A New Class of Integrable Surfaces Related to Bertrand Curves

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Dedications

To my family.

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S. D. G.

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Abstract

A New Class of Integrable Surfaces Related to Bertrand Curves

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In this dissertation we present a new class of integrable surfaces related to Bertrand curves. These surfaces are foliations of constant-torsion curves and are generated by a particular geometric flow on constant-torsion curves. We see that the orbit of this flow traces out Bertrand curves on the surface. The surfaces discussed interpolate two known integrable systems and we establish the connection. We also use tools from soliton theory to generate surface solutions using Bäcklund transformations.

Chapter 1

Introduction

One does not have to look long at the theory of differentiable curves to see that parameterization by arclength is fundamental. Curves are characterized by two functions; but arclength-parameterization is necessary. Therefore, arclength-parameterization is necessary for using invariants as a standard of comparison. While fundamental, the notion is a bit elusive. A survey of texts on the subject will show that a handful of familiar examples are repeated over and over: a line, circle, helix, etc.. Joel Langer writes: "Ironically, in elementary mathematics, arclength-parameterization is mostly an abstraction–one rarely encounters it in the flesh!" [Lan99] And so, in many ways, soliton theory has come to the rescue, filling the void. Two major topics in the intersection of geometry and soliton theory are curve evolution and Bäcklund transformations. We discuss curve evolution in the context of geometric evolution equations, i.e., evolution equations that can be written in terms of the invariants of the curve. The geometry is integrable inasmuch as it is governed by some exactly solvable partial differential equation. One can trace curve evolution back to the pivotal work of Hasimoto and the vortex filament flow [Has72]. Among subsequent developments, Langer and Perline discovered a related infinite hierarchy of integrable geometric flows [LP91]. Rogers and Schief studied evolution equations that preserved particular invariants and geodesics [SR99]. In the category of Bäcklund transformations, Bäcklund and Bianchi produced the classical example of a transposition of solutions to the the sine-Gordon equation. This result amounts to a transformation of Constant Negative Gaussian Curvature (pseudospherical) surfaces. More recently, Schief studied a generalization of this result with Razzaboni surfaces [Sch03]. In the context of curves, Calini and Ivey produced knots by using a Bäcklund transformation on constant-torsion curves [CI98].

Now, we summarize this thesis, chapter by chapter:

Chapter 2: After the introduction, we establish the background of the differential geometry of curves and surfaces. We highlight the invariants of each of these two objects which allow us to characterize the geometry. Curves are characterized by two invariants, curvature and torsion. Surfaces are characterized by the 1st and 2nd fundamental forms. In addition, we look at the frame equations of both curves and surfaces. These are the equations that will allows us to connect the geometry to special PDE systems.

Chapter 3: Next, we look at the classical example of Constant Negative Gaussian Curvature (CNGC) surfaces. We show the connection between these surfaces and the famous sine-Gordon equation. CNGC surfaces provide an example of many tools of soliton theory. We present the Lax pair formulation of the sine-Gordon equation, the use of the Sym-Tafel formula to generate surfaces, the classical Bäcklund transformation on CNGC surfaces, and the dynamical version of CNGC surfaces by way of a geometric flow equation.

Chapter 4: Following CNGC surfaces, we see that the idea of a Bäcklund transformation takes on a wider meaning in the application of Razzaboni surfaces. Razzaboni surfaces, studied by Schief, contain special curves known as Bertrand curves as geodesics.

A Bäcklund transformation is presented for Bertrand curves as well as Razzaboni surfaces. From here, we derive a new flow on Bertrand curves and show that the orbital (or transverse) curves have constant torsion and are arclength-parameterized.

Chapter 5: From the flow on Bertrand curves found at the end of Chapter 4, we set down a geometric flow on arclength-parameterized constant-torsion curves with orbital curves that have the Bertrand property. We derive the governing PDE system and write its Lax pair formulation. Following this, we discuss the Darboux and Bäcklund transformations and generate example surfaces.

Appendix A: We provide additional formulas for our example surfaces in the appendix.

We emphasize that the material up until Section 4.4 is either classical or has appeared in the literature; subsequent material is new to this dissertation.

Chapter 2

Preliminaries

The concept of measurement is at the root of Geometry (as is clear from the word's etymology). Geometry is the study of shape, size, and space and, therefore, is concerned with the properties that are independent of position or coordinates. Geometers are concerned with the properties of objects that are invariant under a rigid body motion such as translation or rotation (or change of coordinates). Our goal in this section is to identify the invariants and characterizations of two geometric structures: curves and surfaces. We start by reviewing necessary preliminaries and outlining the basics from classical differential geometry. Additional details for material in 2.1 can be found in resources such as Struik [Str88], Lipschultz [Lip69], and do Carmo [dC76].

2.1 Curves

Many of the following definitions have extensions to a more general context but, for our purposes, we will be considering objects in Euclidean space $\mathbb{R}^3$. It should also be noted that, for us, *differentiability* (or *smoothness*) is essential so that particular quantities along the curve are defined. Indeed, much will be devoted to the differential equations that govern a curve and (as we will see later) the interaction of those equations with soliton theory. Thus, we begin with an appropriate definition:

Definition 1. *A* parameterized curve of class C^r *is a* C^r *map* $\gamma = \gamma(u) : I \to \mathbb{R}^3$ *where* I

is an open interval in $\mathbb{R}$. *We call* u *the* parameter *of the curve.*

We can think of a curve as the trace of the path of a point in space. Since we are concerned with the path (trace) of the curve, we can identify curves that maintain this path.

To identify the properties of the curve that are invariant under reparameterization, we need the following:

Definition 2. *A* C^r *curve* $\gamma : I \to \mathbb{R}^3$ *(*$r \geq 1$*) is* regular *if* $\gamma'(u) \neq 0$ *for all* $u \in I$.

Intuitively, it should be clear that the definition of length of a curve should be independent of reparameterization. Indeed, that is the case with the following definition:

Definition 3. *The* arclength *of a regular* C^r *curve* $\gamma : I \to \mathbb{R}^3$ *from the point* u_0 *is given by the formula*

$$s(u) = \int_{u_0}^{u} \|\gamma'(w)\| \, dw. \tag{2.1}$$

Definition 4. *For a regular* C^r *curve* γ*, if* $\|\gamma'\| \equiv 1$ *we say that* γ *is* arclength-parameterized *(or* parameterized by arclength*) and denote the parameter of* γ *by* s.

The arclength parameterization is sometimes referred to as the natural parameter.

Lemma 5. *Any regular curve can be parameterized by arclength.*

It must be noted that, while a regular curve *can* be parameterized by arclength, the explicit representation of that curve in terms of arclength depends on the invertibility of the function $s(u)$ given in (2.1) (which is, generally speaking, an intractable problem if one wants explicit formulas).

2.1.1 Invariants – Curvature and Torsion

Definition 6. *For an arclength-parameterized curve* $\gamma = \gamma(s)$*, we define the* tangent vector of $\gamma(s)$ *by* $\mathbf{T}(s) := \gamma'(s)$.

The tangent vector points in the direction of trajectory. By definition, the tangent vector has unit length.

Definition 7. *For an arclength-parameterized curve* γ*, we define the* curvature of γ *by*

$$\kappa(s) := |\mathbf{T}'(s)| = |\gamma''(s)|.$$

The scalar quantity κ measures how much the curve is pulling away from its tangent vector.

Example 8 (A line)**.** *A straight line can be written*

$$\mathbf{r}(u) = u\,\mathbf{a} + \mathbf{c}. \qquad (\mathbf{a}, \mathbf{c} \textit{ constant vectors})$$

The arclength is given by

$$s = \int_0^u ||\mathbf{a}||\, dw = u||\mathbf{a}||.$$

Thus, we can parameterize by arclength:

$$\mathbf{r}(s) = s\frac{\mathbf{a}}{||\mathbf{a}||} + \mathbf{c}$$

Now, the tangent vector $\mathbf{T}(s) = \frac{\mathbf{a}}{||\mathbf{a}||}$ *is a constant vector so the curvature is* $\kappa(s) =$

$||\mathbf{T}'(s)||= ||\mathbf{0}||= 0.$

Definition 9. *If* $\kappa \neq 0$*, we define the* normal vector of γ *by*

$$\mathbf{N}(s) := \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|}.$$

Because

$$(|\mathbf{T}(s)|^2)' = (\mathbf{T}(s) \cdot \mathbf{T}(s))' = 2\mathbf{T}(s) \cdot \mathbf{T}'(s)$$

and

$$(|\mathbf{T}(s)|^2)' = (1)' = 0,$$

we have $\mathbf{T}(s) \cdot \mathbf{T}'(s) = 0$, so $\mathbf{N}(s)$ is normal to $\mathbf{T}(s)$.

Definition 10. *We define the* binormal vector of γ *by* $\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$.

Computing, $\mathbf{B}'(s) = \mathbf{T}'(s) \times \mathbf{N}(s) + \mathbf{T}(s) \times \mathbf{N}'(s) = \mathbf{T}(s) \times \mathbf{N}'(s)$, we see that $\mathbf{B}'(s)$ is parallel to $\mathbf{N}(s)$. Thus,

Definition 11. *We define the* torsion of γ *by* $\tau(s) := -\mathbf{B}'(s) \cdot \mathbf{N}(s)$.

Theorem 12. *The vectors* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *form an orthonormal frame.*

Definition 13. *We call* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *the* Frenet frame.

Theorem 14. *The vectors* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *satisfy the Frenet-Serret equations,*

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Example 15 (Helix)**.** *An example of a helix can be parameterized as*

$$\mathbf{r}(u) = [a \cos u, a \sin u, bu]\,. \tag{2.2}$$

Computing the arclength, we find

$$s = \int_0^u \sqrt{a^2 + b^2} dw = \sqrt{a^2 + b^2} u$$

so that we can reparameterize (2.2) *by arclength:*

$$\mathbf{r}(s) = \left[a \cos\left(\frac{s}{\sqrt{a^2 + b^2}} \right), a \sin\left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} \right].$$

Now, we compute

$$\mathbf{T} = \left[-\frac{a}{\sqrt{a^2 + b^2}} \sin\left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{a}{\sqrt{a^2 + b^2}} \cos\left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{b}{\sqrt{a^2 + b^2}} \right], \tag{2.3}$$

$$\kappa \mathbf{N} = \left[-\frac{a}{a^2 + b^2} \cos\left(\frac{s}{\sqrt{a^2 + b^2}} \right), -\frac{a}{a^2 + b^2} \sin\left(\frac{s}{\sqrt{a^2 + b^2}} \right), 0 \right] \tag{2.4}$$

so that the curvature is

$$\kappa = \|\kappa \mathbf{N}\| = \frac{a}{a^2+b^2} \tag{2.5}$$

and the normal vector is

$$\mathbf{N} = \left[-\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0 \right]. \tag{2.6}$$

Now we complete the Frenet frame and find the torsion:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left[\frac{b}{\sqrt{a^2+b^2}} \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\frac{b}{\sqrt{a^2+b^2}} \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}} \right] \tag{2.7}$$

$$\mathbf{B}_s = \left[\frac{b}{a^2+b^2} \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{a^2+b^2} \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0 \right] \tag{2.8}$$

$$\tau = -\mathbf{B}_s \cdot \mathbf{N} = \frac{b}{a^2+b^2}. \tag{2.9}$$

In conclusion, we find that κ and τ are both constant. Note that in the special case $b = 0$, we obtain a circle. We plot an example with $a = 3$ and $b = 4$ in Figure 2.1.

We make note of an important result regarding torsion in the following theorem:

Theorem 16. *A curve is planar if and only if $\tau \equiv 0$.*

Example 17 (Hasimoto filament)**.** *Here we consider a curve with constant torsion τ.*

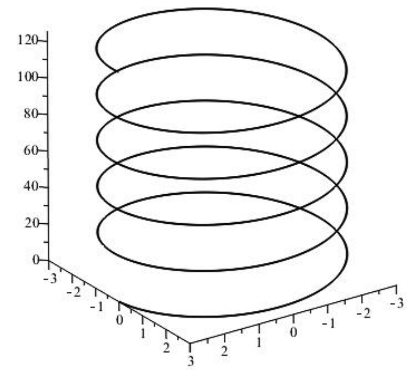


Figure 2.1: Helix

The following is the vortex filament from Hasimoto:

$$\mathbf{r}(s) = \left[s - \frac{2\mu}{\nu} \tanh \eta, r \cos \theta, r \sin \theta \right],$$

where

$$\mu = \frac{\nu^2}{\nu^2 + \tau^2}, \qquad r = \frac{2\mu}{\nu} \operatorname{sech} \eta,$$

$$\eta = \nu(s - 2\tau c), \qquad \theta = \tau s + (\nu^2 - \tau^2)c. \qquad (\nu, c \text{ constant})$$

Then we have

$$\mathbf{T} = \begin{bmatrix} 1 - 2\mu \operatorname{sech} \eta \\ \\ -2\mu \cos\theta \sinh \eta - \frac{2\mu}{\nu} \tau \sin\theta \cosh \eta \\ \\ -r\nu \sin\theta \tanh \eta + r\tau \cos\theta \end{bmatrix}^T$$

$$\mathbf{N} = \begin{bmatrix} 2\mu \tanh\eta \\ \left(\dfrac{\nu^2-\tau^2}{\nu^2+\tau^2} - \dfrac{\nu^2}{2\mu} r^2\right)\cos\theta + \dfrac{2\mu}{\nu}\tau\sin\theta\tanh\eta \\ \left(\dfrac{\nu^2-\tau^2}{\nu^2+\tau^2} - \dfrac{\nu^2}{2\mu} r^2\right)\sin\theta - \dfrac{2\mu}{\nu}\tau\cos\theta\tanh\eta \end{bmatrix}^T$$

$$\mathbf{B} = \begin{bmatrix} \tau r \\ -\dfrac{\nu^2-\tau^2}{\nu^2}\mu\sin\theta + \dfrac{2\mu}{\nu}\tau\cos\theta\tanh\eta \\ \dfrac{\nu^2-\tau^2}{\nu^2}\mu\cos\theta + \dfrac{2\mu}{\nu}\tau\sin\theta\tanh\eta \end{bmatrix}^T$$

with

$$\kappa = 2\nu\,\mathrm{sech}\,\eta.$$

and constant torsion τ*. We plot the example where* $c = 0, \nu = 1,$ *and* $\tau = 1$ *in Figure 2.2.*

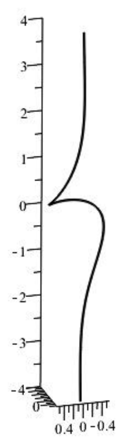


Figure 2.2: Example of the Hasimoto filament

Finally, we present the theorem that allows us to characterize a curve by its invariants κ and τ:

Theorem 18 (The Fundamental Theorem of Curves)**.** *If two continuous functions $\kappa(s)$ and $\tau(s)$, $s > 0$, are given, then there exists one and only one space curve, determined but for its position in space, for which s is the arclength, κ the curvature, and τ the torsion.*

2.1.2 Time Evolution and Variation Formulas

Later we will consider the evolution of a curve in time. Because it is useful to consider how the invariants of a curve evolve, we list necessary formulas here. The details can be found in [LP91].

Lemma 19. *Denote by $\gamma = \gamma(w, u) : (-\epsilon, \epsilon) \times (a, b) \to \mathbb{R}^3$ a one-parameter family of space-curves. If*

$$\mathbf{W} = (\partial\gamma/\partial w)(0, u) = f\mathbf{T} + g\mathbf{N} + h\mathbf{B}$$

is the variation vector field along γ, and γ has speed $v = |\partial\gamma/\partial u|$, curvature κ, and torsion τ, then v, κ, and τ vary according to

$$\mathbf{W}(v) = <\mathbf{W}_s, \mathbf{T}> v = -\alpha v, \quad \alpha = -<\mathbf{W}_s, \mathbf{T}>$$

$$\mathbf{W}(\kappa) = <\mathbf{W}_{ss}, \mathbf{N}> -2 <\kappa\mathbf{W}_s, \mathbf{T}>$$

$$\mathbf{W}(\tau) = <\mathbf{W}_{ss}, \mathbf{B}/\kappa>_s + <\mathbf{W}_s, (\kappa\mathbf{B} - \tau\mathbf{T})> .$$

and the Frenet frame varies acording to

$$\mathbf{W}(\mathbf{T}) = (\kappa f + g_s - \tau h)\mathbf{N} + (\tau g + h_s)\mathbf{B}$$
$$\mathbf{W}(\mathbf{N}) = \frac{1}{\kappa}\left[(-\kappa^2 f - \kappa g_s + \kappa\tau h)\mathbf{T} + (\kappa\tau f + 2\tau g_s - \tau^2 h + \tau_s g + h_{ss})\mathbf{B}\right]$$
$$\mathbf{W}(\mathbf{B}) = \frac{1}{\kappa}\left[(-\kappa\tau g - \kappa h_s)\mathbf{T} + (-\kappa\tau f - 2\tau g_s + \tau^2 h - \tau_s g - h_{ss})\mathbf{N}\right].$$

2.2 Surfaces

Definition 20. *A* parameterized surface $\mathbf{r} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *is a differentiable map* $\mathbf{r}$ *from an open set* $U \subset \mathbb{R}^2$ *into* $\mathbb{R}^3$. *The surface* $\mathbf{r} = \mathbf{r}(u, v)$ *is* regular *if* $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ *for all* $(u, v) \in U$. *A point* $(u, v) \in U$ *where* $\mathbf{r}_u \times \mathbf{r}_v = 0$ *is called a* singluar point *of* $\mathbf{r}$.

The regularity condition in this definition is necessary for ensuring that the following is well-defined:

Definition 21. *Given a regular surface* $\mathbf{r} = \mathbf{r}(u, v)$*, we define the* surface normal *by*

$$\mathbf{n} := \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

As in the case of the curves, surfaces can be reparameterized. Of course, we want a definition of reparameterization that preserves the regularity of the surface:

Definition 22. *Given a surface* $\mathbf{r} = \mathbf{r}(u, v)$*, the transformation*

$$u = u(\bar{u}, \bar{v}), \qquad v = v(\bar{u}.\bar{v})$$

is a regular reparameterization of $\mathbf{r}$ *provided*

$$det\begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{u}} \\ \frac{\partial u}{\partial \bar{v}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix} \neq 0.$$

Definition 23. *Given a surface* $\mathbf{r} = \mathbf{r}(u, v)$*, the lines along the parameters* u *or* v *are called* parameter lines.

2.2.1 Invariants – 1st and 2nd Fundamental Forms

As a curve has curvature and torsion as its invariants, a surface has invariants known as the 1st and 2nd fundamental forms.

Definition 24. *The* 1st fundamental form *and the* 2nd fundamental form *of a surface* $\mathbf{r} = \mathbf{r}(u, v)$ *are defined by*

$$\begin{aligned} \mathbf{I} &:= \quad d\mathbf{r} \cdot d\mathbf{r} = E\,du^2 + 2F\,dudv + G\,dv^2, \\ \mathbf{II} &:= -d\mathbf{r} \cdot d\mathbf{n} = e\,du^2 + 2f\,dudv + g\,dv^2, \end{aligned}$$

respectively, where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v,$$

$$e = \mathbf{r}_{uu} \cdot \mathbf{n}, \quad f = \mathbf{r}_{uv} \cdot \mathbf{n}, \quad g = \mathbf{r}_{vv} \cdot \mathbf{n}.$$

2.2.2 Normal Curvature

Consider a curve on a surface. We can decompose the vector $\mathbf{T}' = \kappa\mathbf{N}$ into parts tangential and normal to the surface.

Definition 25. *The* normal curvature vector *is the component of* $\kappa\mathbf{N}$ *normal to the surface and is denoted by* $\mathbf{k_n}$. *The* normal curvature κ_n *is then defined by* $\mathbf{k_n} = \kappa_n\mathbf{n}$.

Proposition 26. *The normal curvature can be written as*

$$\kappa_n = \frac{e\,du^2 + 2f\,dudv + g\,dv^2}{E\,du^2 + 2F\,dudv + G\,dv^2} = \frac{\mathbf{II}}{\mathbf{I}}.$$

This implies the following:

Theorem 27. *All curves through a point on a surface with the same tangent direction have the same normal curvature.*

Proposition 28. *The maximum and minimum vales of the normal curvature occur in perpendicular directions.*

Definition 29. *The two perpendicular directions for which the values of* κ_n *take on maximum and minimum vales are called the* principal directions, *and the corresponding normal curvatures* κ_1 *and* κ_2 *are called the* principal curvatures.

Definition 30.

2.2.3 The Gaussian and Mean Curvatures

Definition 31. *The* mean curvature *of a surface is defined by*

$$\mathcal{M} := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{Eg - 2fF + eG}{2(EG - F^2)}.$$

Definition 32. *The* Gaussian curvature *of a surface is defined by*

$$\mathcal{K} := \kappa_1\kappa_2 = \frac{eg - f^2}{EG - F^2}.$$

2.2.4 Asymptotic Lines

Definition 33. *The* asymptotic directions *on a surface satisfy*

$$\mathbf{II} = -\mathbf{dr} \cdot \mathbf{dn} = e\,du^2 + 2f\,dudv + g\,dv^2 = 0.$$

Definition 34. *The lines in the direction of asymptotic directions are called* asymptotic lines.

Proposition 35. *If the asymptotic lines are parameter curves, then* $e = 0$ *and* $g = 0$.

Theorem 36. *If the Gaussian curvature of a surface is negative, then the asymptotic lines can be taken as parameter curves.*

2.2.5 Christoffel Symbols and the Gauss Equation

Definition 37. *We define what are known as the* Christoffel symbols of the second kind*:*

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \qquad \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} \qquad \Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

Theorem 38. *On a surface* $\mathbf{r} = \mathbf{r}(u, v)$ *(of class* ≥ 2*) the vectors* $\mathbf{r}_u$*,* $\mathbf{r}_v$*,* $\mathbf{n}$ *satisfy*

$$\mathbf{r}_{uu} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + e\mathbf{n}$$

$$\mathbf{r}_{uv} = \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + f\mathbf{n}$$

$$\mathbf{r}_{vv} = \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + g\mathbf{n}$$

$$\mathbf{n}_u = \frac{fF - eG}{EG - F^2}\mathbf{r}_u + \frac{eF - fE}{EG - F^2}\mathbf{r}_v$$

$$\mathbf{n}_v = \frac{gF - fG}{EG - F^2}\mathbf{r}_u + \frac{fF - gE}{EG - F^2}\mathbf{r}_v$$

The first three equations in this last theorem are called the *Gauss equations*. The last two are called the *Weingarten equations*.

The compatibility conditions $(\mathbf{r}_{uu})_v = (\mathbf{r}_{uv})_u$ and $(\mathbf{r}_{uv}) = (\mathbf{r}_{vv})_u$ become the *Mainardi-Codazzi equations*

$$\left(\frac{e}{H}\right)_v - \left(\frac{f}{H}\right)_u + \frac{e}{H}\Gamma^2_{22} - 2\frac{f}{H}\Gamma^2_{12} + \frac{g}{H}\Gamma^2_{11} = 0$$

$$\left(\frac{g}{H}\right)_v - \left(\frac{f}{H}\right)_v + \frac{e}{H}\Gamma^1_{22} - 2\frac{f}{H}\Gamma^1_{12} + \frac{g}{H}\Gamma^1_{11} = 0$$

where $H = \sqrt{EG - F^2}$ or, equivalently,

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11},$$

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}.$$

It can be shown that the Gaussian curvature has an alternate representation in terms of the 1st fundamental form coefficients alone:

$$\mathcal{K} = \frac{1}{H}\left[\left(\frac{H}{E}\Gamma^2_{11}\right)_v - \left(\frac{H}{E}\Gamma^2_{12}\right)_u\right]. \tag{2.10}$$

With the previous theorems we can give an analogue to the fundamental theorem of curves:

Theorem 39 (The Fundamental Theorem of Surface Theory)**.** *If E, F, G and e, f, g are given as functions of u and v, sufficiently differentiable, which satisfy the Gauss equations and the Mainardi-Codazzi equations, while $EG - F^2 \neq 0$, then there exists a surface which admits as its first and second fundamental forms*

$$\mathbf{I} = E\,du^2 + 2F\,dudv + G\,dv^2 \quad \textit{and}$$

$$\mathbf{II} = e\,du^2 + 2f\,dudv + g\,dv^2$$

respectively. This surface is uniquely determined except for its position in space.

2.2.6 Geodesics

Now we return to the context of a curve on a surface and look at the component of $\mathbf{T}' = \kappa\mathbf{N}$ in the direction tangential to the surface. We find the (signed) length of this vector and define it as follows:

Definition 40. *We define the* geodesic curvature *by* $\kappa_g := (\mathbf{n} \times \mathbf{T}) \cdot \mathbf{T}'$.

Definition 41. *We refer to lines of zero geodesic curvature as* geodesics.

Geodesics can sometimes be used as parameter curves on a surface.

Definition 42. *A surface is parameterized by* geodesic coordinates *if the parameter curves are orthogonal and one of the families are geodesics.*

Chapter 3

Constant Negative Gaussian Curvature Surfaces

In this section we discuss the special class of *Constant Negative Gaussian Curvature Surfaces* (abbreviated as *CNGC surfaces*). We will discuss the connection between CNGC surfaces and the the well-known sine-Gordon equation. The sine-Gordon equation has important historical significance here since it is associated with the classical example of a *Bäcklund transformation*. We review construction by which this transformation takes CNGC surfaces to new CNGC surfaces. We discuss CNGC surfaces for two reasons: Firstly, they provide an example of many essential aspects of the theory of Integrable Geometry. Secondly (and perhaps more importantly), they are directly connected to the main work of this thesis. The text on integrable geometry by Rogers and Schief [RS02] is used as a source. Many additional details may be found there.

3.1 The sine-Gordon Connection

The sine-Gordon equation

$$\phi_{uv} = \frac{1}{\rho^2} \sin \phi$$

has been a well-known partial differential equation since Bour established its connection with CNGC surfaces in 1865 [Bou62]. Bour showed that the sine-Gordon equation comes from the Gauss-Mainardi-Codazzi equations for CNGC surfaces. Therefore, we have a correspondence between solutions to the sine-Gordon equation and CNGC surfaces.

Repeating Rogers and Schief, we review this connection here. First we begin with a theoreom from classical differential geometry:

Theorem 43. *If the Gaussian curvature of a surface is negative, then the asymptotic lines can be taken as parameter curves.*

Recall that when the parameter curves are along asymptotic lines we have $e = g = 0$ in the 2nd Fundamental Form. The Mainardi-Codazzi equations then simplify to become

$$\left(\frac{f}{H}\right)_u + 2\Gamma^2_{12}\frac{f}{H} = 0, \qquad \left(\frac{f}{H}\right)_v + 2\Gamma^1_{12}\frac{f}{H} = 0.$$

We denote the angle between parametric lines by ϕ and write

$$\cos\phi = \frac{F}{\sqrt{EG}}, \qquad \sin\phi = \frac{H}{\sqrt{EG}}.$$

Since the Gaussian curvature is a negative constant we can set

$$\mathcal{K} = -\frac{f^2}{H^2} =: -\frac{1}{\rho^2}. \tag{3.1}$$

Then, because $E, G > 0$ we can take

$$E = \rho^2 a^2, \quad G = \rho^2 b^2. \tag{3.2}$$

Using (3.1), the Mainardi-Codazzi equations above yield

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} = 0, \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)} = 0.$$

Together, these last two equations show that

$$E_v = 0, \qquad G_u = 0. \tag{3.3}$$

As a consequence of the above calculations, the representation (2.10) for $\mathcal{K}$ in terms of the 1st Fundamental Form coefficients reduces as follows:

$$\mathcal{K} = \frac{1}{H}\left[\left(\frac{H}{E}\Gamma^2_{11}\right)_v - \left(\frac{H}{E}\Gamma^2_{12}\right)_u\right] = \frac{1}{H}\left(\frac{H}{E}\Gamma^2_{11}\right)_v. \tag{3.4}$$

By expressing $\mathcal{K}$, H, E, and Γ^2_{11} in terms of ρ, a, and b for the quantities in (3.4), we arrive at the sine-Gordon equation

$$\phi_{uv} = \frac{1}{\rho^2}\sin\phi.$$

Having established our main goal for this section, we now provide the invariants and equations to be used in the following sections. By using (3.2) and (3.3) we see that $a_v = 0$ and $b_u = 0$. We parameterize by arclength along the asymptotic lines by transforming

$$du \to \widetilde{du} = \sqrt{E(u)}du, \quad dv \to \widetilde{dv} = \sqrt{G(v)}dv,$$

so that the fundamental forms become

$$\mathbf{I} = du^2 + 2\cos\phi\, du dv + dv^2,$$
$$\mathbf{II} = \frac{2}{\rho}\sin\phi\, du dv.$$

In addition, the Gauss-Weingarten system is

$$\begin{aligned}
\mathbf{r}_{uu} &= \phi_u \cot\phi\, \mathbf{r}_u - \phi_u \csc\phi\, \mathbf{r}_v,\\
\mathbf{r}_{uv} &= \frac{1}{\rho}\sin\phi\, \mathbf{n},\\
\mathbf{r}_{vv} &= -\phi_v \csc\phi\, \mathbf{r}_u + \phi_v \cot\phi\, \mathbf{r}_v,\\
\mathbf{n}_u &= \frac{1}{\rho}\cot\phi\, \mathbf{r}_u - \frac{1}{\rho}\csc\phi\, \mathbf{r}_v,\\
\mathbf{n}_v &= -\frac{1}{\rho}\csc\phi\, \mathbf{r}_u + \frac{1}{\rho}\cot\phi\, \mathbf{r}_v.
\end{aligned} \tag{3.5}$$

3.2 The Bäcklund Transformation

Given an initial solution ϕ to the sine-Gordon equation, Bäcklund devised a way to generate a new solution $\bar{\phi}$. This classical transformation is known as a *Bäcklund transformation* and is constructed by the following:

$$\bar{\phi}_u = \phi_u + 2a\sin\left(\frac{\bar{\phi}+\phi}{2}\right)$$
$$\bar{\phi}_v = -\phi_v + \frac{2}{a}\sin\left(\frac{\bar{\phi}-\phi}{2}\right).$$

The solutions ϕ and $\bar{\phi}$ correspond to the angle between the parameter lines on two different CNGC surfaces. It is possible (as was shown by Bäcklund and Bianchi [B8̈3, Bia85]) to use the above to provide a formula for a transformation of the CNGC surfaces themselves. Rogers and Schief provide an explanation of this construction.

In order to have a valid transformation of CNGC surfaces, the surface invariants must be preserved. Therefore, we seek a new surface $\bar{\mathbf{r}} = \bar{\mathbf{r}}(u, v)$ with fundamental forms of the appropriate type:

$$\bar{\mathbf{I}} = du^2 + 2\cos\bar{\phi}\, dudv + dv^2,$$
$$\overline{\mathbf{II}} = \frac{2}{\rho}\sin\bar{\phi}\, dudv.$$

For a CNGC surface $\mathbf{r} = \mathbf{r}(u, v)$ with $\mathcal{K} = -1/\rho^2$, we use the orthonormal frame $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$, where

$$\mathbf{A} = \mathbf{r}_u,$$
$$\mathbf{B} = -\mathbf{r}_u \times \mathbf{n} = -\mathbf{r}_u \times \frac{(\mathbf{r}_u \times \mathbf{r}_v)}{\sin\phi},$$
$$\mathbf{C} = \mathbf{n} = \csc\phi\, \mathbf{r}_v - \cot\phi\, \mathbf{r}_u.$$

The Gauss-Weingarten equations (3.5) can now be written as

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix}_u = \begin{pmatrix}0 & -\phi_u & 0\\ \phi_u & 0 & 1/\rho\\ 0 & -1/\rho & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix}_v = \begin{pmatrix}0 & 0 & (1/\rho)\sin\phi\\ 0 & 0 & -(1/\rho)\cos\phi\\ -(1/\rho)\sin\phi & (1/\rho)\cos\phi & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix}.$$

Now, suppose we wish to construct a new CNGC surface

$$\bar{\mathbf{r}} = \mathbf{r} + L(\cos\theta\mathbf{A} + \sin\theta\mathbf{B}) \qquad (L \text{ constant}). \tag{3.6}$$

Constraints must be placed on $\theta = \theta(u, v)$ to ensure that $\bar{\mathbf{r}}$ has 1st and 2nd fundamental forms of the same type as $\mathbf{r}$. So, first we require

$$\bar{\mathbf{r}}_u \cdot \bar{\mathbf{r}}_u = 1, \quad \bar{\mathbf{r}}_v \cdot \bar{\mathbf{r}}_v = 1$$

which results in the relations

$$\theta_u = \phi_u + \frac{1}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right)\sin\theta,$$
$$\theta_v = \frac{1}{L}\left(1 \mp \sqrt{1 - \frac{L^2}{\rho^2}}\right)\sin(\theta - \phi).$$

Checking the compatibility conditions for this pair of equations shows that ϕ satisfies the sine-Gordon equation

$$\phi_{uv} = \frac{1}{\rho^2} \sin\phi.$$

With these conditions we can check that $\overline{F} = \bar{\mathbf{r}}_u \cdot \bar{\mathbf{r}}_v = \cos(2\theta - \phi)$ so that the 1st fundamental form of $\bar{\mathbf{r}}$ is

$$\bar{\mathbf{I}} = du^2 + 2\cos(2\theta - \phi)\, dudv + dv^2.$$

By calculating $\bar{\mathbf{n}}$ and performing the (lengthy) calculations for $\bar{e}$, $\bar{f}$, and $\bar{g}$, we find that the 2nd fundamental form of $\bar{\mathbf{r}}$ is

$$\overline{\mathbf{II}} = \frac{2}{\rho} \sin(2\theta - \phi)\, dudv.$$

This proves that the angle between asymptotic lines on $\bar{\mathbf{r}}$ is given by

$$\bar{\phi} = 2\theta - \phi.$$

Now, eliminating θ and substituting back into the desired form (3.6), we have the Bäcklund transformation of CNGC surfaces:

$$\bar{\mathbf{r}} = \mathbf{r} + \frac{L}{\sin\phi} \left[\sin\left(\frac{\phi - \bar{\phi}}{2}\right) \mathbf{r}_u + \sin\left(\frac{\phi + \bar{\phi}}{2}\right) \mathbf{r}_v \right].$$

The corresponding Bäcklund transformation is

$$\begin{aligned}\left(\frac{\overline{\phi}-\phi}{2}\right)_u &= \frac{\beta}{\rho}\sin\left(\frac{\overline{\phi}+\phi}{2}\right),\\ \left(\frac{\overline{\phi}+\phi}{2}\right)_v &= \frac{1}{\beta\rho}\sin\left(\frac{\overline{\phi}-\phi}{2}\right)\end{aligned} \tag{3.7}$$

where

$$\beta = \frac{\rho}{L}\left(1 \pm \sqrt{1-\frac{L^2}{\rho^2}}\right).$$

3.3 Enneper's Theorem and the Bäcklund Transformation on Constant-Torsion Curves

3.3.1 CNGC Surfaces as Foliation of Constant-Torsion Curves

We now shift our perspective slightly by thinking of CNGC surfaces as foliations of constant-torsion curves. First, we need the following fact from differential geometry:

Theorem 44 (Enneper's Theorem)**.** *The asymptotic lines on a Constant Negative Gaussian Curvature surface have constant torsion given by*

$$\tau^2 = -\mathcal{K}.$$

In addition, there is a formulation by Lamb [GLL77] of a geometric flow on constant-torsion curves that "sweeps out" CNGC surfaces:

Theorem 45. *If γ is a curve with constant torsion τ, then the geometric flow*

$$\gamma_t = \cos\phi\mathbf{T} - \sin\phi\mathbf{N} \tag{3.8}$$

where

$$\phi_s = \kappa$$

preserves constant torsion (i.e., $\mathbf{W}(\tau) = 0$). The variation of curvature is given by

$$\kappa_t = \mathbf{W}(\kappa) = \tau^2 \sin\phi,$$

which implies that ϕ satisfies the sine-Gordon equation

$$\phi_{st} = \tau^2 \sin\phi.$$

We refer to (3.8) *as the* trigonometric flow.

Because the asymptotic lines have constant torsion and the Bäcklund transformation takes asymptotic lines to asymptotic lines, the Bäcklund transformation can be restricted to constant-torsion curves. This Bäcklund transformation was studied by Calini and Ivey [CI98]:

Theorem 46. *Let $\gamma(s)$ be a smooth curve of constant torsion τ in $\mathbf{R}^3$, parameterized by arclength s. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be a Frenet frame, and $\kappa(s)$ the curvature of γ. For any* *constant C, let* $\beta = \beta(s; \kappa(s), C)$ *be a solution of the differential equation*

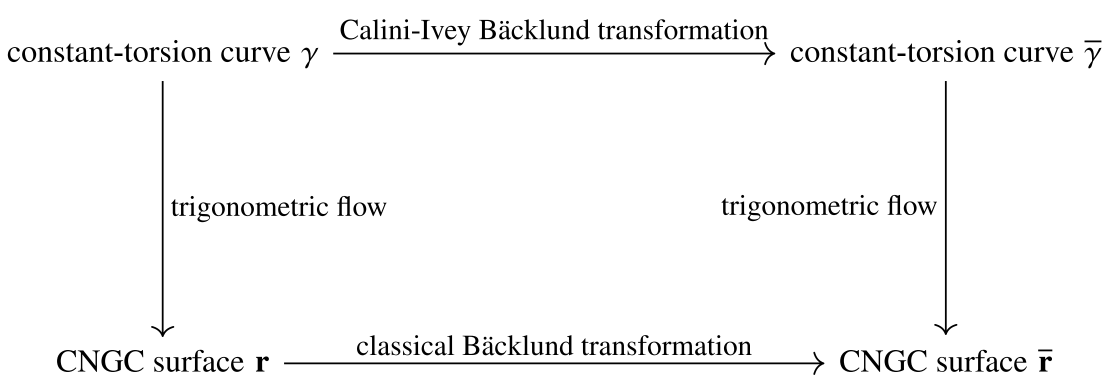
$$\frac{d\beta}{ds} = C \sin\beta - \kappa$$

Then the curve

$$\overline{\gamma}(s) = \gamma(s) + \frac{2C}{C^2 + \tau^2}(\cos\beta \mathbf{T} + \sin\beta \mathbf{B}) \tag{3.9}$$

is a curve of constant torsion τ*, also parameterized by arclength.*

With each of the constant-torsion curves γ and $\overline{\gamma}$ we can trace out a CNGC surface via the trigonometric flow. Thus we have a commutative correspondence as follows:



3.3.2 Derivation of Trigonometric Flow from Bäcklund Transformation

Using the Bäcklund transformation (3.9), Calini and Ivey derive the trigonometric flow as follows: If we let $C \to 0$, the Riccati equation (5.11) becomes

$$\frac{d\beta}{ds} \simeq -\kappa.$$

Therefore,

$$\beta(s) \simeq -\theta(s) = -\int^{s} \kappa(s')ds'$$

and, to first order,

$$\overline{\gamma} = \gamma + \frac{2C}{\tau^2}(\cos\theta\mathbf{T} - \sin\theta\mathbf{N}).$$

This means that the trigonometric flow can be considered as an "infinitesimal version" of the Bäcklund transformation.

3.4 The Lax Pair Formulation and the Sym-Tafel Formula

Soliton equations can often be cast as the compatibility conditions for some pair of partial differential equations

$$\Phi_s = U\Phi, \qquad \Phi_t = V\Phi,$$

where Φ, U, V are matrix-valued functions which depend (in an elementary way) on an auxiliary parameter λ. This is called the *Lax pair* of the system. The formula $F :=$

$\Phi^{-1}\Phi_\lambda$ defines an associated surface (which we call the Sym-Tafel surface [Sym85]). Rather than discussing this construction in its full generality, we illustrate the construction via specific examples.

The Lax pair for the sine-Gordon equation can be formulated in this context:

$$\Phi_s = \frac{1}{2}\left[\begin{pmatrix} 0 & \phi_s \\ -\phi_s & 0 \end{pmatrix} + i\frac{\lambda}{\rho}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\Phi, \tag{3.10a}$$

$$\Phi_t = \frac{i}{2\lambda\rho}\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\Phi. \tag{3.10b}$$

Observe that the matrices in (3.10) are 2×2, skew-Hermitian matrices with trace zero; the set of which is referred to as su(2). We can check that the compatibility condition $U_t - V_s + [U, V] = 0$ is satisfied if

$$\phi_{st} = \frac{1}{\rho^2}\sin\phi.$$

If we take a solution $\Phi(s;\lambda)$ to the spatial part (3.10a) with $\rho = 1$ then the Sym-Tafel formula

$$\gamma := \Phi^{-1}\Phi_\lambda\Big|_{\lambda=\tau}$$

defines an arclength-parameterized curve in su(2) with constant torsion τ and curvature

$\kappa = \phi_s$. The Frenet frame for this curve can be represented by

$$\mathbf{T} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\Phi$$

$$\mathbf{N} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\Phi \tag{3.11}$$

$$\mathbf{B} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi.$$

One can check that this is an orthonormal frame in su(2).

By using the identification in (3.11), the trigonometric flow can be recovered from the Sym-Tafel formulation as follows:

$$\gamma_t = \frac{1}{2}\Phi^{-1}V_\lambda\Phi\Big|_{\lambda=\tau} = \frac{1}{\tau^2}\left(\cos\phi\mathbf{T} - \sin\phi\mathbf{N}\right).$$

In the general setting we have a surface with parameter curves in the directions

$$\mathbf{r}_s = \frac{1}{2}\Phi^{-1}U_\lambda\Phi\Big|_\lambda, \qquad \mathbf{r}_t = \frac{1}{2}\Phi^{-1}V_\lambda\Phi\Big|_\lambda.$$

The Sym-Tafel formula allows us to take a common form of PDE system found in soliton theory and construct a surface (curve) from its solution. This allows us to take tools of soliton theory (Bäcklund transformations, etc.) and apply them to geometric objects.

Chapter 4

Razzaboni Surfaces

Following the example of CNGC surfaces, we see throughout the history of integrable geometry the role of special geometric objects and their corresponding evolution. As the CNGC surfaces were shown to be integrable and to include the special role of constant-torsion curves, Schief presented results on *Razzaboni* surfaces which are foliated by geodesics which are *Bertrand curves* via a given evolution flow. He showed their integrability by way of a Bäcklund transformation and prescribed a dynamical description in the form of a curve evolution flow [Sch03].

As we saw the Bäcklund transformation of constant-torsion curves within the broader context of CNGC surfaces, we now look at special class of curves called Bertrand curves, and, in the broader context, Razzaboni surfaces on which Bertrand curves are geodesics. We see in this context that Bertrand curves have their own type of Bäcklund transformation. Lastly, we derive a new flow from this Bäcklund transformation that preserves Bertrand cuves. This will be the stepping-off point for the main results of this thesis.

4.1 Bertrand Curves

In this section, we consider a generalization of constant-torsion curves called Bertrand curves. We define them as follows:

Definition 47. *A curve* $\mathbf{r}(u) \in \mathbb{R}^3$ *is called a* Bertrand curve *if there exists a curve* $\widetilde{\mathbf{r}}(u)$ *such that the normals of* $\mathbf{r}(u)$ *and* $\widetilde{\mathbf{r}}(u)$ *coincide for any u. The curve* $\widetilde{\mathbf{r}}(u)$ *is called the* Bertrand mate *of* $\mathbf{r}(u)$.

For our purposes,

Theorem 48. *A Bertrand curve is a curve for which there exists a linear relation between curvature and torsion:*

$$\alpha\kappa + \beta\tau = 1 \qquad (\alpha, \beta \text{ constant}).$$

Special cases of Bertrand curves include curves of constant torsion ($\alpha = 0$) and curves of constant curvature ($\beta = 0$).

4.2 Razzaboni Surfaces

We begin with the following fact from classical differential geometry:

Theorem 49. *A curve constitutes a geodesic on a surface if and only if the principal normal of the curve is (anti-)parallel to the normal* $\mathbf{n}$ *to the surface.*

Now we can define the surfaces:

Definition 50. *A surface is termed a* Razzaboni surface *if it is spanned by a one-parameter family of geodesic Bertrand curves associated with the condition*

$$\alpha\kappa + \beta\tau = 1.$$

4.3 Binormal Flow of Bertrand Curves

Theorem 51. *Let* $\mathbf{r} = \mathbf{r}(s,t)$ *be an evolving space curve with curvature* κ*, torsion* τ*, and Frenet frame* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$*. The evolution equation*

$$\mathbf{r}_t = g\mathbf{B}$$

where

$$\kappa_t = -2\tau g_s - \tau_s g, \qquad \tau_t = v_s + \kappa g_s, \qquad g_{ss} = \tau^2 g + \kappa v \tag{4.1}$$

preserves the Bertrand property

$$\sin\sigma\,\kappa + \cos\sigma\,\tau = \frac{1}{a}, \qquad (\sigma \text{ constant})$$

hence $\mathbf{r}(s,t)$ *is a parameterization for a Razzaboni surface.*

4.3.1 Special Cases

At $\sigma = 0$ and $\sigma = \frac{\pi}{2}$ the geometric flow (4.1) describes surfaces foliated by geodesics of constant torsion and constant curvature, respectively. These were also investigated by Schief in [SR99].

Looking in particular at the case of constant torsion, the governing equation has the form

$$q_s = \kappa_t, \quad q_{ss} = c^2 q + \rho\kappa, \quad \rho_s + \kappa q_s = 0. \tag{4.2}$$

Schief refers to this equation as the *extended sine-Gordon equation* as it is an extension of the classical integrable PDE. This system will return in the context of the main results of this thesis.

4.4 Bäcklund Transformation

Theorem 52 (A Bäcklund transformation for Razzaboni surfaces)**.** *Let* $\Sigma : \mathbf{r} = \mathbf{r}(s,t)$ *be a Razzaboni surface parameterized in terms of geodesic coordinates s, t. Then, the position vector of another one-parameter family of Razzaboni surfaces* $\overline{\Sigma}(k)$ *is given by*

$$\bar{\mathbf{r}} = \mathbf{r} + a\cos k\ (\cos\sigma\sin\phi\mathbf{T} + \cos\phi\mathbf{N} + \sin\sigma\sin\phi\mathbf{B}) \tag{4.3}$$

where the function ϕ *is a solution of the compatible Frobenius system*

$$\phi_s = \kappa\cos\sigma - \tau\sin\sigma + \frac{\sin\sigma - \cos k\cos\phi}{a(\cos\sigma + \sin k)}, \tag{4.4a}$$

$$\begin{aligned}\phi_t = &-\frac{\sin k\sin\sigma + \cos k\cos\sigma\cos\phi}{\sin k}v - \frac{\sin k\cos\sigma - \cos k\sin\sigma\cos\phi}{\sin k}\tau g \\ &- \cot k\sin\phi\, g_s - \frac{1 + \sin k\cos\sigma - \cos k\sin\sigma\cos\phi}{a\sin k\,(\cos\sigma + \sin k)}g.\end{aligned} \tag{4.4b}$$

This Bäcklund transformation can be restricted to Bertrand curves:

Theorem 53 (A Bäcklund transformation for Bertrand curves)**.** *Let* $\Gamma : \mathbf{r} = \mathbf{r}(s)$ *be a Bertrand curve parameterized in terms of arclength s. Then, the position vector of* *another one-parameter family of Bertrand curves* $\overline{\Gamma}(k)$ *is given by*

$$\bar{\mathbf{r}} = \mathbf{r} + a\cos k\ (\cos\sigma\sin\phi\mathbf{T} + \cos\phi\mathbf{N} + \sin\sigma\sin\phi\mathbf{B}) \tag{4.5}$$

with $\overline{a} = a$, $\overline{\sigma} = \sigma$, *where the function* ϕ *is a solution of the first-order differential equation*

$$\phi_s = \kappa\cos\sigma - \tau\sin\sigma + \frac{\sin\sigma - \cos k\cos\phi}{a(\cos\sigma + \sin k)}. \tag{4.6}$$

The Bäcklund transformation obeys the constant length property, i.e. the distance $|\bar{\mathbf{r}} - \mathbf{r}| = a|\cos k|$ *between corresponding points on* Γ *and* $\overline{\Gamma}(k)$ *only depends on the Bäcklund parameter k.*

4.5 A New Flow That Preserves Bertrand Curves

We saw in 3.3.2 that a constant-torsion-preserving flow (namely, the trigonometric flow) was derived from the Bäcklund transformation on constant-torsion curves. Similarly, we employ the same technique here, differentiating with respect to the distance parameter to derive a geometric flow. If we set $\epsilon := \cos k$ then the Bäcklund transformation (4.3) can be written

$$\bar{\mathbf{r}} = \mathbf{r} + a\epsilon(\cos\sigma\sin\phi\mathbf{T} + \cos\phi\mathbf{N} + \sin\sigma\sin\phi\mathbf{B}).$$

By taking a first-order approximation with respect to ϵ, we arrive at the geometric flow

$$\mathbf{r}_t = \cos\sigma \sin\phi \mathbf{T} + \cos\phi \mathbf{N} + \sin\sigma \sin\phi \mathbf{B}.$$

Because $\epsilon \to 0$ as $k \to \frac{\pi}{2}$, the governing equation (4.6) reduces to

$$\phi_s = \kappa \cos\sigma - \tau \sin\sigma + \frac{\sin\sigma}{a(\cos\sigma + 1)}.$$

One can check using the variation formulas for $\mathbf{W}(\kappa)$ and $\mathbf{W}(\tau)$ that we have

$$\mathbf{W}(\kappa)\sin\sigma + \mathbf{W}(\tau)\cos\sigma = 0.$$

Thus, this flow preserves the Bertrand property of $\mathbf{r}$. It should be noted, however, that arclength is not preserved. I.e., $\mathbf{W}(v) \neq 0$. Notice that this is NOT the same Bertrand-preserving flow of 4.3, and the flow does not provide a geodesic foliation for the swept out surface.

4.5.1 Orbital Curves

While arclength is not preserved by this new flow, we note that the variation vector field

$$\mathbf{W} = \cos\sigma \sin\phi \mathbf{T} + \cos\phi \mathbf{N} + \sin\sigma \sin\phi \mathbf{B}$$

has unit length. Thus, the orbital curves (along the t parameter) are arclength-parameterized. Let $\tilde{\mathbf{r}}(t) = \mathbf{r}(s,t)$ denote the t-curve along a fixed s which we will refer to as an *orbital curve*. We now compute the geometry of $\tilde{\mathbf{r}}$. We already noted that the tangent vector $\tilde{\mathbf{T}}$ is given by

$$\tilde{\mathbf{T}} = \cos\sigma\sin\phi\mathbf{T} + \cos\phi\mathbf{N} + \sin\sigma\sin\phi\mathbf{B}.$$

In order to compute higher t-derivatives, we need the variation of $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$. This can be computed using the variation formulas:

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}_t = \begin{pmatrix} 0 & \dfrac{\cos\sigma - 1}{a\sin\sigma}\sin\phi & \dfrac{1}{a}\cos\phi \\ -\dfrac{\cos\sigma - 1}{a\sin\sigma}\sin\phi & 0 & -\dfrac{1}{a}\sin\phi \\ -\dfrac{1}{a}\cos\phi & \dfrac{1}{a}\sin\phi & 0 \end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}.$$

Then, we can finish computing the geometry:

$$\begin{aligned}\tilde{\kappa} &= -\frac{\cos\sigma - 1}{a\sin\sigma}\sin\phi - \phi_t,\\ \tilde{\mathbf{N}} &= -\cos\sigma\cos\phi\mathbf{T} + \sin\phi\mathbf{N} - \sin\sigma\cos\phi\mathbf{B},\\ \tilde{\mathbf{B}} &= -\sin\sigma\mathbf{T} + \cos\sigma\mathbf{B},\\ \tilde{\tau} &= -\frac{1}{a}.\end{aligned} \tag{4.7}$$

Thus, $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(t)$ has constant torsion.

To summarize: We have a new flow that preserves the Bertrand property of a curve.

In addition, its orbital curves are arclength-parameterized constant-torsion curves.

Chapter 5

The Interpolatory Flow

5.1 Derivation

At the end of the last chapter we derived a new geometric flow on Bertrand curves. The corresponding orbital curves of this flow were arclength-parameterized and had constant torsion. We now would like to use this result to change our perspective and find a flow that preserves constant torsion and has orbital curves with the Bertrand property. The Frenet frame of the Bertrand curves in terms of the Frenet frame of the constant-torsion curves is

$$\begin{pmatrix} \tilde{\mathbf{T}} \\ \tilde{\mathbf{N}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \cos\sigma\sin\phi & \cos\phi & \sin\sigma\sin\phi \\ -\cos\sigma\cos\phi & \sin\phi & -\sin\sigma\cos\phi \\ -\sin\sigma & 0 & \cos\sigma \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

The flow on constant-torsion curves should be in the direction of **T**. If we solve for **T** in the above system, we find

$$\mathbf{T} = \cos\sigma\sin\phi\tilde{\mathbf{T}} - \cos\sigma\cos\phi\tilde{\mathbf{N}} - \sin\sigma\tilde{\mathbf{B}}.$$

Now we consider a variation vector field **W** in the direction of **T**, i.e.,

$$W = L(s,t)\left(\cos\sigma\sin\phi\tilde{\mathbf{T}} - \cos\sigma\cos\phi\tilde{\mathbf{N}} - \sin\sigma\tilde{\mathbf{B}}\right).$$

From here on, we drop the tilde notation and use $\mathbf{r}, \mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$ to refer to the geometry of the constant-torsion curves. In addition, we swap the role of s and t.

Because the constant-torsion curves are arclength-parameterized, our desired flow should preserve arclength, i.e. $\mathbf{W}(v) = 0$. After some calculation, we see that we can satisfy this condition if

$$L = F(t)\exp\left(\frac{\cos\sigma - 1}{a\sin\sigma}\int \cos\phi\, ds\right). \qquad (F \text{ is a function of } t \text{ only!})$$

In fact, the equation for L is a linear ODE. We switch variables (s and t) in the expression for κ given in (4.7) to arrive at the governing equation for our new flow:

$$\phi_s = -\kappa - \frac{\cos\sigma - 1}{a\sin\sigma}\sin\phi.$$

5.2 A Geometric Flow that Preserves the Property of Constant Torsion

We summarize the results of the previous section: Consider an evolving curve $\mathbf{r}(s,t) \in \mathbb{R}^3$ with curvature κ, torsion τ, and Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

Definition 54. *We call the evolution equation*

$$\mathbf{r}_t = F(t)\exp\left(\frac{\cos\sigma - 1}{a\sin\sigma}\int \cos\phi\, ds\right)(\sin\phi\cos\sigma\mathbf{T} - \cos\phi\cos\sigma\mathbf{N} - \sin\sigma\mathbf{B}) \tag{5.1}$$

where

$$\phi_s = -\kappa - \frac{\cos\sigma - 1}{a\sin\sigma}\sin\phi \qquad (\sigma, a \ \textit{constant}) \tag{5.2}$$

the Interpolatory Flow *(the nomenclature will be explained shortly).*

From here we will refer to (5.1) by *(IF)*. In what follows we check that (IF) generates a one-parameter family of constant-torsion curves.

Theorem 55. *For an initial arclength-parameterized curve* $\mathbf{r}_0 = \mathbf{r}(s, 0)$ *with constant torsion* $\tau_0 = -\frac{1}{a}$*, the Interpolatory Flow preserves arclength and constant torsion.*

Proof. We will use formulas for the evolution of curve invariants. Taking $\mathbf{W}$ = (IF), we find

$$\begin{aligned} v_t &= 0,\\ \kappa_t &= -2F(t)\frac{\cos\sigma - 1}{a^2\sin^2\sigma}\exp\left(\frac{\cos\sigma - 1}{a\sin\sigma}\int\cos\phi\, ds\right)\cos\phi,\\ \tau_t &= 0. \end{aligned}$$

Thus, the evolution (IF) preserves arclength and constant torsion. □

In addition, we can compute the variation of $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along (IF):

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}_t = F(t) \exp\left(\frac{\cos\sigma - 1}{a \sin\sigma} \int \cos\phi \, ds \right) \begin{pmatrix} 0 & \frac{\cos\sigma - 1}{a\sin\sigma} & \frac{\cos\phi}{a} \\ -\frac{\cos\sigma - 1}{a\sin\sigma} & 0 & \frac{\sin\phi}{a} \\ -\frac{\cos\phi}{a} & -\frac{\sin\phi}{a} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

5.3 Interpolatory Flow Surfaces

We have seen how (IF) generates a one-parameter family of constant-torsion curves. For each t in the time evolution, $\mathbf{r}(s, t)$ has constant torsion with respect to the arclength parameter s. Alternatively, we can think of this family of curves as a differentiable *surface* parameterized by s and t.

5.4 The Orbital Curves of (IF)

Now, suppose we consider the curves $\tilde{\mathbf{r}}(t) := \mathbf{r}(s, t)$ along the parameter t, i.e., the curves generated by the orbit of a fixed s-value along (IF). We call these curves the *orbital curves* of (IF). Note that the parameter t is not necessarily an arclength parameter for the curves $\tilde{\mathbf{r}}(t)$.

We denote the curvature and torsion of $\tilde{\mathbf{r}}$ by $\tilde{\kappa}$ and $\tilde{\tau}$, respectively, and we denote the Frenet frame of $\tilde{\mathbf{r}}$ by $\{\tilde{\mathbf{T}}, \tilde{\mathbf{N}}, \tilde{\mathbf{B}}\}$. Thus, starting with

$$\tilde{\mathbf{T}} = \frac{\mathbf{r}_t}{||\mathbf{r}_t||} = \sin\phi\cos\sigma\mathbf{T} - \cos\phi\cos\sigma\mathbf{N} - \sin\sigma\mathbf{B},$$

we use the variation formulas (and extensive symbolic computation) to find

$$\tilde{\kappa} = -\frac{\cos\sigma - 1}{a\sin\sigma} + \frac{1}{F(t)}\cos\sigma \exp\left(-\frac{\cos\sigma - 1}{a\sin\sigma}\int \cos\phi\, ds\right)\phi_t$$

$$\tilde{\tau} = \frac{1}{a} - \frac{1}{F(t)}\sin\sigma \exp\left(-\frac{\cos\sigma - 1}{a\sin\sigma}\int \cos\phi\, ds\right)\phi_t$$

which satisfy

$$\sin\sigma\tilde{\kappa} + \cos\sigma\tilde{\tau} = \frac{1}{a}.$$

Thus, the orbital curves $\tilde{\mathbf{r}}$ are Bertrand curves.

The Frenet frame of $\tilde{\mathbf{r}}$ can be written concisely in terms of the frenet frame of $\mathbf{r}$:

$$\begin{pmatrix} \tilde{\mathbf{T}} \\ \tilde{\mathbf{N}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \cos\sigma\sin\phi & -\cos\sigma\cos\phi & -\sin\sigma \\ \cos\phi & \sin\phi & 0 \\ \sin\sigma\sin\phi & -\sin\sigma\cos\phi & \cos\sigma \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

5.5 Interpolation of Known Systems

As we vary the parameter σ we shall see that (IF) interpolates between two known integrable systems, hence the terminology. The associated sufaces with s-curves of constant torsion and (orbital) t-curves with the Bertrand property assume the properties of different known surfaces under this variation.

5.5.1 (IF) at $\sigma = 0$

As $\sigma \to 0$, the relationship $\sin\sigma\tilde{\kappa} + \cos\sigma\tilde{\tau} = \frac{1}{a}$ becomes $\tilde{\tau} = \frac{1}{a}$ so that the orbital curves have constant torsion and (IF) reduces to

$$\mathbf{r}_t = F(t)\left(\sin\phi\mathbf{T} - \cos\phi\mathbf{N}\right), \qquad \phi_s = -\kappa. \tag{5.3}$$

Note that this is in the direction of the trigonometric flow which, as we have seen in a previous chapter, sweeps out Constant Negative Gaussian Curvature (CNGC) surfaces. If $F(t) \neq 0$, we can reparameterize for t so that (5.3) matches the trigonometric flow. Also, the parameterization by constant-torsion curves is consistent with Enneper's theorem.

5.5.2 (IF) at $\sigma = \frac{\pi}{2}$

At $\sigma = \frac{\pi}{2}$, we have $\tilde{\kappa} = \frac{1}{a}$ so that the orbital curves have constant curvature and (IF) reduces to

$$\mathbf{r}_t = -F(t)\exp\left(-\frac{1}{a}\int\cos\phi\,ds\right)\mathbf{B}, \qquad \phi_s = -\kappa + \frac{1}{a}\sin\phi.$$

This is a subset of solutions to the *binormal motion* of Schief [SR99], which describes a surface spanned by a one-parameter family of geodesics of constant torsion. We note that while the role of orbital curves of constant torsion was known in the case $\sigma = 0$, Schief did not investigate the orbital curves of the binormal motion.

5.6 The Governing System

Here we return to the system discussed in section 4.3.1, the extended sine-Gordon system,

$$q_s = \kappa_t, \quad q_{ss} = c^2 q + \rho\kappa, \quad \rho_s + \kappa q_s = 0.$$

This system is the compatibility condition for the Lax Pair,

$$\Phi_s = U\Phi = \frac{1}{2}\left[\begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} + i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\Phi, \tag{5.4a}$$

$$\Phi_t = V\Phi = \frac{1}{2(1 + c^{-2}\lambda^2)}\left[\frac{i\lambda}{c^2}\begin{pmatrix} -\rho & q_s \\ q_s & \rho \end{pmatrix} + \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}\right]\Phi. \tag{5.4b}$$

In our particular context, given a solution Φ to (5.4a),

$$\mathbf{r} := \Phi^{-1}\Phi_\lambda\Big|_{\lambda=-1/a}$$

defines a curve in su(2) $\equiv \mathbb{R}^3$ with constant torsion $\tau = -1/a$ and Frenet frame

$$\mathbf{r}_s = \mathbf{T} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\Phi\Big|_{\lambda=-1/a},$$

$$\mathbf{N} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\Phi\Big|_{\lambda=-1/a},$$

$$\mathbf{B} = \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi\Big|_{\lambda=-1/a}.$$

Using this correspondence, we find that the t-evolution governed by (5.4b):

$$\begin{aligned}
\mathbf{r}_t &= \Phi^{-1}V_\lambda\Phi\Big|_{\lambda=-1/a} \\
&= \frac{1}{2(c^2+\lambda^2)^2}\Phi^{-1}\begin{pmatrix} -i(c^2-\lambda^2)\rho & -2\lambda c^2 q + i(c^2-\lambda^2)q_s \\ 2\lambda c^2 q + i(c^2-\lambda^2)q_s & i(c^2-\lambda^2)\rho \end{pmatrix}\Phi\Big|_{\lambda=-1/a} \\
&= \frac{1}{(c^2+a^{-2})^2}\left(-(c^2-a^{-2})\rho\,\mathbf{T} + (c^2-a^{-2})q_s\,\mathbf{N} + 2a^{-1}c^2 q\,\mathbf{B}\right)
\end{aligned}$$

We can use the variation formulas to check the following:

Theorem 56. *For q, ρ, and κ satisfying the extended sine-Gordon system, the geometric flow*

$$\mathbf{r}_t = \frac{1}{(c^2+a^{-2})^2}\left(-(c^2-a^{-2})\rho\,\mathbf{T} + (c^2-a^{-2})q_s\,\mathbf{N} + 2a^{-1}c^2 q\,\mathbf{B}\right) \tag{5.5}$$

preserves constant-torsion curves.

Now, to connect this evolution to (IF) we first note that the extened sine-Gordon system admits the integral

$$q_s^2 + \rho^2 - c^2 q^2 = C, \qquad (C \text{ constant}). \tag{5.6}$$

If $C = 0$, we can use the parameterization

$$q_s = cq\cos\phi, \quad \rho = cq\sin\phi.$$

This admits

$$q = G(t)\exp\left(c\int\cos\phi\,ds\right), \tag{5.7}$$

$$\rho = c\,G(t)\exp\left(c\int\cos\phi\,ds\right)\sin\phi. \qquad (G \text{ is a function of } t) \tag{5.8}$$

Then for $G(t) \neq 0$ the curvature and torsion can be computed with extensive symbolic calculation:

$$\begin{aligned}
\tilde{\kappa} &= \frac{1-a^2c^2}{a^2cG(t)}\exp\left(-c\int\cos\phi\,ds\right)\phi_t - c,\\
\tilde{\tau} &= \frac{2}{aG(t)}\exp\left(-c\int\cos\phi\,ds\right)\phi_t + \frac{1}{a}
\end{aligned}$$

so that we have

$$\left(\frac{2ac}{1+a^2c^2}\right)\tilde{\kappa} + \left(\frac{1-a^2c^2}{1+a^2c^2}\right)\tilde{\tau} = \frac{1}{a}.$$

Now, if we let

$$c = \frac{\cos\sigma - 1}{a\sin\sigma}, \qquad G(t) = -\frac{2}{a\sin\sigma}F(t) \qquad (\text{for some function } F \text{ of } t)$$

then

$$q = -\frac{2}{a \sin \sigma} F(t) \exp\left(\frac{\cos \sigma - 1}{a \sin \sigma} \int \cos \phi \, ds \right),$$

$$\rho = -2 \frac{\cos \sigma - 1}{a^2 \sin^2 \sigma} F(t) \exp\left(\frac{\cos \sigma - 1}{a \sin \sigma} \int \cos \phi \, ds \right) \sin \phi$$

is a solution to the extended sine-Gordon system for which the corresponding geometric flow corresponds to (IF). So we have the following:

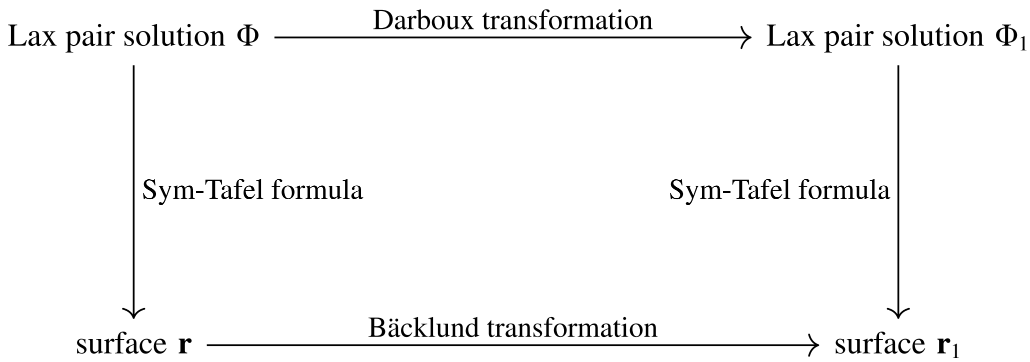
Theorem 57. *The geometric flow* (5.5) *governed by the extended sine-Gordon system with the constraint*

$$q_s^2 + \rho^2 - c^2 q^2 = 0 \tag{5.9}$$

corresponds to (IF).

5.7 The Darboux Transformation and Bäcklund Transformation

We have seen examples of Bäcklund transformations in the context of CNGC surfaces and Razzaboni Surfaces. These took initial surfaces (of respective type) and generated new surfaces. In addition to the transformation on the level of the surface, we would like to know how the underlying Lax pair and associated quantities transform. A transformation on the level of the Lax pair is called a *Darboux transformation*. As a solution to the Lax pair is related to the surface via the Sym-Tafel formula, the Darboux transformation is related to the Bäcklund transformation via the Sym-Tafel formula. The connection is illustrated by the following commutative diagram:



Darboux transformations are well-known for Lax pairs as in (5.4) and take the form of matrix transformations. If we begin with a solution Φ to our Lax pair, we can construct a *Darboux matrix* D such that $D\Phi$ satisfies a Lax pair system of the same form.

The Darboux matrix for (5.4) can be constructed as follows: Let $w = [w_1 \ w_2]^T$ be a vector solution to (5.4) for $\lambda = i\lambda_0$ ($\lambda_0 \in \mathbb{R}$). Then, define the matrix

$$D := \sqrt{\frac{\lambda + i\lambda_0}{\lambda - i\lambda_0}}\left(I - \frac{2i\lambda_0}{\lambda + i\lambda_0}\left(\frac{ww^T}{w^T w}\right)\right). \tag{5.10}$$

Now, if we set $\theta = 2\arctan(w_1/w_2)$, then θ satisfies a pair of equations

$$\theta_s = \kappa - \lambda_0 \sin\theta, \tag{5.11a}$$

$$\theta_t = -\frac{1}{c^2 - \lambda_0^2}(-c^2 q - \lambda_0 \rho \sin\theta + \lambda_0 q_s \cos\theta) \tag{5.11b}$$

and D can be rewritten as

$$D = \frac{1}{\sqrt{\lambda^2 + \lambda_0^2}} \left(\lambda I + i\lambda_0 \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \right).$$

To see how the Lax pair system transforms, we need to find the matrices U_1 and V_1 such that

$$\Phi_{1s} = U_1\Phi_1, \qquad \Phi_{1t} = V_1\Phi_1. \tag{5.12}$$

Using the fact that $\Phi_1 = D\Phi$, the first equation in (5.12) becomes

$$(D\Phi)_s = U_1(D\Phi)$$

for which we can expand derivatives and solve to find

$$U_1 = D_s D^{-1} + DUD^{-1}. \tag{5.13}$$

Similarly, we solve the second equation in (5.12) to find

$$V_1 = D_t D^{-1} + DVD^{-1}. \tag{5.14}$$

We now use (5.13) and (5.14) with (5.11) to find the new Lax pair of the same form:

$$\Phi_{1s} = U_1\Phi_1 = \frac{1}{2}\left[\begin{pmatrix} 0 & \kappa_1 \\ -\kappa_1 & 0 \end{pmatrix} + i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\Phi_1,$$

$$\Phi_{1t} = V_1\Phi_1 = \frac{1}{2(1+c^{-2}\lambda^2)}\left[\frac{i\lambda}{c^2}\begin{pmatrix} -\rho_1 & q_{1s} \\ q_{1s} & \rho_1 \end{pmatrix} + \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix}\right]\Phi_1$$

where

$$\kappa_1 = \kappa - 2\lambda_0 \sin\theta,$$

$$q_1 = \frac{1}{c^2-\lambda_0^2}\Big((c^2+\lambda_0^2)q + (2\lambda_0\sin\theta)\rho - (2\lambda_0\cos\theta)q_s\Big),$$

$$\rho_1 = \frac{1}{c^2-\lambda_0^2}\Big((2\lambda_0 c^2\sin\theta)q + (c^2-\lambda_0^2\cos(2\theta))\rho - (\lambda_0^2\sin(2\theta))q_s\Big).$$

We saw that (IF) corresponded to the geometric flow generated by the extended sine-Gordon system under the restriction

$$q_s^2 + \rho^2 - c^2q^2 = 0.$$

We can check that the Darboux transformation preserves this integral, i.e.,

$$q_{1s}^2 + \rho_1^2 - c^2q_1^2 = q_s^2 + \rho^2 - c^2q^2 = C, \qquad (C \text{ constant}).$$

If we use the representation from before, we have

$$q_1 = \left(\frac{c^2 + \lambda_0^2 - 2c\lambda_0 \cos(\phi + \theta)}{c^2 - \lambda_0^2}\right) F(t) \exp\left(c \int \cos\phi \, ds\right),$$

$$\rho_1 = \left(\frac{2c\lambda_0 \sin\theta - \lambda_0^2 \sin(\phi + 2\theta) + c^2 \sin\phi}{c^2 - \lambda_0^2}\right) cF(t) \exp\left(c \int \cos\phi \, ds\right)$$

Now that we have the Darboux transformation, we can use the Sym-Tafel formula to find the corresponding Bäcklund transformation:

$$\mathbf{r}_1 = \Phi_1^{-1}\Phi_{1\lambda}\Big|_{\lambda=-1/a} = \Phi^{-1}D^{-1}(D\Phi_\lambda + D_\lambda\Phi)\Big|_{\lambda=-1/a} = \left(\Phi^{-1}\Phi_\lambda + \Phi^{-1}D^{-1}D_\lambda\Phi\right)\Big|_{\lambda=-1/a}$$

$$= \Phi^{-1}\Phi_\lambda\Big|_{\lambda=-1/a} - \frac{a^2\lambda_0}{1 + a^2\lambda_0^2}\,\Phi^{-1}\begin{pmatrix} i\cos\theta & -i\sin\theta \\ -i\sin\theta & -i\cos\theta \end{pmatrix}\Phi\Big|_{\lambda=-1/a}$$

$$= \mathbf{r} - \frac{a^2\lambda_0}{1 + a^2\lambda_0^2}(\cos\theta\,\mathbf{T} - \sin\theta\,\mathbf{N}).$$

5.8 Soliton Surface Examples

If we begin with a simple "seed" solution to the extended sine-Gordon system, we can generate some non-trivial examples of (IF) surfaces. The advantage of the Bäcklund transformation is that, given an initial solution, it can be iterated *algebraically*. If we have constructed a new solution $\Phi_1 = D_1\Phi$ via the Darboux transformation procedure, we may then take a vector solution w for Φ_1 and iterate the process to construct D_2 and perform a second Darboux transformation $\Phi_2 = D_2\Phi_1 = D_2D_1\Phi$. Cieslinski has formulas for the single matrix that computes the nth transformation [Cie91]. In the following example we compute 1st and 2nd Bäcklund transformations of an initial (IF) surface to compute new (IF) surfaces. We choose a solution such that both U and V of (5.4) are constant matrices. In particular, we take

$$\kappa = -\frac{b}{a}, \qquad q = \frac{b^2+1}{ab}, \qquad \rho = \frac{b^2+1}{a^2}, \qquad c = \frac{b}{a} \qquad (b \text{ constant}),$$

which is a solution to the extended sine-Gordon and the constraint $q_s^2 + \rho^2 - c^2 q^2 = 0$. Since U and V are constant, we can compute the solution

$$\Phi = e^U e^V$$

and use the Sym-Tafel formula to compute the surface

$$\mathbf{r} = \Phi^{-1}\Phi_\lambda\Big|_{\lambda=-1/a}. \tag{5.15}$$

This solution here is a surface parameterized by helices (and helices have constant torsion and are Bertrand curves). We work with a particular example by setting

$$b = -\frac{3}{4}, \qquad a = 1, \qquad c = \frac{3}{4},$$

which gives us

$$\mathbf{r} = \begin{bmatrix} \frac{18}{125}\ \sin\left(\frac{5}{4}s - \frac{5}{4}t\right) + \frac{8}{25}\,t + \frac{8}{25}\,s \\ -\frac{6}{25}\ \cos\left(\frac{5}{4}s - \frac{5}{4}t\right) + \frac{6}{25} \\ \frac{24}{125}\ \sin\left(\frac{5}{4}s - \frac{5}{4}t\right) - \frac{6}{25}\,t - \frac{6}{25}\,s \end{bmatrix}.$$

(See Figure 5.1).

For the 1st Bäcklund transformation we choose the vector solution $w = \Phi[1,0]^T$ and $\lambda_1 = 3/5$ to compute the Darboux matrix D_1 with formula (5.10). Using the Sym-Tafel formula, we arrive at the surface in Figure 5.2.

The method (set down by Cieslinski) for constructing the product of Darboux matrices $D := D_2 D_1$ is as follows: Take vector solutions w_1 with $\lambda = i\lambda_1$ and w_2 with $\lambda = i\lambda_2$. (We use the special case of Cieslinski's formulas where each initial solution corresponds to λ pure imaginary). Now construct

$$\begin{aligned} C_{jk} &:= -(\lambda_j + \lambda_k) \qquad (\lambda_0 := i\lambda) \\ \Pi_{jk} &:= w_j w_k^T \\ A_{jk} &:= \frac{w_j^T w_k}{C_{kj}} \end{aligned} \tag{5.16}$$

and define A to be the matrix with entries A_{ij}. Then define $B := A^{-1}$ where we will denote the entries of B by B_{jk}. The Darboux matrix is then defined by

$$D := \left(\prod_{k=1}^{2} \frac{\lambda + i\lambda_k}{\sqrt{\lambda^2 + \lambda_k^2}}\right)\left(I - \sum_{j=1}^{2}\sum_{k=1}^{2} \frac{B_{kj}\Pi_{kj}}{C_{0j}}\right). \tag{5.17}$$

Choose parameters b, a, c as we did above, and let $\lambda_1 = 3/5$ and $\lambda_2 = 9/20$. Multiply $D\Phi$ and use the Sym-Tafel formula to compute the 2nd Bäcklund Transformation (see Figure 5.3). We graph an example of a constant-torsion x-parameter curve and an example of a Bertrand t-parameter curve (pictured in Figures 5.4 and 5.5, respectively). Additional formulas for these examples of 1st and 2nd Bäcklund transformations can be found in the appendix.

Figure 5.1: Seed surface: cylinder parameterized by helices

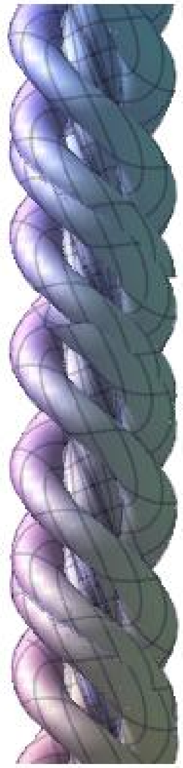


Figure 5.2: (IF) surface generated by Bäcklund transformation

Figure 5.3: (IF) surface generated by double Bäcklund transformation

Figure 5.4: Constant-torsion s-parameter curve on 2nd Bäcklund transformation

Figure 5.5: Bertrand t-parameter curve on 2nd Bäcklund transformation

Bibliography

[B83] A. V. Bäcklund. Om ytor med konstant negativ krökning. *Lunds Universitets Årsskrift*, 1883.

[Bia85] L. Bianchi. Sopra i sistemi tripli ortogonali di Weingarten. *Ann. Matem.*, 1885.

[Bou62] E. Bour. Théorie de la déformation des surfaces. *J. l'École Imperiale Polytech.*, 1862.

[CI98] Annalisa Calini and Thomas Ivey. Bäcklund transformations and knots of constant torsion. *Journal of Knot Theory and Its Ramifications*, 1998.

[Cie91] Jan Cieśliński. An effective method to compute N-fold Darboux matrix and N-soliton surfaces. *Journal of mathematical physics*, 1991.

[dC76] M. P. do Carmo. *Differential geometry of curves and surfaces*. Prentice-hall, 1976.

[GLL77] Jr. G. L. Lamb. Solitons on moving space curves. *J. Math. Phys.*, 1977.

[Has72] H. Hasimoto. A soliton on a vortex filament. *Journal of Fluid Mechanics*, 1972.

[Lan99] Joel Langer. Recursion In Curve Geometry. *New York Journal of Mathematics*, 1999.

[Lip69] M. Lipschultz. *Schaum's Outline of Theory and Problems of Differential Geometry*. McGraw-Hill, 1969.

[LP91] Joel Langer and Ron Perline. Poisson Geometry of the Filament Equation. *Journal of Nonlinear Science*, 1991.

[RS02] C. Rogers and W. K. Schief. *Bäcklund and Darboux transformations: geometry and modern applications in soliton theory*. Cambridge University Press, 2002.

[Sch03] W. K. Schief. On the integrability of Bertrand curves and Razzaboni surfaces. *Journal of Geometry and Physics*, 2003.

[SR99] W. K. Schief and C. Rogers. Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 1999.

[Str88] Dirk J. Struik. *Lectures on Classical Differential Geometry: Second Edition*. Dover, 1988.

[Sym85] A. Sym. Soliton surfaces and their applications. *Lect. Notes Phys.*, 1985.

Appendix A

Soliton Surface Formulas

First Bäcklund Transformation:

$$\mathbf{r} + d_1 \cdot [x_1, x_2, x_3]$$

where

$$s_1 = \frac{1}{20} s, \qquad t_1 = \frac{5}{36} t$$

$$d_1 = \frac{3}{170} \left(-200 \cos(-18 s_1 + 50 t_1) + 800 \cos(-9 s_1 + 25 t_1) - 681\right)^{-1}$$

$$\begin{aligned} x_1 = {} & 2400 \sin(-18 s_1 + 50 t_1) - 2025 \cos(16 s_1 + 16 t_1) - 4800 \sin(-9 s_1 + 25 t_1) \\ & -3600 \cos(-9 s_1 + 25 t_1) - 8550 \sin(-25 s_1 + 9 t_1) - 1200 \sin(-43 s_1 + 59 t_1) \\ & +2550 \sin(7 s_1 + 41 t_1) + 1296 \cos(-25 s_1 + 9 t_1) + 4800 \sin(-34 s_1 + 34 t_1) \\ & -7500 \sin(16 s_1 + 16 t_1) + 2304 \end{aligned}$$

$$\begin{aligned} x_2 = {} & -2000 \cos(-43 s_1 + 59 t_1) - 4250 \cos(7 s_1 + 41 t_1) + 8000 \cos(-34 s_1 + 34 t_1) \\ & -14250 \cos(-25 s_1 + 9 t_1) - 2160 \sin(-25 s_1 + 9 t_1) + 12500 \cos(16 s_1 + 16 t_1) \\ & -3375 \sin(16 s_1 + 16 t_1) \end{aligned}$$

$$\begin{aligned} x_3 = {} & -1800 \sin(-18 s_1 + 50 t_1) - 2700 \cos(16 s_1 + 16 t_1) + 3600 \sin(-9 s_1 + 25 t_1) \\ & +2700 \cos(-9 s_1 + 25 t_1) - 11400 \sin(-25 s_1 + 9 t_1) - 1600 \sin(-43 s_1 + 59 t_1) \\ & +3400 \sin(7 s_1 + 41 t_1) + 1728 \cos(-25 s_1 + 9 t_1) + 6400 \sin(-34 s_1 + 34 t_1) \\ & -10000 \sin(16 s_1 + 16 t_1) - 1728 \end{aligned}$$

Second Bäcklund transformation:

We choose

$$a = 1, \qquad \lambda_1 = \frac{3}{5}, \qquad \lambda_2 = \frac{9}{20}.$$

Then we compute

$$w_1 = \begin{bmatrix} \frac{4}{3} \sin(4\,t_1 - 3\,s_1) + \cos(4\,t_1 - 3\,s_1) \\ \frac{5}{3} \sin(4\,t_1 - 3\,s_1) \end{bmatrix}$$

$$w_2 = \begin{bmatrix} \frac{3}{4} \sin(-4\,s_1 + 3\,t_1) + \cos(-4\,s_1 + 3\,t_1) \\ \frac{5}{4} \sin(-4\,s_1 + 3\,t_1) \end{bmatrix}$$

where

$$s_1 = \frac{3}{40}s, \qquad t_1 = \frac{125}{288}t.$$

$$A_{11} = \frac{40}{27} \cos(-6\,s_1 + 8\,t_1) - \frac{125}{54} - \frac{10}{9} \sin(-6\,s_1 + 8\,t_1)$$

$$A_{12} = -\frac{125}{126} \sin(-7\,s_1 + 7\,t_1) - \frac{5}{18} \sin(s_1 + t_1) - \frac{35}{18} \cos(s_1 + t_1) + \frac{125}{126} \cos(-7\,s_1 + 7\,t_1)$$

$$A_{21} = -\frac{125}{126} \sin(-7\,s_1 + 7\,t_1) - \frac{5}{18} \sin(s_1 + t_1) - \frac{35}{18} \cos(s_1 + t_1) + \frac{125}{126} \cos(-7\,s_1 + 7\,t_1)$$

$$A_{22} = \frac{5}{8} \cos(-8\,s_1 + 6\,t_1) - \frac{125}{72} - \frac{5}{6} \sin(-8\,s_1 + 6\,t_1)$$

Then compute:

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}$$

We then use 5.16 and 5.17 to construct the Darboux matrix D. The surface is then computed via 5.15.

Vita

Jonah Smith graduated from Drexel University with a B.S. in Mathematics in 2008. He then began a Ph.D degree in Mathematics at Drexel University under the supervision of Dr. Ronald Perline. His research is in the area of integrable geometry, specifically a new class of integrable surfaces related to Bertrand curves.