Matching with Restricted Trade

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Abstract

Motivated by various trade restrictions in real-life object allocation problems, we introduce an object allocation with a particular class of trade restrictions model. The set of matchings that can occur through a market-like process under such restrictions is defined, and each such matching is called feasible. We then introduce a class of mechanisms, which we refer to as "Restricted Trading Cycles" (RTC). Any RTC mechanism is feasible, constrained efficient, and respects endowments. An axiomatic characterization of RTC is obtained, with feasibility, constrained efficiency, and a new property that we call hierarchically mutual best. In terms of strategic issues, feasibility, constrained efficiency, and respecting endowments together turns out to be incompatible with strategy-proofness. This in particular implies that no RTC mechanism is strategy-proof. Lastly, we consider a probabilistically restricted trading cycles ($PRTC$) mechanism, which is obtained by introducing a certain randomness to the RTC class. While $PRTC$ continues to be manipulable, compared to RTC, it is more robust to truncations and reshufflings.

JEL classification: C78, D78.

Keywords: Restricted Trading Cycles, Trade Restrictions, Matching, Feasibility, Characterization.

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†I am grateful to the associate editor and the anonymous referees for their comments and suggestions. I thank Utku Ünver for his comments.

1 Introduction

There are sets of agents and objects, which are to be distributed among the agents. Objects can come with multiple copies, and the null-object, which represents receiving no object, is not scarce. Agents have strict preferences over the objects. Each agent is to receive only one object, and there is no medium of exchange, such as money. Objects can be collectively owned, or some (or all) can be owned by particular agents. Real-life examples include public housing allocations, dorm assignments, school choice, and office assignments.

This problem has been well studied in the literature, and one of the main desiderata in the solution has been efficiency, which guarantees that no agent can be assigned to one of his better objects without harming someone else. An important class of efficient mechanisms is based on a market-like process, where agents trade objects with each other as if they own them. In other words, a way to obtain an efficient allocation is to endow the agents with the objects, and then to let them trade. The well-known top trading cycles (attributed to David Gale by Shapley and Scarf (1974); henceforth, TTC), and Papai (2000)'s "hierarchical exchange" are such trading-based mechanisms. However, all such well-studied trading mechanisms work under the supposition of free trade in the sense that no trade pattern is banned. However, in many real-life situations, there are trade restrictions that limit which objects an agent can receive from each other agent. Specifically, under such restrictions, an agent may be allowed to receive an object from someone, but not from someone else. For instance, in course allocations, a student may receive a course from someone for whom the course is elective, but not from someone else for whom the same course is compulsory. Likewise, in school choice, a student may be placed at a school through receiving a non-sibling priority of someone, but not through receiving a sibling priority of some other student. In some public housing assignments, there is a cap on the number of possible house-change. Thus, an agent may receive a house from someone, but he may not be allowed to receive the same type of house from some other if the latter is not allowed to change his house because of the cap.

Motivated by such trade limitations, we incorporate a wide class of trade restrictions into an object allocation problem. More specifically, for each agent i and object c, we introduce a correspondence that specifies the set of agents from whom agent i can receive object c in any trade arrangement. Whenever each of such sets includes all agents, the problem is with free trade. At the other extreme, where each of such sets includes only the associated agent, no agent can receive an object from someone else; hence, no trade would occur. We refer to the collection of those correspondences as "*trade restrictions*."

Because we focus on market-like trading mechanisms, each object is assumed to have a hierarchical endowment structure, which specifies who are initially endowed with the object and who inherit it after its owners receive their object assignment and leave the problem. Our hierarchical endowments structures are a subclass of control right structures of Pycia and Ünver (2011).[1] For ease of exposition, in the main body, we assume that the objects are not privately owned. However, as we mention in Remark 4, with a natural restriction on the hierarchical endowment structures, we can easily adapt our results to the private ownership case as well as to the hybrid ownership case where some, but not all, objects are privately owned.

After formulating the problem, we introduce a sequential trading mechanism that gives us a set of matchings that can occur under the trade restrictions. It is such that, in each step, either a group of agents form a trading cycle and trade their (inherited) endowments, or an agent is assigned to the null-object. Each such agent leaves the problem with his object. The remaining agents inherit the leftover object copies, as governed by the hierarchical inheritance structures. The trade restrictions are incorporated by requiring that an agent can receive someone else's object in a trading cycle only if the former is allowed to receive that object from the latter. Because, in each step, there can be multiple trade possibilities, the sequential trading mechanism produces a set of matchings, each of which is associated

[1]As we allow for multiple-copy objects, our hierarchical endowment structures are slightly more general than those of Papai (2000), and moreover, they do not belong to the family of control right structures of Pycia and Ünver (2017).

with a different trade selection rule. We say that a matching is *feasible* if it can be obtained through the sequential trading mechanism. A mechanism is feasible if it produces a feasible matching for every problem.[2]

Next, we introduce a class of mechanisms that we call "*Restricted Trading Cycles*" (*RTC*). Each *RTC* mechanism is twofold. The first step iteratively matches agents with their top choices once they are endowed with them. Then, in the second step, a certain collection of trading cycles is implemented, while some agents possibly receiving the null-object. While each *RTC* mechanism yields the same assignments in the first stage, the second-stage assignments may differ across *RTC* mechanisms, owing to the possible multiplicity of trading cycle collections. Because of this, each *RTC* mechanism is associated with a different selection rule in its second stage.

We first show that each *RTC* mechanism is feasible. Then, we study the efficiency properties of the *RTC* class. A matching *dominates* another matching if all the agents at least weakly prefer the former to the latter, with this holding strictly for someone. Matching is *efficient* if it is not dominated. Because feasibility entails trade restrictions, an efficient and feasible matching does not always exist. Therefore, as the second-best solution, we say that a matching is *constrained efficient* if it is not dominated by a feasible matching. We show that any *RTC* mechanism is constrained efficient.

To have a better understanding of the *RTC* class, we provide an axiomatic characterization. We say that a matching satisfies *hierarchically mutual best* if any agent receives his top object if he is endowed with it, and this holds iteratively in the reduced problems after such agents receive their assignments and leave the problem. A mechanism satisfies hierarchically mutual best if its outcome satisfies it at every problem. We show that a mechanism is feasible, constrained efficient, and satisfies hierarchically mutual best if and only if it is a *RTC* mechanism.

[2]A mechanism is a function that produces a matching for any problem.

In the final set of results, we study the strategic properties of the RTC class. First, there is no *strategy-proof* mechanism[3] that is feasible, constrained efficient, and that satisfies *respecting endowments*, a desirable property that is weaker than hierarchically mutual best. This implies that any RTC mechanism is vulnerable to preference manipulations. Finally, we let trading cycle selections be uniformly random in the course of RTC. This defines a random mechanism that we call "probabilistically restricted trading cycles" ($PRTC$). While this randomization does not give us a strategy-proof mechanism, it increases the robustness of RTC to certain misreportings.

2 Literature Review

There is a vast body of literature on object allocation problems. Shapley and Scarf (1974) were the first to study a housing market problem where each agent brings his owned house to the market. A variant of this problem, the so-called housing allocation problem, where houses are collectively owned, was first studied by Hylland and Zeckhauser (1979). Abdulkadiroglu and Sönmez (1999) then introduced a more general housing allocation with existing tenants model, where tenants have the right to keep occupied houses.

Papai (2000) introduces hierarchical endowment structures in a unit-copy object allocation problem. She defines a class of hierarchical exchange rules where agents sequentially trade their endowments, and the remaining objects are inherited by the unassigned agents, as governed by the hierarchical endowment structures. Papai (2000) characterizes this mechanism class by group strategy-proofness, efficiency, and reallocation-proofness. Because we allow for multiple copies, our hierarchical endowment structures are slightly more general than hers. In the same setting as Papai (2000), Pycia and Ünver (2017) introduce control right structures that allow agents to be an owner or a broker of an object. They introduce a class of trading cycles mechanisms based on these control right structures and obtain

[3] A mechanism is strategy-proof if no agent can ever profitably misreport his preferences.

that group strategy-proofness and efficiency fully characterize this class.[4] Pycia and Ünver (2011) generalize Pycia and Ünver (2017)'s trading cycles mechanisms to multi-copy object environments.

In contrast to the aforementioned studies, the current study limits trading possibilities. There are some other studies that incorporate various trade restrictions in different models. Pycia (2016) studies a housing market problem with a given network that determines the set of objects each agent can receive from every other agent. This is a unit-copy object study where each agent possesses one object; hence there is no inheritance unlike the present study. Moreover, the trade restrictions under a network are a special case of ours (see Remark 2 for a formal discussion). Papai (2007) considers a general multiple-demand housing market problem, where agents may own more than one house. She introduces the so-called fixed deal exchange market, where possible trades are completely determined a priori. In the same setting as Papai (2007), Papai (2015) imposes responsiveness on preferences in the hope of obtaining a desirable mechanism under more relaxed trade restrictions. There are important differences between the current and these two studies. First, the settings differ in that, as opposed to the current study, both Papai (2007) and Papai (2015) consider only the private ownership case, and agents are allowed to receive multiple objects; hence there is no inheritance in them. Moreover, as we will formally discuss in the model, the respective trade restrictions are different as well.

Another recent study that restricts trades is Dur and Morrill (2015). In a school-choice framework, they introduce restricted priorities that are not allowed to be traded. They incorporate these priorities into the school-choice problem through a property, and then show that this property entails a series of impossibilities. While our approach and theirs to incorporating the restrictions into the respective trading mechanisms are the same, as we discuss formally in the model, the considered trade restrictions and the pursued line of

[4] They also show that in the presence of property rights, a subclass of trading cycles mechanisms, where the associated control right structures reflect the property rights, coincides with the set of group strategy-proof, efficient, and individually rational mechanisms.

results are quite different.

Some well-known trading-based mechanisms have been characterized in the literature. Roth and Postlewaite (1977) show that whenever agents have strict preferences, TTC is the unique core mechanism. Ma (1994) and Svensson (1999) prove that TTC is the unique efficient, strategy-proof, and individually rational mechanism in a housing market. In housing allocation frameworks, TTC has been characterized by others as well, including Abdulkadiroglu and Che (2010), Morrill (2013b), and Dur (2013). Sönmez and Ünver (2010) axiomatize Abdulkadiroglu and Sönmez (1999)'s "you-request-my-house-I-get-your-turn" mechanism. Abdulkadiroglu and Sönmez (1998) obtain that the core from random endowments mechanism is the same as random serial dictatorship.

3 The Model and Results

An object allocation with trade restrictions problem is a tuple $(N, O, P, q, \Gamma, \tau)$. The sets of agents and objects are N and O, respectively. Each agent i has a **strict preference ordering** P_i over O. We write aR_ib only if either aP_ib or $a = b$ ("at-least-as-good-as" relation). The preference profile of the agents is $P = (P_i)_{i \in N}$. For $N' \subset N$, let $P_{N'} = (P_i)_{i \in N'}$. Each object c has a capacity of q_c, and $q = (q_c)_{c \in O}$. The null-object, which is denoted by $\emptyset$ and representing receiving no object, is not scarce. That is, $q_\emptyset = |N|$. Object c is **acceptable** to agent i if $cR_i\emptyset$. Otherwise, it is **unacceptable**. The **hierarchical endowment structure** profile of the objects is $\Gamma = (\Gamma_c)_{c \in O}$, and its formal definition is given later.

The new component is $\tau = (\tau_i)_{i \in N}$. It reflects the trade restrictions as follows. For agent i, $\tau_i : O \rightrightarrows N$ is a correspondence such that for each object c, $\tau_i(c)$ is the set of agents from whom agent i can demand object c. For instance, if $j \in \tau_i(c)$, then agent i can approach (or, in terms of TTC words, "point to") agent j to trade for object c (assuming that agent j is endowed with object c). We assume that for any agent i and object c, $i \in \tau_i(c)$, which means that agent i can obtain object c whenever he is endowed with it. If $\tau_i(c) = \{i\}$, then this means that agent i cannot demand object c from anyone except himself. The other extreme, $\tau_i(c) = N$, means that he can demand object c from anyone.[5] We refer to τ as **trade restrictions profile**. In the rest of the paper, we suppress all the elements but the preferences from the problem notation and simply write P to denote the problem.

For any $N' \subseteq N$, a **submatching** $\mu^{N'}$ is an assignment of the objects to the agents N' such that each agent $i \in N'$ receives one object (possibly the null-object), no agent $j \in N \backslash N'$ receives an object, and every object is assigned to as many agents as up to its capacity. Whenever $N' = \emptyset$, $\mu^{\emptyset}$ is a submatching where no agent is assigned an object. A submatching $\mu^{N'}$ is a **matching** whenever $N' = N$; and in this case, we suppress N from the notation and simply write μ. For a given submatching $\mu^{N'}$ and an agent or an object $k \in N' \cup O$, we write $\mu_k^{N'}$ for the assignment of k. For any pair of group of agents N', N'', and submatchings $\mu^{N'}$ and $\mu^{N''}$, we write $\mu^{N'} \subseteq \mu^{N''}$ if (i) $N' \subseteq N''$, and (ii) for any agent $k \in N'$, $\mu_k^{N'} = \mu_k^{N''}$. Let $\wp$ and $\mathcal{M}$ be the set of all submatchings and matchings, respectively. By definition, $\mathcal{M} \subset \wp$. For any $N' \subset N$ and submatching $\mu^{N'}$, let $\wp(\mu^{N'}) = \{\mu^{N''} \in \wp : \ \mu^{N'} \subseteq \mu^{N''}\}$.[6]

We are now ready to formally define Γ. For any object c, $\Gamma_c : \wp \to 2^N$ is a function such that for any $\mu^{N'} \in \wp$,

$$\Gamma_c(\mu^{N'}) \in \{\hat{N} \subseteq N \setminus N' : \ |\hat{N}| = \min\{|N \setminus N'|, q_c - |\mu_c^{N'}|\}\}.$$

In words, $\Gamma_c(\mu^{N'})$ tells us who inherit object c after the agents in N' receive their assignments at $\mu^{N'}$. We assume that for any pair submatchings $\mu^{N'}$ and $\mu^{N''}$ where $\mu^{N'} \subseteq \mu^{N''}$, if $q_c - |\mu_c^{N''}| \geq |\Gamma_c(\mu^{N'}) \setminus N''|$, then $\Gamma_c(\mu^{N'}) \setminus N'' \subseteq \Gamma_c(\mu^{N''})$. This assumption guarantees that the inheritances are preserved as long as the associated agents and objects are not assigned.[7]

[5] This class of trade restrictions includes a more specific one that specifies which objects an agent can demand from someone else. For instance, if $\tau_i(c) = \{i\}$ for any $c \in O' \subseteq O$, then it means that agent i cannot demand any object in O' from someone else. Hence, he cannot receive any of them unless he is endowed with them. On the other hand, the set of objects that agent i can demand from someone else is given by $\{c \in O : \ \tau_i(c) \cap (N \setminus \{i\}) \neq \emptyset\}$.

[6] By construction, $\mu^{N'} \in \wp(\mu^{N'})$.

[7] A similar assumption is used by Pycia and Ünver (2017), Papai (2000), and Pycia and Ünver (2011). The former refers to it as consistency.

A Specific Class of Hierarchical Endowment Structures: Objects may have a strict ranking over agents, and their hierarchical endowment structures may be induced by them in a natural way. Let $\succ_c$ be object c's ranking order over agents. Below defines Γ_c that is induced by $\succ_c$. For any $N' \subseteq N$ and submatching $\mu^{N'}$:

$$\Gamma_c(\mu^{N'}) = \{\hat{N} \subseteq N \setminus N' : \hat{N} \text{ consists of the best ranked agent group (with respect to } \succ_c) \\ \text{among } N \setminus N' \text{ of size } \min\{|N \setminus N'|, q_c - |\mu_c^{N'}|\}\}.$$

In the proofs of the negative results, we will make use of this ranking induced hierarchical endowment structures. For each agent i and submatching $\mu^{N'}$, $I_i(\mu^{N'}) = \{c \in O : \ i \in \Gamma_c(\mu^{N'})\}$. Note that $I_i(\mu^{N'}) = \emptyset$ for any $i \in N'$, and the null-object always belongs to $I_i(\mu^{N'})$ for any $i \in N \setminus N'$. Moreover, $I_i(\mu^{\emptyset})$ gives the initial endowments of agent i.

In the followings, we will describe the matchings that can take place under the trade restrictions τ.

Definition 1. *A cycle is a collection of distinct objects and agents $C = \{c_1, i_1, c_2, .., c_n, i_n\}$ such that for any $k = 1, .., n$,*

(i) $c_k \neq \emptyset$,

(ii) $i_{k+1} \in \tau_{i_k}(c_{k+1})$, *with* $i_{n+1} = i_1$ *and* $c_{n+1} = c_1$.

Cycles incorporate the trade restrictions through requiring that agent i_k be allowed to demand object c_{k+1} from agent i_{k+1} (Condition *(ii)*).

Definition 2. *A cycle $C = \{c_1, i_1, c_2, .., c_n, i_n\}$ is viable after a submatching $\mu^{N'}$ if for any $k = 1, .., n$,*

(i) $i_k \in N \setminus N'$,

(ii) $i_k \in \Gamma_{c_k}(\mu^{N'})$.

For a cycle C and $i_k \in C$, let $C(i_k) = c_{k+1}$. A **null-pair** consists of an agent i and the null-object, and we write $\Lambda^i = (i, \emptyset)$ to denote it. The implementation of a cycle C yields a submatching that places each agent $i \in C$ at $C(i)$. Similarly, the implementation of Λ^i assigns agent i to the null-object.

Below defines a sequential trading rule, which mimics a market-like process where agents trade their endowments in a sequential fashion. Given problem P,

Step 1. Either pick a cycle C that is viable after $\mu^{\emptyset}$ (if any; and in the case of multiplicity, pick anyone) or select a null-pair and implement it. Let N^1 be the set of assigned agents in this step, and μ^{N^1} be the obtained submatching. If $N^1 = N$, then stop here and the final matching is μ^{N^1}. Otherwise, move to the next step.

In general,

Step k. Let N' and $\mu^{N'}$ be the set of all assigned agents in the previous steps and the obtained submatching, respectively. Either pick a cycle C that is viable after $\mu^{N'}$ (if any; and in the case of multiplicity, pick anyone) or select a null-pair that consists of an unassigned agent and implement it. Let N^k be the set of assigned agents in this step, and μ^{N^k} be the obtained submatching. If $N^k \cup N' = N$, then stop here, and the final matching is μ that is defined by the assignments in the previous steps and the current step. Otherwise, move to the next step.

In the sequential trading rule, at least one agent is assigned to some object in each step, therefore it terminates in a finitely many round. Because the null-object always belongs to the endowment sets of the agents, the sequential trading rule allows any agent to match with the null-object. Note that each null-pair implementation may create new viable cycles through affecting the endowments of the remaining agents. Therefore, the sequential implementation of null-pairs enriches the set of matchings that can emerge through the sequential trading rule.[8]

Proposition 1. *The sequential trading rule produces a non-empty set of matchings, each of which is associated with a different rule of selecting viable cycle and null-pair.*

[8]That is, the set of matchings that would be obtained through a sequential trading rule where null-pairs are implemented simultaneously would be a proper subset of what is obtained through our sequential trading rule above.

Proof. In each step of the sequential trading rule, at least one agent is assigned to some object (possibly the null-object), and the assigned agents leave the problem with their assignments. Hence, in the end, no agent receives more than one object.

It is immediate to see that Step 1 defines a submatching. Let us consider Step k, and write $\mu^{N'}$ for the obtained submatching by the end of previous Step $k-1$. For any object c, $\Gamma_c(\mu^{N'}) \subseteq N \setminus N'$ and $|\Gamma_c(\mu^{N'})| = \min\{|N \setminus N'|, q_c - |\mu_c^{N'}|\}$. This, along with the definition of viable cycles, implies that an agent can receive object c within Step k as long as there is a leftover object c copy. Hence, Step k assignments and those under $\mu^{N'}$ together defines a submatching. These arguments are independent of viable cycles and null-pair selection; hence the sequential trading rule produces a non-empty set of matchings, each of which is associated with a different rule of selecting viable cycle and null-pair. □

Let Ω be the set of matchings that can be obtained through the sequential trading rule. We say that a matching μ is **feasible** if $\mu \in \Omega$.

Remark 1. Dur and Morrill (2015), in a school choice framework, introduce restricted priorities. They limit priority trades through a "limiting trade" property that says the following: a matching limits trade whenever k many students with restricted priorities at a school c are assigned to schools different than school c, each of the top q_c+k priority students at school c cannot receive a worse assignment than school c. Our trade restrictions are more general than theirs in that their restrictions correspond to a particular trade restriction profile τ. Specifically, if a student i has a restricted priority at a school c, then by assuming $i \notin \tau_j(c)$ for any $j \in N \setminus \{i\}$, we are able to incorporate their trade bans in our formulation. However, in general, τ can change across both agents and objects in our model, allowing for $i \in \tau_j(c)$ for some j even though $i \notin \tau_k(c)$ for some other agent k, which is not the case in Dur and Morrill (2015). Another important difference between the formulations is that any trade is allowed under their limiting trade property unless restricted priorities are traded at the expense of certain groups of agents. In other words, trade restrictions are preference based, thereby we can deem them as "soft" constraints. In contrast, we consider our trade bans to be satisfied irrespective of anything else, in other words, they are "hard" constraints.

Remark 2. In a housing market setting where agents own multiple houses, Papai (2007) considers trade restrictions, which are more stringent than ours. In Papai (2007), every possible trade deal is specified a priori, which allows for the full control over trades. More specifically, once a group of agents agree on a trade, then the trade terms (who receive which objects) are fixed a priori, hence non-negotiable. However, in the current study, the same parties can agree on different trade terms. In another related paper, Papai (2015) studies segmented markets in the same setting as Papai (2007). Each object belongs to a certain market segment, and each agent owns a single object from each market segment. Trades across different market segments are precluded, but agents can trade within market segments without any limitation. Neither of the trade restriction classes of the current study and those of Papai (2015) is a special case of the other. For instance, in a two-agent problem, the current setting can preclude any trade. However, it is not the case in Papai (2015) (because each agent owns one object in each market segment, and there is no trade restriction within each market segment). Similarly, again in a two-agent problem, any allowed trade in our study can be carried out irrespective of market segment (because there is no market segment aspect of our trade restrictions), whereas this may not be possible in Papai (2015). In a housing market setting, Pycia (2016) introduces trade restrictions through networks, which limit trades by allowing an agent i to receive an object a from its owner j only if there is a directed link from agent j to i in the given network. His class of restrictions is a special case of ours as by letting $j \in \tau_i(a)$ only if there is such a link, we can have his restrictions. However, the converse is not possible as both our setting and the class of trade restrictions are more general.

Matching μ is **non-wasteful** if for any object c such that $cP_i\mu_i$ for some $i \in N$, $|\mu_c| = q_c$. Matching μ **respects endowments** if for any agent i and object $c \in I_i(\mu^\emptyset)$, $\mu_i R_i c$.[9] Matching μ **dominates** another matching μ' if for any agent $i \in N$, $\mu_i R_i \mu'_i$, with strictly

[9]In the case of private ownership, we will refer to this condition as "individual rationality."

holding for someone. Matching μ is **efficient** if it is not dominated.

Proposition 2. *There does not always exist a feasible and non-wasteful matching.*

Proof. Let $N = \{i, j\}$ and $O = \{a, b\} \cup \{\emptyset\}$, with $q_a = q_b = 1$. Let the preferences and objects' rank orders be as follows:

$$P_i : a, \emptyset, b;\ P_j : b, \emptyset, a.$$ [10]

$$\succ_a: j, i;\ \succ_b:\ i, j.$$

Γ is induced by the rank orders above. The trade restrictions profile τ is such that $\tau_i(a) = \{i\}$. The only non-wasteful matching μ is such that $\mu_i = a$ and $\mu_j = b$. However, it is not feasible (it is easy to verify that any other matching is feasible). □

Efficiency implies non-wastefulness. This, along with Proposition 2, implies the incompatibility of efficiency and feasibility as well.

Corollary 1. *There does not always exist a feasible and efficient matching.*

Matching μ is **constrained non-wasteful** if, for any agent-object pair (i, c) such that $cP_i\mu_i$ and $|\mu_c| < q_c$, the matching that gives object c to agent i while keeping everyone else's assignment the same as under μ is not feasible. Note that constrained non-wastefulness implies that $\mu_i R_i \emptyset$ for any $i \in N$. Matching μ is **constrained efficient** if it is not dominated by a feasible matching. Constrained efficiency implies constrained non-wastefulness whereas the converse is easily not true.

A **mechanism** ψ is a systematic way that produces a matching for every problem P. We write $\psi(P)$ to denote its outcome in problem P. Mechanism ψ is < *feasible, constrained non-wasteful, constrained efficient*> if, for every problem P, $\psi(P)$ is <feasible, constrained non-wasteful, constrained efficient>. Mechanism ψ *respects endowments* if, for each problem P, $\psi(P)$ respects endowments.

[10] The objects are written in decreasing order of the preferences. For instance, object a is the top object of agent i, then the null-object and object b respectively come. This way of writing is used in the object rank orders as well.

3.1 A Class of Feasible Mechanisms: Restricted Trading Cycles

In this section, we propose a class of feasible mechanisms. Let us first introduce some notations. Consider a collection of cycles and null-pairs $\Upsilon = \{C_1, .., C_n, \Lambda^{n+1}, .., \Lambda^{n+m}\}$ and a permutation $\pi : \Upsilon \to \{1, .., n+m\}$. For any $k \in \{1, .., n+m\}$, let $A_k = \{C_\ell \in \Upsilon : \ \pi(C_\ell) < k\} \cup \{\Lambda^\ell \in \Upsilon : \ \pi(\Lambda^\ell) < k\}$. Moreover, $N_k = \{i \in N : \ i \in C$ for some $C \in A_k$ or $\Lambda^i \in A_k\}$.

Definition 3. *Given a problem P, a collection of cycles and null-pairs $\Upsilon = \{C_1, .., C_n, \Lambda^{n+1}, .., \Lambda^{n+m}\}$ is implementable if there exists a permutation $\pi : \Upsilon \to \{1, .., n+m\}$ such that the followings are satisfied.*

(i) Each agent $i \in N$ appears either in only one cycle or in only one null-pair in Υ.

(ii) For any $k \in \{1, .., n\}$ and $C_k \in \Upsilon$, if $\mu^{N_{\pi(C_k)}}$ is the submatching obtained by implementing the cycles and the null-pairs in $A_{\pi(C_k)}$, then C_k constitutes a viable cycle after $\mu^{N_{\pi(C_k)}}$.

(iii) For each agent i that belongs to cycle $C_k \in \Upsilon$ for some $k \in \{1, .., n\}$, his assigned object through implementing C_k is not worse than any object in $I_i(\mu^{N_{\pi(C_k)}})$.

(iv) For each agent i that belongs to null-pair $\Lambda^k \in \Upsilon$ for some $k \in \{n+1.., n+m\}$, the null-object is not worse than any object in $I_i(\mu^{N_{\pi(\Lambda^k)}})$ where $\mu^{N_{\pi(\Lambda^k)}}$ is the submatching obtained by implementing the cycles and the null-pairs in $A_{\pi(\Lambda^k)}$.

In words, a permutation gives us an ordering in which cycles and null-pairs to be implemented. The second condition guarantees that there exists a permutation such that each cycle in the collection becomes viable after the submatching that is obtained by implementing the cycles and the null-pairs that come before it in the ordering induced by the permutation. The third and forth conditions, on the other hand, ascertain that no agent's assignment is worse than any of his endowed objects in the reduced economy that emerges after the leaving of the previously assigned agents along with their assignments.

Let us further clarify the definition and the role of permutation through an example. Let $N = \{i_1, i_2, i_3\}$ and $O = \{c_1, c_2, c_3\} \cup \{\emptyset\}$, each with unit-capacity. Let Γ be induced by the ranking orders of $\succ_{c_1}$: i_1, i_3, i_2; $\succ_{c_2}$: i_2, i_1, i_3; $\succ_{c_3}$: i_3, i_1, i_2. Preferences are as follows: $P_{i_1}: c_2, c_3, \emptyset$; $P_{i_2}: c_1, c_3, c_2, \emptyset$; and $P_{i_3}: c_2, c_1, \emptyset$. Let τ be such that $\tau_{i_1}(c_2) = \{i_1, i_3\}$ and $\tau_\ell(d) = N$ for each other agent-object pair $(\ell, d) \in (N \times O) \setminus \{(i_1, c_2)\}$. Let us first consider $\Upsilon = \{\Lambda^{i_1}, C\}$ where $C = \{c_1, i_3, c_2, i_2\}$. Υ is implementable under π where $\pi(\Lambda^{i_1}) = 1$ and $\pi(C) = 2$. However, it cannot be implementable under π' where $\pi'(C) = 1$ and $\pi(\Lambda^{i_1}) = 2$. This is because cycle C becomes viable only after agent i_1 is matched with the null-object and leaves the problem, showing the role of permutation. Also note that the implementation of Υ would produce matching μ where $\mu_{i_1} = \emptyset$, $\mu_{i_2} = c_1$, and $\mu_{i_3} = c_2$. Another collection of cycles and null-pairs is $\Upsilon' = \{C', C''\}$ where $C' = \{c_1, i_1, c_3, i_3\}$ and $C'' = \{c_2, i_2\}$. It is implementable under any permutation, and its implementation yields matching μ' where $\mu'_{i_1} = c_3$, $\mu'_{i_2} = c_2$, and $\mu'_{i_3} = c_1$. On the other hand, $\Upsilon'' = \{\Lambda^{i_2}, \bar{C}', \bar{C}''\}$ where $\bar{C}' = \{c_2, i_1\}$ and $\bar{C}'' = \{c_1, i_3\}$ is not implementable under any permutation π. This is because agent i_2 is endowed with object c_2, which is better than the null-object for himself, violating the last condition of Definition 3.

The last construction before the mechanism definition is as follows: For any $N' \subseteq N$, submatching $\mu^{N'}$, and object c, we define $\Gamma_c^{\mu^{N'}} : \wp(\mu^{N'}) \to 2^{N \setminus N'}$ such that, for any $\mu^{N''} \in \wp(\mu^{N'})$, $\Gamma_c^{\mu^{N'}}(\mu^{N''}) = \Gamma_c(\mu^{N''})$. Less formally, $\Gamma_c^{\mu^{N'}}$ is the restriction of Γ_c to the set of submatchings $\wp(\mu^{N'})$. Let $\Gamma^{\mu^{N'}} = (\Gamma_c^{\mu^{N'}})_{c \in O}$.

We are now ready to define our class of mechanisms. Given a problem P,

Step 1.

Substep 1.1 Consider any agent i such that his top choice object is in $I_i(\mu^{\emptyset})$. Assign each such agent i to his top object. Let N^1 be the set of matched agents, and μ^{N^1} be the associated submatching. If $N^1 = N$, then the algorithm terminates with the final outcome of μ^{N^1}. Otherwise, if $N^1 \neq \emptyset$, then move to the next substep. On the other hand, if $N^1 = \emptyset$, then move to Step 2.

In general,

Substep 1.k Let μ' be the submatching induced by the assignments in the previous steps (by the definition of Γ, it is easy to verify that μ' defines a submatching). Consider each agent $i \in N \setminus \cup_{t=1}^{k-1} N^t$, and assign agent i to his top choice if it is in $I_i(\mu')$. Let N^k be the set of matched agents in this step. If $N = \cup_{t=1}^{k} N^t$, then the algorithm finishes with the final outcome of μ that is defined by the assignments in the previous steps and the current step. Otherwise, if $N^k \neq \emptyset$, then move to the next substep. If, on the other hand, $N^k = \emptyset$, then move to Step 2.

As everything is finite, the above substeps terminate in some finitely many round T. Let N' and $\mu^{N'}$ be the set of matched agents in the above rounds and the obtained submatching, respectively.

Step 2. Consider the reduced problem $(N \setminus N', O, P_{N \setminus N'}, q, \Gamma^{\mu^{N'}}, \tau)$. Pick a collection of cycles and null-pairs Υ in the reduced problem such that it is implementable, and its implementation produces a matching such that the implementation of no other such collection of cycles and null-pairs produces a matching that dominates the former in the reduced problem. Then, implement each cycle and null-pair in Υ and obtain a matching in the reduced problem. The algorithm then terminates with the final outcome of μ that is induced by the assignments of the agents in Step 1 and Step 2.

In words, Step 1 iteratively matches agents with their top choices once they are endowed with them. Then, in the reduced problem, Step 2 picks an implementable collection of cycles and null-pairs among those whose implementation yields a matching that is not dominated by any other matching induced by such a collection and implements it. While Step 2 produces its assignments depending on the selection of collection of cycles and null-pairs, there is no such multiplicity in Step 1. Because of multiplicity in Step 2, the above rule defines a class of mechanisms, each of which is associated with a different selection of collection of cycles and null-pairs. In what follows, we will continue not to specify any particular selection, and obtain all of our results for any such mechanism. We refer to both this class and each of its mechanisms as "Restricted Trading Cycles" and write *RTC* for short. The example below demonstrates how *RTC* works.

Example. Let $N = \{i_1, i_2, i_3, i_4\}$ and $O = \{c_1, c_2, c_3, c_4\} \cup \{\emptyset\}$, with $q_{c_1} = q_{c_2} = q_{c_3} = q_{c_4} = 1$. Suppose that the objects have a ranking order and Γ is induced by them. Let the preferences and those ranking orders be as follows:

$$P_{i_1}: \ c_1, \emptyset; \ P_{i_2}: \ c_1, c_4, \emptyset; \ P_{i_3}: \ c_3, c_4, \emptyset; \ P_{i_4}: \ c_4, \emptyset.$$

$$\succ_{c_1}: \ i_1, ..; \ \succ_{c_2}: \ \textit{anything}; \ \succ_{c_3}: \ i_2, i_4, ..; \ \succ_{c_4}: \ i_3, ..$$

Let τ be such that $i_2 \notin \tau_{i_3}(c_3)$, $i_4 \in \tau_{i_3}(c_3)$, and $i_3 \in \tau_{i_4}(c_4)$. Let ψ be a *RTC* mechanism. In Step 1 of ψ, only agent i_1 is matched, and his assignment is object c_1. Then, in Step 2, the only implementable collection of cycle and null-pair that yields an undominated matching is $\Upsilon = \{C, \Lambda^{i_2}\}$ where $C = \{c_4, i_3, c_3, i_4\}$ (note that Υ is implementable under permutation π where $\pi(\Lambda^{i_2}) = 1$ and $\pi(C) = 2$).[11] Hence, under the unique *RTC* outcome, say μ, $\mu_{i_1} = c_1$, $\mu_{i_2} = \emptyset$, $\mu_{i_3} = c_3$, and $\mu_{i_4} = c_4$.

Theorem 1. *Each RTC mechanism is feasible, constrained efficient, and satisfies respecting endowments.*

Proof. See the Appendix. □

3.2 A Characterization of RTC

In this section, we provide an axiomatic characterization of the *RTC* class. For agent i with preferences P_i, let us write $top(P_i)$ for his top choice object. For any $k \in \{1, .., |N|\}$, we define $Z_k = \{i \in N : \ top(P_i) \in I_i(\mu^{N_{k-1}})\}$ where (i) $N_{k-1} = \cup_{t=1}^{k-1} Z_t$, (ii) $\mu^{N_{k-1}}$ is the submatching among N_{k-1} that assigns each agent $j \in N_{k-1}$ to his top object, and (iii) for $k = 1$, $\mu^{N_0} = \mu^{\emptyset}$.[12]

A matching μ satisfies **hierarchically mutual best** if for any agent $i \in \cup_{k=1}^{|N|} Z_k$, $\mu_i = top(P_i)$. A mechanism ψ satisfies hierarchically mutual best if, for each problem P, $\psi(P)$ satisfies hierarchically mutual best. In words, hierarchically mutual best guarantees that

[11] It is the unique such collection because of the trade restrictions τ.

[12] By the definition of Γ, it is easy to verify that $\mu^{N_{k-1}}$ defines a submatching for any $k \in \{1, .., |N|\}$.

each agent receives his top object once he is endowed with it or inherits it as others obtain their top choices.[13]

Theorem 2. *A mechanism is feasible, constrained efficient, and satisfies hierarchically mutual best if and only if it is a RTC mechanism.*

Proof. See the Appendix. □

The independence of the axioms is provided in the Appendix.

3.3 An Incentive Analysis

A mechanism ψ is **strategy-proof** if there are no problem P, agent i, and false preferences P_i' for agent i such that $\psi(P_i', P_{-i}) P_i \psi(P)$.[14] Below presents a general incompatibility result.

Proposition 3. *Suppose there exists an agent-object pair $(i, a) \in N \times O$ such that $j \notin \tau_i(a)$ for some agent $j \neq i$. Then, there is no strategy-proof mechanism that is feasible, constrained efficient, and that satisfies respecting endowments.*

Proof. Let $N = \{i, j, ...\}$ and $O = \{a, b, c, ...\} \cup \{\emptyset\}$, with $q_a = q_b = q_c = 1$. Let the preferences and object ranking orders be as follows:

$$P_i : \; a, b, c, \emptyset, ..; \; P_j : \; c, \emptyset; \text{ and } P_k : \; \emptyset \text{ for each } k \in N \setminus \{i, j\}.$$

$$\succ_a = \succ_b : \; j, i, ...; \; \succ_c : \; i, j, ..; \text{ and, for each other object } d \in O \setminus \{a, b, c\}, \succ_d : \; \textit{anything}.$$

Γ is induced by the above ranking orders. Let τ be such that $j \notin \tau_i(a)$, and there is no further assumption on τ.

Let ψ be a mechanism that is feasible, constrained efficient, and that satisfies respecting endowments. Then, at the true preferences P, ψ selects either of the matchings μ and μ'

[13] A less stringent version of this property, so-called "mutual best", is used in Morrill (2013a,b), Toda (2006), and Klaus (2011).

[14] P_{-i} stands for the preference profile of the agents except agent i.

where $\mu_i = b$, $\mu_j = c$, and $\mu'_i = a$, $\mu'_j = \emptyset$ while all the other agents are unassigned under both matchings.

If ψ selects μ at P, then consider $P'_i : a, c, \emptyset$. At the false preference profile $P' = (P'_i, P_{-i})$, the only feasible and constrained efficient matching that also satisfies respecting endowments is μ' above; hence ψ produces μ' at P', benefiting agent i.

If ψ selects μ' at P, then consider $P''_j :\ c, a, \emptyset$. Let $P'' = (P''_j, P_{-j})$. At P'', the only feasible and constrained efficient matching that also satisfies respecting endowments is μ above; hence ψ produces μ at P'', benefiting agent j. This finishes the proof. □

Remark 3. Whenever there is no trade restriction, that is, $\tau_i(a) = N$ for each agent-object pair $(i, a) \in N \times O$, feasibility becomes trivial, hence every matching is feasible. Therefore, TTC is feasible, efficient, strategy-proof, and satisfies respecting endowments. This together with Proposition 3 reveals that a even a minimal trade restriction turns down that positive result.

Corollary 2. *No RTC mechanism is strategy-proof.*

Proposition 3 no longer holds once we refrain from constrained efficiency, as shown below.

Proposition 4. *There exists a feasible and strategy-proof mechanism that also satisfies respecting endowments.*

Proof. Let us order the set of agents and write $N = \{i_1, .., i_n\}$. Consider the following mechanism:

Step 1. By following the ordering, start with i_1. Let him receive his top choice from $I_{i_1}(\mu^{\emptyset})$. Let μ^{i_1} be the associated submatching.

In general,

Step k. Let N' be the assigned agents up to the current step, and $\mu^{N'}$ be the associated submatching. Consider agent i_k, and let him receive his top choice from $I_{i_k}(\mu^{N'})$.

The algorithm terminates whenever every agent is assigned to an object. It is immediate to verify that it is feasible, strategy-proof, and that satisfies respecting endowments. However, it is not constrained efficient.

□

3.4 A Probabilistic RTC Mechanism

So far, we only consider the deterministic RTC mechanisms. However, we can add randomness to the RTC class by letting selections in Step 2 be random. While such a mechanism gives us a (deterministic) RTC outcome ex-post, hence preserving all the properties of RTC class ex-post, as shown in the following, the scope of beneficial preference truncation and reshuffling diminishes under it.

A **random matching** σ is a probability distribution (lottery) over the deterministic matchings $\mathcal{M}$. In what follows, we allow mechanisms to produce random matchings. Let us consider the twofold mechanism that works almost as the same as RTC with the difference that any implementable collection of cycles and null-pairs that arises in Step 2 of RTC is chosen with equal probability. We refer to this mechanism as "Probabilistically Restricted Trading Cycles" and write $PRTC$ for short.

For a random matching σ and a deterministic matching μ, we write $P(\sigma, \mu)$ for the probability attached to μ under σ. For agent i and object c, $\sigma_{i,c} = \sum_{\mu \in \mathcal{M}:\ \mu_i = c} P(\sigma, \mu)$. In words, it is the probability that agent i receives object c under σ.

Given a pair of random matchings σ and σ', the former **first order stochastically dominates** the latter with respect to agent i's preferences P_i if, for any object c, $\sum_{c' \in O:\ c' R_i c} \sigma_{i,c'} \geq \sum_{c' \in O:\ c' R_i c} \sigma'_{i,c'}$, with strictly holding for some object. A mechanism ψ is **strongly manipulable** if there exist a problem P, agent i, and false preferences P'_i such that $\psi(P'_i, P_{-i})$ first order stochastically dominates $\psi(P)$ with respect to P_i. A mechanism ψ is **weakly manipulable** if there exist a problem P, agent i, and false preferences P'_i such that $\psi(P)$

does not first order stochastically dominate $\psi(P'_i, P_{-i})$ with respect to P_i.[15]

The proof of Proposition 3 also shows that $PRTC$ is strongly manipulable.

Corollary 3. *$PRTC$ is strongly manipulable.*

In what follows, we consider some well-known manipulation strategies and obtain that $PRTC$ is only weakly manipulable via these strategies. For a preference list P_i, let $Ac(P_i) = \{c \in O : \; cP_i\emptyset\}$, i.e., the set of acceptable objects, excluding the null-object. A preference list P'_i is a **truncation** of P_i if the relative rankings of the objects in $O \setminus \{\emptyset\}$ are preserved while $Ac(P'_i) \subset Ac(P_i)$. A preference list P'_i is a **reshuffling** of P_i if $Ac(P_i) = Ac(P'_i)$.

As the second stage selection of a RTC mechanism can depend on the submitted preferences, it can be strongly manipulable via truncation and reshuffling. However, below shows that $PRTC$ is only weakly manipulable via truncation and reshuffling.

Proposition 5. *$PRCT$ is only weakly manipulable via truncation and reshuffling.*

Proof. See the Appendix. □

Remark 4. In our analysis above, objects are assumed to be collectively owned. However, we can easily adapt our analysis to the case where some (or all) objects are privately owned, and they are brought to the problem by their owners (this type of problems is called "house allocation with existing tenants"). In this case, individual rationality, which requires no agent to receive a worse object than his owned object, would be a concern. An easy way to address it is to let the object hierarchical endowments structures be such that each agent is endowed with his owned object. Respecting endowments implies individual rationality under that class of hierarchical endowments structures; hence any RTC mechanism is individually rational.

[15] Strong (weak) manipulation implies that strategizing is profitable for every (some) cardinal payoffs profile that is consistent with ordinal preferences.

4 Conclusion

We incorporate a class of trade restrictions into an object allocation problem. We identify the set of feasible matchings, which can occur through a market-like process under those restrictions. Next, the class of RTC mechanisms is introduced and is shown that any RTC mechanism is feasible, constrained efficient, and satisfies respecting endowments. We characterize the RTC class with feasibility, constrained efficiency, and hierarchically mutual best. Any RTC mechanism turns out to be manipulable. Yet, this is not a problem specific to the RTC class, as there is a general incompatibility between feasibility, constrained efficiency, respecting endowments, and strategy-proofness. Nevertheless, adding a randomness to RTC reduces the scope of certain misreporting strategies while preserving the properties of RTC ex-post.

Appendix

Proof of Theorem 1. Let ψ be a RTC mechanism. We first show that it terminates in a finite time. The first stage of ψ iteratively matches agents with their top choices once they are endowed with them and removes the assigned agents as well as their assignments from the problem. It terminates whenever there is no such agent left. From here, because there are finitely many agents, we conclude that Step 1 terminates in a finitely many round and gives us a submatching. Then, the second stage of ψ considers the reduced problem, emerging after removing the assigned agents along with their assignments in the first stage. To show that the second stage gives us a matching in the reduced problem, it is enough to show that there always exists an implementable collection of cycles and null-pairs in the reduced economy. This is what we show in the following.

Let N' and $\mu^{N'}$ be the set of assigned agents and the associated submatching in Step 1 of ψ. Let us first enumerate the remaining agents $N \setminus N' = \{i_1, .., i_n\}$. Then, start with agent i_1. If his favorite object in $I_{i_1}(\mu^{N'})$ is the null-object, then let him form a null-pair. Otherwise, let him form a cycle with his favorite object (note that the null-object always belongs to $I_{i_1}(\mu^{N'})$). Let $\mu^{N'\cup\{i_1\}}$ be the obtained submatching after implementing the null-pair or the cycle over $\mu^{N'}$. Next, let us consider agent i_2 and his favorite object in $I_{i_2}(\mu^{N'\cup\{i_1\}})$, and repeat the same arguments for him, and so on. If we consider the permutation that orders the collection of these cycles and null-pairs as the same as the associated agents' indexes, then the collection becomes implementable under this permutation. Therefore, in Step 2 of ψ, there always exists an implementable collection of cycles and null-pairs, hence a matching in the reduced problem is always obtained. This, along with our finding above, shows that ψ terminates in a finitely many round, with producing a matching.

Next, we show that ψ is feasible. Consider a problem P, and suppose that the set of agents that are assigned within Step 1 of ψ is non-empty. Among them, consider the ones that are matched in the first substep of Step 1. By its definition, if i is such an agent, then he is assigned to his top object, say c, and $c \in I_i(\mu^\emptyset)$, which means that $i \in \Gamma_c(\mu^\emptyset)$. Hence, the agent-object pair (i, c) constitutes a cycle that is viable after $\mu^\emptyset$. Let us now suppose that there exists another such agent-object pair, say (j, c'). That is, agent j is assigned object c' in the first substep of Step 1. As the same as before, it implies that $j \in \Gamma_{c'}(\mu^\emptyset)$; hence (j, c') constitutes a cycle that is viable after $\mu^\emptyset$. On the other hand, by our supposition, agent j continues to keep his object c' endowment even after the submatching formed by the agent i's object c assignment. Therefore, each of such cycles continues to be viable even after the implementation of others in any order.

Let N' be the set of agents that are assigned in the first substep of Step 1, and $\mu^{N'}$ be the associated submatching. Then, in the second substep of Step 1 (if it takes place), only the agents who inherit their top choices are assigned. That is, if i is an assigned agent to some object c, then c is his top choice, and moreover, $c \in I_i(\mu^{N'})$. That is, $i \in \Gamma_c(\mu^{N'})$. Hence, (i, c) constitutes a cycle that is viable after $\mu^{N'}$. Moreover, by the same reasoning as above, any such cycle continues to be viable even after the implementation of the others in any order.

The same arguments for the other substeps of Step 1 show that Step 1 of ψ does not violate feasibility. Once Step 1 is finalized, in Step 2, an implementable collection of cycles and null-pairs is selected. By Definition 3, it is immediate to verify that Step 2 is compatible with feasibility as well. All of these show that ψ is feasible.

Because of the feasibility of ψ and our supposition as to Γ, for any agent i, if $c \in I_i(\mu^{\emptyset})$, then $c \in I_i(\mu^{N'})$ where $i \notin N'$ and $\mu^{N'}$ is any submatching that occurs in the course of ψ. This, along with the definition of implementable collection of cycles and null-pairs, implies that no agent i is assigned with an object which is worse than any object in $I_i(\mu^{\emptyset})$, which means that $\psi(P)$ respects endowments.

All the agents that are assigned in Step 1 receive their top objects. Moreover, by construction, the selected implementable collection of cycles and null-pairs in Step 2 of ψ produces a matching in the associated reduced problem such that no other such collection produces a matching that dominates the former. These two facts together implies that $\psi(P)$ is a constrained efficient matching, which finishes the proof. $\square$

Proof of Theorem 2. **"If" Part.** Let ψ be a *RTC* mechanism. By the definition of Step 1, ψ satisfies hierarchically mutual best. From Theorem 1, ψ is both feasible and constrained efficient.

"Only If" Part. Let ψ be a mechanism that is feasible, constrained efficient, and that satisfies hierarchically mutual best. Consider a problem P. As there is no multiplicity in Step 1 of the *RTC*, any *RTC* mechanism's Step 1 outcome is the same. By its definition, moreover, for any Substep 1.k ($k \in \{1, .., |N|\}$), any agent $i \in \cup_{l=1}^{|N|} Z_l$ is matched with his top object in Step 1. As ψ satisfies hierarchically mutual best, any such agent is matched with his top choice at ψ as well.

Because of the feasibility and constrained efficiency of ψ, the assignments of the rest of the agents, that is, $N \setminus \cup_{k=1}^{k=|N|} Z_k$, under ψ can be obtained by implementing a collection of cycles and null-pairs such that it is implementable in the reduced problem after the agents in $\cup_{k=1}^{k=|N|} Z_k$ receive their top choices, and its induced matching is not dominated by that of any other such collection in the reduced problem. But then, it means that the assignments of the agents in $N \setminus \cup_{k=1}^{k=|N|} Z_k$ at $\psi(P)$ coincide with the Step 2 outcome of some RTC mechanism at P. This, along with the above observation, shows that ψ is a RTC mechanism, which finishes the proof. □

The Independence of the Theorem 2 Axioms

Let $N = \{i, j\}$ and $O = \{a\} \cup \{\emptyset\}$, with $q_a = 1$. Object a has a ranking order, which generates Γ_a. Let the preferences and ranking order be as follows:

$$P_i = P_j : \ a, \emptyset; \succ_a: \ i, j.$$

Suppose that there is no trade restriction, that is, $\tau_k(a) = N$ for any $k \in N$. Consider a mechanism ψ such that at this problem P, $\psi_i(P) = \emptyset$ and $\psi_j(P) = a$. Suppose that at any other problem, ψ gives the same outcome as a RTC mechanism. Consequently, ψ is feasible and constrained efficient. Yet, it does not satisfy hierarchically mutual best (indeed, it does not respect endowments because $a \in I_i(\mu^{\emptyset})$, yet $aP_i\psi_i(P)$).

With the same set of agents, let us now consider $O = \{a, b\} \cup \{\emptyset\}$, with $q_a = q_b = 1$. The preferences and object ranking orders are as follows:

$$P_i : \ a, b, \emptyset; \ P_j : \ b, a, \emptyset.$$
$$\succ_a: \ j, i; \succ_b: \ i, j.$$

There is no trade restriction (that is, $\tau_k(c) = N$ for any $k \in N$ and $c \in O$). Consider a mechanism ψ such that $\psi_i(P) = b$ and $\psi_j(P) = a$. Suppose that at any other problem, ψ gives the same outcome as a RTC mechanism. Hence, ψ is feasible and satisfies hierarchically mutual best, yet it is not constrained efficient.

Let us now introduce a trade restriction to the above problem by letting $\tau_i(a) = \{i\}$. Consider a mechanism ψ such that $\psi_i(P) = a$ and $\psi_j(P) = b$. Suppose that at any other problem, ψ gives the same outcome as a RTC mechanism. In this case, ψ is constrained efficient and satisfies hierarchically mutual best, yet it is not feasible.

Proof of Proposition 5. For ease of notation, let us write ψ for $PRTC$. We first show that ψ is not strongly manipulable via truncation at any problem P. Let P'_i be a truncation of P_i and $P' = (P'_i, P_{-i})$, and let $\sigma = \psi(P)$ and $\sigma' = \psi(P')$. Suppose that σ' first order stochastically dominates σ with respect to P_i.

First of all, at problem P, if agent i receives his assignment in Step 1 of ψ, then this implies that he is assigned to his top object with probability one. Hence, in this case, he cannot be better off via truncating his preferences.

Let us suppose that it is not the case, that is, at problem P, agent i receives his assignment in Step 2 of ψ. Here, we have two cases. Consider that at P', agent i receives his assignment, say c, in Step 1 of ψ. This implies that he inherits (or is initially endowed with) his assigned object c in some substep of Step 1. As $P_{-i} = P'_{-i}$, he inherits object c at P in some substep of Step 1 as well. But then, by the definitions, any implementable collection of cycles and null-pairs that arises in Step 2 of ψ at P gives agent i an object that is not worse than object c. This in turn implies that he does not receive any object that is worse than object c with a positive probability under σ, which shows that he cannot benefit from reporting P'_i.

Let us now consider the other case where at P', agent i receives his assignment in Step 2 of ψ. If the sets of implementable collection of cycles and null-pairs that arise in Step 2 of ψ at P and P' are the same, then agent i's assignment is the same at both problems. Suppose it is not the case. Let us first observe that because $P_{-i} = P'_{-i}$, and agent i is not assigned in Step 1 of ψ at both problems, the Step 1 assignments of ψ are the same at both problems; hence so are the reduced problems in Step 2 of ψ.

Next, observe that in Step 2, any implementable collection of cycles and null-pairs at P that matches agent i with an object in $Ac(P'_i) \cup \{\emptyset\}$ will continue to arise in Step 2 at P'. However, any such collection that matches agent i with an object in $Ac(P_i) \setminus Ac(P'_i)$ will no longer arise in Step 2 at P'. This is because agent i, under P'_i, finds the null-object better than any object in $Ac(P_i) \setminus Ac(P'_i)$. Moreover, as $Ac(P'_i) \subset Ac(P_i)$, any implementable collection of cycles and null-pairs that occurs in Step 2 of ψ at P' and that assigns agent i

to an object in $Ac(P'_i)$ appears at P as well. All of these show that in order for P'_i to be beneficial, there has to exist an implementable collection of cycles and null-pairs that arises at P, but not at P' (because, otherwise, the assignment of agent i would be the same at both P and P' under ψ). By our observations, any such collection gives agent i an object in $Ac(P_i) \setminus Ac(P'_i)$, and as stated above, the only reason for it not to appear at P' is that agent i finds the null-object better than any object in $Ac(P_i) \setminus Ac(P'_i)$. All of these observations imply that $\sigma'_{i,\emptyset} > \sigma_{i,\emptyset}$. This, along with the fact that no agent receives an unacceptable object of himself with a positive probability under ψ,[16] implies that for the least preferred acceptable object (with respect to P_i), say c, we have $\sum_{c' \in O:\ c' R_i c} \sigma_{i,c'} > \sum_{c' \in O:\ c' R_i c} \sigma'_{i,c'}$. This contradicts our supposition that σ' first order stochastically dominates σ with respect to P_i.

Let us now show that ψ is not strongly manipulable via reshuffling. First, we need to introduce some notations. For any object c, $U(P_i, c) = \{c' \in O : c' R_i c\}$ and $SU(P_i, c) = U(P_i, c) \setminus \{c\}$. Assume now that P'_i is a reshuffling of P_i, and let $\sigma = \psi(P)$ and $\sigma' = \psi(P'_i, P_{-i})$. Assume for a contradiction that σ' first order stochastically dominates σ with respect to P_i.

By the same arguments above, if agent i is matched in Step 1 of ψ at either (or both) P or P', then we have the result. Hence, let us consider the case where he receives his assignment in Step 2 of ψ at both problems P and P'. As the same as above, it implies that the reduced problems in Step 2 of ψ are the same at both P and P'.

Let us first observe that swapping the places of a pair of objects at P_i can only be beneficial only if it decreases the number of implementable collection of cycles and null-pairs that arise in Step 2 of ψ. To see this, let us suppose that P'_i is such that (i) $Ac(P_i) = Ac(P'_i)$, (ii) for a particular pair of objects c, c', $c' P'_i c$ whereas $c P_i c'$, and (iii) the ranking of every object remains the same. Any implementable collection of cycles and null-pairs that arises in Step 2 of ψ at P that gives agent i an object $d \in U(P'_i, c')$ continues to arise in Step 2 of ψ

[16] This is due to the facts that ψ satisfies respecting endowments, and the null-object is in $I_i(\mu^{\emptyset})$ for any agent i.

at P'. However, any such collection that gives an agent i an object in $U(P_i', c)\backslash U(P_i', c')$ may disappear. The only reason for it to disappear at P' is that agent i reports object c' better than any object in $U(P_i', c)\backslash U(P_i', c')$; thereby it may no longer produce a constrained efficient matching in the reduced problem. Moreover at P', some new implementable collections of cycles and null-pairs that give object c' to agent i may arise. These, along with the fact that each implementable collection of cycles and null-pairs is chosen with the equal probability in Step 2 of ψ, implies that the only way for P_i' to be profitable to agent i is to have a fewer such collections so that the probabilities of getting objects in $U(P_i, c)$ may increase.

Let us now suppose that P_i' is any reshuffling of P_i. As P_i' can be obtained by iteratively swapping the places of objects at P_i, the above reasoning can be applied iteratively to conclude that the same is true for P_i'. That is, in order for P_i' to be beneficial for agent i, the number of implementable collection of cycles and null-pairs in Step 2 of ψ has to decrease at P'.

Let W be the set of objects such that $c \in W$ whenever there exists an implementable collection of cycles and null-pairs in Step 2 of ψ at P that assigns agent i to object c, yet it does not constitute such a collection in Step 2 of ψ at P'. By our observation above, $W \neq \emptyset$. Let $c' \in W$ such that it is the worst ranked one with respect to P_i. This implies that any implementable collection of cycles and null-pairs in Step 2 of ψ at P and assigns an object which is worse than c' (with respect to P_i) continues to arise in Step 2 of ψ at P'.

Suppose that there are x many implementable collections of cycles and null-pairs in Step 2 of ψ at P that give agent i an object that is not worse than c'. Moreover, suppose that there are in total n_1 many implementable collections of cycles and null-pairs arising in Step 2 of ψ at P. Then, $\sum_{cR_ic'} \sigma_{i,c} = x/n_1$. On the other hand, let k many such collections disappear at P'. Then, we have $\sum_{cR_ic'} \sigma'_{i,c} = (x-k)/(n_1-k)$. It is immediate to verify that $x/n_1 - (x-k)/(n_1-k) \geq 0$. If it is strict, then we are done as it yields a contradiction to our supposition that σ' first order stochastically dominates σ with respect to P_i. Otherwise, that difference is exactly equal to zero if and only if $x = n_1$. But then, it means that agent i

does not receive any object that is worse than object c' with a positive probability under σ, which implies that $\sum_{cR_ic'} \sigma_{i,c} = 1$. However, at P', some implementable collection of cycles and null-pairs that gives object c' to agent i disappears. The only reason for it is that agent i reports an object that is actually worse than object c' as a better alternative at P'. This, along with the fact that c' is the least preferred object in W (with respect to P_i), implies that under σ', agent i receives an object that is worse than object c' with some positive probability. Hence, $\sum_{cR_ic'} \sigma'_{i,c} < 1$. Therefore, $\sum_{cR_ic'} \sigma_{i,c} > \sum_{cR_ic'} \sigma'_{i,c}$, showing that ψ is not strongly manipulable via reshuffling.

In the rest of the proof, we will show that ψ is weakly manipulable via truncation and reshuffling. To see this, let us consider a problem instance where $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $O = \{c_1, c_2, c_3, c_4, c_5\}$, each with unit capacity, except $q_{c_4} = 2$. The preferences and object ranking orders are as follows:

$$P_{i_1}: \; c_1, c_2, c_3, c_4, \emptyset; \; P_{i_2}: \; c_1, c_3, \emptyset; \; P_{i_3}: \; c_2, c_3, c_1, \emptyset; \; P_{i_4} = P_{i_5}: \; c_5, c_4, \emptyset; \; P_{i_6}: \; c_4, c_5, \emptyset.$$

$$\succ_{c_1}: \; i_3, ..; \succ_{c_2}: \; i_1, ..; \succ_{c_3}: \; i_2, ..; \succ_{c_4}: \; i_1, i_4, i_5..; \succ_{c_5}: \; i_6, ...$$

Suppose that for any agent-object pair (i, c), $\tau_i(c) = N$. That is, there is no trade restriction. Then, at the true preference profile P, agent i_1's assignment is as follows: $\psi(P)_{i_1,c_1} = 1/2$ and $\psi(P)_{i_1,c_2} = 1/2$. Now, consider the false preferences $P'_{i_1}: \; c_1, c_4, c_2, c_3, \emptyset$; and let $P' = (P'_{i_1}, P_{-i_1})$. Note that P'_{i_1} is a reshuffling of P_{i_1}. At P', $\psi(P')_{i_1,c_1} = 2/3$ and $\psi(P')_{i_1,c_4} = 1/3$. Hence, ψ is weakly manipulable via reshuffling.

For the weak manipulation via truncation, consider the same set of agents and objects, but each with unit capacity now. Let the preferences and ranking orders be as follows:

$$P_{i_1}: \; c_1, c_2, \emptyset; \; P_{i_2}: \; c_1, c_3, \emptyset; \; P_{i_3}: \; c_2, c_3, c_1, \emptyset; \; P_{i_4}: \; c_2, c_4, c_5, c_6, \emptyset; \; P_{i_5}: \; c_6, c_4, \emptyset;$$
$$P_{i_6}: \; c_6, c_5, \emptyset.$$

$$\succ_{c_1}: \; i_3, ..; \succ_{c_2}: \; i_1, i_4, ..; \succ_{c_3}: \; i_2, ...; \succ_{c_4}: \; i_5, ..; \succ_{c_5}: \; i_6, ..; \succ_{c_6}: \; i_4, i_5, ..$$

There is no trade restriction, as above. Under the true preference profile, $\psi(P)_{i_1,c_1} = 1/2$ and $\psi(P)_{i_1,c_2} = 1/2$. Let us consider the following truncation: $P'_{i_1}: \; c_1, \emptyset$, and let

$P' = (P'_i, P_{-i})$. Then, under P' : $\psi(P')_{i_1,c_1} = 2/3$ and $\psi(P')_{i_1,\emptyset} = 1/3$. Hence, ψ is weakly manipulable via truncation.

□

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