Spectral Density Functions and Their Applications

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Dedications

To my parents

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Abstract

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The Bernstein-Szegő measure moment problem asks when a given finite list of complex numbers form the Fourier coefficients of the spectral density function of a stable polynomial in the one-variable case. Szegő proved that it is possible if and only if the Toeplitz matrix form by these numbers is positive definite (see [Szegö 1919]). Bernstein ([Bernstein 1930]) later proved a real line analog of the problem.

The question remained open in two variables until Geronimo and Woerdeman stated and proved the necessary and sufficient conditions in [Geronimo and Woerdeman 2004]. Unlike the solution in one variable, it does not suffice to write down a single matrix and check whether it is positive definite. A positive definite completion condition is also required.

In this thesis, we further pursue the moment problem in two variables and beyond. We first enhance the two-variable results by identifying the eigenstructure of matrices that arise from the theory. We then create a method that allows us to compute the Fourier coefficients in a given infinite region by using a finite portion of the coefficients. Use is made of determinantal representations of stable polynomials. In addition, we compute the asymptotics for the Fourier coefficients and later generalize the result to higher dimensions. In the final chapter, we draw a connection between offset words and a particular type of spectral density functions and compute the asymptotics of the number of offset words as different parameter changes.

Chapter 1: INTRODUCTION

1.1 Background

The classical moment problem asks to find necessary and sufficient conditions for which an arbitrary sequence of real numbers $m_1, m_2, \dots$ is the sequence of moments $\int_E x^n \, d\mu(x)$ of a real measure μ over a specified range of integration E. (See, for instance, [Akhiezer 1965] and [Shohat and Tamarkin 1943].) One version of this problem is, given complex numbers $c_0 = \overline{c_0}, \dots, c_n = \overline{c_{-n}}$, to find a positive measure σ on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with moments $\widehat{\sigma}(k) = \int_{\mathbb{T}} z^k \, d\sigma(z) = c_k$ for each $k = -n, \dots, n$. The book [Bakonyi and Woerdeman 2011] provides a possible introduction to the study of assorted moment problems and other related topics. In particular, a *classical Bernstein-Szegő measure* is a measure on $\mathbb{T}$ of the form $\mathcal{S}_p(z) = |p(z)|^{-2} \, dz$, where p is a single-variable stable polynomial. These measures were first studied by Szegő in [Szegö 1919]. Bernstein later considered a real line analog in [Bernstein 1930].

The Bernstein-Szegő measure moment problem asks to find an n^{th} degree single-variable stable polynomial p so that the moments $\widehat{\mathcal{S}_p}(k)$ of the classical Bernstein-Szegő measure stemming from the spectral density function $\mathcal{S}_p$ satisfy $\widehat{\mathcal{S}_p}(k) = c_k$ for $k = -n, \dots, n$. Equivalently, the Fourier coefficients $(2\pi)^{-1} \int_0^{2\pi} e^{-ik\theta} \mathcal{S}_p(e^{i\theta}) \, d\theta$ must match the prescribed c_k. This moment problem is solvable if and only if the $(n+1) \times (n+1)$ Toeplitz matrix $T = (c_{i-j})_{i,j=0}^n$ is positive definite. Historical progress towards this solution is grounded in the works of Carathéodory, Toeplitz, and Szegő. Documented accounts and synopses of the subsequent mathematics developed by these three are relayed in [Akhiezer 1965] and [Aheizer and Krein 1962]. In this case, the stable polynomial $p(z) = p_0 + \dots + p_n z^n$ (which is unique when we require $p_0 > 0$) may be found via the Yule-Walker equation

$$\begin{pmatrix} c_0 & \overline{c}_1 & \cdots & \overline{c}_n \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{c}_1 \\ c_n & \cdots & c_1 & c_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\overline{p_0}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proofs of a matrix-valued version and a stronger, operator-valued version of the prior fact are offered in [Delsarte et al. 1978] and [Gohberg et al. 1989], respectively.

In the series of papers [Genin and Kamp 1977], [Delsarte et al. 1979], and [Delsarte et al. 1980], Genin and Kamp, who are later accompanied by Delsarte, initiated research towards establishing a general theory of multivariable orthogonal polynomials and Bernstein-Szegő measures over $\mathbb{T}^d$. In the bivariate setting, complex numbers $c_{k,l}$, $(k,l) \in \{0,\dots,n\} \times \{0,\dots,m\}$ are given. Geronimo and Woerdeman [Geronimo and Woerdeman 2004] stated and proved the following theorem:

Theorem 1.1.1. *Complex numbers $c_{k,l}$, $(k,l) \in \{0,\dots,n\} \times \{0,\dots,m\}$, are given. There exists a stable polynomial*

$$p(z,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} p_{kl} z^k w^l$$

with $p_{00} > 0$ so that its spectral density function

$$f(z,w) := \left(p(z,w)\overline{p(1/z,1/w)}\right)^{-1}$$

has Fourier coefficients $\hat{f}(k,l) = c_{kl}$, $(k,l) \in \{0,\dots,n\} \times \{0,\dots,m\}$, if and only if there exist complex numbers c_{kl}, $(k,l) \in \{1,\dots,n\} \times \{-m,\dots,-1\}$, so that the $(n+1)(m+1) \times (n+1)(m+1)$ doubly indexed Toeplitz matrix

$$\Gamma = \begin{pmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{pmatrix},$$

where

$$C_j = \begin{pmatrix} c_{j0} & \cdots & c_{j,-m} \\ \vdots & \ddots & \vdots \\ c_{jm} & \cdots & c_{j0} \end{pmatrix}, j = -n,\dots,n,$$

and $c_{-k,-l} = c_{kl}$, has the following properties:

1. *Γ is positive definite;*

2. *The* $(n+1)m \times (m+1)n$ *submatrix of* Γ *obtained by removing rows* $1 + j(m+1), j = 0, \ldots, n$ *and columns* $1, 2, \ldots, m+1$, *has rank* nm.

In this case one finds the column vector

$$\begin{bmatrix} p_{00}^2 & p_{00}p_{01} & \cdots & p_{00}p_{0m} & p_{00}p_{10} & \cdots & p_{00}p_{1m} & p_{00}p_{20} & \cdots & p_{00}p_{nm} \end{bmatrix}^T$$

as the first column of the inverse of Γ. *Here* ^{T} *denotes a transpose.*

Note that there are two key differences between this theorem and the solution to the classical Bernstein-Szegő measure moment problem. First, the $c_{k,l}$ with indices of mixed sign are missing and need to be identified. Second, the positive-definiteness of Γ no longer guarantees the existence of a stable polynomial with the desired properties; a low-rank condition must be satisfied as well.

1.2 Definitions and Notations

Throughout this paper, we will focus mainly on two-variable stable polynomials and their intersecting zeros. We will also explore some results in d variables. A d-variable scalar polynomial $p(z_1, \cdots, z_d)$ is called *stable* when p is nonzero for $(z_1, \cdots, z_d) \in \overline{\mathbb{D}}^d$, where $\overline{\mathbb{D}}$ is the closure of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. With p we will associate its adjoint and reverse. In particular, for two-variable scalar polynomial $p(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{ij} z^i w^j$, we will associate its *adjoint* $p^*(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} \bar{p}_{ij} z^i w^j$ and its *reverse* $\overleftarrow{p}(z,w) = z^n w^m \overline{p(1/\bar{z}, 1/\bar{w})} = \sum_{i=0}^{n}\sum_{j=0}^{m} \bar{p}_{ij} z^{n-i} w^{m-j}$.

The notion of intersecting zeros of $p(z,w)$ was introduced in [Geronimo and Woerdeman 2004]. We say (z,w) is an *intersecting zero* of $p(z,w)$ if

$$p(z,w) = \overleftarrow{p}(z,w) = 0.$$

If (z,w) is an intersecting zero of $p(z,w)$, it is straightforward to show that $(1/\bar{z}, 1/\bar{w})$ is also an intersecting zero.

For a stable polynomial $p(z, w)$ we define its *spectral density function* by

$$f(z, w) = \frac{1}{p(z, w)p^*(z^{-1}, w^{-1})}.$$

We are interested in the Fourier coefficients of the spectral density function. Given a function $f(z_1, \ldots, z_d)$, let $\hat{f}(k)$ be the k^{th} Fourier coefficient, where $k \in \mathbb{Z}^d$.

Frequently used symbols in this paper include $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{T}, \mathbb{D}$, and $\mathbb{C}$, which stand for the sets of positive integers, nonnegative integers, integers, complex numbers of modulus one, complex numbers with modulus less than one, and complex numbers, respectively.

Given subsets of $\mathbb{Z}^d$, we frequently order them using *lexicographical ordering*, which in two variables is defined by

$$(k, j) <_{\text{lex}} (k_1, j_1) \iff k < k_1 \text{ or } (k = k_1 \text{ and } l < l_1).$$

Throughout the paper we shall use matrices whose rows and columns are indexed by subsets of $\mathbb{Z}^d$. For example, if $\Lambda = \{(0, 0), (0, 1), (1, 1)\}$, then

$$C = (c_{u-v})_{u,v\in\Lambda}$$

is the 3×3 matrix

$$C = \begin{pmatrix} c_{00} & c_{0,-1} & c_{-1,-1} \\ c_{01} & c_{00} & c_{-1,0} \\ c_{11} & c_{10} & c_{00} \end{pmatrix}.$$

The matrix C may be referred to as an $\Lambda \times \Lambda$ matrix. The first row in this matrix will be referred to the $(0, 0)^{\text{th}}$ row, and similarly for the columns. The entries are referred to according to the row and column index. For example, $c_{1,1}$ is the $((1, 1), (0, 0))$ entry.

Lastly, for polynomials of one or two variables, we may consider ∞ as a root. In one variable, we say

$a(z) = \sum_{i=0}^{n} a_n z^n$ has a root at infinity when $a_n = 0$. Equivalently, ∞ is a root of $a(z)$ if and only if 0 is a root of $\overleftarrow{a}(z)$. In two variables, let

$$p(z, w) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} z^i w^j = \sum_{j=0}^{m} p_j(z) w^j = \sum_{i=0}^{n} \tilde{p}_i(w) z^i$$

be a polynomial of degree (n, m). Then $p(z, \infty) = 0$ corresponds to the statement $p_m(z) = 0$, while $p(\infty, w) = 0$ corresponds to the statement $\tilde{p}_n(w) = 0$. The statement $p(\infty, \infty) = 0$ corresponds to the statement $p_{nm} = 0$.

1.3 Results and Organization

In this dissertation, we set out to expand on the solution to the Bernstein-Szegő moment problem in two variables stated in Section 1.1 and subsequent results such that those summarized in [Bakonyi and Woerdeman 2011]. By doing so, we hoped to gain insights into the generalization of the problem in higher dimensions. While the question remains open, we were able to use the knowledge we gained and applied it to the combinatorical object called abelian squares and developed asymptotics of a generalized object.

In Chapter 2, we began with the examination of the matrices $\Phi_1 \Phi^{-1}$ and $\Phi_2^* \Phi^{-1}$ from Theorem 3.3.1 in [Bakonyi and Woerdeman 2011] (stated as Theorem 2.1.2 in Section 2.1). Mainly, we aimed to determine their eigenstructure. The result is given as Theorem 2.1.3, which completely describe the two matrices using solely the intersecting zeros of the polynomial p and $\overleftarrow{p}$, when the intersecting zeros are distinct. When an intersecting zero has a higher multiplicity, we are able to determine their eigenvectors. The question of how to determine the Jordan structure of $\Phi_1 \Phi^{-1}$ and $\Phi_2^* \Phi^{-1}$ remains open.

Following the above result, we expanded the region of computable Fourier coefficients in Theorem 2.2.1 in [Geronimo and Woerdeman 2004] (stated at Theorem 2.2.1 in Section 2.2) by artificially increasing the degrees of the stable polynomial p. By doing so, we introduced new intersecting zeros and subsequently arrived at Corollary 2.2.3. With this corollary, we are able to compute any Fourier coefficients, as long as we increase the degrees accordingly.

In Section 2.3, we explored the possibility of using determinantal representation of a stable polynomial and the Bernstein-Szegő moment problem. By using permanent expansion of determinantal represenation, we are able to offer an alternate proof to the low rank condition when the degree of the polynomial is $(1, 1)$. We are hopeful this connection will provide new insights when there are more than two variables.

In Section 2.4, we return our focus to the Fourier coefficients of the spectral density function. In particular, we aim to determine asymptotics of the coefficients in any given direction in two variables. In the second and fourth quadrants, we used Theorem 2.2.1 and Jordan decomposition of the matrices in the formula. In the first and third quadrants, we used the theory developed by DeVries, van der Hoeven, and Pemantle in [DeVries et al. 2011] to determine the asymptotics.

In Chapter 3, we presented a conjecture that aim to provide a first step toward an eventual solution to the moment problem in higher dimensions. The conjecture gave sufficient and necessary conditions for a list of complex numbers to be the Fourier coefficients of $p = 1 - \frac{z_1+z_2+z_3}{r}, r > 3$. To prove the conjecture, we only need to prove two recurrence relations, which can be checked numerically. The exploration of this polynomial gives rise to the study of abelian squares and their progenies in Chapter 4.

We also generalized the result in Section 2.4. Given a direction $\mathbf{r} = (r_1, \ldots, r_d), r_i > 0$ or $r_i < 0$ for all $i = 1, \ldots, d$, we found the upper bound for the decay rate. Notice the restriction we placed on the direction $\mathbf{r}$. In this case, the result only applies in the two specific regions. It remains open as to how to determine the asymptotics if the r_i are not all positive or negative.

In Chapter 4, we studied the application of the spectral density function $f = 1/|p|^2, p = 1-x\sum_{i=1}^{d} z_i, x < 1/d$. Through the joint work with Charles Burnette, Jr., we give a combinatorial interpretation of the Fourier coefficients of f by introducing the concept of offset words (See Definition 4.2.2). We then determine the asymptotics of the number of offset words in three ways: as the length of the words goes to infinity without changing the offsetting, as the number of letters in the alphabet goes to infinity with the length of the pre-offset words fixed, and as the length of the offset grows in a give direction. We conclude the chapter and the dissertation by discussing possible future work, the main one involving Bernstein-Szegő moment problem and other possible combinatoric classes. If this is achievable, it may provide a way to explore the moment problem further using combinatoric theories.

Chapter 2: Two-Variable Results

2.1 Eigenstructure of Matrices Associated with Bivariate Bernstein-Szegő Measures

Theorem 2.1.2 in Section 2.1.3 stated that, given c_{ij} in a set $\Lambda = \{0, 1, \ldots, n\} \times \{0, 1, \ldots, m\}$, there exists a stable polynomial $p(z, w) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} z^i w^j$ such that the Fourier coefficients $\hat{f}(i, j)$ of $f(z, w) = 1/|p(z, w)|^2$ equal c_{ij} for $(i, j) \in \Lambda$ if and only if certain conditions that involve the commutativity of two matrices S and $\tilde{S}$ built from the data c_{ij} are met. In this section, we will examine the relationship between the intersecting zeros of $p(z, w)$ and the eigenstructure of the matrices S and $\tilde{S}$ in the scalar case.

2.1.1 Stable Factorization

We will need the notions of left and right stable factorizations of matrix-valued trigonometric polynomials. The following definitions and Proposition 2.1.1 are from [Geronimo and Woerdeman 2004]. A square matrix polynomial $G(z)$ is called *stable* if $\det(G(z))$ is stable. Let $A(z) = \sum_{i=-n}^{n} A_i z^i$ be a matrix-valued trigonometric polynomial that is positive definite on $\mathbb{T}$. In particular, since the values of $A(z)$ on the unit circle are Hermitian, we have $A_i = A_{-i}^*, i = 0, \cdots, n$. The positive matrix function $A(z)$ allows a *left stable factorization*, that is,

$$A(z) = M(z) M(1/\bar{z})^*, z \in \mathbb{C}\backslash\{0\},$$

with $M(z)$ a stable matrix polynomial of degree n. In the scalar case, this is the well-known Fejér-Riesz factorization and goes back to the early 1900's. For the matrix case the result goes back to [Rosenblatt 1958] and [Helson 1964]. When we require that $M(0)$ is lower triangular with positive diagonal entries, the stable factorization is unique, which we shall call *the left stable factor* of $A(z)$. Similarly, we can define right variations of the above notions.

For a square matrix-valued function $G(z)$ we define its *spectrum* by $\Sigma(G) = \{z : \det G(z) = 0\}$. The following proposition plays an important role in the proof of the main theorem.

Proposition 2.1.1. *Let $p(z,w)$ be a stable polynomial of degree (n,m) with $p(0,0)>0$, and let $f(z,w)$ be its spectral density function. Write*

$$p(z,w)=\sum_{i=0}^{m} p_i(z)w^i, f(z,w)=\sum_{i=-\infty}^{\infty} f_i(z)w^i.$$

Put $p_i(z)\equiv 0$ for $i>m$, then the following hold:

i) $T_k(z):=(f_{i-j}(z))_{i,j=0}^{k}>0$ for all $k\in\mathbb{N}_0$ and all $z\in\mathbb{T}$.

ii) For all $k\geq m-1$ and for all z in the domain of T_k with $z\notin\Sigma(T_k)$:

$$T_k(z)^{-1}=\begin{bmatrix} p_0(z) & & 0 \\ \vdots & \ddots & \\ p_k(z) & \cdots & p_0(z)\end{bmatrix}\begin{bmatrix} \bar p_0(1/z) & \cdots & \bar p_k(1/z) \\ & \ddots & \vdots \\ 0 & & \bar p_0(1/z)\end{bmatrix}$$
$$-\begin{bmatrix} \bar p_{k+1}(1/z) & & 0 \\ \vdots & \ddots & \\ \bar p_1(1/z) & \cdots & \bar p_{k+1}(z)\end{bmatrix}\begin{bmatrix} p_{k+1}(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ 0 & & p_{k+1}(z)\end{bmatrix}:=E_k(z). \tag{2.1.1}$$

iii) For $k\geq m-1$, the left stable factors $M_k(z)$ and $M_{k+1}(z)$ of the positive trigonometric matrix polynomials $E_k(z)$ and $E_{k+1}(z)$, respectively, satisfy

$$M_{k+1}(z)=\begin{bmatrix} p_0(z) & 0 \\ col(p_l(z))_{l=1}^{k+1} & M_k(z)\end{bmatrix}. \tag{2.1.2}$$

iv) The spectra of $M_{m-1}, \overleftarrow{M}_{m-1}$ and z^nE_{m-1} are given by

$$\Sigma(M_{m-1})=\{z\in\mathbb{C}_\infty\backslash\overline{\mathbb{D}}:\exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p\},$$

$$\Sigma(\overleftarrow{M}_{m-1}) = \{z \in \overline{\mathbb{D}} : \exists w \textit{ such that } (z,w) \textit{ is an intersecting zero of } p\},$$

$$\Sigma(z^n E_{m-1}) = \{z \in \mathbb{C}_\infty : \exists w \textit{ such that } (z,w) \textit{ is an intersecting zero of } p\} \subset \mathbb{C}_\infty \backslash \mathbb{T}.$$

In particular, p has only a finite number of intersecting zeros. In addition, for $k \geq m$,

$$\begin{aligned}
\Sigma(M_k) &= \Sigma(M_{m-1}) \cup \{z \in \mathbb{C}_\infty : p_0(z) = 0\},\\
\Sigma(\overleftarrow{M}_k) &= \Sigma(\overleftarrow{M}_{m-1}) \cup \{z \in \mathbb{C}_\infty : \overleftarrow{p_0}(z) = 0\},\\
\Sigma(z^n E_k) &= \Sigma(M_k) \cup \Sigma(\overleftarrow{M}_k).
\end{aligned}$$

An analogous proposition can be stated with $p(z,w) = \sum_{j=0}^{n} \tilde{p}_j(w) z^j$, $f(z,w) = \sum_{j=-\infty}^{\infty} \tilde{f}_j(w) z^j$.

2.1.2 Two-Variable Bernstein-Szegő Measures

The following theorem regarding the two-variable matrix valued generalizations of Bernstein-Szegő measures can be found in [Bakonyi and Woerdeman 2011].

Theorem 2.1.2. *Let bounded linear operators $c_{ij} \in \mathcal{L}(\mathcal{H}), (i,j) \in \Lambda := \{-n, \cdots, n\} \times \{-m, \cdots, m\} \backslash \{(n,m), (-n,m), (n,-m), (-n,-m)\}$ be given. There exist stable polynomials*

$$p(z,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} z^i w^j \in \mathcal{L}(\mathcal{H}), r(z,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} r_{ij} z^i w^j \in \mathcal{L}(\mathcal{H}) \tag{2.1.3}$$

with $p_{00} > 0$ and $r_{00} > 0$ such that

$$p(z,w)^{*-1} p(z,w)^{-1} = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} z^i w^j = r(z,w)^{-1} r(z,w)^{*-1}, (z,w) \in \mathbb{T}^2, \tag{2.1.4}$$

for some $c_{ij} \in \mathcal{L}(\mathcal{H}), (i,j) \notin \Lambda$, if and only if

i) $\Phi_1 \Phi^{-1} \Phi_2^* = \Phi_2^* \Phi^{-1} \Phi_1$;

ii) when we put

$$c_{-n,m} = row(c_{k-l})_{\substack{k=(0,m-1), \\ l\in\{1,\cdots,n\}\times\{0,\cdots,m-1\}}} \Phi^{-1} col(c_{k-l})_{\substack{k\in\{0,\cdots,n-1\}\times\{1,\cdots,m\}, \\ l=(n-1,0)}},$$

then the operators

$$(c_{k-l})_{k,l\in\{0,\cdots,n\}\times\{0,\cdots,m\}\backslash\{(n,m)\}} \text{ and } (c_{k-l})_{k,l\in\{0,\cdots,n\}\times\{0,\cdots,m\}\backslash\{(0,0)\}}$$

are positive definite.

Here

$$\Phi = (c_{k-l})_{k,l\in\{0,\cdots,n-1\}\times\{0,\cdots,m-1\}},$$

$$\Phi_1 = (c_{k-l})_{k\in\{0,\cdots,n-1\}\times\{0,\cdots,m-1\}, l\in\{1,\cdots,n\}\times\{0,\cdots,m-1\}},$$

$$\Phi_2 = (c_{k-l})_{k\in\{0,\cdots,n-1\}\times\{0,\cdots,m-1\}, l\in\{0,\cdots,n-1\}\times\{1,\cdots,m\}}.$$

There is a unique choice for $c_{n,m}$ that results in $p_{n,m} = 0$, namely

$$c_{n,m} = (c_{k-l})_{k=(n,m), l\in\{0,\cdots,n\}\times\{0,\cdots,m\}\backslash\{(0,0),(n,m)\}}$$
$$\times[(c_{k-l})_{k,l\in\{0,\cdots,n\}\times\{0,\cdots,m\}\backslash\{(0,0),(n,m)\}}]^{-1}(c_{k-l})_{\substack{k\in\{0,\cdots,n\}\times\{0,\cdots,m\}\backslash\{(0,0),(n,m)\} \\ l=(0,0)}}.$$

When i) and ii) are satisfied, the coefficients of the polynomials $p(z,w)$ and $r(z,w)$ can be found via the equations

$$(c_{k-l})_{k,l\in\{0,\cdots,n\}\times\{0,\cdots,m\}} col(p_{ij})_{i\in\{0,\cdots,n\},\{0,\cdots,m\}} = e_{00}p_{00}^{*-1}, \tag{2.1.5}$$

$$(c_{k-l})_{k,l\in\{0,\cdots,n\}\times\{0,\cdots,m\}} col(r_{n-i,m-j})_{i\in\{0,\cdots,n\},\{0,\cdots,m\}} = e_{00}r_{00}^{*-1}. \tag{2.1.6}$$

If one requires that $p_{00} > 0$ and $r_{00} > 0$, the solutions above are unique.

The matrices that we are interested in are $\Phi_1\Phi^{-1}$ and $\Phi_2^*\Phi^{-1}$. In particular, we will consider the case when $p(z,w)$ is a scalar polynomial and all of its intersecting zeros of are distinct.

2.1.3 Main Result

The following theorem is the main result in this section.

Theorem 2.1.3. *Let c_{ij}, $(i,j) \in \Lambda := \{-n,\cdots,n\}\times\{-m,\cdots,m\} \setminus \{(n,m),(-n,m),(n,-m),(-n,-m)\}$ be such that they satisfy the conditions of Theorem 2.1.2. Suppose the intersecting zeros of the stable polynomial $p(z,w)$ that arose from the c_{ij} are $(z_1,\frac{1}{\bar{w}_1}),\cdots,(z_{nm},\frac{1}{\bar{w}_{nm}})$ and $(\frac{1}{\bar{z}_1},w_1),\cdots,(\frac{1}{\bar{z}_{nm}},w_{nm})$, $z_{1,\ldots,nm},w_{1,\ldots,nm} \in \mathbb{D}$, with multiplicity of 1. Let*

$$v_j = \begin{pmatrix} z_j^{n-1} \\ \vdots \\ z_j \\ 1 \end{pmatrix} \bigotimes \begin{pmatrix} 1 \\ \bar{w}_j \\ \vdots \\ \bar{w}_j^{n-1} \end{pmatrix}, j = 1,\cdots,nm. \tag{2.1.7}$$

Then $\{v_1,\cdots,v_{nm}\}$ are common eigenvectors of $\Phi_1\Phi^{-1}$ and $\Phi_2^*\Phi^{-1}$, with eigenvalues z_j and $\bar{w}_j$, respectively. Furthermore, $\{v_1,\cdots,v_{nm}\}$ are linearly independent.

In order to prove Theorem 2.1.3, we need the following lemma.

Lemma 2.1.4. *Given a stable polynomial $p(z,w)$ of degree (n,m), $\det(E_{m-1}(z)) = 0$ if and only if there exist a w such that (z,w) is an intersecting zero.*

Proof. Let $p(z,w) = \displaystyle\sum_{i=0}^{m} p_i(z)w^i$ be a stable polynomial of degree (n,m). If $\det(E_{m-1}(z)) = 0$, then the lemma follows from Proposition 2.1.1 (iv).

For the converse, when $z \neq 0$, let $q(z,w) = \frac{1}{z^n}\overleftarrow{p}(z,w) = \displaystyle\sum_{i=0}^{m} \bar{p}_{m-i}\left(\frac{1}{z}\right)w^i$. Consider the following resultant matrix

$$R(q,p) = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where

$$A = \begin{bmatrix} \bar{p}_0\left(\frac{1}{z}\right) & \cdots & \bar{p}_{m-1}\left(\frac{1}{z}\right) \\ & \ddots & \vdots \\ & & \bar{p}_0\left(\frac{1}{z}\right) \end{bmatrix}, B = \begin{bmatrix} \bar{p}_m\left(\frac{1}{z}\right) & & \\ \vdots & \ddots & \\ \bar{p}_1\left(\frac{1}{z}\right) & \cdots & \bar{p}_m\left(\frac{1}{z}\right) \end{bmatrix},$$

$$C = \begin{bmatrix} p_m(z) & \cdots & p_1(z) \\ & \ddots & \vdots \\ & & p_m(z) \end{bmatrix}, D = \begin{bmatrix} p_0(z) & & \\ \vdots & \ddots & \\ p_{m-1}(z) & \cdots & p_0(z) \end{bmatrix}.$$

It is simple to check that B and D commute. Therefore, for D invertible,

$$\det(R(q,p)) = \det D \det(A - BD^{-1}C) = \det(DA - BC) = \det(E_{m-1}(z)).$$

For D not invertible, let $\epsilon > 0$ and consider

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix}.$$

Then $D + \epsilon I$ is invertible and

$$\begin{aligned} \det(R(q,p)) &= \lim_{\epsilon \to 0} \det(D + \epsilon I) \det(A - B(D + \epsilon I)^{-1} C) \\ &= \det(DA - BC) = \det(E_{m-1}(z)). \end{aligned}$$

Recall that the resultant is 0 if and only if the two polynomials share a common roots. But (z, w) is an intersecting zero, therefore $\det(R(q,p)) = \det(E_{m-1}(z)) = 0$.

If $z = 0$, $p(z,w)$ reduces to $p(w) = \sum_{j=0}^{m} p_{0j} w^j$ and $\overleftarrow{p}(z,w)$ becomes $\overleftarrow{p}(w) = \sum_{j=0}^{m} \bar{p}_{0j} w^{m-j} = \sum_{j=0}^{m} \bar{p}_{0,m-j} w^j$.

A, B, C, D reduces to

$$A = \begin{bmatrix} \bar{p}_{00} & \cdots & \bar{p}_{0,m-1} \\ & \ddots & \vdots \\ & & \bar{p}_{00} \end{bmatrix}, B = \begin{bmatrix} \bar{p}_{0m} & & \\ \vdots & \ddots & \\ \bar{p}_{01} & \cdots & \bar{p}_{0m} \end{bmatrix}, C = \begin{bmatrix} p_{0m} & \cdots & p_{01} \\ & \ddots & \vdots \\ & & p_{0m} \end{bmatrix}, D = \begin{bmatrix} p_{00} & & \\ \vdots & \ddots & \\ p_{0,m-1} & \cdots & p_{00} \end{bmatrix}.$$

and the rest follows similarly from the case $z \neq 0$. □

We also need the following one-variable result:

Lemma 2.1.5. *Let* $T_m(z) = \sum_{i=-\infty}^{\infty} C_i z^i$, *where* $C_k = \begin{bmatrix} c_{k0} & \cdots & c_{k,-m} \\ \vdots & \ddots & \vdots \\ c_{k,m} & \cdots & c_{k0} \end{bmatrix}$, $k \in \mathbb{Z}$. *Let* $M_m(z) = \sum_{i=0}^{n} M_i z^i$ *be the left stable factor defined in Proposition 2.1.1 (iii). Then* $T_m(z)M_m(z) = M_m(\frac{1}{\bar{z}})^{*-1}$ *and*

$$\begin{bmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix} \begin{bmatrix} M_0 \\ \vdots \\ M_n \end{bmatrix} = \begin{bmatrix} M_0^{*-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.1.8}$$

Similarly, if $R_m(z) = \sum_{i=0}^{n} R_i z^i$ *is the right stable factor of* $E_m(z)$, *then* $T_m(z)R_m\left(\frac{1}{\bar{z}}\right)^* = R_m(z)^{-1}$ *and*

$$\begin{bmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix} \begin{bmatrix} R_n^* \\ \vdots \\ R_0^* \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R_0^{-1} \end{bmatrix}. \tag{2.1.9}$$

Proof. Since $T_m(z) = E_m(z)^{-1} = M_m(\frac{1}{\bar{z}})^{*-1} M_m(z)^{-1}$,

$$T_m(z)M_m(z) = \sum_{i=-\infty}^{\infty} C_i z^i \sum_{j=0}^{n} M_j z^j = M_0^{*-1} + O(1/z),$$

we have

$$C_0M_0 + C_{-1}z^{-1}M_1z + \cdots + C_{-n}z^{-n}M_nz^n = M_0^{*-1}$$

and so forth. Statement 2.1.9 can be proven similarly. □

The above lemma can be proven similarly with w as the variable. Let us now state the proof of Theorem 2.1.3.

Proof. First, consider $\Phi_1\Phi^{-1}$. Notice that

$$\Phi = \begin{bmatrix} \tilde{C}_0 & \cdots & \tilde{C}_{-n+1} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{n-1} & \cdots & \tilde{C}_0 \end{bmatrix}, \Phi_1 = \begin{bmatrix} \tilde{C}_{-1} & \cdots & \tilde{C}_{-n} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{n-2} & \cdots & \tilde{C}_{-1} \end{bmatrix}, \text{ where } \tilde{C}_k = \begin{bmatrix} c_{k0} & \cdots & c_{k,-m+1} \\ \vdots & \ddots & \vdots \\ c_{k,m-1} & \cdots & c_{k0} \end{bmatrix},$$

for $k = -n, \cdots, n$. Therefore, we must have

$$\Phi_1\Phi^{-1} = \begin{bmatrix} * & \cdots & \cdots & * \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix}.$$

Recall from Lemma 2.1.5,

$$\begin{bmatrix} C_0 & \cdots & C_{-n} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix} \begin{bmatrix} M_0 \\ \vdots \\ M_n \end{bmatrix} = \begin{bmatrix} M_0^{*-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Remove the first block row, we have

$$\begin{bmatrix} C_1 & \cdots & C_{-n+1} \\ \vdots & \ddots & \vdots \\ C_n & \cdots & C_0 \end{bmatrix} \begin{bmatrix} M_0 \\ \vdots \\ M_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using the fact

$$C_k = \begin{bmatrix} c_{k0} & row(c_{k,-j})_{j=1}^m \\ col(c_{kj})_{j=1}^m & \tilde{C}_k \end{bmatrix}, M_m(z) = \begin{bmatrix} p_0(z) & 0 \\ col(p_l(z))_{l=1}^m & M_{m-1}(z) \end{bmatrix},$$

we can remove the first row and column from every block matrix. Thus, if we define $M_{m-1}(z) = \sum_{i=0}^{n} \tilde{M}_i z^i$, we have

$$\begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_{-n+1} \\ \vdots & \ddots & \vdots \\ \tilde{C}_n & \cdots & \tilde{C}_0 \end{bmatrix} \begin{bmatrix} \tilde{M}_0 \\ \vdots \\ \tilde{M}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

which can be expressed as

$$\begin{bmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_n \end{bmatrix} \tilde{M}_0 + \begin{bmatrix} \tilde{C}_0 & \cdots & \tilde{C}_{-n+1} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{n-1} & \cdots & \tilde{C}_0 \end{bmatrix} \begin{bmatrix} \tilde{M}_1 \\ \vdots \\ \tilde{M}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_n \end{bmatrix} = - \begin{bmatrix} \tilde{C}_0 & \cdots & \tilde{C}_{-n+1} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{n-1} & \cdots & \tilde{C}_0 \end{bmatrix} \begin{bmatrix} \tilde{M}_1 \\ \vdots \\ \tilde{M}_n \end{bmatrix} \tilde{M}_0^{-1}.$$

By taking the Hermitian transpose, we have

$$\begin{bmatrix} \tilde{C}_{-1} & \cdots & \tilde{C}_{-n} \end{bmatrix} = -\tilde{M}_0^{*-1} \begin{bmatrix} \tilde{M}_1^* & \cdots & \tilde{M}_n^* \end{bmatrix} \begin{bmatrix} \tilde{C}_0 & \cdots & \tilde{C}_{-n+1} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{n-1} & \cdots & \tilde{C}_0 \end{bmatrix}.$$

Thus, we have

$$\Phi_1\Phi^{-1} = \begin{bmatrix} -\tilde{M}_0^{*-1}\tilde{M}_1^* & \cdots & \cdots & -\tilde{M}_0^{*-1}\tilde{M}_n^* \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix}.$$

Now, suppose z is an eigenvalue of $\Phi_1\Phi^{-1}$ with eigenvector $\mathbf{X}$ =col$(X_i)_{i=1}^n$. We then have

$$-\tilde{M}_0^{*-1}\tilde{M}_1^* X_1 - \cdots - \tilde{M}_0^{*-1}\tilde{M}_n^* X_n = zX_1, X_1 = zX_2, \ldots, X_{n-1} = zX_n.$$

But this means that

$$-\tilde{M}_0^{*-1}(\tilde{M}_1^* z^{n-1} + \cdots + \tilde{M}_n^*)X_n = z^n X_n,$$

which simplifies to

$$z^n \left(\tilde{M}_0^* + \frac{1}{z}\tilde{M}_1^* + \cdots + \frac{1}{z^n}\tilde{M}_n^* \right) X_n = 0.$$

But $\tilde{M}_0^* + \frac{1}{z}\tilde{M}_1^* X_n + \cdots + \frac{1}{z^n}\tilde{M}_n^* = M_{m-1}(\frac{1}{\bar{z}})^*$, and $\det(M_{m-1}(\frac{1}{\bar{z}})^*) = 0$ implies that $\det(E_{m-1}(z)) = 0$. Therefore, by Lemma 2.1.4, there exists a w such that $(z, \frac{1}{\overline{w}})$ is an intersecting zero of p. Since $M_{m-1}(z)^*$ is anti-stable, we have that $z, w \in \mathbb{D}$.

It remains to show that $\mathbf{X}$ =col$(z^{n-1-i}X)_{i=0}^{n-1}$ = col$(z^{n-1-i})_{i=0}^{n-1} \bigotimes$col$(\overline{w}^j)_{j=0}^{n-1}$. In other words, we need to show that $M_{m-1}(\frac{1}{\bar{z}})^*col(\overline{w}^j)_{j=0}^{n-1} = 0$. For this, consider again the resultant matrix from Lemma 2.1.4, with $k = m$. We have that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 \\ \bar{w} \\ \vdots \\ \bar{w}^{2m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

But then we have

$$A \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix} + B \begin{bmatrix} \bar{w}^m \\ \vdots \\ \bar{w}^{2m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } C \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix} + D \begin{bmatrix} \bar{w}^m \\ \vdots \\ \bar{w}^{2m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solving for $\mathrm{col}(\bar{w}^i)_{i=m}^{2m-1}$ from the second equality and substituting into the first equation, we have

$$\begin{bmatrix} \bar{w}^m \\ \vdots \\ \bar{w}^{2m-1} \end{bmatrix} = -D^{-1}C \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix}, (A - BD^{-1}C) \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Multiplying D on the left, we get

$$E_{m-1}(z) \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix} = M_{m-1}(z)M_{m-1}(1/\bar{z})^* \begin{bmatrix} 1 \\ \vdots \\ \bar{w}^{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $M_{m-1}(z)$ is stable, it follows that $\mathrm{col}(\bar{w}^i)_{i=0}^{m-1}$ in the kernel of $M_{m-1}(1/\bar{z})^*$, as desired.

Next, consider $\Phi_2^*\Phi^{-1}$. Observe that

$$\Phi_2^*\Phi^{-1} = (Q_{ij})_{i,j=0}^{n-1}, Q_{ij} = \begin{bmatrix} 0 & \delta_{ij}I & & \\ & \ddots & \ddots & \\ & & 0 & \delta_{ij}I \\ * & * & * & * \end{bmatrix},$$

where δ_{ij} is the kronecker delta.To show that $\Phi_2^*\Phi^{-1}$ has the desired properties, let

$$\hat{\Phi} = P\Phi P^{-1} = \begin{bmatrix} \hat{C}_0 & \cdots & \hat{C}_{-m+1} \\ \vdots & \ddots & \vdots \\ \hat{C}_{m-1} & \cdots & \hat{C}_0 \end{bmatrix}, \hat{\Phi}_2 = P\Phi_2 P^{-1} = \begin{bmatrix} \hat{C}_{-1} & \cdots & \hat{C}_{-m} \\ \vdots & \ddots & \vdots \\ \hat{C}_{m-2} & \cdots & \hat{C}_{-1} \end{bmatrix},$$

where P is a permutation matrix such that

$$\hat{C}_k = \begin{bmatrix} c_{0k} & \cdots & c_{-n+1,k} \\ \vdots & \ddots & \vdots \\ c_{n-1,k} & \cdots & c_{0k} \end{bmatrix}.$$

i.e. $\hat{\Phi}$ and $\hat{\Phi}_2$ are in reverse lexicographic order. We then follow the same proof for $\Phi_1\Phi^{-1}$ and have that

$$\hat{\Phi}_2^*\hat{\Phi}^{-1} = \begin{bmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & 0 & I \\ -\tilde{R}_0^{-1}\tilde{R}_m & \cdots & \cdots & -\tilde{R}_0^{-1}\tilde{R}_1 \end{bmatrix},$$

where $R_{n-1}(w) = \sum_{i=0}^{m} \tilde{R}_i w^i$ is the right stable factor of $\tilde{E}_{n-1}(w)$. We can then show $\bar{w}$ is an eigenvalue for $\hat{\Phi}_2^*\hat{\Phi}^{-1}$ with an eigenvector $\mathrm{col}(\bar{w}^i)_{i=0}^{m-1} \bigotimes \mathrm{col}(z^{n-j-1})_{j=0}^{n-1}$. This implies that $\Phi_2^*\Phi^{-1}$ has $\bar{w}$ as an eigenvalue with eigenvector $\mathrm{col}(z^{n-i-1})_{i=0}^{n-1} \bigotimes \mathrm{col}(\bar{w}^j)_{j=0}^{m-1}$, as desired.

To show $v_1, \cdots, v_{nm}$ are linearly independent, suppose

$$a_1 v_1 + \cdots + a_{nm} v_{nm} = 0.$$

Grouping the v_i's based on their eigenvalues z_i's, we have

$$(a_{11}v_{11} + \cdots + a_{1k_1}v_{1k_1}) + \cdots + (a_{j1}v_{j1} + \cdots + a_{jk_j}v_{jk_j}) = 0,$$

where $v_{li}, i = 1, \cdots, k_l$ is an eigenvector for $z_l, l = 1, \cdots, j$. Also, $(a_{l1}v_{l1} + \cdots + a_{lk_l}v_{lk_l})$ is an eigenvector for z_l. Since distinct eigenvalues has linearly independent eigenvectors,

$$a_{l1}v_{l1} + \cdots + a_{lk_l}v_{lk_l} = 0, l = 1, \cdots, j.$$

We also have that v_{li} is an eigenvector for $\bar{w}_{li}$. Since the multiplicity of each intersecting zeros is 1, $\bar{w}_{li} \neq \bar{w}_{l\tilde{i}}$ for $i \neq \tilde{i}$. Therefore, $v_{l1}, \cdots, v_{lk_l}$ are linearly independent, and

$$a_{l1} = \cdots = a_{lk_l} = 0, l = 1, \cdots, j.$$

Therefore, $a_i = 0$ for $i = 1, \cdots, nm$, and $v_1, \cdots, v_{nm}$ are linearly independent. $\square$

We have the following corollary if the multiplicity is higher than 1.

Corollary 2.1.6. *Let c_{ij}, $(i,j) \in \Lambda := \{-n, \cdots, n\} \times \{-m, \cdots, m\} \setminus \{(n,m), (-n,m), (n,-m), (-n,-m)\}$ be such that they satisfy the conditions of Theorem 2.1.2. Suppose $(z_i, 1/\bar{w}_i)$ is a repeated intersecting zeros of the stable polynomial $p(z,w)$ that arose from the c_{ij}. If k is the minimum multiplicity of $1/\bar{w}_i$ as a zero* *of* $p(z_i, w)$ *and* $\overleftarrow{p}(z_i, w)$*, then*

$$\begin{pmatrix} z_i^{n-1} \\ \vdots \\ 1 \end{pmatrix} \bigotimes \frac{d^l}{dw^l} \begin{pmatrix} 1 \\ \vdots \\ w^{m-1} \end{pmatrix} \Bigg|_{\bar{w}_i}, l = 0, \cdots, k-1,$$

are eigenvectors of z_i *of* $\Phi_1\Phi^{-1}$*. Similarly, if* $\tilde{k}$ *is the minimum multiplicity of* z_i *as a zero of* $p(z, 1/\bar{w}_i)$ *and* $\overleftarrow{p}(z, 1/\bar{w}_i)$*, then*

$$\frac{d^l}{dz^l} \begin{pmatrix} z^{n-1} \\ \vdots \\ 1 \end{pmatrix} \Bigg|_{z_i} \bigotimes \begin{pmatrix} 1 \\ \vdots \\ \bar{w}_i^{m-1} \end{pmatrix}, l = 0, \cdots, \tilde{k}-1,$$

are eigenvectors of $\bar{w}_i$ *of* $\Phi_2^*\Phi^{-1}$*.*

Proof. Consider the eigenvectors of z_i for $\Phi_1\Phi^{-1}$. Suppose k is the minimum multiplicity of $1/\bar{w}_i$ as a zero of $p(z_i, w)$ and $\overleftarrow{p}(z_i, w)$. This implies that

$$\frac{d^l}{dw^l} p(z_i, w)\Big|_{\bar{w}_i} = \frac{d^l}{dw^l} \overleftarrow{p}(z_i, w)\Big|_{\bar{w}_i} = 0, l = 0, \cdots, k-1.$$

Therefore, for the matrix $R(q, p)$ from lemma 2.1.4, with $z = z_i$, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \frac{d^l}{dw^l}\Big|_{\bar{w}_i} \begin{bmatrix} 1 \\ \vdots \\ w^{2m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, l = 0, \cdots, k-1.$$

We can prove the statement about the eigenvectors of $\bar{w}_i$ for $\Phi_2^*\Phi^{-1}$ similarly. □

Furthermore, similar statements can be made when we have $\Phi^{-1}\Phi_1$ and $\Phi^{-1}\Phi_2^*$.

Theorem 2.1.7. *Let* $p(z, w) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{ij} z^i w^j$ *be a stable polynomial of degree* (n, m)*, and let* $f(z, w)$ *be its spectral density function. Suppose the intersecting zeros of* $p(z, w)$*are* $(z_1, \frac{1}{\bar{w}_1}), \cdots, (z_{nm}, \frac{1}{\bar{w}_{nm}})$ *and*

$(\frac{1}{\bar{z}_1}, w_1), \cdots, (\frac{1}{\bar{z}_{nm}}, w_{nm})$, $z_{1,\ldots,nm}, w_{1,\ldots,nm} \in \mathbb{D}$, with multiplicity of 1. Then

$$v_j = \begin{pmatrix} 1 & z_j & \ldots & z_j^{n-1} \end{pmatrix} \bigotimes \begin{pmatrix} \bar{w}_j^{m-1} & \ldots & \bar{w}_j & 1 \end{pmatrix} \tag{2.1.10}$$

are common left eigenvectors of $\Phi^{-1}\Phi_1$ and $\Phi^{-1}\Phi_2^*$, with eigenvalues z_j and $\bar{w}_j$, $j = 1, \ldots, nm$, respectively. Furthermore, $\{v_1, \cdots, v_{nm}\}$ are linearly independent.

Corollary 2.1.8. *Let $p(z,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{ij} z^i w^j$ be a stable polynomial of degree (n,m), and let $f(z,w)$ be its spectral density function. Suppose $(z_i, 1/\bar{w}_i)$ is a repeated intersecting zero. If k is the minimum multiplicity of $1/\bar{w}_i$ as a zero of $p(z_i,w)$ and $\overleftarrow{p}(z_i,w)$, then*

$$\begin{pmatrix} 1 & z_j & \ldots & z_j^{n-1} \end{pmatrix} \bigotimes \frac{d^l}{dw^l} \begin{pmatrix} w^{m-1} & \ldots & w & 1 \end{pmatrix} \Big|_{\bar{w}_j}, l = 0, \cdots, k-1,$$

are left eigenvectors of z_i of $\Phi^{-1}\Phi_1$. Similarly, if $\tilde{k}$ is the minimum multiplicity of z_i as a zero of $p(z, 1/\bar{w}_i)$ and $\overleftarrow{p}(z, 1/\bar{w}_i)$, then

$$\frac{d^l}{dz^l} \begin{pmatrix} 1 & z & \ldots & z^{n-1} \end{pmatrix} \Big|_{z_j} \bigotimes \begin{pmatrix} \bar{w}_j^{m-1} & \ldots & \bar{w}_j & 1 \end{pmatrix}, l = 0, \cdots, \tilde{k}-1,$$

are left eigenvectors of $\bar{w}_i$ of $\Phi^{-1}\Phi_2^*$.

Proofs for Theorem 2.1.7 and Corollary 2.1.8 are similar to the previous proofs and will be omitted.

2.1.4 Examples

For all of the numerical examples in this thesis involving Fourier coefficients of spectral density functions, we use the algorithm developed in Section 5 of [Woerdeman et al. 2003].

Example 2.1.9. Suppose $p(z,w) = (z-2)(z-3)(w-2)(w-3)$. Then $\overleftarrow{p}(z,w) = (1-2z)(1-3z)(1-2w)(1-3w)$. The intersecting zeros are $(2,1/2), (2,1/3), (3,1/2), (3,1/3), (1/2,2), (1/2,3), (1/3,2)$, and $(1/3,3)$. We then have

$$\Phi = \frac{1}{36}\begin{bmatrix} 0.1225 & 0.0875 & 0.0875 & 0.0625 \\ 0.0875 & 0.1225 & 0.0625 & 0.0875 \\ 0.0875 & 0.0625 & 0.1225 & 0.0875 \\ 0.0625 & 0.0875 & 0.0875 & 0.1225 \end{bmatrix}, \Phi_1 = \frac{1}{36}\begin{bmatrix} 0.0875 & 0.0625 & 0.0525 & 0.0375 \\ 0.0625 & 0.0875 & 0.0375 & 0.0525 \\ 0.1225 & 0.0875 & 0.0875 & 0.0625 \\ 0.0875 & 0.1225 & 0.0625 & 0.0875 \end{bmatrix},$$

and

$$\Phi_2 = \frac{1}{36}\begin{bmatrix} 0.0875 & 0.0525 & 0.0625 & 0.0375 \\ 0.1225 & 0.0875 & 0.0875 & 0.0625 \\ 0.0625 & 0.0375 & 0.0875 & 0.0525 \\ 0.0875 & 0.0625 & 0.1225 & 0.0875 \end{bmatrix}.$$

Therefore,

$$\Phi_1\Phi^{-1} = \begin{bmatrix} 5/6 & 0 & -1/6 & 0 \\ 0 & 5/6 & 0 & -1/6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \Phi_2^*\Phi^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/6 & 5/6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/6 & 5/6 \end{bmatrix}.$$

$\Phi_1\Phi^{-1}$ and $\Phi_2^*\Phi^{-1}$ both have eigenvalues 1/2 with multiplicity of 2, and 1/3 with multiplicity of 2 with eigenvectors

$$\begin{bmatrix} 1/2 \\ 1/4 \\ 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/6 \\ 1 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/6 \\ 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/9 \\ 1 \\ 1/3 \end{bmatrix}.$$

The following example shows that the two matrices still share an eigenvector of the desired form when there are repeated intersecting zeros.

Example 2.1.10. *Suppose* $p(z,w) = (z-2)^2(w-3)^2$. *Then* $\overleftarrow{p}(z,w) = (1-2z)^2(1-3w)^2$. *The intersecting* *zeros are* $(2, 1/3)$ *and* $(1/2, 3)$*, each with multiplicities* 4*. We then have*

$$\Phi_1\Phi^{-1} = \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -1/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \Phi_2^*\Phi^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/9 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/9 & 2/3 \end{bmatrix}.$$

$\Phi_1\Phi^{-1}$ *has eigenvalue* $1/2$ *with multiplicity* 4*, while* $\Phi_2^*\Phi^{-1}$ *has eigenvalue* $1/3$ *with multiplicity* 4*, as well. One can show that* $[1/2, 1/6, 1, 1/3]$ *is an eigenvector for both matrices, while* $[0, 1/2, 0, 1]$ *is another eigenvector of* $1/2$ *of* $\Phi_1\Phi^{-1}$*, and* $[1, 1/3, 0, 0]$ *is another eigenvector of* $1/3$ *of* $\Phi_2^*\Phi^{-1}$*.*

2.2 Increasing the Degree of the Stable Polynomial and Applications

In [Geronimo and Woerdeman 2004], the following theorem was stated and it provided a formula for calculating Fourier coefficients of the spectral density function $f(z, w)$ in the given region.

Theorem 2.2.1. *Let* $p(z, w)$ *be a stable polynomial of degree* (n, m)*, and let* $f(z, w)$ *be its spectral density function. Then there exists a row vector* $x \in \mathbb{C}^{nm}$*, a column vector* $y \in \mathbb{C}^{nm}$ *and commuting matrices* $S, \tilde{S} \in \mathbb{C}^{nm \times nm}$ *such that*

$$\sigma(S) = \{z \in \mathbb{D} : \exists w \text{ such that } (z, w) \text{ is an intersecting zero of } p\},$$

$$\sigma(\tilde{S}) = \{w \in \mathbb{D} : \exists z \text{ such that } (z, \bar{w}) \text{ is an intersecting zero of } p\}, \tag{2.2.1}$$

and

$$\hat{f}(k, l) = x\tilde{S}^{m+l-1}S^{n-1-k}y, \qquad k \le n-1, \qquad l \ge -m+1. \tag{2.2.2}$$

Choose x, y, S and $\tilde{S}$ as follows:

$$x = row(\hat{f}((n-1,0)-u))_{u\in\Delta}$$

$$y = col\left(\delta_{u+(0,-m+1)}\right)_{u\in\Delta}, S = \Phi^{-1}\Phi_1, \tilde{S} = \Phi^{-1}\Phi_2^*, \tag{2.2.3}$$

where

$$\Phi = (\hat{f}(u-v))_{u,v\in\Delta}, \quad \Phi_1 = \hat{f}(u-v-(1,0)))_{u,v\in\Delta}, \quad \Phi_2 = (\hat{f}(u-v+(0,1))_{u,v\in\Delta}$$

and $\Delta = \{0,\dots,n-1\}\times\{0,\dots,m-1\}$. *In particular the matrix*

$$(\hat{f}(u-v))_{u\in\{\dots,n-2,n-1\}\times\{0,1,\dots\},v\in\{0,1,\dots\}\times\{\dots,m-2,m-1\}} \tag{2.2.4}$$

has rank equal to nm.

In this section, we examine a method to expand the region of computable coefficients by increasing the degree of the stable polynomial. In this section (r,s) indicate the degree we add to p.

2.2.1 Increasing the Degree

For any $(r,s)\in\mathbb{N}_0^2$, let

$$p^{(r,s)}(z,w) = \sum_{k=0}^{n+r}\sum_{l=0}^{m+s} p_{kl}z^k w^l,$$

where $p_{kl}=0$ when $k>n$ or $l>m$. Similarly, we define

$$\overleftarrow{p}^{(r,s)}(z,w) = z^{n+r}w^{m+s}\overline{p}\left(\frac{1}{z},\frac{1}{w}\right) = z^r w^s \overleftarrow{p}^{(0,0)}(z,w).$$

Since we increased the degree, we introduced new intersecting zeros.

Lemma 2.2.2. *Given a stable polynomial $p(z,w)$, the intersecting zeros of $p^{(r,s)}(z,w)$ and $\overleftarrow{p}^{(r,s)}(z,w)$ are:*

a) The intersecting zeros of $p(z,w)$ and $\overleftarrow{p}(z,w)$;

b) $(0,w)$ *and* $\left(\infty, \frac{1}{\bar{w}}\right)$ *such that* $w \in \mathbb{C}\backslash\bar{\mathbb{D}}$ *and* $\tilde{p}_0(w) = 0$*, where* $p^{(r,s)}(z,w) = \sum_{k=0}^{n+r} \tilde{p}_k(w)z^k$ *and* $r \geq 1$*;*

c) $(z,0)$ *and* $\left(\frac{1}{\bar{z}}, \infty\right)$ *such that* $z \in \mathbb{C}\backslash\bar{\mathbb{D}}$ *and* $p_0(z) = 0$*, where* $p^{(r,s)}(z,w) = \sum_{l=0}^{m+s} p_l(z)w^l$ *and* $s \geq 1$*; and*

d) $(0,\infty)$ *and* $(\infty, 0)$*.*

The proof for this lemma is straightforward.

Proof. a) If p and $\overleftarrow{p}$ are 0, then $p^{(r,s)}$ and $\overleftarrow{p}^{(r,s)}$ are also 0. Therefore, the intersecting zeros of p and $\overleftarrow{p}$ are intersecting zeros of $p^{(r,s)}$ and $\overleftarrow{p}^{(r,s)}$.

b) Since $\overleftarrow{p}^{(r,s)}(z,w) = z^r w^s \overleftarrow{p}^{(0,0)}(z,w)$, if $z = 0$ and $r \geq 1$, $\overleftarrow{p}^{(r,s)}(z,w) = 0$ and $p^{(r,s)}(0,w) = \tilde{p}_0(w)$.

c) Since $\overleftarrow{p}^{(r,s)}(z,w) = z^r w^s \overleftarrow{p}^{(0,0)}(z,w)$, if $w = 0$ and $s \geq 1$, $\overleftarrow{p}^{(r,s)}(z,w) = 0$ and $p^{(r,s)}(z,0) = p_0(z)$.

d) Recall that if $p(z,w)$ has degree (n,m), then $p(z,\infty) = 0$ corresponds to the statement $p_m(z) = 0$, while $p(\infty,w) = 0$ corresponds to $\tilde{p}_n(w) = 0$.

□

2.2.2 Main Result

We now have the following corollary to Theorem 2.2.1.

Corollary 2.2.3. *Let* $p(z,w)$ *be a stable polynomial of degree* (n,m)*, and let* $f(z,w)$ *be its spectral density function. Let* $(r,s) \in \mathbb{N}_0^2$*. Then there exists a row vector* $x \in \mathbb{C}^{(n+r)(m+s)}$*, a column vector* $y \in \mathbb{C}^{(n+r)(m+s)}$ *and commuting matrices* $S^{(r,s)}, \tilde{S}^{(r,s)} \in \mathbb{C}^{(n+r)(m+s)\times(n+r)(m+s)}$ *such that*

$$\sigma(S^{(r,s)}) = \{z \in \mathbb{D} : \exists w \text{ such that } (z,w) \text{ is an intersecting zero of } p^{(r,s)}\},$$

$$\sigma(\tilde{S}^{(r,s)}) = \{w \in \mathbb{D} : \exists z \text{ such that } (z,\bar{w}) \text{ is an intersecting zero of } p^{(r,s)}\}, \tag{2.2.5}$$

and

$$\hat{f}(k,l) = x\tilde{S}^{(r,s)^{m+s+l-1}} S^{(r,s)^{n+r-1-k}} y, \qquad k \leq n+r-1, \qquad l \geq -m-s+1. \tag{2.2.6}$$

Choose $x, y, S^{(r,s)}$ *and* $\tilde{S}^{(r,s)}$ *as follows:*

$$x = row(\hat{f}((n+r-1,0)-u))_{u\in\Delta}$$

$$y = col\left(\delta_{u+(0,-m-s+1)}\right)_{u\in\Delta}, S^{(r,s)} = \Phi^{-1}\Phi_1, \tilde{S}^{(r,s)} = \Phi^{-1}\Phi_2^*, \tag{2.2.7}$$

where

$$\Phi = (\hat{f}(u-v))_{u,v\in\Delta}, \quad \Phi_1 = \hat{f}(u-v-(1,0)))_{u,v\in\Delta}, \quad \Phi_2 = (\hat{f}(u-v+(0,1))_{u,v\in\Delta}$$

and $\Delta = \{0,\ldots,n+r-1\} \times \{0,\ldots,m+s-1\}$.

Proof. The proof of the corollary follows the proof of Theorem 2.2.1 by replacing n and m with $n+r$ and $m+s$, respectively, which can be found in [Geronimo and Woerdeman 2004]. □

2.2.3 Examples

From (2.2.9), (2.2.10) and the statement immediately after in the originally proof, we can see that copies of the companion matrices

$$T = \begin{bmatrix} 0 & \cdots & 0 & -\overline{\left(\frac{p_{n0}}{p_{00}}\right)} \\ 1 & \cdots & 0 & -\overline{\left(\frac{p_{n-1,0}}{p_{00}}\right)} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\overline{\left(\frac{p_{10}}{p_{00}}\right)} \end{bmatrix}, \tilde{T} = \begin{bmatrix} -\overline{\left(\frac{p_{01}}{p_{00}}\right)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\left(\frac{p_{0,m-1}}{p_{00}}\right)} & 0 & \cdots & 1 \\ -\overline{\left(\frac{p_{0,m}}{p_{00}}\right)} & 0 & \cdots & 0 \end{bmatrix}$$

are included in $-L_k^{(m+s-1)}$. By following the permutation throughout the proof carefully we can determine where are the entries of T and $\tilde{T}$ in $S^{(r,s)}$ and $\tilde{S}^{(r,s)}$. We illustrate it with the following example.

Example 2.2.4. Let $p(z,w) = 1 - w/4 - w^2/16 - z/4 - zw/16 - z^2w/64$. We have that

$$S = \begin{bmatrix} 0 & 0 & -.005 & -.017 \\ 0 & 0 & -.001 & -.0003 \\ 1 & 0 & .2924 & .1415 \\ 0 & 1 & .0093 & .2536 \end{bmatrix}, \tilde{S} = \begin{bmatrix} .2503 & 1 & -.001 & 0 \\ .0626 & 0 & -.0002 & 0 \\ .1409 & 0 & .2948 & 1 \\ .0172 & 0 & .0684 & 0 \end{bmatrix}, T = \begin{bmatrix} .25 \end{bmatrix}, \tilde{T} = \begin{bmatrix} .25 & 1 \\ .0625 & 0 \end{bmatrix}.$$

We then have

$$S^{(1,0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -0.005 & -0.017 \\ 0 & 1 & 0 & 0 & -0.001 & -0.0003 \\ 0 & 0 & 1 & 0 & 0.2924 & 0.1415 \\ 0 & 0 & 0 & 1 & 0.0093 & 0.2536 \end{bmatrix}, \tilde{S}^{(1,0)} = \begin{bmatrix} .25 & 1 & 0 & 0 & 0 & 0 \\ .0625 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0.2503 & 1 & -0.001 & 0 \\ * & * & 0.0626 & 0 & -0.0002 & 0 \\ * & * & 0.1409 & 0 & 0.2948 & 1 \\ * & * & 0.0172 & 0 & 0.0684 & 0 \end{bmatrix},$$

$$S^{(0,1)} = \begin{bmatrix} 0 & 0 & 0 & S_{1,3} & S_{1,4} & * \\ 0 & 0 & 0 & S_{2,3} & S_{2,4} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & S_{3,3} & S_{3,4} & * \\ 0 & 1 & 0 & S_{4,3} & S_{4,4} & * \\ 0 & 0 & 1 & 0 & 0 & .25 \end{bmatrix}, \tilde{S}^{(0,1)} = \begin{bmatrix} \tilde{S}_{1,1} & \tilde{S}_{1,2} & 0 & \tilde{S}_{1,3} & \tilde{S}_{1,4} & 0 \\ \tilde{S}_{2,1} & \tilde{S}_{2,2} & 1 & \tilde{S}_{2,3} & \tilde{S}_{2,4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{S}_{3,1} & \tilde{S}_{3,2} & 0 & \tilde{S}_{3,3} & \tilde{S}_{3,4} & 0 \\ \tilde{S}_{4,1} & \tilde{S}_{4,2} & 0 & \tilde{S}_{4,3} & \tilde{S}_{4,4} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so forth.

Example 2.2.5. In this example, we will demonstrate how to use the corollary. Again, let

$$p(z,w) = 1 - w/4 - w^2/16 - z/4 - zw/16 - z^2w/64.$$

As the degree is $(2,1)$, Theorem 2.2.1 gives us a formula for any Fourier coefficients $\hat{f}(k,j)$ if $k \le 1$ and

$j \geq 0$, and consequently we have $\hat{f}(-k, -j)$. Now, suppse $(r, s) = (2, 1)$. The corollary then state we can calculate coefficients such as $\hat{f}(2, 0)$. Indeed, with

$$S^{(2,1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & S_{34} \\ 0 & 0 & I_2 & S_{44} \end{bmatrix},$$

where

$$S_{34} = \begin{bmatrix} 0.0047 & 0.0170 \\ 0.0011 & 0.0003 \end{bmatrix}, S_{44} = \begin{bmatrix} 0.2984 & 0.1433 \\ 0.0101 & 0.2538 \end{bmatrix},$$

and

$$\tilde{S}^{(2,1)} = \begin{bmatrix} 0.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.125 & 0 & 0.25 & 1 & 0 & 0 & 0 & 0 \\ 0.0156 & 0 & 0.0625 & 0 & 0 & 0 & 0 & 0 \\ 0.0469 & 0 & 0.1251 & 0 & 0.2503 & 1 & 0.001 & 0 \\ 0.0039 & 0 & 0.0156 & 0 & 0.0626 & 0 & 0.0002 & 0 \\ 0.0127 & 0 & 0.0508 & 0 & 0.1409 & 0 & 0.2948 & 1 \\ 0.0011 & 0 & 0.0043 & 0 & 0.0172 & 0 & 0.0684 & 0 \end{bmatrix},$$

$$x = \begin{bmatrix} 0.0391 & 0.0158 & 0.1194 & 0.0455 & 0.3703 & 0.1295 & 1.2214 & 0.3869 \end{bmatrix}, y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Then

$$\hat{f}(2, 0) = x\tilde{S}^{1+1+0-1}S^{2+2-1-2}y = x\tilde{S}Sy = 0.1194.$$

Remark 2.2.6. The shortcoming of the corollary is that we require more coefficients that we started with since we increased the size of Λ. However, increase the size each time give us infinitely more coefficients.

2.3 Expanding Polynomials Using Permanent and Applications

In [Grinshpan et al. 2013], it was proved that for every bivariate stable polynomial $p(z,w)$ of degree (n,m) with $p(0)=1$, we can construct a determinantal representation of the form

$$p(z,w) = \det(I - KZ),$$

where Z is an $(n+m)\times(n+m)$ diagonal matrix with coordinate variables z and w on the diagonal and K is a contraction. In this section, we will demonstrate the possibility of the connection between determinantal representation and stable polynomial by proving the second statement of Theorem 1.1.1 for $p(z,w)$ of degree $(1,1)$.

2.3.1 Permanent and Permanent Expansion

Given a square matrix, we can define its permanent.

Definition 2.3.1. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, the permanent of A is defined as*

$$\operatorname{perm}(A) = \sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n is the permutation group of n elements.

Example 2.3.2. perm $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = aej + bfg + cdh + afh + bdj + ceg.$

We also have the following identity from [Vere-Jones 1984].

Proposition 2.3.3.

$$\det\left(I - KZ\right)^{-1} = \sum_{(k,l)\in\mathbb{N}_0^2} \operatorname{perm}(K^{(k,l)})\frac{z^n w^m}{n!m!},$$

where

$$K^{(k,l)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, A = \begin{pmatrix} a & \cdots & a \\ \vdots & \ddots & \vdots \\ a & \cdots & a \end{pmatrix} \in \mathbb{C}^{k,k}, B = \begin{pmatrix} b & \cdots & b \\ \vdots & \ddots & \vdots \\ b & \cdots & b \end{pmatrix} \in \mathbb{C}^{k,l},$$

and

$$C = \begin{pmatrix} c & \cdots & c \\ \vdots & \ddots & \vdots \\ c & \cdots & c \end{pmatrix} \in \mathbb{C}^{l,k}, D = \begin{pmatrix} d & \cdots & d \\ \vdots & \ddots & \vdots \\ d & \cdots & d \end{pmatrix} \in \mathbb{C}^{l,l}.$$

When $K \in \mathbb{C}^{(2,2)}$, we have the following identity from [Rubak et al. 2010].

Proposition 2.3.4. *(Proposition A.2.1) For* $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\operatorname{perm}\left(K^{(k,l)}\right) = k!l!a^k d^l \sum_{i=0}^{min(k,l)} \binom{n}{i}\binom{m}{i} \alpha^i, \alpha = \frac{bc}{ad}.$$

2.3.2 Expanding Polynomials Using Permanents

For $f(z,w) = 1/|p(z,w)|^2$, we have that

$$f(z,w) = \sum_{(k,l)\in\mathbb{N}_0^2} \operatorname{perm}(K^{(k,l)}) \frac{z^k w^l}{k!l!} \sum_{(k_1,l_1)\in\mathbb{N}_0^2} \operatorname{perm}(\overline{K}^{(k_1,l_1)}) \frac{1}{z_1^k w_1^l k_1! l_1!}.$$

From this representation, it is clear that the coefficient c_{00} is the sum over $(k,l) = (k_1,l_1)$. By having $\binom{n}{i} = 0$ when $i > n$ and $\binom{n}{-1} = 0$, we have that

$$c_{00} = \sum_{(k,l)\in\mathbb{N}_0^2} \frac{\operatorname{perm}(K^{(k,l)})\operatorname{perm}(\overline{K}^{(k,l)})}{(k!l!)^2} = \sum_{(k,l)\in\mathbb{N}_0^2} (a\bar{a})^k (d\bar{d})^m \sum_{i=0}^{k} \binom{k}{i}\binom{l}{i} \alpha^i \sum_{j=0}^{k} \binom{k}{j}\binom{l}{j} \overline{\alpha}^j.$$

After expressing $c_{-1,0}, c_{0,1}, c_{-1,1}$ similarly, we have

$$\begin{aligned} c_{00}c_{-1,1} - c_{0,1}c_{-1,0} = &\sum_{(k_1,l_1)\in\mathbb{N}_0^2}\sum_{(k_2,l_2)\in\mathbb{N}_0^2} a\bar{d}(a\bar{a})^{k_1+k_2}(d\bar{d})^{l_1+l_2}\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}\binom{l_1}{j_1}\overline{\alpha}^{j_1}\sum_{j_2=0}^{k_2}\binom{k_2}{j_2}\binom{l_2+1}{j_2}\overline{\alpha}^{j_2} \\ &\times\left[\sum_{i_1=0}^{k_1}\binom{k_1}{i_1}\binom{l_1}{i_1}\alpha^{i_1}\sum_{i_2=0}^{k_2+1}\binom{k_2+1}{i_2}\binom{l_2}{i_2}\alpha^{i_2} - \sum_{i_1=0}^{k_1+1}\binom{k_1+1}{i_1}\binom{l_1}{i_1}\alpha^{i_1}\sum_{i_2=0}^{k_2}\binom{k_2}{i_2}\binom{l_2}{i_2}\alpha^{i_2}\right]. \end{aligned} \tag{2.3.1}$$

Since for each $k = k_1 + k_2$ and $l = l_1 + l_2$ the power of a and d differs, there is no overlap. Therefore, to show (2.3.1) $= 0$, we can instead show

$$\begin{aligned} &\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}\binom{l_1}{j_1}\alpha^{j_1}\right)\left(\sum_{j_2=0}^{k-k_1}\binom{k-k_1}{j_2}\binom{l-l_1+1}{j_2}\overline{\alpha}^{j_2}\right) \\ &\times\left[\sum_{i_1=0}^{k_1}\binom{k_1}{i_1}\binom{l_1}{i_1}\alpha^{i_1}\sum_{i_2=0}^{k-k_1+1}\binom{k-k_1+1}{i_2}\binom{l-l_1}{i_2}\alpha^{i_2} - \sum_{i_1=0}^{k_1+1}\binom{k_1+1}{i_1}\binom{l_1}{i_1}\alpha^{i_1}\sum_{i_2=0}^{k-k_1}\binom{k-k_1}{i_2}\binom{l-l_1}{i_2}\alpha^{i_2}\right] \end{aligned} \tag{2.3.2}$$

is equal to 0. By using Pascal's identity $\binom{n+1}{i} = \binom{n}{i-1}\binom{n}{i}$, we have

$$(2.3.2) = \sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}\binom{l_1}{j_1}\overline{\alpha}^{j_1}\right)\left(\sum_{j_2=0}^{k-k_1}\binom{k-k_1}{j_2}\binom{l-l_1+1}{j_2}\overline{\alpha}^{j_2}\right)$$

$$\times\left[\sum_{i_1=0}^{k_1}\binom{k_1}{i_1}\binom{l_1}{i_1}\alpha^{i_1}\sum_{i_2=0}^{k-k_1}\binom{k-k_1}{i_2}\binom{l-l_1}{i_2+1}\alpha^{i_2+1} - \sum_{i_1=0}^{k_1}\binom{k_1}{i_1}\binom{l_1}{i_1+1}\alpha^{i_1+1}\sum_{i_2=0}^{k-k_1}\binom{k-k_1}{i_2}\binom{l-l_1}{i_2}\alpha^{i_2}\right]$$

$$= \sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k-k_1}\binom{k_1}{j_1}\binom{l_1}{j_1}\binom{k-k_1}{j_2}\binom{l-l_1+1}{j_2}\overline{\alpha}^{j_1+j_2}\right)$$

$$\times\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k-k_1}\binom{k_1}{i_1}\binom{k-k_1}{i_2}\left[\binom{l_1}{i_1}\binom{l-l_1}{i_2+1} - \binom{l_1}{i_1+1}\binom{l-l_1}{i_2}\right]\alpha^{i_1+i_2+1}\right). \tag{2.3.3}$$

Let $j = j_1 + j_2$ and $i = i_1 + i_2$. We then have the coefficient in front of $\alpha^{i+1}\overline{\alpha}^j$ in (2.3.3) to be

$$\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1}\binom{k-k_1}{j-j_1}\binom{l-l_1+1}{j-j_1}\right)$$

$$\times\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\left[\binom{l_1}{i_1}\binom{l-l_1}{i-i_1+1}-\binom{l_1}{i_1+1}\binom{l-l_1}{i-i_1}\right]\right). \tag{2.3.4}$$

Using Pascal's identity once again, we have

$$\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1}\binom{k-k_1}{j-j_1}\binom{l-l_1}{j-j_1-1}\right)$$

$$\times\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\left[\binom{l_1}{i_1}\binom{l-l_1}{i-i_1+1}-\binom{l_1}{i_1+1}\binom{l-l_1}{i-i_1}\right]\right). \tag{2.3.5}$$

Similarly, we can express (2.3.4) in term of k_2, l_2, i_2, and j_2.

$$\sum_{k_2=0}^{k}\sum_{l_2=0}^{l}\left(\sum_{j_2=0}^{j}\binom{k_2}{j_2}\binom{l_2}{j_2-1}\binom{k-k_2}{j-j_2}\binom{l-l_2}{j-j_2}\right)$$

$$\times\left(\sum_{i_2=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\left[\binom{l_1}{i_1}\binom{l-l_1}{i-i_1+1}-\binom{l_1}{i_1+1}\binom{l-l_1}{i-i_1}\right]\right). \tag{2.3.6}$$

Relabeling k_2, l_2, i_2, j_2 by k_1, l_1, i_1, j_1 in (2.3.6), respectively, and add (2.3.5) to (2.3.6), we have

$$\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1}\binom{k-k_1}{j-j_1}\binom{l-l_1}{j-j_1-1}\right)\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\binom{l_1}{i_1}\binom{l-l_1}{i-i_1+1}\right) \tag{2.3.7}$$

$$-\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1}\binom{k-k_1}{j-j_1}\binom{l-l_1}{j-j_1-1}\right)\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\binom{l_1}{i_1+1}\binom{l-l_1}{i-i_1}\right) \tag{2.3.8}$$

$$-\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1-1}\binom{k-k_1}{j-j_1}\binom{l-l_1}{j-j_1}\right)\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\binom{l_1}{i_1}\binom{l-l_1}{i-i_1+1}\right) \tag{2.3.9}$$

$$+\sum_{k_1=0}^{k}\sum_{l_1=0}^{l}\left(\sum_{j_1=0}^{j}\binom{k_1}{j_1}\binom{l_1}{j_1-1}\binom{k-k_1}{j-j_1}\binom{l-l_1}{j-j_1}\right)\left(\sum_{i_1=0}^{i}\binom{k_1}{i_1}\binom{k-k_1}{i-i_1}\binom{l_1}{i_1+1}\binom{l-l_1}{i-i_1}\right) \tag{2.3.10}$$

$$=(2.3.5)+(2.3.6). \tag{2.3.11}$$

If we add (2.3.7) to (2.3.10) and (2.3.8) to (2.3.9), we have that

$$(2.3.7)+(2.3.10)=-((2.3.8)+(2.3.9)).$$

Therefore,

$$(2.3.5)+(2.3.6)=(2.3.7)+(2.3.8)+(2.3.9)+(2.3.10)=0.$$

But the sum of two non-negative numbers can only be 0 if both are 0. Therefore, the coefficient in front of $\alpha^{i+1}\overline{\alpha}^j$ equals 0 for all $i, j \geq 0$, and this concludes the proof.

2.4 Asymptotics of the Fourier Coefficients

Recall Theorem 2.2.1, given a stable polynomial $p(z,w)$ of degree (n,m), $f(z,w)$ be its spectral density function, we can compute the Fourier coefficient $\hat{f}(r,s), r \leq n-1, s \geq -m+1$, with the equation

$$\hat{f}(r,s)=x\tilde{S}^{m+s-1}S^{n-1-r}y, \tag{2.4.1}$$

where x and y are specific row vector and column vector of the correct size, respectively. The eigenvalues of S and $\tilde{S}$ are given by

$$\sigma(S) = \{z \in \mathbb{D} : \exists w \text{ such that } (z, w) \text{ is an intersecting zero of } p\},$$

$$\sigma(\tilde{S}) = \{w \in \mathbb{D} : \exists z \text{ such that } (z, \bar{w}) \text{ is an intersecting zero of } p\}.$$

In Section 2.1, we determined the left eigenvectors are in the form

$$\begin{pmatrix} 1 & z_j & \dots & z_j^{n-1} \end{pmatrix} \bigotimes \frac{d^l}{dw^l} \begin{pmatrix} w^{m-1} & \dots & w & 1 \end{pmatrix}\Big|_{\bar{w}_j}, l = 0, \cdots, k-1,$$

and

$$\frac{d^l}{dz^l} \begin{pmatrix} 1 & z & \dots & z^{n-1} \end{pmatrix}\Big|_{z_j} \bigotimes \begin{pmatrix} \bar{w}_j^{m-1} & \dots & \bar{w}_j & 1 \end{pmatrix}, l = 0, \cdots, \tilde{k}-1,$$

for S and $\tilde{S}$, respectively. Furthermore, in Section 2.2, we developed a procedure to compute more coefficients. In this section, we focus on the asymptotics of these coefficients and determine the decay rate for a given direction. Unless indicated otherwise, $\mathbf{r} = (r, s)$ will indicate the direction of interest.

2.4.1 Second and Fourth Quadrants

Since for any (r, s), $\hat{f}(r, s) = \overline{\hat{f}(-r, -s)}$, we only need to concern ourselves with the second quadrant, and the fourth quadrant result will follow. Return to Theorem 2.2.1, if the degree of $p(z, w)$ is (n, m), i.e. $p_n(z) \neq 0$ and $\tilde{p}_m(w) \neq 0$, then for $k \leq 0 \leq n-1$ and $l \geq 0 \geq -m+1$,

$$\hat{f}(r, s) = x\tilde{S}^{m+s-1}S^{n-1-r}y.$$

Suppose the intersecting zeros are distinct. Since the eigenvectors of S and $\tilde{S}$ are linearly independent, they can be expressed as $S = QDQ^{-1}$ and $\tilde{S} = Q\tilde{D}Q^{-1}$, and (2.4.1) can be written as

$$\begin{aligned} \hat{f}(r,s) &= x\tilde{S}^{m+s-1}S^{n-1-r}y \\ &= xQ\tilde{D}^{m+s-1}Q^{-1}QD^{n-1-r}Q^{-1}y. \end{aligned} \tag{2.4.2}$$

Let $L = xQ$ and $R = Q^{-1}y$, where L is a row vector and R is a column vector. Thus, we have

$$\begin{aligned} \hat{f}(r,s) &= L\tilde{D}^{m+s-1}D^{n-1-r}R \\ &= L\begin{bmatrix} w_1^{m+s-1} & & \\ & \ddots & \\ & & w_{nm}^{m+s-1} \end{bmatrix}\begin{bmatrix} z_1^{n-1-r} & & \\ & \ddots & \\ & & z_{nm}^{n-1-r} \end{bmatrix} R \\ &= \left(L_1 w_1^{m+s-1} \dots L_{nm} w_{nm}^{m+s-1}\right)\left(R_1 z_1^{n-1-r} \dots R_{nm} z_{nm}^{n-1-r}\right)^T \\ &= \sum_{\ell=1}^{nm} L_\ell w_\ell^{m+s-1} z_\ell^{n-1-r} R_\ell \\ &= \sum_{\ell=1}^{nm} C_\ell w_\ell^{m+s-1} z_\ell^{n-1-r}, \end{aligned} \tag{2.4.3}$$

where $C_\ell = L_\ell R_\ell$. Without loss of generality, we fix direction (r,s) and assume $|w_1^s z_1^{-r}| \geq |w_\alpha^s z_\alpha^{-r}|$ for all $\alpha = 2, \dots, nm$. By factoring $w_1^s z_1^{-r}$, we can then rewrite (2.4.3) as

$$\hat{f}(r,s) = w_1^s z_1^{-r} \sum_{\ell=1}^{nm} C_\ell \frac{w_\ell^{m+s-1} z_\ell^{n-1-r}}{w_1^s z_1^{-r}}. \tag{2.4.4}$$

Suppose $t > 0, r \leq 0$, and $s \geq 0$. We are interested in the asymptotic behavior of $|\hat{f}(tr, ts)|$ as t approaches infinity. By using (2.4.4),

$$\begin{aligned}
\left|\hat{f}(tr, ts)\right| &= \left| w_1^{ts} z_1^{-tr} \sum_{\ell=1}^{nm} C_\ell \frac{w_\ell^{m+ts-1} z_\ell^{n-1-tr}}{w_1^{ts} z_1^{-tr}} \right| \\
&\leq \left| w_1^{ts} z_1^{-tr} \right| \sum_{\ell=1}^{nm} \left| C_\ell \frac{w_\ell^{ts} z_\ell^{-tr}}{w_1^{ts} z_1^{-tr}} w_\ell^{m-1} z_\ell^{n-1} \right| \\
&\leq \left| w_1^{ts} z_1^{-tr} \right| C,
\end{aligned}$$

where $C = nm \max_\ell \left| C_\ell w_\ell^{m-1} z_\ell^{n-1} \right|$. Thus we have the following theorem.

Theorem 2.4.1. *Let stable polynomial* $p(z, w) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{m} p_{l_1 l_2} z^{l_1} w^{l_2}$ *be given. Suppose the intersecting zeros of* $p(z, w)$ *and* $\overleftarrow{p}(z, w)$ *are* $(z_1, 1/\bar{w}_1), \ldots, (z_{nm}, 1/\bar{w}_{nm})$ *and their reciprocals, where* $z, w \in \mathbb{D}$*, are distinct. Let* r *and* s *be such that* $rs \leq 0$*, and assume* $\left| w_1^{|s|} z_1^{|r|} \right| \geq \left| w_\alpha^{|s|} z_\alpha^{|r|} \right|$*. Then*

$$\left|\hat{f}(tr, ts)\right| = O\left(\left| w_1^{|s|} z_1^{|r|} \right|^t \right), t \to \infty.$$

Proof. The proof for $r \leq 0$ and $s \geq 0$ is given above. Since $\hat{f}(r, s) = \overline{\hat{f}(-r, -s)}$, the case for $r \geq 0$ and $s \leq 0$ follows. □

Clearly, not every stable polynomial has distinct intersecting zeros. Furthermore, we currently do not have a clear understanding of the Jordan structures. However, we can still determine the decay rate along (r, s).

Theorem 2.4.2. *Let stable polynomial* $p(z, w) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{m} p_{l_1 l_2} z^{l_1} w^{l_2}$ *be given. Suppose the intersecting zeros of* $p(z, w)$ *and* $\overleftarrow{p}(z, w)$ *are* $(z_1, 1/\bar{w}_1), \ldots, (z_{nm}, 1/\bar{w}_{nm})$ *and their reciprocals, where* $z, w \in \mathbb{D}$*, not necessarily distinct. Let* r *and* s *be such that* $rs \leq 0$*, and assume* $\left| w_1^{|s|} z_1^{|r|} \right| \geq \left| w_\alpha^{|s|} z_\alpha^{|r|} \right|$*. Then*

$$\left|\hat{f}(tr, ts)\right| = O\left(\left(\left| w_1^{|s|} z_1^{|r|} \right| + \epsilon \right)^t \right), t \to \infty,$$

for all $\epsilon > 0$.

Proof. Let $S = PJP^{-1}$ and $\tilde{S} = \tilde{P}\tilde{J}\tilde{P}^{-1}$, where

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_{\beta_1} \end{bmatrix}, \text{where } J_\alpha = \begin{bmatrix} z_\alpha & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & z_\alpha \end{bmatrix}$$

is a Jordan block for z_α of the correct size l. Notice that

$$J^{n-1-r} = \begin{bmatrix} J_1^{n-1-r} & & \\ & \ddots & \\ & & J_{\beta_1}^{n-1-r} \end{bmatrix},$$

where

$$J_\alpha^{n-1-r} = \begin{bmatrix} z_\alpha^{n-1-r} & \binom{n-1-r}{1} z_\alpha^{n-2-r} & \dots & \binom{n-1-r}{l-1} z_\alpha^{n-l-r} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{n-1-r}{1} z_\alpha^{n-2-r} \\ & & & z_\alpha^{n-1-r} \end{bmatrix},$$

and similarly for $\tilde{J}^{m+s-1}$. Then

$$\begin{aligned} |\hat{f}(tr,ts)| &= |x\tilde{P}\tilde{J}^{m+ts-1}\tilde{P}^{-1}PJ^{n-1-tr}P^{-1}y| \\ &= |L\tilde{J}^{m+ts-1}MJ^{n-1-tr}P^{-1}R| \\ &= \left| \sum_{j_1,j_2,k_1,k_2=1}^{nm} C_{j_1j_2k_1k_2} \tilde{J}_{j_1k_1}^{(m+ts-1)} J_{j_2k_2}^{(n-1-tr)} \right| \end{aligned} \tag{2.4.5}$$

where $J^{(n-1-tr)}_{j_2k_2}$ is the (j_2, k_2) entry of J^{n-1-tr}. If $|z_1^{tr}w_1^{ts}| \geq |z_\alpha^{tr}w_\alpha^{ts}|$,

$$(2.4.5) \leq C\binom{n-1-tr}{l_1-1}\binom{m+ts-1}{l_2-1}\left|z_1^{tr}w_1^{ts}\right|$$
$$\leq C(n-1-tr)^{l_1-1}(m+ts-1)^{l_2-1}\left|z_1^{tr}w_1^{ts}\right|,$$

which is the desired result for direction (r, s) in the second quadrant. □

2.4.2 First and Third Quadrants

Given a rational function $F = G/H$, let V denoted the variety $\{z : H(z) = 0\}$ and assume $H(0) \neq 0$. Then, the Laurent series of F around 0 is given by $F = \sum_r a_{\mathbf{r}} z^{\mathbf{r}}, \mathbf{r} \in \mathbb{Z}^d$. To compute the asymptotic of $a_{\mathbf{r}}$, $\mathbf{r} \in \mathbb{N}^d$, we follow the procedure listed in Section 1.3 of [Pemantle and Wilson 2013], which we will include for completeness. The method is as follow:

Algorithm 2.4.3. (Outline of procedures)

1. Use the multidimensional Cauchy integral to express $a_{\mathbf{r}}$ as an integral over a d-dimensional torus T in $\mathbb{C}^d$.

2. Observe that T may be replaced by any cycle homologous to T in the domain

$$\mathcal{M} := \mathbb{C}^d \backslash \{z : (z_1 \ldots z_d)H(z) = 0\}$$

   of holomorphy of the integrand.

3. Deform the cycle to lower the modulus of the integrand as much as possible; use Morse theoretic methods to characterize the minimax cycle in terms of critical points(see Appendix B and C in [Pemantle and Wilson 2013] for background in Morse Theory). The Minimax cycle is a chain of integration, homologous to T in the domain $\mathcal{M}$, that achieves the least value of $\max_{x \in \mathcal{C}} h(x)$.

4. Use algebraic methods to find the critical points; these are points of V that depend on the direction $\hat{\mathbf{r}}$ of the asymptotics, and are saddle points for the magnitude of the integrand.

5. Use topological methods to locate one or more contributing critical points z_j and replace the integral over T by an integral over quasi-local cycles $C(z_j)$ (see Definition C.3.5 in [Pemantle and Wilson 2013]) near each z_j.

The crudest level at which nontrivial estimation of $a_\mathbf{r}$ normally occurs is the exponential level, namely statements of the form $\log|a_\mathbf{r}| = O(g(\mathbf{r}))$ as $\mathbf{r} \to \infty$ in some specified way. In particular, for a fixed direction $\hat{\mathbf{r}}_* = \mathbf{r}/|\mathbf{r}|$, consider the height function

$$h(\mathbf{z}) := h_{\hat{\mathbf{r}}_*}(\mathbf{z}) = -\sum_{k=1}^{d} \hat{r}_k \log|z_k|.$$

The height function $|\mathbf{r}|h$ is a good surrogate for the log magnitude of the integrand $z^{-r-1}F(z)$ because it captures the part that goes to infinity with $\mathbf{r}$, leaving only the factor $z^{-1}F(z)$ which is bounded on compact subsets of $\mathcal{M}$.

The process to determine critical points are listed in Section 8.3 of [Pemantle and Wilson 2013]. If

$$H(\mathbf{z}) = \prod_{i=1}^{d-k} p_i(\mathbf{z}),$$

the following three formulas are essential:

$$p_i(\mathbf{z}) = 0, i = 1, \ldots, d-k; \tag{2.4.6}$$

$$\det(M_{d-k+i}(\mathbf{z})) = 0, i = 1, \ldots, k; \tag{2.4.7}$$

where M is the $(d-k+1) \times d$ matrix whose row 1 through $d-k$ are the gradients

$$\nabla_{\log} p_i(\mathbf{z}) := (z_1 \partial p_i/\partial z_1, \ldots, z_d \partial p_i/\partial z_d)$$

with respect to $\log \mathbf{z}$ together with $\hat{\mathbf{r}}_*$, and M_{d-k+i} contains the first $d-k$ and the $(d-k+i)^{th}$ columns.

For example, if $H(z_1, z_2, z_3, z_4) = p_1 p_2$, then we have $k = 2$, and M is the matrix

$$M = \begin{pmatrix} z_1 \partial p_1/\partial z_1 & z_2 \partial p_1/\partial z_2 & z_3 \partial p_1/\partial z_3 & z_4 \partial p_1/\partial z_4 \\ z_1 \partial p_2/\partial z_1 & z_2 \partial p_2/\partial z_2 & z_3 \partial p_2/\partial z_3 & z_4 \partial p_2/\partial z_4 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} z_1 \partial p_1/\partial z_1 & z_2 \partial p_1/\partial z_2 & z_3 \partial p_1/\partial z_3 \\ z_1 \partial p_2/\partial z_1 & z_2 \partial p_2/\partial z_2 & z_3 \partial p_2/\partial z_3 \\ r_1 & r_2 & r_3 \end{pmatrix}, M_4 = \begin{pmatrix} z_1 \partial p_1/\partial z_1 & z_2 \partial p_1/\partial z_2 & z_4 \partial p_1/\partial z_4 \\ z_1 \partial p_2/\partial z_1 & z_2 \partial p_2/\partial z_2 & z_4 \partial p_2/\partial z_4 \\ r_1 & r_2 & r_4 \end{pmatrix}.$$

Condition (2.4.7) ensures the span of the $d-k$ gradients contain $\hat{\mathbf{r}}_*$. These points are called multiple points. Smooth critical points are the most common case. Being in the span of $\{\nabla_{\log} p_i\}$ means being parallel to $\nabla_{\log} p_i$, leading to $d-1$ equations for vanishing 2×2 subdeterminants of M:

$$\begin{aligned} p_i(\mathbf{z}) &= 0 \\ r_1 z_2 \frac{\partial p_i}{\partial z_2} &= r_2 z_1 \frac{\partial p_i}{\partial z_1} \\ &\vdots \\ &\vdots \\ r_1 z_d \frac{\partial p_i}{\partial z_d} &= r_d z_1 \frac{\partial p_i}{\partial z_1}. \end{aligned} \tag{2.4.8}$$

For the spectral density function $f(z,w) = 1/p(z,w)\overline{p(z,w)}$ on $\mathbb{T}^2$, we can rewrite it with p and $\overleftarrow{p}$, so

$$f(z,w) = \frac{z^n w^m}{p(z,w)\overleftarrow{p}(z,w)}.$$

There is a Laurent series for f that is true for $\mathbb{T}^2$. The Fourier coefficients equal the Laurent coefficients through the substitution $z = e^{\pi i w}$. With p and $\overleftarrow{p}$, we have $d = 2$, so it follows that $k = 0$ for (2.4.6) and (2.4.7). Therefore, (2.4.7) is vacuously satisfied for intersecting zeros, so the critical points include smooth points, multiple points arise from just p or $\overleftarrow{p}$, and intersecting zeros. As the number of critical points is finite, we have a maximum possible decay rate for the coefficients.

Theorem 2.4.4. *Let $p(z,w) = \sum_{l_1=0}^{n}\sum_{l_2=0}^{m} p_{l_1 l_2} z^{l_1} w^{l_2}$ be a stable polynomial where $p_{l_1 l_2} \neq 0$ for some (l_1, l_2) with $l_1 \geq 1$, $l_2 \geq 1$. Let (z_i, w_i) be points that either satisfy (2.4.8) for direction (r, s), multiple points of p or $\overleftarrow{p}$, or are intersecting zeros of p and $\overleftarrow{p}$, where $rs > 0$. Let k be such that $h(z_k, w_k) < 0$ and for all $j \neq k$, either $h(z_j, w_j) \geq 0$ or $h(z_j, w_j) \leq h(z_k, w_k)$. Then the exponential rate is*

$$\left|\hat{f}(tr, ts)\right| = O\left(\left|w_k^{-s} z_k^{-r}\right|^t\right).$$

Proof. Since the Laurent series converges, the coefficients must go to 0 in any direction $\mathbf{r}$. Therefore, the rate must be negative. Condition (2.4.8) gives critical points for when $H = p$ and $H = \overleftarrow{p}$. By the nature of the condition, the height of these points are negative of each other, these guaranteeing at least one negative critical value. □

While this theorem gives us the greatest possible rate, we want to be able to determine the dominating critical points and the actual decay rate. One thing we can do is to use Algorithm 9.4.7 from [Pemantle and Wilson 2013] to compute the dominating smooth points.

Algorithm 2.4.5. *(Determination of dominating points in the smooth, bivariate case)*

1. *List the critical value in order of decreasing height.*

2. *Set the provisional value of c_* to the highest critical value.*

3. *For each critical point at height c_*:*

   (a) *compute the order k of the critical point;*

   (b) *follow each of the k ascent paths until it is clear whether the z-coordinate or the w-coordinate goes to zero;*

   (c) *add the point to the set E if and only if at least one of the k paths has z-coordinate going to zero and at least one of the k paths has w-coordinate going to zero.*

4. *If E is nonempty then terminate and output c_* and E.*

5. *Else, if c_* is not the least critical value then replace c_* by the next lower critical value and go to step 3.*

6. *Else, if no critical values remain then $c_* = -\infty$, E is empty, and the asymptotics decay super-exponentially.*

Step 3 of the algorithm requires us to compute ascent paths. Given a point z_0 with $g(z_0) \neq 0 \neq g'(z_0)$ for some locally analytic function g, we need to compute another point z_1, such that $|g|$ increases on the line segment from z_0 to z_1. In order to do so and determining whether they go toward $\{z = 0\}$ or $\{w = 0\}$, we require the following six propositions from [DeVries et al. 2011].

Let $\mathcal{B}(c, r)$ be the closed ball centered at $c \in \mathbb{C}$ and radius $r > 0$. Let $\mathcal{S}_{\pi/4}$ be the sector of complex numbers whose arguments are strictly between $\pi/4$ and $-\pi/4$, and $z = x+yi$ is a rational point if x and y are rational.

Proposition 2.4.6. *Let g be locally analytic such that g and g' can be evaluated using ball arithmetic. Let z_0 be a rational point at which $g(z_0) \neq 0 \neq g'(z_0)$.*

1. *For $\epsilon > 0$ sufficiently small, the function*

$$u \mapsto \frac{g'\left(z_0 + u\frac{g(z_0)}{g'(z_0)}\right)}{g'(z_0)} \tag{2.4.9}$$

evaluated at $u = \mathcal{B}(0, \epsilon)$ is contained in $\mathcal{S}_{\pi/4}$.

2. *For such an $\epsilon > 0$, the function $|g|$ is strictly increasing on the line segment from z_0 to $z_0 + \epsilon\frac{g(z_0)}{g'(z_0)}$.*

Proposition 2.4.7. *Let g be locally analytic such that g and its first k derivatives can be evaluated using ball arithmetic. Let z_0 be a rational point at which $g(z_0) \neq 0 \neq g^{(k)}(z_0)$ while $g^{(1)}(z_0) = \cdots = g^{(k-1)}(z_0) = 0$. Define*

$$q(u) := \frac{1}{g(z_0)} g\left(z_0 + u\left[\frac{g(z_0)}{g^{(k)}(z_0)}\right]^{1/k}\right)$$

where any choice of the k^{th} root is allowed. Then when $\epsilon > 0$ is sufficiently small, $q^{(k)}(\mathcal{B}(0, \epsilon)) \subset S_{\pi/4}$ and for such an ϵ, the magnitude of $|g|$ will increase strictly on the line segment from z_0 to $z_0 + \epsilon[g(z_0)/g^{(k)}(z_0)]^{1/k}$.

Furthermore, computing such an ϵ for each choice of $1/k$ power and taking the minimum ensures that $|g|$ increases on all k line segments simultaneously.

Using these propositions, we can compute ascent paths. Let W be the set of points $\mathbf{z} = (z, w)$ in the variety $\mathcal{V}$ but not a critical point, z is a rational point, and the partial derivatives of H with respected to z and w are nonzero.

Proposition 2.4.8. *There is a ball-computable function $\phi : W \to W$ with the following properties. Let $\mathbf{z}_0 = (z_0, w_0) \in W$ and denote $\mathbf{z}_1 := (z_1, w_1) := \phi(\mathbf{z}_0)$. Then the line segment $[z_0, z_1]$ lifts uniquely to a curve in $\mathcal{V}$ connecting $\mathbf{z}_0$ to $\mathbf{z}_1$, along which h is strictly increasing. Furthermore, ϕ may be chosen so that $\mathbf{z}_1$ is not a critical point or a point where $\partial H/\partial z$ vanishes.*

As the proof for this proposition is constructive, we will include the part essential to the calculation.

Proof. Given $\mathbf{z}_0 \in \mathcal{W}$, let w be the locally analytic function such that $w(z_0) = w_0$ and $H(z, w(z)) = 0$. Apply Proposition (2.4.6) to $g(z) := \exp[h(z, w(z))]$ and z_0, obtaining a segment $[z_0, z_1] = [z_0, z_0 + \epsilon_1 g(z_0)/g'(z_0)]$. It is easy, computationally, to choose ϵ_1 always to be at least $\epsilon_0/2$. Define $\phi_1(\mathbf{z}_0) := \mathbf{z}_1 := (z_1, w_1)$. The lifting of $[z_0, z_1]$ to $\mathcal{V}$ is a path along which h increases. $\square$

Similarly, we have the following proposition for when (z_0, w_0) is a critical point.

Proposition 2.4.9. *Let $\mathbf{z}_0 = (z_0, w_0)$ be a critical point. Suppose $\partial H/\partial z$ does not vanish at $\mathbf{z}_0$. Then we may compute a rational $\phi(z_0)$ such that the union of the radial line segments*

$$\{[z_0, z_0 + \phi(z_0)e^{2\pi ij/k}] : 0 \leq j \leq k-1\}$$

lift uniquely to a union of paths from $\mathbf{z}_0$ on $\mathcal{V}$ on each of which h is strictly increasing.

We need to determine if the path is going toward the z-axis or w-axis. Let $\mathcal{V}^{\geq c}$ denote the subset of $\mathcal{V}$ consisting of points $\mathbf{z}$ with $h(\mathbf{z}) \geq c$ and $Z^{\geq c}$ (respectively $W^{\geq c}$) denote the union of those components of $\mathcal{V}^{\geq c}$ containing arbitrarily small values of z (respectively w).

Proposition 2.4.10. *Fix $\hat{\mathbf{r}}$ such that r/s is not a direction of a series solution to $H(z, w(z)) = 0$. Let c_{max} denote the greatest critical value of h and c_{min} denote the least critical value. Let ϵ be small enough so that there are no critical points (z, w) with $|z| < \epsilon$. Then any point $(z, w) \in \mathcal{V}$ with $|z| < \epsilon$ and $h(z, w) > c_{min}$ is in $Z^{>c_{max}}$.*

The above proposition is true when the roles of z and w are switched. Lastly, we need to allow ascent paths to terminate when they come sufficiently near a critical point. Otherwise, it may converge to a critical point along an infinite sequence of ever smaller steps.

Proposition 2.4.11. *Let $c > c_*$ be a critical value of h and let (z_0, w_0) be a critical point at height c in $Z^{\geq c}$. Suppose there is an $\epsilon > 0$ for which the following conditions hold:*

1. *For each z such the $|z - z_0| \leq \epsilon$ there is at most one solution to $H(z, w) = 0$ with $|w - w_0| \leq \epsilon$;*
2. *(z_0, w_0) is the unique critical point (z, w) of h on $\mathcal{V}$ with $|z - z_0|, |w - w_0| \leq \epsilon$.*

Then $|z - z_0|, |w - w_0| \leq \epsilon$ and $H(z, w) = 0$ imply $(z, w) \in Z^{\geq c_*}$.

With Propositions (2.4.8) - (2.4.11), we now have an algorithm to determine if a smooth point contributes.

Algorithm 2.4.12. *(Determining the ascent paths)*

1. *Let ϵ and ϵ' be as in Proposition (2.4.10) for z and w respectively.*
2. *Order the smooth points by their critical values. Starting with the point with the highest critical value, apply Proposition (2.4.9) to ascend from the smooth point, then use Proposition (2.4.8) to continue ascending.*
3. *If the z-component of $\mathbf{z}_i$ has modulus less than ϵ at any iteration, the path approaches the z-axis. Similarly for w.*
4. *If z and w-axis both have at least one path ascending to it, this smooth point dominates the other points. Otherwise, repeat with the next point.*

With this result, we can now state a stronger theorem regarding the asymptotic of the Fourier coefficients.

Theorem 2.4.13. *Let* $p(z,w) = \sum_{l_1=0}^{n}\sum_{l_2=0}^{m} p_{l_1 l_2} z^{l_1} w^{l_2}$ *be a stable polynomial where* $p_{l_1 l_2} \neq 0$ *for some* (l_1, l_2) *with* $l_1 \geq 1$, $l_2 \geq 1$*. Let* (z_i, w_i) *be points that satisfy (2.4.8) for direction* (r,s)*, where* $rs > 0$ *and ordered so that* $h(z_i, w_i) \geq h(z_l, w_l)$ *if* $i \geq l$*. Choose* k *so that* (z_k, w_k) *satisfies the following:*

1. $h(z_k, w_k) < 0$ *and at least one of the ascent paths has* z*-coordinate going to zero and at least one has* w*-coordinate going to zero, and*

2. *for all* $j \neq k$ *such that* $h(z_j, w_j) \leq 0$*, either* $h(z_j, w_j) \geq h(z_k, w_k)$ *but does not satisfy* 1)*, or* $h(z_j, w_j) \leq h(z_k, w_k)$*.*

Then the exponential rate is

$$|\hat{f}(tr, ts)| = \Omega(|w_i^{-s} z_i^{-r}|^t).$$

Proof. As per the proof to the previous theorem, we are guaranteed at least one of these critical points has negative height. We only have to show $\hat{f}(tr, ts)$ are not eventually constant 0. Clearly, if either n or m equal 0, f is reduced to at most one variable, and the Fourier coefficients equal 0 in any direction not on the axis. Therefore, as long as $p_{l_1 l_2} \neq 0$ for some (l_1, l_2) with $l_1 \geq 1$, $l_2 \geq 1$, Algorithim 2.4.5 will terminate and output the dominating smooth point and critical value. □

2.4.3 Examples

In practice, one can bypass calculating the ascent paths in most situations. Since the numbers of critical point is finite, we can compute the height for each and compare it with the actual coefficients. We demonstrate two examples, one with the computation for the ascent paths to demonstrate the result. But first, let us show a result without using ascent paths.

Example 2.4.14. Take stable polynomial $p(z,w) = \frac{11}{6}zw^2 + z - \frac{11}{6}w^2 + \frac{25}{3}w - 11$ from [Geronimo and Woerdeman 2006]. Then $\overleftarrow{p}(z,w) = \frac{11}{6} + w^2 - \frac{11}{6}z + \frac{25}{3}zw - 11zw^2$ and $f(z,w)$ is

$$\begin{aligned} f(z,w) &= \frac{zw^2}{p(z,w)\overleftarrow{p}(z,w)} \\ &= \frac{zw^2}{(\frac{11}{6}zw^2 + z - \frac{11}{6}w^2 + \frac{25}{3}w - 11)(\frac{11}{6} + w^2 - \frac{11}{6}z + \frac{25}{3}zw - 11zw^2)}. \end{aligned}$$

The intersecting zeros are

$$\mathbf{z}_1 = \left(\frac{1}{5}, 2\right), \mathbf{z}_2 = \left(\frac{1}{7}, 3\right), \mathbf{z}_3 = \left(5, \frac{1}{2}\right), \mathbf{z}_4 = \left(7, \frac{1}{3}\right).$$

First, let us consider a direction in the second quadrant. Suppose $(r, s) = (-1, 1)$. The magnitude of $\mathbf{z}_i^{\mathbf{r}}$ are the following:

$\mathbf{z}_i$	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	$\mathbf{z}_4$
$\mathbf{z}_i^{\mathbf{r}}$	10	21	$\frac{1}{10}$	$\frac{1}{21}$

Therefore, the coefficient is exponentially decaying and

$$\hat{f}(-t, t) = O(1/10^t).$$

Using MATLAB, the first 10 coefficients are

$\hat{f}(0,0)$	$\hat{f}(-1,1)$	$\hat{f}(-2,2)$	$\hat{f}(-3,3)$	$\hat{f}(-4,4)$
0.0191	0.0032	0.00038	0.0000042	4.3×10^{-7}
$\hat{f}(-5,5)$	$\hat{f}(-6,6)$	$\hat{f}(-7,7)$	$\hat{f}(-8,8)$	$\hat{f}(-9,9)$
4.3×10^{-8}	4.4×10^{-9}	4.4×10^{-10}	4.4×10^{-11}	4.4×10^{-12}

which are decaying near a rate of $1/10$, as expected. Similarly, for direction such as $(-2, 3)$ or $(2, -3)$, the dominating rate is $1/200$.

Now, let's consider the first quadrant. Suppose $(r, s) = (2, 1)$. Then we have the following:

$$\begin{aligned} h(z, w) &= -2\ln|z| - \ln|w|, \\ \frac{\partial p}{\partial z} &= \frac{11}{6}w^2 + 1, \\ \frac{\partial p}{\partial w} &= \frac{11}{3}zw - \frac{11}{3}w + \frac{25}{3}. \end{aligned} \tag{2.4.10}$$

Then, the points that satisfy (2.4.8) are

$$\mathbf{z}_5 = (1.7957, -6.6599), \mathbf{z}_6 = (0.1301, 2.5107), \mathbf{z}_7 = (9.4588, -0.6829), \mathbf{z}_8 = (7.6155, 0.2866),$$

and the reciprocal of these points satisfy (2.4.8) with $\overleftarrow{p}$ replacing p. Putting $\mathbf{z}_i$ into h, we have the following:

$\mathbf{z}_i$	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	$\mathbf{z}_4$	$\mathbf{z}_5$	$\mathbf{z}_6$	$\mathbf{z}_7$	$\mathbf{z}_8$
$h(\mathbf{z}_i)$	2.5257	2.7932	−2.5257	−2.7932	−3.067	3.1577	−4.1125	−2.8107

Numerically computing the coefficients, we found the decay rate is close to $\exp(-2.5257) = 0.08$, which indicates the dominating point for direction $(2, 1)$ is $\mathbf{z}_3 = (5, 1/2)$.

Example 2.4.15. To have an example with Jordan block, consider $p(z, w)$ of the previous example, and let $q(z, w) = p(z, w)^2$. Clearly, q is stable. Using MATLAB with a tolerence of 5×10^{-13}, we find the rank of the following:

$$\begin{aligned}
\operatorname{rank}\left(S - \frac{1}{5}I\right) &= \operatorname{rank}\left(S - \frac{1}{7}I\right) = 6\\
\operatorname{rank}\left(S - \frac{1}{5}I\right)^2 &= \operatorname{rank}\left(S - \frac{1}{7}I\right)^2 = 5\\
\operatorname{rank}\left(S - \frac{1}{5}I\right)^3 &= \operatorname{rank}\left(S - \frac{1}{7}I\right)^3 = 4.
\end{aligned}$$

Therefore, the matrix J is in the form

$$J = \begin{bmatrix}
1/5 & 1 & & & & & & \\
 & 1/5 & 1 & & & & & \\
 & & 1/5 & & & & & \\
 & & & 1/5 & & & & \\
 & & & & 1/7 & 1 & & \\
 & & & & & 1/7 & 1 & \\
 & & & & & & 1/7 & \\
 & & & & & & & 1/7
\end{bmatrix}.$$

$\tilde{S}$ has the same Jordan structure with 1/2 and 1/3 both with one size 3 block and a size 1 block. Compared to $p(z,w)$, the rate is converging much slower. The ratio

$$\frac{\hat{f}(-15,15)}{\hat{f}(-14,14)} = .1156,$$

is slowly approaching 1/10.

Example 2.4.16. In this example, we will demonstrate the calculation of ascent paths. Let $p(z,w) = 1 - w/2 - z/5 - zw/11$ and $(r,s) = (2,3)$. We then have the following:

$$\begin{aligned} h(z,w) &= -2\ln|z| - 3\ln|w|, \\ \frac{\partial p}{\partial z} &= -\frac{1}{5} - \frac{1}{11}w, \\ \frac{\partial p}{\partial w} &= -\frac{1}{2} - \frac{1}{11}z. \end{aligned} \tag{2.4.11}$$

Then, the points that satisfy (2.4.8) are

$$\mathbf{z}_1 = (1.54535, 1.07876), \mathbf{z}_2 = (-17.79535, -4.07876) \tag{2.4.12}$$

and the reciprocal of these points satisfy (2.4.8) with $\overleftarrow{p}$ replacing p. Putting $\mathbf{z}_i$ into h, we have

$$h(\mathbf{z}_1) = -1.0979, h(\mathbf{z}_2) = -9.97525.$$

Since $\mathbf{z}_1$ has the larger critical value, we begin at that point. Solving $p(z,w) = 0$ for w, we have

$$w(z) = \frac{1 - \frac{z}{5}}{\frac{1}{2} + \frac{z}{11}}.$$

Therefore, $g(z) := \exp[h(z, w(z))] = 1/|z|^2|w(z)|^3$. Taking derivative of $g(z)$, we find that $g^{(1)}(z_1) = 0$ and $g^{(2)}(z_1) \neq 0$, which means the order for $\mathbf{z}_1$ is 2. Let $\epsilon_0 = 0.05$ satisfying Proposition (2.4.7), and $\epsilon_1 = 0.025$.

Then, the two segments from the critical point $\mathbf{z}_1$ are

$$\left[z_1, z_1 + \epsilon_1 \left[\frac{g(z_1)}{g^{(2)}(z_1)}\right]^{1/2}\right] = [1.54535, 1.570]$$

and

$$\left[z_1, z_1 - \epsilon_1 \left[\frac{g(z_1)}{g^{(2)}(z_1)}\right]^{1/2}\right] = [1.54535, 1.5207].$$

For path 1, the endpoint is $(z, w) = (1.570, 1.067)$. For path 2, the endpoint is $(z, w) = (1.5207, 1.0903)$. From Proposition 2.4.10 and the points from 2.4.12, we have $\epsilon = 1.54535$ and $\epsilon' = 1.07876$. Therefore, path 1 approaches the w-axis and path 2 approaches the z-axis, which means $\mathbf{z}_1$ contributes to the asymptotic and

$$\hat{f}(2t, 3t) = \Omega(1.54535^{-2t}1.07876^{-3t}) = \Omega(0.3336^t).$$

The intersecting zeros in this case are

$$\mathbf{z}_3 = (0.3531, 1.7466), \mathbf{z}_4 = (2.8317, 0.5725), \tag{2.4.13}$$

with

$$h(\mathbf{z}_3) = 0.4090, h(\mathbf{z}_4) = -0.4090.$$

At first glance, it is tempting to believe $\mathbf{z}_4$ is the point that dominates. However, after numerically computing the rate, it is in fact $\mathbf{z}_1$ that dominates.

2.4.4 Discussion and Future Work

From our two examples, we can see that, depending on the polynomial and the direction, the intersecting zeros may or may not be a point that contribute to the asymptotic. The question naturally becomes under what condition can we determine if it does? While we have not explore into this question due to the time constrain, the answer to this question will most likely involve topology and Morse theory and is one that will be pursued in the future.

The notion of smooth points and multiple points depend on the variety of $V = \{p = 0\}$ and $\overleftarrow{V} = \{\overleftarrow{p} = 0\}$. We can further decompose the varieties into when they intersect themselves or each other. Multiple points only occur in the intersection, and smooth points occur in the other part of the varieties. This is why intersecting zeros are multiple points, since they are in $V \cap \overleftarrow{V}$.

Chapter 3: D-Variable Results

In this chapter, we will state a three-variable conjecture for the simplest type of stable polynomial, $p(z_1, z_2, z_3) = 1 - \frac{z_1+z_2+z_3}{r}$, where $r > 3$, which we hope to be able to build upon in the future. Furthermore, we will provide a theorem for the asymptotics of the coefficients in d variables analogous to Theorem 2.4.4.

3.1 3-Variable

In [Geronimo and Woerdeman 2004], Theorem 1.1.1 was stated and proved, providing the necessary and sufficient conditions for the Bernstein-Szegő moment problem in two variables. The question remains open when the number of variables is 3 or more. The following proposition and conjecture hopes to provide a starting point to answer the question.

Proposition 3.1.1. *Let real number $r > 3$, and complex numbers $c_j(r)$, $j \in \Lambda = \{0,1\} \times \{0,1\} \times \{0,1\}$ be given. If there exists a stable polynomial of the form*

$$p_r(z_1, z_2, z_3) = 1 - \frac{z_1 + z_2 + z_3}{r}$$

so that its spectral density function $f_r(z_1, z_2, z_3) = 1/|p_r(z_1, z_2, z_3)|^2$ has Fourier coefficients $\hat{f}_r(j) = c_j(r), j \in \Lambda$, then the following conditions are satisfied:

1. $c_{000}(r) = \frac{r^2}{r^2-3} {}_2F_1\left[1/3, 2/3; 1; \frac{27(1-r^2)}{(3-r^2)^3}\right]$, *where ${}_2F_1$ is the hypergeometric function;*
2. $c_j(r) = c_{\sigma(j)}(r)$, *where $\sigma(j)$ is a permutation of j;*
3. $c_{001}(r) = \frac{r(c_{000}(r)-1)}{3}$;
4. $c_{111}(r) = \frac{3c_{011}(r)}{r}$;
5. $\hat{f}_r(0, 1, -1) = \frac{rc_{001}(r) - c_{000}(r)}{2}$; *and*
6. $\hat{f}_r(1, 1, -1) = rc_{011}(r) - 2c_{001}(r)$.

Notice that the polynomials p_r only depend on one parameter, namely r. Thus the coefficients $c_j(r)$ should only depend upon one variable as well.

Also notice that the Fourier coefficient $\hat{f}_r(j,k,l)$ are in the form of

$$\hat{f}_r(j,k,l) = \sum_{n\geq 0}\sum_{\sum n_i = n}\binom{n+j_1+k_1+l_1}{n_1+j_1, n_2+k_1, n_3+l_1}\binom{n+j_2+k_2+l_2}{n_1+j_2, n_2+k_2, n_3+l_2} r^{-2n-|j|-|k|-|l|},$$

where $j = j_1 - j_2, k = k_1 - k_2, l = l_1 - l_2$. One way to arrive at these coefficients is using the determinantal representation

$$p_r(z_1,z_2,z_3) = \det\left(I_3 - \frac{1}{r}\mathbf{1}Z\right),$$

where I_3 is the 3 by 3 identity matrix, $\mathbf{1}$ is the matrix with all entries equals 1, and Z is the diagonal matrix with diagonal z_1, z_2,and z_3. Using the permanent expansion formula in [Vere-Jones 1984], we have

$$\frac{1}{p_r(z_1,z_2,z_3)} = \sum_{n\geq 0}\binom{n}{n_1,n_2,n_3} r^{-n} z_1^{n_1} z_2^{n_2} z_3^{n_3}, \qquad \frac{1}{\overline{p_r(z_1,z_2,z_3)}} = \sum_{n\geq 0}\binom{n}{n_1,n_2,n_3} r^{-n} z_1^{-n_1} z_2^{-n_2} z_3^{-n_3}.$$

Taking the product, we have the desired Fourier expansion.

Proof. To make the notation less cumbersome, let $\hat{f}_{rj} = \hat{f}_r(j)$. Statement 1 follows the definition of hypergeometric function:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where

$$(q)_n = \begin{cases} 1 & n = 0 \\ \frac{(q+n-1)!}{(q-1)!} & n > 0 \end{cases}$$

The hypergeometric function in Statement 1 is equal to

$$\frac{1}{1-3/r^2}\sum_{n=0}^{\infty}\frac{(3n)!1/r^{4n}(1-1/r^2)^n}{n!^3(1-3/r^2)^{3n}},$$

which sum equals the series $\hat{f}_{r_{000}}$. Statement 2 is required for p to be a symmetric function. Statement 3, 4, 5 and 6 can be shown by considering the matrix

$$T = (\hat{f}_{r\,j-k})_{j,k\in\Lambda}.$$

Then we have

$$\begin{pmatrix} \hat{f}_{r0,0,0} & \hat{f}_{r0,0,-1} & \hat{f}_{r0,-1,0} & \hat{f}_{r0,-1,-1} & \hat{f}_{r-1,0,0} & \hat{f}_{r-1,0,-1} & \hat{f}_{r-1,-1,0} & \hat{f}_{r-1,-1,-1} \\ \hat{f}_{r0,0,1} & \hat{f}_{r0,0,0} & \hat{f}_{r0,-1,1} & \hat{f}_{r0,-1,0} & \hat{f}_{r-1,0,1} & \hat{f}_{r-1,0,0} & \hat{f}_{r-1,-1,1} & \hat{f}_{r-1,-1,0} \\ \hat{f}_{r0,1,0} & \hat{f}_{r0,1,-1} & \hat{f}_{r0,0,0} & \hat{f}_{r0,0,-1} & \hat{f}_{r-1,1,0} & \hat{f}_{r-1,1,-1} & \hat{f}_{r-1,0,0} & \hat{f}_{r-1,0,-1} \\ \hat{f}_{r0,1,1} & \hat{f}_{r0,1,0} & \hat{f}_{r0,0,1} & \hat{f}_{r0,0,0} & \hat{f}_{r-1,1,1} & \hat{f}_{r-1,1,0} & \hat{f}_{r-1,0,1} & \hat{f}_{r-1,0,0} \\ \hat{f}_{r1,0,0} & \hat{f}_{r1,0,-1} & \hat{f}_{r1,-1,0} & \hat{f}_{r1,-1,-1} & \hat{f}_{r0,0,0} & \hat{f}_{r0,0,-1} & \hat{f}_{r0,-1,0} & \hat{f}_{r0,-1,-1} \\ \hat{f}_{r1,0,1} & \hat{f}_{r1,0,0} & \hat{f}_{r1,-1,1} & \hat{f}_{r1,-1,0} & \hat{f}_{r0,0,1} & \hat{f}_{r0,0,0} & \hat{f}_{r0,-1,1} & \hat{f}_{r0,-1,0} \\ \hat{f}_{r1,1,0} & \hat{f}_{r1,1,-1} & \hat{f}_{r1,0,0} & \hat{f}_{r1,0,-1} & \hat{f}_{r0,1,0} & \hat{f}_{r0,1,-1} & \hat{f}_{r0,0,0} & \hat{f}_{r0,0,-1} \\ \hat{f}_{r1,1,1} & \hat{f}_{r1,1,0} & \hat{f}_{r1,0,1} & \hat{f}_{r1,0,0} & \hat{f}_{r0,1,1} & \hat{f}_{r0,1,0} & \hat{f}_{r0,0,1} & \hat{f}_{r0,0,0} \end{pmatrix} \begin{pmatrix} 1 \\ -1/r \\ -1/r \\ 0 \\ -1/r \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $c_j(r) = \hat{f}_r(j), j \in \Lambda$, then $c_j(r)$ must satisfy this matrix equation. Therefore, statement 3 follows from the first row, and statement 4 follows from the last row. Notice that we also get the following two equalities:

$$\hat{f}_{r0,1,-1} = \frac{r\hat{f}_{r001} - \hat{f}_{r000}}{2},$$

from row 2, 3, and 5, and

$$\hat{f}_{r1,1,-1} = r\hat{f}_{r011} - 2\hat{f}_{r001}$$

from row 4, 6, or 7. □

We will now state the conjecture that will give us the opposite direction.

Conjecture 3.1.2. *Let real number $r > 3$, and complex functions $c_j(r)$, $j \in \Lambda = \{0,1\} \times \{0,1\} \times \{0,1\}$ be* *given. There exists a stable polynomial of the form*

$$p_r(z_1, z_2, z_3) = 1 - \frac{z_1 + z_2 + z_3}{r}$$

so that its spectral density function $f_r(z_1, z_2, z_3) = 1/|p_r(z_1, z_2, z_3)|^2$ *has Fourier coefficients* $\hat{f}_r(j) = c_j(r), j \in \Lambda$, *if and only if the following conditions are satisfied:*

1. $c_{000}(r) = \frac{r^2}{r^2-3} {}_2F_1 \left[1/3, 2/3; 1; \frac{27(1-r^2)}{(3-r^2)^3}\right]$, *where* ${}_2F_1$ *is the hypergeometric function;*
2. $c_j(r) = c_{\sigma(j)}(r)$, *where* $\sigma(j)$ *is a permutation of* j*;*
3. $c_{001}(r) = \frac{r(c_{000}(r)-1)}{3}$*;*
4. $c_{111}(r) = \frac{3c_{011}(r)}{r}$*;*
5. *let* $x = \frac{1}{r^2}$ *and let* $c_{011}(r) = h(x)$ *and* $c_{000}(r) = g(x)$*, then treating* x *as a variable,*

$$\begin{aligned} 3[x(1-x)(1-9x)h'(x) &+ (1-5x)h(x) - (4x+6x^2)] \\ &= x(1-x)(1-9x)g'(x) + (2-7x+9x^2)g(x) - 2(1+x+9x^2), \end{aligned}$$

   where $h(0) = 0, h'(0) = 2, g(0) = 1$, *and* $g'(0) = 3$*; and*

6. *there exists complex functions* $c_{0,1,-1}(r) = \frac{rc_{001}(r)-c_{000}(r)}{2}$, *and* $c_{1,1,-1}(r) = rc_{011}(r) - 2c_{001}(r)$.

The only difference between the proposition and the conjecture is statement 5. In the conjecture, we do not yet know what $f_r(z_1, z_2, z_3)$ is, so we need a different equation to relate $c_{000}(r)$ and $c_{011}(r)$.

Proof. The first 4 statements follow from the proposition. For statement 5, we need two lemmas and a conjecture that can be checked numerically .

Lemma 3.1.3. *Let* $f^{(n)}_{jkl} = \sum_{\sum n_i = n} \binom{n + j_1 + k_1 + l_1}{n_1 + j_1, n_2 + k_1, n_3 + l_1} \binom{n + j_2 + k_2 + l_2}{n_1 + j_2, n_2 + k_2, n_3 + l_2}$. *Then we have* $f^{(n)}_{1,1,-1} = f^{(n+1)}_{011} - \frac{2}{3} f^{(n+2)}_{000}$.

Proof. Since $\hat{f}_{r1,1,-1} = r\hat{f}_{r011} - 2\hat{f}_{r001}$, we have that

$$\sum_{n\geq 0}\sum_{\sum n_i=n}\binom{n+2}{n_1+1,n_2+1,n_3}\binom{n+1}{n_1,n_2,n_3+1}r^{-2n-3}$$
$$= r\sum_{n\geq 1}\sum_{\sum n_i=n}\binom{n+2}{n_1+1,n_2+1,n_3}\binom{n}{n_1,n_2,n_3}r^{-2n-2} - \frac{2r}{3}\sum_{n\geq 2}\sum_{\sum n_i=n}\binom{n}{n_1,n_2,n_3}\binom{n}{n_1,n_2,n_3}r^{-2n}.$$

Matching the power of r, we get the desired equality. □

Lemma 3.1.4. *Using the above notation, we have*

$$(n+2)^2 f_{000}^{(n+2)} - (10n^2+30n+23)f_{000}^{(n+1)} + 9(n+1)^2 f_{000}^{(n)} = 0.$$

Proof. One method to prove the first recurrence is to use the hypergeometric function in Statement 1 and realize it satisfies the Heun's equation (see [Heun 1888])

$$y'' + \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{z-1/9}\right]y' + \left[\frac{x-1/3}{x(x-1)(x-1/9)}\right]y = 0,$$

which is equal to

$$y'' + \frac{1-20x+27x^2}{x(1-x)(1-9x)}y' + \frac{9x-3}{x(1-x)(1-9x)}y = 0$$

with $y = \sum_{n=0}^{\infty} f_{000}^{(n)} r^{-2n}$. Using the fact that $f_{000}^{(n)} > 0$ for all $n \geq 0$, we match the power of r and arrive at the recurrence. □

To prove the forward direction of Conjecture 3.1.2, we also need the following two recurrence relations, which we confirmed numerically upto $n = 15$.

Conjecture 3.1.5. *Using the above notation, we have*

1. $(n+3)(n+2)f_{011}^{(n+1)} - (10n^2+40n+36)f_{011}^{(n)} + 9n(n+3)f_{011}^{(n-1)} = 0$, *and*

2. $n(n+3)f_{1,1,-1}^{(n)} - (10n^2+20n+6)f_{1,1,-1}^{(n-1)} + 9n(n+1)f_{1,1,-1}^{(n-2)} = 0.$

For statement 5, assuming the conjecture, combining with the above lemma, for $n \geq 1$, we have the equality

$$3\left[(n+3)f_{011}^{(n+1)} - (10+15)f_{011}^{(n)} + 9nf_{011}^{(n-1)}\right] = (n+4)f_{000}^{(n+2)} - (10n+17)f_{000}^{(n+1)} + 9(n+1)f_{000}^{(n)}. \quad (3.1.1)$$

Since $h(x) = \sum_{n=0}^{\infty} f_{011}^{(n)} x^{n+1}$ and $g(x) = \sum_{n=0}^{\infty} f_{000}^{(n)} x^n$, we need to multiply (3.1.1) by x^{n+1} and sum it from 1 to infinity. We then have the equality

$$\sum_{n=1}^{\infty} 3\left[(n+3)f_{011}^{(n+1)} - (10n+15)f_{011}^{(n)} + 9nf_{011}^{(n-1)}\right] x^{n+1} \quad (3.1.2)$$

$$= x\sum_{n=1}^{\infty}\left[(n+4)f_{000}^{(n+2)} - (10n+17)f_{000}^{(n+1)} + 9(n+1)f_{000}^{(n)}\right] x^n. \quad (3.1.3)$$

By manipulating the index, we have the following equations:

$$h(x) = \sum_{n=0}^{\infty} f_{011}^{(n)} x^{n+1}, \qquad h(x) = \sum_{n=0}^{\infty} f_{011}^{(n+1)} x^{n+2} + f_{011}^{(0)} x,$$

$$h'(x) = \sum_{n=0}^{\infty} (n+1) f_{011}^{(n)} x^{n}, \qquad h'(x) = \sum_{n=0}^{\infty} (n+2) f_{011}^{(n+1)} x^{n+1} + f_{011}^{(0)},$$

$$xh'(x) = \sum_{n=0}^{\infty} (n+1) f_{011}^{(n)} x^{n+1}, \qquad x^2h'(x) = \sum_{n=1}^{\infty} n f_{011}^{(n-1)} x^{n+1}.$$

Using these 6 equations, (3.1.2) becomes

$$\begin{aligned}
&\sum_{n=1}^{\infty} 3\left[(n+2)f_{011}^{(n+1)} + f_{011}^{(n+1)} - (10n+10)f_{011}^{(n)} - 5f_{011}^{(n)} + 9nf_{011}^{(n-1)}\right] x^{n+1} \\
&= 3\left[h'(x) - f_{011}^{(0)} - 2f_{011}^{(1)}x + \frac{h(x) - f_{011}^{(0)}x - f_{011}^{(1)}x^2}{x} - 10(xh'(x) - f_{011}^{(0)}x) - 5(h(x) - f_{011}^{(0)}x) + 9x^2h'(x)\right] \\
&= 3\left[(1-10x+9x^2)h'(x) + \left(\frac{1}{x} - 5\right)h(x) - 2f_{011}^{(0)} - 3f_{011}^{(1)}x + 15f_{011}^{(0)}x\right]
\end{aligned}$$

Since $f_{011}^{(0)} = 2$ and $f_{011}^{(1)} = 12$, we have, after factoring $1/x$,

$$\begin{aligned}&\frac{3}{x}\left[x(1-10x+9x^2)h'(x)+(1-5x)\,h(x)-4x-36x^2+30x^2\right]\\&=\frac{3}{x}\left[x(1-10x+9x^2)h'(x)+(1-5x)\,h(x)-(4x+6x^2)\right].\end{aligned}$$

Similarly we have the following for $g(x)$, with $f_{000}^{(0)} = 1$, $f_{000}^{(1)} = 3$, and $f_{000}^{(2)} = 15$:

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} f_{000}^{(n)}x^n = \sum_{n=1}^{\infty} f_{000}^{(n)}x^n + 1 = \sum_{n=1}^{\infty} f_{000}^{(n+1)}x^{n+1} + 1 + 3x = \sum_{n=1}^{\infty} f_{000}^{(n+2)}x^{n+2} + 1 + 3x + 15x^2,\\ g'(x) &= \sum_{n=1}^{\infty} nf_{000}^{(n)}x^{n-1} = \sum_{n=1}^{\infty}(n+1)f_{000}^{(n+1)}x^n + 3 = \sum_{n=1}^{\infty}(n+2)f_{000}^{(n+2)}x^{n+1} + 3 + 30x.\end{aligned}$$

Using these, (3.1.3) becomes

$$\begin{aligned}&x\sum_{n=1}^{\infty}\left[(n+2)f_{000}^{(n+2)}+2f_{000}^{(n+2)}-(10n+10)f_{000}^{(n+1)}-7f_{000}^{(n+1)}+9nf_{000}^{(n)}+9f_{000}^{(n)}\right]x^n\\&=x\left[\frac{g'(x)-3-30x}{x}+2\frac{g(x)-1-3x-15x^2}{x^2}-\left(10(g'(x)-3)+7\frac{g(x)-1-3x}{x}\right)+9xg'(x)+9(g(x)-1)\right]\\&=x\left[\left(\frac{1}{x}-10+9x\right)g'(x)+\left(\frac{2}{x^2}-\frac{7}{x}+9\right)g(x)-\left(\frac{3+30x}{x}+\frac{2+6x+30x^2}{x^2}-\left(30+\frac{7+21x}{x}\right)+9\right)\right].\end{aligned}$$

Factoring $1/x^2$ from each term and simplifying, we have

$$\frac{1}{x}\left[x(1-10x+9x^2)g'(x)+(2-7x+9x^2)g(x)-2(9x^2+x+1)\right].$$

Combining them, we now have statement 5, which conclude the proof for the forward direction.

For the backward direction, statement 2 gives us symmetry in the variables. Statement 1 gives $c_{000}(r)$ that statements 3, 4, 5, and 6 used to define the other Fourier functions. If all statemnts are satisfied, then these values will satisfy the matrix equation in the original proposition and gives the coefficients of $p_r(z_1, z_2, z_3)$. □

In the next chapter, we will see how the study of these polynomials led to a combinatorial application of spectral density function.

3.2 Asymptotics in d Variables

In Section 2.4.2 we derive Theorem 2.4.13, which gave us a range for the asymptotics of the Fourier coefficients for directions in the first and third quadrants. However, the method of ascent paths is only applicable to 2 variables. However, we can still state an analog to Theorem 2.4.4. In this section, let $\mathbf{z} = (z_1, \dots, z_d)$, $\ell = (\ell_1, \dots, \ell_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{r} = (r_1, \dots, r_d)$, and $\mathbf{z}^\ell = z_1^{\ell_1} \dots z_d^{\ell_d}$.

Theorem 3.2.1. *Let $p(\mathbf{z}) = \sum_{\ell} p_\ell \mathbf{z}^\ell, n_i \neq 0$ be given. Let $\mathbf{z}_i$ be points that satisfy either (2.4.6) and (2.4.7) or (2.4.8) for direction $\mathbf{r}$, and r_j are either all positive or all negative. Let k be such that $h(\mathbf{z}_k) < 0$ and for all $j \neq k$, either $h(\mathbf{z}_j) \geq 0$ or $h(\mathbf{z}_j) \leq h(\mathbf{z}_k)$. Then*

$$\left|\hat{f}(t\mathbf{r})\right| = O\left(\left|\mathbf{z}_k^{-\mathbf{r}}\right|^t\right).$$

Proof. The proof is identical to Theorem 2.4.4. □

Example 3.2.2. For simplicity, we use x, y, z in place of z_1, z_2, z_3. Let $p(x, y, z) = 10 - x - 2y - 3z$. If $\mathbf{r} = (a, b, c)$, then the matrix M for (2.4.7) is

$$M = \begin{bmatrix} x & 2y & 3z \\ x(10yz - 2z - 3y) & y(10xz - z - 3x) & z(10xy - y - 2x) \\ a & b & c \end{bmatrix},$$

and

$$\begin{aligned} \det(M) = {} & 20axy^2z - 2ay^2z + 5axyz - 30axyz^2 + 3ayz^2 + 10cx^2yz + 3cxyz \\ & - 3cx^2y - 10bx^2yz - 8bxyz + 2bx^2z + 30bxyz^2 - 20cxy^2z - 6bxz^2 + 6cxy^2. \end{aligned}$$

Suppose $\mathbf{r} = (1, 1, 1)$, then the six points that satisfy (2.4.6) and (2.4.7) are

$$z_1 = (.21552, 3.8022, .726690), \quad z_2 = (.14245, 2.7368, 1.461318), \quad z_3 = (4.643321, 2.1723, .337359),$$

and z_4, z_5, z_6 equal $1/z_1, 1/z_2, 1/z_3$, respectively. With $h(x, y, z) = -\ln(|x|) - \ln(|y|) - \ln(|z|)$, we have

$$\begin{aligned} h(z_1) &= 0.5184, \quad & h(z_2) &= 0.5626, \quad & h(z_3) &= -1.2246, \\ h(z_4) &= -0.5184, \quad & h(z_5) &= -0.5626, \quad & h(z_6) &= 1.2246. \end{aligned}$$

The two points that satisfy (2.4.8) are

$$z_7 = (10/3, 5/3, 10/9), \quad z_8 = (3/10, 3/5, 9/10),$$

with $h(z_7) = -1.8202, h(z_8) = 1.8202$. Therefore, according to the theorem,

$$\left|\hat{f}(t, t, t)\right| \leq O\left(\left|\frac{1}{(5.402)(0.2630)(1.3573)}\right|^t\right) = O(0.5186^t).$$

Similar to the result in two variables, we have not been able to determine a sharp estimate. For example, using MATLAB and the 3 variable extension of the algorithm from Section 5 of [Woerdeman et al. 2003], we found that

$$\frac{\hat{f}(19, 19, 19)}{\hat{f}(18, 18, 18)} = 0.1543.$$

Chapter 4: Abelian Squares and Their Progenies

While searching for a method to prove the conjecture in Section 3.1, we come across the combinatorial object abelian squares. This chapter is the result of the joint work with Charles Burnette, Jr., and it is devoted to the study of abelian squares and offset words and it connections with spectral density function. Due to the nature of this chapter, it is convenient to use somewhat different notations from previous chapters.

4.1 Introduction

For $r = 1, 2, \dots$, the r^{th} *power sum symmetric polynomial* in d variables is

$$p_{r,d}(x_1, \dots, x_d) = \sum_{k=1}^{d} x_k^r.$$

Power sum symmetric polynomials fall within the scope of the theory of symmetric functions and are exposited in various textbooks, for instance, those of [Stanley 1999] and [Macdonald 2015]. Although power sum symmetric polynomials habitually surface in commutative algebra and representation theory, the approach adopted here is purely analytic.

In Section 3.1, we studied the stable polynomial $p[x]^*_{1,3}(z_1, z_2, z_3) = 1 - xp_{1,3}(z_1, z_2, z_3), |x| < 1/3$, along with their spectral density functions $\mathcal{S}_{p[x]^*_{1,3}}$. Treating each $\mathcal{S}_{p[x]^*_{1,3}}$ as a function of x, one can calculate the Maclaurin series expansions of the Fourier coefficients $\widehat{\mathcal{S}_{p[x]^*_{1,3}}}(j, k, l)$. This approach lead us to the notion of abelian squares of length $2n$ over a 3-letter alphabet.

Given a nonempty, finite set of characters Σ, an *abelian square* over Σ is a string in the free monoid Σ^* (the set of finite-length strings that can be generated by concatenating arbitrary elements of Σ allowing the use of the same characters multiple times) of the form ww' where $w \in \Sigma^*$ and w' is a rearrangement of w. Six examples of English abelian squares are `sheesh`, `intestines`, `redder`, `beriberi`, `reappear`, and `aa`, the last being both the shortest and alphabetically first abelian square in the English language. The concept of an abelian square was introduced by Erdős in [Erdős 1961]. Richmond and Shallit expound the combinatorics of abelian squares in [Richmond and Shallit 2009]. Callan vividly describes several enumerative interpretations of the number of abelian squares in [Callan 2008].

It turns out that every Fourier coefficient $\widehat{\mathcal{S}_{p[x]^*_{r,3}}}$ is encoded with combinatorial data. In this chapter, we exhibit multidimensional generalizations of $p[x]^*_{r,3}$, which we shall call stabilized power sum symmetric polynomials, whose spectral density functions are essentially the only $L^2(\mathbb{T}^d)$ functions with Fourier coefficients that are all generating functions for a class of strings satisfying certain constraints. Multiple properties of the coefficients of these generating functions, including recurrent and asymptotic behavior, are deduced afterward. Towards the end of chapter, we show that the rudimentary harmonic analysis underlying our generalization appends a combinatorial counterpart to Parseval's equation.

4.1.1 Notational Conventions

Taking a page from measure theory, we present the following decomposition of integers. If $a \in \mathbb{Z}$, we define $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. Much like other such decompositions, $a = a^+ - a^-$ and $|a| = a^+ + a^-$. Also, for any two integers a and b, there is exactly one integer c such that $a - c^+ = b - c^-$, namely $c = a - b$. This follows easily from the fact that $a - b = (a - b)^+ - (a - b)^-$.

If $\alpha \in \mathbb{Z}^d$, then we set

$$\alpha^+ = (\alpha_1^+, \dots, \alpha_d^+), \qquad \alpha^- = (\alpha_1^-, \dots, \alpha_d^-), \qquad \operatorname{abs}(\alpha) = (|\alpha_1|, \dots, |\alpha_d|)$$

and if $\alpha \in \mathbb{N}_0^d$,

$$|\alpha| = \sum_{k=1}^{d} \alpha_k, \qquad \alpha! = \prod_{k=1}^{d} \alpha_k!, \qquad \binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

It will also be convenient to have a notion of divisibility for integer vectors. We say that $a \mid \alpha$ if $a \mid \alpha_k$ for $k = 1, \dots, d$. Lastly $\| \bullet \|_p$ symbolizes the p-norm.

We shall let $[a : b]$ denote the set of integers between a and b, inclusive. Because the precise lettering of an alphabet is immaterial for our purposes, we will set $\Sigma_d = [1 : d]$ throughout the rest of this document. The length of a string w is the number of characters in the string and is denoted by $|w|$. The *Parikh vector* of a string $w \in \Sigma_d^*$ is given by $\rho(w) = (\rho_1(w), \dots, \rho_d(w))$, where $\rho_j(w)$ denotes the multiplicity of j in w. Clearly

$|w| = \rho_1(w) + \cdots + \rho_d(w)$. (Rohit Parikh, the namesake of this signature for strings, first implemented these vectors in his work on context-free languages in [Parikh 1961].) Lastly, unless stated otherwise, d represents the number of letters in the alphabet and the length of any given vectors.

4.2 Fourier Analysis of the Spectral Density Functions of Stabilized Power Sum Symmetric Polynomials

Definition 4.2.1. *The r^{th} stabilized power sum symmetric polynomial in d variables is*

$$p[x]^*_{r,d}(z_1, \dots, z_d) = 1 - xp_{r,d}(z_1, \dots, z_d),$$

where x is an indeterminate.

In this section, we give a combinatorial interpretation of the Fourier coefficients of $\mathcal{S}_{p[x]^*_{r,d}}$ and show that some harmonic analytic aspects of $\mathcal{S}_{p[x]^*_{r,d}}$ are intertwined with combinatorics on words.

Offset Words

Definition 4.2.2. *For $(n, \xi) \in \mathbb{N}_0 \times \mathbb{Z}^d$, an n^{th} order word offset by ξ is a string in Σ_d^* of length $2n + \|\xi\|_1$ of the form ww' where $w, w' \in \Sigma_d^*$ and $\rho(w) - \rho(w') = \xi$. We denote the set of n^{th} order words offset by ξ by $\mathcal{W}_{(n,\xi)}$, the set of all words offset by ξ (regardless of order) by $\mathcal{W}_\xi$, and set $w_{(n,\xi)} = \left|\mathcal{W}_{(n,\xi)}\right|$.*

Intuitively ξ is a measurement of how removed ww' is from being an abelian square. Evidently an n^{th} order word offset by $\mathbf{0}$ is an abelian square of length $2n$ since a string ww' is an abelian square if and only if $\rho(w) = \rho(w')$.

Example 4.2.3. Consider the alphabet $\Sigma_3 = \{1, 2, 3\}$. Let $w = 13$ and $w' = 12$. Then $ww' = 1312$ is a first order word offset by $(0, -1, 1)$. The string 1312 is also

- a zeroth order word offset by $(-2, -1, -1)$ with $w = \varepsilon$, the empty string, and $w' = 1312$,
- a first order word offset by $(0, -1, -1)$ with $w = 1$ and $w' = 312$,
- a zeroth order word offset by $(2, -1, 1)$ with $w = 131$ and $w' = 2$,

- a zeroth order word offset by $(2,1,1)$ with $w = 1312$ and $w' = \varepsilon$.

Proposition 4.2.4. *Every string $w \in \Sigma_d^*$ is in $|w|+1$ of the sets $\mathcal{W}_{(n,\xi)}$.*

Proof. Suppose $|w| = N$ and let $k \in [0:N]$. Write w as the concatenation of two strings w_k and w'_{N-k} such that $|w_k| = k$ and $\left|w'_{N-k}\right| = N-k$. In setting $\xi = \rho(w_k) - \rho(w'_{N-k})$, we note that $N - \|\xi\|_1$ is even, because

$$\begin{aligned} N - \|\xi\|_1 &= N - \sum_{j=1}^{d} \left|\rho_j(w_k) - \rho_j(w'_{N-k})\right| \\ &\equiv N - \sum_{j=1}^{d} \left(\rho_j(w_k) - \rho_j(w'_{N-k})\right) \pmod 2 = 2\sum_{j=1}^{d} \rho_j(w'_{N-k}) \pmod 2. \end{aligned}$$

Therefore $w \in \mathcal{W}_{(n,\xi)}$ with $\xi = (\rho_1(w_k) - \rho_1(w'_{N-k}), \ldots, \rho_d(w_k) - \rho_d(w'_{N-k}))$ and $n = \frac{1}{2}(N - \|\xi\|_1)$. □

Except for the empty string, which is solely an abelian square, every string in Σ_d^* can be classified as an offset word, in exactly $|w|+1$ way. This may seem to undermine the utility of Definition 4.2.2, but offset words are a natural generalization of abelian squares in the following sense: the generating functions for $w_{(n,\xi)}$ are the Fourier coefficients of one, and essentially only one, function in $L^2(\mathbb{T}^d)$. We will demonstrate this later in the section. First, we have a theorem that gives a formula for $w_{(n,\xi)}$.

Theorem 4.2.5.

$$w_{(n,\xi)} = \sum_{\substack{\nu \in \mathbb{N}_0^d, \\ |\nu| = n}} \binom{|\nu + \xi^+|}{\nu + \xi^+} \binom{|\nu + \xi^-|}{\nu + \xi^-}. \tag{4.2.1}$$

Proof. Set $\nu(ww') := \rho(w) - \xi^+$ for a string $ww' \in \mathcal{W}_{(n,\xi)}$. Since $\rho(w) - \rho(w') = \xi$, we must have $\rho(w) - \xi^+ = \rho(w') - \xi^-$. Observe that

$$\sum_{j=1}^{d} \nu_j(ww') = \frac{1}{2}\sum_{j=1}^{d} [(\rho_j(w) - \xi_j^+) - (\rho_j(w') - \xi_j^-)] = \frac{1}{2}\sum_{j=1}^{d} [\rho_j(ww') - |\xi_j|] = n.$$

This together with the fact that $\rho(w) = \nu(ww') + \xi^+$ and $\rho(w') = \nu(ww') + \xi^-$ means that $w_{(n,\xi)}$ is the number of ways to choose the Parikh vectors of w and w' so that $\rho(w)$ and $\rho(w')$ are weak compositions of $n + |\xi^+|$ and $n + |\xi^-|$, respectively, with $\rho(w) - \xi^+ = \rho(w') - \xi^-$. □

Fourier Coefficients as Generating Functions

We now shift our focus to the polynomial $p[x]^*_{1,d}(z_1, \dots, z_d)$, which is stable precisely when $|x| < 1/d$. Under the additional assumptions that $x \in \mathbb{R}$ and $(z_1, \dots, z_d) \in \mathbb{T}^d$, we can express $(p[x]^*_{1,d})^{-1}$ and its conjugate as geometric sums:

$$\frac{1}{p[x]^*_{1,d}(z_1, \dots, z_d)} = \sum_{n=0}^{\infty} [x p_{1,d}(z_1, \dots, z_d)]^n, \tag{4.2.2}$$

$$\frac{1}{\overline{p[x]^*_{1,d}(z_1, \dots, z_d)}} = \sum_{n=0}^{\infty} [x p_{1,d}(\overline{z_1}, \dots, \overline{z_d})]^n. \tag{4.2.3}$$

Both series in (4.2.2) and (4.2.3) converge absolutely on the assumed domain. Thus, according to Mertens' theorem on the convergence of Cauchy products of series, the restriction of the spectral density function $\mathcal{S}_{p[x]^*_{1,d}}$ to $\mathbb{T}^d$ equals

$$\frac{1}{\left|p[x]^*_{1,d}(z_1, \dots, z_d)\right|^2} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \left(p_{1,d}(z_1, \dots, z_d)\right)^k \left(p_{1,d}(\overline{z_1}, \dots, \overline{z_d})\right)^{n-k} \right] x^n \tag{4.2.4}$$

for $|x| < 1/d$. Furthermore, since $\mathcal{S}_{p[x]^*_{1,d}}\Big|_{\mathbb{T}^d}$ can be presented as a power series in each variable, it converges uniformly on every compact subset of $(-1/d, 1/d) \times \mathbb{T}^d$. (Of course, expressing $\mathcal{S}_{p[x]^*_{1,d}}\Big|_{\mathbb{T}^d}$ as a power series in the z_j requires rearrangement of the terms. This is justified since (4.2.4) holds on all of $(-1/d, 1/d) \times \mathbb{T}^d$, thus rendering that series absolutely convergent.)

We now state the theorem describing the ordinary generating function for $w_{(n,\xi)}$.

Theorem 4.2.6. *The ordinary generating function for $w_{(n,\xi)}$, counted according to the order n, is*

$$W_\xi(x) = \frac{1}{(2\pi)^d x^{\frac{1}{2}\|\xi\|_1}} \widehat{\mathcal{S}_{p[\sqrt{x}]^*_{1,d}}}(\xi) \tag{4.2.5}$$

and has a radius of convergence of $1/d^2$.

Proof. To prove the theorem, we need to compute the Fourier coefficients of $\mathcal{S}_{p[x]^*_{1,d}}$:

$$\widehat{\mathcal{S}_{p[x]^*_{1,d}}}(\xi) = \frac{1}{(2\pi)^d} \int\limits_{[0,2\pi]^d} e^{-i\xi^\mathsf{T}\theta} \mathcal{S}_{p[x]^*_{1,d}}(e^{i\theta_1}, \dots, e^{i\theta_d})\, \mathrm{d}\theta. \tag{4.2.6}$$

Here ξ^T is the transpose of ξ, $\theta = (\theta_1, \dots, \theta_d)$, and the integral is relative to the completion of the d-fold product of the Lebesgue measure on $\mathbb{R}$. By (4.2.4) and the multinomial theorem, $\widehat{\mathcal{S}_{p[x]^*_{1,d}}}(\xi)$ can be written as

$$\frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \left[\int\limits_{[0,2\pi]^d} e^{-i\xi^\mathsf{T}\theta} \sum_{k=0}^{n} \left((p_{1,d}(e^{i\theta_1}, \dots, e^{i\theta_d}) \right)^k \left((p_{1,d}(e^{-i\theta_1}, \dots, e^{-i\theta_d}) \right)^{n-k} \mathrm{d}\theta \right] x^n$$

$$= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \left[\int\limits_{[0,2\pi]^d} \sum_{k=0}^{n} \sum_{\substack{\kappa,\kappa' \in \mathbb{N}_0^d, \\ |\kappa| = k, \\ |\kappa'| = n-k}} \binom{|\kappa|}{\kappa} \binom{|\kappa'|}{\kappa'} \left(\prod_{j=1}^{d} e^{i(\kappa - \kappa' - \xi)^\mathsf{T}\theta} \right) \mathrm{d}\theta \right] x^n. \tag{4.2.7}$$

where the uniform convergence of $\mathcal{S}_{p[x]^*_{1,d}}$ allows us to interchange the summation and the multiple integral with impunity. Fubini's theorem also grants us treatment of these Fourier coefficients as iterated integrals. So since $\int_0^{2\pi} e^{ia\theta}\, \mathrm{d}\theta = 2\pi$ when $a = 0$ and 0 if $a \in \mathbb{Z} - \{0\}$, the only terms in the integrand of (4.2.7) that contribute to the sum are those for which $\kappa - \kappa' - \xi = \mathbf{0}$, which implies that $\kappa - \xi^+ = \kappa' - \xi^-$. Hence, we can simplify the power series for $\widehat{\mathcal{S}_{p[x]^*_{1,d}}}(\xi)$ to

$$\sum_{n=0}^{\infty} \left[\sum_{\substack{\nu \in \mathbb{N}_0^d, \\ |\nu| = n}} \binom{|\nu + \xi^+|}{\nu + \xi^+} \binom{|\nu + \xi^-|}{\nu + \xi^-} \right] x^{2n + \|\xi\|_1}.$$

Replace x with $\sqrt{x}$ in the above power series and then normalize it, we have the desire generating function. $\square$

The Fourier coefficients of $\mathcal{S}_{p[x]^*_{r,d}}$ can be calculated in the same manner by noticing that $p[x]^*_{r,d}(z_1, \dots, z_d) =$

$p[x]^*_{1,d}(z_1^r, \ldots, z_d^r)$. Hence

$$\widehat{\mathcal{S}_{p[x]^*_{r,d}}}(\xi) = \frac{1}{(2\pi)^d} \int\limits_{[0,2\pi]^d} e^{-i\xi^\intercal\theta} \mathcal{S}_{p[x]^*_{1,d}}(e^{ir\theta_1}, \ldots, e^{ir\theta_d})\, \mathrm{d}\theta$$
$$= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \left[\int\limits_{[0,2\pi]^d} \sum_{k=0}^{n} \sum_{\substack{\kappa,\kappa' \in \mathbb{N}_0^d, \\ |\kappa|=k, \\ |\kappa'|=n-k}} \binom{|\kappa|}{\kappa} \binom{|\kappa'|}{\kappa'} \left(\prod_{j=1}^{d} e^{i(r\kappa - r\kappa' - \xi)^\intercal\theta} \right) \mathrm{d}\theta \right] x^n. \tag{4.2.8}$$

Like before, the only terms in the integrand of (4.2.8) that contribute to the sum are those for which $r\kappa - r\kappa' - \xi = \mathbf{0}$. So $\widehat{\mathcal{S}_{p[x]^*_{r,d}}}(\xi) = 0$ unless $r \mid \xi$, in which case we then require $\kappa - \kappa' - r^{-1}\xi = \mathbf{0}$. The analysis now reduces to that of $\widehat{\mathcal{S}_{p[x]^*_{1,d}}}(r^{-1}\xi)$.

Corollary 4.2.7. *For $\xi \in \mathbb{Z}^d$,*

$$\widehat{\mathcal{S}_{p[x]^*_{r,d}}}(\xi) = \widehat{\mathcal{S}_{p[x]^*_{1,d}}}(r^{-1}\xi)\mathbf{1}_{r \mid \xi}, \tag{4.2.9}$$

where

$$\mathbf{1}_{r \mid \xi} = \begin{cases} 1 & \text{if } r \mid \xi, \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem shows the spectral density function $\mathcal{S}_{p[x]^*_{1,d}}$ is essentially the only $L^2(\mathbb{T}^d)$ function with Fourier coefficients that are all generating functions for $W_\xi(x)$.

Theorem 4.2.8. *The spectral density function $\mathcal{S}_{p[\sqrt{x}]^*_{1,d}}$, with $|x| < 1/d^2$, is, up to sets of measure zero, the only parametrized $L^2(\mathbb{T}^d)$ function $f(\cdot\,;x), x \in \mathbb{R}$, that satisfies*

$$x^{\frac{1}{2}\|\xi\|_1} W_\xi(x) = \widehat{f(\cdot\,;x)}(\xi)$$

for all $x \in (-1/d^2, 1/d^2)$.

Proof. $\mathcal{S}_{p[\sqrt{x}]^*_{1,d}} \in L^2(\mathbb{T}^d)$ due to the stability of $p[\sqrt{x}]^*_{1,d}$ for $|x| < 1/d^2$. Parseval's equation thus dictates that

$$\sum_{\xi\in\mathbb{Z}^d} x^{\|\xi\|_1}[W_\xi(x)]^2 = \frac{1}{(2\pi)^d}\int\limits_{[0,2\pi]^d}\left|\mathcal{S}_{p[\sqrt{x}]^*_{1,d}}(e^{i\theta_1},\ldots,e^{i\theta_d})\right|^2 \mathrm{d}\theta < \infty, \tag{4.2.10}$$

for all $x \in (-1/d^2, 1/d^2)$. Uniqueness now follows from Plancherel's theorem. □

4.3 Basic Properties of $w_{(n,\xi)}$

We begin this section by deriving a multiple integral representation of $w_{(n,\xi)}$.

Corollary 4.3.1.

$$w_{(n,\xi)} = \frac{1}{(2\pi)^d}\int\limits_{[0,2\pi]^d} e^{-i\xi^{\mathsf{T}}\theta}\left(p_{1,d}(e^{i\theta_1},\ldots,e^{i\theta_d})\right)^{|\xi^+|} \tag{4.3.1}$$

$$\times\left(p_{1,d}(e^{-i\theta_1},\ldots,e^{-i\theta_d})\right)^{|\xi^-|}\left|p_{1,d}(e^{i\theta_1},\ldots,e^{i\theta_d})\right|^{2n}\mathrm{d}\theta.$$

Proof. Per the discussion preceding Theorem 4.2.6,

$$w_{(n,\xi)} = [x^{2n+\|\xi\|_1}]\widehat{\mathcal{S}_{p[x]^*_{1,d}}}(\xi). \tag{4.3.2}$$

Now work backwards from (4.2.7). Bearing in mind that the $k = n + |\xi^+|$ term of the sum on the left-hand side of (4.2.7) is the only one that contributes to the integral, we see that

$$w_{(n,\xi)} = \frac{1}{(2\pi)^d}\int\limits_{[0,2\pi]^d} e^{-i\xi^{\mathsf{T}}\theta}\left(p_{1,d}(e^{i\theta_1},\ldots,e^{i\theta_d})\right)^{n+|\xi^+|}\left(p_{1,d}(e^{-i\theta_1},\ldots,e^{-i\theta_d})\right)^{n+|\xi^-|}\mathrm{d}\theta.$$

Formula (4.3.1) now follows since $p_{1,d}(e^{-i\theta_1},\ldots,e^{-i\theta_d}) = \overline{p_{1,d}(e^{i\theta_1},\ldots,e^{i\theta_d})}$. □

We naturally wish to find a closed-form expression for $w_{(n,\xi)}$. Unfortunately, the sum in (4.2.1) and the integral in (4.3.1) look wholly intractable. At the time of writing, $w_{(n,\mathbf{0})}$ has defied evaluation for nearly three decades. Richards and Cambanis in [Cambanis and Richards 1987] proposed the problem of calculating what they call $S(n,k) = \sum[\frac{k!}{k_1!k_2!\cdots k_n!}]^2$, where the sum is over all nonnegative integers $k_1,\ldots,k_n$, such that $k_1+\cdots+k_n = k$. Andrews in [Andrews et al. 1988] remarked that the presence of large primes in $S(n,k)$

makes obtaining a closed form finite product representation of $S(n, k)$ implausible. These same heuristic barriers seem to persist in enumerating the n^{th} order words offset by ξ. There is, however, a recursive trait attributable to the $w_{(n,\xi)}$ that we can glean.

Theorem 4.3.2. *The number of n^{th} order words offset by ξ satisfies the recurrence*

$$w_{(n,\xi)} = \sum_{j=0}^{n} \binom{n + |\xi^+|}{j + \xi^+_{s_1} + \cdots + \xi^+_{s_t}} \binom{n + |\xi^-|}{j + \xi^-_{s_1} + \cdots + \xi^-_{s_t}} \times w_{(j,(\xi_{s_1},\ldots,\xi_{s_t}))} w_{(n-j,(\xi_1,\ldots,\widehat{\xi_{s_1}},\ldots,\widehat{\xi_{s_t}},\ldots,\xi_d))}, \quad \textit{for } d \geq 2 \tag{4.3.3}$$

where $t \in [1 : d-1]$, $s_1 < \cdots < s_t$ are t natural numbers selected from Σ_d, and $\widehat{\xi_{s_1}}, \ldots, \widehat{\xi_{s_t}}$ means that the indices $\xi_{s_1}, \ldots, \xi_{s_t}$ are removed from the list $\xi_1, \ldots, \xi_d$.

Proof. Partition Σ_d into two disjoint subsets S and T containing t and $d - t$ characters, respectively. Let $s_1 < \cdots < s_t$ be the elements of S. We now count the number of offset words ww' by conditioning on the value of $\rho_{s_1}(w) + \cdots + \rho_{s_t}(w)$, keeping in mind that w must have at least $\xi^+_{s_1} + \cdots + \xi^+_{s_t}$ characters from S and at least $\xi^+_1 + \cdots + \xi^+_d - (\xi^+_{s_1} + \cdots + \xi^+_{s_t})$ characters from T, and that w' must adhere to the same requirements except with the ξ^+_s replaced by ξ^-_s. So $\rho_{s_1}(w) + \cdots + \rho_{s_t}(w)$ can range from $\xi^+_{s_1} + \cdots + \xi^+_{s_t}$ to $n + \xi^+_{s_1} + \cdots + \xi^+_{s_t}$, inclusive. Choose which $j + \xi^+_{s_1} + \cdots + \xi^+_{s_t}$ of the characters in w and which $j + \xi^-_{s_1} + \cdots + \xi^-_{s_t}$ of the characters in w' come from S. Once the spots designated for S have been selected, they can be filled in $w_{(j,(\xi_{s_1},\ldots,\xi_{s_t}))}$ ways. The remaining spots can be filled in $w_{(n-j,(\xi_1,\ldots,\widehat{\xi_{s_1}},\ldots,\widehat{\xi_{s_t}},\ldots,\xi_d))}$. Summing $\binom{n + \xi^+_1 + \cdots + \xi^+_d}{j + \xi^+_{s_1} + \cdots + \xi^+_{s_t}} \binom{n + \xi^-_1 + \cdots + \xi^-_d}{j + \xi^-_{s_1} + \cdots + \xi^-_{s_t}} w_{(j,(\xi_{s_1},\ldots,\xi_{s_t}))} w_{(n-j,(\xi_1,\ldots,\widehat{\xi_{s_1}},\ldots,\widehat{\xi_{s_t}},\ldots,\xi_d))}$ over j completes the argument. □

We now prove a divisibility property of $w_{(n,\xi)}$ that generalizes a fact about abelian squares conjectured by Andrews and proven by Kolitsch in [Andrews et al. 1988].

Lemma 4.3.3.

$$w_{(n,(m,\ldots,m))} \equiv 0 \pmod{d}, \qquad \textit{for } n \textit{ and } m \textit{ not both } 0.$$

Proof. Set $\mu = (m, \ldots, m)$. Let the cyclic group $\mathbb{Z}_d$ act on the set C of weak compositions of n into exactly d

parts by cyclically permuting the parts of a composition. Pick a representative $\pi^{\mathcal{O}}$ from each orbit $\mathcal{O} \in C/\mathbb{Z}_d$. Then

$$w_{(n,\mu)} = \sum_{\mathcal{O}\in C/\mathbb{Z}_d} |\mathcal{O}| \binom{|\pi^{\mathcal{O}}+\mu^+|}{\pi^{\mathcal{O}}+\mu^+}\binom{|\pi^{\mathcal{O}}+\mu^-|}{\pi^{\mathcal{O}}+\mu^-} \tag{4.3.4}$$
$$= \sum_{\mathcal{O}\in C/\mathbb{Z}_d} |\mathcal{O}| \binom{|\pi^{\mathcal{O}}|}{\pi^{\mathcal{O}}}\binom{|\pi^{\mathcal{O}}+\operatorname{abs}(\mu)|}{\pi^{\mathcal{O}}+\operatorname{abs}(\mu)}$$

since one of m^+ and m^- is 0 while the other is $|m|$.

Now let $\mathcal{O} \in C/\mathbb{Z}_d$ be arbitrary. By the orbit-stabilizer theorem, together with Lagrange's theorem, the size of the orbits divide d, and so

$$\pi^{\mathcal{O}} = (\underbrace{\pi_1^{\mathcal{O}},\ldots,\pi_{|\mathcal{O}|}^{\mathcal{O}}}_{\text{written } d/|\mathcal{O}|\text{times}}).$$

Since $\pi_1^{\mathcal{O}}+|m|,\ldots,\pi_{|\mathcal{O}|}^{\mathcal{O}}+|m|$ are not all 0, their greatest common divisor is nonzero. As a result,

$$\binom{|\pi^{\mathcal{O}}+\operatorname{abs}(\mu)|}{\pi^{\mathcal{O}}+\operatorname{abs}(\mu)} = \binom{(d/|\mathcal{O}|)(\pi_1^{\mathcal{O}}+\cdots+\pi_{|\mathcal{O}|}^{\mathcal{O}}+|\mathcal{O}|\,|m|)}{\underbrace{\pi_1^{\mathcal{O}}+|m|,\ldots,\pi_{|\mathcal{O}|}^{\mathcal{O}}+|m|}_{\text{written } d/|\mathcal{O}| \text{ times}}} \tag{4.3.5}$$
$$\equiv 0 \left(\bmod \frac{(d/|\mathcal{O}|)(\pi_1^{\mathcal{O}}+\cdots+\pi_{|\mathcal{O}|}^{\mathcal{O}}+|\mathcal{O}|\,|m|)}{\gcd(\pi_1^{\mathcal{O}}+|m|,\ldots,\pi_{|\mathcal{O}|}^{\mathcal{O}}+|m|)}\right),$$

in which we have applied Theorem 1 in [Gould 1974]. Yet $\gcd(\pi_1^{\mathcal{O}}+|m|,\ldots,\pi_{|\mathcal{O}|}^{\mathcal{O}}+|m|)$ divides $\pi_1^{\mathcal{O}}+\cdots+\pi_{|\mathcal{O}|}^{\mathcal{O}}+|\mathcal{O}|\,|m|$ so that

$$\frac{(d/|\mathcal{O}|)(\pi_1^{\mathcal{O}}+\cdots+\pi_{|\mathcal{O}|}^{\mathcal{O}}+|\mathcal{O}|\,|m|)}{\gcd(\pi_1^{\mathcal{O}}+|m|,\cdots,\pi_{|\mathcal{O}|}^{\mathcal{O}}+|m|)} \equiv 0 \pmod{d/|\mathcal{O}|}. \tag{4.3.6}$$

Therefore

$$\binom{|\pi^{\mathcal{O}}+\operatorname{abs}(\mu)|}{\pi^{\mathcal{O}}+\operatorname{abs}(\mu)} \equiv 0 \pmod{d/|\mathcal{O}|}, \tag{4.3.7}$$

and so

$$|\mathcal{O}| \binom{|\pi^{\mathcal{O}}|}{\pi^{\mathcal{O}}}\binom{|\pi^{\mathcal{O}}+\operatorname{abs}(\mu)|}{\pi^{\mathcal{O}}+\operatorname{abs}(\mu)} \equiv 0 \pmod{d}, \tag{4.3.8}$$

which indicates that $w_{(n,\mu)} \equiv 0 \pmod{d}$. □

Theorem 4.3.4. *Let $d \geq 2$ and $\xi \in \mathbb{Z}^d - \{\mathbf{0}\}$, and for each $a \in \mathbb{Z}$, let $o_a(\xi)$ be the number of occurrences of a in ξ. Then*

$$w_{(n,\xi)} \equiv 0 \left(\text{mod lcm}(1, 2, \ldots, \max_{a \neq 0} o_a(\xi)) \right). \tag{4.3.9}$$

Proof. Note that $\max_{a \neq 0} o_a(\xi) \geq 1$. By letting a_0 be a nonzero integer with the maximal number of occurrences, we can apply Theorem 4.3.2 to get

$$w_{(n,\xi)} = \sum_{j=0}^{n} \binom{n + |\xi^+|}{j + ta_0^+} \binom{n + |\xi^-|}{j + ta_0^-} w_{(j,(a_0,\ldots,a_0))} w_{(n-j,(\xi_1,\ldots,\widehat{a_0},\ldots,\widehat{a_0},\ldots,\xi_d))} \tag{4.3.10}$$

for each $t \in [1 : \max_{a \neq 0} o_a(\xi)]$. It follows from Lemma 4.3.3 that every summand is divisible by t, as desired. □

4.4 Asymptotics

This section considers the asymptotic behavior of $w_{(n,\xi)}$, first as $n \to \infty$ with ξ and d fixed, then as $\|\xi\|_1 \to \infty$ in a fixed direction in $\mathbb{Z}^d$ with n and d fixed, and then finally as the dimension $d \to \infty$ with n fixed and the components of ξ fixed and all equal.

Coefficient Asymptotics of W_ξ

Recall that $w_{(n,\xi)}$ is defined in Definition 4.2.2. We then have the following asymptotic result when $n \to \infty$.

Theorem 4.4.1.

$$w_{(n,\xi)} \sim d^{2n+d/2+\|\xi\|_1} (4\pi n)^{(1-d)/2} \tag{4.4.1}$$

as $n \to \infty$ with ξ and d fixed.

To prove this result, we need to extract the coefficient asymptotics of W_ξ by using Laplace's method on (4.2.1). Such a summation is well suited for the following version of Laplace's method for sums over lattice point translates as described in [Greenhill et al. 2010].

Theorem 4.4.2 (Greenhill, Janson, Ruciński)**.** *Suppose the following:*

(i) $\mathcal{L} \subset \mathbb{R}^N$ *is a lattice with rank* $r \leq N$.

(ii) $V \subset \mathbb{R}^N$ *is the* r*-dimensional subspace spanned by* $\mathcal{L}$.

(iii) $W = V + w$ *is an affine subspace parallel to* V *for some* $w \in \mathbb{R}^N$.

(iv) $K \subset \mathbb{R}^N$ *is a compact convex set with nonempty interior* K°.

(v) $\phi : K \to \mathbb{R}$ *is a continuous function and the restriction of* ϕ *to* $K \cap W$ *has a unique maximum at some point* $x_0 \in K^\circ \cap W$.

(vi) ϕ *is twice continuously differentiable in a neighborhood of* x_0 *and* $H_\phi(x_0)$ *is its Hessian at* x_0.

(vii) $\psi : K_1 \to \mathbb{R}$ *is a continuous function on some neighborhood* $K_1 \subset K$ *of* x_0 *with* $\psi(x_0) > 0$.

(viii) *For each positive integer* n *there is a vector* $\ell_n \in \mathbb{R}^N$ *with* $\ell_n/n \in W$.

(ix) *For each positive integer* n *there is a positive real number* b_n *and a function* $a_n : (\mathcal{L} + \ell_n) \cap nK \to \mathbb{R}$ *such that, as* $n \to \infty$,

$$a_n(\ell) = O\Big(b_n e^{n\phi(\ell/n)+o(n)}\Big), \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK,$$

and

$$a_n(\ell) = b_n(\phi(\ell/n) + o(1))\, e^{n\pi(\ell/n)}, \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK_1,$$

uniformly for ℓ in the indicated sets.

Then, regarding $-H_\phi$ as a bilinear form on V and provided $\det(-H_\phi|_V) \neq 0$, as $n \to \infty$,

$$\sum_{\ell \in (\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(x_0)}{\det(\mathcal{L})\sqrt{\det\left(-H_\phi|_V(x_0)\right)}} b_n n^{r/2} e^{n\phi(x_0)},$$

where $\det(\mathcal{L})$ is the square root of the discriminant of $\mathcal{L}$.

Proof for Theorem 4.4.1. Write $(n_1, n_2, \ldots, n_d)$ as $(n_1 - n, n_2, \ldots, n_d) + n(1, 0, \ldots, 0)$ so that if $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $n_1 + \cdots + n_d = n$, then $(n_1 - n, n_2, \ldots, n_d)$ is an integer vector whose components sum to 0. Thus, we let $\mathcal{L}$ be the root lattice A_{d-1}, V be the linear span of A_{d-1}, $w = (1, 0, \ldots, 0)$, $W = V + w$, and $\ell_n = nw$. Letting K be the unit hypercube $[0,1]^d$, we have

$$w_{(n,\xi)} = (n + |\xi^+|)!(n + |\xi^-|)! \sum_{\ell \in (A_{d-1}+nw)\cap nK} a_n(\ell) \tag{4.4.2}$$

where

$$a_n(\ell) = \prod_{j=1}^{d} \frac{1}{\ell_j!(\ell_j + |\xi_j|)!}. \tag{4.4.3}$$

Let $x_j = \ell_j/n$ for all $j \in [1:d]$. Applying Stirling's approximation in the form

$$\log(n!) = n\log n - n + \frac{1}{2}\log(\max\{n,1\}) + \frac{1}{2}\log 2\pi + O\left(\frac{1}{n+1}\right), \quad \text{for } n \geq 0 \tag{4.4.4}$$

we obtain, uniformly for $\ell \in (A_{d-1} + nw) \cap nK$ with n sufficiently large,

$$\begin{aligned}
\log(a_n(\ell)) &= -\sum_{j=1}^{d} (\log(\ell_j!) + \log((\ell_j + |\xi_j|)!)) \\
&= -\sum_{j=1}^{d} ((2\ell_j + |\xi_j| + 1)\log n - (2\ell_j + |\xi_j|) + \log 2\pi) \\
&\quad - n\sum_{j=1}^{d} (x_j \log x_j + (x_j + |\xi_j|/n)\log(x_j + |\xi_j|/n)) \\
&\quad - \frac{1}{2}\sum_{j=1}^{d} (\log(\max\{x_j, 1/n\}) + \log(\max\{x_j + |\xi_j|/n, 1/n\})) \\
&\quad - \sum_{j=1}^{d} \left(O\left(\frac{1}{\ell_j + 1}\right) + O\left(\frac{1}{\ell_j + |\xi_j| + 1}\right)\right) \\
&= -(2n + \|\xi\|_1 + d)\log n + 2n + \|\xi\|_1 - d\log 2\pi \\
&\quad - n\sum_{j,\, x_j \neq 0} \big(x_j \log x_j + (x_j + |\xi_j|/n)(\log x_j + |\xi_j|/\ell_j - O(\xi_j^2/\ell_j^2))\big) \\
&\quad - n\sum_{j,\, x_j = 0} (x_j \log x_j + (|\xi_j|/n)\log(|\xi_j|/n)) - \sum_{j,\, x_j = \xi_j = 0} \log(1/n) \\
&\quad - \frac{1}{2}\sum_{j,\, x_j = 0, \xi_j \neq 0} (2\log(1/n) + \log|\xi_j|) - \frac{1}{2}\sum_{j,\, x_j \neq 0} (2\log x_j + O(|\xi_j|/\ell_j)) \\
&\quad - \sum_{j=1}^{d} \left(O\left(\frac{1}{\ell_j + 1}\right) + O\left(\frac{1}{\ell_j + |\xi_j| + 1}\right)\right) \\
&= -(2n + \|\xi\|_1 + d)\log n + 2n - d\log 2\pi \\
&\quad + \sum_{j,\, x_j = 0} \left(|\xi_j| - |\xi_j|\log|\xi_j| - \frac{1}{2}(\log|\xi_j|)\mathbf{1}_{\xi_j \neq 0}\right) - n\sum_{j=1}^{d} 2x_j \log x_j \\
&\quad - \sum_{j=1}^{d} (\log(\max\{x_j, 1/n\}) + |\xi_j|\log(\max\{x_j, 1/n\})) \\
&\quad - \sum_{j=1}^{d}\left(O\left(\frac{1}{\ell_j + 1}\right) + O\left(\frac{1}{\ell_j + |\xi_j| + 1}\right)\right) + \sum_{j,\, x_j \neq 0}\left(O\left(\frac{1}{\ell_j}\right) + O\left(\frac{1}{\ell_j^2}\right)\right).
\end{aligned}$$

We can therefore write

$$a_n(\ell) = b_n \psi(\ell/n) \exp\left(n\phi(\ell/n) + \sum_{j,\, x_j = 0} \left(|\xi_j| - \left(|\xi_j| + \frac{1}{2}\mathbf{1}_{\xi_j \neq 0} \right) \log |\xi_j| \right) \right)$$
$$\times \left(1 + O\left(\frac{1}{\min_j \ell_j + 1} \right) + O\left(\frac{1}{\min_j (\ell_j + |\xi_j|) + 1} \right) \right),$$

where, for $x \in \mathbb{R}^d$,

$$b_n = \frac{e^{2n}}{(2\pi)^d n^{2n + \|\xi\|_1 + d}}, \qquad \psi(x) = \prod_{j=1}^{d} \frac{1}{x_j^{|\xi_j|+1}}, \qquad \phi(x) = -\sum_{j=1}^{d} 2x_j \log x_j,$$

unless if some x_j is 0, in which case we replace it by $1/n$ in the formula for ψ. This indicates that $a_n(\ell) = O(b_n e^{n\phi(\ell/n)+o(n)})$ for $\ell \in (A_{d-1} + nw) \cap nK$. On K°,

$$\sum_{j,\, x_j = 0} \left(|\xi_j| - \left(|\xi_j| + \frac{1}{2}\mathbf{1}_{\xi_j \neq 0} \right) \right) = 0$$

and $\frac{1}{\min \ell_j + 1}, \frac{1}{\min(\ell_j + |\xi_j|)+1} \to 0$ as $n \to \infty$ so that $a_n(\ell) = b_n(\psi(\ell/n) + o(1))e^{n\phi(\ell/n)}$ for $\ell \in (A_{d-1} + nw) \cap nK^\circ$. Also, observe that ψ is continuous and strictly positive on K° and therefore conditions (vii) and (ix) of Theorem 4.4.2 are satisfied, provided that $\phi|_{K \cap W}$ has a unique maximum located in $K^\circ \cap W$.

By Jensen's inequality,

$$\phi(x) = 2\sum_{j=1}^{d} x_j \log\left(\frac{1}{x_j}\right) \leq 2\log d, \tag{4.4.5}$$

for all $x \in K \cap W = \{x \in K : x_1 + \cdots + x_d = 1\}$. Since inequality holds if and only if $x_1 = x_2 = \cdots = x_d$, we see that $\phi_{K \cap W}$ attains its maximum value of $2\log d$ at the point $x_0 = (1/d, \ldots, 1/d)$ only. Finally, the Hessian $H_\phi(x)$ is diagonal with entries $-2/x_j$. Hence $H_\phi(x_0) = -2dI_d$, where I_d is the $d \times d$ identity matrix. All the assumptions of Theorem 4.4.2 have now been verified.

It is well known that A_{d-1} has rank $d-1$ and discriminant d, so all that remains is to calculate

$\det\left(-H_\phi|_V(x_0)\right)$. Let $\{\mathbf{e}_k\}_{k=1}^{d-1}$ be a basis for V. Then

$$\det\left(-H_\phi|_V(x_0)\right) = \frac{\det([\mathbf{e}_i^\intercal(2dI_d)\mathbf{e}_j]_{i,j=1}^{d-1})}{\det([\mathbf{e}_i^\intercal\mathbf{e}_j]_{i,j=1}^{d-1})} = (2d)^{d-1}. \tag{4.4.6}$$

Hence

$$\begin{aligned}\sum_{\ell\in(A_{d-1}+nw)\cap nK} a_n(\ell) &\sim \frac{(2\pi)^{(d-1)/2}d^{d+\|\xi\|_1}}{d^{1/2}(2d)^{(d-1)/2}}\frac{e^{2n}}{(2\pi)^d n^{2n+\|\xi\|_1+d}}n^{(d-1)/2}e^{2n\log d}\\ &= \frac{d^{2n+d/2+\|\xi\|_1}e^{2n}}{2^d\pi^{(d+1)/2}n^{2n+\|\xi\|_1+(d+1)/2}}\end{aligned}$$

and thus

$$\begin{aligned}w_{(n,\xi)} \sim{}& \sqrt{2\pi(n+|\xi^+|)}\left(\frac{n+|\xi^+|}{e}\right)^{n+|\xi^+|}\times\sqrt{2\pi(n+|\xi^-|)}\left(\frac{n+|\xi^-|}{e}\right)^{n+|\xi^-|}\\ &\times\frac{d^{2n+d/2+\|\xi\|_1}e^{2n}}{2^d\pi^{(d+1)/2}n^{2n+\|\xi\|_1+(d+1)/2}}.\end{aligned} \tag{4.4.7}$$

Simplification of (4.4.7) reveals the desired result. □

Stationary Phase Approximation of $w_{(n,\xi)}$

We now seek a leading order estimate of $w_{(n,\xi)}$ as $\|\xi\|_1\to\infty$ in a fixed direction in $\mathbb{Z}^d$ with n fixed. To accomplish this, we will work with

$$\begin{aligned}w_{(n,\lambda\xi)} = \frac{1}{(2\pi)^d}\int\limits_{[-\pi,\pi]^d} & e^{-i\lambda\xi^\intercal\theta}\left(p_{1,d}(e^{i\theta_1},\dots,e^{i\theta_d})\right)^{\lambda|\xi^+|}\\ &\times\left(p_{1,d}(e^{-i\theta_1},\dots,e^{-i\theta_d})\right)^{\lambda|\xi^-|}\left|p_{1,d}(e^{i\theta_1},\dots,e^{i\theta_d})\right|^{2n}\,\mathrm{d}\theta.\end{aligned} \tag{4.4.8}$$

as the positive real parameter λ tends to ∞. Such an oscillatory integral is amenable to the following variation of stationary phase method described in [Pemantle and Wilson 2010].

Theorem 4.4.3 (Pemantle, Wilson)**.** *Let A and φ be complex-valued analytic functions on a compact neighborhood $\mathcal{N}$ of the origin $\mathbb{R}^d$ and suppose that the real part of φ is nonnegative, vanishing only at the origin.*

Suppose that the Hessian matrix H_φ of φ at the origin is nonsingular. Denoting $\mathcal{I}(\lambda) := \int_{\mathcal{N}} A(\mathbf{x}) e^{-\lambda\varphi(\mathbf{x})}\,\mathrm{d}\mathbf{x}$, there is an asymptotic expansion

$$\mathcal{I}(\lambda) \sim \sum_{\ell \geq 0} c_\ell \lambda^{-d/2-\ell}$$

where

$$c_0 = A(\mathbf{0}) \frac{(2\pi)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}}$$

and the choice of sign is defined by taking the product of the principal square roots of the eigenvalues of $H_\varphi(\mathbf{0})$.

We also have the following corollary.

Corollary 4.4.4. *Let A be a real-valued continuous function and φ a complex-valued analytic function on a compact neighborhood $\mathcal{N}$ of the origin $\mathbb{R}^d$ and suppose that $A(\mathbf{0}) \neq 0$ and that the real part of φ is nonnegative, vanishing only at the origin. Suppose that the Hessian matrix H_φ of φ at the origin is nonsingular. Denoting $\mathcal{I}(\lambda) := \int_{\mathcal{N}} A(\mathbf{x}) e^{-\lambda\varphi(\mathbf{x})}\,\mathrm{d}\mathbf{x}$,*

$$\mathcal{I}(\lambda) = A(\mathbf{0}) \frac{(2\pi\lambda)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}} (1 + o(1))$$

as $\lambda \to \infty$. The choice of sign of the square root is defined by taking the product of the principal square roots of the eigenvalues of $H_\varphi(\mathbf{0})$.

Proof. By the Stone-Weierstrass Theorem, there exists a sequence $\{P_n\}_{n=1}^\infty$ of polynomials in $\mathbb{R}[x_1, \ldots, x_d]$ that converges uniformly to A over $\mathcal{N}$. Now for each $n \in \mathbb{N}$, define

$$\mathcal{I}_n(\lambda) := \int\limits_{\mathcal{N}} P_n(\mathbf{x}) e^{-\lambda\varphi(\mathbf{x})}\,\mathrm{d}\mathbf{x}.$$

Then for every $\lambda \in [0, \infty)$,

$$|\mathcal{I}_n(\lambda) - \mathcal{I}(\lambda)| \leq \int\limits_{\mathcal{N}} \left| (P_n(\mathbf{x}) - A(\mathbf{x})) e^{-\lambda\varphi(\mathbf{x})} \right| \mathrm{d}\mathbf{x} \leq \sup_{\mathbf{x} \in \mathcal{N}} |P_n(\mathbf{x}) - A(\mathbf{x})| \int\limits_{\mathcal{N}} \mathrm{d}\mathbf{x} \to 0$$

as $n \to \infty$. Therefore $\mathcal{I}_n$ converges uniformly to $\mathcal{I}$. Since each P_n is analytic, Theorem 4.4.3 tells us that

$$\mathcal{I}_n(\lambda) = P_n(\mathbf{0}) \frac{(2\pi\lambda)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}}(1 + o(1))$$

as $\lambda \to \infty$. We now use the Moore-Osgood Theorem (See Theorem 7.11 in [Rudin 1976]) to conclude that

$$\lim_{\lambda\to\infty} \frac{\mathcal{I}(\lambda)}{\frac{A(\mathbf{0})(2\pi\lambda)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}}} = \lim_{\lambda\to\infty} \lim_{n\to\infty} \frac{\mathcal{I}_n(\lambda)}{\frac{P_n(\mathbf{0})(2\pi\lambda)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}}} = \lim_{n\to\infty} \lim_{\lambda\to\infty} \frac{\mathcal{I}_n(\lambda)}{\frac{P_n(\mathbf{0})(2\pi\lambda)^{-d/2}}{\sqrt{\det(H_\varphi(\mathbf{0}))}}} = 1,$$

as required. □

Corollary 4.4.4 is essential to the main result of this section.

Theorem 4.4.5.

$$w_{(n,\lambda\xi)} = \frac{(2\pi)^{1-3d/2}}{\sqrt{\|\xi\|_1^d(d+1)}} d^{\lambda\|\xi\|_1+2n+d/2+1} \lambda^{-d/2}(1 + o(1)) \tag{4.4.9}$$

as $\lambda \to \infty$ with ξ and n fixed.

Proof. First we invoke the change of variables given by the function $\mathbf{f}_d : \mathbb{R}^d \to \mathbb{R}^d$ with components

$$\theta_j = -\tau_{j-1} + \tau_j + \delta, \quad \text{for } 1 \le j \le d$$

where we have introduced the placeholders $\tau_0 \equiv \tau_d \equiv 0$ in order to avoid messy casework later. (One may envision this as a transformation from the Cartesian coordinate system to a sort-of "signed distance from A_{d-1}" coordinate system where, given a point $\theta_0 \in \mathbb{R}^d$, $-\delta$ gives the value of t for which the hyperplane $\theta_1 + \cdots + \theta_d = 0$ intersects with the normal line $\ell(t) = \theta_0 + t(1, \ldots, 1)$ and $(\tau_1, \ldots, \tau_{d-1})$ are the coordinates of this intersection point relative to the standard basis of A_{d-1}.) The Jacobian matrix of $\mathbf{f}_d$ is

$$\mathbf{J}_{\mathbf{f}_d}(\tau_1,\ldots,\tau_{d-1},\delta)=\begin{bmatrix}\frac{\partial\theta_1}{\partial\tau_1} & \frac{\partial\theta_1}{\partial\tau_2} & \cdots & \frac{\partial\theta_1}{\partial\tau_{d-1}} & \frac{\partial\theta_1}{\partial\delta}\\ \frac{\partial\theta_2}{\partial\tau_1} & \frac{\partial\theta_2}{\partial\tau_2} & \cdots & \frac{\partial\theta_2}{\partial\tau_{d-1}} & \frac{\partial\theta_2}{\partial\delta}\\ \frac{\partial\theta_3}{\partial\tau_1} & \frac{\partial\theta_3}{\partial\tau_2} & \cdots & \frac{\partial\theta_3}{\partial\psi_{d-1}} & \frac{\partial\theta_3}{\partial\delta}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\partial\theta_d}{\partial\tau_1} & \frac{\partial\theta_d}{\partial\tau_2} & \cdots & \frac{\partial\theta_d}{\partial\tau_{d-1}} & \frac{\partial\theta_d}{\partial\delta}\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0 & 1\\ -1 & 1 & \ddots & 0 & 1\\ 0 & -1 & \ddots & 0 & 1\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & -1 & 1\end{bmatrix},$$

a $d\times d$ matrix with 1s on the main diagonal, 1s in the last column, -1s on the subdiagonal, and 0s everywhere else.

We use induction to show that the Jacobian determinant is equal to d. For the base case $d=1$, $\mathbf{f}_1(\theta_1)=\delta$ which trivially has a Jacobian determinant of 1. Now assume the induction hypothesis holds for some integer k, that is, $\det(\mathbf{J}_{\mathbf{f}_k})=k$. Laplace expansion along the first row of $\mathbf{J}_{\mathbf{f}_{k+1}}$ yields

$$\det(\mathbf{J}_{\mathbf{f}_{k+1}})=\det(\mathbf{J}_{\mathbf{f}_k})+(-1)^{k+2}\det(J_{-1,k})$$

where $J_{-1,k}$ is the $k\times k$ Jordan block with eigenvalue -1. We thus have

$$\det(\mathbf{J}_{\mathbf{f}_{k+1}})\underbrace{=}_{\text{ind. hyp.}}k+(-1)^{k+2}(-1)^k=k+1.$$

Since the Jacobian determinant is a nonzero constant, $\mathbf{f}_d$ is invertible and $\mathbf{f}_d^{-1}$ is differentiable. The transformation produces the following integral representation:

$$\begin{aligned}w_{(n,\lambda\xi)}=\frac{d}{(2\pi)^d}\int\limits_{\mathbf{f}_d^{-1}([-\pi,\pi]^d)}&e^{-i\lambda\xi^{\mathsf{T}}\mathbf{f}_d^{-1}(\theta)}\left(p_{1,d}(e^{i(-\tau_0+\tau_1+\delta)},\ldots,e^{i(-\tau_{d-1}+\tau_d\delta)})\right)^{\lambda|\xi^+|}\\ &\times\left(p_{1,d}(e^{-i(-\tau_0+\tau_1+\delta)},\ldots,e^{-i(-\tau_{d-1}+\tau_d+\delta)})\right)^{\lambda|\xi^-|}\\ &\times\left|p_{1,d}(e^{i(-\tau_0+\tau_1+\delta)},\ldots,e^{i(-\tau_{d-1}+\tau_d+\delta)})\right|^{2n}\mathrm{d}\delta\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{d-1}.\end{aligned}$$

Note $e^{i\delta}$ and $e^{-i\delta}$ can be factored out of

$$\left(p_{1,d}(e^{i(-\tau_0+\tau_1+\delta)},\dots,e^{i(-\tau_{d-1}+\tau_d+\delta)}\right)^{\lambda\left|\xi^+\right|}$$

and

$$\left(p_{1,d}(e^{-i(-\tau_0+\tau_1+\delta)},\dots,e^{-i(-\tau_{d-1}+\tau_d+\delta)})\right)^{\lambda\left|\xi^-\right|},$$

respectively, to get

$$\begin{aligned}
w_{(n,\lambda\xi)} = \frac{d}{(2\pi)^d} \int\limits_{\mathbf{f}_d^{-1}([-\pi,\pi]^d)} & e^{-i\lambda\xi^\mathsf{T}\mathbf{f}_d^{-1}(\theta)} e^{i\delta\lambda\left|\xi^+\right|}\left(p_{1,d}(e^{i(-\tau_0+\tau_1)},\dots,e^{i(-\tau_{d-1}+\tau_d)})\right)^{\lambda\left|\xi^+\right|} \\
& \times e^{-i\delta\lambda\left|\xi^-\right|}\left(p_{1,d}(e^{-i(-\tau_0+\tau_1)},\dots,e^{-i(-\tau_{d-1}+\tau_d)})\right)^{\lambda\left|\xi^-\right|} \\
& \times \left|p_{1,d}(e^{i(-\tau_0+\tau_1)},\dots,e^{i(-\tau_{d-1}+\tau_d)})\right|^{2n} \mathrm{d}\delta \mathrm{d}\tau_1\cdots\mathrm{d}\tau_{d-1}.
\end{aligned}$$

Yet

$$\begin{aligned}
e^{-i\lambda\xi^\mathsf{T}\mathbf{f}_d^{-1}(\theta)} &= e^{-i\lambda(\xi_1(-\tau_0+\tau_1+\delta)+\cdots+\xi_d(-\tau_{d-1}+\tau_d+\delta))} \\
&= e^{-i\lambda(\xi_1(-\tau_0+\tau_1)+\cdots+\xi_d(-\tau_{d-1}+\tau_d))} e^{-i\lambda\delta(\xi_1+\cdots+\xi_d)}
\end{aligned}$$

and

$$e^{i\delta\lambda\left|\xi^+\right|}e^{-i\delta\lambda\left|\xi^-\right|} = e^{i\delta\lambda(\xi_1+\cdots+\xi_d)}$$

so that every instance of δ in the integrand cancels out, thus yielding

$$
\begin{aligned}
w_{(n,\lambda\xi)} = \frac{d}{(2\pi)^d} \int\limits_{\mathbf{f}_d^{-1}([-\pi,\pi]^d)} & e^{-i\lambda(\xi_1(-\tau_0+\tau_1)+\cdots+\xi_d(-\tau_{d-1}+\tau_d))} \\
& \times \left(p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})\right)^{\lambda|\xi^+|} \\
& \times \left(p_{1,d}(e^{-i(-\tau_0+\tau_1)},\ldots,e^{-i(-\tau_{d-1}+\tau_d)})\right)^{\lambda|\xi^-|} \\
& \times \left|p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})\right|^{2n} \mathrm{d}\delta \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_{d-1}.
\end{aligned}
$$

Integrate with respect to δ first to get

$$
\begin{aligned}
w_{(n,\lambda\xi)} &= \frac{d}{(2\pi)^d} \int\limits_{\mathcal{N}} (2\pi + m(\tau) - M(\tau)) e^{-i\lambda(\xi_1(-\tau_0+\tau_1)+\cdots+\xi_d(-\tau_{d-1}+\tau_d))} \\
&\quad \times \left(p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})\right)^{\lambda|\xi^+|} \\
&\quad \times \left(p_{1,d}(e^{-i(-\tau_0+\tau_1)},\ldots,e^{-i(-\tau_{d-1}+\tau_d)})\right)^{\lambda|\xi^-|} \\
&\quad \times \left|p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})\right|^{2n} \mathrm{d}\tau \\
&= \frac{d}{(2\pi)^d} \int\limits_{\mathcal{N}} A(\tau) e^{-\lambda\varphi(\tau)} \, \mathrm{d}\tau
\end{aligned}
\tag{4.4.10}
$$

where $\tau = (\tau_1, \ldots, \tau_{d-1})$, $d\tau = d\tau_1 \wedge \cdots \wedge d\tau_{d-1}$, $\mathcal{N} = \mathbf{f}_d^{-1}(\mathrm{span}_{\mathbb{R}}(A_{d-1}) \cap [-\pi,\pi]^d)$,

$$
\begin{aligned}
m(\tau) &= \min\{-\tau_0+\tau_1, \ldots, -\tau_{d-1}+\tau_d\}, \quad M(\tau) = \max\{-\tau_0+\tau_1, \ldots, -\tau_{d-1}+\tau_d\}, \\
A(\tau) &= (2\pi + m(\tau) - M(\tau)) \left|p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})\right|^{2n} \\
\varphi(\tau) &= -\left|\xi^+\right| \log(E(\tau)) - \left|\xi^-\right| \log(\overline{E(\tau)}) + i(\xi_1(-\tau_0+\tau_1) + \cdots + \xi_d(-\tau_{d-1}+\tau_d))
\end{aligned}
$$

and

$$
E(\tau) = p_{1,d}(e^{i(-\tau_0+\tau_1)},\ldots,e^{i(-\tau_{d-1}+\tau_d)})
$$

where the principal branch of the logarithm is taken.

By the triangle inequality,

$$\left|p_{1,d}(e^{i(-\tau_0+\tau_1)}, \ldots, e^{i(-\tau_{d-1}+\tau_d)})\right| \leq p_{1,d}(\left|e^{i(-\tau_0+\tau_1)}\right|, \ldots, \left|e^{i(-\tau_{d-1}+\tau_d)}\right|)$$
$$= p_{1,d}(1, \ldots, 1) = d,$$

and equality occurs if and only if

$$e^{i(-\tau_0+\tau_1)} = \cdots = e^{i(-\tau_{d-1}+\tau_d)}.$$

Referring back to our change of variables, this means that all of the $e^{i\theta_j}$ are equal to each other. But since our region of integration in (4.4.10) demands that we have $\theta \in \operatorname{span}_{\mathbb{R}}(A_{d-1}) \cap [-\pi, \pi]^d$, $\theta_j = 0$ for all j. Hence

$$-\tau_0 + \tau_1 = \cdots = -\tau_{d-1} + \tau_d = 0$$

so that each $\tau_j = 0$ in order for $\left|p_{1,d}(e^{i\tau_1}, \ldots, e^{-i\tau_{d-1}})\right| = d$. The same is true for $p_{1,d}(e^{-i\tau_1}, \ldots, e^{i\tau_{d-1}}) = \overline{p_{1,d}(e^{i\tau_1}, \ldots, e^{-i\tau_{d-1}})}$. Therefore

$$\Re\varphi(\tau) = \left|\xi^+\right| \log \left|p_{1,d}(e^{i(-\tau_0+\tau_1)}, \ldots, e^{i(-\tau_{d-1}+\tau_d)})\right|^{-1}$$
$$+ \left|\xi^-\right| \log \left|p_{1,d}(e^{-i(-\tau_0+\tau_1)}, \ldots, e^{-i(-\tau_{d-1}+\tau_d)})\right|^{-1}$$
$$\geq \left|\xi^+\right| \log d^{-1} + \left|\xi^-\right| \log d^{-1} = -\|\xi\|_1 \log d$$

for every $\tau \in \mathcal{N}$ for which $p_{1,d}(e^{i(-\tau_0+\tau_1)}, \ldots, e^{i(-\tau_{d-1}+\tau_d)}) \neq 0$ with equality occurring precisely when $\tau = \mathbf{0}$. Also, since plugging in $\tau = \mathbf{0}$ begets $p_{1,d}(1, \ldots, 1)) \neq 0$, the phase function φ is analytic on every compact neighborhood of $\mathbf{0}$ contained in $\mathcal{N}$ for which $p_{1,d}(e^{i(-\tau_0+\tau_1)}, \ldots, e^{i(-\tau_{d-1}+\tau_d)}) \neq 0$. (Two things are worth pointing out here. First, $\mathcal{N}$, as the inverse image of a compact set under a continuous map, is itself compact, and since $p_{1,d}$ is continuous, such nonempty compact neighborhoods exist. Second, points τ for which $p_{1,d}$ happens to equal 0 contribute nothing to the integral.) Provided that $H_\varphi(\mathbf{0})$ is invertible, we can apply

Corollary 4.4.4 to

$$w_{(n,\lambda\xi)} = \frac{d^{\lambda\|\xi\|_1+1}}{(2\pi)^d} \int_{\mathcal{N}} A(\tau) e^{-\lambda(\varphi(\tau)+\|\xi\|_1 \log d)} \, \mathrm{d}\tau. \tag{4.4.11}$$

All that remains is to calculate H_φ. We find that, for $1 \le j \le d-1$,

$$\varphi_{\tau_j} = \frac{-\left|\xi^+\right| \left(ie^{i(\tau_j-\tau_{j-1})} - ie^{i(\tau_{j+1}-\tau_j)}\right)}{E(\tau)} - \frac{\left|\xi^-\right| \left(ie^{i(\tau_j-\tau_{j+1})} - ie^{i(\tau_{j-1}-\tau_j)}\right)}{\overline{E(\tau)}} + i(\xi_j - \xi_{j+1}),$$

and so for all j, k with $1 \le j \le k \le d-1$ and $k - j > 1$

$$\begin{aligned}
\varphi_{\tau_j\tau_j} &= -\left|\xi^+\right| \frac{\left(-e^{i(\tau_j-\tau_{j-1})} - e^{i(\tau_{j+1}-\tau_j)}\right)E(\tau) - \left(ie^{i(\tau_j-\tau_{j-1})} - ie^{i(\tau_{j+1}-\tau_j)}\right)^2}{[E(\tau)]^2} \\
&\quad - \left|\xi^-\right| \frac{\left(-e^{i(\tau_j-\tau_{j+1})} - e^{i(\tau_{j-1}-\tau_j)}\right)\overline{E(\tau)} - \left(ie^{i(\tau_j-\tau_{j+1})} - ie^{i(\tau_{j-1}-\tau_j)}\right)^2}{\left[\overline{E(\tau)}\right]^2}, \\
\varphi_{\tau_{j+1}\tau_j} &= -\left|\xi^+\right| \frac{e^{i(\tau_{j+1}-\tau_j)}E(\tau) - \left(ie^{i(\tau_j-\tau_{j-1})} - ie^{i(\tau_{j+1}-\tau_j)}\right)\left(ie^{i(\tau_{j+1}-\tau_j)} - ie^{i(\tau_{j+2}-\tau_{j+1})}\right)}{[E(\tau)]^2} \\
&\quad - \left|\xi^-\right| \frac{e^{i(\tau_j-\tau_{j+1})}\overline{E(\tau)} - \left(ie^{i(\tau_j-\tau_{j+1})} - ie^{i(\tau_{j-1}-\tau_j)}\right)\left(ie^{i(\tau_{j+1}-\tau_{j+2})} - ie^{i(\tau_j-\tau_{j+1})}\right)}{\left[\overline{E(\tau)}\right]^2}, \\
\varphi_{\tau_k\tau_j} &= \left|\xi^+\right| \frac{\left(ie^{i(\tau_j-\tau_{j-1})} - ie^{i(\tau_{j+1}-\tau_j)}\right)\left(ie^{i(\tau_k-\tau_{k-1})} - ie^{i(\tau_{k+1}-\tau_k)}\right)}{[E(\tau)]^2} \\
&\quad + \left|\xi^-\right| \frac{\left(ie^{i(\tau_j-\tau_{j+1})} - ie^{i(\tau_{j-1}-\tau_j)}\right)\left(ie^{i(\tau_k-\tau_{k+1})} - ie^{i(\tau_{k-1}-\tau_k)}\right)}{\left[\overline{E(\tau)}\right]^2}.
\end{aligned}$$

The remaining mixed partial derivatives follow from Clairaut's theorem. Thus

$$\begin{aligned}
\varphi_{\tau_j\tau_j}(\mathbf{0}) &= -\left|\xi^+\right| \frac{-2d}{d^2} - \left|\xi^-\right| \frac{-2d}{d^2} = \frac{2\|\xi\|_1}{d} \\
\varphi_{\tau_j\tau_{j+1}}(\mathbf{0}) = \varphi_{\tau_{j+1}\tau_j}(\mathbf{0}) &= -\left|\xi^+\right| \frac{d}{d^2} - \left|\xi^-\right| \frac{d}{d^2} = -\frac{\|\xi\|_1}{d} \\
\varphi_{\tau_j\tau_k}(\mathbf{0}) = \varphi_{\tau_k\tau_j}(\mathbf{0}) &= 0,
\end{aligned}$$

and so

$$H_\varphi(\mathbf{0}) = -\frac{\|\xi\|_1}{d}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix},$$

which is a $d \times d$ symmetric tridiagonal Toeplitz matrix. Using a formula from [Hu and O'Connell 1996]

$$\begin{aligned} \det(H_\varphi(\mathbf{0})) &= \left(-\frac{\|\xi\|_1}{d}\right)^d \lim_{D \to -2^-} \frac{(-1)^d \sinh\big((d+1)\cosh^{-1}(-D/2)\big)}{\sinh\big(\cosh^{-1}(-D/2)\big)} \\ &= \left(\frac{\|\xi\|_1}{d}\right)^d (d+1). \end{aligned}$$

By Proposition 2.1 of [Kulkarni et al. 1999], the eigenvalues of $\det H_\varphi(\mathbf{0})$ are

$$-\frac{\|\xi\|_1}{d}\left(-2 - 2\cos\frac{k\pi}{d+1}\right) = \frac{4\|\xi\|_1}{d}\cos^2\left(\frac{k\pi}{2(d+1)}\right), \quad \text{for } k = 1, 2, \ldots, d,$$

and so the product of the principal square roots of the eigenvalues of $H_\varphi(\mathbf{0})$ is a nonnegative real number. We thus take the positive square root of $H_\varphi(\mathbf{0})$ and apply Corollary 4.4.4 to get that $w_{(n,\lambda\xi)}$ has an asymptotic series expansion of the form

$$w_{(n,\lambda\xi)} = \frac{d^{\lambda\|\xi\|_1+1}}{(2\pi)^d} c_0 \lambda^{-d/2}(1 + o(1)) \tag{4.4.12}$$

where

$$c_0 = 2\pi d^{2n} \frac{(2\pi)^{-d/2}}{\sqrt{(\|\xi\|_1/d)^d(d+1)}} = \frac{(2\pi)^{1-d/2}}{\sqrt{\|\xi\|_1^d(d+1)}} d^{2n+d/2}, \tag{4.4.13}$$

which is the desired result. □

The Association Between $w_{(n,\xi)}$ and Modified Bessel Functions of the First Kind

The cardinalities $w_{(n,\xi)}$ can be linked to products of modified Bessel functions of the first kind, which are functions given by the Taylor series

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad \text{for } \nu \in \mathbb{C} - \mathbb{Z}_{<0}. \tag{4.4.14}$$

It is customary to keep to the principal branch of $(z/2)^\nu$ so that I_ν is analytic on $\mathbb{C} - (-\infty, 0]$ and two-valued and discontinuous on the cut $\mathrm{Arg} z = \pm\pi$ (if $\nu \notin \mathbb{N}_0$). To facilitate the asymptotic analysis of $w_{(n,(m,\ldots,m))}$, we will instead work with the normalized Bessel function

$$\tilde{I}_\nu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n}, \quad \text{for } \nu \in \mathbb{C} - \mathbb{Z}_{<0}, \tag{4.4.15}$$

which is entire on $\mathbb{C}$.

Set $\mu = (m, \ldots, m) = m \cdot \mathbf{1}_d$, where $\mathbf{1}_d = \underbrace{(1, \ldots, 1)}_{d}$. Observe that

$$w_{(n,\xi)} = (n + \left|\xi^+\right|)!(n + \left|\xi^-\right|)![x^n] \prod_{j=1}^{d} \left(\frac{\tilde{I}_{|\xi_j|}(2\sqrt{x})}{|\xi_j|!} \right). \tag{4.4.16}$$

In particular,

$$w_{(n,\mu)} = \frac{n!(n + d\,|m|)!}{(|m|!)^d} [x^n] \left(\tilde{I}_{|m|}(2\sqrt{x})\right)^d. \tag{4.4.17}$$

Bender, Brody, and Meister proved in [Bender et al. 2003] that

$$\left(\tilde{I}_\nu(z)\right)^d = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)}{n!\Gamma(n+\nu+1)} B_n^{(\nu)}(d) \left(\frac{z}{2}\right)^{2n}, \tag{4.4.18}$$

where $B_n^{(\nu)}(d)$ is a polynomial defined recursively by

$$B_n^{(\nu)}(d) = d\frac{\nu+n}{\nu+1} B_{n-1}^{(\nu)}(d) + \sum_{j=1}^{n} \frac{b_j(\nu)}{n} \frac{\Gamma(\nu+2)}{\Gamma(j+\nu+2)} \binom{\nu+n}{j} B_{n-1}^{(\nu)}(d). \tag{4.4.19}$$

with initial condition $b_0^{(\nu)}(d) = 1$. The sequence $\{b_n(\nu)\}_{n=1}^{\infty}$ is identified by the generating function

$$\sum_{n=1}^{\infty} \frac{b_n(\nu)}{(n-1)!\Gamma(n+\nu+1)} x^n = \frac{x}{\Gamma(\nu+2)} \left(\frac{\sqrt{x}}{\nu+1} \frac{I_\nu(2\sqrt{x})}{I_{\nu+1}(2\sqrt{x})} - 2 \right). \tag{4.4.20}$$

However, in [Moll and Vignat 2014], Moll and Vignat found an alternative characterization of $B_n^{(\nu)}(d)$ in terms of the exponent d. To start, they employ the Hadamard factorization

$$\tilde{I}_\nu(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{j_{\nu,k}^2} \right), \tag{4.4.21}$$

where $\{j_{\nu,k}\}_{k=1}^{\infty}$ is an enumeration of the zeros of $\tilde{I}_\nu(iz)$, from which it follows that

$$\log \tilde{I}_\nu(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta_n(2n) z^{2n}, \tag{4.4.22}$$

where ζ_ν is the Bessel zeta function

$$\zeta_\nu(p) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^p}, \quad \text{for } p > 1. \tag{4.4.23}$$

They then cite the fact that the exponential of a power series is computed via

$$\exp\left[\sum_{n=1}^{\infty} a_n \frac{z^n}{n!} \right] = \sum_{n=0}^{\infty} \mathbf{B}_n(a_1, \ldots, a_n) \frac{z^n}{n!}, \tag{4.4.24}$$

where $\mathbf{B}_n(a_1, \ldots, a_n)$ is the n^{th} complete Bell polynomial. (Read [Riordan 1968], section 5.2, for details.) Multiplying (4.4.22) by d and then plugging it into (4.4.24) leads to the following theorem.

Theorem 4.4.6 (Moll, Vignat)**.** *Define*

$$a_n = a_n^{(\nu)}(d) = (-1)^{n-1}(n-1)!\zeta_\nu(2n)d. \tag{4.4.25}$$

Then $B_n^{(\nu)}(d)$ *is given by*

$$B_n^{(\nu)}(d) = 4^n \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)} \mathbf{B}_n(a_1^{(\nu)}(d), \ldots, a_n^{(\nu)}(d)). \tag{4.4.26}$$

We are now ready to derive the following asymptotic formula.

Theorem 4.4.7. *Given* $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}\backslash\{0\}$ *fixed,*

$$w_{(n,\mu)} = \frac{\sqrt{2\pi\,|m|\,d}}{(|m|!)^d} \left(\frac{|m|\,d^2}{e(|m|+1)}\right)^n \left(\frac{n+|m|\,d}{e}\right)^{|m|d} + O(d^{n-1}) \tag{4.4.27}$$

as $d \to \infty$ *with* n *and* $m \neq 0$ *fixed. If* $m = 0$, *then*

$$w_{(n,\mathbf{0})} = n!d^n - O(d^{n-1}). \tag{4.4.28}$$

In this theorem and the subsequent proof, we restrict $O(d^{n-1})$ to be strictly positive.

Proof. Since $\mathbf{B}_n(a_1^{|m|}(d), \ldots, a_n^{|m|}(d))$ is a polynomial in d, it suffices to find its degree and leading coefficient. This is achieved by invoking the Bell polynomial's determinantal representation

$$\mathbf{B}_n(a_1, \ldots, a_n) = \det \begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & -1 & 0 & \cdots & 0 \\ a_3 & \binom{2}{1}a_2 & a_1 & -1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \binom{n-2}{1}a_{n-2} & \binom{n-2}{2}a_{n-3} & \binom{n-2}{3}a_{n-4} & \ddots & -1 \\ a_n & \binom{n-1}{1}a_{n-1} & \binom{n-1}{2}a_{n-2} & \binom{n-1}{3}a_{n-3} & \cdots & a_1 \end{bmatrix}. \tag{4.4.29}$$

(This determinantal representation is explained in [Collins 2001], albeit with a typographical error.) Each a_i includes exactly one factor of d, and so, because the above matrix is lower triangular save for the superdiagonal comprised of -1s, $\mathbf{B}_n(a_1^{|m|}(d), \ldots, a_n^{|m|}(d))$ is an n^{th} degree polynomial in d with leading coefficient given by the product of the diagonal entries of (4.4.29), which is

$$\left(a_1^{|m|}(d)\right)^n = (d\zeta_{|m|}(2))^n = \left(\frac{d}{4(|m|+1)}\right)^n. \tag{4.4.30}$$

(Here we have used the fact that $\zeta_\nu(2) = \frac{1}{4(\nu+1)}$ which is elaborated on page 502 of [Watson 1995].) Hence

$$w_{(n,\mu)} = \frac{4^n(n+d\,|m|)!}{(|m|!)^d}\left(\left(\frac{d}{4(|m|+1)}\right)^n + O(d^{n-1})\right). \tag{4.4.31}$$

Applying Stirling's approximation on $(n+d\,|m|)!$ completes the proof. □

Formula (4.4.31) suggests that $w_{(n,\mu)}$ is roughly equal to

$$\binom{n+d\,|m|}{|m|\,,\ldots,|m|\,,n}\left(\frac{d}{|m|+1}\right)^n n!$$

when d is substantially large. In the case that $m=0$, we get $d^n n!$, which is the number of ways to construct an n-string w and a permutation w' of w provided that every character in w always differ. This, of course, is untrue, and so $d^n n!$ is a bit of an overestimate for the number of abelian squares of length $2n$. However, if one chooses a character from Σ_d uniform randomly for each letter of w, then the probability that w has no repeated characters is

$$\frac{d(d-1)\cdots(d-n+1)}{d^n} = 1 - O\left(\frac{1}{d^{n-1}}\right).$$

The characters of w are therefore asymptotically almost surely different, and so $d^n n!$ is a satisfactory estimate for $w_{(n,\mathbf{0})}$.

4.5 Discussion and Future Work

A Combinatorial Analogue of Parseval's Equation

Observe that the proof of Theorem 4.2.8 relies only on the quadratic integrability of $\mathcal{S}_{p[x]^*_{1,d}}$ and nothing else. Parseval's equation and Plancherel's theorem handle the rest. The punchline to Theorem 4.2.8 holds generally in any compact abelian group, as we explain next. Details behind the vocabulary and analysis involved herein can be found in [Folland 1995].

Lemma 4.5.1. *Let G be a compact abelian group with Haar measure μ and let $\widehat{G}$ and $\widehat{\mu}$ be the Pontrjagin duals of G and μ, respectively. Let $\mathcal{F}$ be the Fourier transform on $L^2_\mu(G)$, let $\mathfrak{B}$ be an orthonormal basis of $L^2_{\widehat{\mu}}(\widehat{G})$, and let $\mathfrak{G}$ be a family of power series in $\mathbb{C}[[x]]$ that share a common radius of convergence $R > 0$. If there is a parametrized $L^2_\mu(G)$ function $f(\cdot; z)$, $z \in \mathbb{C}$, such that $\mathcal{F}f(\cdot\,; z)$ is a bijection from $\mathfrak{B}$ onto $\mathfrak{G}$ for $|z| < R$, then*

$$\sum_{g \in \mathfrak{G}} |g(z)|^2 = \|\mathcal{F}f(\cdot\,; z)\|_2^2. \tag{4.5.1}$$

Furthermore, if $h(\cdot\,; z) \in L^2_\mu(G)$ and $\mathcal{F}h(\cdot\,; z)|_{\mathfrak{B}} = \mathcal{F}f(\cdot\,; z)|_{\mathfrak{B}}$ for $|z| < R$, then $h(\cdot\,; z) = f(\cdot\,; z)$.

Lemma 4.5.1 is an immediate consequence of the fact that $\mathcal{F}$ is a unitary isometric isomorphism from $L^2_\mu(G)$ onto $L^2_{\widehat{\mu}}(\widehat{G})$. Although G need only be a locally compact abelian group to possess a Fourier transform, compactness means that the Peter-Weyl theorem holds. This equips $L^2_{\widehat{\mu}}(\widehat{G})$ with the inner product necessary to even discuss an orthonormal basis $\mathfrak{B}$. If the binary operation on G is weakened to be noncommutative, then $\mathcal{F}$ takes values as Hilbert space operators, which strikes us as unhelpful; we are dealing with just generating functions after all. Lemma 4.5.1 also suggests that such a family $\mathfrak{G}$ can be "canonically" indexed: simply set $g_{\mathbf{e}}(z) := \mathcal{F}f(\cdot\,; z)(\mathbf{e})$ for each $\mathbf{e} \in \mathfrak{B}$.

One may wonder whether there is any combinatorial significance to (4.5.1). There is, but only under careful circumstances. If $\mathfrak{G}$ is a family of generating functions for unlabeled combinatorial classes, then the L^2 norm on $\widehat{G}$ inherits enumerative meaning, which we illustrate next. Here we incorporate the nomenclature of Theorem I.1 in [Flajolet and Sedgewick 2009].

Theorem 4.5.2. *Let $\mathfrak{A}$ be a family of unlabeled combinatorial classes and let $\mathfrak{G}$ be the corresponding family of ordinary generating functions. If there is a compact abelian group G and a parametrized $L^2(G)$ function $f(\cdot; z), z \in \mathbb{C}$, satisfying the conditions of Lemma (4.5.1), then, for $x \in \mathbb{R}$, $\|\mathcal{F}f(\cdot\,; x)\|_2^2$ is the ordinary generating function for the disjoint union $\bigsqcup_{\mathcal{A} \in \mathfrak{A}} \mathcal{A} \times \mathcal{A}$.*

Proof. If $g(x)$ is the OGF of an unlabeled class $\mathcal{A}$, then $g^2(x)$ is the OGF of $\mathcal{A} \times \mathcal{A}$. Since the sum in (4.5.1) converges for $|x| < R$ and the disjoint union of combinatorial classes translates as the sum of their generating functions, we see that $\sum\limits_{g \in \mathfrak{G}} g^2(x)$ is the OGF for $\bigsqcup\limits_{\mathcal{A} \in \mathfrak{A}} \mathcal{A} \times \mathcal{A}$. $\square$

Example 4.5.3. If $\mathfrak{A} = \{\mathcal{W}_\xi : \xi \in \mathbb{Z}^2\}$, $\mathfrak{G} = \{g_\xi : g_\xi(x) = x^{\frac{1}{2}\|\xi\|_1} W_\xi(x), \xi \in \mathbb{Z}^2\}$, and $G = \mathbb{T}^2$, we recover the context for offset words over a two-letter alphabet. As shown in the proof of Theorem 4.2.6, $\mathcal{F}\mathcal{S}_{p[\sqrt{x}]^*_{1,d}}(\xi) = x^{\frac{1}{2}\|\xi\|_1} W_\xi(x)$ for all $\xi \in \mathbb{Z}^2$, and so, by Theorem 4.5.2, $\|\mathcal{S}_{p[\sqrt{x}]^*_{1,d}(\xi)}\|_2^2$ is the OGF for $\bigsqcup\limits_{\xi \in \mathbb{Z}^2} \mathcal{W} \times \mathcal{W}$. Table 1 displays the correspondence between the first couple Taylor coefficients of $\|\mathcal{S}_{p[\sqrt{x}]^*_{1,2}}\|_2^2$ and ordered pairs of strings in $\bigsqcup\limits_{\xi \in \mathbb{Z}^2} \mathcal{W}_\xi \times \mathcal{W}_\xi$.

In light of Theorem 4.5.2, it is compelling to regard

$$\frac{1}{(2\pi)^d} \int\limits_{[0,2\pi]^d} \frac{1}{|1 - xp_{1,d}(e^{i\theta_1}, \ldots, e^{i\theta_d})|^4} \, \mathrm{d}\theta$$

as the ordinary generating function for the number of strings in Σ_d^* that can be written as the concatenation of two words offset by the same integer vector, counted according to the length of the string, by associating an ordered pair of strings (w, w') with ww'. This interpretation is flawed however. For instance, every abelian square w is the concatenation of two abelian squares in at least two ways, namely εw and $w\varepsilon$, which are counted as different in the preceding framework. (Table 1 even concretely shows that copies of the same ordered pair can emerge in the disjoint union.) Another challenge emanates from the fact that the various classes of offset words are neither counted according to the length of the word nor are they pairwise disjoint as shown in Proposition 4.2.4. We do believe, however, that the above integral still enumerates something involving string concatenations in spite of the aforementioned shortcomings.

Theorem 4.5.2 holds more generally in any inner product space. Indeed, if there is a complex inner product space $\mathcal{H}$, an orthonormal basis $\mathfrak{B}$ of $\mathcal{H}$, and a parametrized $\mathcal{H}$-vector $\mathbf{v}_z, z \in \mathbb{C}$, such that $\langle \mathbf{v}_z, \cdot \rangle$ is a bijection from $\mathfrak{B}$ onto a family $\mathfrak{G}$ of power series in $\mathbb{C}[[x]]$ sharing a common radius of convergence $R > 0$, then

$$\|\mathbf{v}_z\|^2 = \sum_{g \in \mathfrak{G}} |g(z)|^2$$

due to the Pythagorean theorem. We prefer the backdrop of compact abelian groups, however, for two

Table 4.1: Connection between $\|\mathcal{S}_{p[\sqrt{x}]^*_{1,2}}\|_2^2$ and $\bigsqcup_{\xi\in\mathbb{Z}^2}\mathcal{W}_\xi\times\mathcal{W}_\xi$.

n	$[x^n]\|\mathcal{S}_{p[\sqrt{x}]^*_{1,2}}\|_2^2$	length n elements of $\bigsqcup_{\xi\in\mathbb{Z}^2}\mathcal{W}_\xi\times\mathcal{W}_\xi$
0	1	$(\varepsilon,\varepsilon)$
2	8	$(\varepsilon,11),(11,\varepsilon),(\varepsilon,22),(22,\varepsilon)$, $(1,1)\in\mathcal{W}^2_{(1,0)},(1,1)\in\mathcal{W}^2_{(-1,0)}$, $(2,2)\in\mathcal{W}^2_{(0,1)},(2,2)\in\mathcal{W}^2_{(0,-1)}$
4	54	$(\varepsilon,1111),(1111,\varepsilon),(\varepsilon,2222),(2222,\varepsilon)$, $(\varepsilon,1212),(1212,\varepsilon),(\varepsilon,1221),(1221,\varepsilon)$, $(\varepsilon,2112),(2112,\varepsilon),(\varepsilon,2121),(2121,\varepsilon)$, $(1,111)\in\mathcal{W}^2_{(1,0)},(111,1)\in\mathcal{W}^2_{(1,0)}$, $(1,111)\in\mathcal{W}^2_{(-1,0)},(111,1)\in\mathcal{W}^2_{(-1,0)}$, $(1,122),(122,1),(1,221),(221,1)$, $(1,212)\in\mathcal{W}^2_{(1,0)},(212,1)\in\mathcal{W}^2_{(1,0)}$, $(1,212)\in\mathcal{W}^2_{(-1,0)},(212,1)\in\mathcal{W}^2_{(-1,0)}$, $(2,222)\in\mathcal{W}^2_{(0,1)},(222,2)\in\mathcal{W}^2_{(0,1)}$, $(2,222)\in\mathcal{W}^2_{(0,-1)},(222,2)\in\mathcal{W}^2_{(0,-1)}$, $(2,112),(112,2),(2,211),(211,2)$, $(2,121)\in\mathcal{W}^2_{(0,1)},(121,2)\in\mathcal{W}^2_{(0,1)}$, $(2,121)\in\mathcal{W}^2_{(0,-1)},(121,2)\in\mathcal{W}^2_{(0,-1)}$, $(11,11)\in\mathcal{W}^2_{(0,0)},(11,11)\in\mathcal{W}^2_{(2,0)},(11,11)\in\mathcal{W}^2_{(-2,0)}$, $(22,22)\in\mathcal{W}^2_{(0,0)},(22,22)\in\mathcal{W}^2_{(0,2)},(22,22)\in\mathcal{W}^2_{(0,-2)}$, $(11,22),(22,11)$, $(12,12)\in\mathcal{W}^2_{(1,1)},(12,12)\in\mathcal{W}^2_{(1,-1)},(12,12)\in\mathcal{W}^2_{(-1,-1)}$, $(21,21)\in\mathcal{W}^2_{(1,1)},(21,21)\in\mathcal{W}^2_{(-1,1)},(21,21)\in\mathcal{W}^2_{(-1,-1)}$, $(12,21)\in\mathcal{W}^2_{(1,1)},(12,21)\in\mathcal{W}^2_{(-1,-1)}$, $(21,12)\in\mathcal{W}^2_{(1,1)},(21,12)\in\mathcal{W}^2_{(-1,-1)}$

reasons. First, the function $\mathcal{F}f(\cdot;z)$ in Lemma 4.5.1 is, in some sense, a "generating function of generating functions" and acquires both Fourier analytic and enumerative underpinnings. Secondly, we hope that by narrowing our attention to compact abelian groups, previously untapped tools from abstract harmonic analysis may help systematize both the singularity analysis of generating functions that happen to be images of some Fourier transform as well as the symbolic calculus of their corresponding combinatorial classes.

The Bernstein-Szegő Measure Moment Problem

The goal of this thesis is to pursue the question of whether the result from [Geronimo and Woerdeman 2004] can be further generalized to three or more variables. We also wonder if any parallels can be drawn between the Bernstein-Szegő measure moment problem and the problem of determining what conditions must be imposed on a family of unlabeled combinatorial classes and the corresponding family of generating functions to ensure the existence of a compact abelian group G and an $L^2(G)$ function satisfying the conditions of Lemma 4.5.1. Moreover, are there other combinatorial classes whose generating functions are Bernstein-Szegő measures? Perhaps by recognizing crucial facets of such generating functions, one may be able to reverse engineer a generalization of Geronimo and Woerdeman's solution to the two-variable Bernstein-Szegő measure moment problem.

Stabilized Symmetric Functions

It was a wonderful surprise to us that the Fourier coefficients of the stabilized power sum symmetric polynomials are generating functions for classes of offset words. This prompts us to speculate about similar stabilization procedures for symmetric functions as a whole. Can one suitably "stabilize" a generic symmetric function in such a way that the Fourier coefficients of its spectral density function are generating functions for a family of classes of words? Furthermore, would such classes unilaterally categorize all words in some manageable fashion?

We hope that there are still some profound interactions between the theories of symmetric functions and Bernstein-Szegő measures waiting to be unearthed. There is still much remaining to be understood about both areas.

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