CHARACTERIZATION OF THE POTENTIAL SMOOTHNESS OF ONE-DIMENSIONAL DIRAC OPERATOR SUBJECT TO GENERAL BOUNDARY CONDITIONS AND ITS RIESZ BASIS PROPERTY

by

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Abstract

The one-dimensional Dirac operators with periodic potentials subject to periodic, antiperiodic and a special family of general boundary conditions have discrete spectrums. It is known that, for large enough $|n|$ in the disc centered at n of radius $1/4$, the operator has exactly two eigenvalues (counted according to multiplicity) which are periodic (for even n) or antiperiodic (for odd n) and one eigenvalue derived from each general boundary condition. These eigenvalues construct a deviation which is the sum of the distance between two periodic (or antiperiodic) eigenvalues and the distance between one of the periodic (or antiperiodic) eigenvalues and one eigenvalue from the general boundary conditions. We show that the smoothness of the potential could be characterized by the decay rate of this spectral deviation. Furthermore, it is shown that the Dirac operator with periodic or antiperiodic boundary condition has the Riesz basis property if and only if the absolute value of the ratio of these deviations is bounded.

GENEL SINIR KOŞULLARI ALTINDAKİ DİRAC OPERATÖRÜNÜN POTANSİYELİNİN TÜREVLENEBİLİRLİĞİNİN VE RİESZ BAZI ÖZELLİĞİNİN KARAKTERİZASYONU

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Özet

Periyodik, antiperiyodik ve özel olarak tanımlanan genel sınır koşulları ailesine tabi bir boyutlu Dirac operatörünün spektrumu ayrıktır. Yeterince büyük $|n|$ değerleri için n merkezli 1/4 yarıçaplı disklerde operatörün tam olarak iki tane periyodik (n çift ise) veya antiperiyodik (n tek ise) özdeğeri ve bir tane de genel sınır koşullarından gelen özdeğeri vardır. Periyodik (veya antiperiyodik) özdeğerlerin arasındaki mesafe ile bir periyodik ve bir genel sınır koşullarından gelen özdeğer arasındaki mesafenin toplamı bir spektral sapma verir. Potansiyelin türevelenebilirliği bu sapmanın azalma hızıyla karakterize edildiği gösterilmiştir. Dahası, periyodik veya antiperiyodik sınır koşullarına tabi Dirac operatörünün Riesz bazı özelliğinin olmasının ancak ve ancak bu farkların oranın mutlak değerinin sınırlı olmasıyla mümkün olacağı gösterilmiştir.

To Berkay, Kutlay and Cem,

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CHAPTER 1

Introduction

We consider the one-dimensional Dirac operator

$$Ly = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{dy}{dx} + \begin{pmatrix} 0 & \mathcal{P}(x) \\ \mathcal{Q}(x) & 0 \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1.1}$$

where $\mathcal{P}, \mathcal{Q} \in L^2([0,\pi])$, with the following boundary conditions

$$\begin{aligned} &\text{Periodic}: && y_1(0) = y_1(\pi), \quad y_2(0) = y_2(\pi); \\ &\text{Antiperiodic}: && y_1(0) = -y_1(\pi), \quad y_2(0) = -y_2(\pi); \\ &\text{Dirichlet}: && y_1(0) = y_2(0), \quad y_1(\pi) = y_2(\pi). \end{aligned}$$

We also consider a general boundary condition (bc) given by

$$\begin{aligned} y_1(0) + by_1(\pi) + ay_2(0) &= 0 \\ dy_1(\pi) + cy_2(0) + y_2(\pi) &= 0, \end{aligned} \tag{1.2}$$

where a, b, c, d are complex numbers subject to the restrictions

$$b + c = 0, \quad ad = 1 - b^2 \tag{1.3}$$

with $ad \neq 0$. It is well-known that if $\mathcal{P}, \mathcal{Q} \in L^2([0,\pi])$, $\mathcal{P} = \overline{\mathcal{Q}}$ and we extend $\mathcal{P}$ and $\mathcal{Q}$ as π-periodic functions on $\mathbb{R}$, then the operator is self-adjoint and has a band-gap structured spectrum of the form

$$Sp(L) = \bigcup_{n=-\infty}^{+\infty} [\lambda_{n-1}^+, \lambda_n^-],$$

where

$$\cdots \leq \lambda_{n-1}^+ < \lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \cdots .$$

In addition, Floquet theory shows that the endpoints $\lambda_n^{\pm}$ of these spectral gaps are eigenvalues of the same operator considered with $\mathcal{P}, \mathcal{Q} \in L^2([0,\pi])$ under periodic boundary conditions or antiperiodic boundary conditions. Furthermore, the spectrum is discrete for each of the above boundary conditions. Also, for $n \in \mathbb{Z}$ with large enough $|n|$ the disc with center n and radius $1/8$ contains two eigenvalues (counted with multiplicity) λ_n^+ and λ_n^- of periodic (if n is even) or antiperiodic (if n is odd) boundary conditions and as well one eigenvalue μ_n^{Dir} of Dirichlet boundary condition. There is also one eigenvalue $\mu_n^{bc} = \mu_n$ of the general boundary condition (bc) given above (which will be proven in the first section).

There is a very close relationship between the smoothness of the potential and the rate of decay of the deviations $|\lambda_n^+ - \lambda_n^-|$ and $|\mu_n^{Dir} - \lambda_n^+|$. The story of the discovery of this relation was initiated by H. Hochstadt [16, 17] who considered the (self-adjoint) Hill's operator (an analogue of the localization of spectra also holds for Hill's operator; a major difference is that discs are centered at n^2's) and proved that the decay rate of the spectral gap $\gamma_n = |\lambda_n^+ - \lambda_n^-|$ is $O(1/n^{m-1})$ if the potential has m continuous derivatives. Furthermore, he showed that every finite-zone potential (i.e., $\gamma_n = 0$ for all but finitely many n) is a C^∞-function. Afterwards, some authors [20]- [22] studied on this relation and showed that if γ_n is $O(1/n^k)$ for any $k \in \mathbb{Z}^+$, then the potential is infinitely differentiable. Furthermore, Trubowitz [27] showed that the potential is analytic if and only if γ_n decays exponentially fast. In the non-selfadjoint case, the potential smoothness still determines the decay rate of γ_n. However, the decay rate of γ_n does not determine the potential smoothness as Gasymov showed [12]. In this case, Tkachenko [23, 25, 26] gave the idea to consider γ_n together with the deviation $\delta_n^{Dir} = \mu_n^{Dir} - \lambda_n^+$ and obtained characterizations of C^∞-smoothness and analyticity of the potential with these deviations γ_n and δ_n^{Dir}. In addition to these developments, Sansuc and Tkachenko [24] proved that the potential is in the Sobolev space H^m, $m \in \mathbb{N}$, if and only if γ_n and δ_n^{Dir} satisfy

$$\sum (|\gamma_n|^2 + |\delta_n^{Dir}|^2)(1 + n^{2m}) < \infty.$$

The results mentioned above have been obtained by using Inverse Spectral Theory.

Grébert, Kappeler, Djakov and Mityagin studied the relationship between the potential smoothness and the decay rate of spectral gaps for Dirac operators (see [8, 14, 15]).

We recall that a characterization of smoothness of a function can be given by weights $\Omega = \Omega(n)_{n\in\mathbb{Z}}$, where the corresponding weighted Sobolev space is

$$H(\Omega) = \{v(x) = \sum_{k\in\mathbb{Z}} v_k e^{2ikx} : \quad \sum_{k\in\mathbb{Z}} |v_k|^2 (\Omega(k))^2 < \infty\}$$

and the corresponding weighted ℓ^2-space is

$$\ell^2(\Omega, \mathbb{Z}) = \{(x_n)_{n\in\mathbb{Z}} : \sum |x_n|^2 (\Omega(n))^2 < \infty\}.$$

A weight Ω is called sub-multiplicative if $\Omega(n+m) \leq \Omega(n)\Omega(m)$ for each $n, m \in \mathbb{Z}$. It has been proved [5, 8] that for each sub-multiplicative weight $(\Omega(n))_{n\in\mathbb{Z}}$ the following implication holds

$$\mathcal{P}, \mathcal{Q} \in H(\Omega) \implies (\gamma_n) \in \ell^2(\Omega, \mathbb{Z}).$$

As mentioned above, the converse does not necessarily hold. However, a converse of this statement was given [5, 8] in the self-adjoint case, i.e., when $\mathcal{P} = \overline{\mathcal{Q}}$. Furthermore, another converse of this statement is shown in terms of sub-exponential weights and a slightly weaker result is obtained in terms of exponential weights in [19] for Dirac operators with skew-adjoint L^2-potentials. Similar results for Schrödinger operators were obtained in [5, 10, 11, 18].

For the non-self-adjoint case, there is a result in [5] as follows: Let us put

$$\Delta_n = |\lambda_n^+ - \lambda_n^-| + |\lambda_n^+ - \mu_n^{Dir}|,$$

then for each sub-multiplicative weight Ω

$$\mathcal{P}, \mathcal{Q} \in H(\Omega) \implies (\Delta_n)_{n\in\mathbb{Z}} \in \ell^2(\Omega).$$

Moreover, if $\Omega = \Omega(n)_{n\in\mathbb{Z}}$ is a sub-multiplicative weight such that $\log \Omega(n)/n \searrow 0$, then

$$(\Delta_n)_{n\in\mathbb{Z}} \in \ell^2(\Omega) \implies \mathcal{P}, \mathcal{Q} \in H(\Omega)$$

and if $\lim\limits_{n\to\infty} \log \Omega(n)/n > 0$, then

$$(\Delta_n)_{n\in\mathbb{Z}} \in \ell^2(\Omega) \implies \exists\, \epsilon > 0 : \mathcal{P}, \mathcal{Q} \in H(e^{\epsilon|n|}).$$

The proofs are constructed by means of a matrix (for large enough $|n|$)

$$\begin{pmatrix} \alpha_n(z) & \beta_n^+(z) \\ \beta_n^-(z) & \alpha_n(z) \end{pmatrix},$$

which has a very important property that a number $\lambda = n + z$ with $|z| < 1/2$ is a periodic (if n is even) or antiperiodic (if n is odd) eigenvalue if and only if z is an eigenvalue of the matrix (see Lemma 21, [5]). The four entries of the matrix depend analytically on z and V. They are given explicitly in terms of the Fourier coefficients of V. The deviations $|\gamma_n| + |\delta_n^{Dir}|$ are estimated (see Theorem 66 in [5]) by the functionals $\beta_n^{\mp}(z)$ as follows

$$\frac{1}{144}(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|) \leq |\gamma_n| + |\delta_n^{Dir}| \leq 54(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|),$$

where $z_n^* = (\lambda_n^+ + \lambda_n^-)/2 - n$. This shows the significance of these functionals by means of their asymptotic equivalence with the sequence $|\gamma_n| + |\delta_n^{Dir}|$.

The functionals $\alpha_n(z)$ and $\beta_n^{\mp}(z)$ are also crucial in analysing the Riesz basis property of the Dirac operator. P. Djakov and B. Mityagin [7] have proved that the following three claims are equivalent:

(a) The Dirac operator L given by (1.1) with a potential V in $L^2([0,\pi]) \times L^2([0,\pi])$ subject to periodic or antiperiodic boundary conditions has the Riesz basis property.

(b) $0 < \liminf_{\gamma_n \neq 0} \frac{|\beta_n^-(z_n^*)|}{|\beta_n^+(z_n^*)|}$ and $\limsup_{\gamma_n \neq 0} \frac{|\beta_n^-(z_n^*)|}{|\beta_n^+(z_n^*)|} < \infty$.

(c) $\sup_{\gamma_n \neq 0} \frac{|\delta_n^{Dir}|}{|\gamma_n|} < \infty$.

Similar results concerning Riesz basis property are known for Schrödinger operators (see [3, 7, 13]).

In this thesis are obtained new results on potential smoothness and Riesz basis property of one-dimensional Dirac operators.

Theorem 1.1. *If* $V \in L^2([0,\pi]) \times L^2([0,\pi])$, *then*

$$V \in H(\Omega) \implies (\Delta_n^{bc})_{n\in\mathbb{Z}} \in \ell^2(\Omega)$$

for each submultiplicative weight Ω, *where*

$$\Delta_n^{bc} = |\lambda_n^+ - \lambda_n^-| + |\lambda_n^+ - \mu_n^{bc}|.$$

Conversely, if $\Omega = (\Omega(n))_{n\in\mathbb{Z}}$ *is a submultiplicative weight such that* $\frac{\log \Omega(n)}{n} \searrow 0$, *then*

$$(\Delta_n^{bc})_{n\in\mathbb{Z}} \in \ell^2(\Omega) \implies V \in H(\Omega).$$

Furthermore, if Ω *is a submultiplicative weight such that* $\lim_{n\to\infty} \frac{\log \Omega(n)}{n} > 0$, *then*

$$(\Delta_n^{bc})_{n\in\mathbb{Z}} \in \ell^2(\Omega) \implies \exists\, \epsilon > 0 : \; V \in H(e^{\epsilon|n|}).$$

Theorem 1.2. *If* $V \in L^2([0,\pi]) \times L^2([0,\pi])$, *then the Dirac operator (1.1) with periodic or antiperiodic boundary conditions has the Riesz basis property if and only if*

$$\sup_{\gamma_n \neq 0} \frac{|\delta_n^{bc}|}{|\gamma_n|} < \infty$$

holds, where $\delta_n^{bc} = \lambda_n^+ - \mu_n^{bc}$.

Primarily, the following theorem is proven as a generalization of Theorem 66 in [5].

Theorem 1.3. *For* $n \in \mathbb{Z}$ *with large enough* $|n|$, *there are constants* $K_1 > 0$ *and* $K_2 > 0$, *such that*

$$K_1(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|) \leq |\lambda_n^+ - \lambda_n^-| + |\mu_n^{bc} - \lambda_n^+| \leq K_2(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|).$$

Theorem 1.1 and 1.2 are not proven directly. However, their proofs are reduced to the proofs of Theorem 66 in [5] and Theorem 24 in [7], respectively, in which we make use of Theorem 1.3 that gives the asymptotic equivalence of $|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|$ and $|\lambda_n^+ - \lambda_n^-| + |\mu_n^{bc} - \lambda_n^+|$.

CHAPTER 2

General Boundary Conditions

We consider the one-dimensional Dirac operator

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + V(x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

with a potential matrix

$$V(x) = \begin{pmatrix} 0 & \mathcal{P}(x) \\ \mathcal{Q}(x) & 0 \end{pmatrix}, \quad \mathcal{P}, \mathcal{Q} \in L^2([0,\pi]),$$

where $y \in dom(L) \subseteq L^2([0,\pi]) \times L^2([0,\pi])$.

A general boundary condition for the operator L is given by

$$\begin{aligned} a_1 y_1(0) + b_1 y_1(\pi) + a_2 y_2(0) + b_2 y_2(\pi) &= 0, \\ c_1 y_1(0) + d_1 y_1(\pi) + c_2 y_2(0) + d_2 y_2(\pi) &= 0, \end{aligned}$$

where a_i, b_i, c_i, d_i $(i = 1, 2)$ are complex numbers.

Let A_{ij} denote the square matrix whose first and second columns are the i^{th} and j^{th} columns of the matrix

$$\begin{bmatrix} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \end{bmatrix}$$

respectively and let $|A_{ij}|$ be the determinant of A_{ij}. If $|A_{14}| \neq 0$ and $|A_{23}| \neq 0$, then we say that the boundary condition given above is *regular*, if additionally $(|A_{13}|+|A_{24}|)^2 \neq 4|A_{14}||A_{23}|$ holds, it is called *strictly regular*.

Description of a family of special boundary conditions: Consider matrices of the form

$$A = \begin{bmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{bmatrix}, \tag{2.1}$$

where a, b, c, d are complex numbers. For every such matrix, the corresponding boundary condition $bc = bc(A)$ is given by

$$y_1(0) + by_1(\pi) + ay_2(0) = 0,$$
$$dy_1(\pi) + cy_2(0) + y_2(\pi) = 0. \tag{2.2}$$

We consider the family of all such boundary conditions that satisfy also

$$b + c = 0, \quad ad = 1 - b^2 \tag{2.3}$$

with restriction $ad \neq 0$.

We have to get a generalization of Dirichlet boundary condition. Dirichlet boundary condition is a strictly regular boundary condition. Therefore, we try to find general boundary conditions so that the eigenvectors coming from those boundary conditions geometrically behave like those derived from Dirichlet condition, so we seek to choose general boundary conditions among the strictly regular ones. The reason why we choose such a family of boundary conditions will be more clear in Lemma 2.2 except the condition $ad \neq 0$. This condition is necessary in order to control the asymptotic behaviour of the sequence $\beta_n^{\mp}(z^+)$, which will be obvious in the next sections.

2.1 Localization of the spectra

We give the localization of the spectra of Dirac operator subject to three types of boundary conditions which are the general boundary conditions defined by (2.2) and (2.3), periodic and antiperiodic boundary conditions defined as follows

$$\text{Periodic } (bc = Per^{+}): \qquad y(0) = y(\pi), \quad \text{i.e. } y_1(0) = y_1(\pi) \text{ and } y_2(0) = y_2(\pi);$$

$$\text{Antiperiodic } (bc = Per^{-}): \quad y(0) = -y(\pi), \ \text{ i.e. } y_1(0) = -y_1(\pi) \text{ and } y_2(0) = -y_2(\pi).$$

We consider $L_{Per^{\pm}}$ in the domain $dom(L_{Per^{\pm}})$, which consists of all absolutely continuous functions y such that $y' \in L^2([0,\pi]) \times L^2([0,\pi])$ and y satisfy $(Per^{\pm})$. We will write L for the operators $L_{Per^{+}}$ and $L_{Per^{-}}$ and L^0 for the free operators $L^0_{Per^{+}}$ and L^0_{Per-}. We denote by L_{bc} the Dirac operator with general boundary conditions $bc = bc(A)$, where A is given by (2.1), (2.2) and (2.3). Also, we write L^0_{bc} for the corresponding free operator. Furthermore, we consider L_{bc} in the domain $dom(L_{bc})$, which consists of all absolutely continuous functions y such that $y' \in L^2([0,\pi]) \times L^2([0,\pi])$ and y satisfies (bc).

The following theorem is about the localization of the eigenvalues of the Dirac operator $L_{bc} = L^0_{bc} + V$ under the given general boundary conditions defined by (2.1).

Theorem 2.1. *Let A be a matrix of the form (2.1), and let $bc = bc(A)$ be the corresponding boundary condition. If A satisfies (2.3), then the spectrum of the free operator L^0_{bc} is given by $sp(L^0_{bc}) = \mathbb{Z}$. Moreover, for $n \in \mathbb{Z}$ with large enough $|n|$, the disc $D_n = \{z \in \mathbb{C} : |z - n| < 1/2\}$ contains one simple eigenvalue $\mu_n = \mu_n(bc)$ of the operator L_{bc}.*

Proof. First, we consider the equation

$$L^0_{bc}\, y = \lambda y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

It can be written as

$$i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

so, its solution is

$$y = \begin{pmatrix} \xi e^{-i\lambda x} \\ \zeta e^{i\lambda x} \end{pmatrix}.$$

To satisfy the general boundary conditions given by (2.2), (ξ, ζ) must be a solution of the linear system

$$\xi(1 + bz^{-1}) + \zeta a = 0, \qquad (2.4)$$

$$\xi d z^{-1} + \zeta(c + z) = 0, \qquad (2.5)$$

where $z = e^{i\pi\lambda}$.

In order to have a non-zero solution (ξ, ζ) the determinant of the matrix

$$\begin{bmatrix} 1 + bz^{-1} & a \\ dz^{-1} & c + z \end{bmatrix}$$

whose entries are taken from (2.4) and (2.5) has to be zero. So, we obtain

$$z^2 + (b + c)z + bc - ad = 0. \tag{2.6}$$

Together with the restrictions (2.3), we have

$$z^2 = 1 \tag{2.7}$$

which gives $z = \mp 1$. Hence, one may conclude that $sp(L^0_{bc}) = \mathbb{Z}$ since the only solutions to the equation $\mp 1 = e^{i\pi\lambda}$ are integers. If we consider the boundary conditions $bc = bc(A)$ corresponding to the matrix (2.1) subject to the restrictions

$$b + c = 0, \qquad ad = 1 - b^2,$$

then we observe that

$$(|A_{13}| + |A_{24}|)^2 = (b + c)^2 = 0$$

and

$$4|A_{14}||A_{23}| = 4.(bc - ad) = 4.(-b^2 - (1 - b^2)) = -4.$$

This means that $(|A_{13}| + |A_{24}|)^2 \neq 4|A_{14}||A_{23}|$. Hence, $bc(A)$ satisfying (2.3) is strictly regular. The second result comes from Theorem 5.3 of [4] since $bc(A)$ is strictly regular.

□

We already mentioned in the previous section that we consider regular general boundary conditions (2.2) which satisfy (2.3). While knowing that Dirichlet boundary condition is strictly regular and that we try to generalize the Dirichlet boundary condition, one may observe the fact that the general boundary conditions (bc) given by (2.2) and (2.3) are also strictly regular as shown in the proof of Theorem 2.1.

Let us consider the Cauchy-Riesz projections associated with L_{bc}

$$P_{n,bc} = \int_{\partial D_n} (\lambda - L_{bc})^{-1} d\lambda$$

and

$$P^0_{n,bc} = \int_{\partial D_n} (\lambda - L^0_{bc})^{-1} d\lambda,$$

where $D_n = \{z \in \mathbb{C} : |z - n| < 1/2\}$ and ∂D_n is the boundary of D_n. We know that there is an $N \in \mathbb{N}$ such that for all $n \geq N$ the Cauchy-Riesz projections P_{bc} and P^0_{bc} are well defined and

$$dim(P_{bc}) = dim(P^0_{bc}) = 1$$

as well as

$$\lim_{n\to\infty} \|P_{n,bc} - P^0_{n,bc}\| = 0$$

due to Theorem 6.1 in [4].

Similarly, we have the Cauchy-Riesz projections P_n and P^0_n associated with Dirac operator L with periodic boundary conditions if n is even and antiperiodic boundary conditions if n is odd, where

$$dim(P_n) = dim(P^0_n) = 2$$

due to Theorem 18 in [5]. Furthermore,

$$\lim_{n\to\infty} \|P_n - P^0_n\| = 0$$

by Proposition 19 in [5] and for large enough $|n|$, the operator $L = L^0 + V$ has two eigenvalues λ^+_n and λ^-_n (which are periodic for even n and antiperiodic for odd n) such that $|\lambda^\pm_n - n| \leq 1/4$ as a result of Theorem 17 in [5].

2.2 The eigenvectors of the free operators

We want to find the eigenvectors of the free operator $(L_{bc}^0)^*$. In order to obtain these eigenvectors we have to get the adjoint boundary conditions (bc^*) of the general boundary conditions (bc). The adjoint boundary conditions are given by the matrix (see Lemma 3.4 in [4])

$$A^* = \begin{bmatrix} 1 & \tilde{b} & \tilde{a} & 0 \\ 0 & \tilde{d} & \tilde{c} & 1 \end{bmatrix}, \tag{2.8}$$

where

$$\begin{bmatrix} \tilde{b} & \tilde{a} \\ \tilde{d} & \tilde{c} \end{bmatrix} = \left(\begin{bmatrix} b & a \\ d & c \end{bmatrix}^{-1} \right)^*. \tag{2.9}$$

After a calculation by using (2.3) we find that

$$\tilde{a} = \overline{d}, \quad \tilde{b} = -\overline{c}, \quad \tilde{c} = -\overline{b}, \quad \tilde{d} = \overline{a}.$$

Therefore, the adjoint boundary conditions are given by the matrix

$$A^* = \begin{bmatrix} 1 & -\overline{c} & \overline{d} & 0 \\ 0 & \overline{a} & -\overline{b} & 1 \end{bmatrix}. \tag{2.10}$$

Furthermore, we observe that the adjoint boundary conditions are also in the family of general boundary conditions given by (2.2) and (2.3) since

$$\tilde{b} + \tilde{c} = 0 \quad \text{and} \quad \tilde{a}\tilde{d} = 1 - \tilde{b}^2. \tag{2.11}$$

Hence, by Theorem 2.1 we get that the eigenvalues of $(L_{bc}^0)^*$ are integers. As in the proof of Theorem 2.1, we see that every eigenvector is of the form

$$y = \begin{pmatrix} \xi e^{-i\lambda x} \\ \zeta e^{i\lambda x} \end{pmatrix},$$

with $\xi, \zeta \in \mathbb{C}$. Because the eigenvectors satisfy the boundary conditions (2.10) with (2.11), we obtain

$$\xi(1 - \overline{c}z^{-1}) + \zeta\overline{d} = 0, \tag{2.12}$$
$$\xi\overline{a}z^{-1} + \zeta(-\overline{b} + z) = 0, \tag{2.13}$$

where $z = e^{i\pi\lambda}$. Then, for $n \in 2\mathbb{Z}$, the corresponding eigenfunction is

$$\begin{pmatrix} (1 - \overline{b})e^{-inx} \\ -\overline{a}e^{inx} \end{pmatrix} \tag{2.14}$$

and for $n \in 2\mathbb{Z}+1$, the corresponding eigenfunction is

$$\begin{pmatrix} (1+\bar{b})e^{-inx} \\ -\bar{a}e^{inx} \end{pmatrix}. \tag{2.15}$$

Hence, all eigenfunctions corresponding to an eigenvalue $n \in \mathbb{Z}$ are in the form

$$\begin{pmatrix} A_n e^{-inx} \\ B_n e^{inx} \end{pmatrix}, \tag{2.16}$$

where A_n and B_n depend on the general boundary conditions and for convenience we choose

$$A_n = \frac{1-(-1)^n\bar{b}}{\sqrt{|a|^2 + |1-(-1)^n\bar{b}|^2}} \tag{2.17}$$

and

$$B_n = \frac{-\bar{a}}{\sqrt{|a|^2 + |1-(-1)^n\bar{b}|^2}}, \tag{2.18}$$

so that $|A_n|^2 + |B_n|^2 = 1$.

Since the adjoint boundary conditions are also in the family of general boundary conditions given by (2.2) and (2.3), as in the previous section we have

$$\lim_{n\to\infty} \|P_{n,bc^*} - P^0_{n,bc^*}\| = 0, \tag{2.19}$$

where

$$P_{n,bc^*} = \int_{\partial D_n} (\lambda - L_{bc^*})^{-1} d\lambda$$

and

$$P^0_{n,bc^*} = \int_{\partial D_n} (\lambda - L^0_{bc^*})^{-1} d\lambda.$$

Notice that $L^0_{bc^*} = (L^0_{bc})^*$ and $L_{bc^*} = (L_{bc})^*$ (see Lemma 3.4 in [4]).

Furthermore, the spectrum of the free operator L^0 subject to periodic boundary condition is $2\mathbb{Z}$ and each $n \in 2\mathbb{Z}$ is a double eigenvalue and the corresponding eigenvectors are

$$e^1_n(x) = \begin{pmatrix} e^{-inx} \\ 0 \end{pmatrix} \tag{2.20}$$

and

$$e^2_n(x) = \begin{pmatrix} 0 \\ e^{inx} \end{pmatrix}. \tag{2.21}$$

Similarly, the spectrum of the free operator L^0 subject to antiperiodic boundary condition is $2\mathbb{Z}+1$ and each $n \in 2\mathbb{Z}+1$ is a double eigenvalue and the corresponding eigenvectors are obtained by the same formulae (2.20) and (2.21). So, we may write

$$E_n^0 = Span\{e_n^1, e_n^2\}$$

for all $n \in \mathbb{Z}$. Moreover, we may also write

$$E_n = \text{Range}(P_n) \quad \text{and} \quad E_n^0 = \text{Range}(P_n^0)$$

for the eigenspaces of the operators L and L^0, respectively.

2.3 Estimates for $|\mu_n - \lambda_n^+|$

The Dirac operator $L = L^0 + V$ has two eigenvalues λ_n^+ and λ_n^- (periodic for even n and antiperiodic for odd n) in the disc centered at $n \in \mathbb{Z}$ of radius $1/4$ for large enough $|n|$ (Theorem 17 and Theorem 18 in [5]). We denote by λ_n^+ the eigenvalue with larger real part or the one with larger imaginary part if the real parts are equal and we put $\gamma_n = \lambda_n^+ - \lambda_n^-$.

From Lemma 59 given in [5], for sufficiently large $|n|$, there is a pair of vectors $f_n, \varphi_n \in E_n$ such that

1. $\|f_n\| = 1, \|\varphi_n\| = 1, \langle f_n, \varphi_n \rangle = 0$
2. $Lf_n = \lambda_n^+ f_n$
3. $L\varphi_n = \lambda_n^+ \varphi_n - \gamma_n \varphi_n + \xi_n f_n$

and

$$|\xi_n| \leq 4|\gamma_n| + 2\|(z_n^+ - S(\lambda_n^+))P_n\varphi_n\| \tag{2.22}$$

and

$$\|(z_n^+ - S(\lambda_n^+))P_n\varphi_n\| \leq 2(|\xi_n| + |\gamma_n|), \tag{2.23}$$

where $S(\lambda_n^+) : E_n^0 \to E_n^0$ is the operator defined in Lemma 21 of [5], and $z_n^+ = \lambda_n^+ - n$.

Now, let ℓ_0 and ℓ_1 be the functionals from $C([0,\pi]) \times C([0,\pi])$ to $\mathbb{C}$, defined as

$$\begin{aligned} \ell_0(s) &= s_1(0) + bs_1(\pi) + as_2(0), & (2.24)\\ \ell_1(s) &= ds_1(\pi) + cs_2(0) + s_2(\pi), & (2.25) \end{aligned}$$

where

$$s(x) = \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix}.$$

We start with a very crucial lemma which gives us the restrictions on those regular boundary conditions by which Dirichlet condition in [5] could be replaced. Furthermore, this will also lead to an equation that determines the way we estimate $|\mu_n - \lambda_n^+|$.

Lemma 2.2. *If $|n|$ is large enough, then there is vector $G_n \in E_n$ of the form*

$$G_n = s_n f_n + t_n \varphi_n, \quad \|G_n\| = |s_n|^2 + |t_n|^2 = 1 \tag{2.26}$$

such that

$$\ell_0(G_n) = 0, \quad \ell_1(G_n) = 0 \tag{2.27}$$

if and only if the general boundary conditions (2.2) satisfy (2.3).

Proof. It will be enough to prove that the system of linear equations

$$\begin{aligned}\ell_0(s_n f_n + t_n \varphi_n) &= 0 \qquad (2.28)\\ \ell_1(s_n f_n + t_n \varphi_n) &= 0 \qquad (2.29)\end{aligned}$$

has a non-trivial solution if and only if $b + c = 0$ and $ad = 1 - b^2$ hold.

Now, the system can be written as follows

$$\begin{aligned}s_n f_n^1(0) + t_n\varphi_n^1(0) + b(s_n f_n^1(\pi) + t_n\varphi_n^1(\pi)) + a(s_n f_n^2(0) + t_n\varphi_n^2(0)) &= 0\\ d(s_n f_n^1(\pi) + t_n\varphi_n^1(\pi)) + c(s_n f_n^2(0) + t_n\varphi_n^2(0)) + s_n f_n^2(\pi) + t_n\varphi_n^2(\pi) &= 0,\end{aligned}$$

where

$$f_n = \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} \quad \text{and} \quad \varphi_n = \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \end{pmatrix}.$$

Then, since f_n and φ_n satisfy periodic boundary conditions, we can reduce the above system to

$$\begin{pmatrix} (1+b)f_n^1(0) + af_n^2(0) & (1+b)\varphi_n^1(0) + a\varphi_n^2(0) \\ df_n^1(0) + (1+c)f_n^2(0) & d\varphi_n^1(0) + (1+c)\varphi_n^2(0) \end{pmatrix} \begin{pmatrix} s_n \\ t_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To have a non-trivial solution, the determinant must be zero, which is

$$[(1+b)(1+c) - ad].[f_n^1(0)\varphi_n^2(0) - f_n^2(0)\varphi_n^1(0)] = 0. \qquad (2.30)$$

In a similar way from the antiperiodic case of f and φ, we get

$$[(1-b)(1-c) - ad].[f_n^1(0)\varphi_n^2(0) - f_n^2(0)\varphi_n^1(0)] = 0. \qquad (2.31)$$

The equations (2.30) and (2.31) lead to the result after we prove that

$$[f_n^1(0)\varphi_n^2(0) - f_n^2(0)\varphi_n^1(0)] \neq 0$$

for large enough $|n|$. To show this we make use of the notation given in (2.41), (2.42) and (2.43). Then, by Remark 2.6 we have

$$\begin{aligned}f_n^1(0) &= f_{n,1}^0 + O(\kappa_n), \qquad (2.32)\\ f_n^2(0) &= f_{n,2}^0 + O(\kappa_n), \qquad (2.33)\\ \varphi_n^1(0) &= \varphi_{n,1}^0 + O(\kappa_n), \qquad (2.34)\\ \varphi_n^2(0) &= \varphi_{n,2}^0 + O(\kappa_n) \qquad (2.35)\end{aligned}$$

for some sequence (κ_n) converging to zero. Then, due to (2.57), (2.58) and $\|f_n^0\| = 1$ in (2.41) we obtain

$$\begin{aligned}
f_n^1(0)\varphi_n^2(0) - f_n^2(0)\varphi_n^1(0) &= [(f_{n,1}^0 + O(\kappa_n))(\varphi_{n,2}^0 + O(\kappa_n)) \\
&\quad - (f_{n,2}^0 + O(\kappa_n))(\varphi_{n,1}^0 + O(\kappa_n))] \\
&= f_{n,1}^0\varphi_{n,2}^0 - f_{n,2}^0\varphi_{n,1}^0 + O(\kappa_n) \\
&= f_{n,1}^0(-\overline{f_{n,1}^0} + O(\kappa_n)) - f_{n,2}^0(\overline{f_{n,2}^0} + O(\kappa_n)) + O(\kappa_n) \\
&= -(|f_{n,1}^0|^2 + |f_{n,2}^0|^2) + O(\kappa_n) \\
&= -1 + O(\kappa_n).
\end{aligned}$$

Hence, the result follows.

□

As seen by the proof of the previous lemma, we can write G_n as

$$G_n = \tau_n(\ell_0(\varphi_n)f_n - \ell_0(f_n)\varphi_n), \tag{2.36}$$

where

$$\tau_n = \frac{1}{\sqrt{|\ell_0(\varphi_n)|^2 + |\ell_0(f_n)|^2}}.$$

We also write $G_n = s_nf_n + t_n\varphi_n$, where $s_n = \tau_n\ell_0(\varphi_n)$ and $t_n = -\tau_n\ell_0(f_n)$. Now, since G_n is in the domain of L_{bc} and L, we can continue to write

$$\begin{aligned}
L_{bc}G_n &= LG_n = s_n.Lf_n + t_n.L\varphi_n = s_n\lambda_n^+f_n + t_n(\lambda_n^+\varphi_n - \gamma_n\varphi_n + \xi_nf_n) \\
&= \lambda_n^+(s_nf_n + t_n\varphi_n) + t_n(\xi_nf_n - \gamma_n\varphi_n) = \lambda_n^+G_n + t_n(\xi_nf_n - \gamma_n\varphi_n).
\end{aligned}$$

So, we have

$$LG_n = \lambda_n^+G_n + t_n(\xi_nf_n - \gamma_n\varphi_n). \tag{2.37}$$

Let $\tilde{g}_n$ be a unit eigenvector of the adjoint operator $(L_{bc})^*$ corresponding to the eigenvalue $\overline{\mu_n}$, where μ_n is the eigenvalue of L_{bc} in a circle with center n and radius 1/4.

Taking inner products of both sides of the equation (2.37) by $\tilde{g}_n$ we obtain

$$\langle LG_n, \tilde{g}_n\rangle = \lambda_n^+\langle G_n, \tilde{g}_n\rangle + t_n(\xi_n\langle f_n, \tilde{g}_n\rangle - \gamma_n\langle\varphi_n, \tilde{g}_n\rangle). \tag{2.38}$$

We also have

$$\langle LG_n, \tilde{g}_n\rangle = \langle L_{bc}G_n, \tilde{g}_n\rangle = \langle G_n, (L_{bc})^*\tilde{g}_n\rangle = \langle G_n, \overline{\mu_n}\tilde{g}_n\rangle = \mu_n\langle G_n, \tilde{g}_n\rangle. \tag{2.39}$$

The equality of (2.38) and (2.39) leads to

$$(\mu_n - \lambda_n^+)\langle G_n, \tilde{g}_n\rangle = t_n(\xi_n\langle f_n, \tilde{g}_n\rangle - \gamma_n\langle \varphi_n, \tilde{g}_n\rangle). \tag{2.40}$$

The equation (2.40) is crucial because the way we estimate $|\mu_n - \lambda_n^+|$ is determined by this equation. Indeed, our proof of the estimation for $|\mu_n - \lambda_n^+|$ will be based on the approximations for each remaining term in (2.40). In this section, we present the technical lemmas in order to obtain the necessary estimations.

Note that for large enough $|n|$, since $f_n \in E_n$ and P_n is a projection onto E_n we have $P_n f_n = f_n$. So,

$$\|P_n^0 f_n\| = \|P_n f_n - (P_n - P_n^0) f_n\| \geq \|f_n\| - \|P_n - P_n^0\| = 1 - \|P_n - P_n^0\|.$$

Since $\|P_n - P_n^0\|$ is sufficiently small we have that $P_n^0 f_n \neq 0$.

Now, we introduce notations for the projections of the eigenvectors of Dirac operator under periodic (or antiperiodic) boundary conditions and adjoint boundary conditions (bc^*) given by (2.10):

$$f_n^0 = \frac{P_n^0 f_n}{\|P_n^0 f_n\|}, \quad \varphi_n^0 = \frac{P_n^0 \varphi_n}{\|P_n^0 \varphi_n\|}, \quad \tilde{g}_n^0 = \frac{P_{n,bc^*}^0 \tilde{g}_n}{\|P_{n,bc^*}^0 \tilde{g}_n\|}. \tag{2.41}$$

Also, we may put

$$f_n^0 = f_{n,1}^0 e_n^1 + f_{n,2}^0 e_n^2 \tag{2.42}$$

and

$$\varphi_n^0 = \varphi_{n,1}^0 e_n^1 + \varphi_{n,2}^0 e_n^2. \tag{2.43}$$

Lemma 2.3. *There exists a sequence (κ_n) converging to zero such that*

$$\|g_n - \tilde{g}_n^0\| \leq \kappa_n, \quad \|f_n - f_n^0\| \leq \kappa_n, \ \textit{and} \quad \|\varphi_n - \varphi_n^0\| \leq \kappa_n$$

hold.

Proof. Observe that $P_{n,bc^*}\tilde{g}_n = \tilde{g}_n$ since P_{n,bc^*} is a projection onto the one-dimensional eigenspace generated by $\tilde{g}_n$. There exists a sequence $\kappa_n \to 0$ such that

$$\|P_{n,bc^*} - P_{n,bc^*}^0\| \leq \kappa_n$$

by (2.19). Now, we estimate $\|P_{n,bc^*}^0 \tilde{g}_n\|$ as

$$\begin{aligned}
\|P_{n,bc^*}^0 \tilde{g}_n\| &= \|P_{n,bc^*}^0 \tilde{g}_n - P_{n,bc^*}\tilde{g}_n + P_{n,bc^*}\tilde{g}_n\| \\
&\geq \|P_{n,bc^*}\tilde{g}_n\| - \|(P_{n,bc^*} - P_{n,bc^*}^0)\tilde{g}_n\| \\
&\geq \|\tilde{g}_n\| - \|P_{n,bc^*} - P_{n,bc^*}^0\| \\
&\geq 1 - \kappa_n
\end{aligned}$$

and

$$\begin{aligned}\|P^0_{n,bc^*}\tilde{g}_n\| &= \|P^0_{n,bc^*}\tilde{g}_n - P_{n,bc^*}\tilde{g}_n + P_{n,bc^*}\tilde{g}_n\| \\ &\le \|(P_{n,bc^*} - P^0_{n,bc^*})\tilde{g}_n\| + \|P_{n,bc^*}\tilde{g}_n\| \\ &\le \|P_{n,bc^*} - P^0_{n,bc^*}\| + \|\tilde{g}_n\| \\ &\le 1 + \kappa_n.\end{aligned}$$

So, we get that $|\|P^0_{n,bc^*}\tilde{g}_n\| - 1| \le \kappa_n$. Therefore,

$$\begin{aligned}\|\tilde{g}_n - g^0_n\| &\le \|\tilde{g}_n - P^0_{n,bc^*}\tilde{g}_n\| + \|P^0_{n,bc^*}\tilde{g}_n - g^0_n\| \\ &= \|P_{n,bc^*}\tilde{g}_n - P^0_{n,bc^*}\tilde{g}_n\| + \|\|P^0_{n,bc^*}\tilde{g}_n\|.g^0_n - g^0_n\| \\ &\le \|(P_{n,bc^*} - P^0_{n,bc^*})\tilde{g}_n\| + |\|P^0_{n,bc^*}\tilde{g}_n\| - 1|.\|g^0_n\| \\ &\le \kappa_n + \kappa_n = 2\kappa_n.\end{aligned}$$

Similarly we get the other inequalities. □

Lemma 2.4. $|\lambda^+_n - n| \to 0$.

Proof. Consider the eigenfunctions e^1_n and e^2_n of the free operator with periodic condition and antiperiodic condition. Now, we have

$$\lambda^+_n \langle f_n, e^1_n\rangle = \langle Lf_n, e^1_n\rangle = \langle L^0 f_n, e^1_n\rangle + \langle Vf_n, e^1_n\rangle$$

and recalling that L^0 is self-adjoint

$$\langle L^0 f_n, e^1_n\rangle = \langle f_n, L^0 e^1_n\rangle = \langle f_n, n e^1_n\rangle = n\langle f_n, e^1_n\rangle.$$

From these two equalities, we can write

$$(\lambda^+_n - n)\langle f_n, e^1_n\rangle = \langle Vf_n, e^1_n\rangle.$$

In a similar way, we get

$$(\lambda^+_n - n)\langle f_n, e^2_n\rangle = \langle Vf_n, e^2_n\rangle.$$

The last two equalities lead to

$$|\lambda^+_n - n|^2(|\langle f_n, e^1_n\rangle|^2 + |\langle f_n, e^2_n\rangle|^2) = |\langle Vf_n, e^1_n\rangle|^2 + |\langle Vf_n, e^2_n\rangle|^2. \tag{2.44}$$

Now,

$$\langle f_n, e^1_n\rangle = \langle f_n - f^0_n, e^1_n\rangle + \langle f^0_n, e^1_n\rangle.$$

Since

$$|\langle f_n - f^0_n, e^1_n\rangle| \le \|f_n - f^0_n\| \le \kappa_n,$$

by Lemma 2.3, we have

$$\langle f_n, e_n^1\rangle = f_{n,1}^0 + O(\kappa_n).$$

A similar argument gives

$$\langle f_n, e_n^2\rangle = f_{n,2}^0 + O(\kappa_n).$$

We obtain that

$$|\langle f_n, e_n^1\rangle|^2 + |\langle f_n, e_n^2\rangle|^2 \to |f_{n,1}^0|^2 + |f_{n,2}^0|^2 = 1$$

as $n \to \infty$.

Now, if we consider the equation (2.44), it is clear that for large enough $|n|$

$$|\lambda_n^+ - n|^2 \leq 2(|\langle Vf_n, e_n^1\rangle|^2 + |\langle Vf_n, e_n^2\rangle|^2). \tag{2.45}$$

We obtain an estimation for the first term of the right hand side of the above inequality as follows

$$\begin{aligned}
\langle Vf_n, e_n^1\rangle &= \langle V(f_{n,2}^0 e_n^2), e_n^1\rangle + \langle V(f_n - f_{n,2}^0 e_n^2), e_n^1\rangle \\
&= \frac{f_{n,2}^0}{\pi}\int_0^\pi \mathcal{P}(x)e^{-2inx}dx + \langle f_n - f_n^0, V^* e_n^1\rangle + \langle f_n^0 - f_{n,2}^0 e_n^2, V^* e_n^1\rangle.
\end{aligned}$$

Since

$$\langle f_n^0 - f_{n,2}^0 e_n^2, V^* e_n^1\rangle = \langle f_{n,1}^0 e_n^1, V^* e_n^1\rangle = 0$$

and

$$|\langle f_n - f_n^0, V^* e_n^1\rangle| \leq \|f_n - f_n^0\|.\|V^* e_n^1\| \leq \kappa_n.\|\mathcal{P}\|,$$

we have

$$|\langle Vf_n, e_n^1\rangle| \leq |p(n)| + \|\mathcal{P}\|.\kappa_n, \tag{2.46}$$

and in a similar way, we can get

$$|\langle Vf_n, e_n^2\rangle| \leq |q(-n)| + \|\mathcal{Q}\|.\kappa_n, \tag{2.47}$$

where $p(n)$ and $q(n)$ are the Fourier coefficients of $\mathcal{P}$ and $\mathcal{Q}$. Hence, from (2.45),(2.46) and (2.47) we get $|\lambda_n^+ - n| \to 0$ since the Fourier coefficients $p(n)$ and $q(n)$ tend to zero. □

The next proposition gives estimates for $|\ell_i(f_n - f_n^0)|$ and $|\ell_i(\varphi_n - \varphi_n^0)|$, $i = 0, 1$. The technique used in the proof of the proposition is based on a method developed by A. Batal (see Proposition 2.9 in [1] and Proposition 10 in [2]).

Proposition 2.5. *In the notation used above, there exists a sequence (κ_n) such that $\kappa_n \to 0$ and*

$$|\ell_0(f_n - f_n^0)| \leq \kappa_n, \tag{2.48}$$

$$|\ell_0(\varphi_n - \varphi_n^0)| \leq \kappa_n. \tag{2.49}$$

Proof. To obtain a clear notation, we fix and suppress the notation n for the eigenvectors and put

$$f = f_n, \quad f^0 = f_n^0, \quad \varphi^0 = \varphi_n^0$$

and

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad f^0 = \begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \varphi^0 = \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix}.$$

Now, it will be enough to find a sequence (κ_n) converging to zero such that

$$|f_i(0) - f_i^0(0)| \leq \kappa_n \tag{2.50}$$

and

$$|\varphi_i(0) - \varphi_i^0(0)| \leq \kappa_n \tag{2.51}$$

for $i = 1, 2$.

We have $Lf = \lambda_n^+ f$, and $i(f_1^0)' = nf_1^0$. Subtracting these equations, we obtain

$$n(f_1 - f_1^0) = i(f_1 - f_1^0)' + \mathcal{P} f_2 - z_n^+ f_1,$$

where $z_n^+ = \lambda_n^+ - n$.

Now, we assume that n is even, then we have periodic eigenfunctions in the equation. We multiply both sides by $e^{i(n+1)x}$ and apply integration by parts on the first term of the right side. Then, we obtain

$$2i(f_1 - f_1^0)(0) = I_1 + I_2 + I_3, \tag{2.52}$$

where

$$I_1 = -\int_0^\pi e^{i(n+1)x}(f_1 - f_1^0)(x)dx, \quad I_2 = \int_0^\pi e^{i(n+1)x}\mathcal{P}(x) f_2(x) dx$$

and

$$I_3 = -\int_0^\pi z_n^+ e^{i(n+1)x} f_1(x) dx.$$

For I_1, by Cauchy-Schwarz inequality and Lemma 2.3 we have

$$|I_1| \leq \|f_1 - f_1^0\| \leq \|f - f^0\| \leq \kappa_n.$$

To estimate I_2 recall that $f_2^0 = C^0 e^{inx}$ for some constant C^0. Since $\|f^0\| = 1$, we get $|C^0| \leq 1$. Then, we have

$$\begin{aligned} |I_2| &\leq \int_0^\pi |e^{i(n+1)x}\mathcal{P}(x)(P_n - P_n^0)(f_2)(x)|dx + |\int_0^\pi e^{i(n+1)x}\mathcal{P}(x) f_2^0(x)dx| \\ &\leq \|\mathcal{P}\|.\|(P_n - P_n^0)\| + |C^0 \int_0^\pi e^{i(n+1)x} e^{inx}\mathcal{P}(x)dx| \end{aligned}$$

The last term is a Fourier coefficient of the L^2-function $\mathcal{P}(x)$ which tends to zero as $n \to \infty$.

To obtain similar result for I_3, we immediately see that $|I_3| \le |z_n^+| = |\lambda_n^+ - n|$ and $z_n^+ \to 0$ as $n \to \infty$ due to Lemma 2.4. Hence, by (2.52) we get that $|(f_1 - f_1^0)(0)|$ tends to zero.

Estimation method for $(f_2 - f_2^0)(0)$ can be continued by multiplying both sides of the equation

$$n(f_2 - f_2^0) = -i(f_2 - f_2^0)' + Qf_1 - z_n^+ f_2$$

by $e^{-i(n+1)x}$, and all the remaining argument is similar. So, this proves (2.48).

Now, we recall that

$$L\varphi = \lambda^+\varphi - \gamma_n\varphi + \xi_n f$$

and

$$L^0\varphi^0 = n\varphi^0.$$

We can subtract the second equation from the first equation and write the equation of the first components

$$i(\varphi_1 - \varphi_1^0)' + \mathcal{P}\varphi_2 = n(\varphi_1 - \varphi_1^0) + z_n^+\varphi_1 - \gamma_n\varphi_1 + \xi_n f_1. \tag{2.53}$$

After multiplying both sides of (2.53) by $e^{i(n+1)x}$, we integrate and write

$$-2i(\varphi_1 - \varphi_1^0)(0) = J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$J_1 = -\int_0^\pi e^{i(n+1)x}(\varphi_1 - \varphi_1^0)(x)dx, \quad J_2 = -\int_0^\pi e^{i(n+1)x}\mathcal{P}(x)\varphi_2(x)dx,$$

$$J_3 = z_n^+\int_0^\pi e^{i(n+1)x}\varphi_1(x)dx, \quad J_4 = -\gamma_n\int_0^\pi e^{i(n+1)x}\varphi_1(x)dx$$

and

$$J_5 = \xi_n\int_0^\pi e^{i(n+1)x}f_1(x)dx.$$

The estimations for J_1, J_2 and J_3 are very similar to those for I_1, I_2 and I_3 respectively.

Lemma 40 together with Proposition 35 in [5] gives that $\gamma_n \to 0$, additionally Lemma 59, Lemma 60 and Proposition 35 in [5] imply that $\xi_n \to 0$ as n goes to infinity. So, J_4 and J_5 are also dominated by a sequence converging to zero. Hence, $|(\varphi_1 - \varphi_1^0)(0)|$ tends to zero.

To estimate $|(\varphi_2 - \varphi_2^0)(0)|$, we follow similar calculations using $e^{-i(n+1)x}$ instead of $e^{i(n+1)x}$. Furthermore, the way we prove the result for the case when n is odd is also similar. $\square$

Remark 2.6. *In view of (2.50) and (2.51)*

$$|f_n^i(0) - f_{n,i}^0| < \kappa_n \quad \textit{and} \quad |\varphi_n^i(0) - \varphi_{n,i}^0| < \kappa_n \tag{2.54}$$

for $i = 1, 2$, where $\kappa_n \to 0$.

Due to the arguments on pages 699 and 700 in [5], there are functionals $\alpha_n(V;z)$ and $\beta_n^{\pm}(V;z)$ defined for large enough $|n|$, $n \in \mathbb{Z}$ and $|z| < 1/2$ such that $\lambda = n + z$ is (periodic if n is even or antiperiodic if n is odd) eigenvalue of L if and only if z is an eigenvalue of the matrix

$$\begin{pmatrix} \alpha_n(V;z) & \beta_n^-(V;z) \\ \beta_n^+(V;z) & \alpha_n(V;z) \end{pmatrix}.$$

Furthermore, $z_n^{\pm} = \lambda_n^{\pm} - n$ are the only solutions of the basic equation

$$(z - \alpha_n(V;z))^2 = \beta_n^-(V;z)\beta_n^+(V;z),$$

where $|z| < 1/2$.

By Lemma 21 in [5], if $Lf_n = \lambda_n^+ f_n$ then f_n^0 is an eigenvector of the operator $L^0 + S(\lambda_n^+) : E_n^0 \to E_n^0$ with a corresponding eigenvalue λ_n^+, and one may write the following system

$$\begin{pmatrix} z_n^+ - \alpha_n(z_n^+) & -\beta_n^-(z_n^+) \\ -\beta_n^+(z_n^+) & z_n^+ - \alpha_n(z_n^+) \end{pmatrix} \begin{pmatrix} f_{n,1}^0 \\ f_{n,2}^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.55}$$

So, we get

$$(z_n^+ - \alpha_n(z_n^+))^2 = \beta_n^-(z_n^+).\beta_n^+(z_n^+). \tag{2.56}$$

We put

$$\varphi_n^0 = \varphi_{n,1}^0 e_n^1 + \varphi_{n,2}^0 e_n^2.$$

Let

$$\varphi_n^0 = c_1 f_n^0 + c_2 (f_n^0)^{\perp},$$

where

$$(f_n^0)^{\perp} = \overline{f_{n,2}^0} e_n^1 - \overline{f_{n,1}^0} e_n^2.$$

Then,

$$c_1 = \langle \varphi_n^0, f_n^0 \rangle = \langle \varphi_n^0 - \varphi_n, f_n \rangle + \langle \varphi_n^0, f_n^0 - f_n \rangle = O(\kappa_n),$$

and

$$|c_2| = \sqrt{1 - |c_1|^2} = 1 + O(\kappa_n).$$

Hence, without loss of generality we may write

$$\varphi_{n,1}^0 = \overline{f_{n,2}^0} + O(\kappa_n) \tag{2.57}$$

and

$$\varphi^0_{n,2} = -\overline{f^0_{n,1}} + O(\kappa_n). \tag{2.58}$$

We have $\tilde{g}^0_n = e^{i\theta} g^0_n$ for some θ, indeed, without loss of generality, we can put $g^0_n = \tilde{g}^0_n$ because the eigenspace of the free operator under general boundary conditions is one dimensional as stated in Theorem 2.1.

The following two equations are due to Proposition 2.5:

$$\ell_0(f_n) = \ell_0(f^0_n) + O(\kappa_n) \tag{2.59}$$

$$\ell_0(\varphi_n) = \ell_0(\varphi^0_n) + O(\kappa_n) \tag{2.60}$$

By Lemma 2.3, we also obtain other estimations as

$$\begin{aligned}
|\langle f_n, \tilde{g}_n\rangle - \langle f^0_n, g^0_n\rangle| &\leq |\langle f_n - f^0_n, \tilde{g}_n\rangle| + |\langle f^0_n, \tilde{g}_n - g^0_n\rangle| \\
&\leq \|f_n - f^0_n\| . \|\tilde{g}_n\| + \|f^0_n\| . \|\tilde{g}_n - g^0_n\| \\
&\leq 2\kappa_n,
\end{aligned}$$

where $\kappa_n \to 0$. Similarly, we get

$$|\langle \varphi_n, \tilde{g}_n\rangle - \langle \varphi^0_n, g^0_n\rangle| \leq 2\kappa_n.$$

Hence, one may write

$$\langle f_n, \tilde{g}_n\rangle = \langle f^0_n, g^0_n\rangle + O(\kappa_n) \tag{2.61}$$

and

$$\langle \varphi_n, \tilde{g}_n\rangle = \langle \varphi^0_n, g^0_n\rangle + O(\kappa_n). \tag{2.62}$$

Now, let us write g^0_n as

$$g^0_n = A_n e^1_n + B_n e^2_n.$$

Together with (2.59) and (2.60), we have

$$\ell_0(\varphi_n) = (1 + (-1)^n b)\varphi^0_{n,1} + a\varphi^0_{n,2} + O(\kappa_n)$$

and

$$\ell_0(f_n) = (1 + (-1)^n b) f^0_{n,1} + a f^0_{n,2} + O(\kappa_n).$$

So, we conclude that

$$\ell_0(\varphi_n) = (1 + (-1)^n b)\overline{f^0_{n,2}} - a\overline{f^0_{n,1}} + O(\kappa_n) \tag{2.63}$$

$$\ell_0(f_n) = (1 + (-1)^n b) f^0_{n,1} + a f^0_{n,2} + O(\kappa_n) \tag{2.64}$$

$$\langle \varphi_n, \tilde{g}_n\rangle = \overline{A_n f^0_{n,2}} - \overline{B_n f^0_{n,1}} + O(\kappa_n) \tag{2.65}$$

$$\langle f_n, \tilde{g}_n\rangle = \overline{A_n} f^0_{n,1} + \overline{B_n} f^0_{n,2} + O(\kappa_n). \tag{2.66}$$

We now get a nonzero approximation for $\langle G_n, \tilde{g}\rangle$.

Lemma 2.7. *$\tau_n^{-1}\langle G_n, \tilde{g}_n\rangle = C + O(\kappa_n)$ for some constant $C \neq 0$, where C depends on the general boundary conditions given by (2.3) with the restriction $ad \neq 0$.*

Proof. Recall (2.36) as

$$\tau_n^{-1}\langle G_n, \tilde{g}_n\rangle = \ell_0(\varphi_n)\langle f_n, \tilde{g}_n\rangle - \ell_0(f_n)\langle \varphi_n, \tilde{g}_n\rangle.$$

We substitute all the estimations found by (2.63), (2.64), (2.65) and (2.66) into the equation above, and after an easy calculation, obtain

$$\begin{aligned}
\tau_n^{-1}\langle G_n, \tilde{g}_n\rangle &= [(1 + (-1)^n b)\overline{f^0_{n,2}} - a\overline{f^0_{n,1}} + O(\kappa_n)].[\overline{A_n} f^0_{n,1} + \overline{B_n} f^0_{n,2} + O(\kappa_n)] \\
&\quad - [(1 + (-1)^n b) f^0_{n,1} + a f^0_{n,2} + O(\kappa_n)].[\overline{A_n f^0_{n,2}} - \overline{B_n f^0_{n,1}} + O(\kappa_n)] \\
&= [(1 + (-1)^n b)\overline{B_n} - a\overline{A_n}]|f^0_{n,1}|^2 + [(1 + (-1)^n b)\overline{B_n} - a\overline{A_n}]|f^0_{n,2}|^2 + O(\kappa_n) \\
&= (1 + (-1)^n b)\overline{B_n} - a\overline{A_n} + O(\kappa_n).
\end{aligned}$$

If we consider the one-dimensional eigenvectors of the general boundary conditions which have the coefficients given by (2.17) and (2.18), we have

$$\begin{aligned}
\tau_n^{-1}\langle G_n, \tilde{g}_n\rangle &= (1 + (-1)^n b)\overline{B_n} - a\overline{A_n} + O(\kappa_n) \\
&= (1 + (-1)^n b)\frac{\overline{-\overline{a}}}{\sqrt{|a|^2 + |1 - (-1)^n b|^2}} - a\frac{\overline{1 - (-1)^n\overline{b}}}{\sqrt{|a|^2 + |1 - (-1)^n b|^2}} + O(\kappa_n) \\
&= \frac{-2a}{\sqrt{|a|^2 + |1 - (-1)^n b|^2}} + O(\kappa_n).
\end{aligned}$$

So, the result follows because $a \neq 0$. □

Proposition 2.8. *There are constants $D_1, D_2 > 0$ such that for $n \in \mathbb{Z}$ with large enough $|n|$,*

$$|\mu_n - \lambda_n^+| \leq D_1|\gamma_n| + D_2(|B_n^+| + |B_n^-|),$$

where $B_n^\pm = \beta_n^\pm(z_n^+)$.

Proof. We consider the equation (2.40). We assume that $0 < \kappa_n \leq \frac{|C|}{2}$. We multiply both sides of the equation (2.40) by τ_n^{-1} and get

$$\tau_n^{-1}(\mu_n - \lambda_n^+)\langle G_n, \tilde{g}_n\rangle = \tau_n^{-1} t_n(\xi_n\langle f_n, \tilde{g}_n\rangle - \gamma_n\langle \varphi_n, \tilde{g}_n\rangle). \tag{2.67}$$

Lemma 2.7 guarantees that we may divide both sides of (2.67) by $\tau_n^{-1}\langle G_n, \tilde{g}_n\rangle$. We also have the inequality

$$|\xi_n| \leq 4|\gamma_n| + 2(|B_n^+| + |B_n^-|) \tag{2.68}$$

due to Lemma 59 and Lemma 60 in [5].

Note that since $|A_n|^2 + |B_n|^2 = 1$ and $|f_{n,1}^0|^2 + |f_{n,2}^0|^2 = 1$, by Cauchy-Schawrz inequality $|\overline{A_n f_{n,2}^0} - \overline{B_n f_{n,1}^0}| \leq 1$ and $|\overline{A_n} f_{n,1}^0 + \overline{B_n} f_{n,2}^0| \leq 1$. Then, we obtain the following inequality by using the estimations (2.63), (2.65), (2.66) and (2.68):

$$\begin{aligned}
|\mu_n - \lambda_n^+| &= \frac{|\tau_n^{-1}||t_n(\xi_n\langle f_n, \tilde{g}_n\rangle - \gamma_n\langle \varphi_n, \tilde{g}_n\rangle)|}{|\tau_n^{-1}||\langle G_n, \tilde{g}_n\rangle|} \\
&= \frac{|\ell_0(f_n)||(\xi_n\langle f_n, \tilde{g}_n\rangle - \gamma_n\langle \varphi_n, \tilde{g}_n\rangle)|}{|\tau_n^{-1}||\langle G_n, \tilde{g}_n\rangle|} \\
&\leq \frac{(|\ell_0(f_n^0)| + \kappa_n)[|\xi_n|(|\overline{A_n} f_{n,1}^0 + \overline{B_n} f_{n,2}^0| + \kappa_n) + |\gamma_n|(|\overline{A_n f_{n,2}^0} - \overline{B_n f_{n,1}^0}| + \kappa_n)]}{|C| - \kappa_n} \\
&\leq \left(\frac{|(1 + (-1)^n b) f_{n,1}^0 + a f_{n,2}^0| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)|\xi_n| \\
&+ \left(\frac{|(1 + (-1)^n b) f_{n,1}^0 + a f_{n,2}^0| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)|\gamma_n|.
\end{aligned}$$

So, we get

$$\begin{aligned}
|\mu_n - \lambda_n^+| &\leq \left(\frac{1 + |b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)|\xi_n| + \left(\frac{|1 + b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)|\gamma_n| \\
&\leq \left(\frac{1 + |b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)(4|\gamma_n| + 2(|B_n^+| + |B_n^-|)) \\
&+ \left(\frac{1 + |b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)|\gamma_n| \\
&= D_1|\gamma_n| + D_2(|B_n^+| + |B_n^-|),
\end{aligned}$$

where

$$D_1 = 5 \cdot \left(\frac{1 + |b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|)$$

and

$$D_2 = 2 \cdot \left(\frac{1 + |b| + |a| + \frac{|C|}{2}}{\frac{|C|}{2}}\right)(1 + |C|).$$

Apparently, D_1 and D_2 depend on the general boundary conditions. $\square$

2.4 Estimation for $|\mu_n - \lambda_n^+| + |\lambda_n^+ - \lambda_n^-|$

We start with a generalized version of Proposition 63 in [5].

Proposition 2.9. *Let $M > 1$ be a fixed number, then for $n \in Z$ with sufficiently large $|n|$ if*

$$\frac{1}{M}|B_n^-| \leq |B_n^+| \leq M|B_n^-| \tag{2.69}$$

then

$$|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)| \leq \frac{1+M}{\sqrt{M}}|\gamma_n|,$$

where $B_n^\pm = \beta_n^\pm(z_n^+)$ and $z_n^* = (\lambda_n^+ + \lambda_n^-)/2 - n$ in the case of simple eigenvalues and $z_n^* = \lambda_n^+ - n$ otherwise.

Proof. We mainly follow the proof of Propositon 63 in [5].

The case $B_n^+ = B_n^- = 0$ is explained in the proof of Proposition 63 in [5]. Assume $B_n^+ B_n^- \neq 0$ and $\gamma_n \neq 0$. Since $t_n = \frac{|B_n^+|}{|B_n^-|} \in [\frac{1}{M}, M]$, we have

$$\sqrt{M} - \sqrt{t_n} \geq \frac{\sqrt{M} - \sqrt{t_n}}{\sqrt{t_n}.\sqrt{M}}$$

because $1 \geq \frac{1}{\sqrt{t_n M}}\cdot$ Then, we get

$$\frac{1}{\sqrt{M}} + \sqrt{M} \geq \frac{1}{\sqrt{t_n}} + \sqrt{t_n}$$

which leads to

$$\frac{2\sqrt{t_n}}{1+t_n} \geq \frac{2\sqrt{M}}{1+M}.$$

In Lemma 49 in [5] we take

$$\delta_n < \frac{\sqrt{M}}{1+M}$$

for sufficiently large $|n|$, then

$$\begin{aligned}
|\gamma_n| &\geq (\frac{2\sqrt{t_n}}{1+t_n} - \delta_n)(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|) \\
&\geq (\frac{2\sqrt{M}}{1+M} - \frac{\sqrt{M}}{1+M})(|\beta_n^-(z_n^*)| + |\beta_n^+(z_n^*)|),
\end{aligned}$$

which gives us the result.

Now, if $B_n^+ B_n^- \neq 0$ and $\gamma_n = 0$, then z_n^+ is the only root of the equation (2.56) or zero of the function

$$h_n(z) = (\zeta_n(z))^2 - \beta_n^+(z)\beta_n^-(z)$$

in the disc $D = \{z : |z| < 1/8\}$, where $\zeta_n(z) = z - \alpha_n(z)$. The functions h_n is analytic on D since $\beta_n^\mp(z)$ and $\alpha_n(z)$ are analytic funtions on D (see Proposition 28 in [5]). We also have

$$|h_n(z) - z^2| = |z^2 + \alpha_n(z)^2 - 2z\alpha_n(z) - \beta_n^-(z)\beta_n^+(z) - z^2| = |\alpha_n(z)^2 - 2z\alpha_n(z) - \beta_n^-(z)\beta_n^+(z)|.$$

By Proposition 35 in [5], the maximum values of $|\alpha_n(z)|$ and $|\beta_n^{\mp}(z)|$ on the boundary of D converge to zero as $|n| \to \infty$, hence we may write for sufficiently large $|n|$

$$\sup_{\partial D} |h_n(z) - z^2| < \sup_{\partial D} |z^2|.$$

By Rouché's theorem, z^+ is a double root of the equation $h_n(z) = 0$ which leads to $h_n'(z^+) = 0$, so the following holds

$$2\zeta_n(z^+) \cdot (1 - \frac{d\alpha_n}{dz}(z^+)) = \frac{d\beta_n^+}{dz}(z^+) \cdot \beta_n^-(z^+) + \beta_n^+(z^+) \cdot \frac{d\beta_n^-}{dz}(z^+).$$

If we consider the upper bounds

$$\left|\frac{d\alpha_n}{dz}(z^+)\right| \leq \frac{1}{\sqrt{M}(M+1)+1}, \quad \left|\frac{d\beta_n^{\pm}}{dz}(z^+)\right| \leq \frac{1}{\sqrt{M}(M+1)+1}$$

for sufficiently large $|n|$ by Proposition 35 in [5], then triangle inequality gives

$$2|\zeta_n(z^+)|(1 - \frac{1}{\sqrt{M}(M+1)+1}) \leq \frac{1}{\sqrt{M}(M+1)+1}(|B_n^+| + |B_n^-|),$$

where $|\zeta_n(z^+)| = \sqrt{|B_n^+ B_n^-|}$ by the basic equation (2.56). Hence, we have

$$2(\sqrt{M}(M+1)) \leq \frac{|B_n^+| + |B_n^-|}{\sqrt{|B_n^+ B_n^-|}} \leq \frac{\sqrt{M}(M+1)|B_n^-|}{\sqrt{|B_n^- B_n^-|}}$$

by (2.69) and the above inequality, which gives a contradiction $2 \leq 1$. Hence, the proof is complete. □

Now, we consider the complementary cases

$$(i)\ M|B_n^+| < |B_n^-|, \quad (ii)\ M|B_n^-| < |B_n^+|. \tag{2.70}$$

Lemma 2.10. *If (i) in (2.70) is true and $|n|$ is sufficiently large, then the followings hold*

$$|f_{n,2}^0| \leq \frac{1}{\sqrt{M+1}},$$

$$|f_{n,1}^0| \geq \frac{\sqrt{M}}{\sqrt{M+1}},$$

$$|\varphi_{n,2}^0| > \frac{\sqrt{M}}{\sqrt{M+1}}(1 - \kappa_n),$$

$$|\varphi_{n,1}^0| < \frac{1}{\sqrt{M+1}} + 4\sqrt{\kappa_n},$$

where (κ_n) is a sequence of positive numbers such that $\kappa_n \to 0$.

Proof. We follow the proof of Lemma 64 in [5].

Let $M > 1$ and suppose $M|B_n^+| < |B_n^-|$. By (2.55), we have the followings

$$\xi_n^+ f_{n,1}^0 + B_n^- f_{n,2}^0 = 0, \quad (\xi_n^+)^2 = B_n^- B_n^+,$$

which gives

$$|B_n^-||f_{n,2}^0| = |\xi^+||f_{n,1}^0| = \sqrt{|B_n^- B_n^+|}|f_{n,1}^0|.$$

Then, we obtain

$$|f_{n,2}^0| = \frac{\sqrt{|B_n^+|}}{\sqrt{|B_n^-|}}.|f_{n,1}^0| \le \frac{1}{\sqrt{M}}|f_{n,1}^0|$$

and

$$(M+1)|f_{n,2}^0|^2 \le |f_{n,1}^0|^2 + |f_{n,2}^0|^2 = 1,$$

so

$$|f_{n,2}^0| \le \frac{1}{\sqrt{M+1}}. \tag{2.71}$$

And also we get a lower bound for $|f_{n,1}^0|$

$$1 = |f_{n,1}^0|^2 + |f_{n,2}^0|^2 \le |f_{n,1}^0|^2 + \frac{|f_{n,1}^0|^2}{M} = \frac{M+1}{M}|f_{n,1}^0|^2$$

hence

$$|f_{n,1}^0| \ge \frac{\sqrt{M}}{\sqrt{M+1}} \tag{2.72}$$

Now, we need to find bounds for $|\varphi_{n,1}^0|$ and $|\varphi_{n,2}^0|$. Noting that $f_n \perp \varphi_n$, we obtain

$$|\langle f_n^0, \varphi_n^0\rangle| \le |\langle f_n^0 - f_n, \varphi_n^0\rangle| + |\langle f_n, \varphi_n^0 - \varphi_n\rangle| \le \|f_n - f_n^0\|\|\varphi_n^0\| + \|f_n\|\|\varphi_n - \varphi_n^0\| \le 2\kappa_n$$

since $\|f_n - f_n^0\| \le \kappa_n$ and $\|\varphi_n - \varphi_n^0\| \le \kappa_n$ by Lemma 2.3. On the other hand,

$$|\langle f_n^0, \varphi_n^0\rangle| = |f_{n,1}^0\overline{\varphi_{n,1}^0} + f_{n,2}^0\overline{\varphi_{n,2}^0}| \le 2\kappa_n,$$

which leads to

$$|f_{n,1}^0\varphi_{n,1}^0| \le |f_{n,2}^0\varphi_{n,2}^0| + 2\kappa_n.$$

Then, by (2.71) and (2.72),

$$|\varphi_{n,1}^0| \le \frac{|f_{n,2}^0|}{|f_{n,1}^0|}|\varphi_{n,2}^0| + \frac{2\kappa_n}{|f_{n,1}^0|} \le \frac{|\varphi_{n,2}^0|}{\sqrt{M}} + \frac{2\kappa_n\sqrt{M+1}}{\sqrt{M}}$$

and

$$\begin{aligned}
1 = |\varphi_{n,1}^0|^2 + |\varphi_{n,2}^0|^2 &\le \frac{|\varphi_{n,2}^0|^2}{M} + \frac{4\kappa_n^2(M+1)}{M} + \frac{4|\varphi_{n,2}^0|\kappa_n\sqrt{M+1}}{M} + |\varphi_{n,2}^0|^2 \\
&\le \frac{|\varphi_{n,2}^0|^2}{M} + 4\kappa_n\frac{\kappa_n(M+1) + \sqrt{M+1}}{M} + |\varphi_{n,2}^0|^2 \\
&\le \frac{|\varphi_{n,2}^0|^2}{M} + 4\kappa_n\frac{\kappa_n(M+M) + M + M}{M} + |\varphi_{n,2}^0|^2 < \frac{|\varphi_{n,2}^0|^2}{M} + 16\kappa_n + |\varphi_{n,2}^0|^2.
\end{aligned}$$

Now, we have

$$|\varphi_{n,2}^0| > \frac{\sqrt{M}}{\sqrt{M+1}}\sqrt{1-16\kappa_n}. \tag{2.73}$$

On the other hand,

$$1 = |\varphi_{n,1}^0|^2 + |\varphi_{n,2}^0|^2 \geq |\varphi_{n,1}^0|^2 + \frac{M}{M+1}(1-16\kappa_n),$$

which leads to

$$\frac{1+16M\kappa_n^2}{M+1} \geq |\varphi_{n,1}^0|^2.$$

Since we have $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$ for any $x, y \geq 0$, we may write as follows

$$\frac{1}{\sqrt{M+1}} + \frac{4\sqrt{M}\sqrt{\kappa_n}}{\sqrt{M+1}} \geq |\varphi_{n,1}^0|,$$

which gives

$$|\varphi_{n,1}^0| < \frac{1}{\sqrt{M+1}} + 4\sqrt{\kappa_n}. \tag{2.74}$$

□

An analogy of the previous lemma can be given for the case (ii) in (2.70) which has a very similar proof.

Lemma 2.11. *If (ii) in (2.70) is true and $|n|$ is sufficiently large, then the followings hold*

$$|f_{n,1}^0| \leq \frac{1}{\sqrt{M+1}},$$

$$|f_{n,2}^0| \geq \frac{\sqrt{M}}{\sqrt{M+1}},$$

$$|\varphi_{n,1}^0| > \frac{\sqrt{M}}{\sqrt{M+1}}(1-\kappa_n),$$

$$|\varphi_{n,2}^0| < \frac{1}{\sqrt{M+1}} + 4\sqrt{\kappa_n},$$

where (κ_n) is a sequence of positive numbers such that $\kappa_n \to 0$.

The next lemma gives an estimation for the ratio $\frac{|\ell_0(f_n)|}{|\ell_0(\varphi_n)|}$ in the two cases (i) and (ii) in (2.70).

Lemma 2.12. *Suppose $b \neq \pm 1$. If $|n|$ is sufficiently large and one of the cases (i) and (ii) in (2.70) is true for*

$$M > max\{4(|a|/|1+b|)^2, 4(|a|/|1-b|)^2, 4(|1+b|/|a|)^2, 4(|1-b|/|a|)^2\}, \tag{2.75}$$

then there are constants $D_3 > 0$ and $D_4 > 0$ such that

$$D_3 < \frac{|\ell_0(f_n)|}{|\ell_0(\varphi_n)|} < D_4 \tag{2.76}$$

holds.

Proof. We mainly follow the proof of Lemma 64 in [5].

Suppose the case (i) in (2.70) is true. Now, in order to get the inequality (2.76) we have to find lower bounds and upper bounds for both $|\ell_0(f_n)|$ and $|\ell_0(\varphi_n)|$. We easily get upper bounds as

$$|\ell_0(f_n)| \le |\ell_0(f_n^0)| + \kappa_n \le 1 + |b| + |a| + \kappa_n \tag{2.77}$$

and

$$|\ell_0(\varphi_n)| \le |\ell_0(\varphi_n^0)| + \kappa_n \le 1 + |b| + |a| + \kappa_n \tag{2.78}$$

by (2.59) and (2.60).

We continue to get lower bounds for $|\ell_0(f_n)|$ and $|\ell_0(\varphi_n)|$ by using the results coming from Lemma 2.10 as follows

$$\begin{aligned}
|\ell_0(f_n)| &\ge |\ell_0(f_n^0)| - \kappa_n \\
&= |(1+(-1)^n b) f_{n,1}^0 + a f_{n,2}^0| - \kappa_n \\
&\ge ||(1+(-1)^n b) f_{n,1}^0| - |a f_{n,2}^0|| - \kappa_n \\
&\ge |(1+(-1)^n b)||f_{n,1}^0| - |a||f_{n,2}^0| - \kappa_n \\
&\ge |(1+(-1)^n b)|\frac{\sqrt{M}}{\sqrt{M+1}} - |a|\frac{1}{\sqrt{M+1}} - \kappa_n \\
&= \frac{|(1+(-1)^n b)|\sqrt{M} - |a|}{\sqrt{M+1}} - \kappa_n .
\end{aligned}$$

Then, since $M > 4(|a|/|1 \mp b|)^2$ by (2.75) we get

$$|\ell_0(f_n)| \ge \frac{|a|}{\sqrt{M+1}} - \kappa_n .$$

We also obtain the inequality

$$\begin{aligned}
|\ell_0(\varphi_n)| &\ge |\ell_0(\varphi_n^0)| - \kappa_n \\
&= |(1+(-1)^n b)\varphi_{n,1}^0 + a\varphi_{n,2}^0| - \kappa_n \\
&\ge ||(1+(-1)^n b)\varphi_{n,1}^0| - |a\varphi_{n,2}^0|| - \kappa_n \\
&\ge |a|.|\varphi_{n,2}^0| - |(1+(-1)^n b)|.|\varphi_{n,1}^0| - \kappa_n \\
&\ge |a|\frac{\sqrt{M}}{\sqrt{M+1}}(1-\kappa_n) - |(1+(-1)^n b|(\frac{1}{\sqrt{M+1}} + 4\sqrt{\kappa_n}) - \kappa_n .
\end{aligned}$$

Since $M > 4(|1 \mp b|/|a|)^2$ by (2.75) we get

$$|\ell_0(\varphi_n)| \ge \frac{|1+(-1)^n b|}{\sqrt{M+1}} - (|a|\frac{\sqrt{M}}{\sqrt{M+1}} + 1)\kappa_n + 4|1+(-1)^n b|\sqrt{\kappa_n} .$$

In case when n is even, we have

$$|\ell_0(f_n)| \geq \frac{|a|}{2\sqrt{M+1}}$$

and

$$|\ell_0(\varphi_n)| \geq \frac{|1+b|}{2\sqrt{M+1}}$$

for sufficiently large $|n|$. Hence, by (2.77) and (2.78) we may conclude that

$$\frac{\frac{|a|}{2\sqrt{M+1}}}{1+|b|+|a|+\kappa_n} \leq \frac{|\ell_0(f_n)|}{|\ell_0(\varphi_n)|} \leq \frac{1+|b|+|a|+\kappa_n}{\frac{|1+b|}{2\sqrt{M+1}}}$$

which leads to (2.76) with

$$D_3 = \frac{\frac{|a|}{2\sqrt{M+1}}}{1+|b|+|a|+\kappa_n}, \qquad D_4 = \frac{1+|b|+|a|+\kappa_n}{\frac{|1+b|}{2\sqrt{M+1}}}.$$

Similar result holds in the case when n is odd. Also, the proof for the case (ii) in (2.70) is the same as the proof for the case (i) in (2.70).

□

We give an analogue of Proposition 65 in [5] with its similar proof.

Proposition 2.13. *If (i) or (ii) in (2.70) holds for*

$$M > max\{4(|1-b|/|a|)^2, 4(|1+b|/|a|)^2, 4(|a|/|1-b|)^2, 4(|a|/|1+b|)^2\},$$

where $b \neq \mp 1$ (or equivalently $ad \neq 0$), then there are constants $D_8 > 0$ and $D_9 > 0$ such that

$$|B_n^+| + |B_n^-| \leq D_8|\gamma_n| + D_9|\mu_n - \lambda_n^+| \tag{2.79}$$

holds.

Proof. We prove the cases when n is even and n is odd simultanously. Assume the case (i) in (2.70) for the given M in the hypothesis. We know that $\ell_0(G_n) = 0$ where $G_n = s_n f_n + t_n\varphi_n$, so $|\ell_0(f_n)|/|\ell_0(\varphi_n)| = |t_n|/|s_n|$ and by Lemma 2.12

$$0 < D_3 \leq |t_n|/|s_n| \leq D_4$$

and

$$1 = |s_n|^2 + |t_n|^2 \leq \frac{|t_n|^2}{D_3^2} + |t_n|^2,$$

so we obtain

$$|t_n| \geq \sqrt{\frac{D_3^2}{1+D_3^2}} > 0. \tag{2.80}$$

Furthermore, the inequality $M|B_n^+| < |B_n^-|$ and its consequence with Lemma 2.10 give the following estimation to get a lower bound for $|\langle f_n, \tilde{g}_n\rangle|$ by making use of (2.66) and the coefficients in (2.17) and (2.18) as

$$\begin{aligned}
|\langle f_n, \tilde{g}_n\rangle| &\geq |\langle f_n^0, g_n^0\rangle| - \kappa_n \\
&= |\overline{A_n} f_{n,1}^0 + \overline{B_n} f_{n,2}^0| - \kappa_n \\
&\geq \frac{|1-(-1)^n b|}{\sqrt{|a|^2 + |1-(-1)^n b|^2}} |f_{n,1}^0| - \frac{|a|}{\sqrt{|a|^2 + |1-(-1)^n b|^2}} |f_{n,2}^0| - \kappa_n \\
&\geq \frac{|1-(-1)^n b|}{\sqrt{|a|^2 + |1-(-1)^n b|^2}} \cdot \frac{\sqrt{M}}{\sqrt{M+1}} - \frac{|a|}{\sqrt{|a|^2 + |1-(-1)^n b|^2}} \cdot \frac{1}{\sqrt{M+1}} - \kappa_n \\
&\geq \frac{|a|}{\sqrt{|a|^2 + |1-(-1)^n b|^2}.\sqrt{M+1}} - \kappa_n.
\end{aligned}$$

So, for large enough $|n|$ we get

$$|\langle f_n, \tilde{g}_n\rangle| \geq D_5 > 0 \tag{2.81}$$

where

$$D_5 = min\left\{ \frac{|a|}{2\sqrt{|a|^2 + |1-b|^2}.\sqrt{M+1}} \, , \, \frac{|a|}{2\sqrt{|a|^2 + |1+b|^2}.\sqrt{M+1}} \right\}.$$

Similarly, the case (ii) in (2.70) also gives the existence of a positive lower bound for $|\langle f_n, \tilde{g}_n\rangle|$.

Now, we consider the equation (2.40). First, we note that $t_n = -\tau_n.\ell_0(f_n)$ and $|\ell_0(f_n)| \geq |\ell_0(f_n^0)| - \kappa_n \geq C_0 - \kappa_n$, where C_0 is a positive number depending on general boundary conditions and $\kappa_n \leq C_0/2$. We also have that $|\langle f_n, \tilde{g}_n\rangle| \geq D_5 > 0$. Hence, we may divide both sides of the equality (2.40) by $\tau_n^{-1} t_n \langle f_n, \tilde{g}_n\rangle$ and get that

$$|\xi| = \left| \frac{\tau_n^{-1}(\mu_n - \lambda_n^+)\langle G_n, \tilde{g}_n\rangle + \tau_n^{-1} t_n \gamma_n \langle \varphi_n, \tilde{g}_n\rangle}{\tau_n^{-1} t_n \langle f_n, \tilde{g}_n\rangle} \right|.$$

Then, due to (2.65), (2.80), (2.81) we have

$$\begin{aligned}
|\xi_n| &= \left| \frac{\tau_n^{-1}(\mu_n - \lambda_n^+)\langle G_n, \tilde{g}_n\rangle + \tau_n^{-1} t_n \gamma_n \langle \varphi_n, \tilde{g}_n\rangle}{\tau_n^{-1} t_n \langle f_n, \tilde{g}_n\rangle} \right| \\
&\leq \frac{|\mu_n - \lambda_n^+|(|C|+1) + |\gamma_n|.|\ell_0(f_n)|.|\overline{A_n f_{n,2}^0} - \overline{B_n f_{n,1}^0} + 1|}{|-\ell_0(f_n)|.|\langle f_n, \tilde{g}_n\rangle|} \\
&\leq \frac{|\mu_n - \lambda_n^+|(|C|+1) + 3.|\gamma_n|.(1 + |b| + |a| + 1)}{\frac{C_0}{2} D_5}.
\end{aligned}$$

So, we obtain

$$|\xi_n| \leq D_6|\mu_n - \lambda_n^+| + D_7|\gamma_n|, \tag{2.82}$$

where

$$D_6 = \frac{|C|+1}{\frac{C_0}{2} D_5}$$

and
$$D_7 = \frac{3.(2+|b|+|a|)}{\frac{C_0}{2}D_5}.$$

We also have
$$\frac{1}{2}(|B_n^+|+|B_n^-|) \le \|(z_n^+ - S(\lambda_n^+))P_n^0\varphi\| \tag{2.83}$$
for sufficiently large $|n|$ by Lemma 60 in [5]. Moreover, we have
$$\|(z_n^+ - S(\lambda_n^+))P_n^0\varphi\| \le 2(|\xi_n|+|\gamma_n|) \tag{2.84}$$
for sufficiently large enough $|n|$ by Lemma 59 in [5]. Hence, by (2.82), (2.83) and (2.84) we obtain
$$|B_n^+|+|B_n^-| \le 4|\xi_n|+4|\gamma_n| \le D_8|\mu_n-\lambda_n^+|+D_9|\gamma_n|,$$
where $D_8 = 4D_6$ and $D_9 = 4D_7+4$. This completes the proof. □

In the above results, the analogues of Proposition 62, Proposition 63 and Proposition 65 in [5] (which are used in the proof of Theorem 66 in [5]) are given as Proposition 2.8, Proposition 2.9 and Proposition 2.13, respectively. Hence, as in Theorem 66 in [5] we get the proof of Theorem 1.3 which claims that for $n \in \mathbb{Z}$ with large enough $|n|$ there are constants $K_1 > 0$ and $K_2 > 0$ such that
$$K_1(|\beta_n^-(z_n^*)|+|\beta_n^+(z_n^*)|) \le |\lambda_n^+-\lambda_n^-|+|\mu_n^{bc}-\lambda_n^+| \le K_2(|\beta_n^-(z_n^*)|+|\beta_n^+(z_n^*)|).$$

Theorem 1.1 and Theorem 1.2 now follow from Theorem 1.3 due to the asymptotical equivalence of the sequence $(|\beta_n^+(z_n^*)|+|\beta_n^-(z_n^*)|)$ with each of the sequences (Δ_n) and (Δ_n^{bc}).

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