THE HARDY-RAMANUJAN-RADEMACHER EXPANSION FOR THE PARTITION FUNCTION AND ITS EXTENSIONS

by

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Abstract

Partition theory has been studied more extensively during the last century, athough it has been around since Euler. It is not only because its combinatorial or classical analytical aspects, but also because of the opportunities number theorists saw in applications of modular forms in a different and deep view. In this thesis, we study exact formulas for the number of various partitions. For each one, we need to prove a modular transformation formula, and use Farey dissection to avoid the essential singularities of the generating functions. After that, we need to control or estimate the resulting integrals which are rooted from Cauchy integral formula.

In this way, we first study an exact formula for the number of ordinary partitions of any given integer. This formula is a famous result by Ramanujan, Hardy, and Rademacher. Also, we studied another well-known result by Hao, which gives an exact formula for the number of partitions into odd parts. This partition can also be considered for the partitions with distinct parts, thanks to Euler's partition identity. The generating function is a modular form which needs Kloosterman's estimates to handle the integrals.

Next, we propose a result which is aimed at the colored partitions with parts of the form $10t \pm a$ or $2t \pm 1$. This is a continuation of recent works to generalize to partitions into parts in certain symmetric residue classes modulo a given integer. Finally, we will explain about possible future plans to find exact formulas for various other partition functions.

SONLU CİSİMLER ÜZERİNDEKİ İNDİRGENEMEZ POLİNOMLARIN BAZI ALT SINIFLARI ÜZERİNE

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Özet

Tamsayı parçalanışları teorisi Euler'den beri bilinmesine rağmen son yüzyılda daha yoğun çalışılmıştır. Bunun sebebi sadece kombinatorik veya klasik analizden beslenmekle kalmayıp modüler fonksiyonların bu alana farklı ve derin uygulamalarını sayı teorisyenlerinin farketmeleri ile olmuştur. Bu tezde bazı parçalanış fonksiyonlarının kesin formülleri üzerine çalıştık. Her biri için öncelikle bir modüler dönüşüm formülü ispatlamamız gerekti, ve sonrasında Farey ayrışımı kullanarak üreteç fonksiyonların esas tekilliklerinden kaçındık. Bundan sonra Cauchy integral formülünden türeyen integrallerin büyümesini kontrol ettik veya değerlerini tahmin ettik.

Bu şekilde, ilk önce adi parçalanış sayıları için bir kesin formül üzerinde çalıştık. Bu formül Ramanujan, Hardy ve Rademacher'ın bir sonucudur. Bunun yanısıra yine iyi bilinen bir sonuç olan, Hao'nun tek kısımlara parçalanış sayısını veren kesin formülü üzerine çalıştık. Euler'in tamsayı parçalanış özdeşliğine göre bu aynı zamanda farklı kısımlara parçalanış sayısıdır. Burada üreteç fonksiyon modüler bir fonksiyondur ve intregrallerin hesaplamak için Kloosterman'ın tahminleri gerekir.

Bundan sonra, kısımları $10t \pm a$ veya $2t+1$ olan renkli parçalanışlarla ilgili bir sonuç ortaya attık. Bu, eldeki sonuçları simetrik denklik sınıflarından kısımlara parçalanışları ele alan, yakın zamandaki araştırmaların devamı niteliğindedir. Son olarak, çeşitli diğer parçalanış fonksiyonlarına kesin formüller bulmak için ileride yapılabilecek çalışmalardan bahsettik.

To my family

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CHAPTER 1

Introduction

In this chapter, we explain some basic notions, which are useful in this thesis, in a brief way.

1.1. Partitions and their properties

In the first place, we will explain the basic notions of partitions. Assume that $n \in \mathbb{N}$. We want to find the number of ways that n can be writen as a summation of positive integers. For example $p(4) = 4$; since $4 = 1+1+1+1$ or $4 = 1+3$ or $4 = 1+1+2$ or $4 = 2+2$ ($1+3$ and $3+1$ are regarded as the same partitions). The generating function of such partitions is

$$F(q) = \prod_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} q^{mn} \right). \tag{1.1}$$

We can use the equality

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n. \tag{1.2}$$

So

$$F(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.3}$$

For brevity, we define

$$(a;q)_n = \prod_{m=0}^{n-1} (1-aq^m) \qquad \textit{and} \qquad (a;q)_\infty = \prod_{m=0}^{\infty} (1-aq^m) \tag{1.4}$$

For the second case, the definition is for $|q| < 1$, to make it convergent. Also, in order to simplify the explanations, when we write an infinite series or product, we do not mention the proper radius q, but we assume the convergent neighborhood. In fact, $|q| < 1$ makes all series and products in the thesis converge absoulutely (see for example [1] for the proof). So we need to find $p(n)$ which is the nth coefficient of $\frac{1}{(q;q)_\infty}$. We will study a proof of an exact formula for $p(n)$ in chapter 2.

There are other kinds of partitions. One of the most important one is the partition with odd parts (we call it odd partitions). One can see that its generating function F_o is as follows (see [1]).

$$F_o(q) = \frac{1}{(q;q^2)_\infty}. \tag{1.5}$$

We will study an exact formula for this function in chapter 3. Also, one can see that

$$F_o(q) = \frac{1}{(q;q^2)_\infty} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n+1})} = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{(1-q^n)} = \prod_{n=1}^{\infty}(1+q^n). \tag{1.6}$$

So $F_o(q)$ is also the generating function of the partitions into distinct parts. In general, the generating function of partitions with parts $Mt \pm a$ can be writen as

$$F_{M,a}(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{Mn\pm a})} = \frac{1}{(q^a;q^M)_\infty} \frac{1}{(q^{M-a};q^M)_\infty}. \tag{1.7}$$

We can also generalize the partition into the union of a set A of parts in the forms $Mn + \pm a_1$ or $Mn \pm a_2$ or ... or $Mn \pm a_i$. In this way the generating function is

$$F_{M,A}(q) = \prod_{t=1}^{i} \prod_{n=1}^{\infty} \frac{1}{(1-q^{Mn\pm a_t})}. \tag{1.8}$$

We will discuss the properties of the generating function for different kinds of the set A in chapter 4 and 5. Another interesting partition is the one with parts which are not divisible by r for some r. This partition has the following generating function.

$$F_r(q) = \prod_{n=1}^{\infty} \frac{1-q^{rn}}{(1-q^n)} = \prod_{n=1}^{\infty}(1+q^n+q^{2n}+\cdots+q^{(r-1)n}). \tag{1.9}$$

We can show that this $F_i(q)$ is also the generating function of the partitions in which parts are repeated at most $r-1$ times. One of the most general case is to find partitions with parts of the form $M_1 t \pm a_1$ or $M_2 t \pm a_2$ or ... or $M_k t + a_k$ for some k, M_i, a_i. We tried to find a special case of this for $M_1 = 10$, $M_2 = 2$, $a_2 = 1$ in chapter 4.

1.2. Modular forms

Modular functions are a category of functions from upper half complex plane to complex numbers, which can fix the set of translation and rotation. A modular function f of wieght k satisfies the following equation.

$$f(\frac{az+b}{cz+d}) = (cz+d)^{-2k}\omega_{c,d} f(z) \tag{1.10}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$, and $\omega_{c,d}$ is a root of unity, which is called multiplier system. Now we explain the necessary properties of a multiplier system. First, we need to see the transformation $\frac{az+b}{cz+d}$ as a group action. Recall that $PSL_2(\mathbb{Z})$ is the class of 2×2 matrices with determinant 1 and integer entries; when $A \sim B$ for $A = \pm B$. One can see that the following action of $PSL_2(\mathbb{Z})$ over upper half plane is a group action. For every matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.11}$$

we define an action as follows.

$$M.z = \frac{az+b}{cz+d}. \tag{1.12}$$

We can show that it transfers the upper half plane to the uper half plane. Now, if we define a function from $PSL_2(\mathbb{Z})$ to roots of unity with the relation $\omega(M) = \omega_{c,d}$ and the property that $\omega(M_1 M_2) = \omega_{c_1,d_1} \circ \omega_{c_2,d_2}$. We can prove that $PSL_2(\mathbb{Z})$ can be generated by the matrices

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1.13}$$

and

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.14}$$

We call a modular function as a modular form, if f is holomorphic in upper half plane.

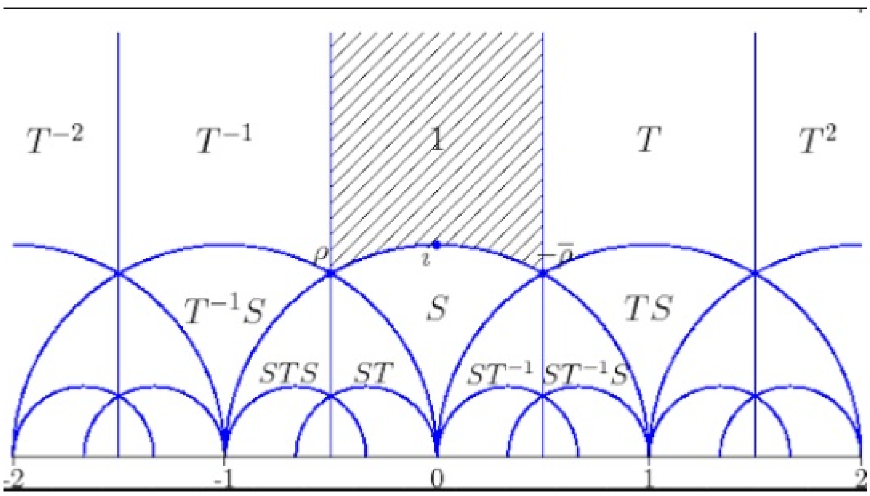

Figure 1.1: Fundamental domain of a modular form

For each modular form, there exists a subset of upper half plane which is invariant over

$PSL_2(\mathbb{Z})$. This region is called as fundumental region and is like the figure 1.1. There are different categories for modular forms. We introduce two important ones. The first one is Eisenstein series which are in the following form.

$$E_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}} \tag{1.15}$$

which have weight k. One can see that this also corresponds to the extended series of an elliptic curve (see [30]). So this category is very important. One of the most famous property of this category is the fact that the set of $\{E_k\}$ forms a finite generated C-algebra. In fact this C-algebra is $C[E_2, E_3]$ (see [30] for a proof).

Another important category is theta functions. First, we define lattices over $\mathbb{Z}$. A lattice L is a subgroup of $\alpha\mathbb{Z} \oplus \beta\mathbb{Z}$, where $\alpha, \beta \in \mathbb{C}$ are linearly independent (see for example [26]). Then one can define a theta function as follows.

$$\Theta(z) = \sum_{\Lambda\in L} e^{\pi i||\Lambda||^2 z} = \sum_{(m,n)\neq(0,0)} e^{\pi i(m^2+n^2)z}. \tag{1.16}$$

This function is in fact equal to $G(z)^2$, where $G(z) = \sum_{m\neq 0} e^{\pi i m^2 z}$. From number-theoretic point of view, this function can be seen in a beatiful way as follows.

$$(G(z))^m = (\sum_{n=1}^{\infty} r_m(n) e^{\pi i n z}). \tag{1.17}$$

where $r_m(n)$ is the number of ways that we can write n as the sum of m squares. This function is extensively studied (for more information, reader can see [19]). There is another famous function which has a very close relationship with G. It is called Dedkind eta function which is defined as follows.

$$\eta(e^{2\pi i\tau}) = e^{\frac{\pi i\tau}{12}} \prod_{n=1}^{\infty} \frac{1}{1 - e^{2\pi i n\tau}}. \tag{1.18}$$

and has the following relation with G.

$$G(e^{2\pi i\tau}) = \frac{\eta^2(e^{\pi i(\tau+1)})}{\eta(e^{2\pi i(\tau+1)})}. \tag{1.19}$$

This function is a half-weight modular form and is very useful in the discussion of this thesis. In fact, a lot of modular forms can be writen based on this function. We can write η as

$$\eta(z) = (\sum_{n=1}^{\infty} l(n) e^{\pi i n z}) \tag{1.20}$$

Then,

$$(\eta(z))^m = (\sum_{n=1}^{\infty} l_m(n) e^{\pi i n z}) \tag{1.21}$$

where $l_m(n)$ is the number of ways that n can be writen as the sum of triangular numbers (i.e. the numbers of the form $\frac{n(n+1)}{2}$). The coefficients l_m are completely identified as follows in [19].

$$l_m(n) = \frac{e^{\frac{-\pi i m}{4}} \pi^{\frac{m}{2}} n^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2})} \sum_{c=1}^{\infty} c^{-\frac{m}{2}} \sum_{0\leq h<2c} \upsilon_m(c,h) e^{\frac{\pi i n h}{c}} \tag{1.22}$$

where $\upsilon_m(c,h)$ is the multiplier system for η as follows. For every $M \in PSL_2(\mathbb{Z})$ with

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.23}$$

According to [19], we have

$$\upsilon_\eta(M) = \begin{cases} \left(\frac{d}{|c|}\right) e^{\frac{\pi i}{12}((a+d)c-bd(c^2-1)-3c)} & c: odd \\ \left(\frac{d}{|c|}\right) (-1)^{\frac{(c-1)(d-1)}{2}} e^{\frac{\pi i}{12}((a+d)c-bd(c^2-1)+3d-3-3cd)} & c: even \end{cases} \tag{1.24}$$

In the same way, we can find the multiplier system of G. For an extensive sdiscussion, please see [19]).

1.3. Kloosterman's sum

In this section, we try to cover basic notions of the Kloosterman's sum. Kloosterman's sum is a generalization of Ramanujan sum which is as follows.

$$K(a,b;m) = \sum_{\substack{0\leq h\leq m-1 \\ gcd(h,m)=1}} e^{\frac{2\pi i}{m}(ah+bh')}. \tag{1.25}$$

where $hh' \equiv 1(modm)$. These sums are very useful to study Bessel functions and also have various applications in Fourier extension of modular forms. Kloosterman sums have multiplicative property. In another words, for $m = m_1m_2$ where m_1, m_2 are coprime, $n_1m_1 \equiv 1(modm_2)$, and $n_2m_2 \equiv 1(modm_1)$, then

$$K(a,b;m) = K(n_2a, n_2b; m_1)K(n_1a, n_2b; m_2). \tag{1.26}$$

So it is enough to find $K(a,b;p^\alpha)$. Sato-Tate conjectured that there is no exact formula for $K(a,b,p)$. But we have a very useful formula for the following special case

$$K(a,a;p) = \sum_{m=0}^{p-1} \left(\frac{m^2-4a^2}{p}\right) e^{\frac{2\pi i m}{p}}, \tag{1.27}$$

where $\left(\frac{m^2-4a^2}{p}\right)$ is Jacobi's symbol. We need some estimations for an incomplete Kloosterman's sum in the thesis. There are different bounds to estimate a Kloosterman's sum. The best known bound for these sums goes back to Weil bound which is

$$|K(a,b;m)| \leq \tau(m)\sqrt{m\,\mathrm{gcd}(a,b,m)}. \tag{1.28}$$

As a result, there is the following straightforward bound for the Kloosterman's sum.

$$|K(a,a;p)| \leq 2\sqrt{p}. \tag{1.29}$$

By an incomplete Kloosterman's sum, we mean $\sum_{h \in A} e^{\frac{2\pi i}{m}(ah+bh')}$, where A is a subset of $\mathbb{Z}*_m$. Thus an incomplete Kloosterman's sum runs over a subset modulo m. One of the best bounds for an incomplete Kloosterman's sum which the number of summands is less than m^ϵ, is $e^{(\log(m))^{\frac{2}{3}+\epsilon}}$. So we can estimate an incomplete Kloosterman's sum $R(a,b;m,\epsilon)$ with length m^ϵ as

$$R(a,b;m,\epsilon) \leq m^{\frac{2}{3}+\epsilon}. \tag{1.30}$$

1.4. Bessel functions and Mellin transformation

Bessel function are nothing but the contious version of Kloosterman's sum. They are the solutions of the forllowing differential equation.

$$x^2 \frac{d^2y}{dy^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0. \tag{1.31}$$

We call α order the Bessel function. There are two different kinds of solutions for each Bessel differential equation. In general, the first kind of the Bessel function of order α is

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}. \tag{1.32}$$

If we view a first kind Bessel function of order zero in the following way,

$$J_0(x) = \frac{1}{2}\int_0^\infty e^{-t+\frac{z^2}{4t}} \frac{d}{dt}, \tag{1.33}$$

then it can be considered as the solution of the analouge of the equation (1.25). As modular forms are connected to Kloosterman's sum, it is natural that Maass forms can be controlled by Bessel functions. They also have a relation with hypergeometric serries; which are a generalization of geometric serries. So it is natural that the geometric serries can be end up to Kloosterman sums.

We also need to take care of the assymptotic properties of Bessel functions. In small amounts of z, Bessel function of order α is behaving like $\left(\frac{z}{2}\right)^\alpha$. In fact, for $0 \leq z << \alpha+1$, we have $J_\alpha(z) = \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha$. Finally, we mention a useful property of Bessel functions.

$$e^{(\frac{x}{2})(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{1.34}$$

Now, we explain Mellin transformation briefly. It can be considered as the multiplicative version of two-sided Laplace transformation. It is mainly because of the following relations with Laplace transformation.

$$M\left(f\left(-\log(t)\right)\right)(s) = L\left(f(t)\right) \tag{1.35}$$

Finally, one of the most important properties of Mellin transformation is that for $c > 0$, $Re(y) > 0$, and y^{-s} on the principle branch, we have

$$e^{-y} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-s} ds. \tag{1.36}$$

1.5. Farey dissection and Lipshitz summation formula

Farey dissection is a recurrence sequences of numbers which are generated as follows. Start with $\frac{a_0}{b_0} = \frac{0}{1}$ and $\frac{c_0}{d_0} = \frac{1}{1}$; and if $\frac{a_n}{b_n}, \frac{c_n}{d_n} \in A_n$, then $\frac{a_{n+1}}{b_{n+1}} = \frac{a_n+c_n}{b_n+d_n} \in A_{n+1}$. So the cardinality of the sequence F_n can be found inductively as follows.

$$|F_n| = |F_{n-1}| + \phi(n). \tag{1.37}$$

So one can see that $|F_n| \sim \frac{3n^2}{\pi^2}$. It has a close relation with Ford circles which can be seen at [].

Now, we introduce Lipschitz summation formula. Let $Im(z) > 0$, $N \in \mathbb{N}$, and $0 \leq \alpha \leq 1$. Then

$$\sum_{l=-N}^{N} \frac{e^{-l\alpha}}{(\tau + l)^p} = \frac{(-2\pi i)^p}{\Gamma(p)} \sum_{m=0}^{\infty} (m+\alpha)^{p-1} q^{m+\alpha} + E(\tau, p, N + \frac{1}{2}), \tag{1.38}$$

where $E(\tau, p, N + \frac{1}{2})$ ia an error term and given by

$$(i(N+\frac{1}{2}))^{1-p} \int_{-\infty}^{\infty} \frac{h(x-i) - h(x+i)}{1 + e^{2\pi x(N+\frac{1}{2})}} dx \tag{1.39}$$

and

$$h(x) = \frac{e^{2\pi x (N+\frac{1}{2})\alpha}}{(x + \frac{\tau}{i(N+\frac{1}{2})})^p}. \tag{1.40}$$

This formula will be very useful for us in the next chapetrs. It can help us to find a way to use analytic continuation of a complex function.

1.6. Organization

In this thesis, the first chapter is for familiarizing the notions and basic constructions like partitions, modular forms, Kloosterman sums, Mellin transforms, and Bessel functions. In the second chapter, after a brief historical explanation, we study Rademacher's proof for the exact formula for the number of partitions. There are three main steps to do this. First, we should find a modular transformation for the generating function of number of partitions. Then we have to take care of singularities of the generating functions, which is adressed by Farey dissection. Finally we will bound the Cauchy integral and find an exact formula for the partitions.

In the second chapter, we plan to study the proof by Hao for the number of partitions into odd parts (or the number of partitions with distinct parts; see [1]). We also need a modular transformation for the first step as well. Then we use Farey dissection to find an incomplete Kloosterman's sum. Then we will control the integral by bounding the Kloosterman's sum. The fourth chapter is on a modular transformation for the number of partitions with parts of the form $10t \pm a$ or $2t + 1$. We will find this transformation by dividing the whole case to four ones based on the $\gcd(10, k)$. Finally, in the last chapter, we suggest some of possible directions for future research.

CHAPTER 2

The Hardy-Ramanujan-Rademacher expansion of $p(n)$

2.1. Introduction

One of the most amazing results in the theory of partitions goes back to the joint efforts of Hardy and Ramanujan to find an asymptotic formula for $p(n)$. At first, it seemed highly unlikely that $p(n)$ has a relationship with modular forms. But in one of the most surprising proofs in the analytic number theory, Hardy and Ramanujan could offer a very technical analytic proof to find an almost exact formula for $p(n)$. The story begins from one of Ramanujan's conjectures. He could find an asymptotic formula as follows for $p(n)$.

$$p(n) \sim \frac{1}{2\sqrt{2}} \sum_{q=1}^{v} \sqrt{q} \sum_{p}^{\prime} \omega_{p,q} e^{\frac{-2np\pi i}{q}} \frac{d}{dx} (\pi \sqrt{\frac{2}{3}} \frac{\sqrt{x}}{q})|_{x=n}. \tag{2.1}$$

where $\omega_{a,b}$ is a root of unity which will be explained explicitly. It is unknown that how Ramanujan was sure that this formula is an almost exact approximate for $p(n)$. But He believed that there is a formula with $O(1)$ for $p(n)$ (see for example [1], [23]). So he tried several times to find the formula. After a joint work with Hardy, they could find a similar formula to (2.1).

They used an indirect way, which is using the Cauchy integral to find the generating function of $p(n)$. In this way, they could use complex analytic techniques. The first problem which occured was to find a proper contour, which can avoid the poles of the generating function of $p(n)$. As one can see, the generator function $F(q) = \sum_{n=0}^{\infty} p(n) q^n$ is $\prod_{n=0}^{\infty} (1-q^n)^{-1}$ (see [1]). This function has infinitely many essential singularities in the unit circle. So the wisest idea seems to avoid this circle. Hardy and Ramanujan avoided this circle and considered a circle inside the unit circle. In this way, they had an integral and after a proper parametrization, they needed to partition the circle. It was because of the fact that they had to avoid rational points in the contour, which are the essential singularities. Hardy and Ramanujan used the function *cosh*, and obtained a divergent series which gives an asymptotic formula. But Rademacher used *sinh* and the Farey partition and obtained a convergent series. This led to an exact formula. This was the main contribution in the Rademacher's proof. Using Farey dissection gave a better shape to the proof. Also, this idea has been used in almost all of the similar results about the partition that came later. The next step was the most important one. Hardy and Ramanujan found out that there is a modular transformation for the generating function $F(q)$. After using different ideas like Lipschitz summation formula ([21]) and analytic continuation, they proved a modular formula. The next necessary step was to control the growth of some integrals. Then they divided the integral to two parts. One of them was significant, which leads to the asymptotic formula. The small integral could be bounded by considering a proper contour.

One of the best ways to prove this theorem can be found in [5]. Berndt proposed a simpler proof, which is similar to the one that we will give in this chapter. Also, he proved similar results for various other modular forms. For example he proved transformation formula for generalized Dedekind eta function and a large class of generalized Eisenstein series. For more information, one can see [**?**, **?**].

One interesting point about the asymptotic formula is its pace in the convergence. In fact, this formula is one of the sharpest formula in the area of analytic number theory and modular forms. To justify this claim, consider the figure **??**. This figure shows the first 10 terms of Ramanujan's estimate for the number of partitions of 200 and $p(200)$. One can see that the difference is negligible even for 10 terms, which is a small portion of the sum in the scale of analytic number theory. Also, the asymptotic formula of partitions needs only 17 steps to have an error less than 1 for $p(750)$.

2.2. The modular transformation formula

In the first place, we try to view the generating function of partitions as a modular function. We aim at finding an asymptotic formula for $p(n)$ which is as follows.

$$F(q) = \prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n. \tag{2.2}$$

We try to view $F(q)$ as a complex function. Also, we consider that $q = e^{2\pi i z}$. So We have $F(z) = \sum_{n=0}^{\infty} p(n)e^{2\pi i n z}$. Before we find the formula, we try to justify that why we need such modular equation. Cauchy integration formula immediately implies that $p(n) = \frac{1}{2\pi i}\int_C \frac{F(s)}{(s)^{n+1}}ds$, where C is a contour inside the unit circle. We choose this to avoid the possible essential singularities over the unit circle (In fact, we are dealing with $\prod_{n=1}^{N}(1-q^n)^{-1}$, so we have to avoid the poles in our computations). To be more precise, F has poles in all points $e^{2\pi i q}$ for large enugh N, where q is a rational number. So we need to avoid computing unnecessary residues. The next natural question which arose here is to find the zeroes of F inside the region for C. This is the first place that modularity of F helps us to estimate place of poles in the integral. After that, we try to avoid the poles by using Farey dissection. Then we need to compute the integral by finding the residues. In order to compute it, we use the modularity again. We will discuss it in more details in the next part.

Now we prove the modularity of F. Let η be the Dedekind eta function defined as follows

$$\eta(z) = \frac{e^{\frac{\pi iz}{12}}}{F(e^{2\pi iz})}. \tag{2.3}$$

Also, for $Im(z) > 0$,

$$A(z,s) = \sum_{m,n\geq 1} n^{s-1} e^{2\pi imnz} \tag{2.4}$$

In the first step, we want to show that $A(z,0) = \frac{\pi iz}{12} - \log(\eta(z))$. One can see

$$\begin{aligned} A(z,0) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi imnz}}{n} = -\sum_{m=1}^{\infty} \log(1-e^{2\pi imz}) \\ &= -\log(\prod_{m=1}^{\infty}(1-e^{2\pi imz})) = -\log(P(e^{2\pi iz})) = \frac{\pi iz}{12} - \eta(z). \end{aligned} \tag{2.5}$$

Let $G(z,s), g(z,s)$ be defined as follows. For all $-\pi \leq arg(s) < \pi$, $Im(z) > 0$, and $Re(s) > 2$.

$$\begin{aligned} G(z,s) &:= \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(mz+n)^s}. \\ g(z,s) &:= \sum_{\substack{m\leq 0 \\ n\in\mathbb{Z} \\ (m,n)\neq(0,0) \\ dm-cn>0}} \frac{1}{(mz+n)^s}. \\ L(z,s) &:= \sum_{j=1}^{c} \int_C \frac{u^{s-1}e^{-(cz+d)\frac{ju}{c}+\{\frac{jd}{c}\}u}du}{(1-e^{-czu-du})(e^u-1)}. \end{aligned} \tag{2.6}$$

There are similar relations for Eisenstein series (See for example [5]).

In order to continue, first we prove the following equality. Consider that arg of every complex number is between $-\pi$ and π. Let $A, B, C, D \in \mathbb{R}$, $w \in \mathbb{C}$, A, B are non zero, $C > 0$, and $Im(w) > 0$. Then

$$arg(\frac{Aw+B}{Cw+D}) = arg(Aw+B) - arg(Cw+D) + 2\pi k \tag{2.7}$$

where k can be defined as follows.

$$k = \begin{cases} 1, & A \leq 0, AD-BC > 0 \\ 0, & otherwise \end{cases} \tag{2.8}$$

So one can see

$$(cz+d)^{-s}G(Vz,s) = G(z,s) + (e^{-2\pi is}-1)g(z,s). \tag{2.9}$$

On the other hand, we know that

$$\begin{aligned} g(z,s) = \sum_{\substack{m\leq 0\\ n\in\mathbb{Z}\\ (m,n)\neq(0,0)\\ dm-cn>0}} \frac{1}{(mz+n)^s} &= \sum_{n=-\infty}^{-1} \frac{1}{n^s} + \sum_{\substack{m<0\\ n\in\mathbb{Z}\\ (m,n)\neq(0,0)\\ n<dm/c}} \frac{1}{(mz+n)^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{(e^{\pi i}n)^s} + \sum_{\substack{m>0\\ n\in\mathbb{Z}\\ n>dm/c}} \frac{1}{(-mz-n)^s} = e^{\pi is}(\zeta(s)+h(z,s)) \end{aligned} \tag{2.10}$$

where $h(z,s) = \sum_{\substack{n\in\mathbb{Z}\\ n>dm/c}} \frac{1}{(mz+n)^s}$ for all $-\pi \leq arg(s) < \pi$, $Im(z) > 0$, and $Re(s) > 2$.

Now we try to find $\Gamma(s)h(z,s)$. We have

$$\Gamma(s)h(z,s) = \int_0^\infty u^{s-1}e^{-u}du \sum_{\substack{m>0\\ n>dm/c}} (mz+n)^{-s} = \int_0^\infty \sum_{\substack{m>0\\ n>dm/c}} (mz+n)^{-s}u^{s-1}e^{-u}du \tag{2.11}$$

Now, we change the parameter u to $\frac{u}{mz+n}$. So

$$\Gamma(s)h(z,s) = \int_0^\infty \sum_{\substack{m>0\\ n>dm/c}} u^{s-1}e^{-u(mz+n)}du \tag{2.12}$$

In order to start the sum with $m,n=0$, we change m to $m-1$ and n to $n-[\frac{md}{c}]-1$. So we will have

$$\Gamma(s)h(z,s) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_0^\infty u^{s-1}e^{-(m+1)zu-(n+1+[\frac{(m+1)d}{c}])u}du. \tag{2.13}$$

Next, we try to simplify the term $[\frac{(m+1)d}{c}]$. So we consider $m = pc+j-1$, $0 \leq p < \infty$ and $1 \leq j \leq c$. Then for $Re(z) > -d/c, Im(z) > 0$,

$$\begin{aligned} \Gamma(s)h(z,s) &= \sum_{j=1}^{c}\sum_{p=0}^{\infty}\sum_{n=0}^{\infty} \int_0^\infty u^{s-1}e^{-(pc+j)zu-(n+1+pd+[\frac{jd}{c}])u}du \\ &= \sum_{j=1}^{c} \int_0^\infty u^{s-1}e^{-(jzu+[\frac{jd}{c}]u+u)} \sum_{p=0}^{\infty}\sum_{n=0}^{\infty} e^{-pczu-(n+pd)u}du \\ &= \sum_{j=1}^{c} \int_0^\infty u^{s-1}e^{-(jzu+[\frac{jd}{c}]u+u)} \sum_{p=0}^{\infty} e^{-pczu-pdu} \sum_{n=0}^{\infty} e^{-nu}du \end{aligned} \tag{2.14}$$

We have to justify the last two equalities. Since $Re(z) > -d/c$, one can see $|e^{-(cz+d)u}| = e^{-Re(cz+d)u} < 1$. So both the series $\sum_{p=0}^{\infty} e^{-pczu-pdu}$ and $\sum_{n=0}^{\infty} e^{-nu}$ are uniformly convergent. So we can change the order of integral and summations. So after using the geometric series formula, one can see

$$\begin{aligned}\Gamma(s)h(s,z) = \sum_{j=1}^{c}\int_0^\infty \frac{u^{s-1}e^{-(jzu+[\frac{jd}{c}]u+u)}du}{(1-e^{-czu-du})(1-e^{-u})} &= \sum_{j=1}^{c}\int_0^\infty \frac{u^{s-1}e^{-(jzu+[\frac{jd}{c}]u)}du}{(1-e^{-czu-du})(e^{u}-1)} \\ &= \sum_{j=1}^{c}\int_0^\infty \frac{u^{s-1}e^{-(cz+d)\frac{ju}{c}+\{\frac{jd}{c}\}u}du}{(1-e^{-czu-du})(e^{u}-1)}\end{aligned} \tag{2.15}$$

where the last equality is followed from $\frac{jd}{c} = [\frac{jd}{c}] + \{\frac{jd}{c}\}$.

So one can see that for $Re(z) > -d/c, Im(z) > 0, Re(s) > 2$

$$\Gamma(s)h(s,z) = (1-e^{2\pi is})^{-1}L(z,s). \tag{2.16}$$

So according to (2.10),

$$g(z,s) = e^{\pi is}\zeta(s) + \frac{e^{\pi is}(1-e^{2\pi is})^{-1}L(z,s)}{\Gamma(s)}. \tag{2.17}$$

Hence, by (2.9)

$$(cz+d)^{-s}G(Vz,s) = G(z,s) + (e^{-2\pi is}-1)\left(e^{\pi is}\zeta(s) + \frac{e^{\pi is}(1-e^{2\pi is})^{-1}L(z,s)}{\Gamma(s)}\right) \tag{2.18}$$

Thus

$$(cz+d)^{-s}G(Vz,s)\Gamma(s) = \Gamma(s)G(z,s) - 2i\sin(\pi s)\Gamma(s)\zeta(s) + e^{-\pi is}L(z,s). \tag{2.19}$$

According to Lipschitz summation formula from chapter 1, we can view $G(z,s)$ as follows.

$$\begin{aligned}G(z,s) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}(mz+n)^{-s} &= \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} n^{-s} + \sum_{m<0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s} + \sum_{m>0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s} \\ &= \sum_{n=1}^{\infty} n^{-s} + \sum_{n=-\infty}^{-1} n^{-s} + \sum_{m<0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s} + \sum_{m>0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s}\end{aligned} \tag{2.20}$$

Now we try to find the values of these series. First

$$\sum_{n=-\infty}^{-1} n^{-s} = \sum_{n=1}^{\infty}(-n)^{-s} = \sum_{n=1}^{\infty} n^{s}e^{\pi is} = e^{\pi is}\zeta(s). \tag{2.21}$$

Second

$$\sum_{m>0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s} = \sum_{m>0}\frac{(-2\pi i)^s}{\Gamma(s)}\sum_{n>0}n^{s-1}e^{2\pi imnz} = \frac{(-2\pi i)^s}{\Gamma(s)}A(z,s). \tag{2.22}$$

Also

$$\sum_{m>0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s}=\sum_{m<0}\sum_{n=-\infty}^{\infty}(-mz-n)^{-s}=\sum_{m>0}\sum_{n=-\infty}^{\infty}(mz+n)^{-s}e^{\pi is}$$
$$=e^{\pi is}\frac{(-2\pi i)^s}{\Gamma(s)}A(z,s). \qquad (2.23)$$

So according to (2.20), (2.21), (2.22), (2.23), for $Re(z)>-d/c, Im(z)>0$, and $Re(s)>2$

$$G(z,s)=(1+e^{\pi is})(\zeta(s)+\frac{(-2\pi i)^s}{\Gamma(s)}A(z,s)). \qquad (2.24)$$

We know that $A(z)$ is analytic for every complex s and $Im(z)>0$. So by analytic continuation, we can define $G(z,s)$ as an analytic function for every s and $Im(z)>0$. Thus by definition of $H(z,s)$

$$G(z,s)=(1+e^{\pi is})\zeta(s)+\frac{(-2\pi i)^s}{\Gamma(s)}H(z,s). \qquad (2.25)$$

Hence, by (2.19) and (2.25)

$$\begin{aligned}(cz+d)^{-s}H(Vz,s)=&H(z,s)-\frac{\Gamma(s)\zeta(s)(cz+d)^{-s}(1+e^{\pi is})}{(-2\pi i)^s}\\&+\frac{\Gamma(s)(1+e^{-\pi is})\zeta(s)}{(-2\pi i)^s}+\frac{e^{-\pi is}L(z,s)}{(2\pi i)^s}\\=&H(z,s)-\Gamma(s)\zeta(s)(cz+d)^{-s}(1+e^{\pi is})(2\pi i)^{-s}\\&+\Gamma(s)(1+e^{\pi is})\zeta(s)(2\pi i)^{-s}+L(z,s)(2\pi i)^{-s}. \qquad (2.26)\end{aligned}$$

Now we put $s=0$. One can see

$$H(Vz,0)=H(z,0)-2\Gamma(0)\zeta(0)+2\Gamma(0)\zeta(0)+L(z,0)=H(z,0)+L(z,0). \qquad (2.27)$$

So the remaining thing is to find $L(z,0)$. We have to find the residue of $\sum_{j=1}^{c}\frac{e^{-(cz+d)\frac{ju}{c}+\{\frac{jd}{c}\}u}}{u(1-e^{-czu-du})(e^u-1)}$ over the contour C. It is shown that the residue of this function is

$$\sum_{j=1}^{c}\frac{A_j}{-12(cz+d)} \qquad (2.28)$$

where

$$\begin{aligned}A_j=&-(cz+d)^2-(-6(-(cz+d)\frac{j}{c}+\{\frac{jd}{c}\})+3)(-cz-d)\\&-6(-(cz+d)\frac{j}{c}+\{\frac{jd}{c}\})^2+6(-(cz+d)\frac{j}{c}+\{\frac{jd}{c}\})-1 \qquad (2.29)\end{aligned}$$

Which after some computations (see for example [1]), it will become as follows.

$$Res(\sum_{j=1}^{c-1} \frac{e^{-(cz+d)\frac{ju}{c}+\{\frac{jd}{c}\}u}}{u(1-e^{-czu-du})(e^u-1)})$$
$$= \left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + \sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right). \quad (2.30)$$

So

$$L(z,0) = \sum_{j=1}^{c}\int_C \frac{e^{-(cz+d)\frac{ju}{c}+\{\frac{jd}{c}\}u}}{u(1-e^{-czu-du})(e^u-1)}$$
$$=2\pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + 2\pi i\sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right). \quad (2.31)$$

This and (2.27) leads to the following

$$H(Vz,0) = H(z,0) + L(z,0)$$
$$=H(z,0) + 2\pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + 2\pi i\sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right). \quad (2.32)$$

Since $H(z,0) = 2A(z,0) = 2(\frac{\pi i z}{12} - \log(\eta(z)))$, one can see that

$$\frac{2\pi i Vz}{12} - 2\log(\eta(Vz)) - \frac{2\pi i z}{12} + 2\log(\eta(z))$$
$$= 2\pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + 2\pi i\sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right). \quad (2.33)$$

Hence,

$$\frac{\pi i}{6}(Vz-z) + 2\log(\frac{\eta(z)}{\eta(Vz)})$$
$$= 2\pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + 2\pi i\sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right). \quad (2.34)$$

Let $s(d,c) = \sum_{j=1}^{c-1}\left(\frac{j}{c}-\frac{1}{2}\right)\left(\frac{jd}{c}-[\frac{jd}{c}]-\frac{1}{2}\right)$. So

$$\eta(z) = \eta(Vz)\exp\left(-\frac{\pi i(Vz-z)}{12} + \pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + \pi i s(d,c)\right) \quad (2.35)$$

Thus

$$P(e^{Vz}) = P(e^z)\exp\left(\pi i\left(\frac{-1}{12c(cz+d)} - \frac{(cz+d)}{12c} - \frac{1}{4}\right) + \pi i s(d,c)\right) \quad (2.36)$$

Now we try to change the parameters. Let $a = -h'$, $b = \frac{hh'-1}{k}$, $c = k$, and $d = -h$. Suppose that $hh' \equiv -1(\pmod{k})$. Also, assume that

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1). \tag{2.37}$$

Now it is easy to see that

$$V\left(\frac{h+iz}{k}\right) = \frac{-h'(\frac{h+iz}{k}) + \frac{-1+hh'}{k}}{k(\frac{h+iz}{k}) - h} = \frac{-ih'z - 1}{kiz} = \frac{-h'z + i}{kz}. \tag{2.38}$$

So according to (2.36), we have

$$P(e^{2\pi i\frac{h+iz}{k}}) = P(e^{2\pi i\frac{-h'+iz^{-1}}{k}})e^{\pi is(h,k)}e^{\frac{-\pi}{12k}(z^{-1}-z)}e^{\frac{-\pi i}{4}}. \tag{2.39}$$

So if $\omega(h,k) = e^{\pi is(h,k)}$, then

$$P(e^{2\pi i\frac{h+iz}{k}}) = P(e^{2\pi i\frac{-h'+iz^{-1}}{k}})\omega(h,k)e^{\frac{\pi}{12k}(z^{-1}-z)}\sqrt{z}. \tag{2.40}$$

2.3. Farey Dissection

As we discussed earlier, we have $F(q) = \sum_{n=0}^{\infty} p(n)q^n$. Cauchy integration formula immediately implies that $p(n) = \frac{1}{2\pi i}\int_C \frac{F(s)}{s^{n+1}}ds$, where C is a contour inside the unit circle. We planned to avoid the essential singularities of F. As we will see, choosing a disk as large as possible inside the unit disk is very helpful to compute the integral. We know that F has essential singularities on rational points $e^{2\pi i\frac{h}{k}}$. According to (2.40),

$$F(e^{2\pi i\frac{h}{k}-2\pi\frac{z}{k}}) = F(e^{2\pi\frac{-ih'-z^{-1}}{k}})\omega(h,k)e^{\frac{\pi}{12k}(z^{-1}-z)}\sqrt{z}. \tag{2.41}$$

one can see for small enough z,

$$e^{\frac{-2\pi ih'}{k}}e^{\frac{-2\pi z^{-1}}{k}} \sim e^{\frac{-2\pi z^{-1}}{k}} \longrightarrow 0. \tag{2.42}$$

So for small z,

$$F(e^{2\pi i\frac{h}{k}-2\pi\frac{z}{k}}) \sim \omega(h,k)e^{\frac{\pi}{12k}(z^{-1}-z)}\sqrt{z}. \tag{2.43}$$

We choose this to avoid the possible essential singularities over the unit circle. Suppose that $s = re^{2\pi it}$ and $0 \le t \le 2\pi$. So if for the rational number $\frac{h}{k}$, the cut of Farey dissection of order N is denoted by $[\theta'_{h,k}, \theta''_{h,k}]$,

$$p(n) = \int_0^1 \frac{F(re^{2\pi it})}{r^n e^{2\pi int}}dt = \frac{1}{r^n}\int_0^1 \frac{F(re^{2\pi it})}{e^{2\pi int}}dt = \frac{1}{r^n}\sum_{\substack{k=1\\(h,k)=1\\0\le h<k}}^{N}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} \frac{F(re^{\frac{2\pi ih}{k}+2\pi it})}{e^{\frac{2\pi inh}{k}+2\pi int}}dt \tag{2.44}$$

Now we choose a proper amount for r to simplify our computations. We know that r should be close enough to 1. We pick $r = e^{-\frac{2\pi}{N^2}}$; so as when N becomes large enough, r becomes close enough to 1. Thus

$$p(n) = \frac{1}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \frac{F(e^{\frac{2\pi ih}{k}+2\pi it-\frac{2\pi}{N^2}})}{e^{\frac{2\pi inh}{k}+2\pi int}} dt \tag{2.45}$$

If we define $z = k(N^{-2} - it)$, then

$$p(n) = \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{F(e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}})}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz. \tag{2.46}$$

Using (2.40), one can see

$$p(n) = \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{F(e^{\frac{2\pi i(h'+iz^{-1})}{k}})z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{k}}}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz \tag{2.47}$$

Since for small enough z, the term $\frac{2\pi i(h'+iz^{-1})}{k}$ tends to zero exponentially, $F(e^{\frac{2\pi i(h'+iz^{-1})}{k}})$ tends to 1 rapidly. So we can consider $F(z) = 1 + F(z) - 1$ so as to have a couple of negligible parts. So

$$\begin{aligned} p(n) = I_1 + I_2 = & \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{k}}}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz \\ & + \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{(F(e^{\frac{2\pi i(h'+iz^{-1})}{k}})-1)z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{k}}}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz. \end{aligned} \tag{2.48}$$

Now we need to find I_1, I_2.

2.4. Integral estimates

So far we have seen that finding the asymptotic formula for $p(n)$ reduced to finding I_1 and I_2. We will prove that the dominant part is I_1 and the integral I_2 is negligible. It was predicted, since $F(z) - 1$ tends rapidly to zero for small enough z.

2.4.1. Estimation of I_2

We prove that I_2 is negligible in this part. First, one can see that

$$|e^{\pi\frac{-z}{12k}}| = |e^{\frac{\pi}{12}(-(N^{-2}-it)}| = e^{\frac{-\pi}{12N^2}} \tag{2.49}$$

Also, since $p(0) = 1$,

$$\begin{aligned} |e^{\pi\frac{z^{-1}}{12k}}(F(e^{\frac{2\pi i(h'+iz^{-1})}{k}}) - 1)| &= |\sum_{m=1}^{\infty} p(m)e^{m\frac{2\pi i(h'+iz^{-1})}{k}+\pi\frac{z^{-1}}{12k}}| \\ &\leq \sum_{m=1}^{\infty} p(m)|e^{m\frac{2\pi i(h'+iz^{-1})}{k}+\pi\frac{-z^{-1}}{12k}}| \\ &= \sum_{m=1}^{\infty} p(m)e^{m\frac{-2\pi Re(z^{-1})}{k}+\pi\frac{Re(-z^{-1})}{12k}} \\ &= \sum_{m=1}^{\infty} p(m)e^{-2\pi\frac{Re(z^{-1})(m-\frac{1}{24})}{k}} \end{aligned} \tag{2.50}$$

So

$$|z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{12k}}\left(F(e^{2\pi i\frac{h'+iz^{-1}}{k}}) - 1\right)| \leq |z|^{\frac{1}{2}}e^{-\frac{\pi}{12N^2}}\sum_{m=1}^{\infty} p(m)e^{-2\pi\frac{Re(z^{-1})(m-\frac{1}{24})}{k}} \tag{2.51}$$

So we need to compute $Re(z^{-1})$. One can see that

$$Re(z^{-1}) = \frac{N^{-2}}{k^2(N^{-4}+t^2)}. \tag{2.52}$$

It is a well-known fact in Farey dissection that $\frac{1}{2kN} \leq \theta_{h,k} < \frac{1}{kN}$. Moreover, $-\theta_{h,k} \leq t \leq \theta''_{h,k}$; so $-\frac{1}{kN} < t < \frac{1}{kN}$

$$Re(z^{-1}) = \frac{N^{-2}}{k^2(N^{-4}+t^2)} > \frac{N^{-2}}{k^2N^{-4}+(kt)^2} > \frac{N^{-2}}{k^2N^{-4}+N^{-2}} = \frac{1}{k^2N^{-2}+1} \geq \frac{k}{2}. \tag{2.53}$$

The last inequality follows from the fact that $k < N^2$. Also,

$$|z|^{\frac{1}{2}} = (k^2(N^{-4}+t^2))^{\frac{1}{2}} < (k^2N^{-4}+N^{-2})^{\frac{1}{2}} \leq \sqrt{2}N^{-1}. \tag{2.54}$$

This and (2.53) lead to

$$|(F(e^{\frac{2\pi i(h'+iz^{-1})}{k}}) - 1)z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{k}}| \leq \sqrt{2}N^{-1}e^{-\frac{\pi}{12N^2}}\sum_{m=1}^{\infty} p(m)e^{-k\pi\frac{(m-\frac{1}{24})}{k}} \tag{2.55}$$

This and (2.50) leads to

$$\begin{aligned} |I_2| &= \left| \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{(F(e^{\frac{2\pi i(h'+iz^{-1})}{k}}) - 1)z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{k}}}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz \right| \\ &\leq \frac{1}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \left|\frac{1}{e^{\frac{2\pi inh}{k}}}\right| \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{\sqrt{2}N^{-1}e^{-\frac{\pi}{12N^2}}\sum_{m=1}^{\infty} p(m)e^{-k\pi\frac{(m-\frac{1}{24})}{k}}}{|e^{2\pi in(i\frac{z}{k}-iN^{-2})}|} dz \end{aligned} \tag{2.56}$$

We need to discuss about $|e^{2\pi in(i\frac{z}{k}-iN^{-2})}| = |e^{-2\pi n(\frac{z}{k}+N^{-2})}|$. One can see

$$|e^{-2\pi n(\frac{z}{k}+N^{-2})}| = e^{\frac{-2\pi n}{N^2}} e^{\frac{-2\pi n Re(z)}{k}} = e^{\frac{-4\pi n}{N^2}} \tag{2.57}$$

So

$$\begin{aligned}
|I_2| &\leq \frac{1}{e^{\frac{-2\pi n}{N^2}}} \sum_{\substack{k=1\\(h,k)=1\\0\leq h<k}}^{N} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \sqrt{2}N^{-1}e^{-\frac{\pi}{12N^2}} \sum_{m=1}^{\infty} p(m) e^{-k\pi\frac{(m-\frac{1}{24})}{k}} dz \\
&= e^{\frac{2\pi n}{N^2}-\frac{\pi}{12N^2}} \sqrt{2}N^{-1} \sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})} \sum_{\substack{k=1\\(h,k)=1\\0\leq h<k}}^{N} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} dz \\
&= e^{\frac{2\pi n}{N^2}-\frac{\pi}{12N^2}} \sqrt{2}N^{-1} \sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})} \sum_{\substack{k=1\\(h,k)=1\\0\leq h<k}}^{N} (k(N^{-2}-i\theta''_{h,k}) - ki\theta'_{h,k} + kN^{-2}))
\end{aligned} \tag{2.58}$$

We know $k(\theta''_{h,k}-\theta'_{h,k}) \leq \frac{1}{2N}$. So

$$|I_2| \leq e^{\frac{2\pi n}{N^2}-\frac{\pi}{12N^2}} \sqrt{2}N^{-1} \sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})} \sum_{\substack{k=1\\(h,k)=1\\0\leq h<k}}^{N} \frac{1}{2N} \tag{2.59}$$

According to [], $p(m) < 2^m$. So $\sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})}$ converges. This implies that there exists $0 < C < \infty$ such that

$$I_2 \leq CN^{-1}e^{\frac{2\pi n}{N^2}}. \tag{2.60}$$

So for a fixed n, I_2 tends to zero as N tends to infinity. This demonstrates that we can control amount of I_2 for large enough N. Thus I_2 will be assumed as a negligible term.

2.4.2. Estimation of I_1

Now, we compute I_1. According to (2.48), we know that

$$I_1 = \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1\\(h,k)=1\\0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{k(i\theta'_{h,k}-N^{-2})}^{k(N^{-2}-i\theta''_{h,k})} \frac{z^{\frac{1}{2}}e^{\pi\frac{z^{-1}-z}{12k}}}{e^{2\pi in(i\frac{z}{k}-iN^{-2})}} dz \tag{2.61}$$

Now assume that $\omega = N^{-2} - it$. So $\omega = \frac{z}{k}$. This implies that

$$
\begin{aligned}
I_1 =& \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{e^{\frac{2\pi i n h}{k}}} \int_{i\theta'_{h,k}-N^{-2}}^{N^{-2}-i\theta''_{h,k}} \frac{(k\omega)^{\frac{1}{2}} e^{\frac{\pi((k\omega)^{-1}-k\omega)}{12k}}}{k e^{2\pi n(-\omega+N^{-2})}} d\omega \\
=& \frac{-i}{e^{\frac{4\pi n}{N^2}}} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{\sqrt{k} e^{\frac{2\pi i n h}{k}}} \int_{i\theta'_{h,k}-N^{-2}}^{N^{-2}-i\theta''_{h,k}} (\omega)^{\frac{1}{2}} e^{\frac{\pi}{12k^2\omega}} e^{2\pi(n-\frac{1}{24})\omega} d\omega
\end{aligned} \tag{2.62}
$$

Let $g(\omega) = (\omega)^{\frac{1}{2}} e^{\frac{\pi}{12k^2\omega}} e^{2\pi(n-\frac{1}{24})\omega}$. In order to find the last integral, we use the residue

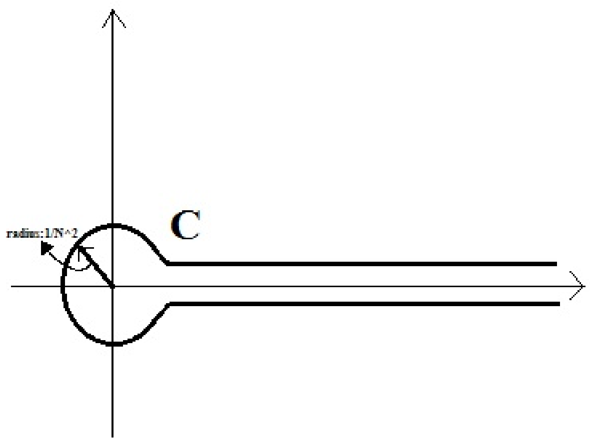

Figure 2.2: The contour of Cauchy integral

theorem for a proper contour. Let C be a contour as in the figure 2.2. For small enough ϵ, one can see that

$$
\begin{aligned}
2\pi i Res(g) =& \int_C g(\omega) d\omega \\
=& \int_{-\infty}^{-\epsilon} g(\omega) d\omega + \int_{-\epsilon}^{-\epsilon-i\theta''_{h,k}} g(\omega) d\omega + \int_{-\epsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} g(\omega) d\omega \\
&+ \int_{N^{-2}-i\theta''_{h,k}}^{N^{-2}+i\theta'_{h,k}} g(\omega) d\omega + \int_{N^{-2}+i\theta'_{h,k}}^{-\epsilon+i\theta'_{h,k}} g(\omega) d\omega + \int_{-\epsilon+i\theta'_{h,k}}^{-\epsilon} g(\omega) d\omega + \int_{-\epsilon}^{-\infty} g(\omega) d\omega
\end{aligned} \tag{2.63}
$$

We consider the main branch for the square root function. So

$$
\begin{aligned}
\int_{N^{-2}-i\theta''_{h,k}}^{N^{-2}+i\theta'_{h,k}} g(\omega) d\omega =& 2\pi i Res(g) - \int_{-\infty}^{-\epsilon} g(\omega) d\omega - \int_{-\epsilon}^{-\epsilon-i\theta''_{h,k}} g(\omega) d\omega \\
&- \int_{-\epsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} g(\omega) d\omega - \int_{N^{-2}+i\theta'_{h,k}}^{-\epsilon+i\theta'_{h,k}} g(\omega) d\omega - \int_{-\epsilon+i\theta'_{h,k}}^{-\epsilon} g(\omega) d\omega - \int_{-\epsilon}^{-\infty} g(\omega) d\omega.
\end{aligned} \tag{2.64}
$$

So

$$\frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1\\(h,k)=1\\0\le h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \int_{N^{-2}-i\theta''_{h,k}}^{N^{-2}+i\theta'_{h,k}} g(\omega)d\omega$$

$$= \frac{-i}{e^{\frac{2\pi n}{N^2}}} \sum_{\substack{k=1\\(h,k)=1\\0\le h<k}}^{N} \frac{1}{e^{\frac{2\pi inh}{k}}} \left(L_{k,h} - J_1 - J_2 - J_3 - J_4 - J_5 - J_6\right). \tag{2.65}$$

Now we bound J_2, J_3, J_4, J_5. We assume that $\epsilon < N^{-2}$ as it is clear in the figure 2.2. In particular, we are only interested in the case that $\epsilon \longrightarrow 0$. First we discuss I_2. Let $\omega = \epsilon + iv$. We know that $|\omega|^2 = \epsilon^2 + v^2$ on this line. Also

$$|e^{\frac{\pi}{12k^2\omega}} e^{2\pi(n-\frac{1}{24})\omega}| = e^{Re(\frac{\pi}{12k^2\omega})} e^{2\pi(n-\frac{1}{24})Re(\omega)} = e^{\frac{-\pi\epsilon}{12k^2(\epsilon^2+v^2)}} e^{2\pi(n-\frac{1}{24})\epsilon}. \tag{2.66}$$

Thus one can see

$$|J_2| \le \left| \int_{-\epsilon}^{-\epsilon-i\theta''_{h,k}} g(\omega)d\omega \right| \le \int_{0}^{-\theta''_{h,k}} (\epsilon^2+v^2)^{\frac{1}{2}} e^{\frac{-\pi\epsilon}{12k^2(\epsilon^2+v^2)}} e^{2\pi(n-\frac{1}{24})\epsilon} dv. \tag{2.67}$$

Since $e^{\frac{-\pi\epsilon}{12k^2(\epsilon^2+v^2)}} < 1$ and $e^{2\pi(n-\frac{1}{24})\epsilon} > 1$, we have

$$|J_2| < (\epsilon^2 + {\theta''_{h,k}}^2)^{\frac{1}{2}} \theta''_{h,k} < (\epsilon^2 + \frac{1}{k^2N^2})^{\frac{1}{2}} \frac{1}{kN}. \tag{2.68}$$

One can show that as $\epsilon \longrightarrow 0$, then $(\epsilon^2 + \frac{1}{k^2N^2})^{\frac{1}{2}} \frac{1}{kN} \longrightarrow k^{-\frac{3}{2}} N^{-\frac{3}{2}}$. In a similar way, we have

$$|J_5| = \left| \int_{-\epsilon+i\theta'_{h,k}}^{-\epsilon} g(\omega)d\omega \right| \le \int_{\theta'_{h,k}}^{0} (\epsilon^2+v^2)^{\frac{1}{2}} e^{\frac{-\pi\epsilon}{12k^2(\epsilon^2+v^2)}} e^{2\pi(n-\frac{1}{24})\epsilon} |dv|$$

$$< \int_{\theta'_{h,k}}^{0} (\epsilon^2+v^2)^{\frac{1}{2}} |dv| = (\epsilon^2 + {\theta'_{h,k}}^2)^{\frac{1}{2}} \theta'_{h,k} < (\epsilon^2 + \frac{1}{k^2N^2})^{\frac{1}{2}} \frac{1}{kN}. \tag{2.69}$$

So as $\epsilon \longrightarrow 0$, $|J_5| < (kN)^{-\frac{3}{2}}$.

For J_3, let $\omega = u - i\theta''_{h,k}$. Then

$$|J_3| = \left| \int_{-\epsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} g(\omega)d\omega \right| \le \int_{-\epsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} (u^2 + {\theta''_{h,k}}^2)^{\frac{1}{2}} e^{\frac{\pi}{12k^2} Re(\frac{1}{u-i\theta''_{h,k}})} e^{2\pi(n-\frac{1}{24})Re(u-i\theta''_{h,k})} du. \tag{2.70}$$

One can see that

$$\frac{1}{k^2} Re(\frac{1}{u-i\theta''_{h,k}}) = \frac{u}{k^2(u^2+{\theta''_{h,k}}^2)} \le \frac{N^{-2}}{k^2{\theta''_{h,k}}^2}. \tag{2.71}$$

By the definition of θ'', one can see that $\frac{N^{-2}}{k^2\theta''_{h,k}{}^2} \leq 4$. So

$$|J_3| < (N^{-4} + \theta''_{h,k}{}^2)e^{\frac{\pi}{3}}e^{\frac{2\pi n}{N^2}}(\epsilon + N^{-2}). \tag{2.72}$$

Since $\theta''_{h,k} < \frac{1}{kN}$, one can see

$$|J_3| < (N^{-4} + (kN)^{-2})^{\frac{1}{4}}e^{\frac{\pi}{3}+\frac{2\pi n}{N^2}}(\epsilon + N^{-2}) \tag{2.73}$$

We know $k < N$; thus

$$|J_3| < (\epsilon + N^{-2})2^{\frac{1}{4}}k^{-\frac{1}{2}}N^{-\frac{1}{2}}e^{\frac{\pi}{3}+\frac{2\pi n}{N^2}} \tag{2.74}$$

If $\epsilon \longrightarrow 0$, then $J_3 \leq 2^{\frac{1}{4}}k^{-\frac{1}{2}}N^{-\frac{5}{2}}e^{\frac{\pi}{3}+\frac{2\pi n}{N^2}}$. For I_4, $\omega = u + i\theta'_{h,k}$ for $-N^{-2} \leq u \leq -\epsilon$, we have

$$\begin{aligned}|J_4| &= \left|\int_{-N^{-2}+i\theta'_{h,k}}^{-\epsilon+i\theta'_{h,k}} g(\omega)d\omega\right| \leq \int_{-N^{-2}}^{-\epsilon}(u^2 + \theta''_{h,k}{}^2)^{\frac{1}{2}}e^{\frac{\pi}{12k^2}Re(\frac{1}{u-i\theta''_{h,k}})}e^{2\pi(n-\frac{1}{24})Re(u-i\theta''_{h,k})}du \\ &= \int_{-N^{-2}+i\theta'_{h,k}}^{-\epsilon+i\theta'_{h,k}}(u^2 + \theta''_{h,k}{}^2)^{\frac{1}{2}}e^{\frac{\pi u}{12k^2(u^2+\theta''_{h,k}{}^2)}}e^{2\pi(n-\frac{1}{24})u}du \end{aligned} \tag{2.75}$$

With similar reasoning as J_3, one can see $|J_4| \leq 2^{\frac{1}{4}}k^{-\frac{1}{2}}N^{-\frac{5}{2}}e^{\frac{\pi}{3}+\frac{2\pi n}{N^2}}$.

Now, we discuss J_1 and J_6. We have

$$\begin{aligned}J_1 + J_6 &= \int_{-\infty}^{-\epsilon}\sqrt{|u|}e^{-\frac{\pi i}{2}}e^{\frac{\pi}{12k^2u}+2\pi(n-\frac{1}{24})u}du + \int_{-\epsilon}^{-\infty}\sqrt{|u|}e^{\frac{\pi i}{2}}e^{\frac{\pi}{12k^2u}+2\pi(n-\frac{1}{24})u}du \\ &= -2i\int_{\epsilon}^{\infty}\sqrt{t}e^{\frac{\pi i}{2}}e^{\frac{\pi}{12k^2t}-2\pi(n-\frac{1}{24})t}dt.\end{aligned} \tag{2.76}$$

Let us denote $H_k = \int_{\epsilon}^{\infty}\sqrt{t}e^{\frac{\pi i}{2}}e^{\frac{\pi}{12k^2t}-2\pi(n-\frac{1}{24})t}dt$. In the next step, we prove that

$$-i\sqrt{k}L_{k,h} + 2\sqrt{k}H_k = \frac{\sqrt{k}}{\pi\sqrt{2}}\left(\frac{d}{dx}\frac{sinh(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})})}{\sqrt{x-\frac{1}{24}}}\right)|_{x=n}. \tag{2.77}$$

We start with $L_{k,h}$. One can see

$$-iL_{k,h} = -i\int_C \omega^{\frac{1}{2}}e^{\frac{\pi}{12k^2\omega}+2\pi(n-\frac{1}{24})\omega}d\omega \tag{2.78}$$

since $e^z = \sum_{s=0}^{\infty}\frac{z^s}{s!}$. Taking $z = \frac{\pi}{12k^2\omega}$, one can see

$$-iL_{k,h} = -i\int_C \omega^{\frac{1}{2}}e^{2\pi(n-\frac{1}{24})\omega}\sum_{s=0}^{\infty}\frac{\left(\frac{\pi}{12k^2\omega}\right)^s}{s!}d\omega = -i\sum_{s=0}^{\infty}\frac{\left(\frac{\pi}{12k^2\omega}\right)^s}{s!}\int_C \omega^{\frac{1}{2}-s}e^{2\pi(n-\frac{1}{24})\omega}d\omega \tag{2.79}$$

Let $z = 2\pi(n - \frac{1}{24})\omega$. Then

$$-iL_{h,k} = 2\pi \sum_{s=0}^{\infty} \frac{(\frac{\pi}{12k^2})^s}{s!} \left(2\pi(n - \frac{1}{24})\right)^{s-\frac{3}{2}} \frac{1}{2\pi i} \int_{C'} e^z z^{-s+\frac{1}{2}}. \tag{2.80}$$

where C' is the corresponding path for z according to the definition of C. Now, Hankel's loop integral formula (see [6]) tells us that

$$\frac{1}{\Gamma(s-\frac{1}{2})} = \frac{1}{2\pi i} \int_C e^z z^{-s+\frac{1}{2}} dz \tag{2.81}$$

for the mentioned contour C. So

$$-iL_{k,h} = \sqrt{\frac{1}{2\pi(n-\frac{1}{24})^3}} \sum_{s=0}^{\infty} \frac{(\frac{\pi^2(n-\frac{1}{24})}{6k^2})^s}{s!\Gamma(s-\frac{1}{2})}. \tag{2.82}$$

Also, we know from the definition of Γ that

$$\Gamma(s - \frac{1}{2}) = 2^{-s+1}\Gamma(\frac{1}{2})(2s-3)(2s-5)\cdots 3 \cdot 1. \tag{2.83}$$

We know that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. So for each Y, we have

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{(\frac{Y^2}{4})^s}{s!\Gamma(s-\frac{1}{2})} =& \sum_{s=0}^{\infty} \frac{(\frac{Y^2}{4})^s}{s!2^{-s+1}\pi^{\frac{1}{2}} \prod_{n=1}^{s-1}(2n-1)} = \sum_{s=0}^{\infty} \frac{Y^{2s}}{4s\sqrt{\pi}} \prod_{n=1}^{s-1} \frac{1}{2n(2n-1)} \\ =& \frac{Y^2}{2\sqrt{\pi}} \sum_{s=1}^{\infty} \frac{(2s-1)Y^{2s-2}}{(2s)!} = \frac{1}{2\sqrt{\pi}} \left(-1 + Y^2 \frac{d}{dY} \sum_{s=1}^{\infty} \frac{Y^{2s-1}}{(2s)!}\right). \end{aligned} \tag{2.84}$$

We know that $\cosh(Y) = \sum_{s=0}^{\infty} \frac{Y^{2s}}{(2s)!}$. So

$$\sum_{s=0}^{\infty} \frac{(\frac{Y^2}{4})^s}{s!\Gamma(s-\frac{1}{2})} = \frac{1}{2\sqrt{\pi}} \left(-1 + Y^2 \frac{d}{dY} \frac{\cosh(Y) - 1}{Y}\right). \tag{2.85}$$

Considering (2.82) and assuming $Y = \frac{\pi^2(n-\frac{1}{24})}{6k^2}$, we have

$$-iL_{k,h} = \sqrt{\frac{1}{2\pi(n-\frac{1}{24})^3}} \left(-1 + (\frac{\pi(n-\frac{1}{24})^{\frac{1}{2}}}{6k})^2 \frac{d}{dY} \frac{\cosh(\frac{\pi(n-\frac{1}{24})^{\frac{1}{2}}}{6k}) - 1}{\frac{\pi(n-\frac{1}{24})^{\frac{1}{2}}}{6k}}\right). \tag{2.86}$$

By applying the chain rule, one can see that

$$-iL_{h,k} = \frac{1}{k\sqrt{3}} \left(\frac{d}{dx} \frac{cosh(\frac{\pi}{k}\sqrt{2/3(x-1/24)})}{\frac{\pi}{k}\sqrt{2/3(x-1/24)}}\right) \tag{2.87}$$

Finally, for H_k, we have for real a, c

$$\int_0^{\infty} e^{-c^2t-a^2\frac{1}{t}}\sqrt{t}dt = 2\int_0^{\infty} e^{-c^2u^2-a^2\frac{1}{u^2}} du = \frac{\sqrt{\pi}}{c} e^{-2ac} = -\frac{\sqrt{\pi}}{2c}\frac{d}{dc}\left(\frac{e^{-2ac}}{c}\right). \tag{2.88}$$

If we consider $a = 2\pi(n - \frac{1}{24})$ and $c = \frac{\pi}{12k^2}$, we will see that

$$H_k = \frac{1}{2\pi\sqrt{2}} \left(\frac{d}{dx} \frac{e^{-\pi\sqrt{2/3}\sqrt{x-1/24}/k}}{\sqrt{x - 1/24}} \right) |_{x=n} \tag{2.89}$$

Now, the equations (2.89), (2.87), (2.77), (2.65), and (2.76) and considering the fact that J_2, J_3, J_4, J_5 are negligible for large N leads to

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh(\frac{\pi}{k}\sqrt{\frac{2}{3}(x - \frac{1}{24})})}{\sqrt{x - \frac{1}{24}}} \right]_{x=n} . \tag{2.90}$$

CHAPTER 3

An asymptotic formula for the number of odd partitions

3.1. Historical background

After what Ramanujan, Rademacher, and Hardy did for the asymptotic formula, a lot of mathematicions tried to find asymptotic formulas for similar functions. These cases, which are almost all of the partitions with parts of the form $at + b$, needs different categories of modular forms. In another words, if we are interested in number of partitions with parts of the form $at+b$, we have to define a proper subgroup of $\Gamma(n)$. So what we need is to prove a modular transformation as well as approximating the integrals. We mention some of the important results here so if the reader wants know more, he can see these works.

First, Hao in [14] proposed an asymptotic formula for the number of odd partitions. He used the same idea which Rademacher suggested for the general case. The main difference are two things. First the modular form is not defined over $\Gamma(1)$. Also, the summation over the Farey dissection leads to a Kloosterman's sum, while it was a simple geometric sum for $p(n)$. After that, Haberzetle in [12], proposed another asymptotic formula for the number of partitions into parts which are not divisible by either p or q; where p, q are two primes such that $24|(p-1)(q-1)$. In order to do this, she wisely chose a proper way to describe generating function for such partitions; which was based on the generating function which proposed by Rademacher. Then she suggests four different modular transformations for the generating function; depending on the $\gcd(pq, k)$, where k is the denominator of Farey dissection. Then she could approximate the integrals using geometric series for different cases of $\gcd(pq, k)$.

The next work is by Lehner in [20], which is an asymptotic formula for the number of partitions into parts that are not divisible by 5. The main issues to solve this problem goes back to the Kloosterman's sum which is not complete. He addresed this problem using Bezout's theorem cleverly and computing a term in two different ways. For the modularity part, he also assumed different cases and find the transformation by the properties of zeta-function and gamma-function. Finally, he approximated the integrals. As an expected natural step, next is the work by Livingood in [22], who proposed an asymptotic formula for the number of partitions into parts which are in the form $pt + a$ for prime p and $0 \leq a \leq p$. What he did is a generalization of the method which was applied by Lehner. He first generalized the Kloosterman's sum for an arbitrary prime number. He also proved the modular transformation. The following steps are almost the same as Lehner's.

After the formula for the partitions of the form $pt + a$, it took more than 10 years to find the asymptotic formula for the number of partitions into parts of the form $Mt + a$. Isako in [17], [15] proposed an approximate for an incomplete Kloosterman's sum in general and an asymptotic formula for the number of partitions into parts of the form $Mt + a$. The method of the Kloosterman's sum needs to change the order of two sums, which he justified. The idea for the asymptotic formula is almost the same as the preceding ones. The same author in [16], proved an asymptotic formula for the partitions into parts which are coprime with a number M. The idea was based on the fact that if $\gcd(a, M) = 1$, then $\gcd(M - a, M) = 1$. The rest is contributed by taking a sum over all such $a < \frac{M}{2}$, and some analytic methods.

As we discussed in the last chapter, Berndt in [3], [4] generalized the results concerning the modular transformation for a bigger category. He also found similar results for a more general class of Eisenstein series. Bringmann and Ono in [7], proposed an exact formula for harmonic Maass forms which are a generalized category of modular forms with negetive weight. They also have found a result for these forms of half weight. This work can be considered as an alternative solution for the generalized problem which Hardy-Ramanujan proposed. Recently, Laughlin in [24] proved an asymptotic formula for the number of partitions into parts which are coprime with both numbers r, s simultanously. It is worth to mention that the main idea is almost the same as in [12] with a very advanced procedures.

In this section, we will discuss the steps of the proof by Hao for the number of odd partitions. In fact the goal is to prove the following exact formula for the partions into odd parts $p_o(n)$ (Note that $p_o(n)$ is also the number of partitions into distinct parts (see for example [5])).

$$p_0(n) = \frac{1}{2} \sum_{k=1,odd}^{\infty} \sum_{\substack{(h,k)=1 \\ 0<h\leq k}} \omega_{h,k} e^{\frac{-2\pi i h n}{k}} \frac{d}{dn} J_0 \left(\frac{i\pi}{k} \sqrt{\frac{2}{3}(n + \frac{1}{24})} \right). \tag{3.1}$$

where J_0 is the Bessel function of the 0th order and $\omega_{h,k}$ is a certain root of unity.

3.2. Farey dissection

The first steps of the proof is almost the same as Rademacher's method. Let $p_o(n)$ be the number of odd partitions. So one can see that

$$f(q) = 1 + \sum_{n=1}^{\infty} p_0(n) q^n = \frac{1}{(q;q^2)_\infty}. \tag{3.2}$$

As $f(q)$ is the generating function, we can use Cauchy integral formula to get

$$p_0(n) = \frac{1}{2\pi i} \int_C \frac{f(x)}{x^{n+1}} dx. \tag{3.3}$$

We again choose a path very close to unit circle; the circle with the center at origin and radius $e^{-2\pi i N}$. This path is inside the unit circle. Again, we have essential singularities on the unit circle at $e^{2\pi i \frac{h}{2k+1}}$. So we need to choose proper Farey dissection to avoid these essentil singularities, when $N \longrightarrow \infty$. Let us use the Farey dissection of order n. Then if we have $\xi(h,k)$ as an arc, then $\xi(h,k)$ is as follows

$$-\vartheta_1(h,k) = \frac{h+h_1}{k+k_1} - \frac{h}{k} \leq \vartheta \leq \frac{h+h_2}{k+k_2} - \frac{h}{k} = \vartheta_2(h,k). \tag{3.4}$$

for three consequtive points of dissection. As we discussed in the previous chapters, we have the following inequalities.

$$-\frac{1}{k(N+k)} \leq \vartheta_1(h,k), \vartheta_2(h,k) \leq \frac{1}{k(N+1)} \tag{3.5}$$

So

$$q(n) = \frac{1}{2\pi i} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\leq k\leq N}} \int_{-\vartheta_1(h,k)}^{\vartheta_2(h,k)} \frac{f(x)}{x^{n+1}} dx. \tag{3.6}$$

We know that $2 \nmid h$; so it is important whether $2 \mid k$. This makes two possible cases: $2 \mid k$ or $2 \nmid k$. Lets call the integral for the first case as I_1 and for the second case as I_2.

In the next step, we need to use the modular transformation to substitute with f in the integrand. It is mainly because of the fact that if the variable of integrand is in the numenator, then we will have an ordinary integral in real variables, when $N \longrightarrow \infty$. That is our motivation for the next section.

3.3. The elliptic modular transformation formula

In this section, we are seeking a modular transformation. The motivation is almost the same as the general case; and fortunately the modularity property is almost saved in the odd partitions. In fact, the generating function $f(z)$ is a modular function. First, we assume that $2 \mid k$. We start from a changing of parameter in the integral in 3.6. One can see in similar way of the previous chapter that

$$x = e^{-\frac{2\pi z}{k}+\frac{2\pi ih}{k}} \quad , \quad x' = e^{-\frac{2\pi}{kz}-\frac{2\pi ih'}{k}} \tag{3.7}$$

where $hh' \equiv 1(modk)$. So $x = M.x'$; where $M \in \Gamma(2)$. Let $\omega_{h,k}$ is defined as follows.

$$\omega(h,k) = \epsilon_{h,k} e^{-\frac{\pi i}{12k}(h+h')}, \tag{3.8}$$

where

$$\epsilon_{h,k} = e^{-\pi i\left(\frac{(h'^2-1)}{8}\left(\frac{1-hh'}{k}-1\right)+\frac{h'(1-hh')}{8k}+\frac{1}{24}\left(\frac{1-hh'}{k}+k\right)(hh'^2-h'-h)\right)}. \tag{3.9}$$

Then we claim that $x^{\frac{1}{24}}f(x)$ is a modular form of weight zero. In fact, the following modular transform holds.

$$f(x) = \omega_{h,k} e^{-\frac{\pi}{12kz}+\frac{\pi z}{12k}} f(x'). \tag{3.10}$$

So if we define $g(x) = x^{\frac{1}{24}}f(x)$, then

$$\begin{aligned} g(e^{-\frac{2\pi z}{k}+\frac{2\pi ih}{k}}) = \omega_{h,k} e^{-\frac{\pi}{12kz}+\frac{\pi z}{12k}} g(e^{-\frac{2\pi}{kz}-\frac{2\pi ih'}{k}}) e^{\frac{2\pi}{24kz}+\frac{2\pi ih'}{24k}-\frac{2\pi z}{24k}+\frac{2\pi ih}{24k}} \\ = \omega_{h,k} g(e^{-\frac{2\pi}{kz}-\frac{2\pi ih'}{k}}) e^{\frac{2\pi ih'}{24k}+\frac{2\pi ih}{24k}}. \end{aligned} \tag{3.11}$$

So

$$g(M.x) = \omega_{h,k} e^{\frac{2\pi ih'}{24k}+\frac{2\pi ih}{24k}} g(x) \tag{3.12}$$

where

$$M = \begin{bmatrix} h' & \frac{1-hh'}{k} \\ -k & h \end{bmatrix} \tag{3.13}$$

This implies that g is a modular form of wieght zero with multiplier system $\omega_{h,k} e^{\frac{2\pi ih'}{24k}+\frac{2\pi ih}{24k}}$. Now we prove the equation (3.12). Let us define

$$x = q^2 = e^{2\pi it} \quad and \quad x' = Q^2 = e^{2\pi iT} \tag{3.14}$$

where

$$t = \frac{h+iz}{k} \quad and \quad T = \frac{-h' + \frac{i}{z}}{k}. \tag{3.15}$$

So

$$M.t = \frac{\frac{1-hh'}{k} + h'\frac{h+iz}{k}}{h - k\frac{h+iz}{k}} = \frac{\frac{1+ih'z}{k}}{iz} = \frac{h' - i/z}{k} = T. \tag{3.16}$$

Now we have to prove that $g(e^{2\pi it})$ has the modularity property. One can see that

$$g(e^{2\pi it}) = e^{\frac{\pi it}{12}} \frac{1}{(e^{2\pi it}; e^{4\pi it})_\infty} = e^{\frac{\pi it}{12}} \prod_{n=1}^{\infty} \frac{(1 - e^{2\pi nit})}{(1 - e^{4\pi int})}. \tag{3.17}$$

But according to [19], we know that the function $h(t) = e^{\frac{\pi it}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi int})$ is a modular function with the following relation

$$h(M.t) = \nu_h(M)\sqrt{-kt+h}h(t) = \nu_h(M)\sqrt{-iz}h(t) \tag{3.18}$$

where

$$\nu_h(M) = i(\frac{k}{h})e^{\frac{\pi i}{12}\left(-(h'+h)k - h\frac{1-hh'}{k}(k^2-1)+3h-3+3hk\right)}. \tag{3.19}$$

Also, one can see that

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{8} e^{-\frac{\pi iM.t}{12}}(1 - e^{2\pi in(M.t)}) =& \frac{1}{8}e^{-\frac{\pi it}{12}} \frac{\phi(e^{\pi it})}{\chi(e^{\pi it})} \\ =& \frac{1}{8}e^{\pi i\left(\frac{1}{8}(h'^2-1)(\frac{1-hh'}{k}-1)+\frac{h'-hh'^2}{8k}-\frac{(-k+\frac{hh'-1}{k})(hh'^2-h'-h)}{24}\right)} \\ & \times e^{\frac{-\pi iM.t}{12}} \frac{\phi(e^{2\pi it})}{\chi(e^{2\pi it})}. \end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned} g(M.t) =& \prod_{n=1}^{\infty} \frac{1}{8}e^{-\frac{\pi iM.t}{12}}(1 - e^{2\pi in(M.t)}) \\ =& e^{\pi i\left(\frac{1}{8}(h'^2-1)(\frac{1-hh'}{k}-1)+\frac{h'-hh'^2}{8k}-\frac{(-k+\frac{hh'-1}{k})(hh'^2-h'-h)}{24}\right)} e^{\frac{\pi}{12k}(\frac{1}{z}-z)} e^{\frac{\pi i}{12k}(h+h')} g(t). \end{aligned} \tag{3.21}$$

This proves (3.12).

Now we need to calculate an incomplete Kloosterman's sum. The Kloosterman's sum in the integrand is based on h and it is bounded by k. We will do this step in the next section.

3.4. Kloosterman's sum

In this case, we want to prove some materials in order to cope with a special Kloosterman's sum. In the first place, we produce an identifier function as follows

$$g(N,\vartheta,h,k)=\begin{cases}1 & -\vartheta_1(h,k)\le\vartheta\le\vartheta_2(h,k)\\ 0 & Otherwise\end{cases} \tag{3.22}$$

Now we find its Fourier transformation. So let $g(N,\vartheta,h,k)=\sum_{r=1}^{k} b_r e^{\frac{2\pi i r h'}{k}}$, where $hh'\equiv 1(modk)$. So the Fourier transformation is itself a Kloosterman's sum. We can show that $|b_r|<log(k)$, according to [21].

The other helpful point for the proof is the following equality.

$$\sum_{\substack{1\le h\le ak\\ \gcd(h,ak)=1\\ h\equiv l(moda)}} e^{\frac{2\pi i}{ak}(nh+mh')}=O(k^{\frac{2}{3}+\epsilon}(\gcd(n,k)^{\frac{1}{3}})). \tag{3.23}$$

The proof of the general case is by Kartsuba at [18].

The next useful point is the following relation, which holds for even k.

$$\sum_{\substack{1\le h\le k\\ \gcd(h,k)=1\\ hh'\equiv 1(modk)}} \omega(h,k)e^{\frac{2\pi i}{k}(nh+mh')} \tag{3.24}$$

Let $k=24t+2r$. Thus

$$\begin{aligned}
&\sum_{\substack{1\le h\le 24t+2r\\ \gcd(h,24t+2r)=1\\ hh'\equiv 1(mod2ht+2r)}} \omega(h,24t+2r)e^{\frac{2\pi i}{24t+2r}(nh+mh')}\\
&\quad=\sum_{\substack{1\le h\le 24t+2r\\ \gcd(h,12t+r)=1, r:odd\\ \gcd(h,2)=1\\ hh'\equiv 1(mod2ht+2r)}} \omega(h,24t+2r)e^{\frac{2\pi i}{24t+2r}(nh+mh')}\\
&\quad+\sum_{\substack{1\le h\le 24t+2r\\ \gcd(h,6t+r/2)=1, r/2:odd\\ \gcd(h,4)=1\\ hh'\equiv 1(mod2ht+2r)}} \omega(h,24t+2r)e^{\frac{2\pi i}{24t+2r}(nh+mh')}\\
&\quad+\sum_{\substack{1\le h\le 24t+2r\\ \gcd(h,3t+r/4)=1, r/4:odd\\ \gcd(h,8)=1\\ hh'\equiv 1(mod2ht+2r)}} \omega(h,24t+2r)e^{\frac{2\pi i}{24t+2r}(nh+mh')}\\
&\quad+\sum_{\substack{1\le h\le 24t\\ \gcd(h,t)=1\\ \gcd(h,24)=1\\ hh'\equiv 1(mod24t)}} \omega(h,24t)e^{\frac{2\pi i}{24t}(nh+mh')}
\end{aligned} \tag{3.25}$$

We only compute the last one. The others can be found from the fact that $\omega(l+2r, 24t+2r)e^{\frac{2\pi i}{24t+2r}(n(l+2r)+mh')} = \omega(l-24t, 24t+2r)e^{\frac{2\pi i}{24t+2r}(n(l-24t)+mh')}$. If $h = 24s+l$, we can show that

$$\sum_{\substack{1\le h\le 24t \\ \gcd(h,t)=1 \\ \gcd(h,24)=1 \\ hh'\equiv 1(mod 24t)}} \omega(h,24t)e^{\frac{2\pi i}{24t}(nh+mh')} = \sum_{\substack{1\le l\le 24 \\ \gcd(l,24)=1}} \sum_{\substack{1\le h\le t \\ \gcd(h,t)=1 \\ h=24s+l \\ hh'\equiv 1(mod 24t)}} \omega(h,24t)e^{\frac{2\pi i}{24t}(nh+mh')}$$

$$= \sum_{\substack{1\le l\le 24 \\ \gcd(l,24)=1}} O(k^{\frac{2}{3}+\epsilon}(\gcd(n,k)^{\frac{1}{3}})) = O(k^{\frac{2}{3}+\epsilon}(\gcd(n,k)^{\frac{1}{3}})). \tag{3.26}$$

If k is odd, since $h = 2s+1$; then

$$\sum_{\substack{1\le 2s+1\le 2t+1 \\ \gcd(2s+1,2t+1)=1 \\ hh'\equiv 1(mod 2t+1), h':odd}} \omega(h,2t+1)e^{\frac{2\pi i}{k}(nh+mh')} = \sum_{\substack{1\le h<2k \\ \gcd(h,2k)=1 \\ hh'\equiv 1(mod 2k)}} \omega(h,k)e^{\frac{2\pi i}{k}(nh+mh')}. \tag{3.27}$$

This implies that the sum is of order $O(\sqrt[2]{4k^2}\sqrt{\gcd(h,k)})$. So the Kloosterman's sum can be controlled by $O(\sqrt[3]{k^2})$ in all of the cases.

3.5. Approximating the integrals

Now we come back to the integral (3.35). As we have discussed

$$\begin{aligned} p_o(n) &= \frac{1}{2\pi i} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k\le N}} \int_{-\vartheta_1(h,k)}^{\vartheta_2(h,k)} \frac{f(x)}{x^{n+1}} dx \\ &= \frac{1}{2\pi i} \sum_{1\le k\le N} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k\le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} g(N,\vartheta,h,k) f(e^{\frac{2\pi i h - 2\pi z}{k}}) e^{\frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} d\vartheta \end{aligned} \tag{3.28}$$

where we change the parameter from $k(N^{-2} - i\vartheta)$ to z. As we discussed previously, and according to (3.12),

$$I_1 = \frac{1}{2\pi i} \sum_{1\le k\le N, 2|k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k\le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} g(N,\vartheta,h,k) f(e^{\frac{2\pi i h - 2\pi z}{k}}) e^{\frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} d\vartheta \tag{3.29}$$

So,

$$I_1 = \frac{1}{2\pi i} \sum_{1 \le k \le N, 2|k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k \le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} g(N, \vartheta, h, k)\omega(h, k) e^{\frac{\pi}{12k}(z-\frac{1}{z})} \times f(e^{\frac{2\pi h' - 2\pi i/z}{k}}) e^{\frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} d\vartheta \qquad (3.30)$$

If we substitute f with its definition, we will have

$$I_1 = \frac{1}{2\pi i} \sum_{1 \le k \le N, 2|k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k \le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} g(N, \vartheta, h, k)\omega(h, k) e^{\frac{\pi}{12k}(z-\frac{1}{z})} \times \left(\sum_{m=0}^{\infty} q(m) e^{m(\frac{2\pi h' - 2\pi i/z}{k})} \right) e^{\frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} d\vartheta \qquad (3.31)$$

We can show that

$$I_1 = \frac{1}{2\pi i} \sum_{1 \le k \le N, 2|k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k \le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} g(N, \vartheta, h, k)\omega(h, k) \times \left(\sum_{m=0}^{\infty} q(m) e^{\frac{\pi}{12k}(z-\frac{1}{z}) + m(\frac{2\pi h' - 2\pi i/z}{k}) + \frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} \right) d\vartheta. \qquad (3.32)$$

Then we replace g with its Fourier series. One can see that

$$I_1 = \frac{1}{2\pi i} \sum_{1 \le k \le N, 2|k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\le k \le N}} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} \sum_{r=1}^{k} e^{\frac{2\pi i r h'}{k}} \omega(h, k) \times \left(\sum_{m=0}^{\infty} q(m) e^{\frac{\pi}{12k}(z-\frac{1}{z}) + m(\frac{2\pi h' - 2\pi i/z}{k}) + \frac{-2\pi i h(n+1)+2\pi(n+1)z}{k}} \right) d\vartheta. \qquad (3.33)$$

Now we start to bound I_1 helping from a formula for b_r in [8]. According to this formula, since $\sum_r |b_r| < \log(4k)$, we have

$$|I_1| \le \frac{1}{2\pi} \sum_{1 \le k \le N, 2|k} \int_{-\frac{1}{k(N+1)}}^{\frac{1}{k(N+1)}} \left(\sum_{m=0}^{\infty} q(m) e^{-\pi(m+\frac{1}{24})} \sum_{r=1}^{k} |b_r| k^{\frac{2}{3}} \right) d\vartheta < \sum_{k=1}^{N} \log(k) k^{\frac{2}{3}} \frac{1}{kN} = O(\frac{1}{N} \sum_{k=1}^{N} k^{\frac{-1}{3}+\varepsilon}). \qquad (3.34)$$

So $|I_1| = O(N^{-1+\frac{2}{3}}) = O(N^{\frac{-1}{3}})$.

Now we will find an aproximation for I_2. We know that

$$I_2 = \frac{1}{2\pi i} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\leq k\leq N \\ 2\nmid k}} \int_{-\vartheta_1(h,k)}^{\vartheta_2(h,k)} \frac{f(x)}{x^{n+1}} dx. \tag{3.35}$$

We substitute $\omega = \frac{1}{N^2} - i\vartheta$. Then,

$$J = \frac{-i}{2\pi i} \sum_{1\leq k\leq N, 2\nmid k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\leq k\leq N}} \int_{-\frac{1}{N^{-2}}-i\vartheta_2}^{-\frac{1}{N^{-2}}+i\vartheta_1} \omega_{h,k} e^{\frac{-2\pi i h n}{k}} e^{2\pi\omega(n+\frac{1}{24})+\frac{\pi}{24}k^2\omega} d\vartheta. \tag{3.36}$$

If we consider a contour as in figure 3.3, then

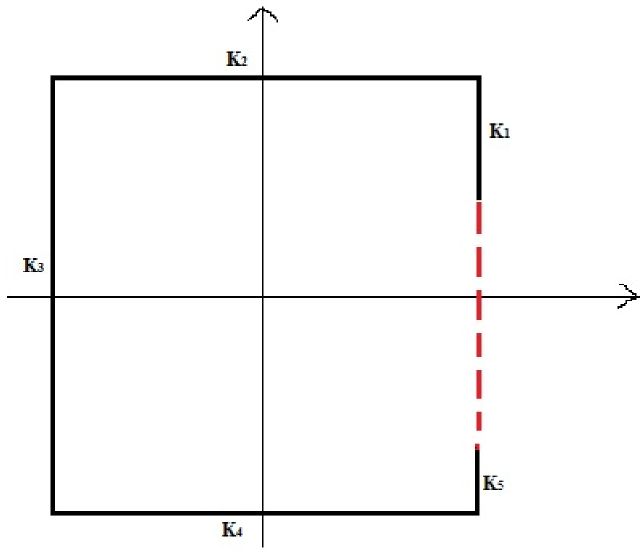

Figure 3.3: The contour of K_1, K_2, K_3, K_4, K_5.

$$\begin{aligned} J = \frac{-i}{2\pi i} \sum_{1\leq k\leq N, 2\nmid k} \sum_{\substack{\gcd(h,k)=1 \\ 0<h\leq k\leq N}} & \int_{N^{-2}+i\vartheta_1}^{N^{-2}+ik^{-1}(N+1)^{-1}} + \int_{N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}+ik^{-1}(N+1)^{-1}} + \int_{-N^{-2}+ik^{-1}(N+1)^{-1}}^{N^{-2}-ik^{-1}(N+1)^{-1}} \\ & + \int_{-N^{-2}-ik^{-1}(N+1)^{-1}}^{N^{-2}+ik^{-1}(N+1)^{-1}} + \int_{N^{-2}-ik^{-1}(N+1)^{-1}}^{N^{-2}-i\vartheta_2} - 2\pi i Res \\ & = K_1 + K_2 + K_3 + K_4 + K_5 + L \end{aligned} \tag{3.37}$$

The integrand for all of K_i is

$$\omega_{h,k} e^{\frac{-2\pi i h n}{k}} e^{2\pi\omega(n+\frac{1}{24})+\frac{\pi}{24}k^2\omega}. \tag{3.38}$$

One can see that

$$|K_1| \le \frac{1}{2\pi i} \sum_{1\le k\le N, 2\nmid k} k^{2/3+\epsilon} \int_{N^{-2}+i\vartheta_1}^{N^{-2}+ik^{-1}(N+1)^{-1}} e^{2\pi\omega(n+\frac{1}{24})Re(\omega)+\frac{\pi}{24k^2}Re(\frac{1}{\omega})} d\omega$$

$$= O(\sum_{k=1}^{N} k^{2/3+\epsilon} e^{-2\pi n N^{-2}} \int_{k^{-1}(N+h)^{-1}}^{k^{-1}(N+1)^{-1}} d\vartheta) = O(N^{-1/3+\epsilon}). \tag{3.39}$$

For K_5, we have

$$|K_5| \le \frac{1}{2\pi i} \sum_{1\le k\le N, 2\nmid k} k^{2/3+\epsilon} \int_{N^{-2}-ik^{-1}(N+1)^{-1}}^{N^{-2}-i\vartheta_2} e^{2\pi\omega(n+\frac{1}{24})Re(\omega)+\frac{\pi}{24k^2}Re(\frac{1}{\omega})} d\omega$$

$$= O\left(\sum_{k=1}^{N} k^{\frac{2}{3}+\epsilon} e^{-2\pi n N^{-2}} \int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-1}} d\vartheta\right) = O(N^{\frac{-1}{3}+\epsilon}). \tag{3.40}$$

Now we have

$$Re(\frac{1}{k^2\omega}) = \frac{N^{-2}}{k^2N^{-2}+N^2} \tag{3.41}$$

So,

$$K_2 = \frac{1}{2\pi i} \sum_{1\le k\le N, 2\nmid k} k^{2/3+\epsilon} \int_{N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}+ik^{-1}(N+1)^{-1}} e^{2\pi\omega(n+\frac{1}{24})Re(\omega)+\frac{\pi}{24k^2}Re(\frac{1}{\omega})} d\omega$$

$$= \frac{1}{2\pi} \int_{N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}+ik^{-1}(N+1)^{-1}} e^{2\pi\omega(n+\frac{1}{24})Re(\omega)+\frac{\pi}{24}\frac{N^{-2}}{N^2+k^2N^{-2}}} d\omega$$

$$= O(\sum_{k=1}^{N} N^{-2} k^{\frac{2}{3}+\epsilon}) = O(N^{-\frac{1}{3}+\epsilon}). \tag{3.42}$$

The same goes for K_4. Finally we have

$$|K_3| \le \frac{1}{2\pi} \sum_{1\le k\le N, 2\nmid k} O(\frac{e^{\frac{-2\pi n}{N^2}}}{kN}) = O(N^{-1/3}). \tag{3.43}$$

Finally, we find L. One can see

$$L = Res\left(\frac{1}{2\pi} \sum_{1\le k\le N, 2\nmid k} \sum_{\substack{\gcd(h,k)=1\\ 0<h\le k\le N}} \omega_{h,k} e^{\frac{-2\pi i h n}{k}} e^{2\pi\omega(n+\frac{1}{24})+\frac{\pi}{24}k^2\omega} d\vartheta)\right)|_0 \tag{3.44}$$

We have

$$e^{2\pi\omega(n+\frac{1}{24})+\frac{\pi}{24}k^2\omega} = \left(\sum_{r=1}^{\infty} \frac{(2\pi\omega(n+\frac{1}{24}))^r}{r!}\right)\left(\sum_{l=1}^{\infty} \frac{(\frac{\pi}{24k^2\omega})^l}{l!}\right) \tag{3.45}$$

So the residue can be found as follows.

$$\sum_{l=1}^{\infty}\frac{1}{l!(l-1)!}(\frac{\pi}{24k^2})^l(2\pi(n+\frac{1}{24}))^{l-1} = \frac{1}{2\pi}\frac{d}{dl}\sum_{l=1}^{\infty}\frac{1}{(l!)^2}\left(\frac{\pi}{24k^2}\right)^l(2\pi(n+\frac{1}{24}))^l$$
$$= \frac{1}{2\pi}\frac{d}{dl}J_0\left(\frac{i\pi}{k}\sqrt{\frac{1}{3}(n+\frac{1}{24})}\right). \tag{3.46}$$

Therefore

$$p_o(n) = \frac{1}{\sqrt{2}}\sum_{1\le k\le N, 2\nmid k}\sum_{\substack{\gcd(h,k)=1\\ 0<h\le k\le N}}\omega_{h,k}e^{\frac{-2\pi i h n}{k}}\frac{d}{dn}J_0\left(\frac{i\pi}{k}\sqrt{\frac{1}{3}(n+\frac{1}{24})}\right)+O(N^{\frac{-1}{3}+\epsilon}). \tag{3.47}$$

And for $N \longrightarrow \infty$, we have

$$p_o(n) = \frac{1}{\sqrt{2}}\sum_{1\le k\le \infty, 2\nmid k}\sum_{\substack{\gcd(h,k)=1\\ 0<h\le k\le \infty}}\omega_{h,k}e^{\frac{-2\pi i h n}{k}}\frac{d}{dn}J_0\left(\frac{i\pi}{k}\sqrt{\frac{1}{3}(n+\frac{1}{24})}\right). \tag{3.48}$$

CHAPTER 4

An attempt for asymptotic formula for another kind of partitions

4.1. Introduction

In the first place, we need to explain about the exact formula that Loughlin proved in [24]. He proved an exact formula for the number of partitions with parts which are coprime with r and s. If $x = e^{2\pi i\tau}$, then the generating function for this aim is as follows.

$$P_{r,s}(x) = \sum_{n} p_{r,s}(n)x^n = \frac{\left(\prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi ik\tau}}\right)\left(\prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi ikrs\tau}}\right)}{\left(\prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi ikr\tau}}\right)\left(\prod_{k=1}^{\infty} \frac{1}{1-e^{2\pi iks\tau}}\right)}. \tag{4.1}$$

We can see that

$$P_{r,s}(e^{2\pi ik\tau}) = e^{-\frac{\pi i(r-1)(s-1)\tau}{12}} \frac{\eta(r\tau)\eta(s\tau)}{\eta(\tau)\eta(rs\tau)} \tag{4.2}$$

where η is the Dedekind eta function as defined in [19]. According to [19],the coefficients of the function η are completely identified. But at the same time, it is not easy to work with them to find the coeffiecints of $P_{r,s}$. That is why this result is interesting. In fact it can lead to a formula on Bessel functions. In this chapter, we try to compute a modular transformation formula for a similar, but colored partition for the case $r = 10$ and $s = 2$. What we mean by colered partition is that some odd parts may appear in two different copies (or colors).

Let $p_a(n)$ be the number of partitions which are in the form $5t \pm a$ or $2t+1$. We know the exact formula for the generating function $G_{M,a}(x)$ for the partitions with parts of the form $Mt \pm a$ from [15]. But in order to find the coefficients for the function $p_a(n)$, we have to use Chinese remainder theorem to find $0 \leq l < 10$ such that $l \equiv a(mod5)$, and $l = 2t + 1$. Then we need to compute

$$\frac{G_{5,a}(x)G_{2,1}(x)}{G(x)G_{10,l}(x)}, \tag{4.3}$$

which is very hard to compute. So $p_a(n)$ has a generating function as follows.

$$\begin{aligned} F_a(x) = 1 + \sum_{n=1}^{\infty} p_a(n)x^n &= \frac{1}{(x^a; x^5)_\infty (x^{5-a}; x^5)_\infty (x; x^2)_\infty} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - x^{5m+a})(1 - x^{5m+5-a})(1 - x^{2m+1})}. \end{aligned} \tag{4.4}$$

We will attempt to find a modular transformation for the generating function of $p_a(n)$.

4.2. The modular transformation

One can see that

$$F_a(e^{2\pi i\tau}) = \frac{e^{\frac{-\pi i\tau}{12}}\eta(\tau)}{G_a(2\tau)} \tag{4.5}$$

where G_a is as defined in [20]. It seems natural to find a modular transformation for $G_a(2\tau)$. One can see that

$$H_a(x) := G_a(x^2) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{x^{2(5m+a)n}}{n} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{x^{2(5m+5-a)n}}{n} \tag{4.6}$$

Let $x = e^{2\pi i(\frac{h+iz}{k})}$ and $x' = T.x = e^{2\pi i\frac{h'+i/z}{k}}$. We have four cases.

4.2.1. Case 1: $10|k$

Let $10|k$. Assume the following definitions.

$$\begin{aligned} 10m \pm 2a =& qk + \mu_a \quad , \quad 0 < \mu_a < k \quad , \quad q \in \mathbb{N} \cup \{0\} \\ n =& rk + \nu \quad , \quad 0 < \nu < k \quad , \quad r \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{4.7}$$

Since $10|k$, it is easy to show that $10|\mu_a \pm 2a$. So

$$H_a(x) = \sum_{\substack{\mu_a \equiv \pm 2a(mod 10)\\ 0<\nu<k}} e^{2\pi i(qk+\mu_a)(rk+\nu)} \sum_{q,r=0}^{\infty} \frac{1}{rk+\nu} e^{\frac{-2\pi z}{k}(qk+\mu)(rk+\nu)}. \tag{4.8}$$

So in the same way as in [20], we have

$$H_a(x) = \frac{1}{4\pi i k^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod 10)\\ 0<\nu<k\\ 0<\lambda\leq k}} \cos(\frac{2\pi h\mu\nu}{k}) \cos(\frac{2\pi\lambda\mu}{k}) \int_{(\frac{3}{2})} \frac{\zeta(1-s,\frac{\lambda}{k})\zeta(1+s,\frac{\nu}{k})ds}{z^s \cos(\frac{\pi s}{2})}$$
$$+ \frac{1}{4\pi k^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod 10)\\ 0<\nu<k\\ 0<\lambda\leq k}} \sin(\frac{2\pi h\mu\nu}{k}) \sin(\frac{2\pi\lambda\mu}{k}) \int_{(\frac{3}{2})} \frac{\zeta(1-s,\frac{\lambda}{k})\zeta(1+s,\frac{\nu}{k})ds}{z^s \sin(\frac{\pi s}{2})}. \tag{4.9}$$

where (a) means the line $a+iy$ for $y \in \mathbb{R}$. Let $hh' \equiv -1(mod k)$. Since $\gcd(h,k)=1$, so $\gcd(h,10)=1$. If $h \equiv \pm 1(mod 10)$, then $h' \equiv \mp 1(mod 10)$. On the other hand, if $h \equiv \pm 3(mod 10)$, then $h' \equiv \mp 3(mod 10)$. If we define b as follows,

$$b = \begin{cases} a & h \equiv \pm 1(mod 10) \\ 3a & h \equiv \pm 3(mod 10). \end{cases} \tag{4.10}$$

then one can see that the equation $\mu_a \equiv h'\mu_a(mod k)$, and $0 < \mu_b < k$ results in the fact that $\mu_b \equiv \mu_a(\mod 10)$, if $h \equiv \pm 1(mod 10)$; and $\mu_b \equiv 3\mu_a(mod 10)$. Since the possible amount for μ_a, μ_b are $2,4,6,8$, then we can conclude that if $h \equiv \pm 1(mod 10)$, then $\mu_a \equiv \pm 2(mod 10)$ iff $\mu_a \equiv \pm 2(mod 10)$ and $\mu_a \equiv \pm 4(mod 10)$ iff $\mu_a \equiv \pm 4(mod 10)$. On the other hand if $h \equiv \pm 3(mod 10)$, then $\mu_a \equiv \pm 2(mod 10)$ iff $\mu_a \equiv \pm 4(mod 10)$ and $\mu_a \equiv \pm 4(mod 10)$ iff $\mu_a \equiv \pm 2(mod 10)$. So in the similar way as in [20], one can see that

$$H_a(x) = H_b(x') - 2\pi i Res. \tag{4.11}$$

where Res is the residue of the path C which is a rectangle with edges $Im(z) = \pm\frac{3}{2}$. So this is the first case modular transformation.

4.2.2. Case 2: $k = 10t+5$

Let $k = 10t+5$ for some $t \in \mathbb{Z}$. Assume again the definitions of (4.7). We know that

$$H_a(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{x^{(10m+2a)n}}{n} + \frac{x^{(10m-2a)n}}{n}. \tag{4.12}$$

Since $qk+\mu_a = 10m \pm 2a$ and $5|k$, we can conclude that $5|\mu_a \pm 2a$. So we can show again that

$$H_a(x) = \sum_{\substack{\mu_a \equiv \pm 2a+5 (mod 10)\\ 0<\nu<k}} \sum_{q,r=0}^{\infty} \frac{e^{\frac{2\pi i h((2q+1)k+\mu_a)(rk+\nu)}{k}}}{rk+\nu} e^{\frac{-2\pi z}{k}((2q+1)k+\mu)(rk+\nu)}$$
$$+ \sum_{\substack{\mu_a \equiv \pm 2a (mod 10)\\ 0<\nu<k}} \sum_{q,r=0}^{\infty} \frac{e^{\frac{2\pi i h(2qk+\mu_a)(rk+\nu)}{k}}}{rk+\nu} e^{\frac{-2\pi z}{k}(2qk+\mu)(rk+\nu)}. \tag{4.13}$$

So by using the Mellin transformation, we have

$$H_a(x) = \sum_{\substack{\mu_a \equiv \pm 2a+5 (mod 10)\\ 0<\nu<k}} \sum_{q,r=0}^{\infty} e^{\frac{2\pi i h \mu_a \nu}{k}} \frac{1}{2\pi i(rk+\nu)} \int_{(\frac{3}{2})} \frac{\Gamma(s) k^s ds}{(2\pi z)^s ((2q+1)k+\mu_a)^s (rk+\nu)^s}$$
$$+ \sum_{\substack{\mu_a \equiv \pm 2a (mod 10)\\ 0<\nu<k}} \sum_{q,r=0}^{\infty} e^{\frac{2\pi i h \mu_a \nu}{k}} \frac{1}{2\pi i(rk+\nu)} \int_{(\frac{3}{2})} \frac{\Gamma(s) k^s ds}{(2\pi z)^s (2qk+\mu_a)^s (rk+\nu)^s}. \tag{4.14}$$

By using the definition of the hurwitz-zeta function and considering the convergence, we will see that

$$H_a(x) = \frac{1}{2\pi i k} \sum_{\substack{\mu_a \equiv \pm 2a (mod 10)\\ 0<\nu<k}} e^{\frac{2\pi i h \mu_a \nu}{k}} \int_{(\frac{3}{2})} \frac{\Gamma(s)}{(4\pi k z)^s} \zeta(s, \frac{\mu+k}{2k}) \zeta(s+1, \frac{\nu}{k}) ds$$
$$+ \frac{1}{2\pi i k} \sum_{\substack{\mu_a \equiv \pm 2a+5 (mod 10)\\ 0<\nu<k}} e^{\frac{2\pi i h \mu_a \nu}{k}} \int_{(\frac{3}{2})} \frac{\Gamma(s)}{(4\pi k z)^s} \zeta(s, \frac{\mu}{2k}) \zeta(s+1, \frac{\nu}{k}) ds \tag{4.15}$$

Now, we want to use the following property of hurwitz-zeta function

$$\zeta(s, \frac{\mu}{2k}) = \frac{2\Gamma(1-s)}{(4\pi k)^{1-s}} \Bigg(\sin(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \cos(\frac{\pi \lambda \mu_a}{k}) \zeta(1-s, \frac{\lambda}{2k})$$
$$+ \cos(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \sin(\frac{\pi \lambda \mu_a}{k}) \zeta(1-s, \frac{\lambda}{2k}) \Bigg) \tag{4.16}$$

and

$$\zeta(s, \frac{\mu+k}{2k}) = \frac{2\Gamma(1-s)}{(4\pi k)^{1-s}} \Bigg(\sin(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \cos(\frac{\pi \lambda \mu_a}{k}) \zeta(1-s, \frac{\lambda}{2k}) (-1)^{\lambda}$$
$$+ \cos(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \sin(\frac{\pi \lambda \mu_a}{k}) \zeta(1-s, \frac{\lambda}{2k}) \Bigg). \tag{4.17}$$

So (4.15) will transform as follows.

$$
\begin{aligned}
H_a(x) = \frac{1}{2\pi ik} \sum_{\substack{\mu_a \equiv \pm 2a(mod10) \\ 0<\nu<k}} e^{\frac{2\pi ih\mu_a\nu}{k}} \int_{(\frac{3}{2})} & \frac{2\Gamma(s)\zeta(s+1,\frac{\nu}{k})\Gamma(1-s)}{(4\pi k)z^s} \\
& \times \Bigg(\sin(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \cos(\frac{\pi\lambda\mu_a}{k})\zeta(1-s,\frac{\lambda}{2k})(-1)^{\lambda} \\
& + \cos(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \sin(\frac{\pi\lambda\mu_a}{k})\zeta(1-s,\frac{\lambda}{2k}) \Bigg) ds \\
+ \frac{1}{2\pi ik} \sum_{\substack{\mu_a \equiv \pm 2a+5(mod10) \\ 0<\nu<k}} e^{\frac{2\pi ih\mu_a\nu}{k}} \int_{(\frac{3}{2})} & \frac{2\Gamma(s)\zeta(s+1,\frac{\nu}{k})\Gamma(1-s)}{(4\pi k)z^s} \\
& \times \Bigg(\sin(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \cos(\frac{\pi\lambda\mu_a}{k})\zeta(1-s,\frac{\lambda}{2k}) \\
& + \cos(\frac{\pi s}{2}) \sum_{\lambda=1}^{k} \sin(\frac{\pi\lambda\mu_a}{k})\zeta(1-s,\frac{\lambda}{2k}) \Bigg) ds.
\end{aligned} \tag{4.18}
$$

We know that sin and cos are orthogonal; so if we change $e^{\frac{2\pi ih\mu_a\nu}{k}}$ to $\cos(\frac{2\pi h\mu_a\nu}{k}) + i\sin(\frac{2\pi h\mu_a\nu}{k})$, then we can use this orthognality and use the relation $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to get the following equation.

$$
\begin{aligned}
H_a(x) = & \frac{1}{4\pi ik^2} \sum_{\substack{\mu_a \equiv \pm 2a+5(mod10) \\ 0<\nu<k \\ 0<\lambda\le k}} \cos(\frac{2\pi h\mu_a\nu}{k})\cos(\frac{\pi\lambda\mu_a}{k}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})(-1)^{\lambda}}{z^s\cos(\frac{\pi s}{2})} ds \\
& + \frac{1}{4\pi k^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod10) \\ 0<\nu<k \\ 0<\lambda\le k}} \sin(\frac{2\pi h\mu_a\nu}{k})\sin(\frac{\pi\lambda\mu_a}{k}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})}{z^s\sin(\frac{\pi s}{2})} ds \\
& + \frac{1}{4\pi ik^2} \sum_{\substack{\mu_a \equiv \pm 2a+5(mod10) \\ 0<\nu<k \\ 0<\lambda\le k}} \cos(\frac{2\pi h\mu_a\nu}{k})\cos(\frac{\pi\lambda\mu_a}{k}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})(-1)^{\lambda}}{z^s\cos(\frac{\pi s}{2})} ds \\
& + \frac{1}{4\pi ik^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod10) \\ 0<\nu<k \\ 0<\lambda\le k}} \sin(\frac{2\pi h\mu_a\nu}{k})\sin(\frac{\pi\lambda\mu_a}{k}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})}{z^s\sin(\frac{\pi s}{2})} ds.
\end{aligned} \tag{4.19}
$$

So

$$H_a(x) = \frac{1}{4\pi i k^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod5) \\ 0<\nu,\lambda<k \\ 0<\lambda\leq k}} \cos(\frac{2\pi h\mu_a\nu}{k})\cos(\frac{\pi\lambda\mu_a}{k}) \times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})(-1)^{\lambda+\mu_a+1}}{z^s\cos(\frac{\pi s}{2})}ds + \frac{1}{4\pi k^2} \sum_{\substack{\mu_a \equiv \pm 2a(mod5) \\ 0<\nu,\lambda<k \\ 0<\lambda\leq k}} \sin(\frac{2\pi h\mu_a\nu}{k})\sin(\frac{\pi\lambda\mu_a}{k}) \times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2k})(-1)^{\lambda+\mu_a+1}}{z^s\sin(\frac{\pi s}{2})}ds. \tag{4.20}$$

Assume again that $\mu_a \equiv h'\mu_b(modk)$ and $hh' \equiv -1(modk)$. If $h \equiv \pm 1(mod5)$ and considering $5|k$, we can conclude that $5|h'\mu_b \mp 2a$. So $5| \mp \mu_b \pm 2a$. This implies that $\mu_b \equiv \pm 2a(mod5)$. Similarly, we can show that if $h \equiv \pm 2(mod5)$, then $\mu_b \equiv \pm a(mod5)$. This shows that

$$b = \begin{cases} a & h \equiv \pm 1(mod5) \\ a* & h \equiv \pm 2(mod5) \end{cases} \tag{4.21}$$

Thus if we change s to $-s$, we have

$$H_a(x) = \frac{1}{4\pi i k^2} \sum_{\substack{0<\nu,\lambda<k \\ \mu_b}} \cos(\frac{2\pi\mu_b\nu}{k})\cos(\frac{\pi\lambda\mu_b h'}{k}) \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2k})(-1)^{\lambda+h'\mu_b+1}}{z^{-s}\cos(\frac{\pi s}{2})}ds + \frac{1}{4\pi k^2} \sum_{\substack{\mu_b,0<\nu,\lambda<k \\ 0<\lambda\leq k}} \sin(\frac{2\pi\mu_b\nu}{k})\sin(\frac{\pi\lambda h'\mu_b}{k}) \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2k})(-1)^{\lambda+h'\mu_b+1}}{z^{-s}\sin(\frac{\pi s}{2})}ds. \tag{4.22}$$

On the other hand, one can see that

$$
\begin{aligned}
H_b(x') = &\frac{1}{4\pi i k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \cos(\frac{2\pi h' \mu_b \nu}{k}) \cos(\frac{\pi \lambda \mu_b}{k}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{k})\zeta(1-s, \frac{\lambda}{2k})(-1)^{\lambda+\mu_b+1}}{z^s \cos(\frac{\pi s}{2})} ds \\
&+ \frac{1}{4\pi k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \sin(\frac{2\pi h' \mu_b \nu}{k}) \sin(\frac{\pi \lambda \mu_b}{k}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{k})\zeta(1-s, \frac{\lambda}{2k})(-1)^{\lambda+\mu_b+1}}{z^s \sin(\frac{\pi s}{2})} ds. \qquad (4.23)
\end{aligned}
$$

Since $\gcd(2,k) = 1$ and $0 < \nu < k$, $0 < \lambda \le k$, we can replace 2ν with ν and $\frac{\lambda}{2}$ with λ. So

$$
\begin{aligned}
H_b(x') = &\frac{1}{4\pi i k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \cos(\frac{\pi h' \mu_b \nu}{k}) \cos(\frac{2\pi \lambda \mu_b}{k}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{2k})\zeta(1-s, \frac{\lambda}{k})(-1)^{2\lambda+\mu_b+1}}{z^s \cos(\frac{\pi s}{2})} ds \\
&+ \frac{1}{4\pi k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \sin(\frac{\pi h' \mu_b \nu}{k}) \sin(\frac{2\pi \lambda \mu_b}{k}) \\
&\int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{2k})\zeta(1-s, \frac{\lambda}{k})(-1)^{2\lambda+\mu_b+1}}{z^s \sin(\frac{\pi s}{2})} ds. \qquad (4.24)
\end{aligned}
$$

If we replace ν, λ and consider that h' is odd, we will get

$$
\begin{aligned}
H_b(x') = &\frac{1}{4\pi i k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \cos(\frac{\pi h' \mu_b \lambda}{k}) \cos(\frac{2\pi \nu \mu_b}{k}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\lambda}{2k})\zeta(1-s, \frac{\nu}{k})(-1)^{2\nu+h'\mu_b+1}}{z^s \cos(\frac{\pi s}{2})} ds \\
&+ \frac{1}{4\pi k^2} \sum_{\substack{\mu_b \equiv \pm 2a (mod 5) \\ 0<\nu,\lambda<k \\ 0<\lambda\le k}} \sin(\frac{\pi h' \mu_b \lambda}{k}) \sin(\frac{2\pi \nu \mu_b}{k}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\lambda}{2k})\zeta(1-s, \frac{\nu}{k})(-1)^{2\nu+h'\mu_b+1}}{z^s \sin(\frac{\pi s}{2})} ds. \qquad (4.25)
\end{aligned}
$$

Now we define

$$H_{a,\lambda}(x) = \frac{1}{4\pi i k^2} \sum_{\substack{\mu_a \equiv \pm 2a (mod 5) \\ 0<\nu<k}} \cos(\frac{2\pi h \mu_a \nu}{k}) \cos(\frac{\pi \lambda \mu_a}{k})$$
$$\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{k})\zeta(1-s, \frac{\lambda}{2k})(-1)^{\mu_a+1}}{z^s \cos(\frac{\pi s}{2})} ds$$
$$+ \frac{1}{4\pi k^2} \sum_{\substack{\mu_a \equiv \pm 2a (mod 5) \\ 0<\nu<k}} \sin(\frac{2\pi h \mu_a \nu}{k}) \sin(\frac{\pi \lambda \mu_a}{k})$$
$$\times \int_{(\frac{3}{2})} \frac{\zeta(s+1, \frac{\nu}{k})\zeta(1-s, \frac{\lambda}{2k})(-1)^{\mu_a+1}}{z^s \sin(\frac{\pi s}{2})} ds. \tag{4.26}$$

Then we have $H_a(x) = \sum_{\lambda=1}^{k} H_{a,\lambda}(x)(-1)^{\lambda}$. Comparing (4.25), (4.20), we have

$$H_{a,\lambda}(x) = H_b(b, \lambda)(x') + 2\pi i Res_{\lambda} \tag{4.27}$$

where Res_{λ} is the residue in the same mentioned path C in case 1. Thus

$$H_a(x) = H_b(x') + 2\pi i \sum_{\lambda=1}^{k} Res_{\lambda}(-1)^{\lambda}. \tag{4.28}$$

4.2.3. Case 3: $\gcd(k,5) = 1$ and k is even

Assume that $5 \nmid k$. Let $K = 5k$ and $H = 5h$. We define

$$\begin{aligned} 10m \pm 2a =& qK + \mu_a \quad , \quad 0 < \mu_a < K \quad , \quad q \in \mathbb{N} \cup \{0\} \\ n =& rk + \nu \quad , \quad 0 < \nu < k \quad , \quad r \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{4.29}$$

Also

$$HH' \equiv -1 (mod k) \quad , \quad hh' \equiv -1 (mod k) \tag{4.30}$$

So we have

$$\begin{aligned} H_a(x) := G_a(x^2) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{x^{(10m+2a)n}}{n} + \sum_{m=0}^{\infty} \frac{x^{(10m+10-2a)n}}{n} \\ &= \sum_{\substack{\mu_a \equiv \pm 2a (mod 10) \\ 0<\nu \leq k \\ 0<\mu_a<K}} e^{2\pi i \frac{(h(qK+\mu_a)(rk+\nu))}{k}} \sum_{q,r=0}^{\infty} \frac{e^{\frac{-2\pi z}{k}(qK+\mu_a)(rk+\nu)}}{rk+\nu}. \end{aligned} \tag{4.31}$$

By using a Mellin transformation

$$H_a(x) = \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k}} e^{2\pi i\frac{h\mu_a\nu}{k}} \sum_{q,r=0}^{\infty} \frac{1}{2\pi i(rk+\nu)} \int_{(\frac{3}{2})} \frac{\Gamma(s)k^s ds}{(qK+\mu_a)^s(rk+\nu)^s(2\pi z)^s}$$

$$= \frac{1}{2\pi ik} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k}} \frac{e^{2\pi i\frac{h\mu_a\nu}{k}}}{rk+\nu} \int_{(\frac{3}{2})} \frac{\Gamma(s)}{(2\pi zK)^s}\zeta(s,\frac{\mu_a}{k})\zeta(s+1,\frac{\nu}{k})ds. \tag{4.32}$$

Again, using the property of hurwitz-zeta function [20], one can see that

$$H_a(x) = \frac{1}{4\pi ikK} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \cos(2\pi\frac{h\mu_a\nu}{k})\cos(\frac{2\pi\lambda\mu_a}{K}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{k})ds}{z^s\cos(\frac{\pi s}{2})}$$

$$+ \frac{1}{4\pi kK} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \sin(2\pi\frac{h\mu_a\nu}{k})\sin(\frac{2\pi\lambda\mu_a}{K}) \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{k})ds}{z^s\sin(\frac{\pi s}{2})} \tag{4.33}$$

If we define $\mu_a \equiv 5H'\mu_b(modk)$, then since $\mu_a \equiv \pm 2a(mod10)$ and k is even, we can show that $\mu_a \equiv 10H'\mu \pm \alpha k$ for some α such that $\alpha k \equiv \pm 2a(mod10)$. Also, since μ_a runs over a residue system modulus k twice, one can see that μ_b runs over the same modulus k. Thus

$$H_a(x) = \frac{1}{4\pi ikK} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k\\ 0<\mu_a<K}} \cos(2\pi\frac{\mu_b\nu}{k}) \sum_{0<\lambda\leq K} \int_{(\frac{3}{2})} \cos(\frac{2\pi\lambda\mu_a}{K})\frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{k})ds}{z^s\cos(\frac{\pi s}{2})}$$

$$+ \frac{1}{4\pi kK} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\nu\leq k\\ 0<\mu_a<K}} \sin(2\pi\frac{\mu_b\nu}{k}) \sum_{0<\lambda\leq K} \int_{(\frac{3}{2})} \sin(\frac{2\pi\lambda\mu_a}{K})\frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{k})ds}{z^s\sin(\frac{\pi s}{2})}. \tag{4.34}$$

By using property of the hurwitz-zeta function again; and replacing s by $-s$, we will have

$$H_a(x) = \frac{1}{2\pi iK} \sum_{\substack{\mu_a\equiv\pm 2a(mod10)\\ 0<\mu_a,\lambda<K}} e^{2\pi\frac{\mu_a\lambda}{K}} \int_{(\frac{-3}{2})} \frac{z^s\Gamma(s)\zeta(s+1,\frac{\lambda}{K})\zeta(s,\frac{\mu_b}{k})ds}{(2\pi k)^s} \tag{4.35}$$

In the first place, we know the following equations.

$$\mu_a \equiv 10H'\mu_b(modk) \quad , \quad \mu_a \equiv \pm 2a(mod10). \tag{4.36}$$

We know that $\gcd(k,10) = 2$; so we can find α' such that $\alpha k \equiv \pm 2a(mod10)$. So it is easy to see that there exists α such that $\mu_a \equiv 10H'\mu_b \pm \alpha k(modK)$. Assume that

$$x = e^{2\pi i\frac{h+iz}{k}} \quad , \quad x'' = e^{\frac{2\pi i(H'+i/z)}{K}} \tag{4.37}$$

If we define J_α as

$$J_\alpha(x) = \sum_{m,n=1}^{\infty} \frac{e^{\frac{2\pi i m\alpha}{10}} x^{(10m+2a)n}}{n} + \sum_{m,n=1}^{\infty} \frac{e^{\frac{-2\pi i m\alpha}{10}} x^{(10m+10-2a)n}}{n}, \tag{4.38}$$

then

$$J_\alpha(x'') = \frac{1}{2\pi i K} \sum_{\substack{\mu_a \equiv \pm 2a (mod 10) \\ 0<\mu_a,\lambda<K}} e^{2\pi \frac{\mu_a \lambda}{K}} \int_{(\frac{3}{2})} \frac{z^s \Gamma(s)\zeta(s+1,\frac{\lambda}{K})\zeta(s,\frac{\mu_b}{k})ds}{(2\pi k)^s}. \tag{4.39}$$

So one can see that

$$H_a(x) = J_\alpha(x'') - 2\pi i Res. \tag{4.40}$$

4.2.4. Case 4: $\gcd(k, 10) = 1$.

Assume that $10 \nmid k$. Let $K = 5k$ and $H = 100h$. We define

$$\begin{aligned} 10m \pm 2a =& qK + \mu_a \quad , \quad 0 < \mu_a < K \quad , \quad q \in \mathbb{N} \cup \{0\} \\ n =& rk + \nu \quad , \quad 0 < \nu < k \quad , \quad r \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{4.41}$$

Also

$$HH' \equiv -1 (mod k) \quad , \quad hh' \equiv -1 (mod k) \tag{4.42}$$

So we have

$$\begin{aligned} H_a(x) := G_a(x^2) =& \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{x^{(10m+2a)n}}{n} + \sum_{m=0}^{\infty} \frac{x^{(10m+10-2a)n}}{n} \\ =& \sum_{\substack{\mu_a \equiv \pm 2a (mod 10) \\ 0<\nu\leq k \\ 0<\mu_a<K}} e^{2\pi i \frac{(h(2qK+\mu_a)(rk+\nu))}{k}} \sum_{q,r=0}^{\infty} \frac{e^{\frac{-2\pi z}{k}(2qK+\mu_a)(rk+\nu)}}{rk+\nu} \\ +& \sum_{\substack{\mu_a \equiv \pm 2a+5 (mod 10) \\ 0<\nu\leq k \\ 0<\mu_a<K}} e^{2\pi i \frac{(h(2qK+K+\mu_a)(rk+\nu))}{k}} \sum_{q,r=0}^{\infty} \frac{e^{\frac{-2\pi z}{k}(2qK+K+\mu_a)(rk+\nu)}}{rk+\nu}. \end{aligned} \tag{4.43}$$

By using a Mellin transformation

$$
\begin{aligned}
H_a(x) = &\sum_{\substack{\mu_a\equiv\pm 2a(mod 10)\\ 0<\nu\leq k}} e^{2\pi i\frac{h\mu_a\nu}{k}} \sum_{q,r=0}^{\infty} \frac{1}{2\pi i(rk+\nu)} \int_{(\frac{3}{2})} \frac{\Gamma(s)k^s ds}{(2qK+\mu_a)^s(rk+\nu)^s(2\pi z)^s} \\
&+ \sum_{\substack{\mu_a\equiv\pm 2a+5(mod 10)\\ 0<\nu\leq k}} e^{2\pi i\frac{h\mu_a\nu}{k}} \sum_{q,r=0}^{\infty} \frac{1}{2\pi i(rk+\nu)} \int_{(\frac{3}{2})} \frac{\Gamma(s)k^s ds}{(2qK+K+\mu_a)^s(rk+\nu)^s(2\pi z)^s} \\
&= \frac{1}{2\pi ik} \sum_{\substack{\mu_a\equiv\pm 2a(mod 10)\\ 0<\nu\leq k}} \frac{e^{2\pi i\frac{h\mu_a\nu}{k}}}{rk+\nu} \int_{(\frac{3}{2})} \frac{\Gamma(s)}{(4\pi zK)^s}\zeta(s,\frac{\mu_a}{2K})\zeta(s+1,\frac{\nu}{k})ds \\
&+ \frac{1}{2\pi ik} \sum_{\substack{\mu_a\equiv\pm 2a+5(mod 10)\\ 0<\nu\leq k}} \frac{e^{2\pi i\frac{h\mu_a\nu}{k}}}{rk+\nu} \int_{(\frac{3}{2})} \frac{\Gamma(s)}{(4\pi zK)^s}\zeta(s,\frac{\mu_a+K}{2k})\zeta(s+1,\frac{\nu}{k})ds. \qquad (4.44)
\end{aligned}
$$

Again, using the property [20] of hurwitz-zeta function, one can see that

$$
\begin{aligned}
H_a(x) = &\frac{1}{4\pi ikK} \sum_{\substack{\mu_a\equiv\pm 2a(mod 10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \cos(2\pi\frac{h\mu_a\nu}{k})\cos(\frac{2\pi\lambda\mu_a}{2K}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2K})ds}{(2z)^s\cos(\frac{\pi s}{2})} \\
&+\frac{1}{4\pi kK} \sum_{\substack{\mu_a\equiv\pm 2a(mod 10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \sin(2\pi\frac{h\mu_a\nu}{k})\sin(\frac{2\pi\lambda\mu_a}{2K}) \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2K})ds}{(2z)^s\sin(\frac{\pi s}{2})} \\
&+\frac{1}{4\pi ikK} \sum_{\substack{\mu_a\equiv\pm 2a+5(mod 10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \cos(2\pi\frac{h\mu_a\nu}{k})\cos(\frac{2\pi\lambda\mu_a}{2K})(-1)^\lambda \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2K})ds}{(2z)^s\cos(\frac{\pi s}{2})} \\
&+\frac{1}{4\pi kK} \sum_{\substack{\mu_a\equiv\pm 2a+5(mod 10)\\ 0<\nu\leq k\\ 0<\mu_a,\lambda<K}} \sin(2\pi\frac{h\mu_a\nu}{k})\sin(\frac{2\pi\lambda\mu_a}{2K})(-1)^\lambda \\
&\times \int_{(\frac{3}{2})} \frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{2K})ds}{(2z)^s\sin(\frac{\pi s}{2})}. \qquad (4.45)
\end{aligned}
$$

Let $10H'\mu_b \equiv \mu_a(modk)$ and $0 \leq \mu_b < k$. Then $-\mu_b \equiv HH'\mu_b \equiv 100hH'\mu_b \equiv 10h\mu_a(modk)$. Also, $10H'\mu_b \equiv \mu_a(modk)$ and $10H'\mu_b \equiv 0(mod10)$. This implies that by the Chinese remainder theorem, $10H'\mu_b \equiv \mu_a (modK)$. So by changing s to $-s$,

$$\begin{aligned}
H_a(x) = \frac{1}{4\pi ikK} & \sum_{\substack{\mu_a \equiv \pm 2a(mod10) \\ 0<\nu\leq k \\ 0<\mu_a,\lambda<K}} \cos(\pi\frac{\mu_b\nu}{K})\cos(\frac{2H'\pi\lambda\mu_b}{k}) \\
& \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})} \\
+\frac{1}{4\pi kK} & \sum_{\substack{\mu_a \equiv \pm 2a(mod10) \\ 0<\nu\leq k \\ 0<\mu_a,\lambda<K}} \sin(\pi\frac{\mu_b\nu}{K})\sin(\frac{2H'\pi\lambda\mu_a}{k}) \\
& \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})} \\
+\frac{1}{4\pi ikK} & \sum_{\substack{\mu_a \equiv \pm 2a+5(mod10) \\ 0<\nu\leq k \\ 0<\mu_a,\lambda<K}} \cos(\pi\frac{\mu_b\nu}{K})\cos(\frac{2\pi\lambda H'\mu_b}{k})(-1)^\lambda \\
& \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})} \\
+\frac{1}{4\pi kK} & \sum_{\substack{\mu_a \equiv \pm 2a+5(mod10) \\ 0<\nu\leq k \\ 0<\mu_a,\lambda<K}} \sin(\pi\frac{\mu_b\nu}{K})\sin(\frac{2\pi\lambda H'\mu_b}{k})(-1)^\lambda \\
& \times \int_{(\frac{-3}{2})} \frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})}.
\end{aligned} \tag{4.46}$$

On the other hand,

$$\begin{aligned}H_b(x') = &\frac{1}{4\pi ikK}\sum_{\substack{\mu_b\equiv\pm ?(mod 10)\\0<\nu\leq k\\0<\mu_b,\lambda<K}}\cos(\pi\frac{\mu_b\nu}{K})\cos(\frac{2H'\pi\lambda\mu_b}{k})\\&\times\int_{(\frac{3}{2})}\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})}\\&+\frac{1}{4\pi kK}\sum_{\substack{\mu_b\equiv\pm ?(mod 10)\\0<\nu\leq k\\0<\mu_a,\lambda<K}}\sin(\pi\frac{\mu_b\nu}{K})\sin(\frac{2H'\pi\lambda\mu_b}{k})\\&\times\int_{(\frac{3}{2})}\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})}\\&+\frac{1}{4\pi ikK}\sum_{\substack{\mu_a\equiv\pm ?(mod 10)\\0<\nu\leq k\\0<\mu_a,\lambda<K}}\cos(\pi\frac{\mu_b\nu}{K})\cos(\frac{2\pi\lambda H'\mu_b}{k})(-1)^\lambda\\&\times\int_{(\frac{3}{2})}\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})}\\&+\frac{1}{4\pi kK}\sum_{\substack{\mu_a\equiv\pm ?(mod 10)\\0<\nu\leq k\\0<\mu_a,\lambda<K}}\sin(\pi\frac{\mu_b\nu}{K})\sin(\frac{2\pi\lambda H'\mu_b}{k})(-1)^\lambda\\&\times\int_{(\frac{3}{2})}\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})ds}{(2z)^{-s}\cos(\frac{\pi s}{2})}.\end{aligned} \tag{4.47}$$

Hence,

$$H_{b,\lambda,\nu}(x')(-1)^\lambda + 2\pi i Res = H_{a,\lambda,\nu}(x). \tag{4.48}$$

So

$$\sum_{\lambda,\nu} H_{b,\lambda,\nu}(x')(-1)^\lambda + \sum_{\lambda,\nu} 2\pi i Res = \sum_{\lambda,\nu} H_{a,\lambda,\nu}(x). \tag{4.49}$$

This is the modular transformation. So we have the following theorem.

Theorem 4.2.1 *Let $H_a(x)$ is defined as* (4.6)*. Then we have four cases.*
If $10|k$, then

$$H_a(x) = H_b(x') + 2\pi i Res\left(\sum_{0<\nu,\lambda\leq k}\frac{\zeta(1-s,\frac{\lambda}{k})\zeta(1+s,\frac{\nu}{k})}{z^s\cos(\frac{\pi s}{2})}\right). \tag{4.50}$$

If $k = 10t+5$, then

$$H_a(x) = H_b(x') + 2\pi i\sum_{\lambda=1}^{k} Res\left(\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2k})(-1)^{\lambda+h'\mu_b+1}}{z^{-s}\cos(\frac{\pi s}{2})}\right). \tag{4.51}$$

If $\gcd(k,5)=1$ *and* k *is even,*

$$H_a(x) = J_\alpha(x'') + 2\pi i Res\left(\cos(\frac{2\pi\lambda\mu_a}{K})\frac{\zeta(s+1,\frac{\nu}{k})\zeta(1-s,\frac{\lambda}{k})}{z^s\cos(\frac{\pi s}{2})}\right). \tag{4.52}$$

where $J_\alpha(x'')$ *is defined in* (4.39)*,* $\mu_a \equiv 10H'\mu_b \pm \alpha k (mod 5k)$*, and* $H=5h$*.*
If $\gcd(k,5)=1$ *and* k *is odd,*

$$\sum_{\lambda,\nu} H_{b,\lambda,\nu}(x')(-1)^\lambda + \sum_{\lambda,\nu} 2\pi i Res\left(\frac{\zeta(-s+1,\frac{\nu}{k})\zeta(1+s,\frac{\lambda}{2K})}{(2z)^{-s}\cos(\frac{\pi s}{2})}\right) = \sum_{\lambda,\nu} H_{a,\lambda,\nu}(x). \tag{4.53}$$

where $K=5k$ *and* $100h=H$*.*

CHAPTER 5

Future research

In this chapter, we will explain an outline of the possible future research. There are two different categories for the continuation of this work. The first class are the ones which deal with a subset of $\mathbb{N}$ containing symmetric residue classes. In the latest work, Loughlin proved an exact formula for the case that the parts are not multiples of r, s for some square-free coprime integers. One can see if $r \nmid n$ and $s \nmid n$, then $-n \nmid r$ and $-n \nmid s$ for $n \in \mathbb{Z}_M$. So this set is a symmetric one. We also had some results about the modularity of partitions with parts of the form $10t \pm a$ or $2t+1$. So in the first step, we can prove an exact formula for the partitions with parts of the form $rt \pm a$ or $st \pm b$ for some square-free coprime integers r, s. This is also a symmetric subset. Also, we can prove an exact formula for the partitions with parts which are not divisible by i square-free and coprime numbers $r_1, \cdots, r_i$. This set is also a symmetric sequence. Also the generating function can be seen as follows.

$$P(\tau) = \prod_{j=0}^{i} (f(\prod_{1 \le l \le j} r_l)\tau)^{(-1)^{j-1}} \tag{5.1}$$

where

$$f(\tau) = \prod_{k} \frac{1}{1 - e^{2k\pi i \tau}}. \tag{5.2}$$

This idea is achievable, since the function f can be computed based on the Dedekind function η. We know the modularity transformation and coefficients of Dedekind functions explicitely. Thus we can use help from the properties of Dedekind functions to show an exact formula for the number of such partitions.

We can also aim for a more general case. We can generalize chapter 4 for more than two moduli. So we need to prove an exact formula for the number of partitions with parts $r_i \pm a_i$ where r_i are square-free and pairwise coprime numbers. Another idea is to find a function to prove an exact formula for the number of partitions with parts $rt \pm a$ and $st \pm b$ where r, s are not coprime. This is also doable, because we can write inclusion-exclusion formula to filter the greatest common divisor and use the formula (5.1) for $i = 2$.

One of the best way to continue this research, which is the second category, is to find the number of partitions with parts in a non symmetric residue classes. The simplest example is to find the number of partitions with parts of the form $Mt + a$. If we can find the number of this partition, then it may be possible to find the number of any non symmetric collection of residue classes. In particular, we can find the number of partitions with residue parts in $\mathbb{Z}_p$. It means that we can count the number of ways to represent $n \in \mathbb{Z}_p$ as $a_1^2 + \cdots + a_k^2$. So we can find out that locally-speaking, how many ways do we have to represent n by the quadratic form $x_1^2 + \cdots + x_k^2$. If we know that this number is positive for every prime number, we can conclude that n can be represented by $x_1^2 + \cdots + x_k^2$ in Q, according to Hasse-Minkowski Theorem.

Theorem 5.0.1 *[30, Hasse-Minkowski Theorem for quadratic forms] In order that f represent 0, it is necessary and sufficient that, for all $v \in V$, the form $f(v)$ represent 0. (In another word, f has a global zero, if and only if f has everywhere local zero).*

So n can be represented in $\mathbb{Z}$ by $x_1^2 + \cdots + x_k^2$. Also, we can compute the number of representing n by $x_1^2 + \cdots + x_k^2$ according to [19, Chapter 5]. So if we compare this number with the same number in every $\mathbb{Z}_p$, we will get a new view about the tightness of Hasse-Minkowski theorem (obviously this number in finite fields is bigger than this number in $\mathbb{Z}$).

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