n+1}. \tag{3.27}$$

*Remark* 3.6. Formulas similar to (3.14)–(3.19) hold for $\tilde{\Phi}_{n,m}$ and will be denoted by ($\tilde{3}$.14)–($\tilde{3}$.19). Throughout the rest of the paper we use the same notation to denote the extension to $\tilde{\Phi}_{n,m}$ of existing formulas stated for $\Phi_{n,m}$.

*Proof.* Equation (3.14) follows from Lemma 3.3, (2.17), (2.36), and (2.37). Likewise (3.15) follows in an analogous manner from (2.22). To prove (3.16) note that, because of the linear independence of the entries of $\Phi_{n,m}$, there is an $m \times (m+1)$ matrix $\Gamma_{n,m}$ such that $\Gamma_{n,m}\Phi_{n,m} - \Phi_{n,m-1} \in \hat{\prod}^{n-1,m}_m$. Furthermore

$$\langle \Gamma_{n,m}\Phi_{n,m} - \Phi_{n,m-1}, z^i w^j\rangle = 0, \qquad 0 \le i \le n-1, \quad 0 \le j \le m-1.$$

Thus Lemma 3.2 implies that

$$\Gamma_{n,m}\Phi_{n,m} - \Phi_{n,m-1} = H_{n,m}\tilde{\Phi}_{n-1,m}.$$

The remaining recurrence formulas follow in a similar manner. ∎

*Remark* 3.7. As indicated in the proof, (3.14) follows from the theory of matrix orthogonal polynomials and so allows us to compute in the $n$ direction along a strip of size $m+1$. This formula does not mix the polynomials in the two orderings. However, to increase $m$ by one for polynomials constructed in the lexicographical ordering, the remaining relations show that orthogonal polynomials in the reverse lexicographical ordering must be used.

Using the orthogonality relations from Lemmas 3.1 and 3.2 and (3.1), we find the following relations.

PROPOSITION 3.8. *The following relations hold between the coefficients in the equations for* $\tilde{\Phi}$ *and* $\Phi$*:*

$$\tilde{\mathcal{K}}_{n,m} = \mathcal{K}^{\dagger}_{n,m},\ \tilde{I}_{n,m} = I^{\dagger}_{n,m}, \tag{3.28}$$

$$\tilde{I}^1_{n,m} = (I^1_{n,m})^T,\ \tilde{\mathcal{K}}^1_{n,m} = (\mathcal{K}^1_{n,m})^T. \tag{3.29}$$

*Also*

$$A_{n,m}A^{\dagger}_{n,m} = I_m - \hat{E}_{n,m}\hat{E}^{\dagger}_{n,m}, \tag{3.30}$$

$$\Gamma_{n,m}\Gamma^{\dagger}_{n,m} = I_m - \mathcal{K}_{n,m}\mathcal{K}^{\dagger}_{n,m}, \tag{3.31}$$

$$\Gamma^1_{n,m}(\Gamma^1_{n,m})^{\dagger} = I_m - \mathcal{K}^1_{n,m}(\mathcal{K}^1_{n,m})^{\dagger}, \tag{3.32}$$

$$I_{n,m}I^{\dagger}_{n,m} + \Gamma^{\dagger}_{n,m}\Gamma_{n,m} = I_{m+1}, \tag{3.33}$$

$$I^1_{n,m}(I^1_{n,m})^{\dagger} + (\Gamma^1_{n,m})^{\dagger}\Gamma^1_{n,m} = I_{m+1}. \tag{3.34}$$

*Remark* 3.9. The matrix $\Gamma_{n,m}$ has a zero in the entries $(i,j), i \geq j$, and has positive $(i,i{+}1)$ entries. Since $\Gamma_{n,m}\Gamma^{\dagger}_{n,m} = \Gamma_{n,m}U^{\dagger}_m U_m \Gamma^{\dagger}_{n,m}$, where $U_m$ is the $m{\times}m{+}1$ matrix given by

$$U_m = \begin{bmatrix} 0, & I_m \end{bmatrix}, \tag{3.35}$$

we see that $\Gamma_{n,m}U^{\dagger}_m$ is the upper Cholesky factorization of the right-hand side of (3.31). From this $\Gamma_{n,m}$ can be obtained once $\mathcal{K}_{n,m}$ is specified. The matrix $\Gamma^1_{n,m}$ has zeros in the entries $(i,j), i > j$, with positive $(i,i)$ entries. The matrix $I_{n,m}$ has the first row and column equal to zero except for a one in the $(1,1)$ entry.

The above recurrence formulas also give pointwise formulas for the recurrence coefficients. In order to obtain these formulas we define the $m \times m+1$ matrix $U^1_m$ as

$$U^1_m = \begin{bmatrix} I_m, & 0 \end{bmatrix}, \tag{3.36}$$

and the $(n+1)(m+1) \times (n+1)(m+1)$ matrix $P^{n,m}_{rl}$, which takes monomials in the lexicographical ordering to those in the reverse lexicographical ordering; i.e.,

$$P^{n,m}_{rl}[z^n w^m, z^n w^{m-1}, \dots, 1]^T = [w^m z^n, w^m z^{n-1}, \dots, 1]^T. \tag{3.37}$$

Analogous equations hold for the $n \times (n+1)$ matrices $\tilde{U}_n$ and $\tilde{U}^1_n$.

PROPOSITION 3.10. *Let*

$$\Phi_{n,m}(z,w) = \Phi^m_n(z) \begin{bmatrix} w^m \\ \vdots \\ 1 \end{bmatrix} \text{ and } \tilde{\Phi}_{n,m}(z,w) = \tilde{\Phi}^n_m(w) \begin{bmatrix} z^n \\ \vdots \\ 1 \end{bmatrix}, \tag{3.38}$$

*where*

$$\begin{aligned} \Phi^m_n(z) &= \Phi^m_{n,n} z^n + \Phi^m_{n,n-1} z^{n-1} + \cdots, \\ \tilde{\Phi}^n_m(w) &= \tilde{\Phi}^n_{m,m} w^m + \tilde{\Phi}^n_{m,m-1} w^{m-1} + \cdots; \end{aligned} \tag{3.39}$$

*then the following relations hold:*

$$\Gamma_{n,m} = \Phi_{n,n}^{m-1} U_m (\Phi_{n,n}^m)^{-1}, \tag{3.40}$$

$$\Gamma_{n,m}^1 = \Phi_{n,n}^{m-1} U_m^1 (\Phi_{n,n}^m)^{-1}, \tag{3.41}$$

$$\mathcal{K}_{n,m} = -\Gamma_{n,m} I_{n,m} \tilde{F}_{n,m}, \tag{3.42}$$

$$\mathcal{K}_{n,m}^1 = -\Gamma_{n,m}^1 \bar{I}_{n,m}^1 \bar{\tilde{F}}_{n,m}^1, \tag{3.43}$$

$$I_{n,m} = (\Phi_{n,n}^m{}^\dagger)^{-1} [I_{m+1}, 0, \dots, 0] C_{n,m}^{-1} P_{rl}^{n,mT} [I_{n+1}, 0, \dots, 0]^T (\tilde{\Phi}_{m,m}^n)^{-1}, \tag{3.44}$$

$$I_{n,m}^1 = (\Phi_{n,n}^m{}^T)^{-1} [0, \dots, 0, J_{m+1}] C_{n,m}^{-1} P_{rl}^{n,mT} [I_{n+1}, 0, \dots, 0]^T (\tilde{\Phi}_{m,m}^n)^{-1}, \tag{3.45}$$

*where* $\tilde{F}_{n,m} = \tilde{\Phi}_{m,m}^n U_n^T (\tilde{\Phi}_{m,m}^{n-1})^{-1}$, *and* $\tilde{F}_{n,m}^1 = \tilde{\Phi}_{m,m}^n (U_n^1)^T (\tilde{\Phi}_{m,m}^{n-1})^{-1}$.

*Proof.* Equation (3.41) follows by equating the coefficients of $z^n$ in (3.17) on the left. The same argument gives (3.41). To show (3.42) multiply (3.18) on the left by $\Gamma_{n,m}$ and then subtract the resulting equation from (3.16). Now equating the coefficients of $w^m$ gives the result. Equation (3.43) follows by taking the transpose of the reverse of (3.17), then multiplying (3.19) on the left by $\bar{\Gamma}_{n,m}^1$, and subtracting the resulting equations. Equating powers of $w^m$ then gives the result. Equation (3.44) follows by equating the highest powers of $w$ in (3.18), and (3.45) follows in a similar manner from (3.19) and the fact that $C_{n,m}$ is a doubly Toeplitz matrix. ∎

*Remark* 3.11. From (3.11) and Lemma 2.5 we see that $(\Phi_{n,n}^m)^\dagger$ is the lower Cholesky factor of $[I_{m+1}, 0, \dots, 0] C_{n,m} [I_{m+1}, 0, \dots, 0]^T$ and a similar relation holds between $\tilde{\Phi}_{m,m}^n$and $\tilde{C}_{n,m}$. Thus (3.44) and (3.45) give the relation between $I_{n,m}$ and $I_{n,m}^1$ and the Fourier coefficients of $\mathcal{L}_{N,M}$. These coupled with (3.42) and (3.43) relate the Fourier coefficients of $\mathcal{L}_{N,M}$ to $\mathcal{K}_{n,m}$ and $\mathcal{K}_{n,m}^1$.

We now give relations between the coefficients in the recurrence formulas at one level in terms of those at previous levels.

LEMMA 3.12 (relations for $\mathcal{K}_{n,m}$). *For* $0 < n, m$,

$$\Gamma_{n,m-1}^1 \mathcal{K}_{n,m} = \mathcal{K}_{n,m-1} (\tilde{A}_{n-1,m}^{-1})^\dagger - \mathcal{K}_{n,m-1}^1 \hat{\tilde{E}}_{n-1,m}^\dagger (\tilde{A}_{n-1,m}^{-1})^\dagger, \tag{3.46}$$

$$\mathcal{K}_{n,m} (\tilde{\Gamma}_{n-1,m}^1)^\dagger = A_{n,m-1}^{-1} \mathcal{K}_{n-1,m} - A_{n,m-1}^{-1} \hat{E}_{n,m-1} \bar{\mathcal{K}}_{n-1,m}^1. \tag{3.47}$$

*Proof.* To show (3.46) multiply (3.22) on the left by $\Gamma_{n,m-1}^1$ and then use (3.17) with $m$ reduced by one to obtain

$$\Gamma_{n,m-1}^1 \mathcal{K}_{n,m} = \langle w\Phi_{n,m-2}, \tilde{\Phi}_{n-1,m} \rangle.$$

Eliminating $\tilde{\Phi}_{n-1,m}$ using ($\tilde{3}.14$) and then applying (3.22) and (3.24) gives (3.46). Equation (3.47) follows in an analogous manner. ∎

LEMMA 3.13 (relations for $\mathcal{K}_{n,m}^1$). *For* $0 < n, m$,

$$\Gamma_{n,m-1} \mathcal{K}_{n,m}^1 = \mathcal{K}_{n,m-1}^1 (\tilde{A}_{n-1,m}^{-1})^T - \mathcal{K}_{n,m-1} (\hat{\tilde{E}}_{n-1,m})^T (\tilde{A}_{n-1,m}^{-1})^T, \tag{3.48}$$

$$\mathcal{K}_{n,m}^1 (\tilde{\Gamma}_{n-1,m})^T = A_{n,m-1}^{-1} \mathcal{K}_{n-1,m}^1 - A_{n,m-1}^{-1} \hat{E}_{n,m-1} \bar{\mathcal{K}}_{n-1,m}. \tag{3.49}$$

*Proof.* To show (3.48) multiply (3.24) on the left by $\Gamma_{n,m-1}$ and then use (3.16) to obtain

$$\Gamma_{n,m-1} \mathcal{K}_{n,m}^1 = \langle w\Phi_{n,m-2}, \overleftarrow{\tilde{\Phi}}_{n-1,m}^T \rangle.$$

Now use ($\tilde{3}.14$) with $n$ reduced by one and then (3.24) and (3.22) to find (3.48). Equation (3.49) follows in a similar manner. $\square$

LEMMA 3.14 (relations for $\hat{E}_{n,m}$). *For* $0 < n, m$,

$$\Gamma_{n-1,m}\hat{E}_{n,m} = A_{n,m-1}\mathcal{K}_{n,m}(I^1_{n-1,m})^\dagger + \hat{E}_{n,m-1}\bar{\Gamma}^1_{n-1,m}, \tag{3.50}$$

$$\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = I_{n-1,m}(\mathcal{K}^1_{n,m})^T A^T_{n,m-1} + \Gamma^\dagger_{n-1,m}\hat{E}_{n,m-1}. \tag{3.51}$$

*Proof.* To establish (3.50) multiply (3.20) on the left by $\Gamma_{n-1,m}$ and then use (3.16) to obtain

$$\Gamma_{n-1,m}\hat{E}_{n,m} = \langle z\Phi_{n-1,m-1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle.$$

With the use of (3.14) to eliminate $z\Phi_{n-1,m-1}$, we find

$$\Gamma_{n-1,m}\hat{E}_{n,m} = A_{n,m-1}\langle\Phi_{n,m-1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle + \hat{E}_{n,m-1}\langle\overleftarrow{\Phi}^T_{n-1,m-1}, \overleftarrow{\Phi}^T_{n-1,m}\rangle.$$

The second inner product on the right-hand side of the above equation evaluates to $\bar{\Gamma}^1_{n-1,m}$, while the first may be evaluated using (3.19) followed by (3.22) to give the claimed equation. To obtain (3.51) multiply (3.20) on the right by $(\Gamma^1_{n-1,m})^T$ and then use (3.17) to get

$$\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = \langle z\Phi_{n-1,m}, \overleftarrow{\Phi}^T_{n-1,m-1}\rangle.$$

Using (3.14) to eliminate $\overleftarrow{\Phi}^T_{n-1,m-1}$ yields

$$\hat{E}_{n,m}(\Gamma^1_{n-1,m})^T = \langle z\Phi_{n-1,m}, \overleftarrow{\Phi}^T_{n,m-1}\rangle A^T_{n,m-1} + \langle\Phi_{n-1,m}, \Phi_{n-1,m-1}\rangle\hat{E}^T_{n,m-1}.$$

Equation (3.23) can be used to evaluate the second inner product on the right-hand side of the above equation, while the reverse transpose of (3.17) and (3.26) can be used to obtain the first inner product. $\square$

LEMMA 3.15 (relation for $\Gamma^1_{n,m}$). *For* $0 < n, m$,

$$\begin{aligned}\Gamma^1_{n,m}\Gamma^\dagger_{n,m} = {} & I_{n,m-1}\tilde{\hat{E}}_{n,m}(I^1_{n,m-1})^T + \Gamma^\dagger_{n,m-1}\Gamma^1_{n,m-1} \\ & + \mathcal{K}^1_{n,m}\bar{\tilde{A}}^{-1}_{n-1,m}\tilde{\hat{E}}^\dagger_{n-1,m}\tilde{A}_{n-1,m}\mathcal{K}^\dagger_{n,m}.\end{aligned} \tag{3.52}$$

*Proof.* To show (3.52) multiply (3.25) on the left by $\Gamma^\dagger_{n,m}$ and use (3.16) to find

$$\Gamma^1_{n,m}\Gamma^\dagger_{n,m} = \langle w\Phi_{n,m-1}, \Phi_{n,m-1}\rangle - \langle w\Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle\mathcal{K}^\dagger_{n,m}. \tag{3.53}$$

Eliminating $w\Phi_{n,m-1}$ in the second term on the right-hand side of the above equation using (3.17) and then applying ($\tilde{3}.15$) gives the third term on the right-hand side of (3.52). In the first term on the right-hand side of the above equation, substitute the reverse transpose of (3.19) to find

$$\langle w\Phi_{n,m-1}, \Phi_{n,m-1}\rangle = \langle w\Phi_{n,m-1}, \overleftarrow{\tilde{\Phi}}^T_{n,m-1}\rangle(I^1_{n,m-1})^T + \Gamma^\dagger_{n,m-1}\Gamma^1_{n,m-1},$$

where (3.23) has been used to obtain the second term on the right-hand side of the above equation. The result may now be obtained by applying (3.18) for $\Phi_{n,m-1}$ and then using ($\tilde{3}.20$). $\square$

LEMMA 3.16 (relations for $I_{n,m}$ and $I^1_{n,m}$).

$$I_{n,m}\tilde{\Gamma}^\dagger_{n,m} = -\Gamma^\dagger_{n,m}\mathcal{K}_{n,m}, \tag{3.54}$$

$$I^1_{n,m} = -\bar{A}^{-1}_{n,m}\hat{E}^\dagger_{n,m}A_{n,m}I_{n,m} + A^T_{n,m}I^1_{n-1,m}\tilde{\Gamma}_{n,m}, \quad 0 < n. \tag{3.55}$$

*Proof.* Equation (3.54) follows by multiplying (3.26) on the right by $\tilde{\Gamma}^\dagger_{n,m}$ and then using ($\tilde{3}.16$) and (3.23). In (3.27) use ($\tilde{3}.18$) and (3.28) to find

$$I^1_{n,m} = \langle \overleftarrow{\Phi}^T_{n,m}, \Phi_{n,m}\rangle I_{n,m} + \langle \overleftarrow{\Phi}^T_{n,m}, \tilde{\Phi}_{n-1,m}\rangle \tilde{\Gamma}_{n,m}.$$

The first inner product on the right-hand side may be evaluated using (3.15). To evaluate the second inner product, eliminate $\overleftarrow{\Phi}^T_{n,m}$ using the reverse transpose of (3.15) and then use (3.27) to obtain the claimed equation. ∎

**4. Christoffel–Darboux formulas.** The Christoffel–Darboux formula plays an important role in the theory of one variable scalar and matrix orthogonal polynomials. Using the connection between two variable orthogonal polynomials and matrix orthogonal polynomials, we derive two variable analogs of the Christoffel–Darboux formula. These will play an important role in the theory of two variable stable polynomials discussed later.

LEMMA 4.1. *Given* $\{\Phi_{n,m}\}$ *and* $\{\tilde{\Phi}_{n,m}\}$,

$$\overleftarrow{\Phi}_{n,m}(z,w)\overleftarrow{\Phi}^\dagger_{n,m}(z_1,w_1) - \bar{z}_1 z\Phi^T_{n,m}(z,w)\Phi^\dagger_{n,m}(z_1,w_1)^T \tag{4.1a}$$

$$= (1-\bar{z}_1 z)\Phi_{n,m}(z,w)^T\Phi^\dagger_{n,m}(z_1,w_1)^T$$

$$+ \overleftarrow{\Phi}_{n-1,m}(z,w)\overleftarrow{\Phi}^\dagger_{n-1,m}(z_1,w_1) - \bar{z}_1 z\Phi^T_{n-1,m}(z,w)\Phi^\dagger_{n-1,m}(z_1,w_1)^T \tag{4.1b}$$

$$= (1-\bar{z}_1 z)\tilde{\Phi}_{n,m}(z,w)^T\tilde{\Phi}^\dagger_{n,m}(z_1,w_1)^T$$

$$+ \overleftarrow{\Phi}_{n,m-1}(z,w)\overleftarrow{\Phi}_{n,m-1}(z_1,w_1)^T - \bar{z}_1 z\Phi_{n,m-1}(z,w)^T\Phi^\dagger_{n,m-1}(z_1,w_1)^T. \tag{4.1c}$$

*Proof.* The equality (4.1a)=(4.1b) follows by subtracting (2.24) with $n$ reduced by one from the original equation and then using Lemma 3.3. The equality (4.1a)=(4.1c) can be obtained in the following manner. Let

$$Z_{n,m}(z,w) = [1, w, \ldots, w^m][I_{m+1}, zI_{m+1}, \ldots, z^n I_{m+1}],$$

and let $\tilde{Z}_{n,m}(z,w)$ be given by a similar formula with the roles of $z$ and $w$ and $n$ and $m$ interchanged. Then from Lemma 2.5, (2.24), and (3.11) we find

$$\frac{\overleftarrow{\Phi}_{n,m}(z,w)\overleftarrow{\Phi}^\dagger_{n,m}(z_1,w_1) - \bar{z}_1 z\Phi^T_{n,m}(z,w)\Phi^\dagger_{n,m}(z_1,w_1)^T}{1-\bar{z}_1 z}$$
$$= Z_{n,m}(z,w)C^{-1}_{n,m}Z_{n,m}(z_1,w_1)^\dagger = \tilde{Z}_{n,m}(z,w)\tilde{C}^{-1}_{n,m}\tilde{Z}_{n,m}(z_1,w_1)^\dagger$$
$$= \tilde{\Phi}^T_{n,m}(z,w)\tilde{\Phi}^\dagger_{n,m}(z_1,w_1)^T + \tilde{Z}_{n,m-1}(z,w)\tilde{C}^{-1}_{n,m-1}\tilde{Z}_{n,m-1}(z_1,w_1)^\dagger.$$

Switching back to the lexicographical ordering in the second term in the last equation and then using Lemma 2.5 yields the result. ∎

As an immediate application of the above lemma we obtain the following.

THEOREM 4.2 (Christoffel–Darboux formula). *Given* $\{\Phi_{n,m}\}$ *and* $\{\tilde{\Phi}_{n,m}\}$,

$$\begin{aligned}
&\frac{\overleftarrow{\Phi}_{n,m}(z,w)\overleftarrow{\Phi}^{\dagger}_{n,m}(z_1,w_1) - \bar{z}_1 z \Phi^T_{n,m}(z,w)\Phi^{\dagger}_{n,m}(z_1,w_1)^T}{1-\bar{z}_1 z} \\
&\qquad = \sum_{k=0}^{n} \Phi^T_{k,m}(z,w)\Phi^{\dagger}_{k,m}(z_1,w_1)^T \\
&\qquad = \sum_{j=0}^{m} \tilde{\Phi}^T_{n,j}(z,w)\tilde{\Phi}^{\dagger}_{n,j}(z_1,w_1)^T.
\end{aligned}$$

In the first line of the above equation, the terms $\bar{z}_1 z$ may be replaced by $\bar{w}_1 w$ if we switch to $\tilde{\Phi}_{n,m}$.

An interesting variant of (4.1c) is the following.

LEMMA 4.3.

$$\begin{aligned}
&\Phi_{n,m}(z,w)^T\Phi^{\dagger}_{n,m}(z_1,w_1)^T - \Phi^T_{n,m-1}(z,w)\Phi^{\dagger}_{n,m-1}(z_1,w_1)^T \\
&= \tilde{\Phi}_{n,m}(z,w)^T\tilde{\Phi}^{\dagger}_{n,m}(z_1,w_1)^T - \tilde{\Phi}^T_{n-1,m}(z,w)\tilde{\Phi}^{\dagger}_{n-1,m}(z_1,w_1)^T.
\end{aligned} \tag{4.2}$$

*Proof.* Equating the sums in the above theorem yields

$$\Phi_{n,m}(z,w)^T\Phi^{\dagger}_{n,m}(z_1,w_1)^T - \sum_{j=0}^{m-1} \tilde{\Phi}^T_{n,j}(z,w)\tilde{\Phi}^{\dagger}_{n,j}(z_1,w_1)^T \tag{4.3}$$

$$= \tilde{\Phi}_{n,m}(z,w)^T\tilde{\Phi}^{\dagger}_{n,m}(z_1,w_1)^T - \sum_{j=0}^{n-1} \Phi^T_{j,m}(z,w)\Phi^{\dagger}_{j,m}(z_1,w_1)^T. \tag{4.4}$$

Switching to the lexicographical ordering in the sum on the left-hand side of the above equation and reverse lexicographical ordering in the sum on the right-hand side, extracting the highest terms, and then using the Christoffel–Darboux formula to eliminate the remaining sums gives the result. ◻

*Remark* 4.4. The above equations can be derived from the recurrence formulas in the previous sections. However, the derivation of (4.1c) is rather tedious.

**5. Algorithm.** In this section we use the relations developed earlier to provide an algorithm that allows us to compute the coefficients in the recurrence formula at higher levels in terms of those at lower levels plus some indeterminates that are equivalent to the moments. This will allow us to construct positive definite doubly Toeplitz matrices. As a byproduct we construct the orthogonal polynomials associated with these matrices. More precisely, at each level we use the new indeterminates and the coefficients on the levels $(n, m-1)$ and $(n-1, m)$ to construct $\mathcal{K}_{n,m}$ and $\mathcal{K}^1_{n,m}$. With this we can construct the other coefficients needed to proceed to the next level. The $\hat{E}_{n,m}$ are closely related to the matrix recurrence coefficients needed to compute $C_{n,m}$. Furthermore $\Phi_{n,m}$ and $\tilde{\Phi}_{n,m}$ can also be computed. In order to construct the above matrices we will have need of the $m \times (m+1)$ matrices $U_m$ and $U^1_m$ given by (3.35) and (3.36), respectively, and the vector $e^m_1 \in \mathbb{R}^m$, which is the vector with one in the first entry and zeros everywhere else. From the definition of $\mathcal{K}^1_{n,m}$ we see that

$$\begin{aligned}\mathcal{K}^1_{n,m} &= \langle w\Phi_{n,m-1}, \overleftarrow{\tilde{\Phi}}^T_{n-1,m}\rangle \\ &= \left\langle w\begin{bmatrix}\phi^{m-1}_{n,m-1}\\ \vdots \\ \phi^0_{n,m-1}\end{bmatrix}, \begin{bmatrix}\overleftarrow{\tilde{\phi}}^{\,n-1}_{n-1,m}\\ \vdots \\ z^n\overleftarrow{\tilde{\phi}}^{\,0}_{n-1,m}\end{bmatrix}\right\rangle \\ &= c_{-n,-m}d_{n,m}e^m_1(e^n_1)^T + R_{n,m},\end{aligned} \tag{5.1}$$

where $R_{n,m}$ is an $m\times n$ matrix containing moments $c_{i,j}, \{|i| \le n, |j| \le m\}\backslash\{(\pm n, \pm m)\}$. Likewise, with the help of (3.38) and its tilde counterpart we find

$$\begin{aligned}\mathcal{K}_{n,m} &= \langle \Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle \\ &= c_{-n,m}\Phi^{n,m-1}_{n,m-1}\begin{bmatrix}0\\ \vdots \\ 1\end{bmatrix}\left(\tilde{\Phi}^{m,n-1}_{m,n-1}\begin{bmatrix}0\\ \vdots \\ 1\end{bmatrix}\right)^\dagger + \hat{R}_{n,m},\end{aligned} \tag{5.2}$$

where $\hat{R}_{n,m}$ contains only moments from lower levels.

We proceed as follows: At level $(0,0)$ we have the parameter $u_{0,0} > 0$, which corresponds to $c_{0,0}$. The polynomials $\Phi_{0,0}$ and $\tilde{\Phi}_{0,0}$ are chosen as $\frac{1}{\sqrt{u_{0,0}}}$. From (3.26) and (3.27) we see that $I_{0,0} = 1 = I^1_{0,0}$. At level $(i,0)$ there is one new parameter $u_{i,0}$, which can be taken to correspond with the one-dimensional recurrence coefficient i.e., $u_{i,0} = \alpha_i = \hat{E}_{i,0}$, corresponding to the $(i,0)$ level and must be less than one in magnitude. From (3.30) and the normalization chosen for the polynomials, $A_{i,0} = \sqrt{1-|\hat{E}_{i,0}|^2}$. This allows us to compute $\Phi_{i,0}$, and $\overleftarrow{\Phi}_{i,0}$. The sizes of the matrices given in (3.22), (3.23), (3.24), and (3.25) show that

$$\mathcal{K}_{i,0} = \tilde{\mathcal{K}}_{i,0} = \Gamma_{i,0} = \mathcal{K}^1_{i,0} = \tilde{\mathcal{K}}^1_{i,0} = \Gamma^1_{i,0} = 0,$$

where (3.28) and (3.29) have also been used. Furthermore (3.33) and Remark 3.9 imply that $I_{i,0} = (e^{i+1}_1)^T = \tilde{I}^\dagger_{i,0}$ where (3.28) has been used. Equation ($\tilde{3}$.31) implies that $\tilde{\Gamma}_{i,0} = U_i$, while (3.55) and (3.29) allow us to compute

$$I^1_{i,0} = (\tilde{I}^1_{i,0})^T = -\begin{bmatrix}\hat{E}^\dagger_{i,0} & A_{i,0}\hat{E}^\dagger_{i-1,0} & \dots & -\prod_{j=1}^i A_{j,0}\end{bmatrix}. \tag{5.3}$$

$\tilde{\Phi}_{i,0}$ can now be computed from ($\tilde{3}$.18).

At level $(0,j)$ there is one new parameter $u_{0,j}$, which as above can be taken to correspond with the one-dimensional recurrence coefficient, i.e., $u_{0,j} = \alpha_j = \tilde{\hat{E}}_{0,j}$, corresponding to the $(0,j)$ level and must be less than one in magnitude. The analysis for the $(i,0)$ level can be carried over with the roles of the lexicographical and reverse lexicographical orderings interchanged. Thus from ($\tilde{3}$.30) and the normalization chosen for the polynomials, $\tilde{A}_{0,j} = \sqrt{1-|\tilde{\hat{E}}_{0,j}|^2}$, which allows us to compute $\tilde{\Phi}_{0,j}$. Again

$$\tilde{\mathcal{K}}_{0,j} = \mathcal{K}_{0,j} = \tilde{\Gamma}_{0,j} = \tilde{\mathcal{K}}^1_{0,j} = \mathcal{K}^1_{0,j} = \tilde{\Gamma}^1_{0,j} = 0.$$

Likewise $\tilde{I}_{0,j} = (e^{j+1}_1)^T = I^\dagger_{0,j}$ and $\Gamma_{0,j} = U_j$. Equations ($\tilde{3}$.55) and (3.29) allow us to compute $\tilde{I}^1_{0,j}$ as above with $i$ and $j$ interchanged as well as the orderings. Equation (3.18) now allows us to compute $\Phi_{0,j}$.

At level $(n, m)$, with $n, m > 0$, there are two new parameters $u_{n,m}$ and $u_{-n,m}$ since $u_{-n,-m} = \overline{u}_{n,m}$ and $u_{n,-m} = \overline{u}_{-n,m}$. These along with the coefficients on the $(n-1, m)$ and $(n, m-1)$ level will be used to compute $\mathcal{K}_{n,m}$ and $\mathcal{K}^1_{n,m}$. This will be sufficient to compute the remaining coefficients on level $(n, m)$. We begin with the following.

*Computation of* $\mathcal{K}_{n,m}$. If $n = 1, m = 1$, then (3.22) shows that $\mathcal{K}_{1,1}$ is a scalar which we choose as $\bar{u}_{1,-1}$. If $m > 1$, we see from (3.41) and (3.36) that

$$\Gamma^1_{n,m-1}\Phi^{m-1}_{n,n}e^m_m = 0,$$

where $e^m_m$ is the $m$-dimensional vector with zeros in all its entries except the last, which is one. Since $\Phi^{m-1}_{n,n} = A^{-1}_{n,m-1}\dots A^{-1}_{1,m-1}\Phi^{m-1}_{0,0}$ is an upper triangular invertible matrix, we find

$$\Gamma^1_{n,m-1}\Phi^{m-1}_{n,n}((U^1_{m-1})^TU^1_{m-1} + e^m_m(e^m_m)^T) = \Gamma^1_{n,m-1}\Phi^{m-1}_{n,n}(U^1_{m-1})^TU^1_{m-1},$$

and from (3.41) $\Gamma^1_{n,m-1}\Phi^{m-1}_{n,n}(U^1_{m-1})^T = \Phi^{m-2}_{n,n}$. Thus (3.46) can be written as

$$\begin{aligned}
&U^1_{m-1}(\Phi^{m-1}_{n,n})^{-1}\mathcal{K}_{n.m}((\tilde{\Phi}^{n-1}_{m,m})^\dagger)^{-1}\\
&= (\Phi^{m-2}_{n,n})^{-1}(\mathcal{K}_{n,m-1}(\tilde{A}^{-1}_{n-1,m})^\dagger - \mathcal{K}^1_{n,m-1}\hat{\tilde{E}}^\dagger_{n-1,m}(\tilde{A}^{-1}_{n-1,m})^\dagger)((\tilde{\Phi}^{n-1}_{m,m})^\dagger)^{-1}\\
&= (\Phi^{m-2}_{n,n})^{-1}(\mathcal{K}_{n,m-1} - \mathcal{K}^1_{n,m-1}\hat{\tilde{E}}^\dagger_{n-1,m})((\tilde{\Phi}^{n-1}_{m-1,m-1})^\dagger)^{-1}\\
&= H_{n,m-1}.
\end{aligned}$$

In the last equality we have used the fact that $\tilde{A}_{n-1,m}\tilde{\Phi}^{n-1}_{m,m} = \tilde{\Phi}^{n-1}_{m-1,m-1}$. Likewise,

$$\begin{aligned}
&(\Phi^{m-1}_{n,n})^{-1}\mathcal{K}_{n,m}((\tilde{\Phi}^{n-1}_{m,m})^{-1})^\dagger(U^1_{n-1})^T\\
&= (\Phi^{m-1}_{n-1,n-1})^{-1}(\mathcal{K}_{n-1,m} - \hat{E}_{n,m-1}\bar{\mathcal{K}}^1_{n-1,m})((\tilde{\Phi}^{n-2}_{m,m})^\dagger)^{-1}\\
&= \tilde{H}_{n-1,m}.
\end{aligned}$$

If

$$(e^m_m)^T(\Phi^{m-1}_{n,n})^{-1}\mathcal{K}_{n,m}((\tilde{\Phi}^{n-1}_{m,m})^\dagger)^{-1}e^n_n = \bar{u}_{n,-m}, \tag{5.4}$$

then $\mathcal{K}_{n,m}$ can be solved for as

(5.5)

$$\begin{aligned}
&\mathcal{K}_{n,m}\\
&= \Phi^{m-1}_{n,n}\left(u_{-n,m}e^m_m(e^n_n)^T + (U^1_{m-1})^TH_{n,m-1} + e^m_m(e^m_m)^T\tilde{H}_{n-1,m}(U^1_{n-1})\right)(\tilde{\Phi}^{n-1}_{m,m})^\dagger.
\end{aligned}$$

A necessary condition in order to be able to continue is that $||\mathcal{K}_{n,m}|| < 1$.

*Computation of* $\Gamma_{n,m}$. Since $\mathcal{K}_{n,m}$ is presumed to be a contraction, Remark 3.9 shows that $\Gamma_{n,m}$ and $\tilde{\Gamma}_{n,m}$ may be computed from the upper Cholesky factor of $I - \mathcal{K}_{n,m}\mathcal{K}^\dagger_{n,m}$ and $I - \mathcal{K}^\dagger_{n,m}\mathcal{K}_{n,m}$, respectively.

*Computation of* $\mathcal{K}^1_{n,m}$. In $\mathcal{K}^1_{n,m}$ we see from (5.1) that the only new entry is $(\mathcal{K}_{n,m})_{1,1}$. If $n = 1, m = 1$, set $\mathcal{K}_{1,1} = \bar{u}_{1,1}$. If $m > 1$, we will show that all of the rows except the first can be obtained from (3.48). The structure of $\Gamma_{n,m-1}$ implies that $\Gamma_{n,m-1}e^m_1 = 0$ so that

$$\Gamma_{n,m-1} = \Gamma_{n,m-1}(U^T_{m-1}U_{m-1} + e^m_1(e^m_1)^T) = \Gamma_{n,m-1}U^T_{m-1}U_{m-1}.$$

But $\Gamma_{n,m-1}U_{m-1}^T$ is an invertible matrix, which allows us to rewrite (3.48) as follows:

$$U_{m-1}\mathcal{K}_{n,m}^1 = (\Gamma_{n,m-1}U_{m-1}^T)^{-1}(\mathcal{K}_{n,m-1}^1(\tilde{A}_{n-1,m}^{-1})^T - \mathcal{K}_{n,m-1}\hat{\tilde{E}}_{n-1,m}^T(\tilde{A}_{n-1,m}^{-1})^T).$$

This gives all of the entries in $\mathcal{K}_{n,m}^1$ except the first row.

Similarly, if $n > 1$ we can write

$$\tilde{\Gamma}_{n-1,m} = \tilde{\Gamma}_{n-1,m}(U_{n-1}^T U_{n-1} + e_1^n(e_1^n)^T) = \tilde{\Gamma}_{n-1,m}U_{n-1}^T U_{n-1},$$

i.e., $\tilde{\Gamma}_{n-1,m}^T = U_{n-1}^T U_{n-1}\tilde{\Gamma}_{n-1,m}^T$, and (3.49) can be rewritten as

$$\mathcal{K}_{n,m}^1 U_{n-1}^T = (A_{n,m-1}^{-1}\mathcal{K}_{n-1,m}^1 - A_{n,m-1}^{-1}\hat{E}_{n,m-1}\bar{\mathcal{K}}_{n-1,m})(U_{n-1}\tilde{\Gamma}_{n-1,m}^T)^{-1}. \tag{5.6}$$

Thus the $m \times (n-1)$ matrix $\mathcal{K}_{n,m}^1 U_{n-1}^T$, which is obtained from $\mathcal{K}_{n,m}^1$ by deleting the first column, is known from the previous levels. This allows us to compute all entries in the first row of $\mathcal{K}_{n,m}^1$ except $(\mathcal{K}_{n,m}^1)_{1,1}$, and we put

$$(\mathcal{K}_{n,m}^1)_{1,1} = \bar{u}_{n,m}. \tag{5.7}$$

A necessary condition on the parameters in order to be able to continue is that $||\mathcal{K}_{n,m}^1|| < 1$, which implies that $|u_{n,m}| < 1$.

*Computation of* $\hat{E}_{n,m}$. We begin by taking the transpose of (3.51), using the fact that $\hat{E}_{n,m}$ is symmetric, and then multiplying on the left by the matrix $e_1^{m+1}(e_1^m)^T$. Now multiply (3.50) by $U_m^T$ and add the resulting equations. If $\hat{\Gamma}_{n-1,m}$ is the $(m+1) \times (m+1)$ matrix obtained by stacking the first row of $\Gamma_{n-1,m}^1$ on $\Gamma_{n-1,m}$, we find

$$\begin{aligned}\hat{\Gamma}_{n-1,m}\hat{E}_{n,m} =& U_m^T(A_{n,m-1}\mathcal{K}_{n,m}(I_{n-1,m}^1)^\dagger + \hat{E}_{n,m-1}\bar{\Gamma}_{n-1,m}^1)\\ &+ e_1^{m+1}(e_1^m)^T(A_{n,m-1}\mathcal{K}_{n,m}^1 I_{n-1,m}^T + \hat{E}_{n,m-1}\bar{\Gamma}_{n-1,m}).\end{aligned} \tag{5.8}$$

From the structure of $\Gamma^1$ and $\Gamma$ we see that $\hat{\Gamma}_{n-1,m}$ is an upper triangular matrix with positive diagonal entries and is hence invertible. Thus $\hat{E}_{n,m}$ can be computed from the above equation. If $||\hat{E}_{n,m}|| < 1$, then $\hat{\tilde{E}}_{n,m}$ may be computed from ($\tilde{3}$.50) and ($\tilde{3}$.51). We may also compute $A_{n,m}$, $\tilde{A}_{n,m}$, and the polynomials $\Phi_{n,m}$ and $\tilde{\Phi}_{n,m}$. While the condition that $\hat{E}_{n,m}$ be a contraction is necessary and sufficient to be able to continue, it is not optimal in the sense that it does not take into account the redundancy inherent in the equations giving $\hat{E}_{n,m}$. This will be taken into account in the computation of $\Gamma_{n,m}^1$.

*Computation of* $\Gamma_{n,m}^1$. As above we see that (3.52) gives

$$\begin{aligned}\Gamma_{n,m}^1 U_m^T =& (I_{n,m-1}\hat{\tilde{E}}_{n,m}(I_{n,m-1}^1)^T + \Gamma_{n,m-1}^\dagger\Gamma_{n,m-1}^1\\ &+ \mathcal{K}_{n,m}^1\bar{\tilde{A}}_{n-1,m}^{-1}\hat{\tilde{E}}_{n-1,m}^\dagger\tilde{A}_{n-1,m}\mathcal{K}_{n,m}^\dagger)(U_m\Gamma_{n,m}^\dagger)^{-1},\end{aligned} \tag{5.9}$$

which allows the computation of all of the entries of $\Gamma_{n,m}^1$ except the $(1,1)$ entry. Since $(e_1^m)^T I_{n,m-1} = (e_1^m)^T$, $(e_1^m)^T\Gamma_{n,m-1}^\dagger = 0$, and likewise, with $I_{n,m-1}$ and $\Gamma_{n,m-1}^\dagger$ replaced by $\tilde{I}_{n,m}$ and $\tilde{\Gamma}_{n,m-1}^T$, respectively, we find with the help of (5.8)

$$(e_1^m)^T\Gamma_{n,m}^1 U_m^T = (e_1^m)^T H_{n,m}^2 + (e_1^m)^T\mathcal{K}_{n,m}^1 H_{n,m}^1, \tag{5.10}$$

where

$$H^2_{n,m} = I_{n,m-1}((I^1_{n,m-1})^\dagger \bar{\mathcal{K}}_{n,m}\tilde{A}^T_{n-1,m} + (\tilde{\Gamma}^1_{n,m-1})^\dagger \hat{\tilde{E}}_{n-1,m}) \\ \times U_n(\hat{\tilde{\Gamma}}^T_{n,m-1})^{-1}(I^1_{n,m-1})^T(U_m\Gamma^\dagger_{n,m})^{-1}, \tag{5.11}$$

and

$$H^1_{n,m} = \tilde{A}^T_{n-1,m}e^n_1(e^{n+1}_1)^T(\hat{\tilde{\Gamma}}^T_{n,m-1})^{-1}(I^1_{n,m-1})^T(U_m\Gamma^\dagger_{n,m})^{-1} \\ + \bar{\tilde{A}}^{-1}_{n-1,m}\hat{\tilde{E}}^\dagger_{n-1,m}\tilde{A}_{n-1,m}\mathcal{K}^\dagger_{n,m}(U_m\Gamma^\dagger_{n,m})^{-1}. \tag{5.12}$$

Thus the first entry $\Gamma^1$ can be computed using the first row of (5.10) and (3.32), which gives

$$|(\Gamma^1_{n,m})_{(1,1)}|^2 = 1 - (e^m_1)^T H^3_{n,m}(e^m_1), \tag{5.13}$$

where

$$H^3_{n,m} = (H^2_{n,m} + \mathcal{K}^1_{n,m}H^1_{n,m})(H^2_{n,m} + \mathcal{K}^1_{n,m}H^1_{n,m})^\dagger + \mathcal{K}^1_{n,m}(\mathcal{K}^1_{n,m})^\dagger. \tag{5.14}$$

*Computation of the remaining coefficients.* Using the arguments above we see that the relevant part of $I_{n,m}$ may be computed from (3.54) and $I^1_{n,m}$ may be computed from (3.55). The matrix $\tilde{\Gamma}^1_{n,m}$ can be computed in the same manner as $\Gamma^1_{n,m}$.

**6. Construction of a positive linear functional.** The above algorithm allows us to find a linear functional given the coefficients in the recurrence formulas. More precisely, it is as follows.

THEOREM 6.1. *Given parameters $u_{i,j} \in \mathbb{C}$, $0 \le i \le n, |j| \le m, u_{-i,j} = \bar{u}_{i,-j}$ we construct*

- *scalars $\hat{E}_{i,0}$, $i = 1, \dots, n$, and $\hat{\tilde{E}}_{0,j}$, $j = 1, \dots, m$;*
- *$i \times j$ matrices $\mathcal{K}_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$; and*
- *$i \times j$ numbers $(e^j_1)^T H^3_{i,j} e^j_1$, $i = 1, \dots, n$, $j = 1, \dots, m$.*

*If*

$$u_{0,0} > 0,\ |\hat{E}_{i,0}| < 1, |\hat{\tilde{E}}_{0,j}| < 1,\ ||\mathcal{K}_{i,j}|| < 1,\ \text{and } {e^j_1}^T H^3_{i,j} e^j_1 < 1, \tag{6.1}$$

*then there exists a positive linear functional $\mathcal{L}$ on $\prod^{n,m}$ such that*

$$\mathcal{L}(\Phi_{i,m}\Phi^\dagger_{j,m}) = \delta_{i,j}I_{m+1} \text{ and } \mathcal{L}(\tilde{\Phi}_{n,i}\tilde{\Phi}^\dagger_{n,j}) = \delta_{i,j}I_{n+1}. \tag{6.2}$$

*The conditions* (6.1) *are also necessary.*

*Proof.* We construct the linear functional by induction. First, if $n = m = 0$ we set

$$\mathcal{L}(1) = u_{0,0} \text{ and } \Phi_{0,0} = \tilde{\Phi}_{0,0} = \frac{1}{\sqrt{u_{0,0}}},$$

and thus $\mathcal{L}(\Phi_{0,0}\Phi^\dagger_{0,0}) = \mathcal{L}(\tilde{\Phi}_{0,0}\tilde{\Phi}^\dagger_{0,0}) = 1$.

If $m = 0$, we construct $A_{i,0} = \sqrt{1 - |\hat{E}_{i,0}|^2}$, where $\hat{E}_{i,0} = u_{i,0}$. The polynomials $\Phi_{i,0}, i = 0, \dots n$, are now computed using (3.14), and then we define

$$\mathcal{L}(\Phi_{i,0}\Phi^\dagger_{j,0}) = \delta_{i,j}.$$

This gives a well-defined positive linear functional on $z^j$ for $|j| \le n$.

Likewise, if $n=0$, we construct $\tilde{\Phi}_{0,k}$ using ($\tilde{3}.14$) and define

$$\mathcal{L}(\tilde{\Phi}_{0,i}\tilde{\Phi}_{0,j}^{\dagger}) = \delta_{i,j},$$

which gives the linear functional on $w^j$ for $|j| \le m$. Thus (6.2) will hold if $m=0$ or $n=0$.

Assume now that the functional $\mathcal{L}$ is well defined and positive for all levels $0 \le i \le n-1$, $0 \le j \le m$ and $0 \le i \le n$, $0 \le j \le m-1$ before $(n,m)$. To ease notation we will use the bracket given in (3.8) with $\mathcal{L}_{N,M}$ replaced by $\mathcal{L}$. We first extend $\mathcal{L}$ so that

$$\langle \Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle = \mathcal{K}_{n,m}. \tag{6.3}$$

To check that the above equation is consistent with how $\mathcal{L}$ is defined on the previous levels, note that from (3.46)

$$\langle \Gamma^1_{n,m-1}\Phi_{n,m-1}, \tilde{\Phi}_{n-1,m}\rangle = \Gamma^1_{n,m-1}\mathcal{K}_{n,m}, \tag{6.4}$$

which follows from the construction of $\mathcal{K}_{n,m}$ and the definition of $\mathcal{L}$ on the previous levels (see Lemma 3.12). Similarly, using the second defining relation of $\mathcal{K}_{n,m}$ (i.e., the last row of (3.47)) we see that

$$\langle \Phi_{n,m-1}, \tilde{\Gamma}^1_{n-1,m}\tilde{\Phi}_{n-1,m}\rangle = \mathcal{K}_{n,m}(\tilde{\Gamma}^1_{n-1,m})^{\dagger}. \tag{6.5}$$

Equations (6.4) and (6.5) show that most of (6.3) is automatically true. We now define $\mathcal{L}(z^n w^{-m})$ so that (5.4) holds, which completes (6.3).

Using an analogous argument we can use the construction of $\mathcal{K}^1_{n,m}$ to extend the functional to $z^n w^m$ so that

$$\mathcal{K}^1_{n,m} = \langle w\Phi_{n,m-1}, \overleftarrow{\tilde{\Phi}}{}^T_{n-1,m}\rangle. \tag{6.6}$$

This completes the extension of $\mathcal{L}$. What remains to be shown is that (6.2) holds. This is accomplished by first constructing $\tilde{\hat{E}}_{n,m}$ from ($\tilde{5}.8$). The condition on $(e_1^m)^T H^3_{n,m} e_1^m$ and (5.9) shows that the first row of $\Gamma^1_{n,m}$ may be computed and that we may choose

$$(\Gamma^1_{n,m})_{1,1} > 0.$$

With the first row of $\Gamma^1_{n,m}$ and all of $\Gamma_{n,m}$ (which is calculated from the Cholesky factorization of $\mathcal{K}_{n,m}\mathcal{K}^{\dagger}_{n,m}$), $\Phi_{n,m}$ may be constructed from (3.16) and (3.17). Equations (6.3) and (6.6), coupled with (3.16), (3.17), and the orthogonality relations on the previous levels show that

$$\langle \Gamma_{nm}\Phi_{n,m},\ \tilde{\Phi}_{n-1,k}\rangle = 0, \quad k = 0,1,\ldots,m,$$

and

$$\left\langle (e_1^m)^T\Gamma^1_{nm}\Phi_{n,m},\ w^k \begin{bmatrix} z^{n-1} \\ \vdots \\ 1 \end{bmatrix} \right\rangle = 0, \quad k = 1,2,\ldots,m. \tag{6.7}$$

Equations (6.7) and (3.17) show

$$0 = \langle (e_1^m)^T\Gamma^1_{n,m}\Phi_{n,m},\ \overleftarrow{\tilde{\Phi}}{}^T_{n-1,m}\rangle = \left\langle (e_1^m)^T\Gamma^1_{nm}\Phi_{n,m},\ \begin{bmatrix} z^{n-1} \\ \vdots \\ 1 \end{bmatrix} \right\rangle.$$

The fact that $\overleftarrow{\tilde{\Phi}}^T_{n-1,m}$ has an invertible coefficient multiplying

$$\begin{bmatrix} z^{n-1} \\ \vdots \\ 1 \end{bmatrix}$$

has been used to obtain the second equality in the above equation. The above implies that

$$\langle \Phi_{n,m},\ \tilde{\Phi}_{n-1,k}\rangle = 0, \qquad k = 0, 1, \dots, m,$$

which in turn implies that

$$\langle \Phi_{n,m},\ \Phi_{j,m}\rangle = 0, \qquad j = 0, 1, \dots, n-1.$$

To show that

$$\langle \Phi_{n,m},\ \Phi_{n,m}\rangle = I_{m+1}$$

we note that (3.16), (3.17), and (3.52) imply that

$$\langle (e_1^m)^T \Gamma^1_{nm}\Phi_{n,m},\ \Gamma_{nm}\Phi_{n,m}\rangle = (e_1^m)^T \Gamma^1_{n,m}\Gamma^{\dagger}_{n,m}$$

and (3.32) implies that

$$\langle (e_1^m)^T \Gamma^1_{nm}\Phi_{n,m},\ (e_1^m)^T \Gamma^1_{nm}\Phi_{nm}\rangle = (e_1^m)^T \Gamma^1_{nm}(\Gamma^1_{nm})^{\dagger}(e_1^m).$$

Thus $\mathcal{L}$ is a positive linear functional. The orthogonality relations for the polynomials $\tilde{\Phi}_{i,j}$ now follow. ∎

Let $C(\mathbb{T}^2)$ denote the set of continuous functions on the bicircle; above the theorem now allows the following.

THEOREM 6.2. *Given parameters $u_{i,j} \in \mathbb{C}$, with $u_{i,-j} = \bar{u}_{-i,j}$, if* (6.1) *holds for all $0 \le i, j$, then there exists a positive measure $\mu$ supported on the bicircle such that for any $f \in C(\mathbb{T}^2)$*

$$\mathcal{L}(f) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} f(\theta, \phi)\, d\mu(\theta, \phi).$$

*Proof.* From the hypotheses imposed above, Theorem 6.1 shows that $C_{n,m}$ is positive definite for all $n$ and $m$, so the result follows from Bochner's theorem [18, section 1.4.3]. ∎

*Remark* 6.3. The above construction gives a criterion for the existence of a one step extension of the functional. That is, given moments so that there exists a positive linear functional on $\prod^{n-1,m} \cup \prod^{n,m-1}$, any set

$$\{u_{n,m}, u_{-n,m}\},\ u_{-n,-m} = \bar{u}_{n,m}, \quad u_{n,-m} = \bar{u}_{-n,m},$$

that satisfies (6.1) can be used to extend the functional to $\prod^{n,m}$. However it is not difficult to construct examples where no extension exists. See section 8.

**7. Two variable stable polynomials and Fejér–Riesz factorization.** In this section we study the consequences of $\mathcal{K}_{n,m} = 0$. This will make a connection with the results in [8] on stable polynomials and the Fejér–Riesz factorization theorem.

We say that a polynomial $p(z,w)$ is stable if $p(z,w) \neq 0$, $|z| \leq 1$, $|w| \leq 1$. A polynomial $p$ is of degree $(n,m)$ if

$$p(z,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} k_{i,j} z^i w^j,$$

with $k_{n,m} \neq 0$. Finally we say that the polynomial $p_{n,m}$ of degree $(n,m)$ has the spectral matching property (up to $(n,m)$) if

$$\mathcal{L}(z^k w^j) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{z^k w^j}{|p_{n,m}(z,w)|^2} d\theta d\phi, \quad z = e^{i\theta},\ w = e^{i\phi},$$

for $|k| \leq n$, $|j| \leq m$.

LEMMA 7.1. *Suppose that $\mathcal{L}$ is a positive definite linear functional on $\prod^{n,m}$ and $\mathcal{K}_{n,m} = 0$; then*

$$\begin{aligned} &\overleftarrow{\phi}^{m}_{n,m}(z,w)\overline{\overleftarrow{\phi}^{m}_{n,m}(z_1,w_1)} - \phi^m_{n,m}(z,w)\overline{\phi^m_{n,m}(z_1,w_1)} \\ &\quad = (1-w\bar{w}_1)\overleftarrow{\Phi}_{n,m-1}(z,w)\overleftarrow{\Phi}^{\dagger}_{n,m-1}(z_1,w_1) \\ &\qquad + (1-z\bar{z}_1)\tilde{\Phi}_{n-1,m}(z,w)^T\tilde{\Phi}^{\dagger}_{n-1,m}(z_1,w_1). \end{aligned} \tag{7.1}$$

*Proof.* If $\mathcal{K}_{n,m} = 0$, then (3.16) shows that $\Gamma_{n,m}\Phi_{n,m}(z,w) = \Phi_{n,m-1}(z,w)$. Thus $\overleftarrow{\Phi}_{n,m}(z,w)\Gamma^{\dagger}_{n,m} = w\overleftarrow{\Phi}_{n,m-1}(z,w)$. Also (3.31) implies that $(\Gamma_{n,m})_{(i,i+1)} = 1$, $i = 1,\ldots,m$ with all other entries zero. Thus we find

$$\begin{aligned} &\overleftarrow{\Phi}_{n,m}(z,w)\overleftarrow{\Phi}_{n,m}(z_1,w_1)^{\dagger} \\ &= \overleftarrow{\phi}^{m}_{n,m}(z,w)\overline{\overleftarrow{\phi}^{m}_{n,m}(z_1,w_1)} + \overleftarrow{\Phi}_{n,m}(z,w)\Gamma^{\dagger}_{n,m}\Gamma_{n,m}\overleftarrow{\Phi}_{n,m}(z_1,w_1)^{\dagger} \\ &= \overleftarrow{\phi}^{m}_{n,m}(z,w)\overline{\overleftarrow{\phi}^{m}_{n,m}(z_1,w_1)} + w\bar{w}_1\overleftarrow{\Phi}_{n,m-1}(z,w)\overleftarrow{\Phi}_{n,m-1}(z_1,w_1)^{\dagger}. \end{aligned} \tag{7.2}$$

From (4.1c) observe that

$$\begin{aligned} &\overleftarrow{\phi}^{m}_{n,m}(z,w)\overline{\overleftarrow{\phi}^{m}_{n,m}(z_1,w_1)} - z\bar{z}_1\phi^m_{n,m}(z,w)\overline{\phi^m_{n,m}(z_1,w_1)} \\ &\quad = (1-w\bar{w}_1)\overleftarrow{\Phi}_{n,m-1}(z,w)\overleftarrow{\Phi}^{\dagger}_{n,m-1}(z_1,w_1) \\ &\qquad + (1-z\bar{z}_1)\tilde{\Phi}_{n,m}(z,w)^T\tilde{\Phi}^{\dagger}_{n,m}(z_1,w_1). \end{aligned}$$

Using ($\tilde{3}.16$) and the fact that $\tilde{\phi}^n_{n,m}(z,w) = \phi^m_{n,m}$ gives the result. ∎

We now have the following.

THEOREM 7.2. *Suppose that $\mathcal{L}$ is a positive definite linear functional on $\prod^{n,m}$ and $\mathcal{K}_{n,m} = 0$; then $\overleftarrow{\phi}^{m}_{n,m}(z,w)$ is stable and*

$$\mathcal{L}(e^{-ik\theta}e^{-il\phi}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} \frac{e^{-ik\theta}e^{-il\phi}}{|\phi^m_{n,m}(e^{i\theta},e^{i\phi})|^2} d\theta d\phi, \quad |k| \leq n,\ |l| \leq m.$$

*Conversely if $\pi_{n,m}(z,w)$ is a polynomial of degree $(n,m)$ such that $\overleftarrow{\pi}_{n,m}$ is stable and*

$$\mathcal{L}(e^{-ik\theta}e^{-il\phi}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} \frac{e^{-ik\theta}e^{-il\phi}}{|\pi_{n,m}(e^{i\theta},e^{i\phi})|^2} d\theta d\phi, \quad |k| \leq n,\ |l| \leq m,$$

*then $\mathcal{K}_{n,m} = 0$.*

*Proof.* If $\mathcal{L}$ is positive definite and $\mathcal{K}_{n,m}$ is equal to zero, then Lemma 7.1 shows that $\overleftarrow{\phi}^{n}_{n,m}(z,w)$ satisfies (7.1). The first part of the result now follows from the proof of Theorem 2.3.1 in [8]. To show the second part, let $f(z,w) = 1/|\overleftarrow{\pi}_{n,m}(z,w)|^2, |z| = 1 = |w|$ be the spectral density function associated with $\pi_{n,m}$. Then from (2.1.5) in [8] and Lemma 3.3 we find that

$$\overleftarrow{\Phi}_{n,m}(z,w) = [\overleftarrow{\pi}_{n,m}(z,w), w\overleftarrow{\Phi}_{n,m-1}(z,w)].$$

But this implies that $\Phi_{n,m}(z,w) = [\pi_{n,m}(z,w), \Phi_{n,m-1}(z,w)^T]^T$. Hence from (3.16) $\mathcal{K}_{n,m} = 0$. □

This leads to the following alternative proof of the two-variable Fejér–Riesz theorem in [8].

THEOREM 7.3. *Suppose that* $f(z,w) = \sum_{k=-n}^{n}\sum_{l=-m}^{m} f_{kl}z^k w^l$ *is positive for* $|z| = |w| = 1$. *Then there exists a polynomial*

$$p(z,w) = \sum_{k=0}^{n}\sum_{l=0}^{m} p_{kl}z^k w^l,$$

*with* $p(z,w) \neq 0$ *for* $|z|, |w| \leq 1$, *and* $f(z,w) = |p(z,w)|^2$ *if and only if* $\mathcal{K}_{n,m} = 0$.

*Proof.* For $g \in C(\mathbb{T}^2)$ let $\mathcal{L}(g) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{g(\theta,\phi)}{|p(e^{i\theta},e^{i\phi})|^2} d\theta d\phi$. Then $\mathcal{L}$ is a positive definite linear functional on $\mathbb{T}^2$. The necessary part of the above result now follows from Theorem 7.2. The sufficiency also follows from the above theorem and the maximal entropy condition [1]. □

An alternative approach for finding a factorization as above may be done using the notion of intersecting zeros (see [9]). Also, the question of factorizing a nonnegative trigonometric polynomial as a modulus square of an outer polynomial was addressed in [6], allowing for generalizations in the operator valued case. When such a factorization of the desired degree does not exist, one can approximate the trigonometric polynomial with one that does have the desired factorization. This question was pursued in [12].

The vanishing of $\mathcal{K}_{n,m}$ has the following geometric interpretation.

LEMMA 7.4. *Suppose* $\mathcal{L}$ *is positive definite on* $\prod^{n,m}$; *then* $\mathcal{K}_{n,m} = 0$ *if and only if for* $\Phi_{n,m-1}$ *constructed as in* (3.3)

$$\langle \Phi_{n,m-1}, z^i w^m \rangle = 0, \quad 0 \leq i \leq n-1. \tag{7.3}$$

*Proof.* The definition of $\Phi_{n,m-1}$ shows that it is already orthogonal to $z^i w^j$, $0 \leq i < n,\ 0 \leq j \leq m-1$. The remaining orthogonality conditions show that $\Phi_{n,m-1}$ is orthogonal to all of the monomials in $\tilde{\Phi}_{n-1,m}$. Thus the sufficiency part of the theorem follows from (3.22). To see the necessary part note that from the definition of $\Phi_{n,m-1}$

$$\mathcal{K}_{n,m} = \left\langle \Phi_{n,m-1}, \tilde{\Phi}^{n-1}_{n-1,m}\begin{bmatrix} z^{n-1} \\ \vdots \\ 1 \end{bmatrix} w^m \right\rangle, \tag{7.4}$$

with $\tilde{\Phi}^{n-1}_{n-1,m}$ an invertible matrix. Thus (7.3) follows. □

Unfortunately at this point we are unable to see what the condition $\mathcal{K}_{n,m} = 0$ implies for $u_{i,j},\ |i| \leq n,\ |j| \leq m$, except for $u_{n,-m} = 0$, which follows from (5.4). We can, however, get a partial characterization for when a positive measure on the bicircle can be written as the reciprocal of the magnitude square of a stable polynomial. We begin with the following auxiliary result.

LEMMA 7.5. *If $\hat{E}_{i,j} = 0$, then the first column of $K^1_{i,j}$ is equal to zero, in particular $u_{i,j} = 0$. If $\hat{E}_{i,j}$, $\mathcal{K}_{i,j}$, and $\mathcal{K}_{i-1,j}$ are zero, then so is $\hat{E}_{i,j-1}$. Conversely if $\mathcal{K}_{i,j}$, $K_{i-1,j}$, $\hat{E}_{i,j-1}$, and $u_{i,j}$ are zero, then $\hat{E}_{i,j} = 0$. In both cases $\mathcal{K}^1_{i,j} = [0, \mathcal{K}^1_{i-1,j}]$. Likewise if $\hat{\hat{E}}_{i,j} = 0$, then the first row of $K^1_{i,j}$ is equal to zero. If $\hat{\hat{E}}_{i,j}$, $\mathcal{K}_{i,j}$, and $\mathcal{K}_{i,j-1}$ are zero, then so is $\hat{\hat{E}}_{i,j-1}$. Conversely if $\mathcal{K}_{i,j}$, $K_{i,j-1}$, $\hat{\hat{E}}_{i-1,j}$, and $u_{i,j}$ are zero, then $\hat{\hat{E}}_{i,j} = 0$. In both cases $\mathcal{K}^1_{i,j} = [0, (\mathcal{K}^1_{i,j-1})^T]^T$.*

*Proof.* If $\hat{E}_{i,j} = 0$, then (3.51) and Remark 3.9 show that the first column of $\mathcal{K}^1_{i,j}$ is zero. If $\mathcal{K}_{i-1,j}$ is equal to zero, then (3.54) shows that all of the entries of $I_{i-1,j}$ are zero except for a one in the first entry. Thus (3.50) and (3.51) imply that if $\hat{E}_{i,j} = 0$, $\mathcal{K}_{i,j} = 0$, and $\mathcal{K}_{i-1,j} = 0$, then $\hat{E}_{i,j-1}\bar{\Gamma}^1_{i-1,j} = 0$ and $\hat{E}_{i,j-1}\bar{\Gamma}_{i-1,j} = 0$. Following the argument in the construction of $\hat{E}_{n,m}$ we see that $\hat{E}_{i,j-1} = 0$. The above hypothesis on $\mathcal{K}_{i-1,j}$ shows that $\tilde{\Gamma}_{i-1,j} = U_{i-1}$; thus (5.6) and the fact that the first column of $\mathcal{K}^1_{i,j}$ is zero gives $\mathcal{K}^1_{i,j} = [0, \mathcal{K}^1_{i-1,j}]$. The converse statement follows from (5.8). The remaining statements follow in an analogous fashion using Proposition 3.8. ∎

LEMMA 7.6. *Let $\mu$ be a positive measure on the bicircle. Then $\mu$ is purely absolutely continuous with respect to the Lebesgue measure and*

$$d\mu(\theta, \phi) = \frac{1}{|p_{n,m}|^2} d\theta d\phi,$$

*where $p_{n,m}$ is a polynomial of degree $(n, m)$, with $\overleftarrow{p}_{n,m}(z, w)$ stable if and only if*

$$\mathcal{K}_{i,j} = 0, \ \hat{E}_{i+1,j} = 0, \ \text{and } \hat{\hat{E}}_{n,j+1} = 0, \ i \geq n, j \geq m.$$

*Proof.* Suppose that $d\mu = \frac{1}{|p_{n,m}(z,w)|^2} d\theta d\phi$, with $\overleftarrow{p}_{n,m}$ stable; then the sequence

$$\{\psi_{i,j}(z, w)\} \quad \psi_{i,j}(z, w) = z^{i-n} w^{j-m} p_{n,m}(z, w), \ i \geq n, \ j \geq m,$$

is a set of polynomials with degrees $(i, j)$, respectively, such that $\overleftarrow{\psi}_{i,j} = \overleftarrow{p}_{n,m}$ are stable and have the spectral matching property. Thus Theorem 7.2 implies that $\mathcal{K}_{i,j} = 0$ for $i \geq n, \ j \geq m$. Since $\overleftarrow{\psi}_{i+1,j} = \overleftarrow{\psi}_{i,j}, \ i \geq n, \ j \geq m$, we see from (2.1.5) in [8] that $\overleftarrow{\Phi}_{i+1,j} = \overleftarrow{\Phi}_{i,j}$ for $i \geq n, \ j \geq m$. This implies that $A_{i+1,j} = I_{j+1}$ so that (3.14) shows that $\hat{E}_{i+1,j} = 0, \ i \geq n, j \geq m$. Since $\overleftarrow{\tilde{\phi}}^{\,i}_{i,j+1} = \overleftarrow{\tilde{\phi}}^{\,i}_{i,j}, \ i \geq n, \ j \geq m$, the preceding argument shows that $\tilde{E}_{i,j+1} = 0, \ i \geq n, j \geq m$. This proves the necessary part.

To prove sufficiency note that if $\mathcal{K}_{i,j} = 0, i \geq n, \ j \geq m$, there exist polynomials $\psi_{i,j}$ of degree $(i, j)$ where $\overleftarrow{\psi}_{i,j}$ is a stable polynomial which has the spectral matching property. In order to show that $\overleftarrow{\psi}_{i,j} = \overleftarrow{\psi}_{n,m}$ we note that since $\hat{E}_{i+1,j} = 0$ (3.14) implies that $\Phi_{i+1,j} = \Phi_{i,j}, \ i \geq n, \ j \geq m$. Furthermore $\hat{\hat{E}}_{n,j+1} = 0, \ j \geq m$, implies that $\tilde{\Phi}_{n,j+1} = \tilde{\Phi}_{n,m}$. Since $\psi_{i,j} = \phi^j_{i,j} = \tilde{\phi}^i_{i,j}$ for $i \geq n, \ j \geq m$, the result follows. ∎

The conditions on $\mathcal{K}_{i,j}$, $E_{i,j}$, and $\tilde{E}_{n,j}$ given in Lemma 7.6 are not optimal since they are redundant. Some of this redundancy is removed in the next theorem.

THEOREM 7.7. *Let $\mu$ be a positive measure on the bicircle. Then $\mu$ is purely absolutely continuous with respect to the Lebesgue measure and $d\mu = \frac{d\theta d\phi}{|p_{n,m}|^2}$, where $p_{n,m}$ is a polynomial of degree $(n, m)$ with $\overleftarrow{p}_{n,m}$ stable if and only if*

(a) $\mathcal{K}_{n,j} = 0$, $\hat{\tilde{E}}_{n-1,j+1} = 0$, *and* $u_{n,j+1} = 0$, $j \geq m$;
(b) $\mathcal{K}_{i,m} = 0$, $\hat{E}_{i,m-1} = 0$, *and* $u_{i,m} = 0$, $i > n$;
(c) $u_{|i|,j} = 0, i > n,\ j > m$.

*Remark* 7.8. Equation (5.4) and Lemma 7.5 show that $u_{-n,j}$, $u_{-i,m}$, $u_{n-1,j+1}$, and $u_{i,m-1}$ are also equal to zero for $j \geq m, i > n$.

*Proof.* If $\mu$ has the form indicated in the hypotheses, then Lemma 7.6 says that $\mathcal{K}_{i,j} = 0,\ i \geq n,\ j \geq m$, which coupled with (5.4) implies that $u_{-i,j} = 0$, $i \geq n,\ j \geq m$; the remaining conditions on the coefficients follow from Lemma 7.5. If the coefficients obey (a)–(c), then Lemma 7.5 shows that $\hat{E}_{i+1,m}$ and $\hat{\tilde{E}}_{n,j+1}$ are equal to zero for $i \geq n$ and $j \geq m$. Since $\hat{E}_{n,m+1} = 0$, $\hat{\tilde{E}}_{n+1,m} = 0$, and by hypothesis $u_{-n-1,m+1} = 0$, (5.5) shows that $\mathcal{K}_{n+1,m+1} = 0$. With this Lemma 7.5 shows that $\hat{E}_{n+1,m+1} = 0$ and $\hat{\tilde{E}}_{n+1,m+1} = 0$. The result now follows by induction. □

It is possible to modify slightly the hypotheses of Theorem 7.7 to obtain a statement just on the coefficients in the recurrence formulas.

Theorem 7.9. *Suppose* $u_{i,j}$ *are given so that* (6.1) *is satisfied for* $0 \leq i \leq n$ , $|j| \leq m$, *and so that (a)–(c) in Theorem* 7.7 *hold. Then for* $f \in C(\mathbb{T}^2)$

$$\mathcal{L}(f) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} f(\theta, \phi) d\mu(\theta, \phi),$$

*where* $\mu$ *is absolutely continuous with respect to the Lebesgue measure with* $d\mu = \frac{d\theta d\phi}{|p_{n,m}|^2}$. *Here* $p_{n,m}$ *is a polynomial of degree* $(n, m)$ *with* $\overleftarrow{p}_{n,m}$ *stable.*

*Proof.* From Theorem 6.1 there exists a positive definite linear functional on $\prod^{n,m}$ with the above parameters, and from Theorem 7.2 the functional has the representation

$$\mathcal{L}(e^{-ik\theta}e^{-il\phi}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} \frac{e^{-ik\theta}e^{-il\phi}}{|p_{n,m}(e^{i\theta}e^{i\phi})|^2} d\theta d\phi, \quad |k| \leq n,\ |l| \leq m,$$

with $p_{n,m}$ a polynomial of degree $(n, m)$ with $\overleftarrow{p}_{n,m}$ stable. The result now follows from Theorem 7.7. □

**8. Examples.** We now give some examples that illustrate various aspects of the results presented earlier. We begin with the case $n = 1, m = 1$, with $u_{0,0} = 1$, $\mathcal{K}_{1,1} = u_{-1,1} = 0$, and $\mathcal{K}^1_{1,1} = \bar{u}_{1,1}$. From Theorem 6.1 we see that we must choose $|u_{0,1}| < 1$ and $|u_{1,0}| < 1$. Since $\mathcal{K}_{1,1} = 0$ the only remaining condition for $\mathcal{L}$ to be a positive linear functional on $\prod^{1,1}$ is for ${e^1_1}^T \tilde{H}^3_{1,1} e^1_1 < 1$. From (5.9) we see that $\Gamma^1_{1,1} U^T_1 = I_{1,0} \hat{\tilde{E}}_{1,1} (I^1_{1,0})^T$. The construction of $I_{1,0}$, $I^1_{1,0}$, and (5.8) shows that ${e^1_1}^T \tilde{H}^3_{1,1} e^1_1 < 1$ is given by

$$a|u_{1,1}|^2 + b(\bar{u}_{1,1}\bar{u}_{0,1}u_{1,0} + u_{1,1}u_{0,1}\bar{u}_{1,0}) + c < 1,$$

with $a = \frac{1-|u_{0,1}u_{1,0}|^2}{1-|u_{1,0}|^2}$, $b = \frac{\sqrt{1-|u_{0,1}|^2}}{\sqrt{1-|u_{1,0}|^2}}$, and $c = |u_{0,1}|^2$. This simplifies to

$$|\hat{u}_{1,1}| < 1,$$

where

$$\hat{u}_{1,1} = \frac{(1 - |u_{0,1}u_{1,0}|^2)u_{1,1}}{\sqrt{1 - |u_{0,1}|^2}\sqrt{1 - |u_{1,0}|^2}} + u_{0,1}\bar{u}_{1,0}.$$

Thus from Theorems 6.1 and 7.2 we see that with $u_{0,0} = 1$

$$\mathcal{L}(e^{-ik\theta}e^{-ij\phi}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} \frac{e^{-ik\theta}e^{-ij\phi}}{|\phi_{1,1}(e^{i\theta}, e^{i\phi})|^2} d\theta d\phi, \ |k| \le 1, \ |j| \le 1,$$

where $\phi_{1,1}$ constructed using (3.16) and the top row of (3.17) is a polynomial of degree (1,1) with $\overleftarrow{\phi}_{1,1}$ stable if and only if $|u_{0,1}| < 1$, $|u_{1,0}| < 1$, $u_{-1,1} = 0$, and $|\hat{u}_{1,1}| < 1$. Furthermore if we set $u_{j,0}$, $u_{0,j}$, $u_{i,j}$ equal to zero for $i > 1, |j| > 1$, then Theorem 7.9 shows that the above representation for $\mathcal{L}$ extends to all continuous functions on $\mathbb{T}^2$.

We can also use the previous results to investigate contractive Toeplitz matrices. In this case we find

$$C_{1,1} = \begin{bmatrix} I & C_{-1} \\ C_1 & I \end{bmatrix}, \tag{8.1}$$

where $C_{-1} = C_1^\dagger$ is a $2 \times 2$ Toeplitz matrix. In this case $u_{0,0} = 1$ and $u_{0,1} = 0$ so that $\tilde{E}_{0,1} = 0$, $\tilde{A}_{0,1} = 1$. Since $\mathcal{K}_{1,1} = u_{-1,1}$, we find $\Gamma_{1,1} = [0, \sqrt{1 - |u_{-1,1}|^2}]$. This plus the computation of $\tilde{\Gamma}^1_{1,0}$ described in the construction of $\mathcal{L}$ yields

$$\begin{aligned} I &= (e_1^1)^T \tilde{H}^3_{1,1} (\tilde{H}^3_{1,1})^\dagger (e_1^1) \\ &= (1+d)|u_{1,1}|^2 + d(u_{1,1}u_{-1,1} + \bar{u}_{1,1}\bar{u}_{-1,1}) + d|u_{-1,1}|^2 < 1, \end{aligned} \tag{8.2}$$

where

$$d = \frac{|u_{1,0}|^2}{(1 - |u_{1,0}|^2)(1 - |u_{-1,1}|^2)}.$$

By completing the square this can be simplified to

$$|\hat{u}_{1,1}| < 1,$$

where

$$\hat{u}_{1,1} = (1+d)\sqrt{1 - |u_{1,0}|^2} u_{1,1} + d_1 \bar{u}_{-1,1},$$

and

$$d_1 = \frac{|u_{1,0}|^2}{(1 - |u_{-1,1}|^2)\sqrt{1 - |u_{1,0}|^2}},$$

which puts constraints on $u_{-1,1}$. Thus we find that the conditions for $\mathcal{L}$ to be a positive linear functional and hence $C_1$ to be a contractive Toeplitz matrix are $|u_{1,0}| < 1$, $|u_{-1,1}| < 1$, and $|\hat{u}_{1,1}| < 1$. These constraints may not be strong enough to allow $\mathcal{L}$ to be extended. To see this suppose $n = 1, m = 2$, $u_{0,2} = 0$, and $u_{1,0} = 0$. It is not difficult to see then that $\hat{E}_{1,1} = \operatorname{diag}(\bar{u}_{1,1}, u_{-1,1})$. With $u_{1,0} = 0$ the constraint on $\hat{u}_{1,1}$ above reduces to $|u_{1,1}| < 1$. However,

$$K_{1,2} = \begin{pmatrix} \frac{u_{-1,1}}{(1-|u_{1,1}|^2)^{1/2}} \\ \frac{u_{-1,2}}{(1-|u_{-1,1}|^2)^{1/2}} \end{pmatrix},$$

so we see that in order for $K_{1,2}$ to be a contraction $\frac{|u_{-1,1}|}{\sqrt{1-|u_{1,1}|^2}} < 1$, which may not be satisfied.

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