Proceedings of the 7th International Conference on HydroScience and Engineering

Philadelphia, USA September 10-13, 2006 (ICHE 2006)

ISBN: 0977447405

Drexel University

College of Engineering

TECHNIQUES FOR MESH DENSITY CONTROL

Yaoxin Zhang[1], Yafei Jia[2] and Sam S.Y. Wang[3]

ABSTRACT

Mesh generation is crucial in computational fluids dynamic (CFD) analysis, which solves a set of partial differential equations (PDE) based on a computational mesh. The success of solving these equations depends to a large extent on the mesh quality. In addition to the orthogonality and the smoothness, the mesh density distribution is the key to desirable mesh. The objective of the current research is to develop methods which make the control of mesh density simple, easy and effective. With these, the resulting mesh is near-orthogonal but more desirable for the numerical simulation.

In this study, two new techniques for mesh density control are proposed. The first one is a three-parameter stretching function which can not only stretch the node in two directions but also control the location of the distribution. The second method is a modified RL system (Ryskin and Leal, 1983) in which the distortion function is evaluated by the averaged scale factors and the scale factors controlled by weighting functions.

1. INTRODUCTION

In addition to the orthogonality and the smoothness, the quality of a mesh is also determined by the mesh density distribution in the computational fluid dynamic analysis. Since the solution accuracy is proportional to the mesh density, the computational region with high gradient variations usually needs higher mesh density.

To obtain the desired mesh density distribution, two kinds of methods have been developed. The first one is through the stretching function which can control the nodal distribution not only along the boundaries but also the interior grid. It is widely used in the algebraic mesh generation. Many stretching functions have been developed in the past. In Chuang (2002) and Thompson et al. (1985a), some of them were summarized. Eiseman (1979) used a two-parameter stretching function which can stretch nodes to one direction.

The second method generates meshes by solving the P.D.Es and the nodal clustering is handled by the control functions. One typical example is a Poisson equation system, proposed by Thompson et al. (1977), with a set of such functions which can control the nodal distribution effectively. Another example is a Laplacian variational system developed by Brackbill and Saltzman

[1] Research Scientist, Center for Computational Hydroscienec and Engineering, The University of Mississippi, MS 38677, USA (yzhang@ncche.olemiss.edu)
[2] Research Professor, , Center for Computational Hydroscience and Engineering, The University of Mississippi, MS 38677, USA (jia @ ncche.olemiss.edu)
[3] Frederick A.P. Barnard Professor and Director, Center for Computational Hydroscience and Engineering, The University of Mississippi, MS 38677, USA (wang @ ncche.olemiss.edu)

(1982) for adaptive mesh generation, which can not only control the mesh density, but also the mesh orthogonality and smoothness. However, the formulation of the system is complicated, and it is difficult to use.

In this paper, two methods are proposed for the density control. The first one is a three-parameter stretching function which can control the nodal distribution along the boundary. In this stretching function, there are three parameters, namely, exponential parameter, deviation parameter, and scale parameter. The exponential parameter controls the contraction and repulsion of the distribution; the deviation parameter determines the location of the contraction and repulsion; and the scale parameter controls the effects of the distribution. The second method is based on the RL system (Ryskin and Leal, 1983) which is well-known for orthogonal mapping. In current study, this system is modified to generate meshes with density control. The distortion function is evaluated by the averaged scale factors and the scale factors controlled by weighting functions which are constructed according to variable distribution, such as water depth, bed slope, or transport concentration, etc. With the averaged scale factors (mesh smoothness) and the weighted scale factors (mesh adptivity), the proposed system is able to produce meshes with a good combination of orthogonality, adaptivity and smoothness. The proposed methods are demonstrated by test example and application.

2. STRETCHING FUNCTION

The stretching function is widely used in the algebraic mesh generation. In this study, a more flexible and powerful two-direction stretching function EDS is proposed.

$$s_j = \sum_{1}^{j-1}\left[\frac{2}{\exp(\phi)+\exp(-\phi)}\right]^E \Big/ \sum_{1}^{N-1}\left[\frac{2}{\exp(\phi)+\exp(-\phi)}\right]^E , \tag{1a}$$

$$\phi = \left[\frac{j-1}{N} - D\right] \times S , \tag{1b}$$

where s_j is the relative location; j is the label of one point; N is the total number of points along a mesh line; E (= -1, 0, 1) is the exponential parameter; D ($0 \le D \le 1$) is the deviation parameter; S (>0) is the parameter used to control the degree of stretching, called scale parameter.

With this stretching function, the location of any node in one line AB (Figure 1) is calculated by

$$x_j = x_A + (x_B - x_A) \cdot s_j \tag{2}$$

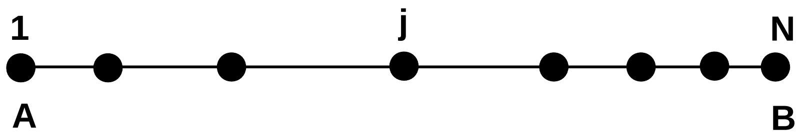


Figure 1 Nodal distributions on one line

Figure 2 illustrates the effects of these three parameters, E, D, and S. The exponential parameter determines the characteristic of the distribution: contraction to a point, repulsion from a point, or uniformity. If E = -1, the distribution is contracting to the point; if E = 1, the distribution is repulsing from the point; and if E = 0, the distribution is uniform. The deviation parameter provides the relative location of this point along AB. For example, if D = 0.5, this point is located at the center. The scale parameter S controls the degree of stretching. The larger S is, the more the distribution is stretched. If S = 0, the distribution is uniform. Note that in this EDS stretching function, the three parameters can be of any values. The reference values provided here would make it easy for using the function.

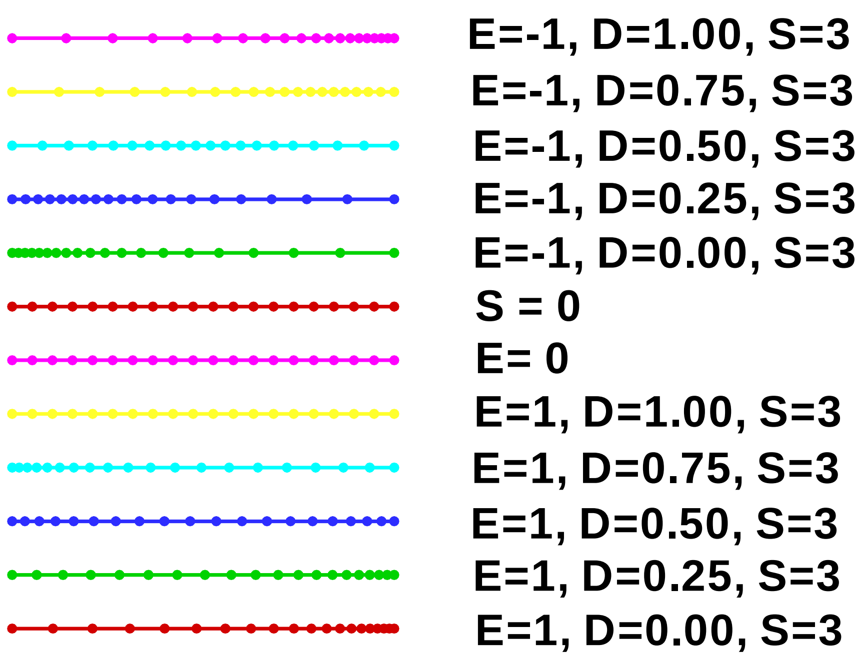


Figure 2 Effects of E, D, and S

In the current study, the EDS stretching function is applied for the adaptive mesh generation. Because the deviation parameter D can control the relative location of the distribution, a 2D adaptive mesh for a natural river can be easily obtained. For example, to generate a mesh for a river channel with high nodal density distributed along the thalweg, in each transverse cross section, the mesh nodes should be contracted to the thalweg. The deviation parameter D can be calculated by

$$D_j = \frac{N_{z\min}}{N_j} \tag{3}$$

where $N_{z\min}$ is the number of the node with the minimum bed elevation in the cross section j; and N_j is the total number of nodes in this cross section.

An adaptive algebraic mesh generator can be established based on Equations (1), (2) and (3) with the help of the standard Laplacian smoothing technique described as follows:

$$x_i^n = \sum_{k=1}^{N(P_i)} x_k / N(P_i) \tag{4a}$$

$$y_i^n = \sum_{k=1}^{N(P_i)} y_k / N(P_i) \tag{4b}$$

where $N(P_i)$ is the number of nodes around P_i; the superscript "*n*" means the new value.

The solution process is as follows:

- Generate an initial mesh with E = 0 or S = 0.
- Interpolate the bed elevation for all the mesh nodes.
- Evaluate the deviation parameter D in each cross section using Equation (3).
- Choose proper values for E and S. For contraction, E = -1; and for repulsion, E = 1.
- Generate another mesh using the E, D, and S from the previous steps.
- Smooth the mesh using the standard Laplacian smoothing scheme.
- Interpolate the bed elevation for the final mesh.

3. MODIFIED RL SYSTEM

In the elliptic mesh generation system, the mesh density can be controlled through the control functions. Thompson et al. (1977) proposed a set of control functions which can control the grid clustering effectively. Another method---a Laplacian variational system was proposed by Brackbill and Saltzman (1982). In this paper, a simpler method based on the well-known orthogonal mapping system RL developed by Ryskin and Leal (1983) is proposed.

In the RL system, the orthogonal mapping between the physical coordinates ($x^i(\equiv x, y), i = 1, 2$) and the computational coordinates ($\xi^i(\equiv \xi, \eta), i = 1, 2$) is described using the following covariant Laplace equations:

$$\frac{\partial}{\partial \xi}(f\frac{\partial x}{\partial \xi}) + \frac{\partial}{\partial \eta}(\frac{1}{f}\frac{\partial x}{\partial \eta}) = 0 \tag{5a}$$

$$\frac{\partial}{\partial \xi}(f\frac{\partial y}{\partial \xi}) + \frac{\partial}{\partial \eta}(\frac{1}{f}\frac{\partial y}{\partial \eta}) = 0 \tag{5b}$$

where the distortion function *f* (also called aspect ratio) is defined as the ratio of the scale factors in ξ and η directions (h_ξ and h_η):

$$f = \frac{h_\eta}{h_\xi} = (\frac{x_\eta^2 + y_\eta^2}{x_\xi^2 + y_\xi^2})^{1/2} \tag{6a}$$

$$h_\xi = g_{11}^{1/2}, \quad h_\eta = g_{22}^{1/2} \tag{6b}$$

and the metric tensor g_{ij} is defined as follows:

$$g = \begin{vmatrix} (x_\xi^2 + y_\xi^2) & (x_\xi x_\eta + y_\xi y_\eta) \\ (x_\xi x_\eta + y_\xi y_\eta) & (x_\eta^2 + y_\eta^2) \end{vmatrix} \tag{7}$$

and, $x_\xi = \partial x / \partial \xi$ and so forth.

The RL system has been the objective of many researchers (see Zhang et al. (2004, 2006a, and 2006b)) and the focus of these researches was the determination of the distortion function *f*, which generally cannot be prescribed arbitrarily. Zhang et al. (2006b) proposed a method to directly control the distortion function using the averaged scale factors to improve the mesh smoothness. For one typical mesh node (i, j), their method can be described as follows:

$$f_{i,j} = \frac{(\overline{h_\eta})_j \cdot s_\eta + (h_\eta)_{i,j} \cdot (1 - s_\eta)}{(\overline{h_\xi})_i \cdot s_\xi + (h_\xi)_{i,j} \cdot (1 - s_\xi)} \tag{8a}$$

$$(\overline{h_\xi})_i = \frac{1}{N_j - 2}\sum_{j=2}^{N_j-1}(h_\xi)_{i,j} \tag{8b}$$

$$(\overline{h_\eta})_j = \frac{1}{N_i - 2}\sum_{i=2}^{N_i-1}(h_\eta)_{i,j} \tag{8c}$$

where $(\overline{h_\xi})_i$ and $(\overline{h_\eta})_j$ are the global averaged scale factors at $\xi = i$ line and at $\eta = j$ line, respectively; N_i and N_j are the total number of mesh lines in ξ and η directions; and, s_η and s_ξ are two adjustable parameters within the range of [0, 1] to control the ratio between the averaged scale factors and the local scale factors and further to control the local balance of mesh orthogonality and smoothness.

Since the scale factors h_ξ and h_η are defined as the cell length in ξ and η directions, respectively, an intuitive idea is to calculate them using some weighting function, so the cell length and further the mesh density will be controlled. Thus, one can obtain:

$$(h_\xi^*)_{i,j} = (L_\xi)_i \cdot (w_\xi)_{i,j} \tag{9a}$$

$$(h_\eta^*)_{i,j} = (L_\eta)_j \cdot (w_\eta)_{i,j} \tag{9b}$$

$$(L_\xi)_i = \sum_{j=1}^{N_j}(h_\xi)_{i,j} \tag{9c}$$

$$(L_\eta)_j = \sum_{i=1}^{N_i}(h_\eta)_{i,j} \tag{9d}$$

where $(L_\xi)_i$ and $(L_\eta)_j$ are the total length of lines at $\xi = i$ and at $\eta = j$, respectively; and, w_ξ and w_η are the weighting functions in ξ and η directions.

Note that Equation (9) can be used to control the mesh density only in a single direction. That is,

$$\text{In } \xi \text{ direction: } f_{i,j}^* = \frac{(h_\eta)_{i,j}}{(h_\xi^*)_{i,j}} \tag{10a}$$

$$\text{In } \eta \text{ direction: } f_{i,j}^* = \frac{(h_\eta^*)_{i,j}}{(h_\xi)_{i,j}} \tag{10b}$$

If the distortion function is evaluated by both Equations (9a) and (9b)---the weighted scale factors, their weighting effects will cancel each other, and in results, the mesh density cannot be controlled as expected.

As pointed out in Zhang et al. (2004, 2006a and 2006b), the RL system is lack of emphasizes on mesh smoothness and serious mesh distortion and overlapping may occur in geometrically complex domains. To improve mesh smoothness, a simplified version of Equation (8) that the two parameters s_η and s_ξ are assumed equal is adopted in the current study. Therefore, for mesh density controls, the distortion function is evaluated by

$$f_{i,j}^* = \frac{(h_\eta)_{i,j} \cdot (1 - r_a) + (\overline{h_\eta})_j \cdot r_a}{(h_\xi^*)_{i,j}} \quad \text{(in } \xi \text{ direction)} \tag{11a}$$

$$f_{i,j}^* = \frac{(h_\eta^*)_{i,j}}{(h_\xi)_{i,j} \cdot (1 - r_a) + (\overline{h_\xi})_i \cdot r_a} \quad \text{(in } \eta \text{ direction)} \tag{11b}$$

where r_a $(= s_\xi$ or $s_\eta)$ is the smoothness parameter.

4. EXAMPLES

The proposed two methods are demonstrated by two examples. For both methods, only the Dirichlet boundary condition is applied. The mesh quality is evaluated quantitatively by the standard academic criterions, such as Maximum Deviation Orthogonality (MDO), Averaged Deviation from Orthogonality (ADO), Maximum grid Aspect Ratio (MAR), and Averaged grid Aspect Ratio (AAR). ADO and MDO are used to measure the orthogonality, while AAR and MAR measure the global smoothness. These four indicators are defined as follows:

$$MDO = \max(\theta_{i,j}) \tag{12a}$$

$$ADO = \frac{1}{(N_i - 2)}\frac{1}{(N_j - 2)}\sum_{2}^{N_i - 1}\sum_{2}^{N_j - 1}\theta_{i,j} \tag{12b}$$

$$MAR = \max[\max(f_{i,j}, \frac{1}{f_{i,j}})] \tag{13a}$$

$$AAR = \frac{1}{(N_i - 2)}\frac{1}{(N_j - 2)}\sum_{2}^{N_i - 1}\sum_{2}^{N_j - 1}\max(f_{i,j}, \frac{1}{f_{i,j}}) \tag{13b}$$

where θ is defined as

$$\theta_{i,j} = \arccos(\frac{g_{12}}{h_\xi h_\eta})_{i,j} \tag{14}$$

Rectangular Domain

The first example is a rectangular domain, in which the control of mesh density using the EDS stretching function is illustrated. As shown in Figure 3, the mesh size in this domain is 35×35, and it is required that: (1) contraction or repulsion occurs at the node A, B, C, and D; (2) contraction or repulsion occurs at the line EF and GH.

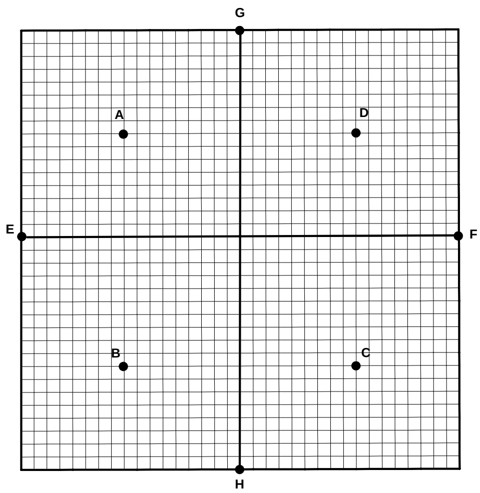


Figure 3 Rectangular domain

Each boundary is divided into four sections, and in each section the EDS stretching function is applied, so the nodes along the boundaries are not equally spaced. The parameters are shown in Tables 1 and 2. The ordering of the *I* and *J* indices is from left to right in *x* direction (*I*) and from top to bottom in *y* direction (*J*).

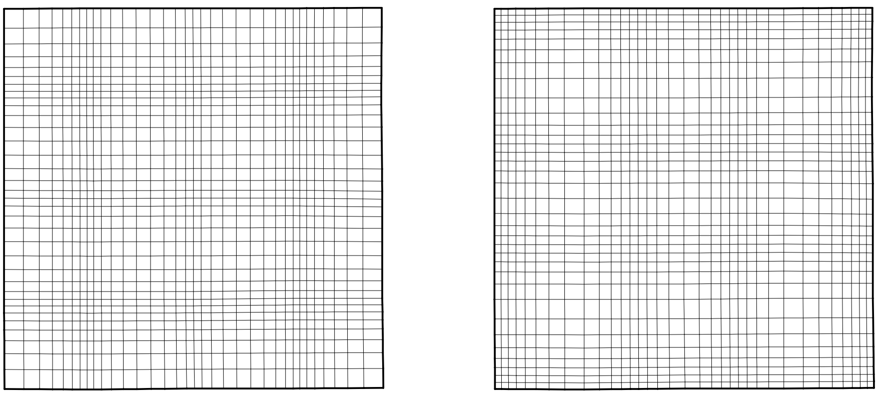
Table 1 Parameters of EDS stretching function for contraction

Boundary	Section No.	No. of Nodes	E	D	S
Left & right	1	9	-1	1	2
	2	10	1	0.5	3
	3	10	1	0.5	3
	4	9	-1	0	2
Top & bottom	1	9	-1	1	2
	2	10	1	0.5	3
	3	10	1	0.5	3
	4	9	-1	0	2

Table 2 Parameters of EDS stretching function for repulsion

Boundary	Section No.	No. of Nodes	E	D	S
Left & right	1	9	-1	0	2
	2	10	-1	0.5	3
	3	10	-1	0.5	3
	4	9	-1	1	2
Top & bottom	1	9	-1	0	2
	2	10	-1	0.5	3
	3	10	-1	0.5	3
	4	9	-1	1	2

Figure 4 shows the resulting meshes. The EDS stretching function works well to contract or repulse mesh lines with smooth transition. However, it can only be applied to one line because it is one-dimensional.



(a) Contraction (b) Repulsion

Figure 4 Meshes using EDS stretching function

Natural River

The second example is a curved natural river with an island as shown in Figure 5. In this domain, the main channel needs more mesh lines than the floodplains. The EDS stretching function and the modified RL system are used to generate the depth-adaptive meshes for this channel.

For this case, only in ξ direction (the transverse direction) the mesh lines are controlled, so Equation (11b) is used to calculate the distortion function. The modified RL system used the following weighting function for mesh generation.

$$(w_\xi)_{i,j} = \frac{1}{\sqrt{1+(Z_\xi)_{i,j}^2} + Z_{\max} - Z_{i,j}} \tag{15}$$

where $Z_\xi = \partial Z / \partial \xi$.

Figure 6 shows the comparisons of adaptive meshes using the EDS stretching function with different scale factor S. With the scale factor increasing, the mesh lines become more squeezed to the main channel. However, the island was not identified due to the fact that only one deviation point is provided along one line. To remedy this problem, the multi-block concept can be borrowed and the domain can be split into two blocks in the transversal direction. In each block, the EDS stretching function is applied.

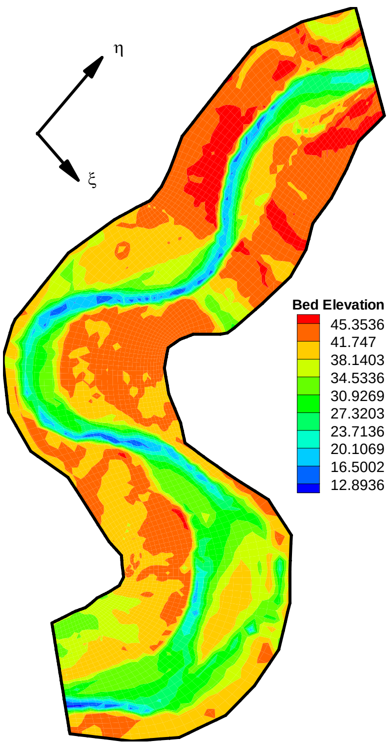
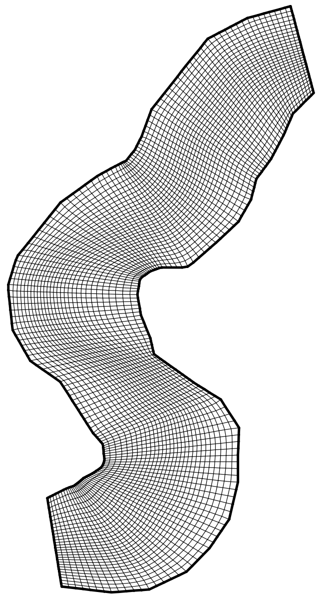
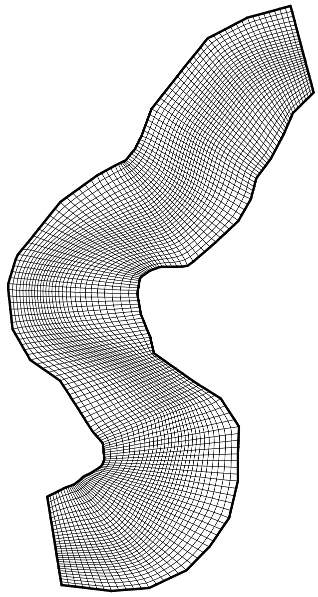
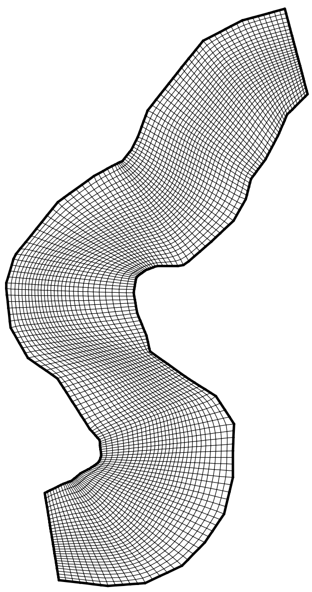
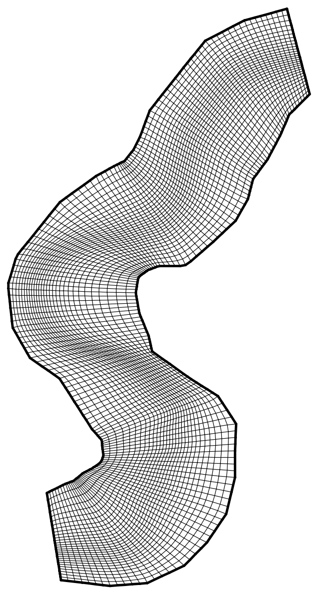


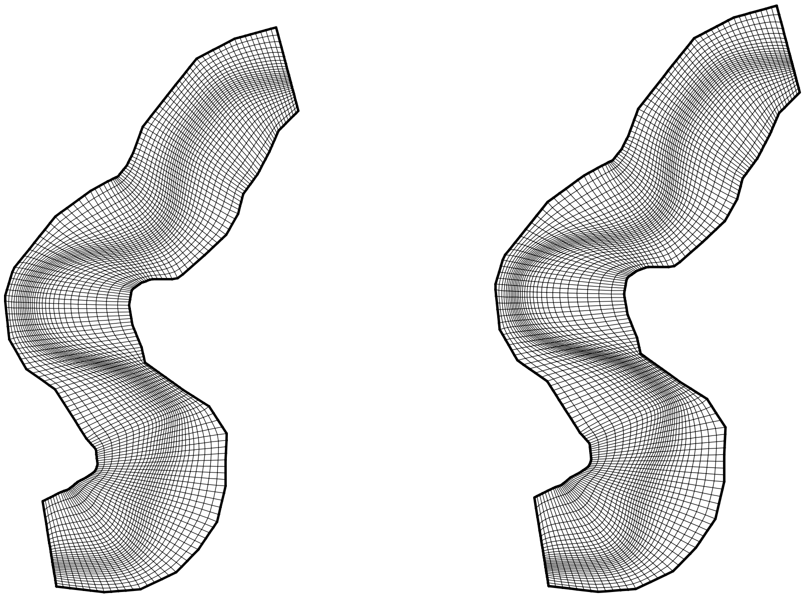
Figure 5 Natural river


(a) S = 2.5


(b) S = 3.0


(c) S = 3.5


(d) S = 4.0



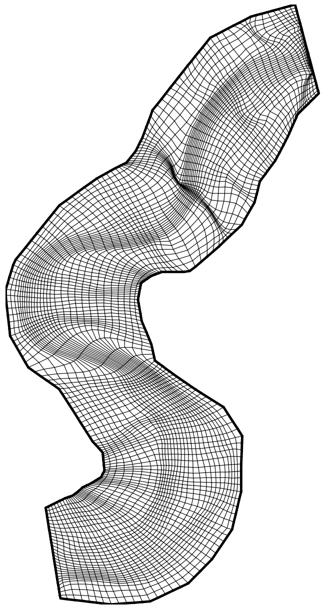
(e) S = 4.5 (f) S = 5.0

Figure 6 Meshes using EDS stretching function

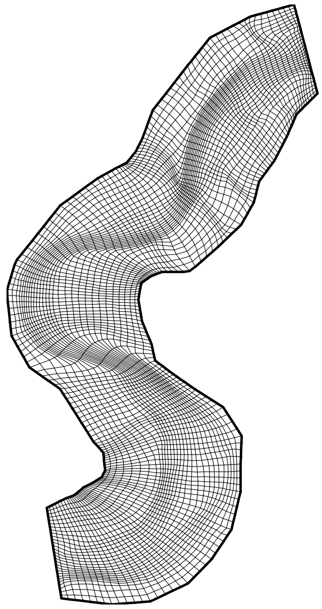
Figure 7 shows the comparison of adaptive meshes using the modified RL system with different smoothness parameter r_a and Table 3 summarizes the mesh quality. As can be seen, serious mesh distortion exists in the domain, as shown in Figure 7(a), and with considering mesh smoothness ($r_a > 0$), the overall mesh quality was significantly improved. The larger the smoothness parameter r_a is, the more mesh smoothness was gained.

Table 3 Evaluation of Meshes Using Modified RL System

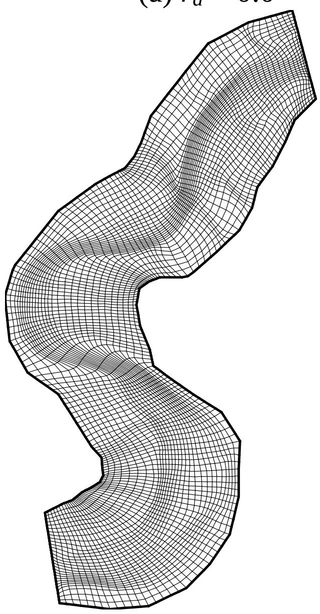
Domain	Case	Size	ADO	MDO	AAR	MAR	r_a
Natural River	A	30×150	2.60	12.19	2.52	91.9	0
	B	30×150	2.61	10.36	2.17	13.56	0.1
	C	30×150	2.63	9.90	2.12	12.44	0.15
	D	30×150	2.64	9.72	2.09	11.61	0.2
	E	30×150	2.65	9.73	2.06	10.89	0.25
	F	30×150	2.67	9.82	2.04	10.04	0.3
	G	30×150	2.70	10.19	1.96	8.39	0.5
	H	30×150	2.71	10.41	1.88	7.10	0.7



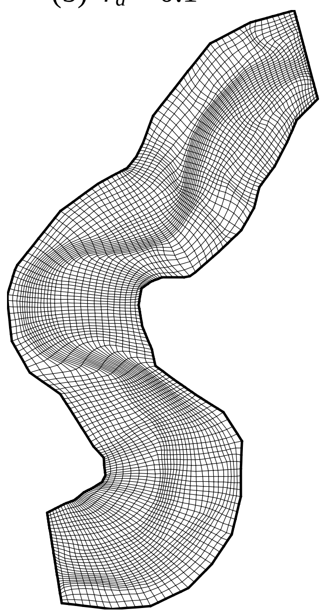
(a) $r_a = 0.0$



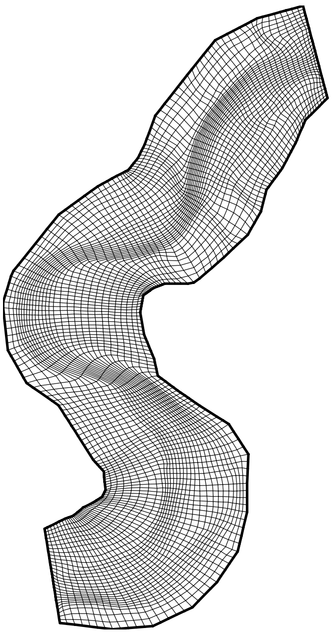
(b) $r_a = 0.1$



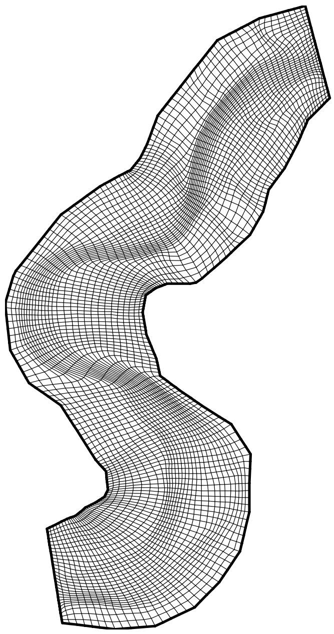
(c) $r_a = 0.15$



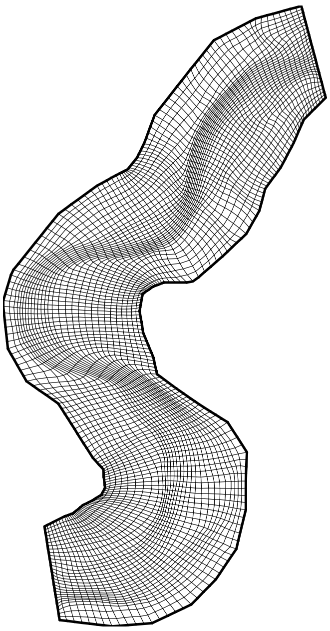
(d) $r_a = 0.2$



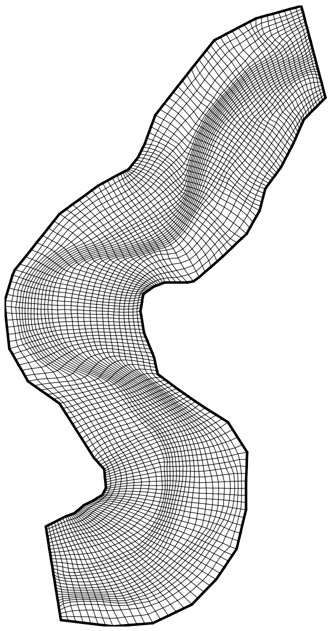
(e) r_a = 0.25



(f) r_a = 0.3



(g) r_a = 0.5



(h) r_a = 0.7

Figure 7 Depth-adaptive meshes using modified RL

5. CONCLUSIONS

In this paper, a three-parameter stretching function and a modified RL system are proposed to control the mesh density. It is shown that this stretching function is more flexible and powerful; the modified RL system is capable of producing near-orthogonal mesh with the effective control of mesh density. The proposed methods are simple, effective and easy to use.

REFERENCES

Brackbill, J.U., and Saltzman, J.S. (1982), "Adaptive zoning for singular problems in two dimensions", Journal of Computational Physics, Vol. 46: pp342-368.

Chuang, T.J. (2002), "Computational Fluid Dynamics: Automatic Grid Generation, Adaptive Methods, and Computing Techniques", Cambridge University Press, 2002.

Eiseman, P.R. (1979), "A Multi-surface method of coordinate generation", Journal of Computational Physics, Vol. 33: pp. 118-150.

Ryskin, G., and Leal, L.G. (1983) "Orthogonal Mapping", Journal of Computational Physics, Vol. 50, pp. 71-100.

Thompson J.F., Thames F.C., and Mastin C.W.(1977), TOMCAT—A code for numerical generation of boundary-fitted curvilinear coordinate system on fields containing any number of arbitrary two-dimensional bodies. *J. Comput. Phys*. 1977; 24:274-302

Thompson, J.F., Warsi, Z.U.A., and Mastin, C.W. (1985), *Numerical Grid Generation: Foundation and Application*, North-Holland, New York, 1985.

Zhang, Y.X., Jia, Y.F., and Wang, Sam S.Y. (2004), "2D Nearly Orthogonal Mesh Generation." *Int. J. Numer. Methods Fluids*, 2004; Vol. 46: pp. 685-707.

Zhang, Y.X, Jia Y.F., and Wang, Sam S. Y. (2006a), "Structured Mesh Generation with Smoothness Controls", *Int'l Journal for Numerical Methods in Fluids*, 2006; Vol. 51: pp. 1255-1276.

Zhang, Y.X., Jia, Y.F. and Wang, S.Y. (2006b), "2D Nearly Orthogonal Mesh Generation with Controls on Distortion Function", Journal of Computational Physics, 2006 (in press).