Assignment Maximization*

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Abstract

We evaluate the goal of maximizing the number of individually rational assignments. We show that it implies incentive, fairness, and implementation impossibilities. Despite that, we present two classes of mechanisms that maximize assignments. The first are Pareto efficient, and undominated – in terms of number of assignments – in equilibrium. The second are fair for unassigned students and assign weakly more students than stable mechanisms in equilibrium. We provide comparisons with well-known mechanisms through computer simulations. Those show that the difference in number of matched agents between the proposed mechanisms and others in the literature is large and significant.

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1 Introduction

Maximizing the number of assignments in discrete assignment problems is an important and natural design objective in many practical domains. One domain where this

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takes place is that of school choice. Abdulkadiroğlu et al. (2005) describe the change in New York City's high schools' matching program. One of the main problems identified was that the normal process would leave a large proportion of the students unmatched, and would end up assigning them via an administrative process to schools which were not necessarily among those stated in their preferences. In fact, they show that 30,000 out of 100,000 students were assigned in this way in 2002.[1] Data from the New Orleans OneApp, another centralized school choice program, show that an average of 20% of the applicants remained unmatched after the main assignment round.(Harris et al., 2015) Having to go through the additional processes used to assign students who are not matched in the main process can also cause frustration and emotional stress, as shown in the quote below:

> *"(...)The High School application process is a nerve wrecking nightmare and extremely unfair to single parents, new immigrant families and any other families who simply cannot put in the countless hours it takes to attend Open Houses, tours and fairs. We got lucky and our daughter got into a school of her choice, but my heart goes out to the families who have to go through this process twice." (Tine Kindermann)* [2]

From the perspective of policymakers, leaving students unassigned, even temporarily, may have serious consequences. In 2013, for example, the city of São Paulo (Brazil) was ordered by a state court to pay restitution to 943 parents who had to put their children in temporary private childcare, as a result of remaining unmatched by the city's assignment process.[3] Maximizing the number of assignments might in fact be the primary objective of the assignment process, as indicated by the following quote from the Frankfurt secondary school district and North Rhine-Westphalia secondary school district:

> *"The organization of the "Frankfurt School Mechanism" is shared between State, city and school. Its primary goal is to give as many applicants as possible one of their preferred schools. Each school decides for itself which students to admit..." (Basteck et al., 2015)*

Maximizing the number of organ transplantations is perhaps the most important objective of organ exchange programs, as evidenced by the recent literature on those types

[1] Even after a change in the mechanism, proposed by the authors, the number of students who remained unmatched was still about 7,600, requiring additional elicitation of preferences over what are supposedly undesired schools.

[2] *Source: http://www.nytimes.com/2011/05/08/nyregion/in-applying-for-high-school-some-8th-graders-find-a-maze.html (NYT selected comments, accessed 09/11/2017.)*

[3] Source: http://g1.globo.com/sao-paulo/noticia/2013/05/mae-ganha-direito-de-indenizacao-apos-ficar-sem-vaga-para-o-filho-em-creche.html (in Portuguese, accessed *09/11/2017.)*

of mechanisms. For both kidney exchange (Roth et al., 2005) and lung exchange *(Ergin et al., 2017)*, the objective of maximizing the number of matchings (and therefore transplants), is put first and foremost in the design of their mechanisms.

Another area in which maximizing the number of assignments is relevant and has raised significant interest on the part of market designers is in the matching of asylum seekers to countries or states. Andersson and Ehlers (2016), for example, propose an algorithm to find maximum mutually acceptable matchings[4] which are also stable. Other examples of applications in which matching maximization is relevant include the matching of babies to nurseries (Sasaki and Ura, 2016) and public housing.

We evaluate the general objective of maximizing the number of matches from a market design perspective. The objective of maximizing the number of (individually rational) matchings has been tackled mostly from its mathematical and algorithmic perspectives. In this paper, we consider the economic problems faced by a policymaker who wants to produce maximal matchings.

Consider the problem of assigning students to schools.[5] The reason why efficiency and stability (or equivalently, fairness) may conflict with maximizing the number of matches is that some schools may be deemed unacceptable to some students. As a result, there may be some Pareto efficient and/or stable matchings that do not maximize assignments. Consider, for example, the case in which there are two schools (A and B), each with only one seat, and two students (1 and 2). Student 1 only deems A as acceptable, whereas student 2 simply prefers A to B. In this case, student 2 being matched to A and 1 remaining unmatched is a Pareto efficient assignment. Moreover, if student 2 has higher priority at school A than student 1, that is also the unique stable assignment. However, student 1 being assigned to A and student 2 to B is a Pareto efficient assignment that only matches students to acceptable schools. Therefore, there may typically be Pareto efficient and stable mechanisms that can be significantly improved upon in terms of the number of assignments.[6]

In this paper, we set the maximization of the number of assignments as our primary design goal. We show that maximizing the number of assignments is incompatible not only with fairness, but also with strategy-proofness (Proposition 1), and that no mechanism is maximal in equilibrium (Proposition 6). While these can be interpreted as strong negative results, we present a large set of proposals and analyses.

First, we design a family of mechanisms, denoted *Efficient Assignment Maximizing Mechanisms* (EAMs), that are Pareto efficient and maximal in terms of the number of as-

[4]A refugee family and a landlord are *mutually acceptable* if they have a language in common and the number of beds offered by the household exceeds the number of beds needed by the refugee family.

[5]While for the remainder of the text we will frame the problems in terms of school choice, the entire analysis applies to the more general problem of priority-based object allocation.

[6]The efficiency cost of stability has been pointed out before in the literature. See Abdulkadiroglu and Sönmez (2003) and Kesten (2010).

signments (Theorem 1). Due to the impossibility above EAMs are not strategy-proof, but we characterize the unique Nash equilibrium outcome, which is Pareto efficient (Proposition 5). Moreover, EAMs are not dominated (in terms of the number of assignments) by any other mechanism in equilibrium (Theorem 3).

While assignment maximality and fairness are incompatible, we show that a weaker version of fairness is compatible. We say that an outcome is *fair for unassigned students* if there is no situation in which an unassigned student justifiably envies the assignment of some other agent. We define another family of mechanisms, denoted *Fair Assignment Maximizing mechanisms* (FAMs), which maximize the number of assignments and are fair for unassigned students (Theorem 2). Interestingly, a tradeoff between fairness and efficiency also emerges for this weaker notion of fairness (Proposition 4). Moreover, while EAMs are also Pareto efficient in equilibrium, we show that FAMs produce at least the same number of assignments as the problem's stable matchings in equilibrium (Proposition 7).

We also provide results regarding how well-known mechanisms compare in terms of the number of assignments made. We show that there is no dominance relation between four mechanisms used in practice and the literature (Proposition 2): Gale-Shapley Deferred Acceptance (DA),[7] Boston Mechanism (BM), Top-Trading Cycles (TTC), and Serial Dictatorship (SD).

To test the relevance of our theoretical results and see how much EAMs/FAMs improve upon well-known mechanisms in terms of number of assignments, we conduct a simulation analysis comparing the number of assignments produced by five different mechanisms – DA, BM, TTC, SD, and EAMs/FAMs. Two facts from the simulation analysis stand out: (1) the difference between EAMs/FAMs and other mechanisms in terms of number of assignments is large and significant, (2) for any choice of parameters, the number of matched students in DA, BM, TTC, and SD are very similar. These simulations reinforce the appropriateness of our proposals to the problem presented, by showing that under true preferences the improvements are very significant, and that the results are at least as good as the alternatives when students behave strategically under EAMs.

The remainder of the paper is organized as follows. In section 2 we introduce the model, the mechanisms we propose, and its properties. In section 3 we show the equilibrium behavior and outcomes induced by those mechanisms, and in section 4 we present the result of computer simulations comparing mechanisms outcomes. Proofs absent from the main text can be found in the appendix.

[7]Or any other stable mechanism.

2 Model

A **school choice problem** consists of the following elements:

- A finite set of students $I = \{i_1, ..., i_n\}$,
- a finite set of schools $S = \{s_1, ..., s_m\}$,
- a strict priority structure for schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is a linear order over I,
- a capacity vector $q = (q_{s_1}, ..., q_{s_m})$ where q_s is the number of available seats at school s,
- a profile of strict preference of students $P = (P_i)_{i \in I}$, where P_i is student i's preference relation over $S \cup \{\emptyset\}$ and $\emptyset$ denotes the option of being unassigned. We denote the set of all possible preferences for a student by $\mathcal{P}$. Let R_i denote the *at-least-as-good-as* preference relation associated with P_i, that is: $sR_is^{'} \Leftrightarrow sP_is^{'}$ or $s = s^{'}$. A school s is **acceptable** to i if $sP_i\emptyset$, and **unacceptable** otherwise.

In the rest of the paper, we consider the tuple $(I, S, \succ, q)$ as the commonly known primitive of the problem and refer to it as the **market**. We suppress all those from the problem notation and simply write P to denote the problem. A **matching** is a function $\mu : I \rightarrow S \cup \{\emptyset\}$ such that for any $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A student i is **assigned** under μ if $\mu(i) \neq \emptyset$. For any $k \in I \cup S$, we denote by μ_k the assignment of k. Let $|\mu|$ be the total number of students assigned under μ.

A matching μ is **individually rational** if, for any student $i \in I$, $\mu_i R_i \emptyset$. A matching μ is **non-wasteful** if for any school s such that $sP_i\mu_i$ for some student $i \in I$, $|\mu_s| = q_s$. A matching μ is **fair** if there is no student-school pair (i, s) such that $sP_i\mu_i$, and for some student $j \in \mu_s$, $i \succ_s j$. A matching μ is **stable** if it is individually rational, non-wasteful, and fair.

In the rest of the paper, we will consider only individually rational matchings. Therefore, whenever we refer to a matching, unless explicitly stated, we refer to an individually rational matching. Let $\mathcal{M}$ be the set of matchings.

A matching μ **dominates** another matching μ' if, for any student $i \in S$, $\mu_i R_i \mu'_i$, and for some student j, $\mu_j P_j \mu'_j$. A matching μ is **efficient** if it is not dominated by any other matching. Note that efficiency implies both individual rationality and non-wastefulness. We say that a matching μ **size-wise dominates** another matching μ' if $|\mu| > |\mu'|$. A matching μ is **maximal** if it is not size-wise dominated.

A **mechanism** ψ is a systematic way of selecting a matching for every problem, that is, it is a function from $\mathcal{P}^{|I|}$ to $\mathcal{M}$. A mechanism ψ is [stable, efficient, fair, individually rational] if, for any problem $P \in \mathcal{P}^{|I|}$, $\psi(P)$ is [stable, efficient, fair, individually rational]. A Mechanism ψ is **strategy-proof** if there exist no problem P, and student i with a false preference P'_i such that $\psi_i(P'_i, P_{-i}) P_i \psi_i(P)$.[8]

At first sight, the natural objective of a designer would be to find a mechanism that is fair, maximal, and strategy-proof.

Proposition 1. *Regarding maximal mechanisms:*
(i) No fair mechanism is maximal.
(ii) No strategy-proof mechanism is maximal.

Proposition 1 sets the stage for the rest of the paper. Not only there is no strategy-proof mechanism that is maximal, but even without considering incentives, there exists a fundamental incompatibility between fairness and maximality.

Since we will focus on the number of students matched to schools, we also make use of a method for comparing mechanisms with respect to that dimension. A mechanism ψ **size-wise dominates** another mechanism ϕ if, for any problem P, $\phi(P)$ does not size-wise dominate $\psi(P)$, while, for some problem P', $\psi(P')$ size-wise dominates $\phi(P')$. A mechanism ψ is maximal if it is not size-wise dominated by any other mechanism.

2.1 A Size-Wise Domination Comparison Among Well-Known Mechanisms

Here we compare well-known mechanisms in terms of the number of assigned students. Namely, we consider the Gale-Shapley deferred acceptance (DA), Top Trading Cycles (TTC), Boston (BM), and serial dictatorship (SD) mechanisms. Their definitions are given in the Appendix.

Proposition 2. *There is no size-wise domination between any pair of mechanisms among the DA, TTC, BM, and SD.*

As a consequence of the rural hospitals theorem (Roth, 1984), every stable matching assigns the same number of students to schools, and so we have the following more general result.

Corollary 1.

(i) There is no size-wise domination between any pair of mechanisms among the class of stable mechanisms, the TTC, the BM, and the SD.

(ii) None of stable, TTC, the BM, and SD mechanisms are maximal.

[8] P_{-i} is the preference profile of all students except student i.

The results above take place based on the fact that some students might not rank all of the schools as acceptable. When that is not the case, the only reason for a student to be unassigned under these mechanisms is that all schools have been filled up, and therefore they all assign the same number of students.

Remark 1. If every school is acceptable to every student, then DA, TTC, BM, and SD all match the same number of students in any problem, consisting of the total sum of schools' capacities.

2.2 A Class of Efficient Maximal Mechanisms

In what follows, we first introduce two concepts which will be critical to the class of mechanisms in this section.

Definition 1. A matching μ admits an **improvement chain** at problem P if there are distinct students and schools $\{i_1, ..., i_n, c_1, c_2, .., c_{n+1}\}$ such that $|\mu_{c_{n+1}}| < q_{c_{n+1}}$ and for every $k = 1, ..n$,
(i) $\mu_{i_k} = c_k$,
(ii) $c_{k+1} P_{i_k} c_k$.

Definition 2. A matching μ admits an **improvement cycle** in problem P if there are distinct students and schools $\{i_1, ..., i_n, c_1, c_2, .., c_n, c_{n+1}\}$ such that $c_{n+1} = c_1$ and for every $k = 1, ..n$,
(i) $\mu_{i_k} = c_k$,
(ii) $c_{k+1} P_{i_k} c_k$.

We are now ready to introduce the class of mechanisms. Given a problem P and an enumeration of the students in I $(i_1, ..i_n)$,

Step 0. Let $\xi^0 = \mathcal{M}$.

Step 1.

Substep 1.1. Define the set $\xi^1 \subseteq \xi^0$ as follows:

$$\xi^1 = \begin{cases} \{\mu \in \xi^0 : \ \mu_{i_1} \neq \emptyset\} & \text{If } \exists \mu \in \xi^0 \text{ such that } \mu_{i_1} \neq \emptyset \\ \xi^0 & \text{otherwise} \end{cases}$$

In general, for every $k \leq n$,

Substep 1.k. Define the set $\xi^k \subseteq \xi^{k-1}$ as follows:

$$\xi^k = \begin{cases} \{\mu \in \xi^{k-1} : \ \mu_{i_k} \neq \emptyset\} & \text{If } \exists \mu \in \xi^{k-1} \text{ such that } \mu_{i_k} \neq \emptyset \\ \xi^{k-1} & \text{otherwise} \end{cases}$$

Step 1 ends with the selection of a matching $\mu \in \xi^n$.

Step 2.

Substep 2.1. If the matching μ does not admit an improving chain or cycle, then the algorithm ends with the final outcome of μ. Otherwise, pick a chain or cycle, and obtain a new matching by assigning each student in the chosen chain (cycle) to the school she prefers in the chain (cycle), and move to the next substep.

In general:

Substep 2.k. Let $\tilde{\mu}$ be the matching obtained in the previous round. If $\tilde{\mu}$ does not admit an improving chain or cycle then the algorithm ends with the final outcome of $\tilde{\mu}$. Otherwise, pick such a chain or cycle, and obtain a new matching by assigning each student in the chosen chain (cycle) to the school he prefers in the chain (cycle), and move to the next substep.

As everything is finite and, in every substep of Step 2, students are all weakly better off with at least one being strictly better off, Step 2 terminates after finitely many substeps. The matching obtained in the final round of Step 2 is the outcome of the algorithm. This algorithm defines a class of mechanisms, each of which is associated with different selections of the student ordering, the matching in the end of Step 1, and chains and cycles in the course of Step 2. We refer to this class of mechanisms as "Efficient Assignment Maximizing" (EAM) mechanisms.

The first step of the EAM mechanisms is a "priority mechanism", introduced by Roth et al. (2005) in the context of the pairwise kidney exchange problem. The authors show that this process finds a maximal matching. Though it may seem counterintuitive that this simple process yields a maximal matching, the intuition behind it is simple. At each step, the set of outcomes is restricted to outcomes that will match the student being considered to an acceptable school. Each one of these may lead to at most one other student remaining unmatched. Therefore, following the enumeration above and trying to match each student leads to a maximal matching.

The matching produced, however, may not be efficient. To fix this, the second stage implements improving chains and cycles. As these chains and cycles are welfare-improving, the second stage preserves the maximality of the first stage outcome while benefiting the students. Consequently, every EAM mechanism is maximal and efficient.

Theorem 1. *Every EAM mechanism is maximal and efficient.*

From Proposition 1, fairness and maximality are incompatible. This, along with Theorem 1, implies that no EAM mechanism is fair. However, since maximality aims to assign as many students as possible, we may be able to satisfy a weaker notion of fairness. We say that a matching μ is **fair for unassigned students** if there is no student-school pair (i, s) where $\mu_i = \emptyset$ and $i \succ_s j$ for some $j \in \mu_s$. A mechanism ψ is fair for unassigned students if, for any problem P, $\psi(P)$ is fair for unassigned students.

Proposition 3. *No EAM mechanism is fair for unassigned students.*

Proof. Let $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let ψ be any EAM mechanism where the student ordering starts with i. Let the priorities be such that $\succ_a$: j, i and $\succ_b$: i, j. Let us first consider the following preferences: $P_i : a, \emptyset$ and $P_j : a, \emptyset$. Then, $\psi_i(P) = a$ and $\psi_j(P) = \emptyset$, violating fairness for unassigned students.

Next, consider any EAM mechanism, say ϕ, such that the student ordering starts with j. Let us now consider the preferences where $P_i : b, \emptyset$ and $P_j : b, \emptyset$. Then, $\phi_i(P) = \emptyset$ and $\phi_j(P) = b$, violating fairness for unassigned students. □

In the next subsection we show, however, that this weaker notion of fairness is compatible with assignment maximization, and we provide a mechanism that produces those outcomes.

2.3 A Class of Maximal and Fair for Unassigned Students Mechanisms

Below is a description of how each mechanism in this class works. Given a problem P,

Step 1. Pick an EAM mechanism ψ, and let $\psi(P) = \mu$.

Step 2.

Substep 2.1. If μ is fair for unassigned students then the algorithm terminates with the final outcome of μ. Otherwise, pick a student-school pair (i, s) such that $sP_i\emptyset$, $\mu_i = \emptyset$, and $i \succ_s j$ for some $j \in \mu_s$. Place student i at school s, and let the lowest priority student in μ_s be unassigned (note that since μ is maximal, we have $|\mu_s| = q_s$), while keeping everyone else's assignment the same. Let μ' be the obtained matching, and move to the next substep.

In general,

Substep 2.k. Let $\tilde{\mu}$ be the matching obtained in the previous step. If $\tilde{\mu}$ is fair for unassigned students, the algorithm terminates with the outcome $\tilde{\mu}$. Otherwise, pick a student-school pair (i, s) such that $sP_i\emptyset$, $\tilde{\mu}_i = \emptyset$, and $i \succ_s j$ for some $j \in \tilde{\mu}_s$. Place student i at school s, and let the lowest priority student in $\tilde{\mu}_s$ be unassigned, while keeping everyone else's assignment the same. Note that as in each substep the number of assigned students is preserved, $\tilde{\mu}$ is maximal. Hence, we have $|\tilde{\mu}_s| = q_s$. Let $\hat{\mu}$ be the obtained matching, and move to the next substep.

As, in every substep, a higher priority student is placed at a school while a lower priority one is displaced from the school, and both the students and schools are finite, the algorithm terminates in finitely many rounds. The above procedure defines a class of mechanisms, each of which is associated with different selections of the first stage

EAM mechanism as well as the student-school pairs in the course of Step 2. We refer to this class of mechanisms as "Fair Assignment Maximizing" (FAM) mechanisms.

The procedure above is similar to the Deferred Acceptance with Arbitrary Input (DAAI) in Blum et al. (1997). Its fundamental difference from our proposal is that in the second step of a FAM, only unmatched students may fulfill their justified envies, whereas under the DAAI, students who are matched may also fulfill their justified envies. In fact, while outcomes of the DAAI mechanism are always stable, outcomes of a FAM may not be.

Theorem 2. *Every FAM mechanism is fair for unassigned students and maximal.*

Proof. Let ψ be a FAM mechanism, and μ be the outcome of its first step. As μ is the outcome of an EAM mechanism, and in Step 2 of ψ, no student is assigned to one of his unacceptable choices, ψ is individually rational. Because μ is maximal and the number of assigned students is preserved as $|\mu|$ in the course of Step 2, ψ is maximal. Moreover, as ψ does not stop until no student-school pair violates fairness for unassigned students, ψ is fair for unassigned students as well. □

An important downside of the FAM class is the lack of efficiency, in that no FAM mechanism is efficient. However, this is not a problem specific to the FAM class as there exists a general incompatibility between efficiency and fair for unassigned students, as shown below.

Proposition 4. *No mechanism is efficient and fair for unassigned students.*

Proof. Let $I = \{i, j, k\}$ and $S = \{a, b\}$, each with unit capacity. Consider the following preferences and priorities:

P_i : $a, b, \emptyset$; P_j : $b, a, \emptyset$; P_k : $b, \emptyset$.

$\succ_a$: j, i, k; $\succ_b$: i, k, j.

Let ψ be an efficient mechanism, and $\psi(P) = \mu$. By the efficiency of μ, exactly one student is left unassigned.

Case 1. Suppose $\mu_k = \emptyset$. Then, by efficiency of μ, $\mu_i = a$ and $\mu_j = b$. However, as $k \succ_b j$, μ cannot be fair for unassigned students.

Case 2. Suppose $\mu_j = \emptyset$. Then, by efficiency of μ, $\mu_i = a$ and $\mu_k = b$. However, as $j \succ_a i$, μ cannot be fair for unassigned students.

Case 3. Suppose $\mu_i = \emptyset$. By efficiency of μ, $\mu_j = a$ and $\mu_k = b$. However, as $i \succ_b k$, μ cannot be fair for unassigned students. □

3 Incentives and Equilibrium Analysis

As shown in Proposition 1, there is no mechanism which is maximal and strategy-proof. Hence, in particular, none of the EAM and FAM mechanisms are strategy-proof.

Corollary 2. *None of the EAM and FAM mechanisms are strategy-proof.*

In this section we show, however, that the mechanisms in the classes *EAM* and *FAM* have surprisingly regular properties in terms of equilibrium outcomes. We also present some results comparing equilibrium outcomes between mechanisms. Consider the preference reporting game induced by a mechanism ψ. At problem P, a preference submission $P' = (P'_i)_{i\in I}$ is a (Nash) **equilibrium** of ψ if for every student i, $\psi_i(P')\, R_i \psi_i(P''_i, P'_{-i})$ for any $P''_i \in \mathcal{P}$. Let Ω be the set of mechanisms that admit an equilibrium in any problem $P \in \mathcal{P}^{|I|}$. In the rest of this section, we consider only the mechanisms in Ω.

The first result relates to the equilibria of *EAM* and *FAM* mechanisms.

Proposition 5. *Every EAM and FAM mechanism is in Ω. Moreover, for any problem, an EAM mechanism has a unique equilibrium outcome that is equivalent to the outcome of the serial dictatorship where the student ordering is the same as that used in that EAM mechanism.*

Proposition 5 shows, therefore, that equilibrium outcomes of *EAM* are not only Pareto efficient, but will match as many students as a commonly used strategy-proof mechanism.

A mechanism ψ is **maximal in equilibrium** if, at any problem P and any equilibrium submission P' under ψ, $\psi(P')$ is maximal.

Proposition 6. *No mechanism is maximal in equilibrium.*

Corollary 3. *No EAM and FAM mechanism is maximal in equilibrium.*

Our next question is how mechanisms compare, in terms of the number of assignments, in equilibrium. For that, we define the concept of **size-wise domination in equilibrium.**

Definition 3. For a given market $(I, S, \succ, q)$, a mechanism ψ **size-wise dominates** another mechanism ϕ **in equilibrium** if, for any problem P and for every equilibria P', P'' under ψ and ϕ, respectively $|\psi(P')| \geq |\phi(P'')|$, and there exists a problem P^* such that for every equilibria $\hat{P}, \tilde{P}$ under ψ and ϕ, respectively $|\psi(\hat{P})| > |\phi(\tilde{P})|$.

What is needed, therefore, for a mechanism ψ to size-wise dominate mechanism ϕ in equilibrium in given a market, is that in every problem ψ assigns at least as many students as ϕ regardless of the equilibrium selection that is made, and that there is at least one problem in which those differences are strict.

Theorem 3. *In any market $(I, S, \succ, q)$, no EAM mechanism is size-wise dominated by an individually rational mechanism in equilibrium.*

Notice that the fact that size-wise domination is defined in terms of a given market makes Theorem 3 stronger: it is not enough to show that the result is true for a specific market. The theorem instead shows that for any set of students, schools, capacities and priorities there is no individually rational mechanism that dominates any EAM in equilibrium.

While we do not have a similar result to above for the FAM mechanisms, we are able to compare the number of assigned students under the FAM in equilibrium and the weakly dominant strategy equilibrium of the DA, which is truth-telling.

Proposition 7. *Regarding the FAM mechanisms:*

(i) For any problem P and any stable matching for P μ^*, for every equilibrium P' of a FAM mechanism ψ, $|\psi(P')| \geq |\mu^*|$.

(ii) There exist a FAM mechanism ψ, problem P, and an equilibrium profile P' of ψ at P such that $|\psi(P')| > |\mu^{**}|$, where μ^{**} is any stable matching for P.

4 Simulations

While we have shown that the EAM family of mechanisms[9] dominate any individually rational mechanism under true preferences and that they also produce good outcomes in equilibrium, one may wonder whether in practice the magnitude of the difference in the actual number of students assigned justifies the proposal of a new mechanism. To provide an answer to that question, in this section we describe and analyze simulation results in which we compare the number of students matched under five mechanisms: EAM, DA, BM, TTC, and SD.

The construction of the problems to be simulated follows a method similar to that applied in Hafalir et al. (2013). Each problem contains a set of students $I = \{i_1, \ldots, i_n\}$, a set of schools $S = \{s_1, \ldots, s_m\}$ and their capacities $Q = \{q_1, \ldots, q_m\}$. Students have strict preferences $\{P_{i_1}, \ldots, P_{i_n}\}$ over $S \cup \{\emptyset\}$ and schools have strict priorities $\{P_{s_1}, \ldots, P_{s_m}\}$ over $I \cup \{\emptyset\}$. Those ordinal preferences and priorities are derived from utilities that each student and school have over the other side of the market. Let us first consider a student $i \in I$. Her utility from being assigned to school $s \in S$ is the following:

$$U_i(s) = \begin{cases} \alpha\Theta^s + (1-\alpha)\,\Theta_i^s & \text{if } \alpha\Theta^s + (1-\alpha)\,\Theta_i^s \geq \lambda_i \\ -\infty & \text{otherwise} \end{cases}$$

[9] For simplicity, in this section we refer only to EAM mechanisms. Since the number of assignments is the same under any EAM and FAM mechanism, however, unless explicitly stated, all the results below hold for both families of mechanisms.

The interpretation of the parameters goes as follows. The utility that a student i derives from being assigned to a school s is a combination of a value that is shared by all students (Θ^s) and an idiosyncratic value that is unique to a student-school pair (Θ^s_i). The value of Θ^s could therefore be the widespread understanding of the quality of the school and Θ^s_i incorporate, for example, the distance of the school to the student's house and whether the extra-curricular activities fit the student's taste. For each problem, and for each values of $s \in S$ and $(s,i) \in S \times I$, Θ^s and Θ^s_i are independently drawn from the normal distribution with mean zero and variance 1. The value of α, which represents the correlation of preferences between students, is exogenously set in the range $[0,1]$.

Remark 1 showed that when every student deems every school as acceptable and no student is unacceptable to any school, every mechanism among those being evaluated assign the same number of students. We therefore allow for students to have outside options and for schools to deem some students unacceptable.

Each student i has an outside option which yields utility λ_i. Therefore, a student would only accept being matched to a school if the utility that she derives from that school exceeds the value of λ_i.[10] The value of those outside options are also a combination of common and idiosyncratic values:

$$\lambda_i = \gamma\overline{\Theta} + (1-\gamma)\,\overline{\Theta_i}$$

For each problem and $i \in I$, $\overline{\Theta}$ and $\overline{\Theta_i}$ are independently drawn from the normal distribution with a mean of zero and variance 1. The exogenous parameter $\gamma \in [0,1]$ represents how correlated the value of the outside options are between students.

Schools' priorities over students follow a similar model. The ordinal priorities of school s over the students are derived from utility functions:

$$U_s(i) = \begin{cases} \beta\Theta^i + (1-\beta)\,\Theta^i_s & \text{if } \beta\Theta^i + (1-\beta)\,\Theta^i_s \geq \lambda_s \\ -\infty & \text{otherwise} \end{cases}$$

Here once again, for each problem, each value of Θ^i and Θ^i_s is independently drawn from the normal distribution with a mean of zero and variance 1. The concept of acceptability here, however, is not related to the presence of some "outside option" for the school. We interpret λ_s, instead, as an eligibility criterion. In exam schools, for example, it could be a minimum exam score for admission. For schools which give distance-based priority it could be a maximal distance requirement, and so on. For

[10] Although it may seem extreme to define the utility of being matched to any school with value below λ_i to be $-\infty$, that choice is inconsequential when we translate those utilities to ordinal preferences. That is, for any i,s such that $U_i(s) = -\infty$, it will simply be the case that school s is unacceptable to i: $\emptyset \; P_i \; s$.

each $s \in S$, λ_s is drawn independently from the normal distribution with mean λ^* and variance 1. Therefore, when $\lambda^* = -\infty$, no student is unacceptable to any school. Moreover, $\beta \in [0,1]$ is an exogenous parameter which represents the degree of correlation between schools' priority rankings. Notice that the case in which students may be unacceptable to schools is not considered in the theoretical analysis, and therefore those simulations should be taken as an additional experiment on the outcomes of those mechanisms under true preferences.

In each simulation performed, we set the values of the parameters $(n, m, Q, \alpha, \gamma, \beta, \lambda^*)$ and generated 100 problems, each representing different draws for values of the random variables. More specifically, in all simulations shown below, $n = 400$, $m = 20$ and every school had capacity $q = 20$. Every combination of the values of the parameters α, β and γ, in steps of 0.1, were used. In other words, every $(\alpha, \beta, \gamma) \in [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]^3$ was simulated.

For each problem generated, we produced the matching outcome for each of the five mechanisms : EAM, DA, BM, TTC, and SD, and recorded the number of students who remained unassigned.[11]

4.1 Case 1: No Student is Unacceptable to Any School

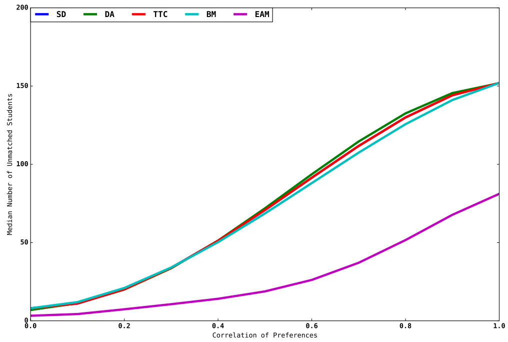
In this case we set the value of λ^* to be low enough such that no student is deemed unacceptable to any school.[12] This is often the case in school choice problems. Figure 1 shows the median value of the number of unmatched students across simulations, for each value of the indicated correlation parameter.[13] Two facts clearly stand out. One is that the median number of unmatched students, for any choice of fixed parameter among α, β and γ, is very similar between the DA, BM, TTC, and SD mechanisms. The second is how significant the difference is in the number of unmatched students between EAM and all the other mechanisms. When combining all the simulations performed in case 1, the DA, BM, TTC, and SD mechanisms had a median number of unmatched students of 60 or 61, while for EAM the value was 21, a reduction of 65% in the number of students unmatched.

In fact, when performing two-sided T-tests testing the null hypotheses that the number of unmatched students is the same between any two mechanisms, we are not able to reject the null hypothesis of them being equal at the 0.01 significance level for a wide range of parameters for the DA, BM, TTC, and SD mechanisms. That is not the case for any value of those parameters for any two-sided comparison between EAM

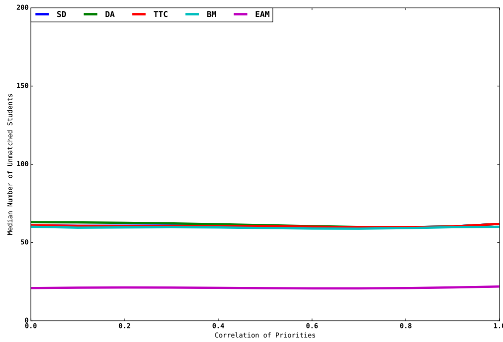
[11] For SD, following the principle behind the equilibrium results of EAM, the ordering of students that was used was drawn from a uniform distribution, independently of the schools' priorities.

[12] More specifically, the value of λ^* was set to -1.797×10^{308}, the lowest technically possible.

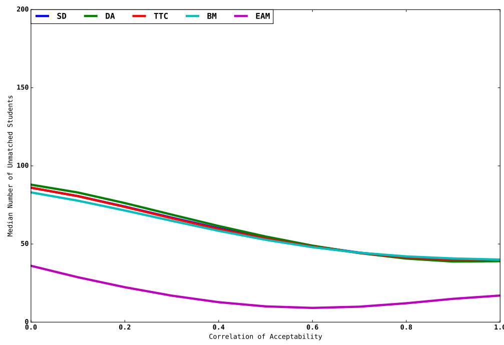
[13] For the purpose of presentation, the graphs in this section were generated by polynomial fitting of the simulation results.



Median Number of Unmatched Students as a function of α



Median Number of Unmatched Students as a function of β



Median Number of Unmatched Students as a function of γ

Figure 1: Median Number of Unmatched Students as a function of correlation parameters

and the other four mechanisms. Table 4.1 shows the precise results for all combinations of two mechanisms.

The values of the median and variance of the number of students unmatched for each mechanism and each value of α, β and γ can be found in the appendix.

		SD	DA	TTC	BM	AMM
SD	α					
	β	$---$				
	γ					
DA	α	$[0.0, 1.0]$				
	β	$[0.0, 1.0]$	$---$			
	γ	$[0.0, 1.0]$				
TTC	α	$[0.0, 1.0]$	$[0.0, 1.0]$			
	β	$[0.0, 1.0]$	$[0.0, 1.0]$	$---$		
	γ	$[0.0, 1.0]$	$[0.0, 1.0]$			
BM	α	$[0.0, 0.6] \cup [0.9, 1.0]$	$[0.0, 0.5] \cup [1.0]$	$[0.0, 0.6] \cup [0.9, 1.0]$		
	β	$[0.0, 1.0]$	$[0.0, 1.0]$	$[0.0, 1.0]$	$---$	
	γ	$[0.2, 1.0]$	$[0.5, 1.0]$	$[0.3, 1.0]$		
AMM	α	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
	β	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$---$
	γ	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

Table 1: Ranges of values for α, β and γ for which we cannot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms, at the 0.01 significance level (Case 1)

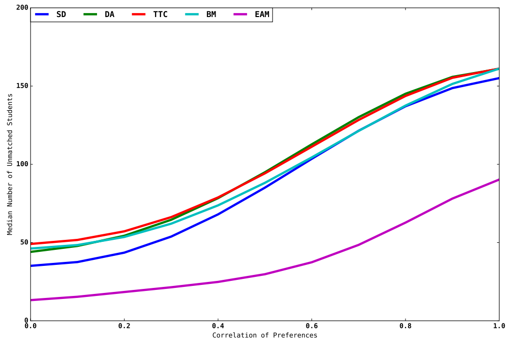
In light of propositions 5 and 7, which characterize the equilibrium outcome of EAMs and establishes a lower-bound on the number of assignments in equilibrium for FAMs, the simulation results are also informative about equilibrium results. Ergin and Sönmez (2006) showed that, under the assumptions that we used, every Nash equilibrium in undominated strategies for the BM is stable and therefore have the same number of assignments as DA.

So, to sum up, in equilibrium, EAMs have the same number of assignments as SD, BM the same as DA, and FAMs have at least the same number as DA. The results in table 4.1 imply, therefore, that there is no statistically significant difference between equilibrium outcomes of DA, BM, TTC, SD and AMMs, in terms of the number of assignments, for any of the combinations of parameters considered. Moreover, those results together with proposition 7 do not allow us to reject the hypothesis that equilibrium outcomes of FAMs are also indistinguishable from those outcomes as well.

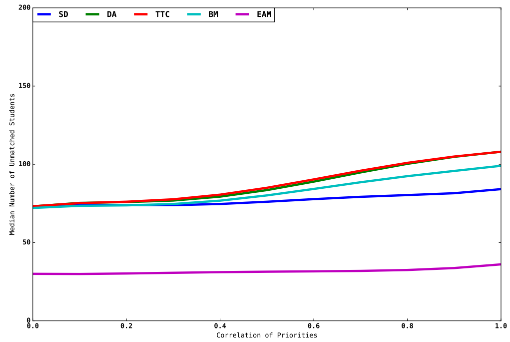
4.2 Case 2: Students May be Unacceptable

In this case we set $\lambda^* = -1$, that is, schools may find some students unacceptable. Figure 2 shows the median value of the number of unmatched students across simulations, for each value of the indicated correlation parameter.

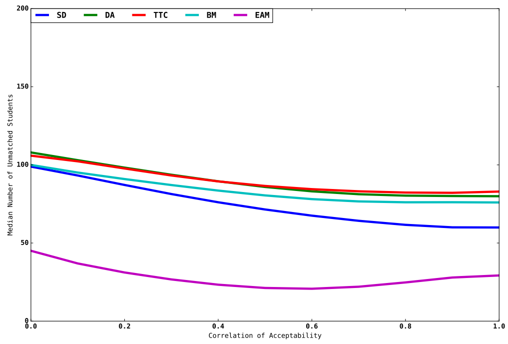
Similarly to case 1, EAM mechanisms perform significantly better than all other mechanisms in terms of the number of students matched, in all configurations of parameters evaluated. When combining all the simulations performed in case 2, the



Median Number of Unmatched Students as a function of α



Median Number of Unmatched Students as a function of β



Median Number of Unmatched Students as a function of γ

Figure 2: Median Number of Unmatched Students as a function of correlation parameters

mechanisms had more distinct performances, with SD, DA, TTC, BM, and EAM having a median number of unmatched students 77, 91, 91, 85, 32, respectively. Table 2 shows, for each pair of distinct mechanisms, the ranges of values for α, β and γ for which we cannot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms at the 0.01 significance level.

The values of the median and variance of the number of students unmatched for each mechanism and each value of α, β and γ can be found in the appendix.

		SD	DA	TTC	BM	AMM
SD	α					
	β	$---$				
	γ					
DA	α	$\emptyset$				
	β	$[0.0, 0.3]$	$---$			
	γ	$\emptyset$				
TTC	α	$\emptyset$	$[0.4, 1.0]$			
	β	$[0.0, 0.2]$	$[0.0, 1.0]$	$---$		
	γ	$\emptyset$	$[0.0, 1.0]$			
BM	α	$[0.7, 0.8]$	$[0.1, 0.3] \cup [1.0]$	$[1.0]$		
	β	$[0.0, 0.3]$	$[0.0, 0.5]$	$[0.0, 0.3]$	$---$	
	γ	$\emptyset$	$[0.6, 1.0]$	$[0.9, 1.0]$		
AMM	α	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
	β	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$---$
	γ	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

Table 2: Ranges of values for α, β and γ for which we canmot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms at the 0.01 significance level (Case 2)

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Appendix

Proofs

Proposition 1

(i). Let ψ be a fair mechanism. Consider a problem where $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, b, \emptyset; P_j : a, \emptyset.$$

$$\succ_a = \succ_b = i, j.$$

The unique maximal matching is μ' where $\mu'_i = b$ and $\mu'_j = a$. However, μ' is not fair, showing that no fair mechanism is maximal.

(ii). Assume for a contradiction that ψ is a strategy-proof and maximal mechanism. Consider a problem where $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let the priorities be such that $\succ_a = \succ_b$: i, j. Consider the problem P where $P_i : a, b, \emptyset$ and $P_j : a, \emptyset$.

As ψ is maximal, $\psi_i(P) = b$ and $\psi_j(P) = a$. Let $P'_i : a, \emptyset$ and $P' = (P'_i, P_j)$. Due to the strategy-proofness of ψ, $\psi_i(P') = \emptyset$ and $\psi_j(P') = a$. The latter is because ψ is maximal.

Let us now consider $P''_j : a, b, \emptyset$ and $P'' = (P'_i, P''_j)$. As ψ is maximal, $\psi_i(P'') = a$ and $\psi_j(P'') = b$. This, along with the fact that $\psi_j(P') = a$, implies that student

j profitably reports false preferences P'_j whenever the true preferences are P''. This, however, contradicts the strategy-proofness of ψ, which finishes the proof.

Proposition 2

Let us consider a problem consisting of $I = \{i, j, k\}$ and $S = \{a, b, c\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, \emptyset; P_j : \ a, b, c, \emptyset; P_k : \ b, a, c, \emptyset.$$

$$\succ_a: \ k, j, i; \succ_b: \ i, j, k; \succ_c: \ j, i, k.$$

In the above problem, the DA and BM produce the same matching, say μ, and it is such that $\mu_i = \emptyset$, $\mu_j = a$, and $\mu_k = b$. That is, $|\mu| = 2$. On the other hand, the TTC outcome, say μ', is such that $\mu'_i = a$, $\mu'_j = c$, and $\mu'_k = b$. That is, $|\mu'| = 3$. Hence, neither the DA nor the BM dominate the TTC.

Let us now consider $I = \{i, j, k, h\}$ and $S = \{a, b, c, d\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, b, \emptyset; P_j : \ a, \emptyset; P_k : \ d, b, c, \emptyset; P_h : d, \emptyset.$$

$$\succ_a: \ k, j, i; \succ_b: \ i, j, k; \succ_c: \ j, i, k; \succ_d: \ h, i, j, k.$$

The DA and BM outcomes are the same, say μ, where $\mu_i = b$, $\mu_j = a$, $\mu_k = c$, and $\mu_h = d$. On the other hand, the TTC outcome, say μ', is such that $\mu'_i = a$, $\mu'_j = \emptyset$, $\mu'_k = b$, and $\mu'_h = d$. Hence, $|\mu| > |\mu'|$, showing that the TTC does not dominate either of the DA and the BM.

For the non-existence of a domination relation between the DA and the BM, consider $I = \{i, j, k\}$ and $S = \{a, b, c\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, c, \emptyset; P_j : \ b, a, \emptyset; P_k : \ b, \emptyset.$$

$$\succ_a: \ k, j, i; \succ_b: \ k, i, j; \succ_c: \ j, i, k.$$

In the above problem, the DA outcome, say μ, is such that $\mu_i = c$, $\mu_j = a$, and $\mu_k = b$. On the other hand, the BM outcome, say μ', is such that $\mu'_i = a$, $\mu'_j = \emptyset$, and $\mu'_k = b$. Hence, $|\mu| > |\mu'|$, showing that the BM does not dominate the DA. Next, for the converse, consider the following preferences and priorities:

$$P_i : b, a, c, \emptyset; P_j : \ a, \emptyset; P_k : \ b, \emptyset.$$

$$\succ_a: \ k, i, j; \succ_b: \ k, i, j; \succ_c: \ j, i, k.$$

In the above problem, the DA outcome, say μ, is such that $\mu_i = a$, $\mu_j = \emptyset$, and $\mu_k = b$. On the other hand, the BM outcome, say μ', is such that $\mu'_i = c$, $\mu'_j = a$, and $\mu'_k = b$. Hence, $|\mu| < |\mu'|$, showing that the DA does not dominate the BM. This finishes the proof.

For the non-existence of a domination relation between the SD and the other mechanisms, consider $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, b; P_j : a, \emptyset.$$

$$\succ_a : j, i; \succ_b : j, i.$$

Let us consider the SD mechanism where student i comes first in the student ordering. Then, the SD outcome μ is such that $\mu_i = a$ and $\mu_j = \emptyset$. On the other hand, all the DA, TTC, and BM outcomes are the same, say μ', and it is such that $\mu'_i = b$ and $\mu'_j = a$. Hence, the SD mechanism does not size-wise dominate the DA, TTC, and BM.

Let us now consider the following preferences, with the same priorities as above.

$$P_i : a, \emptyset, P_j : a, b, \emptyset.$$

At the above problem, the SD outcome μ is such that $\mu_i = a$ and $\mu_j = b$. All the DA, TTC, and BM outcomes are the same, say μ', and it is such that $\mu'_i = \emptyset$ and $\mu'_j = a$. Hence, none of DA, TTC, and BM size-wise dominate the SD mechanism.

In the above market, the symmetric arguments easily show that there is no size-wise domination relation between the other SD mechanism where student j comes first, and the other mechanisms. This finishes the proof.

Theorem 1

We will use the following Lemma:

Lemma. *A maximal matching μ is efficient if and only if it does not admit an improving chain or cycle.*

Proof. **"Only If" Part.** Let μ be an efficient matching. If it admits an improving chain or cycle then we can implement it and obtain a new matching. By the definitions of improving chain and cycle, that matching dominates μ, contradicting our starting supposition.

"If" Part. Let μ be a maximal matching such that it does not admit an improving chain or cycle. We want to show that it is efficient. Assume for a contradiction that there exists a matching μ' that dominates μ.

Let $W = \{i \in I : \mu'_i P_i \mu_i\}$. By the supposition, $W \neq \emptyset$. Note that for any student i with $\mu_i \neq \emptyset$, we have $\mu'_i \neq \emptyset$. This, along with the maximality of μ, implies that $|\mu'| = |\mu|$. Hence, for any student i with $\mu_i = \emptyset$, $\mu'_i = \emptyset$.

Let us enumerate the students in $W = \{i_1, .., i_n\}$ and write $\mu'_{i_k} = c_k$ for any $k = 1, .., n$. If $|\mu_{c_k}| < q_{c_k}$ for some k then the pair $\{i_k, c_k\}$ would constitute an improving chain, which would yield a contradiction.

Let us suppose that $|\mu_{c_k}| = q_{c_k}$ for any $k = 1, .., n$. As school c_1 does not have excess capacity at μ, and $\mu'_{i_1} = c_1$, we have another student in W, say i_2, such that $\mu_{i_2} = c_1$. Then, consider student i_2, and as c_2 does not have excess capacity at μ and $\mu'_{i_2} = c_2$, we have another student in W, say i_3, such that $\mu_{i_3} = c_2$. If we continue to apply the same arguments to the other students in W, as W is finite, we would eventually obtain an improving cycle, which yields a contradiction. □

We can now proceed to the proof of the theorem. Let ψ be an EAM mechanism, and μ and μ' be its first stage and final outcome, respectively. As students are not assigned to one of their unacceptable schools in the course of Step 1 of ψ, μ is individually rational. Moreover, as Step 1 tries to match as many students as possible while preserving individual rationality, it is immediate to see that μ is maximal.

In Step 2 of ψ, new matchings are obtained by implementing improving chains and cycles (if any). By their definitions, in the course of Step 2, no student receives a worse school than his assignment μ. This, along with the individual rationality of μ, implies that μ' is maximal. The efficiency of μ' directly comes from the Lemma above.

Proposition 5

Let ψ be an EAM mechanism. By its definition, the first student in the ordering in Step 0 of the EAM obtains his top choice by reporting it as the only acceptable choice, irrespective of the other students' preference submissions. By the same reasoning, the second student in the ordering can obtain his top choice among the schools with available seats (after the capacity of the first student assignment is decreased by one) by reporting that school as his only acceptable choice, irrespective of the other students' preference submissions. Once we repeat the same arguments for every other student, we not only find an equilibrium of ψ, but also conclude that it is the unique equilibrium outcome, which coincides with the outcome of serial dictatorship with the ordering as the same as that in Step 0 of ψ.

Let ϕ be a FAM mechanism. Let μ be a stable matching at P. Consider the preferences submission P' under which for any student i, the only acceptable school is μ_i. Any unassigned student at μ reports no school acceptable at P'. It is easy to verify that $\phi(P') = \mu$.

Next, we claim that P' is an equilibrium submission under ϕ. Suppose for a contradiction that there exist a student i and P''_i such that $\phi_i(P''_i, P'_{-i}) P_i \phi_i(P')$. For ease of writing, let $\phi_i(P''_i, P'_{-i}) = s$ and $\phi_i(P') = s'$. As μ is stable, $|\mu_s| = q_s$. This, along with the definition of P' and $\phi_i(P''_i, P'_{-i}) = s$, implies that there exists a student $j \neq i$ such that $\mu_j = s$ and $\phi_j(P''_i, P'_{-i}) = \emptyset$. Moreover, from the stability of μ, we also have $j \succ_s i$. These altogether contradict the fairness for unassigned students of ϕ, showing that P' is equilibrium of ϕ.

Proposition 6

Let $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Assume for a contradiction that $\psi \in \Omega$ such that it is maximal in equilibrium.

Consider the preferences where $P_i : a, \emptyset$ and $P_j : a, b, \emptyset$. In any equilibrium at P, ψ places student i and student j at school a and b, respectively.

Consider the problem P' where $P'_i : a, \emptyset$ and $P'_j : a, \emptyset$. If there exists an equilibrium of ψ at P' under which student j is assigned to school a, then this submission constitutes an equilibrium at P as well. This, however, contradicts ψ being maximal in equilibrium. Hence, under any equilibrium at P', student i is assigned to school a while student j is unassigned.

Let us now consider the problem P'' where $P''_i : a, b, \emptyset$ and $P''_j = P'_j : a, \emptyset$. As ψ is maximal in equilibrium, under any equilibrium at P'', student i and student j have to be placed at school b and school a, respectively. Moreover, it is easy to see that any equilibrium at P' is also an equilibrium at P''. Hence, there exists an equilibrium at P'' under which student i is assigned to school a, and student j is unassigned. This, however, contradicts ψ being maximal in equilibrium, finishing the proof.

Theorem 3

In the proof, we will use the following lemma.

Lemma. *Let ψ be an EAM and ϕ be an individually rational mechanism. In any market $(I, S, \succ, q)$ and problem P, if $|\psi(P')| < |\phi(P'')|$ where P' and P'' are equilibria under ψ and ϕ, respectively, then there exists a student i such that $\psi_i(P') P_i \phi_i(P'') P_i \emptyset$.*

Proof. In a market $(I, S, \succ, q)$ and problem P, let $|\psi(P')| < |\phi(P'')|$ where P' and P'' are equilibria under ψ and ϕ, respectively. This implies that for some school s, $|\psi_s(P')| < |\phi_s(P'')| \leq q_s$. Hence, let $i \in \phi_s(P'') \setminus \psi_s(P')$. By the individual rationality of ϕ and P'' being equilibrium under ϕ, we have $sP_i\emptyset$, where $\phi_i(P'') = s$. As the unique equilibrium outcome of ψ coincides with the (truthtelling) outcome of a SD mechanism (Proposition 5), we have $\psi(P') = SD(P)$. Hence, school s has an excess capacity under $SD(P)$. Moreover, from above, $\psi_i(P') = SD_i(P) \neq s$. Hence, by the non-wastefulness of SD, i must be matched to a school strictly better than s and therefore $\psi_i(P') = SD_i(P) P_i \phi_i(P'') P_i \emptyset$, which finishes the proof. □

Let now $(I, S, \succ, q)$ be a market and ψ be an EAM mechanism. Assume for a contradiction that an individually rational mechanism ϕ size-wise dominates ψ in equilibrium. This in particular implies that for some problem P, $|\psi(P')| < |\phi(P'')|$ for every equilibria P' and P'' under ψ and ϕ, respectively. In what follows, we will fix one such pair P', P''. We prove the result in two steps.

Step 1. By the Lemma above, there exists a student i such that $\psi_i(P')\, P_i \phi_i(P'')\, P_i \emptyset$. Let $\bar{P}_i$ be the preference relation that keeps the relative rankings of the schools the same as under P_i, while reporting any school that is worse than $\psi_i(P')$ as unacceptable. In other words, $\bar{P}_i$ truncates P_i below $\psi_i(P')$. Let us write $\bar{P} = (\bar{P}_i, P_{-i})$. Recall that the unique equilibrium outcome of ψ always coincides with the truthtelling outcome of a SD mechanism (Proposition 5). Moreover, by the construction of $\bar{P}$, $SD(P) = SD(\bar{P})$. This in turn implies that $\psi(P') = \psi(\bar{P}')$ for every equilibrium $\bar{P}'$ under ψ in problem $\bar{P}$.

We next consider problem $\bar{P}$. If there exists no student j such that $\psi_j(\bar{P}')\, \bar{P}_j \phi_j(\bar{P}'')\, \bar{P}_j \emptyset$ for some equilibria $\bar{P}'$ and $\bar{P}''$ under ψ and ϕ, respectively, then we move to Step 2. Otherwise, we pick such student j. Note that because of the definition of $\bar{P}_i$ states that any outcome below $\psi_i(\bar{P}')$ is unacceptable for i and ϕ is individually rational, $\psi_j(\bar{P}')\, \bar{P}_j \phi_j(\bar{P}'')\, \bar{P}_j \emptyset$ cannot hold for $j = i$, therefore $j \neq i$. Then, as the same as above, let $\bar{P}_j$ be the preference list that truncates P_j below $\psi_j(\bar{P}')$. Let us write $\tilde{P} = (\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. By the same reason as above, $\psi(P') = \psi(\tilde{P}')$ for any equilibrium $\tilde{P}'$ under ψ in problem $\tilde{P}$.

We next consider problem $\tilde{P}$. If there exists no student k such that $\psi_k(\tilde{P}')\, \tilde{P}_k \phi_k(\tilde{P}'')\, \tilde{P}_k \emptyset$ for some equilibria $\tilde{P}'$ and $\tilde{P}''$ under ψ and ϕ, respectively, then we move to Step 2. Otherwise, we pick such a student k. By the same reason as above, student k is different than both i and j. Then, we follow the same arguments above and obtain a new preference profile. In each iteration, we have to consider a different student. But then, since there are finitely many students, this case cannot hold forever. Hence, we eventually obtain a problem, say $\hat{P}$, in which there exists no student h such that $\psi_h(\hat{P}')\hat{P}_h\phi_h(\hat{P}'')\, \hat{P}_h \emptyset$ for some equilibria $\hat{P}'$ and $\hat{P}''$ under ψ and ϕ, respectively, and move to Step 2. We also have $\psi(P') = \psi(\hat{P}')$ for any equilibrium $\hat{P}'$ under ψ in problem $\hat{P}$.

Step 2. By the Lemma above, in problem $\hat{P}$, we have $|\psi(\hat{P}')| \geq |\phi(\hat{P}'')|$ for any equilibria $\hat{P}'$ and $\hat{P}''$ under ψ and ϕ, respectively. If it holds strictly for some equilibria, then we reach a contradiction. Suppose $|\psi(\hat{P}')| = |\phi(\hat{P}'')|$ for any equilibria $\hat{P}'$ and $\hat{P}''$.

We now claim that $\hat{P}''$ is an equilibrium under ϕ in problem P. Suppose it is not, and let student k have a profitable deviation, say $\ddot{P}_k$, from $\hat{P}''_k$. This means that $\phi_k(\ddot{P}_k, \hat{P}''_{-k})\, P_k \phi_k(\hat{P}'')$. But then, by construction above, $\hat{P}_k$ preserves the relative rankings under P_k. This implies that $\phi_k(\ddot{P}_k, \hat{P}''_{-k})\, \hat{P}_k \phi_k(\hat{P}'')$, contradicting $\hat{P}''$ being an equilibrium under ϕ in problem $\hat{P}$.

Recall that $\psi(P') = \psi(\hat{P}')$. Hence, this, along with $|\psi(\hat{P}')| = |\phi(\hat{P}'')|$ and our above finding, implies that in problem P, $|\psi(P')| = |\phi(\hat{P}'')|$ where P' and $\hat{P}''$ are equilibria under ψ and ϕ, respectively. Therefore, we constructed an equilibrium pair for problem P where ψ matches as many students as ϕ, contradicting our assumption that this does not hold in problem P.

Proposition 7

(i). First, by the Rural Hospital Theorem (Roth, 1984), the number of assignments in any stable matching is the same as that of DA. Let ψ be a *FAM* mechanism. Assume for a contradiction that there exist a problem P and an equilibrium profile P' under ψ such that $|\psi(P')| < |DA(P)|$. For ease of writing, let $DA(P) = \mu$ and $\psi(P') = \mu'$.

We now claim that for some student i, $\mu_i = s$ for some school s whereas $\mu'_i = \emptyset$ and, moreover, $|\mu'_s| < q_s$. To prove this claim, let us define $W = \{i \in I : \mu_i = s$ and $\mu'_i = \emptyset\}$. By our supposition that $|DA(P)| > |\psi(P')|$, we have $W \neq \emptyset$. Suppose that for each $i \in W$ with $\mu_i = s$, $|\mu'_s| = q_s$. But then this implies that $|\mu'| \geq |\mu|$, contradicting our initial supposition, which finishes the proof of the claim.

Let $i \in I$ such that $\mu_i = s$, $\mu'_i = \emptyset$, and $|\mu'_s| < q_s$. Now, consider the following preferences P'':

$$P''_k = \begin{cases} P'_k & \text{If } k \neq i \\ s, \emptyset & \text{If } k = i \end{cases}$$

First, observe that there exists a (individually rational) matching at P'' that assigns $|\mu'| + 1$ many students (to see this, keep the assignment of everyone except student i the same as at μ', and place student i at school s). Therefore, due to the maximality of ψ, we have $|\psi(P'')| \geq |\mu'| + 1$. If student i is assigned to school s at $\psi(P'')$ then this contradicts P' being equilibrium under ψ. Hence, $\psi_i(P'') = \emptyset$. But then, by the definition of P'', $\psi(P'')$ is individually rational at P'. This, along with the maximality of ψ, implies that $|\psi(P')| \geq |\psi(P'')|$, contradicting our previous finding that $|\psi(P'')| \geq |\psi(P')| + 1$, which finishes the proof of the first part.

(ii). Let us consider $I = \{i, j, k, h\}$ and $S = \{a, b, c\}$, each with unit capacity. The preferences and the priorities are given below.

$$P_i : a, b, \emptyset;\ P_j : c, a, \emptyset;\ P_k : c, a, \emptyset;\ P_h : c, \emptyset.$$

$$\succ_a : k, i, j, h;\ \succ_b : k, h, j, i;\ \succ_c : k, h, i, j.$$

Let ψ be a *FAM* mechanism with the student ordering k, j, i, h. Mechanism ψ is such that it produces matching μ at P where $\mu_i = b$, $\mu_j = a$, $\mu_k = c$, and $\mu_h = \emptyset$. For any $P'_i \in \mathcal{P}$ with $bP'_i\emptyset$, let $\psi(P'_i, P_{-i}) = \mu'$ where $\mu'_i = b$, $\mu'_j = \emptyset$, $\mu'_k = a$, and $\mu'_h = c$.

Moreover, for any $P'_i \in \mathcal{P}$ with $\emptyset P'_i b$, $\psi(P'_i, P_{-i}) = \mu''$ where $\mu''_i = \emptyset$, $\mu''_j = \emptyset$, $\mu''_k = a$, and $\mu''_h = c$. And, for any $P'_h \in \mathcal{P}$, let $\psi(P_{-h}, P'_h) = \mu$.

Note that student j can never get school c under ψ by misreporting because otherwise student h would be unassigned, and he has higher priority at school c. It is immediate to see that the above matchings can be obtained in the course of FAM through particular selection. All of these show that under ψ, truth-telling is an equilibrium at P, and $|\psi(P)| = 3$. On the other hand, $DA(P)$ is such that $DA_i(P) = a$, $DA_k(P) = c$, and $DA_h(P) = DA_j(P) = \emptyset$. Hence, $|\psi(P)| > |DA(P)|$, finishing the proof of the second part.

Description of mechanisms

The Deferred Acceptance Mechanism (DA)

Step 1. Each student applies to her favorite acceptable school. Each school tentatively accepts the students among its applicants one at a time following its priority order up to its capacity, and rejects the rest.

In general,

Step k. Each rejected student in the previous step applies to her next favorite acceptable school. Each school tentatively accepts the students among its current step applicants and the tentatively accepted ones in the previous step one at a time following its priority order, and rejects the rest.

The algorithm terminates whenever any student is tentatively accepted by a school or has all acceptable applications rejected. The tentative assignments in the terminal step become the final DA assignments.

The Top Trading Cycles Mechanism (TTC)

Step 1. Each student points to her favorite acceptable school. Each school points to the highest priority student. As both the sets of students and schools are finite, there exists a cycle. Assign each student in a cycle to the school he is pointing to, and decrease the capacity of each school appearing in a cycle by one.

In general,

Step k. Each unassigned student points to her favorite acceptable school with remaining capacity. Each school with an empty seat points to the highest priority unassigned student. As there are finitely many unassigned students and schools with remaining capacity, there exists a cycle. Assign each student in a cycle to the school he is pointing to, and decrease the remaining capacity of each school appearing in a cycle by one.

The algorithm terminates whenever any student is assigned or all of his acceptable schools exhaust their capacities.

Boston Mechanism (*BM*)

Step 1. Each student applies to her best acceptable school. Each school permanently accepts the students among its applicants one at a time following its priority order up to its capacity, and rejects the rest.

In general,

Step k. Each rejected student applies to her next best acceptable school. Each school with remaining capacity permanently accepts the students among its current step applicants one at a time following its priority order up to its remaining capacity, and rejects the rest.

The algorithm terminates whenever any student is assigned or all of his acceptable schools exhaust their capacities.

Serial Dictatorship (*SD*)

Step 0. Enumerate the students $I = \{i_1, .., i_n\}$.

Step 1. Start with the first student i_1, and let him choose his top acceptable school with an available seat. Decrease the capacity of his assigned school by one while keeping the capacity of every other school the same. If there is no acceptable school with an available seat, then leave him unassigned.

In general,

Step k. Let student i_k choose his top acceptable school among those with an available seat. Decrease the capacity of his assigned school by one while keeping the capacity of every other school the same. If there is no acceptable school with an available seat then leave him unassigned.

The algorithm terminates by the end of Step n. The above description indeed defines a class of mechanisms, each member of which is associated with a different enumeration in Step 0. We call any mechanism in this class serial dictatorship (SD).

Simulation results

Case 1: No student is unacceptable to any school

α	SD	DA	TTC	BM	AMM
0.0	8 (28.49)	7 (28.65)	8 (28.5)	8 (28.41)	3 (28.55)
0.1	11 (32.93)	11 (33.13)	11 (32.92)	12 (32.79)	5 (32.85)
0.2	20 (41.86)	20 (42.17)	20 (41.87)	21 (41.68)	7 (41.06)
0.3	34 (50.85)	34 (51.19)	34 (50.84)	34 (50.56)	10 (49.18)
0.4	51 (57.47)	51 (57.82)	51 (57.46)	50 (57.14)	14 (55.81)
0.5	71 (65.87)	72 (66.14)	71 (65.88)	69 (65.58)	20 (66.31)
0.6	91 (71.85)	93 (71.99)	91 (71.88)	88 (71.71)	26 (76.3)
0.7	112 (75.68)	115 (75.64)	112 (75.68)	107 (75.71)	36 (84.76)
0.8	130 (76.52)	133 (76.31)	130 (76.47)	126 (76.66)	52 (89.18)
0.9	144 (76.96)	145 (76.81)	144 (76.95)	141 (77.12)	68 (92.52)
1.0	152 (75.82)	152 (75.85)	152 (75.85)	152 (75.85)	81 (92.84)

Table 3: Median and standard deviation for the number of unmatched students, varying α from 0.0 to 1.0 (Case 1)

β	SD	DA	TTC	BM	AMM
0.0	61.0 (81.28)	63.0 (82.22)	61.0 (81.3)	60.0 (80.69)	21.0 (76.73)
0.1	61.0 (80.11)	63.0 (81.11)	61.0 (80.16)	60.0 (79.53)	21.0 (75.38)
0.2	60.0 (80.31)	62.0 (81.21)	60.0 (80.3)	59.0 (79.68)	21.0 (75.25)
0.3	61.5 (81.42)	63.0 (82.17)	61.0 (81.39)	60.0 (80.8)	22.0 (76.9)
0.4	61.0 (81.67)	62.0 (82.32)	61.0 (81.66)	60.0 (81.03)	21.0 (77.28)
0.5	60.0 (80.08)	60.0 (80.59)	60.0 (80.07)	59.0 (79.46)	20.0 (75.22)
0.6	60.0 (80.89)	61.0 (81.22)	60.0 (80.88)	59.0 (80.3)	21.0 (76.42)
0.7	60.0 (80.57)	60.0 (80.77)	60.0 (80.57)	59.0 (79.95)	21.0 (75.95)
0.8	60.0 (80.13)	60.0 (80.22)	60.0 (80.13)	59.0 (79.53)	21.0 (75.38)
0.9	60.0 (80.4)	60.0 (80.44)	60.0 (80.41)	60.0 (79.82)	21.0 (75.77)
1.0	62.0 (82.15)	62.0 (82.14)	62.0 (82.14)	60.0 (81.55)	22.0 (77.85)

Table 4: Median and standard deviation for the number of unmatched students, varying β from 0.0 to 1.0 (Case 1)

γ	SD	DA	TTC	BM	AMM
0.0	86.0 (47.46)	88.0 (48.12)	86.0 (47.49)	83.0 (46.52)	36.0 (20.18)
0.1	81.0 (49.51)	83.0 (50.19)	81.0 (49.55)	78.0 (48.51)	29.0 (21.78)
0.2	73.0 (53.1)	76.0 (53.76)	73.0 (53.09)	71.0 (52.13)	22.0 (27.64)
0.3	68.0 (57.83)	69.0 (58.48)	67.0 (57.85)	65.0 (56.85)	17.0 (36.49)
0.4	60.0 (63.05)	62.0 (63.63)	60.0 (63.02)	59.0 (62.03)	13.0 (45.51)
0.5	53.0 (70.3)	54.0 (70.93)	53.0 (70.29)	52.0 (69.38)	10.0 (58.25)
0.6	49.0 (79.21)	49.0 (79.77)	48.0 (79.15)	48.0 (78.35)	9.0 (71.71)
0.7	45.0 (89.01)	45.0 (89.54)	45.0 (89.0)	45.0 (88.31)	10.0 (86.04)
0.8	41.0 (101.12)	40.0 (101.66)	41.0 (101.13)	41.5 (100.61)	12.0 (101.22)
0.9	40.0 (110.93)	39.0 (111.43)	40.0 (110.94)	41.0 (110.51)	15.0 (113.0)
1.0	40.0 (120.95)	39.0 (121.41)	40.0 (120.94)	40.0 (120.54)	17.0 (123.78)

Table 5: Median and standard deviation for the number of unmatched students, varying γ from 0.0 to 1.0 (Case 1)

Case 2: Students may be unacceptable

α	SD	DA	TTC	BM	AMM
0.0	35 (32.29)	44 (35.9)	49 (34.26)	46 (32.99)	13 (33.26)
0.1	38 (38.43)	48 (40.8)	52 (39.22)	49 (38.4)	16 (40.12)
0.2	43 (44.95)	54 (46.15)	57 (44.71)	53 (44.27)	18 (46.55)
0.3	54 (53.29)	65 (53.13)	66 (51.85)	62 (51.93)	21 (54.53)
0.4	68 (60.06)	78 (58.78)	79 (57.7)	74 (58.2)	25 (61.98)
0.5	85 (65.82)	95 (63.75)	94 (62.92)	88 (63.76)	30 (69.18)
0.6	104 (71.33)	113 (68.62)	112 (68.08)	105 (69.41)	38 (78.28)
0.7	121 (74.43)	130 (71.38)	128 (71.15)	121 (72.62)	48 (84.72)
0.8	137 (74.87)	145 (71.72)	143 (71.65)	137 (73.14)	62 (88.18)
0.9	149 (75.16)	156 (72.12)	156 (72.11)	152 (73.18)	79 (91.14)
1.0	155 (74.32)	161 (71.58)	161 (71.58)	161 (71.58)	90 (90.68)

Table 6: Median and standard deviation for the number of unmatched students, varying α from 0.0 to 1.0 (Case 2)

β	SD	DA	TTC	BM	AMM
0.0	73 (77.73)	73 (79.29)	73 (77.36)	72 (76.55)	30 (77.02)
0.1	75 (77.36)	76 (78.89)	76 (76.89)	74 (76.19)	30 (77.02)
0.2	73 (77.85)	75 (78.97)	75 (76.89)	73 (76.47)	30 (77.82)
0.3	74 (78.05)	77 (78.58)	78 (76.33)	75 (76.37)	31 (78.26)
0.4	75 (75.7)	80 (75.24)	81 (73.1)	77 (73.55)	31 (75.9)
0.5	76 (77.95)	83 (75.98)	85 (74.03)	80 (75.04)	31 (78.78)
0.6	78 (75.93)	89 (72.36)	90 (70.86)	84 (72.24)	32 (77.14)
0.7	79 (75.45)	95 (70.05)	96 (69.09)	89 (70.87)	32 (76.94)
0.8	80 (74.65)	100 (67.84)	101 (67.37)	92 (69.41)	32 (76.76)
0.9	82 (74.47)	105 (66.57)	105 (66.45)	96 (68.71)	34 (76.87)
1.0	84 (74.9)	108 (66.4)	108 (66.4)	99 (68.69)	36 (77.65)

Table 7: Median and standard deviation for the number of unmatched students, varying β from 0.0 to 1.0 (Case 2)

γ	SD	DA	TTC	BM	AMM
0.0	99.0 (42.3)	108.0 (42.4)	106.0 (41.52)	100.0 (40.95)	45.0 (21.06)
0.1	93.0 (44.3)	103.0 (44.46)	102.0 (43.42)	95.0 (42.82)	37.0 (24.0)
0.2	87.0 (47.8)	98.0 (47.72)	98.0 (46.52)	91.0 (45.91)	31.0 (29.75)
0.3	82.0 (52.92)	94.0 (52.47)	93.0 (51.1)	87.0 (50.73)	27.0 (39.01)
0.4	76.0 (58.68)	90.0 (58.06)	90.0 (56.53)	84.0 (56.27)	23.0 (49.05)
0.5	71.0 (66.66)	85.0 (65.49)	86.0 (63.82)	80.0 (63.78)	21.0 (61.25)
0.6	67.0 (75.38)	83.0 (73.8)	84.0 (72.01)	78.0 (72.28)	21.0 (74.42)
0.7	65.5 (85.64)	82.0 (83.49)	84.0 (81.64)	77.0 (82.2)	23.0 (88.56)
0.8	61.0 (95.31)	80.0 (92.64)	82.0 (90.66)	76.0 (91.56)	23.0 (100.71)
0.9	60.0 (107.81)	80.0 (104.5)	82.0 (102.54)	76.0 (103.71)	29.0 (115.26)
1.0	60.0 (114.66)	80.0 (111.05)	83.0 (109.05)	76.0 (110.38)	29.0 (122.64)

Table 8: Median and standard deviation for the number of unmatched students, varying γ from 0.0 to 1.0 (Case 2)