ESSAYS IN MICROECONOMIC THEORY

by

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ABSTRACT

ESSAYS IN MICROECONOMIC THEORY

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Keywords Refinements of Nash Equilibrium; Random Assignment Problem; Aggregate Efficieny; R1 Mechanism; Information Acquisition.

This thesis consists of three independent chapters. Each of them represents an area of my research interests. The first chapter of thesis contributes to the Game Theory. We propose a complexity measure and an associated refinement based on the observation that best responses with more variations call for more precise anticipation. The variations around strategy profiles are measured by considering the cardinalities of players' pure strategy best responses when others' behavior is perturbed. After showing that the resulting selection method displays desirable properties, it is employed to deliver a refinement: the tenacious selection of Nash equilibrium. We prove that it exists; does not have containment relations with perfection, properness, persistence and other refinements; and possesses some desirable features.

The second chapter of this thesis contributes to the random assignment problem literature. We introduce aggregate efficiency (AE) for random assignments (RA) by requiring higher expected numbers of agents be assigned to their more preferred choices. It is shown that the realizations of any aggregate efficient random assignment (AERA) must be an AE permutation matrix. While AE implies ordinally efficiency, the reverse does not hold. And there is no mechanism treating equals equally while satisfying weak strategyproofness and AE. But, a new mechanism, the reservation-1 (R1), is identified and shown to provide an improvement on grounds of AE over the probabilistic serial mechanism of Bogomolnia et al. (2001). We prove that R1 is weakly strategyproof, ordinally efficient, and weak envy–free. Moreover, the characterization of R1 displays that it is the probabilistic serial mechanism updated by a principle decreed by the Turkish parliament concerning the random assignment of new doctors.

In the third chapter, we consider a NIRMP matching marketplace consisting of ordered set of doctors and hospitals, and two-stage Interviewing and Preference

Reporting Game where hospitals acquire information through interviews and submit contingent rankings to a center enforcing university-optimal matching. In this setting, we provide a 'simple' example in which there exist no pure strategy Nash equilibrium. Then, we characterize a domain (of doctors' preferences) where each hospital's interview set forms a 'ladder'.

ÖZET

ESSAYS IN MICROECONOMIC THEORY

Zeynel Harun Alioğulları

Ekonomi Doktora Tezi, Ocak 2015

Danışman: Prof. Ahmet Alkan

Anahtar Kelimeler Nash Dengesi; Rassal Atama Problemi; Toplam Verimlilik; R1 Mekanizması; Bilgi Edinimi

Bu tez çalışması üç bağımsız kısımdan oluşmakta ve bu kısımların her biri bir araştırma alanıma girmektedir. Birinci kısım Oyun Teorisi'ne katkıda bulunmaktadır. Bu kısımda en iyi tepki fonksiyonlarının değişkenliğini baz alarak bir komplekslik ölçütü ve bunun üzerinden bir Nash dengesi seleksiyonu önermekteyiz. Strateji profillerinin etrafındaki varyasyon, her oyuncu için diğer oyuncuların davranışları değiştiğinde en iyi tepki fonksiyonunun içerdiği pür stratejilerin kardinalitesi ile ölçülmektedir. Buradan ortaya çıkan seçim metodunun istenilen özellikleri sağladığını gösterdikten sonra, bir Nash dengesi düzeltmesi olarak Nash dengesinin direngen seçilimini sunuyoruz. Sonra ise, her oyun için var olduğunu, diğer bilinen Nash dengesi düzeltmeleri ile herhangi bir mantıksal içerim ilişikisinde olmadığını gösteriyor ve bir dengede aranan bazı özellikleri taşıdığını gösteriyoruz.

Bu tezin ikinci kısmı rassal eşleşme literatürüne katkıda bulunmaktadır. Rassal Eşleşmeler (RE) için kişilerin ilk tercihlerine yerleşme oranı üzerinden hesaplanan Toplam Verimlilik (TV) kavramını tanıtıyor, herhangi bir toplam verimli rassal eşleşmenin gerçekleşmelerinin her birinin TV permutasyon matrisi olduğunu gösteriyoruz. Sırasal verimliliğin TV'yi kapsadığını fakat tersinin doğru olmadığını, eşitlere eşit davranıp zayıf manipülasyona-kapalı ve TV bir mekanizmanın var olmadığını gösteriyor ve yeni bir mekanizma olarak R1 mekanizmasını öneriyoruz. Bu mekanizma yaygın Seri Olasılıksal (SO) (Bogomolnia et al. 2001) mekanizmadan TV olarak daha iyi rassal eşleşmeler önermektedir. Bunun yanında R1 zayıf manipülasyona-kapalı, sırasal verimli ve zayıf kıskançlıktan-muaf bir mekanizmadır. R1'nın karakterizasyonu ise Türkiye'de doktor atamalarında kullanılan bir kuralın SO mekanizmanın karakterizasyonuna uygulanması ile yapılmaktadır.

Üçüncü kısımda, sıralı doktorlar ve hastanelerin olduğu bir eşleşme marketini ele alıp, burada hastanelerin doktorlarla yaptıkları mülakatlar ile bilgi elde ettiği iki aşamalı bir mülakat ve tercih bildirimi oyununu inceliyoruz. Burada tercihler bir merkeze bildirilmekte ve merkez hastane-optimal eşleşmeyi uygulamaktadır. Bu

ortamda, basit bir örnekle pür strateji Nash dengesinin olmadığı durumların varlığını gösteriyor, daha sonra ise hastanelerin mülakatlarının bir "merdiven özelliği" taşıdığı doktor tercihleri kümesini karakterize ediyoruz.

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The others are in mind.

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CHAPTER 1

TENACIOUS SELECTION OF NASH EQUILIBRIUM

1.1 Introduction

The concept of Nash equilibrium (henceforth to be abbreviated as NE) is central in the theory of games, and as put by Myerson (1978), "it is one of the most important and elegant ideas in game theory". On the other hand, Nash's pointwise stability may create multiplicity of equilibria some of which do not satisfy local stability and produce outcomes that can be criticized on grounds of not corresponding to intuitive notions about how plausible behavior should look like. In order to alleviate these problems, important refinements of NE have been developed: perfection by Selten (1975), properness by Myerson (1978), and persistence by Kalai and Samet 1984, among others, have been standards in the theory of games.

However, complex equilibrium anticipation may still be needed. The following game with three players has three NE, $s^1 = (I, I, I)$, $s^2 = (II, II, I)$, and $s^3 = (II, II, II)$:

3

	I				II	
$1\backslash 2$	I	II		$1\backslash 2$	I	II
I	$(1,1,1)$	$(0,0,0)$		I	$(0,0,0)$	$(0,0,0)$
II	$(0,0,0)$	$(0,0,0)$		II	$(0,0,0)$	$(1,1,0)$

Only s^1 and s^2 are perfect and proper as s^3 involves a weakly dominated strategy. The behavior in s^2 corresponds to a coordination failure, hence, is undesirable; and "very specific set of trembles is needed to justify" this equilibrium (Kalai and Samet 1984): for II to appear in player 1's perturbed best response, player 1 has to anticipate that the mistake of player 2 about choosing I instead of II has to be strictly less than the mistake of player 3 about choosing II instead of I.[1] This is a clear display of the serious requirements imposed on players' anticipation capacities: even when every player is making mistakes about his own choices, his assessment about the magnitudes of others' mistakes needs to be correct.

1. Considering a perturbation around s^2 with $s^2_\varepsilon = (\varepsilon_1 I + (1-\varepsilon_1)II, \varepsilon_2 I + (1-\varepsilon_2)II, (1-\varepsilon_3)I + \varepsilon_3 II)$ with $\varepsilon_i > 0$ for $i = 1,2,3$, one can observe that s^2_ε is an ε–perfect equilibrium only if $\varepsilon_3 > \varepsilon_2$ and $\varepsilon_3 > \varepsilon_1$.

On the other hand, s^1 is more desirable on account of involving less complex anticipation: every players' only best response to any one of the others' strategies that are sufficiently close and possibly equal to the one given by s^1, is as given by s^1. Hence, when the approximation is sufficiently precise, the local behavior of each player's best response around s^1 does not involve any variations. Then, each player needs only minimal anticipation capacities. The numbers of actions that appear in best responses around s^2 are given by 2 for player 1, 2 for player 2, and 1 for player 3 even with arbitrarily precise approximation; and these numbers are given by 1 for player 1, 1 for player 2, and 2 for player 3 when considering s^3.

In order to formalize these ideas, the current study proposes the notion of *tenacious selection*: given any strategy profile and any player, we consider the number of pure strategies that may appear in that player's best response when others may choose a strategy vector that is either arbitrarily close or equal to the one specified. By employing the upper hemi continuity of best responses, we show that this integer attains a limit, a lower bound greater or equal to one, before the approximation terms reach zero. We refer to this as the t–index of the given strategy and player, and the t–index of a strategy profile is a vector of t–indices where each coordinate is associated with the t–index of the corresponding player of that given strategy profile. Given any set of strategies, one of its elements belongs to its tenacious selection whenever there is no other element of the same set which has a t–index less than or equal to and not equal to that of the strategy under consideration.

The method of tenacious selection is a low–cost notion of complexity aversion. A higher t–index of a given strategy and player implies that player's optimal plan of action displays more variations around that strategy, hence, demands more accurate anticipation of others' behavior. So strategy profiles involving lower t–indices are more appealing on grounds of complexity aversion.[2] The identification of such strategies involves the simple act of counting the relevant actions while more complicated methods are also available. Indeed, the demonstration that this low–cost method displays a solid performance, we think, is noteworthy.

Tenacious selection of strategies with the best response property is of particular interest, and leads us to the *tenacious selection of Nash equilibrium* (TSNE hereafter): every NE in the TSNE involves less complex anticipation by all the players and this holds strictly for least one of the players when compared with those of the NE that are not in the TSNE.

2. Our notion involves complexity of implementation rather than that of computation, and philosophical aspects of various complexity formulations are not addressed in the current study.

After proving that tenacious selection of any nonempty set of strategy profiles exists, we analyze the TSNE of finite normal–form games and display that it is an idiosyncratic refinement of NE as it does not have any containment relations with the notions of perfection, properness, persistence, among other refinement concepts.[3] In fact, the TSNE equals the set of strict NE whenever there is one.[4] In such cases, apart from containing neither mixed nor weakly dominated NE while being lower hemi continuous, the TSNE does not display a weaker refinement performance in comparison with perfection and properness and persistence and settledness because it is their subset and this relation may be strict. And our further findings indicate that the TSNE is not logically related to these notions even when attention is restricted to games that have no strict NE and neither redundant nor weakly dominated actions. Moreover, we show that the TSNE does not get affected by the elimination of strictly dominated strategies, hence, it is immune to the criticisms of Myerson known as imperfections of perfection which were directed to perfection (Myerson 1978). However, when there is no strict NE, both a pure NE and a mixed NE may be in the TSNE; it may contain a weakly dominated NE, and is not lower hemi continuous.[5]

The notion that is most closely related with the TSNE is persistent equilibrium (PE, henceforth). When there is no strict NE interesting distinctions between these notions surface. The TSNE involves "local" considerations: whether or not the behavior in a specified NE is plausible is judged only with pure strategies which can appear in players' best responses when fine perturbations are considered. On the other hand, the minimality requirement of the essential Nash retracts in the definition of the PE implies that considerations of whether or not equilibrium behavior is plausible may have to incorporate the whole game, hence, they are rather "global".[6] We take the stand that the actions considered to be relevant in the determination of the plausibility of behavior in an equilibrium should involve only the pure strategies that can appear in players' best responses when fine perturbations are considered.

3. These are regular equilibrium (Harsanyi 1973b), essential equilibrium (Wu and Jia-He 1962), strongly stable equilibrium (Kojima, Okada, and Shindoh 1985), and settled equilibrium (Myerson and Weibull 2013).

4. A NE is strict if and only if deviations strictly hurt the deviators. Clearly, strict NE must be pure.

5. In order to dismiss weak domination, one may consider the notion of tenacious selection of undominated NE. Indeed, using our techniques, it is easy to show that all our results continue to hold. Another alternative is to consider the tenacious selection of perfect (alternatively, proper) equilibrium.

6. For the details and formal presentation please see the discussion following example 4 on page 9.

This enables us to present the notion of the TSNE not only as a concept based on complexity aversion, but also as one that has a similar motivation as the PE but with the novel feature of evaluating plausibility of equilibria through local considerations. But while global considerations help persistence to tackle weak domination, the local evaluation measure of the TSNE does not discriminate between weakly dominated NE and mixed NE. As a result, weakly dominated NE may be elements of the TSNE.

It is useful to emphasize that the trembles employed in the current paper are due to players' imprecise anticipation of their opponents' actions. Hence, our approach is immune to the arguments of (Kreps 1990) advocating that classical refinements literature is flawed because there is no explanation for the trembles (see also (Fudenberg, Kreps, and Levine 1988) and (Dekel and Fudenberg 1990)). Moreover, while considerations with approximate common knowledge (Monderer and Samet 1989) and employing incomplete information settings to formulate higher order beliefs (Kajii and Morris 1997a) (Kajii and Morris 1997b) are very interesting, the current study lies within the framework of common knowledge and complete information.

The next section presents the preliminaries and the method of tenacious selection and section 3 the important properties of the TSNE. Section 4 concludes.

1.2 Definitions and Auxiliary Results

Let $\Gamma = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite normal–form game where A_i is a finite nonempty set of actions (alternatively, pure strategies) of player $i \in N$ and $u_i : \times_{i\in N} A_i \to \mathbb{R}$ is agent i's von Neumann Morgenstern utility function. We keep the standard convention that $A = \times_{i\in N} A_i$ and $A_{-i} = \times_{j\neq i} A_j$. A *mixed strategy* of player i is represented by $s_i \in \Delta(A_i) \equiv S_i$ where $\Delta(A_i)$ denotes the set of all probability distributions on A_i and $s_i(a_i) \in [0,1]$ denotes the probability that s_i assigns to $a_i \in A_i$ with the restriction that $\sum_{a_i\in A_i} s_i(a_i) = 1$. $\mathring{S}_i$ denotes the interior of S_i and its members are referred to as *totally mixed strategies*. A strategy profile is denoted by $s \in \times_{i\in N} S_i \equiv S$. We let $S_{-i} \equiv \times_{j\neq i} S_j$, $\mathring{S} \equiv \times_{i\in N} \mathring{S}_i$, and $\mathring{S}_{-i} \equiv \times_{j\neq i} \mathring{S}_j$. We say that *a game has no redundant actions* whenever for all $i \in N$ we have $(u_i(a_i, a_{-i}))_{a_{-i}\in A_{-i}} \neq (u_i(a'_i, a_{-i}))_{a_{-i}\in A_{-i}}$ for any $a_i, a'_i \in A_i$ with $a_i \neq a'_i$. Let $\mathfrak{G}$ be the set of finite normal–form games, and $\mathfrak{G}_R \subset \mathfrak{G}$ be those without redundant actions.

The *best response* of player i to s_{-i} is defined by $\mathcal{BR}_i(s_{-i}) \equiv \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}),\ \text{for all } s'_i \in S_i\}$. s^* is a NE if for every $i \in N$, $s^*_i \in \mathcal{BR}_i(s^*_{-i})$.

The set of NE of Γ is denoted by $\mathcal{N}(\Gamma) \subset S$. A NE, s^*, is *strict* whenever $u_i(s_i^*, s_{-i}^*) > u_i(s_i', s_{-i}^*)$ for all $i \in N$ and for all $s_i' \in S_i \setminus \{s_i^*\}$. $\mathcal{N}_s(\Gamma) \subset A$ denotes the set of strict NE of Γ.[7]

Let F be a correspondence mapping X into Y where X and Y are both finite dimensional Euclidean spaces. We say that F is *lower hemi continuous* if for all $x \in X$ and all $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ that converges to x and for every $y \in F(x)$ there exist $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ with $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ and $y_n \to y$. Insisting on the additional requirement that Y is compact and F is a nonempty and compact valued correspondence, we say that F is *upper hemi continuous* if for all $x \in X$ and all $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $x_n \to x$ and every $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ with $y_n \to y$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ implies $y \in F(x)$.

An action $a_i \in A_i$ is *strictly dominated* for player i, if there exists $a_i' \in A_i \setminus \{a_i\}$ with $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$ for all $a_{-i} \in A_{-i}$. The game obtained from Γ by the elimination of strictly dominated strategies is referred to as the *strict dominance truncation* of Γ and is denoted by $\mathcal{D}(\Gamma)$. We say that s in Γ and $\tilde{s}$ in $\mathcal{D}(\Gamma)$ are equivalent under strict domination, and denote it by $s \overset{\mathcal{D}}{=} \tilde{s}$, whenever for all $i \in N$ it must be that $s_i(a_i) = \tilde{s}_i(a_i)$ for any $a_i \in A_i$ that is not strictly dominated. Moreover, for a given $K \subset S$ in Γ and $\tilde{K} \subset \tilde{S}$ in $\mathcal{D}(\Gamma)$, we say that $K \overset{\mathcal{D}}{=} \tilde{K}$ whenever for every $s \in K$ there exists $\tilde{s} \in \tilde{K}$ with $s \overset{\mathcal{D}}{=} \tilde{s}$ and for every $\tilde{s}' \in \tilde{K}$ there exists $s' \in K$ with $s' \overset{\mathcal{D}}{=} \tilde{s}'$. Clearly, $\mathcal{N}(\Gamma) \overset{\mathcal{D}}{=} \mathcal{N}(\mathcal{D}(\Gamma))$.

An action $a_i \in A_i$ is *weakly dominated* for player i if there exists $a_i' \in A_i$ with $u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i})$ for all $a_{-i} \in A_{-i}$ and this inequality holds strictly for some $a_{-i} \in A_{-i}$. A strategy profile $s \in S$ is *undominated* if $s_i(a_i) = 0$ for any $a_i \in A_i$ that is weakly dominated.

For any given $\varepsilon > 0$, a totally mixed strategy $s \in \overset{\circ}{S}$ is an *ε–perfect equilibrium* if for all $i \in N$ and $a_i \in A_i$, $a_i \notin \mathcal{BR}_i(s_{-i})$ implies $s_i(a_i) \leq \varepsilon$. On the other hand, a totally mixed strategy $s \in \overset{\circ}{S}$ is an *ε–proper equilibrium* if for all $i \in N$ and $a_i, a_i' \in A_i$, $u_i(a_i, s_{-i}) < u_i(a_i', s_{-i})$ implies $s_i(a_i) \leq \varepsilon s_i(a_i')$. s^* is *perfect* (*proper*) if there exists $\{\varepsilon^k\} \subset (0,1)$ and $\{s^k\}$ with the property that $\lim_k \varepsilon^k = 0$ and s^k an ε^k–perfect (ε^k–proper, respectively) equilibrium for each k and $\lim_k s^k = s^*$. It is well–known that every perfect equilibrium must be proper.[8]

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7. A related solution concept, proposed by Harsanyi (1973b), is *quasi–strict equilibrium*: A NE s^* is quasi–strict if for all $i \in N$ and for all $a_i, a_i' \in A_i$ with $s_i^*(a_i) > 0$ and $s^*(b_i) = 0$, $u_i(a_i, s_{-i}^*) > u_i(b_i, s_{-i}^*)$. That is, all pure strategy best responses are required to be chosen with strictly positive probabilities.

8. The notion of regularity implies strong stability and the latter essentiality which in turn implies strict perfection, hence, perfection (Kojima, Okada, and

R is a *retract* of S if $R = \times_{i\in N} R_i$ where for any $i \in N$, R_i is a nonempty convex and closed subset of S_i. For any given $K \subset S$, it is said that R *absorbs* K if for every $s \in K$ and for any $i \in N$ it must be that $\mathcal{BR}_i(s_{-i}) \cap R_i \neq \emptyset$. Any retract absorbing itself is a *Nash retract*, and a retract is an *essential Nash retract* if it absorbs a neighborhood of itself. It is said to be a *persistent retract* if it is an essential Nash retract and is minimal with respect to this property. $s \in S$ is a PE if it is a NE contained in a persistent retract.

For any given $s \in S$ and $\varepsilon > 0$, define $\mathbf{B}_\varepsilon(s_{-i}) \equiv \{s'_{-i} \in S_{-i} : |s'_{-i} - s_{-i}| < \varepsilon\}$. Moreover, let $\mathbb{S}_{\varepsilon,i}(s) \equiv \{a_i \in A_i : a_i \in \mathcal{BR}_i(s'_{-i})$ for some $s'_{-i} \in \mathbf{B}_\varepsilon(s_{-i})\}$, and $\mathbb{S}_\varepsilon(s) \equiv (\mathbb{S}_{\varepsilon,i}(s))_{i=1}^N$; $\mathbb{T}_{\varepsilon,i}(s) \equiv |\mathbb{S}_{\varepsilon,i}(s)|$, and $\mathbb{T}_\varepsilon(s) \equiv (\mathbb{T}_{\varepsilon,i}(s))_{i=1}^N$. The linearity of the expected utility functions, the upper hemi continuity of the best responses, and that $\mathbb{T}_\eta(s)$ is bounded below delivers:

LEMMA 1 *For any $s \in S$ and for any $\eta, \eta' > 0$ with $\eta < \eta'$, it must be the case that $\mathbb{S}_\eta(s) \subset \mathbb{S}_{\eta'}(s)$, hence, $\mathbb{T}_\eta(s) \leq \mathbb{T}_{\eta'}(s)$. Moreover, for any $s \in S$, there exists $\bar{\eta} > 0$ such that for all $\eta, \eta' \in (0, \bar{\eta})$, $\mathbb{S}_\eta(s) = \mathbb{S}_{\eta'}(s)$, hence, $\mathbb{T}_\eta(s) = \mathbb{T}_{\eta'}(s)$.*

This enables us to present the following:

DEFINITION 1 *For any given strategy profile $s \in S$, we define the t–*index *of s by $\mathbb{T}(s) = \mathbb{T}_\eta(s)$ where $\eta \in (0, \bar{\eta})$ and $\bar{\eta}$ is as given in Lemma 1. Moreover, for any nonempty $K \subseteq S$, s is said to be in the* tenacious selection *of K, denoted by $\mathcal{T}(K)$, if there is no $s' \in K$ with $\mathbb{T}(s) \geq \mathbb{T}(s')$ and $\mathbb{T}(s) \neq \mathbb{T}(s')$.*

Next, we provide an existence result without the need of any compactness requirements:

THEOREM 1 *$\mathcal{T}(K)$ is nonempty for any given nonempty $K \subset S$.*

Proof. For any given $K \subset S$, let $s \in K$ and notice that $\mathbb{T}_i(s) \leq |A_i|$, and hence, $V \equiv \cup_{s\in K} \mathbb{T}(s)$ is a finite set in $\mathbb{N}^N$, so $\mathcal{T}(K)$ is nonempty. ■

An observation that may be helpful when employing the method of tenacious selection as a bounded rationality measure involves the requirements on the knowledge of rationality among players: all that is needed is that every player knows that

Shindoh 1985). And properness is implied by strong stability (van Damme 1991, Section 2.4) and by settledness (Myerson and Weibull 2013). Moreover, due to Jansen (1981, Theorem 7.4) and van Damme (1991, Theorem 3.4.5) a NE of a finite two–player game is regular if and only if it is essential and all players utilize each of their pure strategy best responses.

he himself is rational.[9]

1.3 Tenacious Selection of Nash equilibrium

This section presents our findings about the TSNE which exists due to Theorem 1.

1.3.1 Idiosyncrasy

We establish that even when attention is restricted to games without redundant and weakly dominated actions the TSNE does not involve any containment relations with perfection, properness, and persistence.[10]

EXAMPLE 2 *Let* $N = \{1, 2\}$*;* $A_i = \{I, II\}$*; and* $u_i(a) = 1$ *if* $a_i = a_j$*, and 0 otherwise,* $i, j = 1, 2$ *with* $i \neq j$*. The set of NE is* $\{(I, I), (II, II), (1/2I + 1/2II, 1/2I + 1/2II)$*; and the TSNE equals* $\{(I, I), (II, II)\}$ *because the t–index of every pure NE is given by* $(1, 1)$ *and that of the totally mixed NE by* $(2, 2)$*.*

While the mixed NE of this well–known coordination game has been employed by Kalai and Samet (1984) to display the lack of strong neighborhood stability, the current study associates this issue with complexities in players' anticipation as well.[11]

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9. When the use of a societal ranking on the variability of prescribed actions is plausible, t–indices may be "aggregated": for any given $f : \mathbb{N}^N \to \mathbb{R}$, the *aggregation function*, we let the f*–induced aggregate* t*–order*, denoted by $\succsim_T^f \subset S \times S$, be defined by $s \succsim_T^f s'$ if and only if $f(\mathbb{T}(s)) \leq f(\mathbb{T}(s'))$. For any nonempty $K \subset S$, s is said to be in the f*–induced aggregate tenacious selection* of K if $s \succsim_T^f s'$ for all $s' \in K$. And, the set of f–induced aggregate tenacious selection of $K \subset S$ is denoted by $\mathcal{T}_A^f(K)$. Note that for any $K \subset S$ and for any strictly increasing $f : \mathbb{N}^N \to \mathbb{R}$, $\mathcal{T}_A^f(K) \subset \mathcal{T}(K)$. The choice of $f : \mathbb{N}^N \to \mathbb{R}$ determines $\mathcal{T}_A^f(\cdot)$, and insisting on equilibria with less variations calls for f to be strictly increasing, and the following may be used when a symmetric treatment is desired: for any $x \in \mathbb{N}^N$, $f(x) = \sum_{i \in N} x_i$. Also, $\succsim_T^f \subset S \times S$ is complete and continuous preorder whenever f is monotone (either nondecreasing or nonincreasing). Theorem 1 extends to this setting without the need of using any monotonicity requirements: $\mathcal{T}_A^f(K)$ is nonempty for any nonempty $K \subset S$ and any $f : \mathbb{N}^N \to \mathbb{R}$. This follows from: $f(V) \subset \mathbb{R}$, $V \equiv \cup_{s \in K} \mathbb{T}(s)$, is finite, thus, it possesses a minimal element, $f(\mathbb{T}(s^*))$, $s^* \in K$. So, $s^* \succsim_T^f s$ for all $s \in K$, thus, $s^* \in \mathcal{T}_A^f(K)$.

10. Weakening any one of these requirements makes the identification of desired examples easier.

11. Adopting rationalistic interpretation in normal form games (for a formal discussion of these ideas, see Aumann and Brandenburger (1995) and Kuhn (1996))

Example 3 *In the following game player 1 chooses rows, 2 columns, and 3 matrices:*

3

I		
1\2	I	II
I	$(1,1,0)$	$(1,0,1)$
II	$(1,1,1)$	$(0,0,1)$

II		
1\2	I	II
I	$(1,0,1)$	$(0,1,0)$
II	$(0,1,0)$	$(1,0,0)$

The NE are $s^p = ((1-p)I+pII, I, I)$ *where* $p \geq 1/2$ *and* $s^2 = (I, 1/2I+1/2II, 1/2I+1/2II)$ *(while only* s^2 *is perfect and proper).*[12] *Both* s^1 *and* s^2 *are in the TSNE because* $\mathbb{T}(s^p)$ *equals* $(2,1,1)$ *if* $p > 1/2$ *and* $(2,1,2)$ *when* $p = 1/2$*, and* $\mathbb{T}(s^2) = (1,2,2)$.

The game of example 3, possessing neither any redundant actions nor weakly dominated actions nor a strict NE, also displays that the TSNE is not an impassable barrier to mixed strategies: both a pure NE and a mixed NE are in the TSNE.[13]

Example 4 *The following is a coordination game where one of the pure actions in which players are not coordinated is replaced by a matching pennies:*

1\2	I	II	III
I	$(1,1)$	$(2,-2)$	$(-2,2)$
II	$(1,1)$	$(-2,2)$	$(2,-2)$
III	$(0,0)$	$(1,1)$	$(1,1)$

delivers a counterintuitive observation associated with the mixed NE in coordination games which is elegantly described by Harsanyi (1973a, p.1) as follows: "Equilibrium points in mixed strategies seem to be unstable, because any player can deviate without penalty from his equilibrium strategy even if he expects all other players to stick to theirs." Kalai and Samet (1984) observes these local variations around the mixed NE in coordination games are due to the lack of strong neighborhood stability; Young (2009) eliminates such mixed NE by employing a learning procedure, interactive trial and error learning, that selects only the pure NE in this game.

12. Perfection follows because (1) regardless of the magnitudes of player 2 and 3's strictly positive mistakes around mixing I and II with equal probabilities action I for player 1 is the only best response; and (2) every finite normal form game has to possess a PE which has to be Nash.

13. The observations in footnote 8 imply further conclusions of idiosyncrasy when comparing the TSNE with the notions of regularity, essentiality, and strong stability. This is because the totally mixed strategy NE in example 2 is regular but not in the TSNE. And s^1 of example 3 is in the TSNE but not perfect.

Here, the NE, perfect equilibria, proper equilibria, and the PE coincide: $s^1 = (1/4I + 1/4II + 1/2III, 1/2I + 1/4II + 1/4III)$*,* $s^2 = (III, 1/4II + 3/4III)$*,* $s^3 = (III, 3/4II + 1/4III)$*,* $s^4 = (3/4I + 1/4II, I)$*,* $s^5 = (1/4I + 3/4II, I)$*. Because that the t–index of* s^1 *is given by* $(3,3)$ *and the others' by* $(2,2)$*,* s^1 *is not in the TSNE.*

This game has no redundant and weakly dominated actions and no strict NE.[14] Additionally, it displays an important distinction between persistence and our concept: the former, but not the latter, entails that whether or not behavior in a specified equilibrium is plausible may depend on the presence or absence of pure strategies that do not appear in players' best responses when fine perturbations are considered. In other words, while the TSNE employs "local" performance measures when evaluating the performances of NE, the method of evaluation of persistence is rather "global". To see this, it suffices to restrict attention to s^1 and s^2. First, observe that s^1, while being a PE but not in the TSNE, is a totally mixed NE and the persistent retract it is contained in is S. Moreover, $\mathbb{S}_i(s^1) = \{I, II, III\}$, $i = 1, 2$. Second, with persistence (unlike the TSNE) s^1 is not eliminated by s^2 because of the following: $R = \times_{i=1,2} R_i$ and $R_i = \{s_i^2\}$ for $i = 1, 2$, is a Nash retract but not essential because it cannot absorb a neighborhood of itself which is due to both players being indifferent between II and III in s^2. Indeed, $\mathbb{S}_i(s^2) = \{II, III\}$, $i = 1, 2$. Yet, the Nash retract defined by $R' = \times_{i=1,2} R'_i$ with $R'_i = \{(0, x_i, 1 - x_i) : x_i \in [0, 1]\}$ is not essential (due to the inherent matching pennies feature) because for $\varepsilon > 0$ sufficiently small $(\varepsilon, 1 - 2\varepsilon, \varepsilon)$ is a point in the neighborhood of R'_2 to which player 1's corresponding best response calls for $(1, 0, 0)$ and $(1, 0, 0) \cap R'_1 = \emptyset$. Hence, R' is not a persistent retract due to the pure strategy I even though $I \notin \mathbb{S}_1(s^2)$. So with persistence s^1 is not eliminated by s^2 due to I, an action which does not appear in player 1's best responses when the other is choosing a strategy either close or equal to s_2^2. Similarly, s^k, $k = 3, 4, 5$, do not eliminate s^1 with persistence.

Example 5 *Consider the following four player game: Players 1 and 2 play the game on the left in the following table independent of the choices of players* 3 *and* 4*; players* 3 *and* 4 *play the game in the middle when players 1 and 2 choose* (I, I) *or* (II, II) *and the game on the right when 1 and 2 choose* (I, II) *or* (II, I)*.*

14. It should be noted that the pure NE of the coordination game are strict. That is why the game given in example 4 is the one that we employ when dealing with the formal relation between perfection/properness and the TSNE because it has no redundant and weakly dominated actions and no strict NE.

1\2	I	II
I	$(1,1)$	$(0,0)$
II	$(0,0)$	$(1,1)$

3\4	III	IV
III	$(1,0)$	$(0,1)$
IV	$(0,1)$	$(1,0)$

3\4	III	IV
III	$(2,2)$	$(2,0)$
IV	$(0,2)$	$(0,0)$

Here, $s = (1/2I + 1/2II, 1/2I + 1/2II, III, III)$ is in the TSNE, but is not persistent.[15]

Therefore, not every element in the TSNE is a PE even when there are neither redundant nor weakly dominated actions and no strict NE. Moreover, the essence of the distinction between persistence and the TSNE in the context of this game is the very same as that in the context of the game of example 4. However, this time global considerations of persistence help to eliminate s which is not eliminated by employing local concerns of the TSNE.

Our idiosyncrasy result, due to examples 3–5, is:

THEOREM 6 *Even when attention is restricted to games without redundant and weakly dominated actions, the TSNE does not have any containment relations with perfection and properness and persistence whenever there is no strict NE.*[16]

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15. In order to observe that s is in the TSNE, note that there is no pure strategy NE in this game (hence, the set of strict NE is empty). Note further that there is no NE where only one player mixes among his strategies. Therefore, in all NE at least two players randomize. Hence, this game involves t–indices with at least two numbers strictly exceeding 1. Now, considering s we have that $\mathbb{T}_i(s) = 2$, $i = 1, 2$, and $\mathbb{T}_j(s) = 1$, $j = 3, 4$: players $i = 1, 2$ are randomizing in NE, and $j = 3, 4$ are choosing III and if player 1 and player 2 make small mistakes player 3 and 4's best responses will still be III (due to the strict dominance in the game on the right). Hence, $\mathbb{T}(s) = (2, 2, 1, 1)$ which is the best t–index that can be achieved in this game. But s is not a PE. For any $\varepsilon > 0$, the best response of player 1 against the following perturbation is II: $(((1/2-\varepsilon)I + (1/2+\varepsilon)II), ((1/2-\varepsilon)I + (1/2+\varepsilon)II), (1-\varepsilon)III + \varepsilon IV, (1-\varepsilon)III + \varepsilon IV)$. Similarly, I is player 1's only best response when this perturbation is reversed. Moreover, for the retract defined by $\Delta(\{I, II\})$ for players 1 and 2 and III for the others is not persistent. Because when players 1 and 2 choose (I, I) (or (II, II)), the persistent retract in the middle game (of the above table) is $(\Delta(\{III, IV\}))^2$. Therefore, there is no persistent retract which includes this NE other than the whole game. Also, note that there is a persistent retract: neighborhoods around I for player 1 and 2, and $\Delta(\{III, IV\})$ for players 3 and 4. For any strategy in this retract, the best response of first and second players are still I. Third and fourth players best responses to this tremble will be trivially be in this retract as well. So there is a persistent retract other than the whole game which contains the NE given by $(I, I, 1/2I + 1/2II, 1/2I + 1/2II)$, but not s.

16. It is useful to point out that considering example 3, a game that does not have a strict NE, delivers a similar idiosyncrasy result concerning the TSNE and the quasi–strict equilibrium: (1) $s^{0.50}$ is a quasi–strict equilibrium that is not in the TSNE, and (2) s^1 is in the TSNE but is not a quasi–strict equilibrium.

1.3.2 Domination and Strict NE

The following presents properties of the TSNE in relation with domination and strict NE:

Theorem 7 *The following hold:*

1. *The TSNE does not change with strict dominance truncations.*
2. *Suppose that the game possesses a strict NE. Then, the TSNE*
   (a) *equals the set of strict NE;*
   (b) *contains neither weakly dominated NE nor mixed NE;*
   (c) *is a subset of the lower hemi continuous selection of NE;*
   (d) *is a subset of the set of perfect equilibrium, proper equilibrium, and PE. And there are games possessing a strict NE but neither redundant nor weakly dominated actions where this relation is strict.*
3. *If the game does not have a strict NE, then the following hold:*
   (a) *mixed strategy NE and pure NE may both be in the TSNE even if the game at hand is one that does not have any redundant and weakly dominated actions;*
   (b) *the TSNE may contain a weakly dominated NE even if the game under analysis does not have any redundant actions;*
   (c) *the TSNE is not lower hemi continuous even when the game at hand does not have redundant and weakly dominated actions.*

We wish to discuss direct implications of and issues about this theorem before its proof.

First, it should be emphasized that when evaluating the performance of our notion against strict domination we do not encounter the type of problems often cited in the discussion of "imperfections of perfection" (see Myerson (1978)).[17] To see this, consider the following:

Example 8 *First, consider the strict dominance truncation of the following game:*

17. Kohlberg's Example is also based on a similar observation (Kohlberg and Mertens 1986).

$1\backslash 2$	I	II	III
I	$(1,1)$	$(0,0)$	$(1,-1)$
II	$(0,0)$	$(0,0)$	$(2,-1)$
III	$(-1,1)$	$(-1,2)$	$(-1,-1)$

The NE are $s = (I, I)$ and $s' = (II, II)$, but only s is perfect.[18] *But considering strictly dominated strategies as well results in II not being weakly dominated. NE and perfect equilibria coincide and are equal to $s = (I, I)$ and $s' = (II, II)$. But in both cases $\mathbb{T}(s) = (1, 1)$ and $\mathbb{T}(s') = (2, 2)$, so the TSNE equals $\{s\}$.*

The second point concerns our finding that the TSNE exhibits stronger refinement powers than the other refinements of NE when the game at hand possesses a strict NE. To see that the associated containment relation with perfection and properness may be strict in such situations, consider example 2. On the other, the following example performs the same task in conjunction with persistence under the same restrictions:

EXAMPLE 9 *This game is one that has a strict NE but no redundant and weakly dominated actions, and the TSNE is a strict subset of the set of PE.*

$1\backslash 2$	I	II	III
I	$(10,10)$	$(0,0)$	$(0,0)$
II	$(0,0)$	$(3,1)$	$(1,3)$
III	$(0,0)$	$(1,3)$	$(3,1)$

Here, $s^1 = (I, I)$ is a strict NE which, therefore, is not empty. Thus, the TSNE equals the set of strict NE, hence, does not contain the mixed strategy NE, $s^2 = (1/2II + 1/2III, 1/2II + 1/2III)$. But s^2 is a PE: $R = \times_{i=1,2} R_i$ defined by $R_i = \{(0, x, 1-x) : x \in [0, 1]\}$, $i = 1, 2$, is an essential Nash retract because (1) for $\varepsilon > 0$ sufficiently small the best-responses of agents against $(\varepsilon, x, 1-x-\varepsilon)$ do not contain I; (2) it is minimal.

The third remark about Theorem 7 concerns the performance of the TSNE when there is no strict NE. Example 3, a game with no redundant and weakly dominated

18. A similar conclusion holds in the following extensive–form game with an "incredible threat": player 1 chooses first. If his choice is I, the game ends and player 1 obtains a return of 1 and player 2 a payoff of 2. When player 1 chooses II, players obtain a payoff vector of $(2, 1)$ if player 2's choice is I and $(0, 0)$ otherwise. The NE are (II, I) and $(I, \alpha I + (1-\alpha)II)$, $\alpha \leq 1/2$. For $s = (II, I)$ we have $\mathbb{T}(s) = (1, 1)$, and $\mathbb{T}(s') = (1, 2)$ for any other NE s'. So the TSNE equals (II, I) eliminating all the NE involving incredible threats.

actions, shows (1) both pure strategy and a mixed strategy NE may be in the TSNE, and (2) this notion may not be lower hemi continuous.[19] Theorem 7 also contains a warning even without redundant actions: the TSNE may contain weakly dominated NE. This is due to the following:

EXAMPLE 10 *This game has no strict NE and no redundant actions, but two NE:* $s = (II, II, II)$ *and* $s' = (1/2I + 1/2II, 1/2I + 1/2II, I)$.

3

	I			II	
$1\backslash 2$	I	II	$1\backslash 2$	I	II
I	$(1,0,1)$	$(0,1,1)$	I	$(0,0,0)$	$(0,0,0)$
II	$(0,1,0)$	$(1,0,0)$	II	$(0,0,0)$	$(1,0,0)$

s*, involving a weakly dominated action by player 3, is in the TSNE:* $\mathbb{T}(s) = (1,1,2)$ *and* $\mathbb{T}(s') = (2,2,1)$.

The TSNE may not eliminate weak domination because when there is no strict NE it may not be able discriminate between weak domination and randomization: in example 10, $\mathbb{T}_3(s) = 2$ because $s_3 = II$ is a weakly dominated action for player 3; $\mathbb{T}_i(s') = 2$ because $s'_i = 1/2I + 1/2II$ for $i = 1, 2$ is totally mixed. But weak domination is not permitted with persistence (Kalai and Samet 1984, Theorem 4, p.139)) due to the minimality requirement of essential Nash retracts. Therefore, the global evaluation measure embedded in persistence results in the elimination of weak domination, while the local means of evaluation with the TSNE does not suffice towards this regard.

This observation is why our analysis can be extended to the tenacious selection of undominated NE, the TSUNE. It is important to point out that in all the games handled previously, with the exception of the last one, the TSNE coincides with the TSUNE. Moreover, the other items of Theorem 7 hold with the TSUNE as well.

Proof of Theorem 7. The first item of the above theorem stated formally is: For any $\Gamma \in \mathfrak{G}$, $\mathcal{T}(\mathcal{N}(\Gamma)) \stackrel{\mathcal{D}}{=} \mathcal{T}(\mathcal{N}(\mathcal{D}(\Gamma)))$. Because that for any Γ we have that

19. The lack of lower hemi continuity follows from the example 3 which has two NE $s^1 = (II, I, I)$ and $s^2 = (I, 1/2I+1/2II, 1/2I+1/2II)$, and both s^1 and s^2 are in the TSNE. I is player 1's only best response whenever one considers a strategy profile arbitrarily close to s^1 with the requirement that players 2 and 3 are choosing action II with strictly positive probabilities. Hence, we can come up with a sequence of games (each of which does not have any redundant and weakly dominated actions) converging to the one given in example 3 for which the unique TSNE would be only around s^2.

$\mathcal{N}(\Gamma) \stackrel{\mathcal{D}}{=} \mathcal{N}(\mathcal{D}(\Gamma))$, we prove that for any $s \in \mathcal{N}(\mathcal{D}(\Gamma))$ and $s' \in \mathcal{N}(\Gamma)$ with $s' \stackrel{\mathcal{D}}{=} s$ it must be that $\mathbb{T}_i^{\Gamma}(s') = \mathbb{T}_i^{\mathcal{D}(\Gamma)}(s)$ for all $i \in N$. Then, the definition of the TSNE implies that $s \in \mathcal{T}(\mathcal{N}(\mathcal{D}(\Gamma)))$ if and only if $s' \in \mathcal{T}(\mathcal{N}(\Gamma))$ where $s' \stackrel{\mathcal{D}}{=} s$.

Let $\mathbb{T}_i^{\mathcal{D}(\Gamma)}(s) = k$. That means, s may involve one of k pure strategy best responses for player i in the game $\mathcal{D}(\Gamma)$. Now, as all the pure strategies in $\mathcal{D}(\Gamma)$ are available in Γ and s', obtained from s through assigning 0 probabilities to strictly dominated actions in Γ (i.e. $s' \stackrel{\mathcal{D}}{=} s$), is such that $s'_i \in \mathcal{BR}_i^{\Gamma}(s'_{-i})$ for all $i \in N$, it cannot be that $\mathbb{T}_i^{\Gamma}(s) < k$. Hence, suppose $\mathbb{T}_i^{\Gamma}(s') = \ell > k$. Due to $\mathbb{T}_i^{\mathcal{D}(\Gamma)}(s) = k$ we know that there exists $\tilde{\varepsilon} > 0$ such that for all $\tilde{\eta}, \tilde{\eta}' < \tilde{\varepsilon}$ we have $\mathbb{S}_{\tilde{\eta},i}^{\mathcal{D}(\Gamma)}(s) = \mathbb{S}_{\tilde{\eta}',i}^{\mathcal{D}(\Gamma)}(s)$ and $|\mathbb{S}_{\tilde{\eta},i}^{\mathcal{D}(\Gamma)}| = k$. So due to the upper hemi continuity of the best response correspondence it must be that the support of $\mathcal{BR}_i^{\mathcal{D}(\Gamma)}(s_{-i})$ is equal to $\mathbb{S}_{\tilde{\eta},i}^{\mathcal{D}(\Gamma)}(s)$ for $\tilde{\eta} < \tilde{\varepsilon}$. Similarly, the observation that $\mathbb{T}_i^{\Gamma}(s') = \ell$ implies that there exists $\varepsilon > 0$ such that for all $\eta, \eta' < \varepsilon$ we have $\mathbb{S}_{\eta,i}^{\Gamma}(s') = \mathbb{S}_{\eta',i}^{\Gamma}(s')$ and $|\mathbb{S}_{\eta,i}^{\Gamma}(s')| = \ell$. Therefore, because of the upper hemi continuity of the best responses the support of $\mathcal{BR}_i^{\Gamma}(s'_{-i})$ equals $\mathbb{S}_{\eta,i}^{\Gamma}(s')$ for $\eta < \varepsilon$. Letting $\bar{\varepsilon} < \min\{\varepsilon, \tilde{\varepsilon}\}$, these imply that there exists $a_i^* \in A_i$ such that $a_i \in \mathcal{BR}_i^{\Gamma}(s'_{-i})$ but $a_i \notin \mathcal{BR}_i^{\mathcal{D}(\Gamma)}(s_{-i})$. This is a contradiction because $\mathcal{BR}_i^{\Gamma}(s'_{-i}) \stackrel{\mathcal{D}}{=} \mathcal{BR}_i^{\mathcal{D}(\Gamma)}(s_{-i})$: $\mathcal{D}(\Gamma)$ is a strict dominance truncation of Γ, and on account of being a NE s_{-i} and s'_{-i} do not assign strictly positive probabilities to strictly dominated strategies, and player i cannot assign strictly positive probabilities to strictly dominated actions in his best response.

In order to prove item 2a we show the following: Let $\Gamma \in \mathfrak{G}$ be such that $\mathcal{N}_s(\Gamma) \neq \emptyset$; then, $\mathcal{T}(\mathcal{N}(\Gamma)) = \mathcal{N}_s(\Gamma)$. This follows from (1) the observation that for any strict NE, s^*, it must be that $\mathbb{T}_i(s^*) = 1$ for all $i \in N$; (2) for any NE that is not strict, s', there exists $j \in N$ such that $\mathbb{T}_j(s') > 1$. Both 2b and 2c are immediate consequences of 2a.[20] Regarding the proof of item 2d, notice that the fact that every TSNE must be a strict NE implies that for any s^* in the TSNE it must be that there exists $\bar{\eta} > 0$ such that for all $\eta \in (0, \bar{\eta})$ we have $\mathbb{S}_{\eta,i}(s^*) = \{s_i^*\}$ (in turn, implying that $\mathbb{T}_i(s^*) = 1$) for all $i \in N$. This supplies the strictly positive payoff slack/buffer with which such equilibria can withstand sufficiently fine perturbations. Hence, every strict NE must be perfect and proper. Moreover, because of the same reasons R, defined by $R = \times_{i \in N} R_i$ with $R_i = \{s_i^*\}$ for all $i \in N$, is a persistent retract and this implies that s^* a PE. Examples 2 and 9 show that this containment relation of the TSNE concerning perfection and properness and persistence may be strict even when the game at hand is in $\mathfrak{G}_R$ and has no weakly dominated actions.

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20. It is appropriate to point out that, in these cases there may be members of the lower hemi continuous selection of NE that are not in the TSNE. To see this, consider the coordination game of example 2 and notice that the mixed NE of that game is in the lower hemi continuous selection of NE but not in the TSNE.

Part 3 follows from examples 3 and 10. This finishes the proof of Theorem 7. ■

1.4 Concluding Remarks

Our first remark concerns the evaluation of the performances of the TSNE and persistence when attention is restricted to the *unanimity games.* Let the set of actions of every player be given by a finite set C, i.e. $A_i = C$ for all $i \in N$. An action profile $\bar{c} \in C^N$ is called *diagonal* if it is of the form $(c, c, \ldots, c)$ for some $c \in C$. It is assumed that $u_i(a) = 0$ for every $i \in N$ and for all $a \in C^N$ that is not diagonal. And $a' \in C^N$ is *positive* if $u_i(a') > 0$ for every $i \in N$. Naturally, if an action profile is positive, then it is diagonal. Kalai and Samet (1984, Theorem 6) establishes that an action vector is persistent if and only if it is positive provided that the unanimity game at hand has a positive action profile.

Item 2a of Theorem 7 delivers additional insight with the help of Theorem 6 of Kalai and Samet (1984): if the unanimity game has a positive action vector, then the TSNE and the PE and positive action profiles coincide: If $a' \in C^N$ is positive, then it is a strict NE because for every $i \in N$ it must be that $u_i(a') > 0 = u_i(a_i, a'_{-i})$ for every $a_i \in A_i \setminus \{a'_i\}$. So the TSNE equals the set of strict NE, and it is not difficult to see that the set of strict NE equals the set of positive action vectors.

The second remark involves the relation of the TSNE with a recent and elegant refinement, the notion of settled equilibrium due to Myerson and Weibull (2013) (MW hereafter). It is aimed to exclude uncoordinated NE "for more games than persistence, while maintaining general existence of a refined equilibrium that is also proper." Due to space considerations, the definition of this equilibrium notion is omitted and we refer the reader to MW. Even though our desiderata is similar with MW's, below we display that these refinement concepts are idiosyncratic.

When the game under analysis has a strict NE, it is not surprising to observe that the TSNE is a subset of the set of fully settled equilibrium. Moreover, example 9 shows that this relation may be strict: Both s^1 and s^2 are fully settled while the TSNE equals $\{s^1\}$. Meanwhile, the next example establishes that when the given game does not have a strict NE, then the TSNE and the settled equilibrium are not logically related.

Example 11 *This game has no strict NE and neither redundant nor weakly dominated actions, and is obtained by combining two "blocks" consisting of rescaled versions of example 4 of MW and a rock–scissor–paper.*

$1\backslash 2$	A	B	C	D	I	II	III
A	$(0,5)$	$(1,4)$	$(0,3)$	$(1,0)$	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$
B	$(1,0)$	$(0,3)$	$(1,4)$	$(0,5)$	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$
I	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(1,1)$	$(2,0)$	$(0,2)$
II	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(0,2)$	$(1,1)$	$(2,0)$
III	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(-1,-1)$	$(2,0)$	$(0,2)$	$(1,1)$

It can be verified that here $s^1 = (1/2A + 1/2B, 1/2B + 1/2C)$ *is not fully settled while it is in the TSNE; and* $s^2 = (1/3I + 1/3II + 1/3III, 1/3I + 1/3II + 1/3III)$ *is fully settled but not in the TSNE.*

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CHAPTER 2

AGGREGATE EFFICIENCY IN RANDOM ASSIGNMENT PROBLEMS

2.1 Introduction

Random assignment problems are allocation problems allotting some number of distinct indivisible alternatives among a population of agents with the use of a randomization device, e.g. the flip of a coin or the use of a dice, but without the use of monetary transfers. They constitute a non-negligible and often important aspect in our everyday life. Indeed, in recent years the surge of the use of random assignment methods by market designers and social planners has been significant. Relevant examples include student placement in public schools at various levels of education, organ transplantation, and the assignment of dormitory rooms. While many of these applications are implemented all over the world, Turkey, the country of our residence, features another important example: In the fields of medicine and education and justice, recent graduates are assigned to their places of duty via a random allotment arrangement.[21]

Among random assignment mechanisms, rules associating any (reported) preference profile with a stochastic distribution of alternatives to the agents, the *random priority* mechanism (henceforth, to be referred to as RP) is one of the most widely used and it has been analyzed extensively in Abdulkadiroglu and Sonmez (1998). It is also called the *random serial dictatorship* mechanism and defined as follows: A priority ranking of agents is selected uniformly, and following that rank every agent sequentially receives his favorite alternative among the ones that were not chosen by higher ranked agents. That study shows that even though the particular form of this mechanism is surprisingly simple, it is *strategyproof* (i.e. reporting the true preferences is a dominant strategy) and *ex–post efficient* (i.e. it can be represented by a

21. We refer the reader to Roth and Sotomayor (1992) for a classic source on the subject. On the other hand, for more details on random assignment problems, we cite to (Hylland and Zeckhauser 1979), (Abdulkadiroglu and Sonmez 1998), Abdulkadiroglu and Sonmez (1999), Bogomolnaia and Moulin (2001), Bogomolnaia and Moulin (2002), Chen, Sonmez, and Unver (2002), Abdulkadiroglu and Sonmez (2003), Bogomolnaia and Moulin (2004), Roth, Sonmez, and Unver (2004), Ergin and Sonmez (2006), Katta and Sethuraman (2006), Kesten (2009), Kojima (2009), Yilmaz (2009), Yilmaz (2010), Kesten and Unver (2011), Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming).

probability distribution over efficient deterministic assignments). Another efficiency notion may be used when the problem at hand features von Neumann–Morgenstern utilities: A random assignment is *ex–ante efficient* if it is Pareto optimal with respect to the profile of von Neumann–Morgenstern utilities. Bogomolnaia and Moulin (2001) (henceforth, BM) shows that by using only the ordinal preference rankings some of the random assignments that are not ex–ante efficient may be identified even if agents' utility functions are not given. To that regard that study proposes *ordinal efficiency* which necessitates the consideration of (first order) *stochastic dominance.* A random assignment stochastically dominates another one whenever for all agents the probability of being allocated one of the top k ranked alternatives under the former is weakly higher than the one under the latter for all $k = 1, \ldots, K$ where K denotes the total number of available alternatives. A random assignment is ordinally efficient for a given profile of preferences if there is no random assignment stochastically dominating it for that given profile of preferences. BM shows that ex–ante efficiency implies ordinal efficiency and ordinal efficiency implies ex–post efficiency. The reverse directions of these two relations do not hold. Due to McLennan (2002), it is also known that if a random assignment is ordinally efficient then there is a profile of von Neumann–Morgenstern utilities such that that particular random assignment is ex–ante efficient. Motivated by its key finding that RP is not an ordinally efficient mechanism BM introduces and analyzes the *probabilistic serial* (henceforth, PS) mechanism. The outcome of the PS mechanism is identified using BM's *simultaneous eating algorithm* (SEA): Each object is considered as a continuum of probability shares. Agents "eat away" from their favorite objects simultaneously and at the same speed, and once the favorite object of an agent is gone he turns to his next favorite object, and so on. The amount of an object eaten away by an agent in this process is interpreted as the probability with which he is assigned this object under the PS mechanism. BM shows that PS satisfies ordinal efficiency but is not strategyproof. It satisfies the following weaker version: A random assignment mechanism is *weak strategyproof* whenever the random allocation sustained by an agent misrepresenting his preferences stochastically dominates the one he obtains under truthful revelation implies that the two allocations are the same. This shortcoming concerning incentives is made up by some gains in terms of envy–freeness, another relevant notion to judge the value–added of a random assignment mechanism. A random assignment mechanism is *envy–free* if it associates every profile of preferences with a random assignment in which the prescribed random allocation for any agent stochastically dominates that for another agent evaluated with the former's preferences. Meanwhile, relaxing this notion delivers *weak envy–freeness* by requiring that the prescribed random allocation for any agent satisfying the following: The random allocation of another agent stochastically dominating that of the agent at hand implies that the two random allocations are the same. The same study proves that while the PS mechanism involves envy–freeness, the RP rule is weakly envy–free (but not envy–free).

Insisting on ordinal efficiency may create unappealing features. When assigning 100 objects among a population of 100 both of the following assignments may be efficient: The first allocating 1 person to his best and 99 to their second best, and the second allotting 99 to their best and 1 to their second best.[22] Indeed, there are many instances where social planners and market designers evaluate a mechanism by considering how many agents are located into their first best, how many into their second choice, and so on. Often some statistics about how many agents are allocated their higher ranking choices is announced as a positive indicator of the performance of the system.[23]

The current paper introduces a new notion of efficiency, *aggregate efficiency*, tailored for situations in which social planners and market designers value the expected number of agents assigned to their higher ranked choices: We say that a random assignment *aggregate stochastic dominates* another whenever the expected number of agents placed into one of their top k choices under the former is weakly higher than that of the latter for $k = 1, \ldots, K$. Moreover, a random assignment is aggregate efficient whenever another random allocation aggregate stochastic dominating the one under consideration implies that both of them assign the same expected number of agents into any one of their top k choices for $k = 1, \ldots, K$.

We establish that the notion of aggregate efficiency implies ordinal efficiency. Yet

22. Consider a situation where there are 100 agents and 100 objects denoted by $\{a_j\}_{j=1}^{100}$, on which the strict preference relations are as follows: Agent 1 strictly prefers a_1 to a_{100}, and a_{100} to any other alternative, and all other alternatives are ranked strictly lower and arbitrarily. Every other agent $i \neq 1$ strictly prefers a_{i-1} to a_i, and a_i to any other alternative, and all other alternatives are ranked strictly lower and arbitrarily. In this setting assigning each agent i to alternative a_i is (ordinally) efficient, and creates a situation in which one player (agent 1) gets his first best while all the other 99 players obtain their second ranked choice. On the other hand, assigning agent 1 to his second best alternative a_{100} and any other agent i to alternative a_{i-1} is also (ordinally) efficient and causes one agent to obtain his second best while 99 of them are allotted their first ranked choice.

23. OSYM, the Turkish government agency responsible of administering the nation-wide university admission examination and allocating students to programs, includes the percentage of students allocated to one of their top three choices in their press conferences. Moreover, Featherstone (2011), an independent study that was brought to our attention when the final draft of this paper was being prepared, observes that reports by NYC Department of Education 2009 and San Fransisco Unified School District 2011 also include such aspects.

the reverse does not hold and there are no logical relations between ex–ante efficiency and aggregate efficiency. After proving the existence of aggregate efficiency, we show that Gale's conjecture, the incompatibility of strategyproof and efficient mechanisms treating equals equally, takes a new form: The search for an aggregate efficient and weak strategyproof mechanism treating equals equally is futile.

On the other hand, we prove that there is a weak strategyproof, weak envy–free, and ordinally efficient mechanism, the reservation–1 mechanism (henceforth, R1), that displays a better performance on grounds of aggregate efficiency when compared to the PS mechanism. The outcome of the R1 mechanism is also identified using the SEA with an important modification that provides agents reservation rights for their most favorite alternatives. That is, the algorithm starts with agents "eating away" from their favorite objects simultaneously all at the same speed while no agent (who is finished with his favorite alternative) is allowed to start eating an alternative that is a favorite for some other agent. Once these favorite objects are gone, the algorithm proceeds exactly as the unmodified SEA does. Naturally, the amount of an object eaten away by an agent in this process is interpreted as the probability with which he is assigned this object under the R1 mechanism.

A characterization of the R1 mechanism is provided along the lines of a recent important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming). This establishes that the R1 mechanism is nothing but the PS mechanism modified to satisfy a principle decreed by the Turkish parliament on the issue of the random assignment of new doctors to their places of duty.

The organization of the paper is as follows: The next section provides intuition and motivation for the efficiency notion proposed and contains an elucidative discussion of our results. Then section 3 presents the model. In section 4 we analyze aggregate efficiency and obtain some impossibility results. Section 5 introduces and contains the detailed analysis and full characterization of the R1 mechanism.

2.2 Aggregate Efficiency and the R1 Mechanism

In order to facilitate an easier reading and more motivation we wish to introduce the notion of aggregate efficiency and present our results in the context of the following simple example with 3 agents and 3 alternatives. The set of players is $N = \{1, 2, 3\}$ and the set of alternatives $A = \{a, b, c\}$. The preferences of agents are given by $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, and $b \succ_3 a \succ_3 c$, where $x \succ_i y$ denotes agent i strictly preferring x to y.

$I.$	a	b	c
1	1	0	0
2	0	1	0
3	0	0	1

$II.$	a	b	c
1	0	1	0
2	1	0	0
3	0	0	1

$III.$	a	b	c
1	1	0	0
2	0	0	1
3	0	1	0

$IV.$	a	b	c
1	0	0	1
2	1	0	0
3	0	1	0

TABLE I
The deterministic efficient assignments.

$N\backslash A$	a	b	c
1	1/2	1/6	1/3
2	1/2	1/6	1/3
3	0	2/3	1/3

$N\backslash A$	a	b	c
1	1/2	0	1/2
2	1/2	0	1/2
3	0	1	0

TABLE II
Two random assignments.

The deterministic efficient assignments are given in table I. In fact, in matrices I and II player 3 is assigned to c, his least preferred alternative, and one of players 1 and 2 get his favorite alternative a while the other consumes his second best, alternative b. Therefore, one player is given his most favorite one his second best and one his worst. On the other hand, in permutation matrices III and IV two players are achieving their first best while one player has to bear his least preferred alternative.

When the society values the number of agents allocated to their higher ranked alternatives, the dismissal of the efficient matrices I and II can be justified on grounds of an "aggregate" efficiency notion. Consequently, both III and IV can be labeled as aggregate efficient deterministic assignments because there are no other permutation matrices that beat them on grounds of this efficiency notion. Moreover, when one extends this analysis to random assignment settings, this notion implies that no strictly positive weights should be given to permutation matrices I and II. Indeed, in this example any convex combination of III and IV would be aggregate efficient.[24]

On the other hand, when one employs the RP rule and/or the PS mechanism the resulting random assignments coincide and are given by the table on the left hand side of table II. It should be pointed out that under the RP and PS mechanisms the permutation matrices I and II are realized with a probability of 1/6 each and III and IV with a probability of 1/3 each. Hence, the expected number of agents ranked into their top choices is 5/3 and the top two 2 and, naturally, the top three

24. In general, convex combinations of aggregate efficient deterministic assignments are not necessarily aggregate efficient.

3. Noticing that the same figures are given by 2,2, and 3 for the aggregate efficient random assignment, this example establishes that both the RP and PS are not aggregate efficient as they are aggregate stochastic dominated. This follows from $(2,2,3) \geq (5/3,2,3)$ and $(2,2,3) \neq (5/3,2,3)$. The same example also shows that there are ordinally efficient random assignments, the one given by RP and PS, which are not aggregate efficient.

After proving the existence of an aggregate efficient random assignment, we show that the set of aggregate efficient random assignments is a subset of the set of ordinally efficient random allotments and that aggregate efficient random assignments are decomposed only to aggregate efficient permutation matrices. These establish that in any realized state of the world the outcome of an aggregate efficient random assignment must be not only be efficient but also aggregate efficient. Furthermore, ex–ante efficiency and aggregate efficiency are not logically related, i.e. these two notions of efficiency do not have any containment relations between each other. In general, there are von Neumann–Morgenstern utility profiles for which the first of two ordinally efficient random allotments is aggregate efficient and not ex–ante efficient and the second ex–ante efficient but not aggregate efficient.[25] On the other hand, it needs to be mentioned that using McLennan (2002) and our result that aggregate efficiency implies ordinal efficiency it can be concluded that for every aggregate efficient random assignment there exists a profile of von Neumann–Morgenstern utilities with which that particular random assignment is ex–ante efficient.

These findings, naturally, makes one wonder about aggregate efficient and strategyproof mechanisms. Yet one should not forget Gale's conjecture about the incompatibility of efficiency and strategyproofness. It is useful to remind the reader that considering deterministic environments Zhou (1990) proves that efficiency and strategyproofness cannot be simultaneously satisfied by a mechanism treating equals equally. BM extends this result to random assignment problems and prove that there is no mechanism treating equals equally and satisfying ordinal efficiency and strategyproofness. Thus, the mechanism they propose, the PS mechanism, being weak strategyproof is of significance.

In the current study we show that Gale's conjecture takes a new form: We prove that there is no mechanism treating equals equally and satisfying aggregate efficiency and weak strategyproofness. Moreover, another impossibility result involves a weaker notion of envy–freeness and a stronger efficiency concept: There is no mechanism satisfying aggregate efficiency and weak envy–freeness.

25. We refer the reader to example 16 in the proof of Theorem 15 which is obtained from the above example by a particular choice of von Neumann–Morgenstern utilities.

While these results ensure that the search for an aggregate efficient and weak strategyproof mechanism satisfying the equal treatment property is futile, they do not rule out the possibility of an improvement upon the PS mechanism in terms of the notion of aggregate efficiency. Indeed, it turns out that a relevant and interesting observation can be found in Turkey in the context of the random assignment mechanism used in the allotment of new doctors to their specific places of duty. The Turkish lawmaker decrees that the following principle has to be obeyed: (1) whenever a new doctor is the only one ranking a place of duty as the highest, then he is allocated that particular place of duty; and (2) if there are more than one new doctors ranking a particular place of duty as their highest, then one of them is selected with a random draw.[26] This requirement, which we name *condition T*, results in the bistochastic matrix on the right hand side of table II.

The above example establishes that the RP and PS do not satisfy condition T and are both not aggregate efficient. Meanwhile, it also shows that there are ordinally efficient random assignments that are not aggregate efficient.[27] While condition T produced an aggregate efficient allocation in this example, in general we also show that there are situations in which there exists an ordinally efficient random allocation satisfying condition T but not aggregate efficiency, and there is an aggregate efficient random allotment that do not satisfy condition T.[28]

On the other hand, imposing condition T on the PS mechanism produces a weak strategyproof rule that is weak envy-free and outperforms the PS mechanism in terms of aggregate efficiency: the R1 mechanism. We prove that this mechanism aggregate stochastic dominates the PS mechanism and preserves all of the important properties of the PS mechanism with the exception of envy–freeness: The R1 mechanism is weak strategyproof and ordinally efficient and weak envy–free (but not envy–free).

Imposing condition T in the characterization of the R1 mechanism involves the modification of two axioms of a recent and important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming) (HHKKU, hereafter). These two axioms, ordinal fairness and non-wastefulness, fully characterize the PS mechanism. As elegantly put by some of these authors in the working paper version of this study (Kesten, Kurino, and Unver 2011), *ordinal fairness* follows "whenever an agent is

26. We refer the reader to the Official Journal of Republic of Turkey 16 November 1996 issue number 22819.

27. Considering the example given in BM (Bogomolnaia and Moulin 2001, p.298), one can easily show that the resulting random assignments of the PS and the aggregate efficiency coincide while both are different from the outcome of the RP.

28. See the first example in the proof of Theorem 17.

assigned some object with positive probability, his surplus at this object is no greater than that of any other agent at the same object"; and *non-wastefulness* whenever "the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object".[29]

The current study provides a full characterization of the R1 mechanism by employing versions of these axioms modified to make them satisfy condition T. Indeed, the imposition of condition T on the PS is obtained as follows: HHKKU's axiom concerning efficiency (non-wastefulness) is modified to satisfy the first part of condition T and their fairness axiom (ordinal fairness) is updated by the second part. Consequently, our axioms are T-ordinal fairness and T-non-wastefulness are obtained. A random assignment is *T-ordinally fair* if each favorite object has to be assigned with equal probabilities to agents preferring it as the first choice, and whenever an agent is assigned with positive probability some object that is not a favorite by any one of the agents then his surplus at this object is no greater than that of any other agent at the same object. On the other hand, a random assignment is *T-non-wasteful* if each one of the favorite alternatives are fully assigned to those agents preferring it as their first choice, and the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object.

Why not R2? Naturally this is a relevant follow-up question. That is why not allow agents to have two reservations, not just one. We prove that doing so eliminates weak strategyproofness a key property that we do not wish to sacrifice.

A recent and independent study, Featherstone (2011), was brought to our attention when the final draft of this paper was being prepared.[30] It deserves special emphasis. We should point out that that study is also concerned with aggregate efficiency (which it refers to as the *rank efficiency*) and some important parts of our results involving the analysis of aggregate efficiency are common. On the other hand, the two papers differ extensively after developing this efficiency notion. We restrict attention to the identification and characterization of a tangible weakly strategyproof and ordinally efficient mechanism with better aggregate efficiency performances than the PS mechanism (while not completely giving up envy–freeness). On the other hand, Featherstone (2011) analyzes and characterizes aggregate efficient mechanisms (at the expense of weak strategyproofness) and concentrates on special cases given by low information environments, like the ones given in Roth and Rothblum (1999). Moreover, an empirical analysis about costs of strategyproofness

29. The surplus of an agent in a random allotment for a given object is the cumulative probability that he is assigned an alternative at least as good as the current one.

30. We thank Umut Mert Dur in that regard.

is considered and it is established that "it would be a mistake to not at least consider using a rank efficient mechanism" even at the expense of strategyproofness.

2.3 The Model

Let A be a finite set of indivisible objects and $N = \{1, 2, ..., n\}$ be a finite set of agents, with the requirement that $|A| \geq |N|$. A *random assignment* (alternatively, an allocation) $P = [p_{ia}]_{i \in N, a \in A}$ is a matrix where $p_{ia} \in [0, 1]$ denotes the probability of agent i being allocated an object a. $\sum_{a \in A} p_{ia} = 1$ and $\sum_{i \in N} p_{ia} \leq 1$. Let the set of all random assignments be denoted by $\mathcal{P}$. On the other hand, a *preference profile* is denoted with $\succ \equiv (\succ_i)_{i \in N}$, where $\succ_i$ is the strict preference relation of agent i on A. Let $\succeq_i$ denote the weak preference relation induced by $\succ_i$. We assume that preferences are linear orders, i.e., for all $a, b \in A$, $a \succeq_i b \Leftrightarrow a = b$ or $a \succ_i b$. We denote the set of all such preference relations of agent i by Π_i, and the set of all such preference profiles by Π. For any $\succ$ in Π, we define *favorite alternatives* as $F(\succ) = \{a \in A | \exists i \in N : a \succeq_i b, \forall b \in A\}$. For all agents $i \in N$, define *most preferred alternative of agent* i as $F_i(\succ) = \{a \in A | a \succeq_i b, \forall b \in A\}$ and define *the set of agents preferring alternative* a *as their first choices*, $F_a(\succ) = \{i \in N | a \succeq_i b, \forall b \in A\}$. Given a preference profile $\succ$ in Π, define the *weak upper contour set* of agent $i \in N$ at object $a \in A$ by $U(a, \succ_i) = \{b \in A : b \succeq_i a\}$ and given $P \in \mathcal{P}$ let $U(a, P, \succ_i) = \sum_{b \succeq_i a} P_{ib}$ denote the *surplus of agent* i *at* a *under* P, i.e. the probability that i is assigned an object at least as good as a under P_i.

Next, we define *ex-ante efficiency* and *ex-post efficiency*: Let $(u_i)_{i \in N}$ be a profile of von Neumann-Morgenstern utility functions, where each individual one is a real valued function on A and the corresponding preferences over $\mathcal{P}$ is obtained by the comparison of expected utilities where $u_i(P_i) = \sum_{a \in A} p_{ia} u_i(a)$. Given a profile of preferences $\succ$ in Π and an associated profile of von Neumann-Morgenstern utilities $u = (u_i)_{i \in N}$, we say that a random assignment $P \in \mathcal{P}$ is (1) ex-ante efficient at u if and only if P is Pareto optimal in $\mathcal{P}$ at u; and (2) ex-post efficient at $\succ$ whenever its decomposition involves only efficient deterministic assignments.

Given two allocations P and Q, we say that P *stochastically dominates* Q *for agent* i, and denote it by $P_i \succ_i^{sd} Q_i$, if and only if $U(a, P_i, \succ_i) \geq U(a, Q_i, \succ_i)$ for all $a \in A$. Moreover, P *stochastically dominates* Q if and only if $P_i \succ_i^{sd} Q_i$ for all $i \in N$. Furthermore, given preference profile $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is said to be *ordinally efficient* if and only if for any given $P' \in \mathcal{P}$, $P' \succ^{sd} P$ implies $P' = P$.

We say that an allocation $P \in \mathcal{P}$ is *envy-free* for a given preference profile $\succ$ if and only if we have that for all $i, j \in N$, $P_i \succ_i^{sd} P_j$. Moreover, it is *weakly envy-free* if and only if $P_j \succ_i^{sd} P_i$ implies $P_i = P_j$.

A *mechanism* is a function mapping preference profiles to random assignments. Given a mechanism $\varphi : \Pi \to \mathcal{P}$, we say φ is *strategy-proof* if for all $\succ$ in Π and for all $i \in N$ we have $\varphi_i(\succ) \succ_i^{sd} \varphi_i(\succ_i', \succ_{-i})$ for all $\succ_i'$ in Π_i. Furthermore, φ is *weakly strategy-proof* if for all $\succ$ in Π and for all $i \in N$, $\varphi_i(\succ_i', \succ_{-i}) \succ_i^{sd} \varphi_i(\succ)$ implies $\varphi_i(\succ_i', \succ_{-i}) = \varphi_i(\succ)$ for all $\succ_i'$ in Π_i.

Next we introduce *aggregate efficiency*: For a given preference profile $\succ$ in Π, define r_{ik} as the most preferred k objects in A by agent $i \in N$. Moreover, for a given random assignment $P \in \mathcal{P}$ let an aggregate efficiency vector be defined by $w^P = (w_1^P, ..., w_{|A|}^P)$ in $\mathbb{R}^{|A|}$ where $w_k^P = \sum_{i \in N} \sum_{a \in r_{ik}} P_{ia}$. Consequently, given a preference profile $\succ$ in Π, we say that a random assignment P *aggregate stochastically dominates* Q, if $w^P \geq w^Q$, and we denote this by $P \succ^{asd} Q$. Finally, given a preference profile $\succ$ in Π, a random assignment $P \in \mathcal{P}$ is *aggregate efficient* whenever $P' \succ^{asd} P$ for some $P' \in \mathcal{P}$ implies $w^P = w^{P'}$.

Moreover, aggregate stochastic domination between mechanisms is defined as follows: A *mechanism* $\varphi : \Pi \to \mathcal{P}$ *aggregate stochastically dominates another mechanism* $\varphi' : \Pi \to \mathcal{P}$, if for all $\succ$ in Π we have $\varphi'(\succ) \succ^{asd} \varphi(\succ)$ implies $w^{\varphi'(\succ)} = w^{\varphi(\succ)}$, and there exist $\succ^*$ such that $\varphi(\succ^*) \succ^{asd} \varphi'(\succ^*)$ and $w^{\varphi'(\succ^*)} \neq w^{\varphi(\succ^*)}$.

2.4 Aggregate Efficiency and Impossibility Results

Due to BM it is well known that ex-ante efficiency implies ordinal efficiency which in turn implies ex-post efficiency, demanding that every possible realization of the random assignment has to be an efficient deterministic assignment. On the other hand, McLennan (2002) establishes that if a random assignment is ordinally efficient then there exists a profile of von Neumann-Morgenstern utilities for which this random assignment is ex-ante efficient.

After handling the existence question in Theorem 12, we prove that every aggregate efficient random assignment has to be ordinally efficient, (hence, ex-post efficient), and there are ordinally efficient random assignments that are not aggregate efficient (Theorem 14). Moreover, we also show that an aggregate efficient random assignment resolves only into aggregate efficient deterministic assignments (Theorem 13). Therefore, combining these two results formally establishes that every possible realization of an aggregate efficient random assignment can involve only an aggregate efficient deterministic assignment. While aggregate efficiency has these useful properties, an interesting finding emerges when considering the relation of ex-ante efficiency with aggregate efficiency: Ex-ante efficiency does not imply aggregate efficiency (Theorem 15). On the other hand, due to Theorem 14 and McLennan (2002), we know that if a random assignment is aggregate efficient then there exists a profile of von Neumann-Morgenstern utilities for which this random assignment is ex-ante efficient.

Next, we consider the relation between strategyproofness and aggregate efficiency. Zhou (1990) proves Gale's conjecture about the incompatibility of Pareto efficiency and strategyproofness in one–sided deterministic matching problems: "When there are n objects to be assigned to n agents, for $n \geq 3$, there exits no mechanism that satisfies symmetry (equal treatment of equals), Pareto optimality, and strategyproofness." Moreover, BM shows that this incompatibility arises in random allocation problems as a tradeoff between ordinal efficiency and strategyproofness: They show that there is no mechanism treating equals equally which satisfies ordinal efficiency and strategyproofness. We, therefore, ask whether or not similar conclusions hold with the stronger efficiency concept introduced in the current study. Indeed, in Theorem 17 we show that the inevitable trade–off between efficiency and strategyproofness concepts (when attention is restricted to mechanisms treating equals equally) prevails: Aggregate efficiency and weak–strategy proofness are incompatible with the equal treatment property. In other words, when one strengthens the efficiency notion and weakens the strategyproofness concept, there are no changes regarding this impossibility result. This, in turn, points to an updated version of the classic trade–off between efficiency and strategyproofness, this time between aggregate efficiency and weak strategyproofness. Moreover, Theorem 18 points to another impossibility: Aggregate efficiency and the property of weak envy-free are not compatible.

The existence of aggregate efficient random assignments follows from the Theorem 12 which is presented without a proof. This is because the result is an immediate consequence of the acyclicity of the order on random assignments defined in $\mathbb{R}^{|A|}$ and compactness of $\mathcal{P}$, the set of all random assignments on A.

Theorem 12 *Given any preference profile $\succ$ in Π, there exist an aggregate efficient random assignment P in $\mathcal{P}$.*

Theorem 12 also establishes the existence of an aggregate efficient mechanism: For every given preference structure, using Theorem 12 one can simply construct an aggregate efficient mechanism by picking an aggregate efficient allocation for each possible preference structure.

The following 3 Theorems present the results discussed above.

Theorem 13 *An aggregate efficient random assignment can only be decomposed into aggregate efficient permutation matrices.*

Proof. The Von-Neumann Birkhoff Theorem tells us that a matrix is identifying a random assignment if and only if it can be written as a convex combination of permutation matrices. Therefore, any aggregate efficient random assignment $P \in \mathcal{P}$, can be written as a convex combination of permutation matrices $(P_1, P_2, ..., P_k)$, as $P = \sum_{\ell=1}^{k} \lambda_\ell P_\ell$. We claim that all these permutation matrices have to be aggregate efficient. Suppose not, then there exists a permutation matrix P_c for some $c \in \{1, 2, ..., k\}$ that is not aggregate efficient; so, P'_c aggregated stochastically dominates P_c. Hence, P' defined through $(P_1, P_2, ..., P'_c, ..., P_k)$ as the following convex combination $\lambda_c P'_c + \sum_{\ell \neq c} \lambda_\ell P_\ell$ aggregate stochastically dominates P. Consequently, P is not aggregate efficient. ■

Theorem 14 *The set of aggregate efficient random assignments is a subset of the set of ordinally efficient random assignments. Moreover, this containment relation may be strict.*

Proof. For any $P \in \mathcal{P}$ that is not ordinally efficient, it must be that there exists $P' \neq P$ and $i \in N$ such that $P_i \neq P'_i$ and $P'_i \succ_i^{sd} P_i$, which is if and only if $\sum_{b \succeq_i a} P'_{ib} \geq \sum_{b \succeq_i a} P_{ib}$ for all $a \in A$ and there exists $a' \in A$ such that this inequality holds strictly. Hence, $\sum_{a \in r_{jk}} P'_{ja} \geq \sum_{a \in r_{jk}} P_{ja}$, for all $k \leq |A|$ and for all $j \in N$, yet there exists $m \leq |A|$ such that this inequality holds strictly for agent $i \in N$, because otherwise $P' = P$. Therefore, $\sum_{j \in N} \sum_{a \in r_{jk}} P'_{ja} \geq \sum_{j \in N} \sum_{a \in r_{jk}} P_{ja}$ for all k and this inequality holds strictly for $k = m$. Hence, P' aggregated stochastically dominates P, implying that P is not aggregate efficient.

In order to see the second part, consider the example supplied both in the introduction and as 22 in the proof of Theorem 21. ■

Theorem 15 *Given any preference profile $\succ$ in Π and associated von Neumann-Morgenstern utility profile $u = (u_i)_{i \in N}$, an ex-ante efficient random assignment does not need to be aggregate efficient. Moreover, for every aggregate efficient random assignment P in $\mathcal{P}$, there exists a von Neumann-Morgenstern utility profile $\tilde{u} = (\tilde{u}_i)_{i \in N}$ such that P is ex-ante efficient at $\tilde{u}$.*

Proof. The first part is due to the following example which is a cardinal version of the example supplied both in section and as Example 22 in the proof of Theorem 21:

$W(\succ)$	a	b	c	d
1	0.5	0	0.5	0
2	0.5	0	0.5	0
3	0	1	0	0
4	0	0	0	1

$W(\succ')$	a	b	c	d
1	0.5	0	0.5	0
2	0.5	0	0.5	0
3	0	0.5	0	0.5
4	0	0.5	0	0.5

TABLE III
Aggregate efficient allocations for $\succ$ and $\succ'$.

EXAMPLE 16 *Let* $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ *and* $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, $b \succ_3 a \succ_3 c$. *Let* $u_i = (u_{ia}, u_{ib}, u_{ic})$ *be the vector which denotes the utilities of agent* $i \in \{1, 2, 3\}$ *from getting objects* $\{a, b, c\}$ *respectively. Suppose* $u_1 = u_2 = (10, 8, 1)$, *and* $u_3 = (8, 10, 6)$.

Then any aggregate efficient allocation P *can be denoted by a number* $x \in [0, 1]$ *such that* $P_1(\succ) = (x, 0, 1-x)$, $P_2(\succ) = (1-x, 0, x)$ *and* $P_3(\succ) = (0, 1, 0)$. *So, the sum of expected utilities of agents under any aggregate efficient allocation is equal to* 21. *However, let* R *be an allocation such that* $R_1 = (1, 0, 0)$, $R_2 = (0, 1, 0)$ *and* $R_3 = (0, 0, 1)$. *Then, sum of expected utilities of agents under* R *is equal to* 24.

The second part of the Theorem follows from Theorem 14 and McLennan (2002).

■

The following Theorem establishes the incompatibility of aggregate efficiency with weak strategyproofness and equal treatment of equals:

THEOREM 17 *Suppose that* $N \geq 4$. *Then, there is no mechanism treating equals equally and satisfies aggregate efficiency and weak strategyproofness.*

Proof. Consider the following example: $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d\}$ where $a \succ_i c \succ_i b \succ_i d$, for $i = 1, 2$, and $b \succ_3 c \succ_3 d \succ_3 a$ and $b \succ_4 d \succ_4 c \succ_4 a$. The unique aggregate efficient allocation satisfying the equal treatment property is given in first table in table III.

Now, consider $\succ'$ where $\succ'_i = \succ_i$ for $i \in \{1, 2, 3\}$ and $\succ'_4 = \succ_3$. That is to say, $\succ'$ is preference structure constructed with the deviation of fourth agent to third agents' preferences. Then a mechanism satisfying aggregate efficiency and equal treatment of equals should assign the allocation on the right. This deviation is profitable for player 4 because it results in an ordinally better allocation for the fourth agent when compared with stating his true preference.

This example can be embedded into other problems with more than four agents as follows: Suppose there are n agents and n objects. Let, first four agents' are called old agents and their first four preferences are exactly like the above example and the order of preferences of other objects are not important for old agents. Also, any new object is declared as a first choice by exactly one new agent. Then a mechanism satisfying welfare efficiency and equal treatment of equals should assign old agents exactly the allocation given above and assign each new agent his most preferred object. Hence, the same deviation is still profitable for fourth agent as in example above. ■

In the following Theorem, we establish that aggregate efficiency and the property of weak envy-free are not compatible without the need to employ the equal treatment property:

THEOREM 18 *Suppose that $N \geq 4$. Then, there is no mechanism satisfying aggregate efficiency and weak envy-freeness.*

Proof. Consider the example in the proof of Theorem 17 and notice that any aggregate efficient mechanism must assign object b to the third agent and object d to the fourth. Then clearly forth agent strictly envies the third. As discussed in the proof of Theorem 17, this example can be generalized to other problems with more agents and more alternatives. ■

2.5 The R1 Mechanism

Theorem 17 tells us that there is no mechanism treating equals equally and satisfying weak strategyproofness and aggregate efficiency. On the other hand, BM shows that the PS mechanism satisfies ordinal efficiency and weak strategyproofness and equal treatment of equals (while it is shown not to be strategyproof). Therefore, the PS mechanism cannot satisfy aggregate efficiency whenever $N \geq 4$.

Following the same of thinking as in BM, it is a plausible question to ask whether or not the PS mechanism can be beaten on grounds of aggregate efficiency when one restricts attention to weakly strategyproof mechanisms treating equals equally. It turns out that the answer to this question is positive.

Theorems 20 and 21 establish that there exists a mechanism, the R1, treating equals equally and satisfying weak strategyproofness while displaying a better performance with respect to aggregate efficiency. Moreover, this mechanism is weakly envy-free.

The *R1 mechanism* is defined via the following algorithm: Given a problem $(\succ, A)$, each alternative $a \in A$ is interpreted as infinitely divisible with total supply of 1 unit. Agents can eat one object at a time and the eating speed values of agents are all equal. On the other hand, a distinct feature is that each agent has a reservation right for his most preferred object. That is, each agent $i \in N$ starts to eat away from $F_i(\succ)$ until it is depleted. When agent i's best alternative is exhausted, then he starts to eat from their most preferred object in $A \setminus F(\succ)$, until that one is also depleted. Then, he continues with the next best among the nondepleted ones in $A \setminus F(\succ)$. That is, once $F(\succ)$ is depleted, the R1 mechanism behaves exactly as the PS mechanism. The algorithm terminates when each agent has eaten exactly 1 total unit of objects. The allocation of an agent i by R1 is then given by the amount of each object he has eaten until the algorithm terminates. Let $R1(\succ) \in \mathcal{P}$ denote the random assignment obtained as a result of R1 for a preference profile given by $\succ$ in Π.

A natural follow-up questions is about why we are not allowing agents to have two reservations. In fact, why not R2? The interesting finding is that, doing so eliminates weak strategyproofness. Hence, clearly this provides sufficiently strong reasons for the dismissal of the R2 mechanisms. The formal execution is in Appendix **??**.

A further interesting and motivating observation emerges when one considers the characterization of the R1 mechanism: It is nothing but the PS mechanism modified to satisfy a principle set forth by the Turkish parliament. This principle, which we call *condition T* is outlined in the Official Journal of Republic of Turkey 16 November 1996 issue number 22819, and it decrees that: (1) whenever a new doctor (an agent) is the only one ranking a place of duty (an alternative) as the highest, then he is allocated that particular place of duty; and (2) if there are more than one new doctors ranking a particular place of duty as their highest, then one of them is selected with a random draw.

In a recent and important study, Hashimoto, Hirata, Kesten, Kurino, and Unver (forthcoming) provides a full characterization for PS mechanism by using only two axioms, one related to fairness and the other one to efficiency. These are namely ordinal fairness and non-wastefulness. Ordinal fairness follows "whenever an agent is assigned some object with positive probability, his surplus at this object is no greater than that of any other agent at the same object"; and non-wastefulness whenever "the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object".

Theorem 19 of the current study provides a full characterization of the R1 mechanism which employs versions of these axioms modified to make them satisfy condition T. These axioms are T-ordinal fairness and T-non-wastefulness:

Definition 2 *Given* $\succ$ *in* Π*, a random assignment* $P \in \mathcal{P}$ *is T-non-wasteful at* $\succ$ *if* $\sum_{i \in F_a(\succ)} P_{i,a} = 1$ *for all* $a \in F(\succ)$*; and, for all* $i \in N$ *and for all* $a \in A$ *such that* $P_{i,a} > 0$ *we have* $\sum_{j \in N} P_{j,b} = 1$ *for all* $b \in A$ *with* $b \succ_i a$*.*

Definition 3 *Given* $\succ$ *in* Π*, a random assignment* $P \in \mathcal{P}$ *is T-ordinally fair at* $\succ$ *if* $a = F_i(\succ)$ *with* $i \in N$ *and* $a \in A$ *implies* $U(a, P, \succ_i) \leq U(a, P, \succ_j)$ *for all* $j \in F_a(\succ)$*, and for all* $i, j \in N$ *and for all alternatives* $a \neq F_i(\succ)$ *with* $P_{i,a} > 0$ *it must be that* $U(a, P, \succ_i)) \leq U(a, P, \succ_j)$*.*

The next Theorem the proof of which is deferred to the Appendix, renders a full characterization of the R1 mechanism:

Theorem 19 *A mechanism is T-ordinally fair and T-non-wasteful if and only if it is R1.*

In what follows, we provide some important properties of the R1 mechanism. In fact, it is useful to point out that under the R1 mechanism any alternative $a \in F(\succ)$ will be allocated only to agents in $F_a(\succ)$, and with equal probabilities. That is, the R1 mechanism obeys condition T while satisfying the important properties of weak strategyproofness, ordinal efficiency, and weak envy-freeness. These are stated in the following Theorem the proof of which is in the Appendix.

Theorem 20 *R1 mechanism satisfies condition T, weak strategyproofness, ordinally efficency and weak envy-freeness.*

It is important to emphasize that due to Theorem 17 we know that there exists no mechanism treating equals equally and satisfying aggregate efficiency and weak strategyproofness. Moreover, due to Theorem 20 we know that the R1 mechanism is weak strategyproof and ordinally efficient, while being weakly envy-free (which clearly implies the equal treatment property). Therefore, it is not aggregate efficient. On the other hand, recalling that the PS mechanism is weakly strategyproof and ordinally efficient and envy-free, one may wonder whether or not the "slack" created by relaxing envy-freeness to weak envy-freeness is useful for some other property. The answer is affirmative, and the property that gets strengthened concerns aggregate efficiency. To be precise, the R1 mechanism satisfies condition T, and moreover, we show that the R1 mechanism aggregate stochastically dominates the PS mechanism.

Theorem 21 *The R1 mechanism aggregate stochastic dominates the probabilistic serial mechanism.*

Proof. We prove this Theorem by showing that there is no preference profile where $PS(\succ) \succ^{asd} R1(\succ)$ and $w^{PS(\succ)} \neq w^{R1(\succ)}$, where $PS : \Pi \to \mathcal{P}$ denotes the PS mechanism. Moreover, using the example given in the introduction we establish that there exists $\tilde{\succ} \in \Pi$ with $R1(\tilde{\succ}) \succ^{asd} PS(\tilde{\succ})$ and $w^{PS(\tilde{\succ})} \neq w^{R1(\tilde{\succ})}$.

In order to show that there does not exist a preference profile $\succ$ in Π such that $PS(\succ) \succ^{asd} R1(\succ)$ and $w^{PS(\succ)} \neq w^{R1(\succ)}$, suppose (for a contradiction) that there exists a $\succ^*$ for which $PS(\succ^*) \succ^{asd} R1(\succ^*)$. By the definition of aggregate stochastic domination, there should be a strict difference between the allocations, therefore $PS(\succ^*) \neq R1(\succ^*)$. Note that if $PS(\succ^*)_{ia} = R1(\succ^*)_{ia}$ for all $a \in F(\succ^*)$, then $PS(\succ^*) = R1(\succ^*)$, since the R1 algorithm proceeds exactly the same as PS after the favorite alternatives are allocated (i.e. R1 behaves the same as PS for all $a \notin F(\succ^*)$). Therefore, there exist $i \in N$ and $a = F_i(\succ^*)$ such that $PS(\succ^*)_{ia} \neq R1(\succ^*)_{ia}$. Hence, by the defining property of R1, $PS(\succ^*)_{ia} \neq R1(\succ^*)_{ia}$ implies $R1(\succ^*)_{iF_i(\succ^*)} > PS(\succ^*)_{iF_i(\succ^*)}$ (since the reservation right can lead only to an increase in the allocation of a good to an agent who prefers it as his favorite object), so $\sum_{i=1}^{N} PS(\succ^*)_{i,F_i(\succ^*)} < \sum_{i=1}^{N} R1(\succ^*)_{i,F_i(\succ^*)}$ and $PS(\succ^*) \not\succ^{asd} R1(\succ^*)$, delivering the desired contradiction.

EXAMPLE 22 *Let* $N = \{1,2,3\}$, $A = \{a,b,c\}$ *and* $a \succ_1 b \succ_1 c$, $a \succ_2 b \succ_2 c$, $b \succ_3 a \succ_3 c$. *Then* $R1_1(\succ) = R1_2(\succ) = [1/2, 0, 1/2]$ *and* $R1_3(\succ) = [1, 0, 0]$. *Where as* $PS_1(\succ) = PS_2(\succ) = [1/2, 1/6, 1/3]$ *and* $PS_3(\succ) = [2/3, 0, 1/3]$. *When we compare these two allocation,* $R1$ *allocates* 2 *alternatives to agents viewing them as the first choice preferences and* 1 *alternative to an agent ranking it the least while* PS *distributes one to each. That is to say,* $w^{R1(\succ)} = (2,2,3)$ *and* $w^{PS(\succ)} = (1,2,3)$.

This finishes the proof of Theorem 21. ■

2.6 Appendix: Proofs

2.6.1 Proof of Theorem 19

First, we propose an eating algorithm that can be used for any allocation at any preference structure $\succ$ in Π, which will be key to the proof.

Fix a preference structure $\succ$. Then any allocation $P(\succ)$ can be simulated by an eating function defined as follows:

Think each object as an infinitely divisible good with a quota 1. Each agent eats away from his most preferred object among the objects that are assigned him with a positive probability. When an agent eats his assigned probability from some object, he starts to eat away from his next preferred object that is assigned him with a positive probability. Each agent eats with a same speed until the algorithm ends at time 1 when each agent has eaten exactly 1 total unit of objects. Therefore we define the eating function $f : [0,1) \times N \rightarrow A$ such that for all $f^P_{\succ}(t,i) = \{a \in U(t) | a \succeq_i b, \forall b \in U(t)\}$ where $U(t) = \{a \in A : U(a, P, \succ_i) > t\}$. This function identifies the object that agent i eats at time t.

Proof. First we will show that there exist an allocation which satisfies T-ordinal fairness and aggregate non-wastefulness for any preference $\succ$. In particular, we will show that it is $R1(\succ)$.

For all $a \in F(\succ)$, $R1(\succ)_{ia} = 1/|F_a(\succ)|$ for all $i \in F_a(\succ)$. Therefore it must be that $\sum_{i \in F_a(\succ)} R1(\succ)_{ia} = 1$ for all $a \in F(\succ)$. Then, let i be any player in N and a, b be any objects in A such that $P_{i,a} > 0$ and $b \succ_i a$. Then, from the ordinal efficiency of $R1(\succ)$ (due to Theorem 20), $\sum_{j \in N} P_{j,b} = 1$. Hence, $R1(\succ)$ is aggregate non-wasteful.

For all $i \in N$ and $a \in A$ such that $a = F_i(\succ)$, we have $U(a, P, \succ_i) \leq U(a, P, \succ_j)$ for all $j \in F_a(\succ)$ since $R1(\succ)_{ja} = R1(\succ)_{ia} = 1/|F_a(\succ)|$ (by aggregate non-wastefulness of $R1(\succ)$). Now, we have to show that for each object $a \notin F(\succ)$ and all $i, j \in N$ with $P_{ia} > 0$, we have $U(a, P, \succ_i) \leq U(a, P, \succ_j)$. Suppose not, then there exists $t^* \in [0, U(a, P, \succ_i))$ such that $a \succ_j f^{R1}_{\succ}(t^*, j)$. However, this means that agent j has eaten from an object less preferred to a while the unfavorite object a is not exhausted, contradicting with $R1$ mechanism. Therefore $R1(\succ)$ is T-ordinally fair.

We have shown that there exists an aggregate non-wasteful and T-ordinally fair allocation for every preference structure. Now we will show that there is no other allocation than $R1$ satisfying these two properties by showing that if there is an allocation satisfying these properties, it should be characterized by the same eating function with $R1$.

Fix a preference profile $\succ$, and let $P \in \mathcal{P}$ be any T-ordinally fair and aggregate non-wasteful allocation at $\succ$. We will show that $f^P_{\succ}(t,i) = f^{R1}_{\succ}(t,i)$ for all $t \in [0,1)$ and $i \in N$. Suppose not, then there exists a time, say t^*, where the eating functions proceed same until t^*, but starts to differ at t^*. Formally, $t^* = \max\{t \in [0,1) : f^P_{\succ}(t', i) = f^{R1}_{\succ}(t', i)$ for all $t' < t$ for all $i \in N\}$. Then there exist an agent i such that $f^P_{\succ}(t^*, i) \neq f^{R1}_{\succ}(t^*, i)$. Let $f^{R1}_{\succ}(t^*, i) = a$ and $f^P_{\succ}(t^*, i) = b$.

Aggregate non-wastefulness restricts that all favorite goods has to be assigned completely, $\sum_{i \in F_a(\succ)} P_{ia} = 1$, and must be assigned equally among the agents who prefers that object as his favorite good, $P_{iF_i(\succ)} = 1/|F_{F_i(\succ)}(\succ)|$ follows from the definition of T-ordinal fairness. Therefore, the eating functions should be exactly same for favorite goods, hence a and b cannot be in $F(\succ)$. Note that, the eating functions are same until t^*, therefore with the fact that b is not exhausted at t^* and agent i eats from a in $R1$ at t^* where $b \notin F(\succ)$, $a \succ_i b$. Then, there are two cases to consider: (i) There exist $j \in N$ and $t > t^*$ such that $f^P_\succ(t,j) = a$. Then $U(a, P, \succ_j) \geq t > t^* \geq U(a, P, \succ_i)$ contradicting with T-ordinal fairness of P, since $a \notin F(\succ)$ and $P_{ja} > 0$. (ii) Otherwise, $\sum_{k\in N} P_{ka} < \sum_{k\in N} R1_{ka} = 1$ and $P_{ib} > 0$ contradicts with aggregate non-wastefulness of P.

Therefore there exist not such $t^* \in [0,1)$ and hence $f^P_\succ(t,i) = f^{R1}_\succ(t,i)$ for all $t \in [0,1)$ implying $P = R1(\succ)$. ■

2.6.2 Proof of Theorem 20

In the eating algorithm given in proof of Theorem 19, define $t^\succ(a)$ to be time at which object a is exhausted at preference $\succ \in \Pi$ (under $R1$ mechanism), i.e.

$$t^\succ(a) = \begin{cases} \sup\{t \in [0,1) : f^{R1}_\succ(t,i) = a\} & \text{if } \sum_i R1(\succ)_{ia} = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Claim 23 *R1 mechanism satisfies ordinal efficiency.*

Proof. Fix a preference profile $\succ$. Suppose $R1(\succ)$ is not ordinally efficient. Then there exists an allocation $Q \neq R1(\succ)$ such that $Q \succ^{sd} R1(\succ)$. First of all, we claim that there do not exist an agent $i \in N$ and objects $a, b \in A$ such that $R1(\succ)_{ib} > 0$ and $a \succ_i b$, $\sum_{j\in N} R1(\succ)_{ja} < 1$. If otherwise, a cannot be in $F(\succ)$ since R1 mechanism assigns favourable objects completely. On the other hand, object a must be in $F(\succ)$ since agent i starts to eat away from object b when object a is available in the simultaneous eating algorithm for R1 mechanism. End of claim.

$Q \succ^{sd} R1(\succ)$ and $Q \neq R1(\succ)$ implies that there exist an agent $i_1 \in N$ such that $Q_{i_1} \succ^{sd}_{i_1} R1(\succ)_{i_1}$ and $Q_{i_1} \neq R1(\succ)_{i_1}$. Then, there are objects $a_1, a_2 \in A$ such that $a_2 \succ_{i_1} a_1$ and $Q_{i_1a_2} > R1(\succ)_{i_1a_2}$ and $Q_{i_1a_1} < R1(\succ)_{i_1a_1}$. Since, $\sum_{j\in N} R1(\succ)_{ja_2} = 1$, there exist an agent $i_2 \neq i_1$ such that $Q_{i_2a_2} < R1(\succ)_{i_2a_2}$. Since $Q_{i_2} \succ^{sd} R1(\succ)_{i_2}$ and $Q_{i_2a_2} < R1(\succ)_{i_2a_2}$, then there exist an object $a_3 \in A$ such that $Q_{i_2a_3} > R1(\succ)_{i_2a_3}$. Hence, we can successively define sets $\{i_1, i_2, .., i_n\}$ and $\{a_1, a_2, .., a_n, a_{n+1}\}$ where $a_{n+1} = a_m$ for some $m < n$.

Now consider the agents $(i_m, i_{m+1}, ..., i_n)$ and objects $(a_m, a_{m+1}, ..., a_{n+1})$ such that $a_{k+1} \succ_{i_k} a_k$ for every $m \leq k \leq n$ and $a_{n+1} = a_m$. Any object a_k is not in $F(\succ)$, since $a_{k+1} \succ_{i_k} a_k$ and $R1(\succ)_{i_ka_k} > 0$. Then $t^\succ(a_k) > t^\succ(a_{k+1})$ for all $m \leq k \leq n$, since object a_{k+1} must be unavailable when agent i_k eats away from object a_k. Therefore $t^{\succ}(a_m) > t^{\succ}(a_{m+1}) > ... > t^{\succ}(a_{n+1}) = t^{\succ}(a_m)$. Contradiction. ■

Claim 24 *R1 mechanism satisfies weak strategy proofness.*

Proof. Fix a preference structure $\succ$ and take an agent $i \in N$ and let $\succ_i$: $a_1 \succ a_2 \succ ... \succ a_n$. Suppose there exists a deviation $\succ'_i$ such that $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$, where $\succ' = (\succ'_i, \succ_{-i})$. We will show that $R1_i(\succ) = R1_i(\succ')$ with induction, first by showing that for any given deviation, $R1(\succ)$ and $R1(\succ')$ coincides on the interval $[0, t^{\succ}(a_1))$ in the basis step. Then we will assume that for any a_m such that $1 \leq m < n$ and $R1(\succ)_{ia_m} > 0$, eating algorithms coincide on the interval $[0, t^{\succ}(a_m))$ such that $t^{\succ}(a_m) < 1$. Set $f_{\succ}^{R1}(t^{\succ}(a_m), i) = a_l$. We will prove that eating algorithms coincide on the interval $[0, \min(t^{\succ}(a_l), 1))$, which completes this proof in an inductive manner.

Basis: Note that if $R1(\succ)_{ia_1} = 1$, then it is clear that $R1(\succ) = R1_i(\succ')$. If $R1(\succ)_{ia_1} < 1$, then there exist $j \in N \setminus \{i\}$ such that $j \in F_{a_1}(\succ)$. So if $i \notin F_{a_1}(\succ')$, then $R1(\succ')_{ia_1} = 0 < R1(\succ)_{ia_1}$. Contradiction. Hence, i cannot misreport his favorite object, and so $F_j(\succ') = F_j(\succ)$ for all $j \in N$. So, the eating algorithms coincide on the interval $[0, t^{\succ}(a_1))$.

Inductive Step: Suppose for some a_m such that $1 \leq m < n$ and $R1(\succ)_{ia_m} > 0$, eating algorithms coincide on the interval $[0, t^{\succ}(a_m))$ such that $t^{\succ}(a_m) < 1$. Set $f_{\succ}^{R1}(t^{\succ}(a_m), i) = a_l$, and note that $a_l \succeq b$ for all b such that $b \notin F(\succ)$ and $t^{\succ}(b) > t^{\succ}(a_m)$, that is to say a_l is the most preferred object for agent i among the available ones at $t^{\succ}(a_m)$. We will prove that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

Note that, we assumed that $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$, so it must be true that $R1(\succ)_{ia_l} \leq R1(\succ')_{ia_l}$ and therefore $\min\{t^{\succ}(a_l), 1\} \leq \min\{t^{\succ'}(a_l), 1\}$. If there is no agent $j \in N \setminus \{i\}$ and time $t', t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ such that $f_{\succ}^{R1}(t', j) = a_l$, then the surplus of agent i at object a_l is equal to 1, hence $R1(\succ)_i = R1(\succ')_i$. So, there exist an agent $j \in N \setminus \{i\}$, and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f_{\succ}^{R1}(t', j) = a_l$.

If there is no agent $j, j \neq i$ and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f_{\succ}^{R1}(t', j) = a_l$ and $f_{\succ'}^{R1}(t', j) = b$, where $b \neq a_l$, then there are two cases,

(i) if $t^{\succ}(a_l) < 1$, we claim $t^{\succ'}(a_l) = t^{\succ}(a_l)$. First note that $t^{\succ'}(a_l) < t^{\succ}(a_l)$, contradicts with the assumption $R1_i(\succ') \succ_i^{sd} R1_i(\succ)$. Also, if $t^{\succ'}(a_l) > t^{\succ}(a_l)$, then for all $j \in N \setminus \{i\}$ such that $R1(\succ)_{ja_l} > 0$, $R1(\succ')_{ja_l} > R1(\succ)_{ja_l}$, implying $R1(\succ')_{ia_l} < R1(\succ)_{ia_l}$ and hence $R1_i(\succ') \not\succ^{sd} R1_i(\succ)$. Contradiction. So, $t^{\succ'}(a_l) = t^{\succ}(a_l)$ and hence $f_{\succ}^{R1}(t, i) = f_{\succ'}^{R1}(t', i) = a_l$ for all t' such that $t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ implying that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

(ii) if $t^{\succ}(a_l) \geq 1$, then $\sum_{b \succeq a_l} R1(\succ)_{ib} = 1$. Hence $R1(\succ') \succ^{sd} R1(\succ)$ implies $\sum_{b \succeq a_l} R1(\succ')_{ib} = 1$. Therefore, $f^{R1}_{\succ}(t,i) = f^{R1}_{\succ'}(t',i) = a_l$ for all t' such that $t^{\succ}(a_m) \leq t' < 1$ implying that eating algorithms coincide on the interval $[0, \min\{t^{\succ}(a_l), 1\})$.

We have shown that there exist an agent $j \in N \backslash \{i\}$ and a time $t' \in [t^{\succ}(a_m), t^{\succ}(a_l))$ such that $f^{R1}_{\succ}(t',j) = a_l$ and $f^{R1}_{\succ'}(t',j) = b$, where $b \neq a_l$. Then, $b \succ_j a_l$ and note that b and a_l cannot be in either $F(\succ)$ or $F(\succ')$. Hence b is not available at t' under $\succ$ but available under $\succ'$, so $t^{\succ}(b) < t^{\succ'}(b)$. Let G be the set of objects such that $g \neq a_l$ such that $t^{\succ}(g) < t^{\succ'}(g)$. Note that G is nonempty (b is in G) and let y be the object in G with minimal $t^{\succ}(y)$. Note that $t^{\succ}(a_m) \leq t^{\succ}(y) < t^{\succ}(a_l)$, since the algorithm proceeds same until $t^{\succ}(a_m)$ and $t^{\succ}(y) \leq t^{\succ}(b)$ from the definition of y and $t^{\succ}(b) < t^{\succ}(a_l)$ from $b \succ_j a_l$ and $R1(\succ)_{ja_l} > 0$.

Suppose there exist an agent k and a time $t', t' < t^{\succ}(y)$ such that $f^{R1}_{\succ}(t',k) = y$ and $f^{R1}_{\succ'}(t',k) = c$, where $c \neq y$. (Otherwise, verify that $t^{\succ}(y) = t^{\succ'}(y)$. Contradiction.) Note that $k \neq i$ since agent i eats object a_l for all t', $t^{\succ}(a_m) \leq t' < t^{\succ}(a_l)$ at $\succ$ since $t^{\succ}(y) < t^{\succ}(a_l)$.Then, $c \succ_k y$ and $t^{\succ}(c) < t' < t^{\succ}(y)$. Also, $t^{\succ}(c) < t^{\succ'}(c)$ since $c \succ_k y$ and $f^{R1}_{\succ}(t',k) = y$ and $f^{R1}_{\succ'}(t',k) = c$. Then c must be the minimal in G. A contradiction.

Therefore there cannot be any agent starting to eat from any other object between $t^{\succ}(a_m)$ and $t^{\succ}(a_l)$, implying that each algorithm proceed same until $\min\{t^{\succ}(a_l), 1\}$.

■

Claim 25 *R1 mechanism satisfies weak envy-freeness.*

Proof. Fix a preference structure $\succ$. Take $i, j \in N$ such that $R1(\succ)_j \succ^{sd}_i R1(\succ)_i$, we will show that $R1(\succ)_j = R1(\succ)_i$. If $F_i(\succ) \neq F_j(\succ)$, then $R1(\succ)_{jF_i(\succ)} = 0 < R1(\succ)_{iF_i(\succ)}$. A contradiction. So, $f^{R1}_{\succ}(t',i) = f^{R1}_{\succ}(t',j) = F_i(\succ)$ for all t' such that $0 \leq t' < t^{\succ}(F_i(\succ))$.

Suppose $f^{R1}_{\succ}(t',i) = f^{R1}_{\succ}(t',j)$ for all t',$0 \leq t' < t^{\succ}(a) < 1$ for some object $a \in A$ with $R1(\succ)_{ia} > 0$. Then let $f^{R1}_{\succ}(t^{\succ}(a),i) = b$ and $f^{R1}_{\succ}(t^{\succ}(a),j) = c$, where $b \neq c$. Note that $b, c \notin F(\succ)$. Then $b \succ_i c$ and $f^{R1}_{\succ}(t',i) = b$ for all t' such that $t^{\succ}(a) \leq t' < \min(1, t^{\succ}(b))$. Hence, clearly $R1(\succ)_{ib} > R1(\succ)_{jb}$ and so $\sum_{m \succeq_i b} R1(\succ)_{im} > \sum_{m \succeq_i b} R1(\succ)_{jm}$. A contradiction. Hence we can conclude that b and c should be the same good, inductively implying that $R1(\succ)_i = R1(\succ)_j$ ■

2.7 Appendix: R2 Mechanism

We can define R2 mechanism as follows:

A plausible extension of R1 mechanism is to give reservation right to agents not only for their top choice but also to their second most preferred object. Similar to the reservation right for top choice in R1 mechanism, an agent can put reservation to an object if he ranks that object as his second choice in R2 mechanism. This reservation right can be introduced in various ways but we will prefer to define it as general as possible as follows: An agent can put reservation on his second choice as much as his remaining quota if he is the first person who eat from that object through the eating algorithm. We leave the following questions about the reservation rights in R2 mechanism open, since our counter example covers all possible versions of reservation rights for the second most preferred object: "Can an agent put reservation when he is not the first one that eats from an object even he ranks that object as his second choice?" or "what happens if there are more than one agent who can put reservation?" or "Can a group of agents put reservation, when another group is eating from that object?".

Note that, we can extend R1 algorithm by giving two reservation rights to agents where they will put these reservations to not only their top two choices, but the first two goods they eat through the algorithm. Again, the following example will show that this mechanism is also not weak strategy proof.

EXAMPLE 26 *Let* $N = \{1,2,3,4,5\}$ *and* $A = \{a,b,c,d,e\}$*, and the preferences are given by* $a \succ_1 b \succ_1 c \succ_1 d \succ_1 e$*,* $a \succ_2 c \succ_2 b \succ_2 d \succ_2 e$*,* $a \succ_3 c \succ_3 d \succ_3 e \succ_3 b$ *and* $\succ_3 = \succ_4 = \succ_5$*.*

All possible version of R2 allocation is given by:

N/A	a	b	c	d	e
1	1/5	4/5	0	0	0
2	1/5	1/5	1/4	1/10	1/4
3	1/5	0	1/4	3/10	1/4
4	1/5	0	1/4	3/10	1/4
5	1/5	0	1/4	3/10	1/4

When the second agent reports his ranking as $a \succ'_2 c \succ'_2 d \succ'_2 b \succ'_2 e$*, the R2 allocation, for all possible versions, for reported preferences is given by:*

N/A	a	b	c	d	e
1	1/5	4/5	0	0	0
2	1/5	1/5	1/4	1/4	1/10
3	1/5	0	1/4	1/4	3/10
4	1/5	0	1/4	1/4	3/10
5	1/5	0	1/4	1/4	3/10

Therefore, there is a profitable deviation where the second agent gets an allocation that stochastically dominates his previous allocation in truthful submission.

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CHAPTER 3

INFORMATION ACQUISITION IN TWO-SIDED MATCHING MARKETS

3.1 Introduction

Centralized university student matching institutions, in Turkey, China and many other countries, operate essentially as in the Gale–Shapley *college admissions* model: Students submit their orderings for university seats they have in mind, universities order students based on their scores in national exams and secondary school grade point averages, a stable matching is computed and enforced. A particular criticism directed at this practice is that, not having the opportunity for any close look at individual students beyond what their scores reveal, university orderings disregard much relevant information, leading in fact to deterioration in pre-university education. A remedial institution is a matching after *two–stage interviewing and preference reporting*, whereby in the first stage students and university seats are each matched with a *shortlist* of, say k, candidates, utilizing a stable *multipartner* matching procedure but with the coarse university orderings based on scores. In the second stage, universities and students are allowed to take a closer look at their potential mates in the shortlists and submit orderings subsequently. A stable matching μ that assigns each student to at most one university is then computed and enforced.

We analyze this remedial institution with an NIRMP matching marketplace. National Intern Resident Matching Program is a similar institution utilized for the matching of hospitals and doctors in US.

The matching literature generally also assumes agents know their preferences before entering a match. Therefore there are small number of article which is related to this study: (Das and Kamenica 2005), (Chakraborty, Citanna, and Ostrovsky 2010), (Josephson and Shapiro 2008), and (Hoppe, Moldovanu, and Sela 2009) explore the role of information acquisition in matching markets in different settings. (Das and Kamenica 2005) consider sequential learning in the context of dating markets where universities and students repeatedly go on dates to learn their preferences; Chakraborty et al. investigate the stability of matching mechanisms with interdependent values over partners; Hoppe et. al examine the effectiveness of signalling in an assortative matching environment; and Josephson and Shapiro study the role of adverse selection when firms sequentially interview workers in a decentralized matching process. Although the literature on search seems related since it also explores the role of frictions in matching markets, it is quite dissimilar.

In contrast to related literature, the notions of assortative matching will not be an issue of consideration in this study. The crucial difference is the strategic interview decisions that creates externalities on interviewers. The only model, which have similar concerns is (Lee and Schwarz 2009). However, there is a crucial difference: the importance of homogeneity of ex-ante preferences are not considered in (Lee and Schwarz 2009). Hence, none of the articles have substantial intersection with this study.

3.2 Setup and Definitions

We consider an *NIRMP Matching Marketplace* consisting of an ordered set of doctors D and hospitals H. Each doctor $d_k \in D$ has a strict ranking R^k over hospitals and in addition a quality q_k, which can take two values 1 or 0 with equal probabilities $(1/2, 1/2)$ and known to any hospital only upon 'interview'. Each hospital have an ex-ante preference $\succ^0$ on set of doctors such that d_k is preferred to d_{k+1} for all $d_k \in D$. Note that there is a unique stable matching μ^* implied by $\big((R^j)_{d_j \in D}, (R^0)_{h_j \in H}\big)$. Each hospital $h_j \in H$ is indexed such that $\mu_j^*(d_j) = h_j$.

We call $(q_k)_{d_k \in D} = q \in Q$ is a quality profile *state* where Q denotes the set of each realization. Each hospital has a micro-preference on doctors such that d_k is preferred to d_{k+1} unless $q_k = 0$ and $q_{k+1} = 1$. Each hospital can update his ex-ante ranking $\succ^0$ to $\succ^{I_j}$ through interview with $I_j \subset D$, that is $d_{k+1} \succ^{I_j} d_k$ if $q_k = 0$ and $q_{k+1} = 0$, where $d_k, d_{k+1} \in I_j$. Moreover, consistent with micro-preferences, each hospital h_j have the same partial preference ordering $\succ$ on *state-contingent match* $M \in \mathcal{M} = D^{2^{|D|}}$, where $M_q = d_k$ represents the match for hospital when the state is q.

In particular, let $M_q = d_k$ and $M'_q = d_{k'}$, $M \succ M'$ if and only if for all $q \in Q$, either (i) $k \leq k'$ or (ii) $k = k' + 1$ when $q_{k'} = 0$ and $q_k = 1$, and for some $q \in Q$ either (i) holds with strict inequality or (ii) holds.

Also, if there exists $M, M' \in \mathcal{M}$ and $\hat{q}, \bar{q}$ such that $M_q = M'_q$ for all $q \neq \hat{q}$ and $q \neq \bar{q}$, and $M_q = d_{k+1}$ where $M'_q = d_k$ for $q = \hat{q}$ or $q = \bar{q}$, where $\bar{q}^l = \hat{q}^l$ where $l \neq k$ and $\bar{q}^k = 1$, $\hat{q}^k = 0$ and $\hat{q}^{k+1} = \bar{q}^{k+1} = 1$, then $M \succ M'$. This condition simply says that d_{k+1} is not preferred to the d_k where $q_{k+1} = 1$ and q_k is not known.

Moreover, we assume that a hospital is indifferent between to match with d_{k+1} instead of d_k when $(q_k, q_{k+1}) = (0, 1)$ and to match with $d_{k'+1}$ instead of $d_{k'}$ when $(q_{k'}, q_{k'+1}) = (0, 1)$. We call this uniformity assumption.

3.2.1 Interviewing and Preference Reporting Game

In an NIRMP Matching Marketplace, we consider a two-period Interviewing and Preference Reporting (IRP) game where the players are hospitals: Each hospital $h_j \in H$, in period 1 chooses a set $I_j \subset D$ where $|I_j| \leq 2$, namely a set consisting of at most two doctors, for interviewing and in period 2 - contingent on the information gathered through interviewing, that is the realization of $q_{I_j} = (q_k)_{d_k \in I_j}$ - submits a (strict) ranking R_j over the set of all doctors D to the Center who implements the doctor-optimal matching μ. Thus, for each hospital h_j, a strategy is a 2-tuple $(I_j, R_j(q_{I_j}))$ where I_j is the set of doctors that h_j interviews and $R_j(q_{I_j})$ is the "contingent ranking" of doctors that h_j chooses. Naturally, since $|I_j| \leq 2$, the number of states in q_{I_j} is less than four, that is $R_j(q_{I_j})$ consists of at most four rankings.

Moreover, each hospital h_j have ordinal preferences $\succ$ on the matches $\mu_j(S)$, that is profile of final match for h_j when the strategy profile is S. We call $\mu_j(S)_q = d_k$, the final match of h_j is d_k when the strategy profile of hospitals is S and the realized state is q. Note that each strategy profile S implies a state-contingent match M.

Moreover each hospital pays the cost of interviewing $\varepsilon|I_j|$ where ε is sufficiently small.

If $\mu(s_j, s_{-j}) \succ \mu(s'_j, s_{-j})$ for all $s_{-j} \in S_{-j}$, then s_j **strictly dominates** s'_j. If s_j strictly dominates each $s'_j \in S_j$, then s_j is **strict dominant strategy** for h_j. We call s is a **strict dominant equilibrium** iff s_j is strict dominant strategy for all h_j in H. Note that since $\succ$ is not complete, the definition is too strict.

We call $s_j \in \mathcal{BR}_j(s_{-j})$ if there exist not $s'_j \in S_j$ such that $\mu(s'_j, s_{-j}) \succ \mu(s_j, s_{-j})$. Hence we define $s = (s_j)_{h_j \in H}$, where $s_j = (I_j, R_j(q_{I_j}))$ is a **Nash Equilibrium**, if there exist not $s'_j \in S_j$ such that:

$$\mu(s'_j, s_{-j}) \succ \mu(s'_j, s_{-j}) \quad \forall h_j \in H.$$

Note that $\succ$ is defined on state-contingent match and it is not complete. Hence, the definition is coarse.

3.3 Results

3.3.1 Non-Existence of Pure Strategy Nash Equilibrium

Consider an IRP game, Γ^* in an NIRMP Matching Marketplace with three hospitals and five doctors where R^j below:

	d_1	d_2	d_3	d_4	d_5
h_1	1	1	x	x	x
h_2	x	2	x	x	x
h_3	2	x	1	2	x
h_4	x	x	2	1	1
h_5	x	3	x	x	x

PROPOSITION 1 *There is not Pure Strategy Nash Equilibrium of IRP Game,* Γ^*.

Proof. Our proof is by contradiction. Suppose there is a pure strategy Nash equilibrium $s = (s_1, s_2, s_3, s_4)$. Let μ be the state-contingent matching at s.

Step 1 : $\mu_{h_1}(q) = d_2$ when $(q_1, q_2) = (0, 1)$ and $\mu_{h_1}(q) = d_1$ otherwise.

Note that μ_{h_1} gives h_1 its most preferred doctor in all realizations and therefore is strictly better than any other state-contingent match. Since h_1 is the top-ranked hospital for both d_1 and d_2, h_1 is able to ensure μ_{h_1} by "interviewing d_1, d_2 and reporting its top-ranked doctor as d_2 when $(q^1, q^2) = (0, 1)$ and as d_1 otherwise". Since interviewing cost is small, this is, in fact, a strictly dominant strategy for h_1. End of step.

Step 2 : $\mu_{d_2}(q) = h_1$ when $(q_1, q_2) = (0, 1)$ and $\mu_{d_2}(q) = h_2$ otherwise.

From Step 1, $\mu_{d_2}(q) = h_1$ when $(q_1, q_2) = (0, 1)$. Suppose to the contrary that $\mu_{h_2}(q) \neq d_2$ for some q such that $(q_1, q_2) \neq (0, 1)$. In this case d_1 is matched with h_1. Then h_2 must have proposed to a doctor other than d_1, d_2. In fact h_2 must have proposed to and get matched with d_3 (for otherwise h_2 would be better off by proposing to and get matched with d_2.) So it must be that neither h_3 or h_4 has proposed to d_3. Moreover, d_2 must have been matched with h_5. Therefore either hospital h_3 or h_4 is matched with d_5. Then this hospital must have reported d_5 above d_3. However this hospital could by reporting d_3 above d_5 get to propose to and be matched d_3. Contradiction. End of step.

Step 3 : $\mu_{h_3}(q) \neq d_4$ at any q such that $(q_1, q_2) \neq (0, 1)$.

Suppose to the contrary that there exists a q^* such that $\mu_{h_3}(q^*) = d_4$ and $(q_1, q_2) \neq (0, 1)$. Then, for q^*, in s_3 hospital h_3 must have reported d_4 above d_3 and in s_4 hospital h_4 must have reported d_3 above d_4. In fact, $\mu_{h_4}(q^*) = d_3$ (otherwise $\mu_{h_4}(q^*) = d_5$, so h_4 must have reported d_5 above d_3 for q^* in s_4, but then h_4 would be better off by putting d_3 right above d_5 in any contingent ranking of s_4 and consequently getting matched with d_3. Note that, with this change in strategy, at any q either the match of h_4 do not change or h_4 matches with d_3 instead of d_5.

Hence there is a better strategy for h_4 than s_4, contradicting with the fact that s is a Nash equilibrium.)

Case (i) Suppose, $(q_3^*, q_4^*) = (0,1)$ and there exists no q' such that $(q_3', q_4') \neq (0,1)$ and $\mu_{h_4}(q') = d_3$. In this case, h_4 would be better off by only putting d_4 right above in any contingent ranking of s_4 thereby getting matched with d_4. Note that with this strategy change, at any q either the match of h_4 do not change or h_4 matches with d_4 instead of d_3 only when $(q_3, q_4) = (0,1)$. Hence this deviation from s_4 is profitable, contradicting the fact that s is a Nash equilibrium.

Case (ii) $(q_3^*, q_4^*) = (0,1)$ and there exists q' such that $(q_3', q_4') \neq (0,1)$ and $\mu_{h_4}(q') = d_3$. This is not possible because h_3 would be better off by interviewing d_3, d_4 and reporting d_3 above d_4 at q' and following same strategy with s_3 for other q. Note that, with this deviation from s_3, h_3 matches with d_3 instead of d_4 where at q' and contradicting the fact that s is a Nash equilibrium.

From Cases (i) and (ii), it follows that $(q_3^*, q_4^*) \neq (0,1)$. But, then, h_3 would be better off by only reversing the ranking of d_3, d_4 in s_3 at q^* and thereby getting matched with d_3. Contradiction since s is a Nash equilibrium.

Step 4 : h_3 does not interview any doctor and $\mu_{h_3}(q) = d_1$ when $(q_1, q_2) = (0,1)$ and $\mu_{h_3}(q) = d_3$ otherwise.

It follows from Step 1 that when $(q_1, q_2) = (0,1)$, $\mu_{h_1}(q) = d_2$ and h_3 can match with d_1 by ranking d_1 as its top-ranked doctor. Moreover, (from Step 2) $\mu_{h_2}(q) \neq d_2$ and (from Step 3) $\mu_{h_3}(q) \neq d_4$ at any q such that $(q_1, q_2) \neq (0,1)$. Hence, $\mu_{h_3}(q) = d_1$ when $(q_1, q_2) = (0,1)$ and $\mu_{h_3}(q)$ is not equal to d_2 or d_4 when $(q_1, q_2) \neq (0,1)$. Note that $\mu_{h_3} \neq d_5$ since otherwise h_3 would be better of by submitting d_3 right above d_5 at any q. Then, $\mu_{h_3}(q) = d_3$ when $(q_1, q_2) \neq (0,1)$. Finally, ε-cost of interview implies h_3 does not interview any doctor because of the fact that to not interview and submitting ex-ante preferences gives $\mu_{h_3}(q)$ at any q. End of step.

Step 5: h_4 interviews d_4, d_5 and $\mu_{h_4}(q) = d_3$ when $(q_1, q_2) = (0,1)$ and otherwise if $(q_4, q_5) = (0,1)$ $\mu_{h_4}(q) = d_5$ and $\mu_{h_4}(q) = d_4$ otherwise.

From steps above, $\mu(h_4)(q)$ is not equal to d_1 or d_2 at any q. Then $\mu_{h_4}(q)$ is one of d_3, d_4, d_5. Hence, it is not individually rational for h_4 to interview d_1, d_2. Then h_4 does not rank d_5 above d_3 at any q in s_4. (otherwise, there exist a q such that in the contingent ranking at q, d_5 is above d_3 and $(q_1, q_2) = (0,1)$ since h_4 do not interview d_1 or d_2 in s_4. Then, at this q, h_4 matches d_5 instead of d_3 and gets worse off contradicting with s is Nash equilibrium) Then, there are three possible contingent "effective" ranking that can be appear in s_4 are: d_3 above d_4 above d_5, d_4 above d_3 above d_5, d_3 above d_5 above d_4. To identify s_4, we present a table of final match of h_4 in each q for each contingent ranking.

q_1	q_2	(d_3, d_4, d_5)	(d_4, d_3, d_5)	(d_3, d_5, d_4)
0	1	d_3	d_4	d_3
oth	oth	d_4	d_4	d_5

Moreover we claim, in s_4 hospital h_4 interviews with two doctors. For otherwise, if h_4 does not interview in s_4: h_4 would be better off by deviation from s_4 to interviewing d_4, d_5 and ranking d_5 above d_4 when $(q_4, q_5) = (0, 1)$ and d_4 above d_5 otherwise. If in s_4 hospital h_4 interviews only d_3, then to interview with d_3, d_4 and ranking d_4 right above d_3 when $(q_3, q_4) = (0, 1)$ and d_3 above d_4 otherwise gives a better stage-contingent match. Similarly, if h_4 interviews only d_4 in s_4, then interview with d_5 or if h_4 interviews only with d_5 then to interview with d_4 and ranking d_5 right above d_4 if $(q_4, q_5) = (0, 1)$ and d_4 above d_5 otherwise makes h_4 better off, contradicting with s is a Nash equilibrium. Therefore, h_4 interviews two doctors in s_4.

Then in s_4, hospital h_4 interviews d_4, d_5 or d_3, d_4 or d_3, d_5.

If h_4 interviews d_3, d_5 in s_4, then he does not submit (d_4, d_3, d_5) at any (q_3, q_5) realization (otherwise, since there exists two q where in one $q_4 = 1$ and in the other $q_4 = 0$, h_4 could be better of by submitting (d_3, d_4, d_5) and consequently matching with d_3 instead of d_4. Note that, a hospital prefers to match with d_3 when $q_3 = 0$ to the lottery of a match with d_4 when $q_4 = 0$ or $q_4 = 1$ has equal probabilities.) With a similar argument in s_4 hospital h_4 does not submit (d_3, d_5, d_4) at any (q_3, q_5) realization. Hence, at any q he submits (d_3, d_4, d_5). Then ε-cost of interviewing implies it is not individually rational to interview d_3, d_5, contradicting with s is a Nash equilibrium.

If in s_4 hospital h_4 interviews d_3, d_4, with a similar argument used above, h_4 submits (d_4, d_3, d_5) if $(q_3, q_4) = (0, 1)$ and (d_3, d_4, d_5) otherwise. Moreover, if h_4 interviews d_4, d_5 in s_4, then h_4 submits (d_3, d_5, d_4) if $(q_4, q_5) = (0, 1)$ and (d_3, d_4, d_5) otherwise. The stage-contingent match of these two strategies are given below:

q_1	q_2	q_3	q_4	q_5	(d_4, d_5)	(d_3, d_4)
0	1	0	0	-	$d_3(-)$	$d_3(-)$
0	1	0	1	-	$\mathbf{d_3}(-)$	$\mathbf{d_4}(+)$
0	1	1	0	-	$d_3(+)$	$d_3(+)$
0	1	1	1	-	$d_3(+)$	$d_3(+)$
oth	oth	-	0	0	$d_4(-)$	$d_4(-)$
		-	0	1	$\mathbf{d_5}(+)$	$\mathbf{d_4}(-)$
		-	1	0	$d_4(+)$	$d_4(+)$
		-	1	1	$d_4(+)$	$d_4(+)$

Then these two strategies only differ when $(q_1, q_2, q_3, q_4) = (0, 1, 0, 1)$ or $(q_1, q_2) \neq (0, 1), (q_4, q_5) = (0, 1)$. Note that the probability of first case is $p^2(1-p)^2$ which is always less than of the second $(1 - p(1 - p))p(1 - p)$. By uniformity assumption, when a hospital compares to match with d_4 instead of d_3 when $(q_3, q_4) = (0, 1)$ and to match with d_5 instead of d_4 when $(q_4, q_5) = (0, 1)$, he prefers the event with higher probability. Then, h_4 interviews d_4, d_5 in s_4. End of step.

Note that we have shown, in s_4 hospital h_4 interviews d_4, d_5 and ranks d_3 above d_4 and d_5 at any q, and h_3 does not interview in s_3. However, h_3 would be better of by interviewing d_3, d_4 and ranking d_4 above d_3 when $(q_3, q_4) = (0, 1)$, contradicting with s is a Nash equilibrium. Hence, there exists no pure strategy equilibrium of Γ^*. ■

The non-existence of pure strategy equilibrium result we gave above is robust in some senses: First, it can be embedded into preference structures where there are greater number of agents and the result will still hold. Moreover, this result does not depend on the cardinal utilities of the hospitals, but it is enough to just assume the essentials of the model about the preferences. Besides, we have assumes that $p = 1/2$, but for a greater range of p, the result still holds when it assumed that d_{k+1} when $q_{k+1} = 1$ is not preferred d_k where q_k is not known. Here as one can see, range of p can be much wider.

3.3.2 Two Sided Homogenous Market

To show the preference structure $(R^j)_{\{j \in D\}}$, we prefer to use a matrix. The entry k in the d_i column and h_j row indicates h_j is the d_i's kth most-preferred hopsital. However, some entries of this matrix will not be meaningful. For instance, it is for sure that h_1 will be matched with either d_1 and d_2 at the end of the second stage.

So, the entries of the first row on the ith column where $i \geq 3$ are junk. They can be deleted and the numbers of the matrix can be updated according after this deletion. Also, the same thing can be done for h_2, with the updated preference structure: if it is guaranteed that any h_i will not be matched with d_j in the second stage. Then the entry on the ith row and jth column can be deleted. We call this updation algorithm, sequential update algorithm.

For example, consider the following matrix on the left, this matrix will turn to matrix on right after the algorithm.

	d_1	d_2	d_3	d_4
h_1	1	1	1	1
h_2	2	2	2	2
h_3	3	3	3	3

	d_1	d_2	d_3	d_4
h_1	1	1	X	X
h_2	2	2	1	1
h_3	3	3	2	2

	d_1	d_2	d_3	d_4
h_1	1	1	X	X
h_2	2	2	1	X
h_3	X	3	2	1

From now on, we will use the updated preferences matrices in the examples.

After showing that even with 5 hospitals we can find preference structures where pure strategy Nash Equilibrium does not exists, and the equilibrium is too hard to characterize, we will take a close look into a special preference domain, two-sided homogenous market. This special domain is defined as the domain where the doctors' preferences are identical. We will show that in this domain IRP game has a strictly dominant strategy equilibrium in which each hospital h_j interviews d_i, d_{i+1}.

To see this, consider the following preference structure,

	d_1	d_2	..	d_4	d_5
h_1	1	1	..	1	1
h_2	2	2	..	2	2
..	..	..	..	..	..
h_n	n	n	..	n	n

We define ladder property before finding the strict dominant strategy of IRP Game where doctors' preferences are identical.

DEFINITION 4 *A preference structure represented by the preference matrix P has ladder property, if the IRP game with this preference structure, has an equilibrium in which each university h_i interviews d_i, d_{i+1}.*

THEOREM 27 *IRP Game has strict dominant equilibrium s^* when doctors' preferences are identical, where in s_i^* hospital h_i interviews d_i, d_{i+1}.*

Proof. For the equilibrium of IRP game here, we should first look at h_1's strategies. Since, h_1 is the first choice of all doctors, he will interview with d_1 and d_2 and will submit his preferences truthfully. Then, if $q_1 = 0$ and $q_2 = 1$, h_1 will be matched with d_2 and h_2 will be matched with d_1, and h_2 does not need to make an interview anyone. Otherwise, h_1 will be matched with d_1 and h_2 will be the first choice of all students when we exclude h_1 and d_1 out of the market. Then the same argument will work for h_2 and he will interview d_2 and d_3. Therefore, this process will make for h_i to interview d_i and d_{i+1} dominant strategy. ■

Note that we have shown that there is not pure strategy equilibrium of the IRP Game for a given preference structure with 5 hospitals and five doctors. Moreover, this result is robust in the sense that it relies on minimal assumption of the model. Besides, in the effectively homogenous preference domain, we show strict dominant strategy equilibrium where an order (the ladder property) is observed. Note that, with a small change in the preferences of the effectively homogenous domain one can achieve non-existence of the pure strategy equilibrium. Hence, this shows the intractability of the IRP Game. That is, a center can find an algorithm to compute the interview schedule for hospitals only with further assumptions on the preferences. In a decentralized fashion hospitals may randomize over the interviews. Therefore, when the interviewing and preference reporting stages are considered together, we see in some sense the complex and chancy nature of the information acquisition process.

3.3.3 Effectively Homogenous Market

After defining this property of the equilibrium of the IRP game with two-sided homogenous market, we characterize the *effectively homogenous preference domain*, the domain of preference structures where the equilibrium has *ladder* property.

DEFINITION 5 *A preference structure P is effectively homogenous iff $\forall\ j > 1$, $p_{(j-1)j} < p_{jj}$.*

THEOREM 28 *A preference structure has ladder property if and only if it is effectively homogenous.*

Proof. We start with finding the university who will interview d_1 and d_2. With the observation h_1 will either be matched with d_1 or d_2 in the final matching, no one but him can acquire any information by interviewing d_1 and d_2. So, h_1 is the one who had to interview d_1 and d_2.

We also should check whether h_1 has incentive to interview d_1 and d_2. Now, suppose that h_1 is not the first choice of d_2. Then, d_2 will either be matched with her first choice, or she will be rejected by her first choice. If she is matched with her first choice, there is no possibility of getting her; if she is rejected by her first choice, then it means she is not good. In both cases, there is no value of information for h_1 to interview d_1 and d_2. So, h_1 should be the first choice of d_1 to have strong ladder property.

	d_1	d_2	
h_1	1	1	
h_2		2	

	d_1	d_2	
h_1	1	n_1	
h_2		1	

n_1 should be equal to 1

Now we have to find the university who will interview d_2 and d_3. Note that, d_2 can be matched with h_1. In this case no value of information in (d_2, d_3) interview for anyone. If h_1 will not match with d_2, h_2 will match with d_2 if h_2 does not reject d_2. If he rejects it means d_2 is negative and again there is no value of information in (d_2, d_3) interview for anyone but h_2. If h_2 is matched with d_2 again no value of information in (d_2, d_3) interview for anyone. That means, the only one who has value of information on (d_2, d_3) interview is h_2.

	d_1	d_2	d_3
h_1	1	1	
h_2		2	n_2

First, note that h_1 will get either d_1 or d_2 in the second stage. So, at most two university, except h_1, can reject d_3 one to have either d_1 or d_2, one to have d_4 if she is better than d_3. Therefore, $n_2 \leq 3$. If $n_2 = 3$, d_3 can propose to h_2, if one university rejected d_3 to have d_4 and one university rejected d_3 to have d_1 or d_2. This means d_2 has already proposed to someone and will not be rejected. That is to say interview with d_2 and d_3 is meaningless. If $n_2 = 2$, d_3 proposes h_2 if d_{31} rejects d_3. This can be only if d_{31} rejects d_3 either, to have d_4 which means d_3 cannot be better than d_2 and it is meaningless for h_2 to interview with d_2, d_3 pair, or to have d_1 which means h_1 got d_2 which implies again it is not individually rational to interview d_2, d_3 pair. Hence, we conclude $n_2 = 1$.

	s_1	s_2	s_3	
u_1	1	1		
u_2		2	1	
u_3			2	n_3

The same procedure, goes on for all h_i and completes the proof.

■

3.4 References

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