A SURVEY OF PLURISUBHARMONIC FUNCTIONS

by

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Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Degree

Sabancı University
January 2015

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DATE OF APPROVAL: 05.01.2015

He, who knows nothing, loves nothing. He, who can do nothing, understands nothing. He who understands nothing is worthless. But he who understands also loves, notices, sees.... The more knowledge is inherent in a thing, the greater the love ... Anyone who imagines that all fruits ripen at the same time as the strawberries knows nothing about grapes.

PARACELSUS.

A Survey of Plurisubharmonic Functions

Çiğdem Çelik

Mathematics, Master Thesis, 2015

Thesis Supervisor: Prof. Dr. Aydın Aytuna

Keywords: Subharmonic function, Dirichlet problem, plurisubharmonic function, Gluing lemma, Fusion lemma, maximal plurisubharmonic function, Zaharyuta-Siciak extremal function, pluricomplex green function

Abstract

Plurisubharmonic functions have been introduced by Lelong and Oka in 1942, play a major role in the theory of several complex variables. The richness of their properties and, most importantly, their close connection with holomorphic functions have assured these functions a indelible place in several variables.
In this thesis, we present a survey of plurisubharmonic functions as one of the generalization of subharmonic functions. Thereby, in the first part of this study, after giving a short brief of topological notions, we recall the main definitions and theorems of subharmonic functions in one dimensional case. In the rest parts, we focus on multidimensional case and we aim to give the main principles of the theory of plurisubharmonic functions.

Çoklu-altharmonik Fonksiyonlar Üzerine Bir Derleme

Çiğdem Çelik

Matematik, Yüksek Lisans Tezi, 2015

Tez Danışmanı: Prof. Dr. Aydın Aytuna

Anahtar Kelimeler: Altharmonik fonksiyon, Dirichlet problemi, çokluealtharmonik fonksiyon, Gluing lemma, Fusion lemma, maksimal çoklu-altharmonik fonksiyon, green fonksiyonu

Özet

1942 senesinde Lelong ve Oka tarafından tanımlanan çoklu-altharmonik fonksiyonlar çok kompleks değişkenli teoride büyük bir role sahiptir. Zengin özelliklerin oluşu ve en önemlisi anatilik fonksiyonlarla olan yakın ilişkisi, bu fonksiyonlarn çok değişkenli teoride hatırı sayılır bir yere sahip oluşunu garanti altına almaktadır.
Bu tezde, tek değişkenli altharmonik fonksiyonların bir genellemesi olarak, çoklu-altharmonik fonksiyonlar üzerine bir derleme sunduk. Bu bağlamda, çalışmanın ilk kısmında, gerekli topolojik kavramlar hakkında kısa bir özet verdikten sonra, bir kompleks değişkenli altharmonik fonksiyonlarn temel tanım ve teoremlerini hatırlattık. Diğer bölümlerde ise çok değişkenli duruma odaklanarak, çoklu-altharmonik fonksiyonların ana prensiplerini vermeyi amaçladık.

ACKNOWLEDGEMENTS

Foremost, I would like to express my deepest gratitude to my thesis advisor Prof. Aydın Aytuna for his support. This study would not have been possible without his patience and encouragement.

I also would like to express my sincere thanks and best wishes to Prof. Vyacheslav Zaharyuta. He has encouraged me with the confidence in his shining eyes. It is really a pleasure for me to get acquainted with him.

I thank Mert Çağlar for getting me interested in analysis when I was an undergraduate student.

A big thank Gamze, Nazlı, Elif, Buket, Sibel and Berkay for their mental support, friendship and sincerity.

Finally, thanks my family for being to support in my life.

Contents

Introduction

Plurisubharmonic functions have been introduced independently by Lelong[32] and Oka[41] while studying properties of domain of holomorphy and the entire and meromorphic functions, in France and Japan, respectively.

Oka's paper was received by *Tôhoku Mathematical Journal* on October 25, 1941 and published in May, 1942. Lelong's definition emerged in a note in the *Comptes Rendus* presented on November 3, 1942. Even if Oka's definition is in two variables, it agrees with the one in use even since. The generalization to n arose in [42]
Lelong defined a plurisubharmonic function as one that take finite values or minus infinity and is bounded from above on relatively compact domains. The function is not allowed to be minus infinity identically unlike Oka's definition and he did not impose the upper semicontinuity. In the common definition today upper semicontinuity is imposed, just as Oka did.

In this thesis, we present a survey of plurisubharmonic functions which appear in complex analysis as logarithms of moduli of holomorphic functions and analogues of potentials and play a major role in the theory of functions of several complex variable. It is our ambition to set the foundation of the theory of plurisubharmonic functions. and, in particular, to make this text comprehensible for who is in the primary stage of the study of several complex variable.

For the convenience of the reader, we devote the prologue to the topological preliminaries and present the main tools and theorems of subharmonic functions in one dimensional case that will be used throughout this study.

In the first chapter, we start with the elementary properties of plurisubharmonic functions in $\mathbb{C}^n$ as a generalization of subharmonic functions in $\mathbb{C}$. The framework of this part is based on [43]. Then we continue with polar and pluripolar sets that have been defined by Lelong [33] and [34] and have a remarkable importance in pluripotential theory. Lastly, we mention the relation between convex functions in $\mathbb{R}^n$ and plurisubharmonic functions in $\mathbb{C}^n$. This relation is quite natural since the plurisubharmonic functions are in many ways analogous to convex functions. Indeed they relate to subharmonic functions of one complex variable as convex functions of several variables do to convex functions of one real variable.

In the second chapter, we focus on construction new plurisubharmonic functions. Accordingly, we give some fruitful instruments that can be used for that purpose.

In the last part of this study, we take a look at the theory of maximal plurisubharmonic functions, or, roughly, the plurisubharmonic solutions to the complex Monge-Ampère equation $(dd^c u)^n = 0$ where u is a plurisubharmonic function. Finally, we present extremal functions as most important examples of maximal plurisubharmonic functions.

Prologue

0.1 Some Topological Preliminaries

0.1.1 Upper semicontinuous Functions

Definition 1. Let X be topological space. We say that a function $u : X \to [-\infty, \infty)$ is *upper semicontinuous* if the set $\{x \in X : u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$.
Also $v : X \to (-\infty, \infty]$ is called *lower semicontinuous* if $-v$ is upper semicontinuous.

Clearly, one can show that u is upper semicontinuous if and only if

$$\limsup_{y \to x} u(y) := \inf_{\delta > 0} \sup_{0 < \|y - x\| \leq \delta} u(y) \leq u(x) \quad (x \in X).$$

We shall make frequent use of following basic properties of upper semicontinuous functions.

Proposition 0.1.1. *[44]*

(i) If u, v are upper semicontinuous functions and $\lambda \geq 0$, then $u + v, \lambda u$ and $\max\{u, v\}$ are upper semicontinuous functions.

(ii) If (u_i) is a collection of upper semicontinuous functions, then $\inf_i u_i$ is upper semicontinuous. In particular, if $u_1 \geq u_2 \geq \ldots$ then $u(x) := \lim_n u_n(x)$ is upper semicontinuous.

(iii) *If u be an upper semicontinuous function on a metric space (X, d), and u is bounded above on X, then there exists continuous functions $\phi_n : X \to \mathbb{R}$ such that $\phi_1 \geq \phi_2 \geq \ldots \geq u$ on X and $\lim_{n\to\infty} \phi_n = u$*

(iv) *If u be an upper semicontinuous functions on a topological space X, and K be a compact subset of X, then u is bounded above on K and attains its bound.*

Proof. [44]

(i) Let $\alpha \in \mathbb{R}$ and $Y = \{x : u(x) + v(x) < \alpha\}$. To show that Y is open, let $x_0 \in Y$ and $\varepsilon = \alpha - u(x_0) - v(x_0) > 0$. Then $U = \{x : u(x) < u(x_0) + \frac{\varepsilon}{2}\}$ and $V = \{x : v(x) < v(x_0) + \frac{\varepsilon}{2}\}$ are neighbourhoods of x_0 and hence $U \cap V$ is a neighbourhood of x_0 contained of Y.
Next, $\{x : \lambda u(x) < \alpha\} = \{x : u(x) < \frac{\alpha}{\lambda}\}$ for all $\lambda \geq 0$ and $\alpha \in \mathbb{R}$ hence λu is upper semicontinuous if $\lambda \geq 0$.
Furthermore, $\{x : \max\{u(x), v(x)\} < \alpha\} = \{x : u(x) < \alpha\} \cap \{v(x) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$.

(ii) If $\alpha \in \mathbb{R}$, then $\{x : \inf_i u_i(x) < \alpha\} = \cup_i \{x : u_i(x) < \alpha\}$ is open.

(iii) We can suppose that $u \not\equiv -\infty$ (otherwise just take $\phi_n \equiv -n$). For $n \geq 1$, define $\phi_n : X \to \mathbb{R}$ by

$$\phi_n(x) = \sup_{y \in X} (u(y) - nd(x, y)) \quad (x \in X).$$

Then for each n we have

$$|\phi_n(x) - \phi_n(x')| \leq nd(x, x') \quad (x, x' \in X),$$

so ϕ_n is continuous on X. Clearly, also $\phi_1 \geq \phi_2 \geq \ldots \geq u$. So in particular $\lim_{n\to\infty} \phi_n \geq u$. Writing $B(z, \rho)$ for the ball $\{y \in X : d(x, y) \leq \rho\}$, we have

$$\phi_n \leq \max\left(\sup_{B(z,\rho)} u, \sup_X u - n\rho\right) \quad (x \in X, \rho > 0).$$

As u is upper semicontinuous and bounded from above, letting $\rho \to 0$ gives $\lim_{n\to\infty} \phi_n \leq u$.

(iv) The sets $\{x \in X : u(x) < n\} \quad (n \geq 1)$ form an open cover of K, so have a finite subcover. Hence u is bounded from above on K. Let $M = \sup_K u$. Then the open sets $\{x \in X : u(x) < M - 1/n\} \quad (n \geq 1)$ cannot cover K, because they have no finite subcover. Hence $u(x) = M$ at least one $x \in K$.

□

Let X be a topological space and let $(u_\alpha)_{\alpha\in I}$ be a family of upper semicontinuous functions $X \to [-\infty, \infty)$. We assume that (u_α) is locally bounded from above. Then the upper envelope

$u(x) := \sup_\alpha\{u_\alpha(x)\}$

need not to be upper semicontinuous, so we may consider its *upper semicontinuous regularization*:

$$u^*(z) := \lim_{\varepsilon\to 0} \sup_{w\in B(z,\varepsilon)} u(w).$$

It is easy to check that $u^*(z) \geq u(z)$ and u^* is the smallest upper semicontinuous function which is greater than u. Our goal is to show that u^* can be computed with a countable subfamily of (u_α).

Lemma 0.1.1. ***(Choquet's Lemma)*** *[9]. Let X be a second countable metric space. If (u_α) is a family of upper semicontinuous functions defined on X, then it has a countable subfamily $(v_j) = \left(u_{\alpha(j)}\right)$ whose upper envelope satisfies $v \leq u \leq u^* = v^*$.*

Proof. [9] Let $B(z_j, \varepsilon_j)$ be a countable basis of the topology of X. For each j, let (z_{jk}) be a sequence of points in $B(z_j, \varepsilon_j)$ such that

$$\sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u,$$

and for each pair (j,k), let $\alpha(j,k,l)$ be a sequence of indices $\alpha \in I$ such that $u(z_{jk}) = \sup_l u_{\alpha(j,k,l)}(z_{jk})$. Set

$$v = \sup_{j,k,l} u_{\alpha(j,k,l)}.$$

Then $v \le u$ and $v^* \le u^*$. On the other hand

$$\sup_{B(z_j,\varepsilon_j)} u \ge \sup_k v(z_{jk}) \ge \sup_{k,l} u_{\alpha(j,k,l)}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u.$$

As every ball $B(z,\varepsilon)$ is a union of balls $B(z_j,\varepsilon_j)$, we easily conclude that $v^* \ge u^*$, hence $v^* = u^*$. □

0.1.2 Partition of Unity

Now, we introduce a tool of extreme importance in analysis.

Theorem 0.1.1. *[54] Let $A \subset \mathbb{R}^n$ and let $\mathscr{O}$ be an open cover of A. Then there is a collection of Φ of C^∞ functions φ defined in an open set containing A, with the following properties;*

(i) For each $x \in A$ we have $0 \le \varphi(x) \le 1$.

(ii) For each $x \in A$ there is an open set V containing x such that all but finitely many $\varphi \in \Phi$ are 0 on V.

(iii) For each $x \in A$ we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

(iv) For each $\varphi \in \Phi$ there is an open set U in $\mathscr{O}$ such that $\varphi = 0$ outside of some closed set containing in U.

A collection Φ satisfying (i) to (iii) is called C^∞- *partition of unity* for A. If Φ also satisfies (iv), it is said to said to be *subordinate* to the cover $\mathscr{O}$.

Proof. [54] **Case 1**. A is compact.
For each point $x \in A$ choose balls $C_x \subset \bar{C}_x \subset B_x$ such that $x \in C_x$ and $B_x \subset U_\alpha$ for some α. By compactness, finitely many of the balls C_x suffice to cover A; call them $C_1, C_2, \ldots, C_n$.

Having constructed the sets $\bar{C}_1, \ldots, \bar{C}_n$, let ψ_i be a nonnegative C^∞ function which is positive on $\bar{C}_i$ and 0 outside of some closed set contained in U_i (see [54], p. 29). Since $\{\bar{C}_1, \ldots, \bar{C}_n\}$ covers A, we have that $\psi_1 + \ldots + \psi_n > 0$ for all x in some open set U containing A. Define the function φ_i on U as

$$\varphi_i(x) = \frac{\psi_i(x)}{\psi_1 + \ldots + \psi_n}.$$

If $f : U \to [0,1]$ is a C^∞ function which is 1 on A and 0 outside of some closed set in U, then $\Phi = \{f \cdot \varphi_1, \ldots, f \cdot \varphi_n\}$ is the desired partition of unity.

Case 2. $A = \bigcup_i A_i$, where each A_i is compact and $A_i \subset A^0_{i+1}$.

For each i let $\mathscr{O}_i$ consist of all $U \cap (A^0_{i+1} \setminus A_{i-2})$ for U in $\mathscr{O}$. Then $\mathscr{O}_i$ is an open cover of the compact set $B_i = A_i \setminus A^0_{i-1}$. By case 1 there is a partition of unity for Φ_i for B_i, subordinate to $\mathscr{O}_i$. For each $x \in A$ and each i the sum

$$\sigma(x) = \sum_{\varphi \in \Phi_i} \varphi(x)$$

is a finite sum in some open set containing x, since $x \in A_i$ we have $\varphi(x) = 0$ for $\varphi \in \Phi_j$ with $j \geq i + 2$. For each φ in each Ψ_i, define $\varphi' = \frac{\varphi(x)}{\sigma(x)}$. The collection of all φ' is the desired partition of unity.
Case 3 A is open
Let

$$A_i = \{x \in A : |x| \leq i \quad d(x, \partial A) \geq 1/i\},$$

and apply case 2.
Case 4 A is arbitrary

Let B be the union of all U in $\mathscr{O}$. By case 3 there is a partition of unity for B; this is also a partition of unity for A.

$\square$

0.1.3 Smoothing

Now we will give some smoothing theorems. Let Ω be an open subset of $\mathbb{R}^n$ and for $r > 0$ define

$$\Omega_r := \{x \in \Omega : d(x, \Omega^c) > r\}.$$

Let $u : \Omega \to [-\infty, \infty)$ be a locally integrable function, and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $\mathrm{supp}(\phi) \subset B(0, r)$. Then their *convolution* is the function

$$u * \phi(x) = \int_{\mathbb{C}} u(x - w)\phi(w)dV(w) \quad x \in \Omega_r.$$

After a change of variable we also have

$$u * \phi(x) = \int_{\mathbb{C}} u(w)\phi(x - w)dV(w) \quad x \in \Omega_r.$$

This shows that if $\phi \in C^\infty$, then also $u * \phi \in C^\infty$ since we can differentiate under the integral sign arbitrarily many times. Now we shall show that convolutions are useful in smoothing of functions.
Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by the formula

$$\mathcal{H}(\Omega)(t) := \begin{cases} exp(-1/t) & \text{for } t > 0 \\ 0, & \text{for } t \leq 0. \end{cases}$$

Then it is easy to see that $h \in \infty(\mathbb{R})$. Now define $\chi : \mathbb{R}^n \to \mathbb{R}$ so that

$$\chi(x) = h(1 - ||x||^2)/K$$

where

$$K = (\int_{B(0,1)} h(1 - ||x||^2)dV(x))$$

and

$$\chi \in C^\infty, \quad \chi \geq 0, \quad \text{supp}\chi \subset B(0,1), \quad \int_{\mathbb{C}} \chi dV = 1.$$

For $\varepsilon > 0$ define

$$\chi_\varepsilon = \frac{1}{\varepsilon^2}\chi(\frac{x}{\varepsilon}) \quad x \in \mathbb{R}^n.$$

The function χ_ε is called a *mollifier* or *standard smoothing kernel*. If Ω is an open set in $\mathbb{R}^n$ let $C_0^\infty(\Omega)$ denote the family of all C^∞-functions on Ω whose support is compact subset of Ω.

If $\varphi \in C^\infty(\mathbb{R}^n)$, then by using continuity of φ at the origin, we get

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \chi_\varepsilon(x)\varphi(x - w)dV(w) = \varphi(0)$$

Proposition 0.1.2. *[44]. Let $u \in \mathrm{L}^1_{loc}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is open. Then For any compact set $K \subset \Omega$, if $u \in C(\Omega)$, then $u * \chi_\varepsilon \to u$ uniformly on K as $\varepsilon \searrow 0$.*

Proof. [44] Take a compact set $K \in \Omega$ and fix $\varepsilon_0 > 0$ such that $K_{\varepsilon_0} \subset \Omega$, where

$$K_\varepsilon = \{x \in \mathbb{R}^n : d(x, \Omega^c) \leq \varepsilon\}, \quad \varepsilon > 0.$$

Let $0 < \varepsilon < \varepsilon_0$. We have;

$$(u * \chi_\varepsilon - u)(x) = (\chi_\varepsilon * u - u)(x) = \int \chi_\varepsilon(x - w)(u(w) - u(x))dV(w).$$

Therefore

$$\|u * \chi_\varepsilon - u\|_K = \sup_{x \in K} \sup_{w \in B(z,\varepsilon)} |u(x) - u(w)|.$$

The right-hand side goes to zero as $\varepsilon \searrow 0$, because u is uniformly continuous on K_{ε_0}. $\square$

0.1.4 Distribution

In classical differential calculus historically there were some difficulties due to the existence of functions which are not differentiable. In 1945 Schwartz introduced the theory of distributions and by his work allowed us to extend differentiability properties to a more general class of functions [51]. Here we want to give a short review of test functions and distributions.

Definition 2. Let Ω be an open subset of $\mathbb{C}$, then the space of *test functions* $\mathcal{D}(\Omega)$ is the vector space of functions φ with compact support of class $\mathcal{C}^{\infty}$.

We give the topology to the space $\mathcal{D}(\Omega)$ which gives following notion of a convergence of sequences: A sequence of test functions $\varphi_n \in \mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to φ if,

(i) there is a compact set $K \subset \Omega$ with $supp(\varphi_n) \subset K$ for all n,

(ii) all derivatives $\partial^i \varphi_n$ converge uniformly to $\partial^i \varphi$.

Definition 3. A *distribution* T is a continuous linear functional on $\mathcal{D}(\Omega)$, and whenever $\varphi_n \in \mathcal{D}(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ then $T(\varphi_n) \to T(\varphi)$. The space of all distributions is the space of topological dual of $\mathcal{D}(\Omega)$. Equipped with the w^*-topology, this space will be shown by $\mathcal{D}'(\Omega)$. In other words, a sequence of distributions $T_j \in \mathcal{D}'(\Omega)$ converges in $\mathcal{D}'(\Omega)$ to $T \in \mathcal{D}'(\Omega)$ if for every $\varphi \in \mathcal{D}(\Omega)$, $T_j(\varphi)$ converges to $T(\varphi)$.

Fundamentally, convergence of sequence is not enough to define the topology on $\mathcal{D}(\Omega)$. However, we can control the continuity of test functions only by sequences. For the full version of the definition of the topology on $\mathcal{D}(\Omega)$, you may see Chapter 6 in [46].

Example 1. Let f be a function in $\mathrm{L}^1_{loc}(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, we define

$$T_f(\varphi) := \int_\Omega f(x)\varphi(x)dx.$$

For $\varphi_n - \varphi$, suppose K is the compact set which containing support of $\varphi_n - \varphi$ then we have

$$\begin{aligned}|T_f(\varphi) - T_f(\varphi_n)| &= \left|\int_\Omega (\varphi(x) - \varphi_n(x))\, f(x)dx\right| \\ &\leq \sup_{x\in K} |\varphi(x) - \varphi_n(x)| \int_K |f(x)|dx,\end{aligned}$$

which goes zero as $n \to \infty$. Moreover T_f is linear and hence it defines a distribution.

If a distribution T is given by $T_f(\varphi) := \int_\Omega f\varphi dx$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, then we will identify T_f with f. This identification makes sense $T_f = T_g$ if and only if $f = g$ almost everywhere.
A *distribution* T is called positive if $T(\varphi) \geq 0$ for all $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi(x) \geq 0$ for all $x \in \Omega$. Given a positive distribution T, we can consider it is a positive linear operator on the space $C_0(\Omega)$ can regarded as a positive measure by Riesz Representation Theorem. (For Riesz Representation Theorem see [44])

0.2 Subharmonic Functions

Subharmonic functions and the foundations of the associated classical potential theory are sufficiently well exposed in the literature, and so we introduce here only a few fundamental results which we require. More detailed expositions can be found in the monograph of Ransford[44]. See also Hörmander [17] and Vladimirov [55]. See also Brelot [7], where a history of the development of the theory of subharmonic functions is given.

0.2.1 Harmonic Function

Let Ω be an open subset of $\mathbb{C}$. A function $h : \Omega \to \mathbb{R}$ is called a *harmonic* if $h \in C^2(\Omega)$ and $\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ on Ω.

If we write the function h as $h(z) = h(z, \bar{z}) = h(x, y) = h(r\cos\theta, r\sin\theta)$, then one can gets the following formulas for Laplacian operator

$$\Delta h = 4\partial^2 h/\partial z \partial \bar{z}$$

and

$$\Delta h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} = \frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}.$$

The space of harmonic functions on a domain Ω form a vector space since the Laplace operator is linear. This space will be denoted by $\mathcal{H}(\Omega)$. The following basic result provides a useful tool in deriving elementary properties of harmonic functions from holomorphic functions.

Proposition 0.2.1. *[44] Let Ω be a domain in $\mathbb{C}$.*

(i) If f is holomorphic on Ω, then $\mathrm{Re} f$ and $\mathrm{Im} f$ are harmonic.

(ii) If h is harmonic on Ω, and if Ω is simply connected, then $h = \mathrm{Re} f$ for some f holomorphic on Ω. Moreover f is unique up to an additive constant.

Proof. [44]

(i) Writing $f = h + ik$, where u, k are real valued functions on Ω, the Cauchy-Riemann equations give

$$\frac{\partial h}{\partial x} = \frac{\partial k}{\partial y} \quad \frac{\partial h}{\partial y} = -\frac{\partial k}{\partial x},$$

since f is holomorphic. Therefore,

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 k}{\partial y \partial x} - \frac{\partial^2 k}{\partial x \partial y} = 0.$$

Then $h := \mathrm{Re} f$ is harmonic on Ω. Harmonicity of $\mathrm{Im} f$ is proved in a similar fashion.

(ii) If $h = \mathrm{Re} f$ for some holomorphic function f, say $f = h + ik$, then

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x} + i\frac{\partial k}{\partial x} = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}. \tag{0.2.1}$$

Hence, if f exists, then $\frac{\partial f}{\partial x}$ is completely determined by h, and so f is unique up to adding a constant.
Equation 0.2.1 also suggests how we may construct such a function f. Now, define $\phi : \Omega \to \mathbb{C}$ by

$$\phi = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}.$$

Then $\phi \in C^1(\Omega)$ and ϕ satisfies the Cauchy-Riemann equations. Therefore, ϕ is holomorphic on Ω. Fix $z_0 \in \Omega$, and define $f : \Omega \to \mathbb{C}$ by

$$f(z) = h(z_0) + \int_{z_0}^{z} \phi(w)dw,$$

the integral being taken over any path in Ω from z_0 to z. As Ω is simply connected, Cauchy's theorem provides that the integral is independent of the particular path chosen. Then f is holomorphic on Ω and

$$\frac{\partial f}{\partial x} = \phi = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}.$$

Writing $\tilde{h} = \mathrm{Re} f$, we have

$$\frac{\partial \tilde{h}}{\partial x} - i\frac{\partial \tilde{h}}{\partial y} = \frac{\partial f}{\partial x} = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y},$$

such that $\frac{(\partial \tilde{h} - h)}{\partial x} \equiv 0$ and $\frac{(\partial \tilde{h} - h)}{\partial y} \equiv 0$. It follows that $\tilde{h} - h$ is constant on Ω, and putting $z = z_0$ shows that the constant is zero. Thus, $h = \mathrm{Re} f$ indeed.

□

Corollary 0.2.1. *[44] If h is a harmonic function on an open subset Ω of $\mathbb{C}$, then $h \in C^{\infty}(\Omega)$.*

Another foremost consequence of relation between harmonic and holomorphic functions is following:

Theorem 0.2.1. ***(Mean-Value Property)****[44]. Let h be a harmonic on an open neighbourhood of the disc $\bar{B}(z, \rho)$. Then*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + \rho e^{i\theta}) d\theta$$

Proof. [44] Choose $\rho^{'} > \rho$ such that h is harmonic on $B(z, \rho^{'})$. Applying the previous proposition, there exists f on $B(z, \rho^{'})$ so that $h = \mathrm{Re} f$ there. Then by using Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \rho} \frac{f(\zeta)}{\zeta - z} = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta}) d\theta.$$

The result comes from upon taking real parts of both sides. □

Theorem 0.2.2. ***(Identity principle)****[44]. Let h and k are harmonic functions on a domain Ω in $\mathbb{C}$. If $h = k$ on a non-empty open subset U of Ω, then $h = k$ throughout Ω.*

Proof. [44]. Assume that $k = 0$ and set $g = h_x - ih_y$. Then g is holomorphic on Ω and also $g = 0$ on U since $h = 0$ there. By identity principle for holomorphic function, $g = 0$ on whole Ω and hence $h_x = 0$ and $h_y = 0$ on Ω. Therefore h is constant on Ω and since $h = 0$ on U, it must be zero. □

Theorem 0.2.3. ***(Maximum Principle)****[44]. Let h be a harmonic function on a domain Ω in $\mathbb{C}$.*

(i) If f attains a local maximum on Ω, then h is constant.

(ii) If h extends continuously $\overline{\Omega}$ and $h \leq 0$ on $\partial\Omega$, then $h \leq 0$ on Ω.

Proof. [44]

(i) Suppose that h has a local maximum at $w \in \Omega$. Then for some $r > 0$ we have $h \leq h(w)$ on $D(w, r)$. By Proposition 0.2.1 (ii), $Ref = h$ for some holomorphic function f on $D(w, r)$. Then $|e^f|$ attains a local max at w, so e^f must be constant. Therefore h is constant on $D(w, r)$, and hence on the whole Ω by identity principle.

(ii) Since $\bar{\Omega}$ is compact ,h must attain a maximum at some point $w \in \bar{\Omega}$. If $w \in \partial\Omega$, then $h(w) \leq 0$ by assumption, and so $h \leq 0$ on Ω. On the other hand, if $w \in \Omega$, then by part (a) h is constant on Ω and hence on $\bar{\Omega}$, and so $h \leq 0$ on Ω.

□

Corollary 0.2.2. *[44]. Let Ω be a domain in $\mathbb{C}$ and $h : \bar{\Omega} \to \mathbb{R}$ be a continuous function where $\bar{\Omega}$ is compact. Assume that h is harmonic on Ω. Then h attains it maximum M and minimum m on the boundary $\partial\Omega$ of Ω. Further if u and v are two continuous real valued functions on $\bar{\Omega}$, both harmonic in Ω, and if $u = v$ on $\partial\Omega$ then $u = v$ on $\bar{\Omega}$*

Proof. [44] If h is constant then clearly h attains its maximum and minimum on the boundary of Ω. Assume therefore that h is not constant. Since h is

continuous on compact $\bar{\Omega}$, it attains its maximum and minimum in $\bar{\Omega}$. Since h is non constant and harmonic in Ω then it attains M and m on $\partial\Omega$.
To prove the second part put $h = u - v$ which is continuous in $\bar{\Omega}$, harmonic in Ω and vanishes on $\partial\Omega$. By the first part of the corollary, the maximum and minimum of h are zero in Ω hence $u = v$ in Ω. □

Definition 4. Let Ω be o subdomain of $\mathbb{C}$, and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. The *Dirichlet problem* is to find a harmonic function h on Ω such that $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial\Omega$

The theorem of uniqueness is easily settled.

Theorem 0.2.4. ***(Uniqueness Theorem)****[44]. With the notations of above definition, there is at most one solution of Dirichlet problem.*

Proof. [44] Suppose h_1 and h_2 are both solutions. Then $h_1 - h_2$ is harmonic on Ω, extends continuously $\bar{\Omega}$, and zero on $\partial\Omega$. Applying the maximum principle to $\pm(h_1 - h_2)$, we conclude that $h_1 - h_2 = 0$. □

If we interpret the Laplacian classically, we must require that harmonic functions be a priori C^2. However, even if we interpret the definition in the sense of distributions, harmonic functions are still smooth.
A function $\Omega \to \mathbb{R}$ is called *weakly harmonic* if it satisfies Laplace equation in distribution sense. The Dirichlet problem is the problem of finding a harmonic function on a domain with prescribed boundary values. It is one of the great advantages of harmonic functions over holomorphic ones that for 'nice' domains, a solution to the Dirichlet problem always exists. This is a capable tool with many applications.

Definition 5. (a) The Poisson kernel $P : B(0,1) \times \partial B(0,1) \to \mathbb{R}$, is defined by

$$P(z,\zeta) := \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2} \quad (|z| < 1, |\zeta| = 1).$$

(b) If $B = B(w, \rho)$ and $\phi : \partial B \to \mathbb{R}$ is Lebesgue integrable function, then its *Poisson integral* $P_B\phi : B \to \mathbb{R}$ is defined by

$$P_B\phi(z) := \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-w}{\rho}, e^{i\theta}\right)\phi(w+\rho e^{i\theta})d\theta \quad (z \in B).$$

More explicitly, if $r < \rho$ and $0 \le t < 2\pi$, then

$$P_B\phi(w+re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\phi(w+\rho e^{i\theta})d\theta.$$

The following result is fundamental.

Theorem 0.2.5. *[44]. With the notions of Definition 5:*

(i) P_B *is harmonic on* B*;*

(ii) if ϕ *is continuous at* $\zeta_0 \in \partial B$*, then* $\lim_{z\to\zeta_0} P_B\phi(z) = \phi(\zeta_0)$.
In particular, if ϕ *is continuous on the whole* ∂B*, then* $h = P_B\phi$ *solves the Dirichlet problem on* B*.*

Proof. See [44] □

Corollary 0.2.3. ***(Poisson Integral Formula)*** *[44] If* h *is a harmonic on an open neighbourhood of the disc* $\bar{B}(w, \rho)$*, then for* $r < \rho$ *and* $0 \le t < 2\pi$

$$h(w+re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\phi(w+\rho e^{i\theta})d\theta.$$

Proof. See [44] □

Note that this result is a generalization of the mean-value property, which is just the case $r = 0$. It allows us to recapture the values of h everywhere on B from knowledge of h on ∂B.
The following theorem indicates that the Poisson integral formula enable to derive some useful inequalities for positive(non-negative) harmonic functions.

Theorem 0.2.6. ***(Harnack's Inequality)****[44]. Let h be a positive harmonic function on the disc $B(z,\rho)$. Then for $r<\rho$ and $0\le t<2\pi$,*

$$\frac{\rho-r}{\rho+r}h(z)\le h(z+re^{it})\le\frac{\rho+r}{\rho-r}h(w).$$

Proof. [44]. Let s be a positive number with $r<s<\rho$ and by using Poisson integral formula applied to h on $\overline{B}(z,s)$,

$$\begin{aligned} h(z+re^{it}) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{s^2-r^2}{s^2-2sr\cos(\theta-t)+r^2}h(z+se^{i\theta})d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{s+r}{s-r}h(z+se^{i\theta})d\theta \\ &= \frac{s+r}{s-r}h(z), \end{aligned}$$

the last equality being just the mean-value property for h. Letting $s\to\rho$, we deduce that

$$h(z+re^{it})\le\frac{\rho+r}{\rho-r}h(z),$$

which is desired upper bound. The lower bound is proved in a similar way. □

Corollary 0.2.4. ***(Liouville Theorem)****[44]. Every harmonic function on $\mathbb{C}$ which is bounded from above or below is constant.*

Proof. [44]. It is enough to show that every positive harmonic function h is constant. Given $z\in\mathbb{C}$, put $r=|z|$ and let $\rho>r$. Applying Harnack's inequality to h on $B(0,\rho)=B(\rho)$ gives

$$h(z)\le\frac{\rho+r}{\rho-r}h(0).$$

Hence $h(z)\le h(0)$ as $\rho\to\infty$. Thus h attains its maximum at 0, and by Theorem 0.2.1 this implies h is constant. □

The mean-value property actually characterizes harmonic functions. This is given in the next theorem, which also illustrates well the value of being able to solve Dirichlet problem.

Theorem 0.2.7. ***(Converse to Mean Value Property)****[44]. Let $h : \Omega \to \mathbb{R}$ be a continuous function on an open set Ω of $\mathbb{C}$, and suppose that it posseses the local mean-value property, i.e. given $z \in \Omega$, there exists $\rho > 0$ such that*

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} h(z + re^{i\theta})d\theta \quad (0 \leq r < \rho). \tag{0.2.2}$$

Then h is harmonic on Ω.

Proof. [44]. It is enough to show that h is harmonic on each open disk B with $\bar{B} \subset \Omega$. Fix such a B, and define $\nu : \overline{B} \to \mathbb{R}$ by,

$$\nu = \begin{cases} h - \mathbf{P}_B h, & \text{on } B, \\ 0, & \text{on } \partial B. \end{cases}$$

Then ν is continuous on $\overline{B}$ and has a local mean-value property on B. As $\overline{B}$ is compact, ν attains a maximum value M at some point of $\overline{B}$. Define

$$\nabla_1 := \{w \in B : \nu(w) < M\} \quad \text{and} \quad \nabla_2 := \{w \in B : \nu(w) = M\}.$$

Then ∇_1 is open, since ν is continuous. Also ∇_2 is open, for $\nu(z) = M$, then the local mean-value property forces to ν to be equal to M on all sufficiently small circles around z. As ∇_1 and ∇_2 are partition of the connected set B, either $\nabla_1 = B$, in which case ν attains its maximum on ∂B and so $M = 0$, or else $\nabla_2 = B$ in which case $\nu \equiv M$ and again $M = 0$. Thus $\nu \leq 0$, and a similar argument shows that $\nu \geq 0$. Hence $h = \mathbf{P}_B h$ on B, and since $\mathbf{P}_B h$ is harmonic there, so is h. □

A useful feature of Theorem (0.2.7) is that one only needs to check that the mean-value property holds locally(i.e. the value ρ can depend upon z).

0.2.2 Subharmonic Functions

Definition 6. A function defined on an open set $\Omega \subset \mathbb{C}$ and with values $[-\infty, \infty)$ is called *subharmonic* if,
(i) u is upper semicontinuous,
(ii) For every compact set $K \subset \Omega$ and every continuous function h on K which is harmonic in the interior of K and $h \geq u$ on the boundary of K we have $h \geq u$ in K.
Additionally, $v : \Omega \to (-\infty, \infty]$ is called *superharmonic* if $-v$ is subharmonic.

A function f is harmonic if and only if it is both subharmonic and superharmonic. Indeed: f is continuous since f and $-f$ are upper semicontinuous. By using converse mean value property with the fact that f and $-f$ satisfy submean value condition f is a harmonic function.

The space of all subharmonic functions on Ω will be denoted by $\mathcal{SH}(\Omega)$. By our definition the function which is $-\infty$ identically is subharmonic; some authors exclude this function in the definition. Other equivalent definitions of subharmonic functions are often useful:

Theorem 0.2.8. *[17] Let Ω be an open subset of $\mathbb{C}$ and $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function.Then each of the following conditions is necessary and sufficient for u to be subharmonic:*

(i) If D is a closed disc in Ω and f is an analytic polynomial such that $u \leq \mathrm{Re} f$ on ∂D, it follows that $u \leq \mathrm{Re} f$ in D.

(ii) If $\Omega_\delta : \{z : d(z, \Omega^c) > \delta\}$, we have

$$u(z) 2\pi \int d\mu(r) \leq \int_0^{2\pi} \int u(z + re^{i\theta}) d\theta d\mu(r), \quad z \in \Omega_\delta, \qquad (0.2.3)$$

for every positive measure $d\mu$ on the interval $[0, \delta]$.

(iii) For every $\delta > 0$ and every $z \in \Omega_\delta$ there exists some positive measure $d\mu$ with support in $[0, \delta]$ has some mass outside the origin and (0.2.3) *is valid.*

Proof. [17] Definition 6 implies (i), and it is also trivial that (ii) implies (iii). Thus we only have to prove (i) $\Rightarrow$ (ii) and (iii) implies that u is subharmonic.
(i) $\Rightarrow$ (ii) Let $z \in \Omega_\delta$ and $0 < r \leq \rho$. Set $D = \{\zeta : |\zeta - z| \leq r\} \subset \Omega$. If $\varphi(\theta) = \sum_k a_k e^{ik\theta}$ is a trigonometric polynomial such that

$$u(z + re^{i\theta}) \leq \varphi(\theta)$$

for all θ, the polynomial $f(\zeta) = a_0 + 2\sum_{k>0} a_k(\zeta - z)^k/r^k$ has a real part which is an upper bound for u on ∂D. Hence $u \leq \mathrm{Re} f$ in D and in particular

$$u(z) \leq a_0 = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)d\theta. \tag{0.2.4}$$

Now if ϕ is an arbitrary continuous function such that $u(z + re^{i\theta}) \leq \phi(\theta)$, by Weierstrass approximation theorem, for every $\varepsilon > 0$ we can find a trigonometric polynomial φ with $\phi \leq \varphi \leq \phi + \varepsilon$ and conclude that (0.2.4) is valid with φ replaced $\varphi + \varepsilon$. Hence (0.2.4) for every continuous function φ which is an upper bound for $u(z + re^{i\theta})$, and by definition of the integral of a semicontinuous function this proves that

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta.$$

Integration with respect to $d\mu(r)$ gives (0.2.3).
(iii) implies that u is subharmonic. Let K be a compact subset of Ω and h is a continuous function which is harmonic in the interior of K, and assume that $h \geq u$ on ∂K. If $M := \sup_K v = u - h$ is positive, the semicontinuity of v shows that $v = M$ on a non-empty compact subset F of the interior of K. Let z_0 be a point in F with minimal distance to ∂K. If the distance is bigger than δ, then every circle $|z_0 - z| = r$, $r \leq \delta$, contains points where $v(z) < M$

and, in fact, a whole arc, since v is upper semicontinuous. This implies that

$$\int \int v(z_0 + re^{i\theta}) d\mu(r) < M2\pi \int d\mu(r) = v(z_0)2\pi \int d\mu(r)$$

if $d\mu$ is a measure with the properties list in (iii). But this contradicts the hypothesis (iii) and the fact that (0.2.3) is valid with the equality for harmonic functions. □

Note that the integrals in (ii) exist and are not ∞ since u is upper semicontinuous.

Corollary 0.2.5. *[17] A function u defined on an open set $\Omega \subset \mathbb{C}$ is subharmonic if every point in Ω has a neighbourhood where u is subharmonic.*

In other words, subharmonicity is a local property.

Theorem 0.2.9. *[45] If u is subharmonic on Ω, and if φ is a monotonically increasing convex function on $\mathbb{R}$, then $\varphi \circ u$ is subharmonic.*

Proof. [45] First, $\varphi \circ u$ is upper semicontinuous since φ is increasing and continuous. Next, if $\bar{B}(z, r) \subset \Omega$ we have,

$$\varphi(u(z)) \leq \varphi\left(\int_0^{2\pi} u(z + re^{i\theta}) d\theta\right) \leq \int_0^{2\pi} \varphi(u(z + re^{i\theta})) d\theta.$$

The first of the inequalities holds since u is subharmonic and φ is increasing; the second follows from the Jensen's inequality for real convex functions. □

Proposition 0.2.2. *[45] If f is a holomorphic function on a domain Ω in $\mathbb{C}$, then $\log |f|$ is subharmonic.*

Proof. [45] If $f(z) \equiv 0$ then the result is trivial. If $f(z) \not\equiv 0$, we assume w.l.o.g $z = 0$ and apply the classical Jensen formula:
Let f be a holomorphic function on $B(0, r)$ and $0 < r < R$. Let the zeros of f in $\bar{B}(0, r)$ be $\alpha_1, \ldots, \alpha_N$ with repeated multiplicity. Then

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \log|\frac{r^N}{\alpha_1\ldots\alpha_N}|$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta.$$

□

Proposition 0.2.3. *[45] Let $(s_\alpha)_{\alpha\in A}$ be a family of subharmonic functions on a domain $\Omega \subset \mathbb{C}$. If $s := \sup s_\alpha$ is upper semicontinuous and finite everywhere, then s is subharmonic.*

Proof. If $s \leq h$ on $\partial\Omega$, where $D \subset\subset \Omega$ and $h : \bar{D} \to \mathbb{R}$ is continuous and harmonic on D, then $s_\alpha \leq h$ on ∂D for every $\alpha \in A$. Since the s_α are harmonic, it follows that $s_\alpha \leq h$ on D for every $\alpha \in A$. But then $s \leq h$ on D as well. □

Theorem 0.2.10. *[44] Let $u \in \mathcal{SH}(\Omega)$ and not $-\infty$ identically in any component of Ω. Then $u \in \mathrm{L}^1_{loc}(\Omega)$, which implies $u > -\infty$ almost everywhere.*

Proof. [44] If $z \in \Omega$, $u(z) > -\infty$, and D is a closed disc with center z contained in Ω, we obtain from (0.2.3) and the fact that u is bounded from above on D that u is integrable over D. If E is the set of all z such that u is integrable over a neighborhood of z, it follows that $u = -\infty$ in a neighborhood of every point in $\Omega \setminus E$. Hence both E and $\Omega \setminus E$ are open so that $\Omega \setminus E$ is a union of components of Ω, all of must be empty by the hypothesis since $u = -\infty$ on $\Omega \setminus E$. □

Now we can give another description of subharmonic functions in distribution sense.

Since subharmonic functions are locally integrable, their Laplacians can be evaluated in the sense of distributions.

Theorem 0.2.11. *[17] Let $\Omega \subset \mathbb{C}$ be open and $u \in \mathcal{SH}(\Omega)$, then $\Delta u \geq 0$ in the sense of distribution, i.e.*

$$\int_\Omega u(x)\Delta\varphi(x)d\Lambda(x) \geq 0$$

for any non-negative test function $\varphi \in \mathcal{C}_0^\infty$. *Conversely, if* $v \in \mathrm{L}^1_{loc}(\Omega)$ *is such that* $\Delta v \geq 0$ *in* Ω *in the sense of distribution, then the function* $u = \lim_{\varepsilon \to 0}(v * \chi_\varepsilon)$ *is well-defined, subharmonic in* Ω *and equal to* v *almost everywhere in* Ω.

Since the proof of the theorem is quite similar to n-dimensional case in section 1 Theorem 1.1.1 so we skip it.

Lemma 0.2.1. *[44] Let* u, v *are subharmonic functions on a domain* Ω *in* $\mathbb{C}$ *with* $u, v \neq -\infty$. *If* $\Delta u = \Delta v$, *then* $u = v + h$ *where* h *is harmonic on* Ω.

Proof. [44] Let $(\chi_r)_{r=1}^{\infty}$ be the functions used in smoothing theorem, and for $r > 0$ write

$$\Omega_r = \{z \in \Omega : d(z, \Omega^c) > r\}.$$

Then $u * \chi_r \in C^\infty(\Omega_r)$, and for $z \in \Omega_r$ we have

$$\begin{aligned} \Delta(u * \chi_r)(z) &= \int u(w) \Delta_z \chi_r(z - w) dA(w) \\ &= \int u(w) \Delta_w \chi_r(z - w) dA(w) \\ &= \int \varphi \Delta u. \end{aligned}$$

where $\varphi = \chi_r(z - w) \in C_c^\infty(\Omega)$. Since $\Delta u = \Delta v$, with the same calculation for v, we get $\Delta(u * \chi_r) = \Delta(v * \chi_r)$ on Ω_r. Therefore there exist a harmonic function h_r on Ω_r with

$$u * \chi_r = v * \chi_r + h_r \quad \text{on} \quad \Omega_r.$$

Now, applying the smoothing lemma to $\pm h_r$, we have $h_r * \chi_s = h_r$ on Ω_{s+r} for each $s > 0$ and hence

$$h_r = h_r * \chi_s = (u - v) * \chi_r * \chi_s = h_s * \chi_r = h_s \quad \text{on} \quad \Omega_{r+s}$$

Therefore, there is a single harmonic function h on Ω_r so that for every $r > 0$

$$u * \chi_r = v * \chi_r + h_r \quad \text{on} \quad \Omega_r$$

Letting $r \searrow 0$ with using smoothing theorem we deduced $u = v + h$ on Ω. $\square$

0.3 Generalized Dirichlet Problem

As we mentioned before, if Ω is a disk then a solution always exists and we even have a formula for it. But, for a general domain Ω, the situation is more complicated. In this case, the Dirichlet problem may have no solution. For example, take $\Omega = \{z : 0 < |z| < 1\}$, and let $\phi : \partial\Omega \to \mathbb{R}$ be given by

$$\phi(\zeta) = \begin{cases} 0, & |\zeta| = 1, \\ 1, & |\zeta| = 0. \end{cases}$$

Then, any solution h would have a removable singularity at 0, and the maximum principle would then imply that $h(0) \leq 0$, violating the condition that $\lim_{z \to 0} \phi(0) = 1$.

Henceforth, we shall consider conditions under which a solution does exist, and also, even more importantly, derive a natural reformulation of Dirichlet problem which always has a solution. Firstly, we shall allow Ω to be any proper subdomain of $\mathbb{C}_\infty$. The other generalization will be consider arbitrary bounded functions $\phi : \partial\Omega \to \mathbb{R}$, rather than continuous ones. Although certainly no solution to the Dirichlet problem is possible if ϕ is discontinuous, it is nevertheless useful to allow this extra freedom.
The key idea, called the Perron method, is in the following definition.

Definition 7. Let Ω be a proper subdomain of $\mathbb{C}_\infty$, and let $\phi : \partial\Omega \to \mathbb{R}$ be a bounded function. The associated *Perron function* $H_\Omega \phi : \Omega \to \mathbb{R}$ is defined by

$$H_\Omega \phi = \sup_{u \in \mathcal{U}} u,$$

where $\mathcal{U}$ denotes the family of all subharmonic functions u on Ω such that $\limsup_{z \to \zeta} u(z) \leq \phi(\zeta)$ for each $z \in \Omega$ and $\zeta \in \partial\Omega$.

The motivation for this definition is that, if the Dirichlet problem has a solution at all, then $H_\Omega \phi$ is it! Indeed, if h is such a solution, then $h \in \mathcal{U}$, and so $h \leq H_\Omega \phi$. On the other hand, by the maximum principle, if $u \in \mathcal{U}$, then $u \leq h$ on Ω, and so $H_\Omega \phi \leq h$. Hence $H_\Omega \phi = h$.
Our first result is that, regardless of whether the Dirichlet problem has a solution, $H_\Omega \phi$ is a always a bounded harmonic function.

Lemma 0.3.1. ***(Poisson Modification)****[44]. Let Ω be a domain in $\mathbb{C}$, let B be an open disc with $\bar{B} \subset \Omega$, and let u be a subharmonic function on Ω with $u \not\equiv -\infty$. If we define $\tilde{u}$ on Ω by*

$$\tilde{u} = \begin{cases} P_B u, & \text{on } B, \\ u, & \text{on } \Omega \setminus B. \end{cases}$$

then $\tilde{u}$ is subharmonic on Ω, harmonic on B, and $\tilde{u} \geq u$ on Ω.

Proof. [44] Since u is Lebesgue integrable on ∂B, $P_B u$ makes sense and harmonic on B, thus $P_B u \geq u$ there. It thus remains to show that $\tilde{u}$ is subharmonic on Ω, and by the gluing theorem for subharmonic functions this will follow provided that for all $\zeta \in \partial B$,

$$\limsup_{z \to \zeta} P_B u(z) \leq u(\zeta).$$

To prove this inequality, choose continuous functions ϕ_n on ∂B such that $\psi_n \searrow u$ there. Then for all $\zeta \in \partial B$ we have,

$$\limsup_{z \to \zeta} P_B u(z) \leq \lim_{z \to \zeta} P_B \psi_n(z) = \psi_n(\zeta),$$

and by letting $n \to \infty$ we get the desired conclusion. $\square$

Theorem 0.3.1. *[44] Let* Ω *be a proper subdomain of* $\mathbb{C}_\infty$*, and let* $\phi : \partial\Omega \to \mathbb{R}$ *be a bounded function. Then the function* $H_\Omega\phi$ *is harmonic on* Ω*, and*

$$\sup_{\Omega} |H_\Omega \phi| \leq \sup_{\partial\Omega} |\phi|. \tag{0.3.1}$$

Proof. [44] Let $\mathcal{U}$ be the family defined in Definition 7. Set $M = \sup_{\partial\Omega} |\phi|$, then $-M \in \mathcal{U}$, so $H\Omega\phi \geq -M$. Also given $u \in \mathcal{U}$, by maximum principle $u \leq M$ and hence $H_\Omega\phi \leq M$. So we have the required inequality.
For harmonicity of $H_\Omega\phi$ on Ω, we need to prove harmonicity on each open disk B with $\bar{B} \subset \Omega$. Fix such a B and a point $w_0 \in B$. We can find $(u_n)_{n=1}^{\infty} \in \mathcal{U}$ such that $u_n(w_0) \to H_\Omega\phi(w_0)$ by definition of $H_\Omega\phi$. We may further suppose that $u_1 \leq u_2 \leq$ on Ω by replacing u_n by $\max(u_1, \ldots, u_n)$. Now, for each n let $\tilde{u}_n$ denote the Poisson modification of u_n. Then we also have $\tilde{u}_1 \leq \tilde{u}_2 \leq \ldots$ on Ω, and we claim that $\tilde{u} := \lim_{n\to\infty} \tilde{u}_n$ satisfies:

(i) $\tilde{u} \leq H_\Omega\phi$ on Ω;

(ii) $\tilde{u}(w_0) = H_\Omega\phi(w_0)$;

(iii) $\tilde{u}$ is harmonic on Ω.

Indeed, by previous lemma each $\tilde{u}_n$ is subharmonic on Ω and for all $\zeta \in \partial\Omega$,

$$\limsup_{z\to\zeta} \tilde{u}_n(z) = \limsup_{z\to\zeta} u_n(z) \leq \phi(\zeta)$$

such that $\tilde{u}_n \in \mathcal{U}$. Hence $\tilde{u}_n \leq H_\Omega\phi$ for all n, and so $\tilde{u}_n \leq H_\Omega\phi$, which gives (i).
Again by the previous lemma, $\tilde{u}_n \geq u_n$, so

$$\tilde{u}(w_0) = \lim_{n\to\infty} \tilde{u}_n(w_0) \geq \lim_{n\to\infty} u_n(w_0) = H_\Omega\phi(w_0).$$

Since the reverse inequality follows from (i), this proves (ii). finally, each $\tilde{u}_n$ is subharmonic on B, so by Harnack's theorem the same is true for $\tilde{u}$, which gives (iii). So, to finish to proof, we would show that $\tilde{u} = H_\Omega\phi$ on B.
Take an arbitrary point $w \in B$, and choose $(v_n)_{n=1}^{\infty} \in \mathcal{U}$ such that $v_n(w) \to H_\Omega(\phi)(w)$. With replacing v_n by $\max(u_1, \dots, u_n, v_1, \dots, v_n)$, we can suppose that $v_1 \le v_2 \le$ and $v_n \ge u_n$ on Ω. Let $\tilde{v}_n$ denote the Poisson modification of v_n. Then $\tilde{v_n} \nearrow \tilde{v}$, where:

(i) $\tilde{v} \le H_\Omega\phi$ on Ω;

(ii) $\tilde{v}(w) = H_\Omega\phi(w)$;

(iii) $\tilde{v}$ is harmonic on Ω.

In particular, (i) implies that

$$\tilde{v}(w_0) \le H_\Omega\phi(w_0) = \tilde{u}(w_0).$$

On the other hand, for each n, $\tilde{v}_n \ge \tilde{u}_n$ so $\tilde{v} \ge \tilde{u}$. Hence $\tilde{u} - \tilde{v}$ is harmonic on B and attains its maximum value of 0 at w_0. In particular, ,t follows

$$\tilde{u}(w) = \tilde{v}(w) = H_\Omega\phi(w).$$

Therefore, $\tilde{u} = H_\Omega\phi$ on B since w is arbitrary.

□

From the definition of $H_\Omega\phi$, one may expect that $\lim_{z\to\zeta} H_\Omega\phi(z) = \phi(\zeta)$ at each point $\zeta \in \partial\Omega$. But if $\Omega = \{z : 0 < |z| < 1\}$ then this cannot be true, because, as we have seen, the Dirichlet problem may have no solution. It is explanatory to see exactly what goes wrong.
First let

$$\phi(\zeta) = \begin{cases} 0, & |\zeta| = 1, \\ 1, & |\zeta| = 0. \end{cases}$$

If $u \in \mathcal{U}$, then by maximum principle $u \leq 0$ on Ω, and so $H_\Omega \phi \leq 0$. Since $0 \in \mathcal{U}$, in fact $H_\Omega \phi \equiv 0$ on Ω.
Now let

$$\phi(\zeta) = \begin{cases} 0, & |\zeta| = 1, \\ -1, & |\zeta| = 0. \end{cases}$$

The same argument as before shows that $H_\Omega \phi \leq 0$. In this case $0 \notin \mathcal{U}$. However, it is true that $\varepsilon \log |z| \in \mathcal{U}$ for each $\varepsilon > 0$ and so once again $H_\Omega \phi = 0$ on Ω.

Definition 8. Let Ω be a proper subdomain of $\mathbb{C}_\infty$, and $\zeta_0 \in \partial\Omega$. A *barrier* at ζ_0 is a subharmonic function b defined on $\Omega \cap N$, where N is an open neighbourhood of ζ_0, satisfying

$$b < 0 \quad \text{on} \quad \Omega \cap N \qquad \text{and} \qquad \lim_{z \to \zeta_0} b(z) = 0.$$

A boundary point at which a barrier exists is called *regular*, otherwise it is *irregular*. If every $\zeta \in \partial\Omega$ is regular, then Ω is called a *regular domain.*

Lemma 0.3.2. *[44] If Ω is a proper subdomain of $\mathbb{C}_\infty$ and $\phi : \partial\Omega \to \mathbb{R}$ is a bounded function, then on the domain Ω*

$$H_\Omega \phi \leq -H_\Omega(-\phi)$$

Proof. [44] Let $\mathcal{U}$ be the family of subharmonic function as the Definition 7 and let $\mathcal{V}$ be the corresponding family for $-\phi$. Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, their sum is subharmonic on Ω and for $\zeta \in \partial\Omega$ satisfies

$$\limsup_{z \to \zeta} (u + v)(z) \leq \phi(\zeta) - \phi(\zeta) = 0$$

Hence by maximum principle $u + v \leq 0$ 0n Ω. So, $H_\Omega \phi + H_\Omega(-\phi) \leq 0$ on Ω. □

The following lemma enables us to globalize barrier functions.

Lemma 0.3.3. ***(Bouligand's Lemma)****[44] Let ζ_0 be a regular boundary point of a domain Ω, and let N_0 be an open neighborhood of ζ_0. Then, given $\varepsilon > 0$, there exists a subharmonic function b_ε on Ω so that*

$$b_\varepsilon < 0 \quad on \quad \Omega, \qquad b_\varepsilon \leq -1 \quad on \quad \Omega \setminus N_0 \quad and \quad \liminf_{z\to\zeta} b_\varepsilon(z) \geq -\varepsilon.$$

Proof. [44] Since ζ_0 is regular, there exists a neighborhood N of ζ_0 and a barrier b on $D \cap N$ as in Definition 8.
Now, let $B = B(\zeta_0, \rho)$, with ρ satisfying $\bar{B} \subset N \cap N_0$. Then normalized Lebesgue measure on ∂B is a regular measure, so we can find a compact set $K \subset \Omega \cap \partial B$ such that $\Upsilon := (\Omega \cap \partial B) \setminus K$ has measure less than ε. Since Υ is open in ∂B, it follows that for all $z \in \Omega$ and $\eta \in \Upsilon$

$$\lim_{z\to\eta} P_B\chi_\Upsilon(z) = 1.$$

Set $m = -\sup_K b$, such that $m > 0$. Then for $\eta \in \Omega \cap \partial B$ and $z \in \Omega \cap B$

$$\limsup_{z\to\eta} \big(\frac{b(z)}{m} - P_B\chi_\Upsilon(z)\big) \leq \Theta(\eta) \leq 1$$

where

$$\Theta(\eta) = \begin{cases} b(\eta)/m, & \text{if } \eta \in K, \\ -1, & \text{if } \eta \in \Upsilon. \end{cases}$$

Therefore, if we define b_ε on D by

$$b\varepsilon = \begin{cases} \max(-1, (b/m - P_B\chi_\Upsilon), & \text{on } \Omega \cap B, \\ -1, & \text{on } \Omega \setminus B. \end{cases}$$

then by gluing theorem for subharmonic functions b_ε is subharmonic on Ω. Obviously,

$$b_\varepsilon < 0 \quad \text{on} \quad \Omega \qquad \text{and} \qquad b_\varepsilon \leq -1 \quad \text{on} \quad \Omega \setminus N_0.$$

Lastly, we have

$$\liminf_{z\to\zeta_0} b_\varepsilon(z) \geq \lim_{z\to} \big(\frac{b(z)}{m} - P_B 1_\Upsilon(\zeta_0)\big) = 0 - P_B 1_\Upsilon(\zeta_0) > -\varepsilon,$$

the last inequality coming from the fact that, as ζ_0 is the center of B, the value of $P_B \chi_\Upsilon(\zeta_0)$ is exactly the normalized Lebesgue measure of Υ. □

Theorem 0.3.2. *[44] Let Ω be a proper subdomain of $\mathbb{C}_\infty$, and let ζ_0 be a regular boundary point of Ω. If $\phi : \partial\Omega \to \mathbb{R}$ is a bounded function which is continuous at ζ_0, then*

$$\lim_{z\to\zeta_0} H_\Omega \phi(z) = \phi(\zeta_0).$$

Proof. [44] Let $\varepsilon > 0$. Since ϕ is continuous at ζ_0, there exists an open neighborhood N_0 of ζ_0 so that $|\phi(\zeta) - \phi(\zeta_0)| < \varepsilon$ for all $\zeta \in \partial\Omega \cap \bar{N}_0$. Let b_ε be as in previous lemma and set

$$u = \phi(\zeta_0) - \varepsilon + (M + \phi(\zeta_0)) b_\varepsilon,$$

where $M = \sup_{\partial\Omega} |\phi|$. Then u is subharmonic on Ω, and if $\zeta \in \partial\Omega$ then

$$\limsup_{z\to\zeta} u(z) \leq \Phi(\zeta) \leq \phi(\zeta),$$

where

$$\Phi(\zeta) = \begin{cases} \phi(\zeta_0) - \varepsilon, & \text{if } \zeta \in \partial\Omega \cap \bar{N}_0, \\ \phi(\zeta_0) - \varepsilon - (M + \phi(\zeta_0)), & \text{if } \zeta \in \partial\Omega \setminus \bar{N}_0. \end{cases}$$

Hence from definition of Perron function $u \leq H_\Omega \phi$ on Ω. In particular,

$$\liminf_{z\to\zeta_0} H_\Omega \phi(z) \geq \liminf_{z\to\zeta_0} u(z) \geq \phi(\zeta_0) - \varepsilon(1 + M + \phi(\zeta_0)).$$

Since ε is arbitrary, it follows that

$$\liminf_{z\to\zeta_0} H_\Omega \phi(z) \geq \phi(\zeta_0).$$

Repeating the same procedure for $-\phi$, we also have

$$\liminf_{z \to \zeta_0} H_\Omega(-\phi)(z) \geq -\phi(\zeta_0).$$

Hence by Lemma 0.3.2, $H_\Omega \leq -H_\Omega(-\phi)$, and so it follows that

$$\limsup_{z \to \zeta_0} H_D\phi(z) \leq \phi(\zeta_0).$$

□

Putting together what we have learned, we obtained the following result.

Corollary 0.3.1. ***(Solution of Dirichlet Problem)****[44] Let Ω be a regular domain, and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function h on Ω such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial\Omega$.*

Proof. [44] Uniqueness of the solution was proved in Theorem 0.2.4., and the existence comes from the Theorem 0.3.1 and Theorem 0.3.2. with taking $h = H_\Omega\phi$. □

Chapter 1

Plurisubharmonic Functions

1.1 Plurisubharmonic Functions and Elementary Properties

Let $\Omega \subset \mathbb{C}^n$ be open. For $u : \Omega \to [-\infty, \infty)$, $a \in \Omega$, and $X \in \mathbb{C}^n$, we define

$$\Omega_{a,X} = \{\lambda \in \mathbb{C} : a + \lambda X \in \Omega\}, \quad \Omega_{a,X} \ni \lambda \overset{u_{a,X}}{\longmapsto} u(a + \lambda X).$$

Definition 9. A function $u : \Omega \to [-\infty, \infty)$ is called *plurisubharmonic* (briefly *psh*; $u \in \mathcal{PSH}(\Omega)$) if
(i) u is uppersemicontinuous on Ω
(ii) For every $a \in \Omega$ and $X \in \mathbb{C}^n$ the function $u_{a,X}$ is subharmonic in a neighborhood of zero.

We say that a function $u : \Omega \to [-\infty, \infty)$ is *logaritmically plurisubharmonic* if $\log u \in \mathcal{PSH}(\Omega)$.
Notation We denote a point in $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$ and use the standard notion

$$z_j = x_j + iy_j, \quad \bar{z}_j = x_j - iy_j,$$

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j,$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

We denote by dV the standard Euclidean volume form on $\mathbb{C}^n$, i.e.

$$\begin{aligned} dV &= (dx_1 \wedge dy_1) \wedge \ldots \wedge (dx_n \wedge dy_n) \\ &= \left(\frac{i}{2}\right)^n (dz_1 \wedge d\bar{z}_1) \wedge \ldots \wedge (dz_n \wedge d\bar{z}_n). \end{aligned}$$

Consider an $\mathbb{R}$-linear mapping $L : \mathbb{C}^n \to \mathbb{C}^m$. It can be split in a unique manner, into a $\mathbb{C}$-linear part and an anti $\mathbb{C}$-linear part:

$$L(z) = \underbrace{\frac{1}{2}(L(z) - iL(iz)}_{\mathbb{C}-linear} + \underbrace{\frac{1}{2}(L(z) + iL(iz))}_{\text{anti} \quad \mathbb{C}-linear} \quad (z \in \mathbb{C}^n)$$

In particular, if $f : \Omega \to \mathbb{C}$ is differantiable at $a \in \Omega \subset \mathbb{C}$, then the differential $d_a f$ can be split into the $\mathbb{C}$-linear part $\partial_a f$ and the anti $\mathbb{C}$-linear part $\overline{\partial}_a f$:

$$df = \partial f + \bar{\partial} f,$$

and we have that,

$$\begin{aligned} df &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j\right), \\ \partial f &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \\ \bar{\partial} z_j &= \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j. \end{aligned}$$

If $f \in C^2(\Omega)$ and $a \in \Omega$ then the *complex Hessian* of f at point a is defined as the matrix

$$\left[\frac{\partial^2 u(a)}{\partial z_j \partial \bar{z}_k}\right]_{j,k}^{n}.$$

We shall denote its transpose by $\mathscr{L}u$. Let

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j$$

for $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$. If $u : \Omega \to \mathbb{R}$ is twice $\mathbb{R}$-differentiable at a point $a \in \Omega$, then we define the *Levi form of u at a*:

$$\mathscr{L}u(a; X) := \langle \mathscr{L}u(a)X, X\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \bar{X}_k, \quad X = (X_1, \dots, X_n) \in \mathbb{C}^n. \tag{1.1.1}$$

Example 2. Let $f(z) := \|z\|^2 = \sum_{i=1}^{n} z_i \bar{z}_i$. Then $\mathscr{L}f(a; X) = \|X\|^2$ for every $a \in \mathbb{C}^n$.

Proposition 1.1.1. *[43] Let $\in C^2(\Omega, \mathbb{R})$. Then $u \in \mathcal{PSH}(\Omega)$ if and only if $\mathscr{L}u(a; X) \geq 0$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$.*

Proof. First, fix $a \in \Omega$ and $X \in \mathbb{C}^n$. Since u is plurisubharmonic, $u_{a,X}$ is subharmonic. Therefore $\Delta u_{a,X}(\lambda)|_{\lambda=0} = \frac{1}{4} \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \overline{X}_k \geq 0$.
Conversely assume that $\mathscr{L}u(a; X) \geq 0$ for all $a \in \Omega, X \in \mathbb{C}^n$. Then $\Delta u_{a,X}(0) = \frac{1}{4} \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \overline{X}_k$ which is nonnegative by assumption. Hence, $u_{a,X}$ is subharmonic and thus u is plurisubharmonic on Ω. □

Plurisubharmonicity can also be characterized in terms of distributional derivatives

Theorem 1.1.1. *[24] If $\Omega \subset \mathbb{C}^n$ is open and $u \in \mathcal{PSH}(\Omega)$, then for each $X = (X_1, \dots, X_n) \in \mathbb{C}^n$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \geq 0$$

in Ω, in the sense of distributions, i.e.

$$\int_\Omega u(z)\langle \mathscr{L}\varphi(z)X, X\rangle dV(z) \geq 0$$

for any non-negative test function $\varphi \in C_0^\infty(\Omega)$. Conversely, if $v \in \mathrm{L}^1_{loc}(\Omega)$ is such that $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \geq 0$$

in Ω, in the sense of distributions then there is a plurisubharmonic function u on Ω that is equal to v almost everywhere in Ω.

Proof. [24] Let $u \in \mathcal{PSH}(\Omega)$, and let $u_\varepsilon = u * \chi_\varepsilon$ for $\varepsilon > 0$. For a non-negative test function $\varphi \in C_0^\infty(\Omega)$ and a vector $X = (X_1, \ldots, X_n)$ we have

$$\begin{aligned}\int_\Omega u(z)\langle \mathscr{L}\varphi(z)X, X\rangle dV(z) &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(z)\langle \mathscr{L}\varphi(z)X, X\rangle dV(z)\\ &= \lim_{\varepsilon\to 0}\langle \mathscr{L}u_\varepsilon(z)X, X\rangle\varphi(z)dV(z) \geq 0.\end{aligned}$$

where the first equality is obtained by Lebesgue's dominated convergence theorem and the second one is obtained by using the integration by parts formula twice for smooth functions and φ has compact support. It is positive since u_ε is plurisubharmonic and smooth.

Conversely assume $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, and $\sum_{j,k=1}^n \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \geq 0$ satisfied. Let $v_\varepsilon = v * \chi_\varepsilon$ for $\varepsilon > 0$. Then, $\int_\Omega v_\varepsilon(z)\langle \mathscr{L}\varphi(z)X, X\rangle dV(z) \geq 0$. Therefore, v_ε is plurisubharmonic in the sense of distribution. Further, v_ε is smooth hence, it is plurisubharmonic in the usual sense. For $\varepsilon_2 > \varepsilon_1 > 0$ and $x \in \Omega$, we have

$$\begin{aligned}v_{\varepsilon_2} &= \lim_{\delta\to 0}(v * \chi_{\varepsilon_2}) * \chi = \lim_{\delta\to 0}(v * \chi_\varepsilon * \chi_{\varepsilon_2}\\ &\geq \lim_{\delta\to 0}(v * \chi_\delta) * \chi_{\varepsilon_1} = \lim_{\delta\to 0}(v * \chi_{\varepsilon_1}) * \chi_\delta = v_{\varepsilon_1}.\end{aligned}$$

Hence, the limit function u is plurisubharmonic.

$\square$

The following properties of plurisubharmonic functions directly related to the theory of subharmonic functions.

For $a \in \mathbb{C}^n$ and $r > 0$, the *polydisc* with center a and radius r is the set

$$\mathbb{P}(a; r) := \{z \in \mathbb{C}^n : \|z - a\|_\infty < r\}$$

where $\|.\|_\infty$ is the maximum norm in $\mathbb{C}^n$.

Proposition 1.1.2. ***(Maximum principle)****[43]. Let $D \in \mathbb{C}^n$ be a domain and let $u \in \mathcal{PSH}(D)$. If $u \leq u(a)$ for some points $a \in D$, then $u \equiv u(a)$.*

Proof. [43] Let $D_0 = \{x \in D : u(x) = u(a)\}$. Observe that the set

$$D \setminus D_0 = \{x \in D : u(x) < u(a)\}$$

is open, therefore, D_0 is closed in D. Let $z_0 \in D_0$. Applying the maximum principle for subharmonic function to each of the functions $u_{z_0,X}$ with $\|X\|_\infty = 1$, we conclude that $\mathbb{P}(z_0, d(z_0, D^c)) \subset D_0$. Thus D_0 is open and therefore $D = D_0$. $\square$

Proposition 1.1.3. *[43] Let $\Omega \subset \mathbb{C}^n$. For an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ the following conditions are equivalent:*

(i) $u \in \mathcal{PSH}(\Omega)$

(ii) For all $a \in \Omega$ and $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, there exists an R such that $0 < R \leq d(a, \Omega^c)$ satisfying,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta} X) d\theta, \quad 0 < r < R;$$

(iii) *For all $a \in \Omega$ and $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, there exists an R, $0 < R \leq d(a, \Omega^c)$ such that,*

$$u(a) \leq \frac{1}{\pi r^2} \int_{B(r)} u(a + \xi X) d\Lambda_2(\xi), \quad 0 < r \leq R;$$

(iv) *For all $a \in \Omega$ and $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, there exists an R, $0 < R \leq d(a, \Omega^c)$ such that, if $u(a + \lambda X) \leq \mathrm{Re} f(\lambda)$ for $|\lambda| = r$, then $u(a) \leq \mathrm{Re} f(0)$ for all $0 < r < R$ and $f \in \mathcal{P}(\mathbb{C})$ where $\mathcal{P}(\mathbb{C})$ denotes the spaces of all complex polynomials of one complex variable;*

(v) *For all $a \in \Omega$ and $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, there exists R, $0 < R \leq d(a, \Omega^c)$ such that, if $u_{a,X}(\lambda) \leq h(\lambda)$ for $|\lambda| = r$, then $u(a) \leq h(0)$ for all $0 < r < R$ and $h \in \mathcal{H}(B(r)) \cap C(\bar{B}(r))$ where $\mathcal{H}(B(r))$ denotes the spaces of all real-valued harmonic functions on $B(r)$;*

(vi) *for any $a \in \Omega$ and $X \in \mathbb{C}^n$ the function $u_{a,X}$ is subharmonic in $\Omega_{a,X}$.*

Proof. Let u be an upper semicontinuous function on Ω. The implication (i) $\Rightarrow$ (ii) directly comes from definition. For (ii) $\Rightarrow$ (iii), by hypothesis we have

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

Now, by integrating both sides with respect to R for $0 < r < R$,

$$\begin{aligned} \int_0^R u(a) R dR &\leq \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(a + re^{i\theta}) d\theta R dR \\ &= \frac{0}{2\pi} \int_{B(r)} u(a + \xi X) d\Lambda_2(\xi). \end{aligned} \tag{1.1.2}$$

this result in

$$u(a) \leq \frac{1}{\pi r^2} \int_{B(r)} u(a + \xi X) d\Lambda_2(\xi).$$

Since all polynomials are holomorphic functions, for any $f \in \mathcal{P}(\mathbb{C})$, real part of f is a harmonic function. So the implication (ii) $\Rightarrow$ (iv)and hence (iv) $\Rightarrow$ (v) comes from the Poisson formula. For (v) $\Rightarrow$ (vi) this is direct consequence of translation invariance of the Lebesgue measure. The last implication is trivial.

$\square$

Proposition 1.1.4. *[43] Let* $\Omega \subset \mathbb{C}^n$.

(i) If $(u_v)_{v=1}^{\infty} \subset \mathcal{PSH}(\Omega)$ *and* $u_v \searrow u$ *pointwise on* Ω*, then* $u \in \mathcal{PSH}(\Omega)$ *In particular, if* $(u_v)_{v=1}^{\infty} \in \mathcal{PSH}(\Omega)$ *and* $u_v \leq 0$ *for all* $v \in \mathbb{N}$*, then* $\sum_{v=1}^{\infty} u_v \in \mathcal{PSH}(\Omega)$.

(ii) If $(u_v)_{v=1}^{\infty} \in \mathcal{PSH}(\Omega)$ *and* $u_v \to u$ *locally uniformly in* Ω*, then* $u \in \mathcal{PSH}(\Omega)$.

Proof. (i) The upper semicontinuity of u comes from Proposition 0.1.1. For any $v \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{v\to\infty} u_v(a) &\leq \frac{1}{2\pi} \lim_{v\to\infty} \int_0^{2\pi} u_v(a + re^{i\theta}X)d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}X)d\theta,\end{aligned}$$

where the equality comes from the monotone convergence theorem. Hence $u \in \mathcal{PSH}(\Omega)$.
In particular, if we define $\alpha_n := \sum_{v=1}^{n} u_v$ for $u_v \in \mathcal{PSH}(\Omega)$ with $u_v \leq 0$ for all v. Then $\alpha_n \searrow \alpha$ hence $\alpha \in \mathcal{PSH}(\Omega)$.

(ii) The upper semicontinuity again follows from Proposition 0.1.1. On one hand, the similar procedure as in part (i) with the dominated converges theorem gives us the mean value inequality for u.

$\square$

Our next aim is to characterize plurisubharmonic functions in terms of mean value inequalities. To this end, let $a = (a_1, \dots, a_n) \in \mathbb{C}^n, r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$.

If $u : \partial_0 \mathbb{P}(a, r) \to [-\infty, \infty)$ is bounded from above and measurable, then we define

$$\mathbf{P}(u; a, r; z) := \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \left(\prod_{j=1}^{n} \frac{r_j^2 - |z_j - a_j|^2}{|r_j e^{i\theta_j} - (z_j - a_j)|^2} \right) u(a + r.e^{i\theta}) d\Lambda_n(\theta),$$

for $z = (z_1, \dots, z_n) \in \mathbb{P}(a, r)$ and,

$$\mathbf{M}(u; a, r) := \mathbf{P}(u; a, r; a) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} u(a + r.e^{i\theta}) d\Lambda_n(\theta).$$

If $u : \mathbb{P}(a, r) \to [-\infty, \infty)$ is bounded from above and measurable then we define

$$\mathbf{A}(u; a, r) := \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{\mathbb{P}(a,r)} u d\Lambda_{2n} = \frac{1}{(\pi)^n} \int_{(\mathbb{D})^n} u(a + r.w) d\Lambda_{2n}(w)$$

Observe that

$$\begin{aligned}\mathbf{A}(u; a, r) &= \frac{2}{r_1^2} \dots \frac{2}{r_n^2} \int_0^{r_1} \dots \int_0^{r_n} \mathbf{M}(u; a, (\tau_1, \dots, \tau_n)) \tau_1 \dots \tau_n d\tau_1 \dots d\tau_n \\ &= 2^n \int_0^1 \dots \int_0^1 \mathbf{M}(u; a, (\tau_1 r_1, \dots, \tau_n r_n)) \tau_1 \dots \tau_n d\tau_1 \dots d\tau_n.\end{aligned}$$

In light of this observation, we may give the following:

Proposition 1.1.5. *[43] Let $\Omega \subset \mathbb{C}^n$ be open and let $u \in \mathcal{PSH}(\Omega)$, $a \in \Omega$. Then*

$$\boldsymbol{M}(u; a, r) \searrow u(a) \quad \text{when} \quad r \searrow 0, \quad \boldsymbol{A}(u; a, r) \searrow u(a) \text{when} \quad r \searrow 0$$

Proof. [43] According to observation above, it is enough to consider only $\mathbf{M}(u; a, \cdot)$. First, we prove that $\mathbf{M}(u; a, r') \leq \mathbf{M}(u; a, r'')$ for $r' = (r'_1, \dots, r'_n), r'' =$

$(r_1^{''},\dots,r_n^{''}), 0 < r_j^{'} \leq r_j^{''} < d(a,\Omega^c), j=1,\dots,n.$
To see this in the case $n=1$, let $r^{'}$ and $r^{''}$, where $r^{'} < r^{''}$ be arbitrary numbers in $(0, d(a,\Omega^c))$, and let $U^{\star}(z)$ be the smallest harmonic majorant of the function $u(z)$ in the circle $|z-a| < r^{''}$. Then,

$$\mathbf{M}(u;a,r^{'}) \leq \mathbf{M}(U^{\star};a,r^{'}) = \mathbf{M}(U^{\star};a,r^{''}) = \mathbf{M}(u;a,r^{''}),$$

which shows that the function $\mathbf{M}(u;a,r)$ is an increasing function of r in $(0,d(a,\Omega^c))$ [See [55], p. 59]. Hence, for $j=1,\dots,n$

$$\mathbf{M}(u(z^{'},\dots,z^{''});a_j,r_j^{'}) \leq \mathbf{M}(u(z^{'},\dots,z^{''});a_j,r_j^{''}), \quad (z^{'},a_j,z^{''}) \in \mathbb{P}(a,d(a,\Omega^c)).$$

Consequently, using a finite induction, one can easily get the desired inequality.
By Fatou's lemma we have

$$\begin{aligned} u(a) &\leq \lim_{r\to 0} \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} u(a + r.e^{i\theta}) d\Lambda_n(\theta) \\ &\leq \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \limsup_{r\to 0} u(a + r\cdot e^{i\theta}) d\theta \\ &\leq u(a), \end{aligned}$$

which proves that $\mathbf{M}(u;a,r) \searrow u(a)$ when $r \searrow 0$. □

Corollary 1.1.1. *[43] Let* $u_1, u_2 \in \mathcal{PSH}(\Omega)$*. If* $u_1 = u_2$ Λ_{2n}*-almost everywhere in* Ω*, then* $u_1 \equiv u_2$*.*

Proof. [43] Fix an $a \in \Omega$. Since $u_1 = u_2$ Λ_{2n}-almost everywhere, for $0 <$

$r < d(a, \Omega^c)$, we get

$$\begin{aligned}\mathbf{A}(u_1, a, r) &= \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)} \int_{\mathbb{P}(a,r)} u_1 d\Lambda_{2n} \\ &= \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)} \int_{\mathbb{P}(a,r)} u_2 d\Lambda_{2n} \\ &= \mathbf{A}(u_2, a, r)\end{aligned}$$

Therefore, by the above proposition, $u_1(a) = u_2(a)$. □

Plurisubharmonic functions, as a several complex variable counterpart of subharmonic functions, have many properties that can be deduced from the theory of one-dimensional case. For example;

Proposition 1.1.6. *[43] Let $\Omega \subset \mathbb{C}^n$ be open, let $u \in \mathcal{PSH}(\Omega)$, and let $\bar{\mathbb{P}}(a, r) \subset \Omega \quad (r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0})$. Then for $z \in \mathbb{P}(a, r)$*

$$u(z) \leq \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \left(\prod_{j=1}^{n} \frac{r_j^2 - |z_j - a_j|^2}{|r_j e^{i\theta_j} - (z_j - a_j)|^2} \right) u(a + r.e^{i\theta}) d\Lambda_n(\theta), \quad (1.1.3)$$

$$u(z) \leq \frac{1}{(0, 2\pi)^n} \int_{[0,2\pi]^n} u(a + r.e^{i\theta}) d\Lambda_n(\theta) \quad (1.1.4)$$

$$u(z) \leq \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)} \int_{\mathbb{P}(a,r)} u d\Lambda_{2n} = \frac{1}{(\pi)^n} \int_{(\mathbb{D})^n} u(a + r.w) d\Lambda_{2n}(w) \quad (1.1.5)$$

Proof. [43] Inequality (1.1.3) is well known for $n = 1$. In particular,

$$u(w^{'}, z_j, w^{''}) \leq \mathbf{P}(u(w^{'}, \cdot, w^{''}); a_j, r_j; z_j), \quad (w^{'}, z_j, w^{''}) \in \mathbb{P}(a, r) \quad j = 1, \dots, n. \quad (1.1.6)$$

Hence, after a finite induction, we get (1.1.3). Inequalities (1.1.4) and (1.1.5) can be shown in a similar way. □

Corollary 1.1.2. *[43] Let $D \Subset \mathbb{C}^n$ be a domain. If $u \in \mathcal{PSH}(D)$ and $u \not\equiv -\infty$, then $u \in \mathrm{L}^1_{loc}(D)$; in particular $\Lambda_{2n}(u^{-1}(-\infty)) = 0$.*

Proof. [43] Suppose that there exists a point $a \in D$ such that $\int_U u d\Lambda_{2n} = -\infty$ for every neighbourhood U of a. Let $2r := d(a, D^c)$. Observe that $\int_{\mathbb{P}(z,r)} u\Lambda_{2n} = -\infty$ for any $z \in \mathbb{P}(a,r)$. Consequently,

$$u(z) \leq \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)} \int_{\mathbb{P}(a,r)} u d\Lambda_{2n} = -\infty, \quad z \in \mathbb{P}(a,r).$$

Hence $u = -\infty$ in $\mathbb{P}(a,r)$. Let

$$D_0 := \{z \in D : u = -\infty \quad \text{in a neighbourhood of} \quad z\}.$$

We have proved that $D_0 \neq \emptyset$. The same method of proof shows that D_0 is closed in D. Thus $D_0 = D$ is a contradiction. □

Now we will give some theorems about the smoothing of plurisubharmonic functions. This is the generalization of the main approximation theorem of subharmonic functions in one-dimensional cases.

Let $\Omega \subset \mathbb{C}^n$, set

$$\Omega_\varepsilon := \{z \in \Omega : d(z, \Omega^c) > \varepsilon\}, \quad \varepsilon > 0.$$

For every function in $u \in \mathrm{L}^1_{loc}(\Omega)$ and $\Phi \in C_0^\infty(\mathbb{C}^n, \mathbb{R}^+)$, define

$$\begin{aligned} u_\varepsilon(z) &:= \int_\Omega u(w)\Phi_\varepsilon(z-w)d\Lambda_{2n}(w) \\ &= \int_{\mathbb{D}^n} u(z+\varepsilon w)\Phi(w)d\Lambda_{2n}(w), \quad z \in \Omega_\varepsilon. \end{aligned} \tag{1.1.7}$$

where

This function u_ε is called the ε-*regularization* of u.

Proposition 1.1.7. *[43] If $u \in \mathcal{PSH}(\Omega)$, $u \not\equiv -\infty$, then $u_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \searrow u$ pointwise in Ω when $\varepsilon \searrow 0$.*

Proof. [43] It is clear that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Take an $a \in \Omega_\varepsilon$. By the second part of (1.1.7), we get

$$u_\varepsilon(a) = (2\pi)^n \int_0^1 \dots \int_0^1 \mathbf{M}(u; a, \varepsilon(\tau_1, \dots, \tau_n))\Phi(\tau_1, \dots, \tau_n)\tau_1, \dots, \tau_n d\tau_1, \dots, d\tau_n.$$

Consequently, $u_\varepsilon \searrow u$. It remains to show that u_ε is plurisubharmonic. Now, if we fix $a \in \Omega_\varepsilon$, $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, and $0 < r < d(a, \Omega_\varepsilon^n)$, then

$$\begin{aligned}\frac{1}{2\pi}\int_0 2\pi u_\varepsilon\left(a\ + re^{i\theta}X\right)d\theta \\ &= \int_{\mathbb{D}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta}X + \varepsilon w)d\theta\right)\Phi(w)d\Lambda_{2n}(w) \\ &\geq \int_{\mathbb{D}^n} u(a + \varepsilon w)\Phi(w)d\Lambda_{2n}(w) = u_\varepsilon(a).\end{aligned}$$

□

1.2 Polar and Pluripolar Sets

Let M be a subset of a domain Ω in $\mathbb{C}$, we shall say that it is *polar* if there is a subharmonic function u in Ω which is not identically minus infinity and such that $E \subset u^{-1}(-\infty)$.

Lelong called a set *polar* if it is contained in $u^{-1}(-\infty)$ for some plurisubharmonic function u (a global definition) in 1945 [33], but later, in 1957 [34] he changed the definition to a local one as follows.

Definition 10. A set $M \subset \mathbb{C}^n$ is called *pluripolar* if any point $a \in M$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a \subset v_a^{-1}(-\infty)$.

By Corollary 1.1.1, pluripolar sets are of Lebesgue measure zero. The problem of whether an arbitrary pluripolar set can be described by one global plurisubharmonic function was solved by B. Josefson in 1978.

Theorem 1.2.1. ***(Josefson theorem)****[19] If $M \subset \mathbb{C}^n$ is pluripolar, then there exist a a $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

Proposition 1.2.1. *[43] Let $M_j \subset \mathbb{C}^n$ be pluripolar, $j \in \mathbb{N}$. Define*

$$M := \bigcup_{j=1}^{\infty} M_j.$$

Then M is pluripolar.

Proof. [43] By Josefson's theorem, for each $j \in \mathbb{N}$ there exists a $v_j \in \mathcal{PSH}(\mathbb{C}^n)$, $v_j \not\equiv -\infty$, such that $M_j \subset v_j^{-1}(-\infty)$. Since, for each j the set $v_j^{-1}(-\infty)$ is of measure zero, there exists a point $b \in \mathbb{D}^n$ such that $v_j(b) > -\infty$ for all j. We may assume that $v_j \leq 0$ on $\mathbb{P}(0, j)$ and $v_j(b) \geq -2^{-j}$, $j \in \mathbb{N}$. Define $v := \sum_{j=1}^{\infty} v_j$. Then $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v(b) \geq 1$, and $M \subset v^{-1}(-\infty)$. □

Definition 11. A subset E of an open set $\Omega \subset \mathbb{C}^n$ is called *negligible* if $E \subset \{u < u^*\}$, where $(u_i)_{i \in I} \subset \mathcal{PSH}(\Omega)$ locally bounded from above and $u = \sup u_i$.

Here the family u_i can be chosen to be countable by Chouquet's lemma. It is easy to see that if $E \subset \{v = -\infty\}$ for $v \in \mathcal{PSH}(\Omega)$ then E is negligible since $E \subset \{u < u^*\}$ for $u = \sup_{j \in \mathbb{N}} v/j$. In other words, pluripolar sets are negligible. The question whether the converse is true, posed by Lelong in 1966 [35] , remained unanswered until 1982, when Bedford and Taylor showed that this was indeed in case [2]. Cartan had solved the corresponding problem for subharmonic functions in 1942 [8].

Theorem 1.2.2. ***(Bedford-Taylor theorem)****[24] Negligible sets are pluripolar.*

Theorem 1.2.3. ***(Removable singularities of plurisubharmonic functions)****[43] Let M be a closed pluripolar set in Ω.*

(i) Let $u \in \mathcal{PSH}(\Omega \setminus M)$ be locally bounded from above in Ω. For $w \in \Omega \setminus M$ and $z \in \Omega$, define

$$\tilde{u}(z) := \limsup_{w \to z} u(w)$$

Then $\tilde{u} \in \mathcal{PSH}(\Omega)$.

(ii) For every function $u \in \mathcal{PSH}(\Omega)$ and $w \in \Omega \setminus M$ we have

$$u(z) = \limsup_{w \to z} u(w), \quad z \in \Omega.$$

(iii) If Ω is a domain, then the set $\Omega \setminus M$ is connected.

Proof. [43]

(i) The result has a local character. Then we may assume Ω is connected, $u \leq 0$ in $\Omega \setminus M$ and $M \subset v^{-1}(-\infty)$ with $v \in \mathcal{PSH}(\Omega)$, $v \leq 0$, $v \not\equiv -\infty$. For $i \in \mathbb{N}$, put

$$u_i = \begin{cases} u + (1/i)v, & \text{on } \Omega \setminus M, \\ -\infty, & \text{on } M. \end{cases}$$

Then $u_i \in \mathcal{PSH}(\Omega)$. If we define $u_0 = \sup_{i \in \mathbb{N}}$, we get $u_0 = u$ on $\Omega \setminus P$ and $u_0 = -\infty$ on P where $P := v^{-1}(-\infty)$. The envelope of u_0, $(u_0)^* \in \mathcal{PSH}(\Omega)$. Since the set $A := \{z \in \Omega : u_0(z) \leq (u_0)^*(z)\}$ is of measure zero, $(u_0)^* = u_0 = u$ on $\Omega \setminus (P \cup A)$. Therefore, by proposition 1.1.5, $(u_0)^* = u$ on $\Omega \setminus M$.
It remains to prove $(u_0)^* = \tilde{u}$. Clearly, $(u_0)^* = u = \tilde{u}$ on $\Omega \setminus M$. Take an $a \in M$, then

$$\begin{aligned} \tilde{u}(a) &= \limsup_{\Omega \setminus M \ni z \to a} u(z) = \limsup_{\Omega \setminus M \ni z \to a} (u_0)^*(z) \leq \limsup_{z \to a} (u_0)^*(z) = (u_0)^*(a) \\ &= \limsup_{z \to a} u_0(z) \leq \limsup_{\Omega \setminus P \ni z \to a} u_0(z) = \limsup_{\Omega \setminus P \ni z \to a} u(z) \\ &\leq \limsup_{\Omega \setminus P \ni z \to a} u(z) = \tilde{u}(a). \end{aligned}$$

(ii) Let

$$\tilde{u}(a) := \limsup_{\Omega\setminus M \ni w \to a} u(w), \quad z \in \Omega.$$

Then by part (a), $\tilde{u} \in \mathcal{PSH}(\Omega)$. Further, $\tilde{u} = u$ on $\Omega \setminus M$. Now, since $\Lambda_{2n}(M) = 0$, by applying proposition 1.1.5, we get the required result.
(c) Suppose that $\Omega \setminus M = U_1 \cup U_2$ where U_1 and U_2 are disjoint and nonempty open sets. Define $u : \Omega \setminus M \to [-\infty, \infty)$ by

$$u = \begin{cases} 0 & \text{on } U_1, \\ -\infty, & \text{on } U_2. \end{cases}$$

Then, in view of part (a), u has a plurisubharmonic extension to the whole Ω. Since u is locally integrable, if $U_2 \neq \emptyset$, then $u = -\infty$ on Ω and so $U_1 = \emptyset$. Hence, $\Omega \setminus M$ is connected.

□

We finish this section with one of the most important results about plurisubharmonic functions; the Hartogs lemma.

Lemma 1.2.1. *(**Hartogs Lemma**)[43] Let $(u_v)_{v=1}^{\infty} \subset \mathcal{PSH}(\Omega)$ be a sequence locally bounded from above. Assume that for some $m \in \mathbb{R}$,*

$$\limsup_{v\to\infty} u_v \leq m.$$

Then for every compact subset $K \subset \Omega$ for every $\varepsilon > 0$, there exists a v_0 such that

$$\max_K u_v \leq m + \varepsilon, \quad v \geq v_0.$$

Proof. [43] Take an $\varepsilon > 0$. It is sufficient to show that for every $a \in \Omega$ there exist $\delta(a) > 0$ and $v(a)$ such that $u_v \leq m + \varepsilon$ in $\mathbb{P}(a, \delta(a))$ for $v \geq v(a)$. Fix a and $0 < R < d(a, \Omega^c)/2$. We may assume that $u_v \leq 0$ in $\mathbb{P}(a, 2R)$ for any

$v \geq 1$, and $m < 0$. Let $0 < \delta < R/2$. Then

$$\begin{aligned}
\limsup_{v\to\infty} \sup_{z\in\mathbb{P}(a,\delta)} u_v(z) &\leq \limsup_{v\to\infty} \sup_{z\in\mathbb{P}(a,\delta)} \frac{1}{\pi^n(R+\delta)^{2n}} \int_{\mathbb{P}(z,R+\delta)} u_v d\Lambda_{2n} \\
&\leq \limsup_{v\to\infty} \frac{R^{2n}}{(R+\delta)^{2n}} \frac{1}{\pi^n R^{2n}} \int_{\mathbb{P}(z,R)} u_v d\Lambda_{2n} \\
&\leq \frac{R^{2n}}{(R+\delta)^{2n}} \frac{1}{\pi^n R^{2n}} \int_{\mathbb{P}(z,R)} \limsup_{v\to\infty} u_v d\Lambda_{2n} \\
&\leq \frac{R^{2n}}{(R+\delta)^{2n}} \frac{1}{\pi^n R^{2n}} \int_{\mathbb{P}(z,R)} m d\Lambda_{2n} \\
&\leq \frac{R^{2n}}{(R+\delta)^{2n}} m < m + \varepsilon,
\end{aligned}$$

provided that δ is sufficiently small. □

1.3 Relation between convex functions and plurisubharmonic functions

Convex functions constitute an important part of plurisubharmonic functions. Therefore, a certain number of propositions in the theory of convex functions follow from the corresponding assertions regarding plurisubharmonic functions. For more detailed information on convex functions, see, for example Hörmander [18], and Bremermann [4].

Definition 12. A real valued function $u(x)$ of real variable x is said to be *convex* in an interval (a, b) if, for all x and x' in (a, b) and for all $\lambda \in [0, 1]$, it satisfies

$$u(\lambda x + (1-\lambda)x') \leq \lambda u(x) + (1-\lambda)u(x').$$

A function $u(x)$, where $x = (x_1 \ldots, x_n)$ is said to be *convex* in a domain $U \subset \mathbb{R}^n$ if, for all $x^0 \in U$ and b such that $|b| = 1$, the function $u(x^0 + tb)$ is convex with respect to t in every interval contained in the open set

$$U_{x^0,b} = \{t : x^0 + tb \in U\}.$$

The properties of convex functions are analogues of continuous plurisubharmonic functions and they follow from the properties of convex functions of one variable like as the most of the properties of plurisubharmonic functions follow from the properties of subharmonic functions of one complex variable case.

Let $U \subset \mathbb{R}^N$ is open and $v \in C^2(U, \mathbb{R})$, then we define the *real Hessian*

$$\mathscr{H}v(x;\xi) := \sum_{j,k=1}^{N} \frac{\partial^2 v}{\partial x_j \partial x_k}(x)\xi_j\xi_k, \quad x \in U, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.$$

Then the function v is said to be *convex* in U if $\mathscr{H}v$ is positive semidefinite in U, i.e. $\mathscr{H}v \geq 0$ for all $x \in U$ and $\xi \in \mathbb{R}^n$ ([55], section 11.3).

Proposition 1.3.1. *[55] Let If a real valued function $u(z) = u(x, y)$ is convex in a domain $U \subset \mathbb{C}^n \approx \mathbb{R}^n + i\mathbb{R}^n$, it is plurisubharmonic in that domain.*

Proof. [55] If we take $\xi_j = \alpha_j + i\beta_j$, we have,

$$\begin{aligned} \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} &= \frac{1}{4}\sum_{j}(\frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2})(\alpha_j^2 + \beta_j^2) \\ &+ \frac{1}{4}\sum_{j \neq k}[(\frac{\partial^2 u}{\partial x_j \partial x_k} + \frac{\partial^2 u}{\partial y_j \partial y_k})(\alpha_j\alpha_k + \beta_j\beta_k) \\ &+ (\frac{\partial^2 u}{\partial x_j \partial y_k} - \frac{\partial^2 u}{\partial x_k \partial y_j})(\beta_j\alpha_k - \beta_k\alpha_j)]. \end{aligned} \tag{1.3.1}$$

That is, we have

$$4\mathscr{L}u((x + iy);(\xi)) = \mathscr{H}((x, y);(\alpha, \beta)) + \mathscr{H}((x, y),(\beta, -\alpha))$$

which gives the desired.

□

Proposition 1.3.2. *[43] Let U be a domain in $\mathbb{R}^n$ and let $v : U \to [-\infty, \infty)$. Define*

$$\widetilde{U} := U + i\mathbb{R}^n \subset \mathbb{C}^n, \quad \tilde{v}(x+iy) := v(x), \quad x+iy \in \widetilde{U}.$$

Then $\tilde{v} \in \mathcal{PSH}(\widetilde{U})$ if and only if v is convex on U.

Proof. [55] First consider the case where v is of class $C^2(U)$. Since $\frac{\partial v}{\partial y_j} = 0$ for $j = 1, \dots, n$ by the equation (1.3.1), we get

$$4\mathscr{L}\tilde{v}(x+iy; a+ib) = \mathscr{H}v(x;a) + \mathscr{H}v(x;b),$$

which implies the desired result.
In general case, assume that $\tilde{v}$ is plurisubharmonic and let $\tilde{v}_\varepsilon$ denote the ε-regularization of $\tilde{v}$. Observe that $\widetilde{U}_\varepsilon + i\mathbb{R}^n = \widetilde{U}_\varepsilon$. Hence $\widetilde{U}_\varepsilon = U^\varepsilon + i\mathbb{R}^n$ for an open set $U_\varepsilon \subset \mathbb{R}^n$. Moreover,

$$\begin{aligned}\tilde{v}_\varepsilon(z+it) &= \int_{\mathbb{D}^n} \tilde{v}(z+it+\varepsilon w)\Phi(w)d\Lambda_{2n}(w)\\ &= \int_{\mathbb{D}^n} \tilde{v}(z+\varepsilon w)\Phi(w)d\Lambda_{2n}(w) = \tilde{v}_\varepsilon(z), \quad z \in \widetilde{U}_\varepsilon, t \in \mathbb{R}^n\end{aligned}$$

Hence $\tilde{v}_\varepsilon(x+iy) = v^\varepsilon(x)$, $x+iy \in \widetilde{U}_\varepsilon$, where $v^\varepsilon : U^\varepsilon \to \mathbb{R}$. Note that $v^\varepsilon \searrow v$. By the first part of the proof, v^ε is convex in U^ε for any $\varepsilon > 0$. Consequently, v is convex.

□

The following can be seen as consequences of the above proposition.

Proposition 1.3.3. *[36] Suppose Ω is a domain in $\mathbb{C}^n$ with the following property: for $z = (z_k) = (x_k + iy_k) \in \Omega$ and $0 \le t \le 1$, we have $z' = (x_k + ity_k) \in \Omega$. Then if $\varphi \in \mathcal{PSH}(\Omega)$ depends only on x_k, it is continuous convex function of $x = (x_1, \dots, x_n)$*

Proof. [36] Let π be the natural projection onto the real coordinate, i.e. $\pi(z) = x$ for $z = x + iy$. Then φ extends in a natural way to a plurisubharmonic function on $\Omega' = \omega \times \mathbb{R}^n$, where $\omega = \phi(\Omega)$. Let $\varepsilon > 0$ and $\Omega_\varepsilon = \{z \in \Omega' : d(z, \Omega'^c) > \varepsilon\}$. Then $\varphi_\varepsilon \in \mathcal{PSH}(\Omega') \cap C^\infty(\Omega'_\varepsilon)$ and φ_ε depends only on x. Further, $\mathscr{L}\varphi_\varepsilon(\cdot, X) = \sum_{j,k=1}^n \frac{\partial^2 \varphi_\varepsilon}{\partial x_j \partial x_k} X_j \bar{X}_k$, and if $X \in \mathbb{R}^n$, φ_ε is seen to be convex. Since a decreasing sequence of convex functions is convex ϕ is convex, and since a convex function locally bounded from above is continuous([55], Section 11.2), φ is continuous. □

Corollary 1.3.1. *[36] Let $\Omega \Subset \mathbb{C}^n$ be a domain $\Omega = \{z : 0 \leq r'_j \leq |z_k| \leq r''_j\}$. A function $\varphi(r)$, $r = (r_1, \dots, r_n)$, $r_j = |z_j|$ defined in Ω is in $\mathcal{PSH}(\Omega)$ if and only if it is a convex function of the variable $v = (v_1, \dots, v_n)$, where $v_j = \log r_j$.*

Proof. [36] Let $z = (z_1, \dots, z_n) \in \Omega$. Then we can find a neighborhood ω_z of z such that we can define a branch $\log z_k = v_k + iv'_k$ of z_k in ω_z for every k. For $\varphi \in \mathcal{PSH}(\Omega)$, $\psi(v_k) = \tilde{\psi}(v_k + iv'_k) = \varphi(e^{v_1}, \dots, e^{v_n})$ is plurisubharmonic function of the variable $(v_1 + iv'_1, \dots, v_n + iv'_n)$. By above proposition, it is a convex function of $v = (v_1, \dots, v_n)$. Conversely, if $\psi(v)$ is defined in an open set $\omega = \{v : \log r'_j \leq v_j \leq \log r''_j\}$ and is convex function of the variable v, we extend ψ as a convex function on $\omega + i\mathbb{R}^n$ by $\psi(v_k) = \tilde{\psi}(v_k + iv'_k)$. Hence $\tilde{\psi} \in \mathcal{PSH}(\omega + i\mathbb{R}^n)$. □

Chapter 2

Construction New Plurisubharmonic Functions

In this chapter we will review certain instruments of obtaining new plurisubharmonic functions from the given ones.

2.1 New Plurisubharmonic Functions From Old

In this section we will give some elementary but convenient ways to get new plurisubharmonic functions from old ones. Up to now we have encountered some instruments of forming plurisubharmonic functions. For example, it is well known that the family $\mathcal{PSH}(\Omega)$ is a positive cone, i.e. If α and β are non-negative numbers and u, v are plurisubharmonic functions on Ω, then $\alpha u+\beta v$ is plurisubharmonic on Ω. We have already mentioned in Proposition 1.2.1 that if f is holomorphic then $\log|f|$ is plurisubharmonic. If $(u_j)_{j\in\mathbb{N}}$ is a decreasing sequence of plurisubharmonic functions then $\lim_{j\to\infty} u_j = u(z)$ is plurisubharmonic hence if $(v_j)_{j\in\mathbb{N}}$ is a sequence of negative plurisubharmonic functions then $v := \sum_{j=1}^{\infty} v_j$ is plurisubharmonic.

The upper semicontinuiy of the above functions follows from Proposition 0.2.1. In addition to these ones, upper semicontinuous regularization of supremum of a sequence of plurisubharmonic functions is plurisubharmonic. Besides of these, a very useful tool is the following.

Proposition 2.1.1. *[24] Let $\Omega \subset \mathbb{C}^n$ and $u \in \mathcal{PSH}(\Omega)$. If ψ is a real valued increasing convex function, then $\psi \circ u$ is plurisubharmonic.*

Proof. [24] Since convex functions are continuous on intervals $\phi \circ u$ is obviously upper semicontinuous. Also if $\bar{B}(z, \rho) \subset \Omega$,

$$\psi \circ u(z) \leq \psi \left(\frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta} X) d\theta \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi \circ u(z + re^{i\theta} X) d\theta,$$

where the second inequality comes from Jensen's Inequality([44] Theorem 2.6.2) for convex functions. Hence $\phi \circ u$ is a plurisubharmonic function. $\square$

Proposition 2.1.2. *[36] Let Ω be a domain in $\mathbb{C}^n$, and $(\Omega_i)_{i=1}^{\infty}$ be a covering of Ω by domains Ω_i. If u is defined on Ω and $u \in \mathcal{PSH}(\Omega_i)$ for every i, then $u \in \mathcal{PSH}(\Omega)$.*

Proof. [36] Since u is upper semicontinuous on domains Ω_i for all i, it is also upper semicontinuous on $\Omega := \bigcup_{i=1}^{\infty} \Omega_i$ and submean value inequality holds also for Ω since it is a local property, it is satisfying on each connected component Ω_i. $\square$

2.2 Pushout and Pullback

Let $u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega)$ and $F : \Omega' \to \Omega$ be holomorphic, where $\Omega' \subset \mathbb{C}^m$ is open. Then by using the definition of Levi form, for $b \in \Omega'$ and $Y \in \mathbb{C}^n$ we have

$$\mathscr{L}(u \circ F)(b; Y) = \mathscr{L}u(F(b); F'(b)(Y)).$$

Indeed, from the chain rule we have

$$\begin{aligned}\frac{\partial^2(u\circ F)}{\partial z_j\partial\bar{z}_j} &= \frac{\partial}{\partial z_j}\Big(\sum_{r=1}^{m}\frac{\partial u}{\partial\bar{w}_r}\frac{\partial\bar{f}_r}{\partial\bar{z}_k}\Big)\\ &= \Big(\sum_{r,s=1}^{m}\frac{\partial^2 u}{\partial w_s\partial\bar{w}_r}\frac{\partial f_s}{\partial z_j}\overline{\frac{\partial f_r}{\partial z_k}}\Big)+\sum_{r=1}^{m}\frac{\partial u}{\partial\bar{w}_r}\frac{\partial}{\partial z_j}\Big(\frac{\partial\bar{f}_r}{\partial\bar{z}_k}\Big)\end{aligned}$$

and the last sum vanish because f is holomorphic. Hence

$$\mathscr{L}(u\circ F)=\sum_{r,s=1}^{m}\frac{\partial^2 u}{\partial w_s\partial\bar{w}_r}\frac{\partial f_s}{\partial z_j}\overline{\frac{\partial f_r}{\partial z_k}}\geq 0$$

Proposition 2.2.1. *[24] Let $\Omega^{'}\subset\mathbb{C}^n$ be open and let $F\in\mathcal{O}(\Omega^{'},\Omega)$. Then $u\circ F\in\mathcal{PSH}(\Omega^{'})$ for all $u\in\mathcal{PSH}(\Omega)$.*

Proof. [24] We may assume that $u\in\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. If $u\in C^2$ then the result follows from the above formula. In general case, let u_ε denote the ε-regularization of u. Put $\Omega^{'}:=F^{-1}(\Omega_\varepsilon)$, then $u\circ F\in\mathcal{PSH}(\Omega^{'}_\varepsilon)$ and $u_\varepsilon\circ F\searrow u\circ F$. Consequently, $u\circ F\in\mathcal{PSH}(\Omega)$. □

The following proposition gives tools for defining new plurisubharmonic functions by using special holomorphic functions. Before stating it, we will describe some basic facts concerning proper holomorphic mappings and the notion of analytic cover. We refer to reader [47] and [16], for detailed information.

Definition 13. Let X and Y be topological spaces. A continuous map $f:X\to Y$ is said to be *proper* if $f^{-1}(K)$ is compact in X for every compact set $K\subset Y$.

We shall study proper holomorphic maps $f:\Omega\to\Omega^{'}$ where Ω and $\Omega^{'}$ are domains in $\mathbb{C}^n$. In this context, the compactness of $f^{-1}(K)$ for every compact $K\subset\Omega^{'}$ is equivalent to the following requirement: if $\{\alpha_i\}$ is a sequence of in Ω that has no limit point in Ω, then $\{f(\alpha_i)\}$ has no limit point in $\Omega^{'}$.

Some elementary facts. Let Ω and $\Omega^{'}$ are domains in $\mathbb{C}^n$ and suppose that $f : \Omega \to \Omega^{'}$ is holomorphic and proper.
If $w = (w_1, \ldots, w_n) \in \Omega^{'}$, then $f^{-1}(w)$ is a subvariety of Ω, being the intersection of zero sets $f_i - w_i$, where f_i is ith component of f and $f^{-1}(w)$ is compact since f is proper. Hence $f^{-1}(w)$ is a finite set (Theorem 14.3.1 in [47]).
Let $M := \{J = 0\}$, J being the Jacobian of f. Its image f(M) is called the *critical set* of f. Each $w \in f(M)$ is a critical value of f. Every other point of $f(\Omega)$ is called a *regular value* of f.
Since f is proper, it is a closed map: if E is closed in Ω then $f(E)$ is closed in $\Omega^{'}$. In particular, $f(M)$ and $f(\Omega)$ are closed in $\Omega^{'}$, and the set of regular values of f form an open set.

Let Ω be a domain in $\mathbb{C}^n$. By a covering of Ω, we mean that a domain $\Omega^{'}$ and proper map

$\pi : \Omega^{'} \to \Omega$ that is a covering space in the topological sense of word. This means by definition that every point $z \in \Omega$ has a neigborhood U_z with $\pi^{-1}(U_z)$ is a disjoint union of open sets $\{V_j\}$ such that $\pi|_{V_j} : V_j \to \Omega$ is a homeomorphism onto U_z.

A topological surface $\Omega^{'}$, one can put a complex structure on $\Omega^{'}$ by using π. In this case $\pi|_{V_j} : V_j \to \Omega$ becomes a biholomorphism onto U_z, by the notation above. More information about topological covering can be found in [12] and [13].

Definition 14. A mapping $f : X \to Y$ is called *light* if $f^{-1}(y)$ consists of only a discrete set of points, for all $y \in Y$.

In view of the above material, we introduce the notion of analytic cover.

Definition 15. An *analytic cover* is a triple (X, π, Ω) such that

(i) X is locally compact Hausdorff space;

(ii) Ω is a domain in $\mathbb{C}^n$;

(iii) π is proper, light, continuous mapping of X onto Ω;

(iv) there is a negligible set $A \subset \Omega$, and an integer λ such that π is a λ-sheeted covering map from $X \setminus \pi^{-1}(A)$ onto $\Omega \setminus A$;

(v) $X \setminus \pi^{-1}(A)$ is dense in X.

The following theorem is a useful instrument for constructing new plurisubharmonic functions by pullback, which is our purpose and the reason that why we mentioned above procedure.

Theorem 2.2.1. *[16]. Let Ω be a domain in $\mathbb{C}^n$. Suppose $\Omega^{'}$ is a domain in and $f : \Omega \to \Omega^{'}$ is proper. Then $(\Omega, f, \Omega^{'})$ is an analytic cover.*

Finally, we give our construction by pullback.

Proposition 2.2.2. *[25] Let $f : \Omega \to \Omega^{'}$ be a proper holomorphic surjection between two open sets in $\mathbb{C}^n$. If $u \in \mathcal{PSH}(\Omega)$, then the formula*

$$v(z) = \max\{u(w) : w \in f^{-1}(z)\} \quad (z \in \Omega^{'})$$

defines a plurisubharmonic function.

Proof. [24][25] Without lost of generality, assume that $\Omega^{'}$ is connected. Denote by A the zero locus of the Jacobian of f, i.e. A is the critical set of f which is a subvariety of Ω. Then, by proper mapping theorem ([16] p.162), $f(A)$ is a subvariety in $\Omega^{'}$, hence it is negligible.

In view of the main approximation theorem, it is enough to show that the proposition is true for continuous function. Assume that $u \in \mathit{\Omega} \cap \mathcal{PSH}(\Omega)$. By previous theorem, the proper mapping f is a analytic cover. Hence f is open [16] and closed. Let a and b are real numbers with $a < b$, then

$$v^{-1}\left((a,b)\right) = f^{-1}\left(u^{-1}\left((a,\infty)\right)\right) \setminus f^{-1}\left(u^{-1}\left([b,\infty)\right)\right).$$

Consequently, v is continuous on $\Omega^{'}$. Now, by proposition of holomorphic functions (Theorem 1.3.1 in[24]), the proper surjection

$$f|_{f^{-1}(\Omega'\setminus f(A))} : \Omega^{'} \setminus f(A) \to \Omega^{'} \setminus f(A)$$

is locally biholomorphic. Hence, there is a natural number k such that for each $z \in \Omega^{'} \setminus f(A)$ there exist a neighborhood $V \subset \Omega^{'} \setminus f(A)$ of z, and mutually disjoint neighborhoods $U_1, \ldots, U_k$ of $w_1, \ldots, w_k$, respectively where $\{w_1, \ldots, w_k\} = f^{-1}(z)$ such that $f|_{U_j} : U_j \to V$ is a biholomorphic mapping for each j and $f^{-1}(V) = \bigcup_{j=1}^{k} U_j$.
Consequently, $v \in \mathcal{PSH}(\Omega^{'} \setminus f(A))$ and since v is continuous, $f(A)$ is pluripolar. By removable singularities theorem, $v \in \mathcal{PSH}(\Omega^{'})$, as required. $\square$

2.3 Gluing Lemma

In many situations we need to glue two plurisubharmonic functions together and next lemma help us with this.

Lemma 2.3.1. *(**Gluing lemma**)[43] Let $G \subset \Omega \subset \mathbb{C}^n$ be open and let $v \in \mathcal{PSH}(G)$ and $u \in \mathcal{PSH}(\Omega)$. Assume that*

$$\limsup_{G \ni z \to \xi} v(z) \leq u(\xi) \quad \xi \in \Omega \cap \partial G.$$

Put

$$\tilde{u}(z) := \begin{cases} \max\{v(z), u(z)\}, & \text{for } z \in G \\ u(z), & \text{for } z \in \Omega \setminus G. \end{cases}$$

Then $\tilde{u} \in \mathcal{PSH}(\Omega)$.

Proof. [43] The upper semicontinuity of $\tilde{u}$ is obvious from assumption. Obviously $\tilde{u}$ is plurisubharmonic on $\Omega \setminus \partial G$. Take a point $a \in \Omega \cap \partial G$, a vector

$X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, and $0 < r < d(a, \Omega^c)$. Then

$$\tilde{u}(a) = u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u_{a,X}(re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_{a,X}(re^{i\theta}) d\theta.$$

Then $\tilde{u} \in \mathcal{PSH}(\Omega)$. $\square$

Notice that, if $u \leq v$ on G, then $\tilde{u}$ can be seen as a plurisubharmonic extension of u to whole Ω. In view of that observation, we may give the following corollaries of Gluing lemma.

Proposition 2.3.1. *Let Ω_1 and Ω_2 are domains in $\mathbb{C}^n$ such that $\Omega := \Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that $u_1 \in \mathcal{PSH}(\Omega_1)$ and $u_2 \in \mathcal{PSH}(\Omega_2)$ such that*

$$\liminf_{z \to \zeta} u_1(z) \leq u_2(\zeta) \quad \zeta \in \partial\Omega \cap \Omega_2$$

and

$$\liminf_{z \to \xi} u_2(z) \leq u_1(\xi) \quad \xi \in \partial\Omega \cap \Omega_1.$$

Define a function Ψ on $\Omega_1 \cup \Omega_1$ as

$$\Psi(z) := \begin{cases} u_1 & \text{for } z \in \Omega_1 \setminus \Omega \\ \max\{u_1(z), u_2(z)\}, & \text{for } z \in \Omega \\ u_2 & \text{for } z \in \Omega_2 \setminus \Omega\ . \end{cases}$$

Then $\Psi \in \mathcal{PSH}(\Omega_1 \cup \Omega_2)$

Proof. Since $\Omega \subset \Omega_1$, applying Gluing lemma for $u_1 \in \mathcal{PSH}(\Omega_1)$ and $u_2 \in \mathcal{PSH}(\Omega)$, we obtain a new plurisubharmonic function, Ψ_1, on Ω_1. Likewise, for $u_2 \in \mathcal{PSH}(\Omega_2)$ and $u_1 \in \mathcal{PSH}(\Omega)$, we get $\Psi_2 \in \mathcal{PSH}(\Omega_2)$. Now, since $\Psi_1|_\Omega = \Psi_2|_\Omega$, we may define Ψ as a plurisubharmonic extension of Ψ_1 on $\Omega_1 \cup \Omega_2$ such that $\Psi(z) = \Psi_2(z)$ for all $z \in \Omega_2 \setminus \Omega_1$. $\square$

Proposition 2.3.2. *Let Ω_1 and Ω_2 are disjoint open sets and K be a compact set in $\mathbb{C}^n$ and let $u_1 \in \mathcal{PSH}^c(\Omega_1 \cup K)$ and $u_2 \in \mathcal{PSH}^c(\Omega_2 \cup K)$ where*

$\mathcal{PSH}^c(\Omega)$ denotes the set of continuous plurisubharmonic functions on Ω. Choose a neighborhood of V of K such that $u_j \in \mathcal{PSH}(V)$ for $j = 1, 2$. If $u_1|_{\Omega_2 \cap V} = u_2$, then u_1 has an extension, u, on $\Omega_1 \cup K \cup \Omega_2$ such that $u(z) = u_2(z)$ on Ω_2. Similarly, if $u_2|_{\Omega_1 \cap V} = u_1$, then u_2 has a plurisubharmonic extension, v, on $\Omega_1 \cup K \cup \Omega_2$ such that $v(z) = u_1(z)$ on Ω_1.

2.4 Fusion Lemma

In the conclusions above, we mentioned that, under which conditions, we may extend a plurisubharmonic functions by applying Gluing lemma. Strictly speaking, we are given a function u on a set G and we seek to approximate u by a function v plurisubharmonic on a larger open set Ω containing G. This type of approximation is related to the problem of plurisubharmonic extensions. In fact, if we find a plurisubharmonic function v on Ω which is actully agrees with u on G, then, without any doubt, v is a very good approximation indeed, for the error function $u - v$ is identically is zero on G, as the derived corollaries above. But, what if the error function is nonzero? The Fusion lemma gives allows to approximate two functions simultaneously.

Lemma 2.4.1. *(**Fusion Lemma**)[14] Let δ be a strictly plurisubharmonic function on $\mathbb{C}^n$. Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}^n$ with $\bar{\Omega_1} \cap \bar{\Omega_2} = \emptyset$ and $\bar{\Omega_2}$ compact. Then, there is a constant C such that if $K \subset \mathbb{C}^n$ is any compact set, $\varepsilon > 0$, and u_j are functions such that $u_j \in \mathcal{PSH}^c(\Omega_j \cup K)$ for $j = 1, 2,$. Then there exists $v \in \mathcal{PSH}^c(\Omega_1 \cup K \cup \Omega_2)$ such that, for all $z \in \Omega_j \cup K$,*

$$|v(z) - u_j(z)| \leq C \cdot \max\{|\delta(z)|, 1\} \cdot \max\{\|u_1 - u_2\|_K, \varepsilon\}. \tag{2.4.1}$$

for $j = 1, 2$.

Proof. Let $\chi_1 \in C_0^\infty(\mathbb{C}^n)$ such that $-1 \leq \chi_1 \leq 0$, $\chi_1 = -1$ on $\bar{\Omega_2}$ and $\chi_1 = 0$ on $\bar{\Omega_1}$. Set $\chi_2 = -1 - \chi_1$. Choose λ positive and so small that $\delta + \lambda\chi_j$ are both plurisubharmonic for $j = 1, 2$. Choose a neighborhood V of K so that

$u_j \in \mathcal{PSH}^c(\bar{V})$ and

$$\|u_1 - u_2\|_V < 2\max\{\varepsilon, \|u_1 - u_2\|_K\}.$$

We define a positive constant η by

$$\lambda\eta := 2\max\{\varepsilon, \|u_1 - u_2\|_K\}.$$

Now set

$$f_j = u_j + \eta(\delta + \lambda\chi_j)$$

on $\Omega_j \cup V$ and $f_j \equiv -\infty$ elsewhere. Finally, we set

$$v = \max\{f_1, f_2\}.$$

Obviously, v is plurisubharmonic and continuous on $(\Omega_1 \cup V \cup \Omega_2) \setminus \partial V$. Assume that $z_0 \in \partial V \cap \Omega_1$. Since $\chi_1 = 0$ on Ω_1 and $\chi_2 = -1$ on Ω_1, if z is near z_0 and $z \in \bar{V}$, then

$$f_2(z) = u_2(z) + \eta(\delta - \lambda) = f_1(z) + [u_2(z) - u_1(z)] - \lambda\eta < f_1(z).$$

Since $f_2(z) = -\infty$ on $\Omega_1 \setminus \bar{V}$, we have that $f_2(z) \leq f_1(z)$ for all z near z_0. The point z_0 was arbitrary point of $\partial V \cap \Omega_1$ ans so it follows that v is continuous and plurisubharmonic on a neighborhood of $\partial V \cap \Omega_1$ and, by similar argument, on a neighborhood of $\partial V \cap \Omega_2$. Thus, $v \in \mathcal{PSH}(\Omega_1 \cup V \cup \Omega_2)$.

There remains to verify that v has required approximation property.

On $\Omega_1 \setminus \bar{V}$,

$$\begin{aligned}|v(z) - u_1(z)| &= |f_1(z) - u_1(z)| = \eta|\delta(z) + \lambda\chi_1(z)| \\ &= \eta|\delta(z)| = 2\lambda^{-1}\max\{\varepsilon, \|u_1 - u_2\|_K\}\delta(z).\end{aligned}$$

On $\Omega_1 \cap \bar{V}$, we consider two cases. First of all, if $|v(z) - u_1(z)| = v(z) - u_1(z)$, then, since $v \leq \max\{u_1, u_2\} + \eta(|\delta| + \lambda)$, we have

$$
\begin{aligned}
|v(z) - u_1(z)| &\leq \max\{(u_1(z), u_2(z)) - u_1(z) + \eta(|\delta(z)| + \lambda)\} \\
&\leq 2\max\{\varepsilon, \|u_1 - u_2\|_K\} + \lambda^{-1} 2\max\{\varepsilon, \|u_1 - u_2\|_K\}(|\delta(z)| + \lambda).
\end{aligned}
$$

On the other hand, if $|v(z) - u_1(z)| = u_1(z) - v(z)$, then

$$
\begin{aligned}
|v(z) - u_1(z)| &= u_1(z) - \max\{f_1(z), f_2(z)\} \\
&\leq u_1 - \max\{u_1(z) - \eta(|\delta(z)| + \lambda), u_2(z) - \eta(|\delta(z)| + \lambda)\} \\
&= u_1(z) - \max\{u_1(z), u_2(z)\} + \eta(|\delta(z)| + \lambda)
\end{aligned}
$$

which yields the same estimate as in the first case. Thus, for $C = 4 + 2\lambda^{-1}$, 2.4.1 holds on $\Omega_1 \cap \bar{V}$. The same estimates on $\Omega_2 \cap \bar{V}$ are similar. This completes the proof of the fusion lemma. □

Note that, if we are interested in finding v plurisubharmonic in $\Omega_1 \cup K^0 \cup \Omega_2$ rather than $\Omega_1 \cup K \cup \Omega_2$, then, in case $\|u_1 - u_2\|_K = 0$, the inequality 2.4.1 is trivial, for of course u_1 and u_2 are then plurisubharmonic extensions of each other and we may set $v = u_j$ on $\Omega_j \cup K^0$, as in proposition above. Moreover, the formula 2.4.1 can be simplified for most applications. Indeed, if

(i) $\|u_1 - u_2\|_K \neq 0$, then $|v(z) - u_j(z)| \leq C \cdot \max\{|\delta(z)|, 1\} \cdot \|u_1 - u_2\|_K$;

(ii) $\|u_1 - u_2\|_K = 0$, then $|v(z) - u_j(z)| \leq \varepsilon \cdot \max\{|\delta(z)|, 1\}$;

(iii) δ is bounded on $\Omega_1 \cup K \cup \Omega_2$, then $|v(z) - u_j(z)| \leq C \cdot \max\{\|u_1 - u_2\|_K, \varepsilon\}$;

(iv) $\|u_1 - u_2\|_K \neq 0$ and δ is bounded on $\Omega_1 \cup K \cup \Omega_2$, then

$$
|v(z) - u_j(z)| \leq C \cdot \|u_1 - u_2\|_K;
$$

(v) $\|u_1 - u_2\|_K = 0$ and δ is bounded on $\Omega_1 \cup K \cup \Omega_2$, then

$$|v(z) - u_j(z)| \leq \varepsilon.$$

In particular, from (iii) we see that if we assured of existence of a bounded strictly plurisubharmonic function on $\Omega_1 \cup K \cup \Omega_2$, then we may omit any mention of δ in lemma.

Example 3. Let Ω_1, Ω_2 and K as in the above lemma with the property that $\Omega_1 \cap K = \Omega_2 \cap K = \emptyset$. Then we may construct the function v as

$$v(z) := \begin{cases} u_1(z) & \text{for } z \in \Omega_1 \cup V \\ u_2(z), & \text{for } z \in \Omega_2. \end{cases}$$

or

$$v(z) := \begin{cases} u_1(z) & \text{for } z \in \Omega_1 \\ u_2(z), & \text{for } z \in \Omega_2 \cup V. \end{cases}$$

Example 4. Let Ω_1, Ω_2 and K be as in the Fusion lemma. Assume that $\Omega_1 \cap K \neq \emptyset$ and $\Omega_2 \cap K = \emptyset$. In this case, we may define the function v with the following fashion

$$v(z) := \begin{cases} u_1(z) & \text{for } z \in \Omega_1 \cup V \\ u_2(z), & \text{for } z \in \Omega_2. \end{cases}$$

Chapter 3

Maximal Plurisubharmonic Functions

As mentioned before, plurisubharmonic function are generalizations of the subharmonic functions. A natural counterpart to the class of harmonic functions is the class of so called maximal plurisubharmonic functions which will be discussed in this chapter.

In $\mathbb{C}^n$, if $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k$, $k \leq n$, using multi-index notation, we define

$$dz^\alpha := dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_k}, \qquad d\bar{z}^\alpha := d\bar{z}^{\alpha_1} \wedge \ldots \wedge d\bar{z}^{\alpha_k}.$$

Now we will define another very important operator

$$d^c := i(\bar{\partial} - \partial)$$

noting that

$$dd^c = 2i\partial\bar{\partial},$$

and, if $u \in C^2(\Omega)$, then

$$dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

From this formula and the fact that for any $b, c \in \mathbb{C}^n$,

$$dz_j \wedge d\bar{z}_k(b, c) = b_j \bar{c}_k - \bar{b}_k c_j,$$

we deduce that

$$(dd^c u)(a)(b, c) = -4\text{Im} < \mathscr{L}u(a)b, c >$$

for any $a \in \Omega$ and $b, c \in \mathbb{C}^n$.

3.1 Maximal Plurisubharmonic Functions

In the definition below, we use the same terminology of Sadullaev ([49] and [50]).
Let Ω be an open subset of $\mathbb{C}^n$.

Definition 16. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u \in \mathcal{PSH}(\Omega)$ is said to be *maximal* if for any open set $G \subset\subset \Omega$ and $v \in \mathcal{PSH}(G)$ such that $\limsup_{z \to p} v(z) \leq u(p)$ for all $p \in \partial G$ it follows that $v \leq u$ in G.

We shall be using the symbol $\mathcal{MPSH}(\Omega)$ to denote the family of all maximal plurisubharmonic functions on Ω.
Notice that if $n = 1$, then $\mathcal{MPSH}(\Omega) = \mathcal{H}(\Omega)$. Indeed, let u be a continuous maximal function on Ω and $\hat{u}$ is a harmonic function with $u = \hat{u}$ on ∂B for an arbitrary ball $B \subset\subset \Omega$. Then since $\hat{u} \leq u$ on ∂B, by maximality of u, we have $\hat{u} \leq u$ on B. On the other hand, since $\hat{u} \geq u$ on ∂B, by maximum principle for subharmonic functions, $\hat{u} \geq u$ on Ω. Hence, $u = \hat{u}$ meaning u is a harmonic function. If u is not continuous, by the suitable approximation arguments one can get the result.
Unlike the classical case $n = 1$, where every maximal subharmonic function is smooth, the maximal plurisubharmonic function in $\mathbb{C}^n$, $n > 1$, not need be even continuous. The next proposition is useful in many applications.

Proposition 3.1.1. *[5] Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathcal{PSH}(\Omega)$. The followings are equivalent:*

(i) u is maximal.

(ii) For every relatively compact open subset G of Ω and for every function $v \in \mathcal{PSH}(G)$, if

$\liminf_{z\to p}(u(z) - v(z)) \geq 0$ *for all $p \in \partial G$, then $u \geq v$ in G.*

(iii) For each open set $G \subset\subset \Omega$ and $v \in \mathcal{PSH}(\Omega)$ such that $\liminf_{G\ni z\to p}(u(z) - v(z)) \geq 0$ *for all $p \in \partial G$ it follows $u \geq v$ in G.*

(iv) For each $v \in \mathcal{PSH}(\Omega)$ which has the property that for each $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $u - v \geq -\varepsilon$ in $\Omega \setminus K$ then $u \geq v$ in Ω.

(v) For each open set $G \subset\subset \Omega$ and $v \in \mathcal{PSH}(\Omega)$ such that $v(p) \leq u(p)$ for all $p \in \partial G$ it follows $u \geq v$ in G.

Proof. [5] Assume that u is maximal and let v, G be as in (ii). Then $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$. Indeed, let $\{z_k\} \subset G$ be such that $z_k \to p$ and let $L := \limsup_{z\to p} v(z) = \limsup_{k\to\infty} v(z_k)$. Then

$$0 \leq \liminf_{z\to p}(u(z)-v(z)) \leq \liminf_{k\to\infty}(u(z_k)-v(z_k)) \leq \limsup_{k\to\infty} u(z_k)-L \leq u(p)-L.$$

Thus $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$ and by (i) $v \leq u$ in G, which gives (ii).

Since $\mathcal{PSH}(\Omega)|_G = \mathcal{PSH}(G)$, it is obvious to see that (ii) implies (iii). Now assume (iii) holds. Let $v \in \mathcal{PSH}(\Omega)$ with the property that for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $u - v \geq -\varepsilon$ in $\Omega \setminus K$. To get a contradiction, assume that there exists a $a \in \Omega$ such that $u(a) \leq v(a) - \delta$ for some $\delta > 0$. By hypothesis, there exists a compact set $K \subset \Omega$ so that

$u(z) - v(z) \geq -\delta/2$ for all $z \in \Omega \setminus K$. Notice that $a \in K$. Let $G \subset\subset \Omega$ be an open set such that $K \subset G$. Then $\liminf_{G \ni z \to p}(u(z) - v(z) + \delta/2) \geq 0$ for all $p \in \partial G$. Since $(v - \delta/2) \in \mathcal{PSH}(\Omega)$ then (iii) implies that $u \geq v - \delta/2$ in G and in particular than $u(a) \geq v(a) - \delta/2$, which is a contradiction. Thus (iii) gives (iv).

Assume (iv) holds. Let $G \subset\subset \Omega$ be an open set and $v \in \mathcal{PSH}(\Omega)$ be so that $u(p) \geq v(p)$ for all $p \in \partial G$. Define

$$w(z) := \begin{cases} \max\{u(z), v(z)\}, & z \in G, \\ u(z), & z \in \Omega \setminus G. \end{cases}$$

By Gluing lemma, $w \in \mathcal{PSH}(\Omega)$. By construction, for all $\varepsilon > 0$ it follows that $0 = u(z) - w(z) \geq -\varepsilon$ for all $z \in \Omega \setminus \bar{G}$. By (iv) it follows that $u \geq w$ in Ω and thus $u \geq v$ in G as desired.

Finally, if (v) holds, given $G \subset\subset \Omega$ an open set and $v \in \mathcal{PSH}(G)$ such that $\limsup_{z \to p} v(z) \leq u(p)$ for all $p \in \partial G$ we define w as above. Then $w \in \mathcal{PSH}(\Omega)$, $w \leq u$ on ∂G and by (v) it follows that $w \leq v$ in G, proving that $v \leq u$ in G and then (i). □

3.1.1 Characterization of maximal plurisubharmonic functions of class C^2

In this subsection, we characterize maximal plurisubharmonic functions of class C^2 by means of their Levi form. In several variables *the Pluricomplex Dirichlet problem* asks to find an upper-semicontinuous function u on $\tilde{\Omega}$, $u : \tilde{\Omega} \to \mathbb{R}$ for a given extended real valued function f defined in $\partial\Omega$ such that $(u|_{\Omega}) \in \mathcal{MPSH}$ and $u|_{\partial\Omega} \equiv f$. Recall that the *complex Monge-Ampére operator* in $\mathbb{C}^n$ is defined as the n-th exterior power of dd^c, namely

$$(dd^c)^n = \underbrace{dd^c \wedge \ldots \wedge dd^c}_{n-times}.$$

Let $\Omega \subset \mathbb{C}^n$ and let $u \in C^2(\Omega)$ then we have the following equality

$$(dd^c u)^n = 4^n n! det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] dV,$$

where

$$dV = \left(\frac{i}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n \tag{3.1.1}$$

is the usual volume form in $\mathbb{C}^n$. In particular if $n = 1$,then $(dd^c)^n$ becomes usual Laplacian times the area form in $\mathbb{R}^2$.

For an arbitrary plurisubharmonic function u, it is well known that $dd^c u$ is a positive $(1, 1)$ current [24]. But it is not clear that the powers of $dd^c u$ are well defined. In fact, examples indicate that it is not possible to define $(dd^c u)^n$ as a distribution for all plurisubharmonic function u. If $u \in \mathrm{L}^\infty_{loc}$ then Bedford-Taylor was able to define $(dd^c u)^n$ as a measure. For more information, we refer to reader to [1].

One of the important features of the Monge Ampére operators is the fact that the maximality of plurisubharmonic functions can be characterized in terms of equations.

Theorem 3.1.1. *[24] Let $u \in C^2(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is open.*
If $u \in \mathcal{MPSH}(\Omega)$ then

$$\det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right]_{1 \le j,k \le n} \equiv 0 \quad in \quad \Omega$$

Proof. [24][5] Assume that $u \in C^2 \cap \mathcal{PSH}(\Omega)$ is maximal and assume that there exists $a \in \Omega$ so that $\det \left[\frac{\partial^2 u(a)}{\partial z_j \partial \bar{z}_k} \right] > 0$ in Ω. This implies that $\mathscr{L}u(a; X)$ is positive definite for each $X \in \mathbb{C}^n$ with $\|X\| = 1$.

In view of the continuity of the second order derivatives of u, one can find a ball $B(a, r) \subset\subset \Omega$ and $C > 0$ such that $\mathscr{L}u(z; X) \ge C$ for all $X \in \mathbb{C}^n$ such that $\|X\| = 1$ and $z \in B(a, r)$. Then $u(z) + C(r^2 - \|z\|^2) \in \mathcal{PSH}(B(a, r))$

since $\mathscr{L}(u+C(r^2-\|.\|^2))(z;X)=\mathscr{L}u(z;X)-C\|X\|^2\geq 0$ by construction. Define

$$v(z)=\begin{cases} u(z), & z\in\Omega\setminus\bar{B}(a,r),\\ u(z)+c(r^2-\|z-a\|^2), & z\in B(a,r).\end{cases}$$

By Gluing lemma, $v\in\mathcal{PSH}(\Omega)$. Moreover $v=u$ on $\partial B(a,r)$ and $v(a)=u(a)+Cr^2>u(a)$ against to maximality of u. $\square$

Theorem 3.1.2. *[24] Let Ω be an open subset of $\mathbb{C}^n$, and let $u\in C^2(\Omega)\cap\mathcal{PSH}(\Omega)$. Then u is maximal in Ω if and only if $(dd^cu)^n=0$ in Ω.*

Proof. [5] Assume that $(dd^cu)^n=0$. Let $G\subset\subset\Omega$ be an open set and $v\in\mathcal{PSH}(\Omega)$ such that $v(p)\leq u(p)$ for all $p\in\partial G$. What we need to show that is $v\leq u$ in G and by proposition 3.1.1 and by the arbitrariness of v implies that u is maximal. Seeking for a contradiction, we assume that there exists $a\in G$ such that $0<v(a)-u(a)\leq\sup_{z\in G}(v-u)(z)$. Let $\delta>0$ satisfying $v(a)-\delta>u(a)$. Then $v(z)-\delta\in\mathcal{PSH}(\Omega)$ and $v(p)-\delta<u(p)$ for all $p\in\partial G$. Thus, if $\{v_\varepsilon\}$ is the decreasing sequence of regularizing plurisubharmonic functions for $v-\delta$, there exists $\varepsilon>0$ such that $G\subset\subset\Omega_\varepsilon$, $v_\varepsilon\in C^\infty(\Omega_\varepsilon)\cap\mathcal{PSH}(\Omega_\varepsilon)$, $v_\varepsilon(a)>u(a)$ and $v_\varepsilon(p)\leq u(p)$ for all $p\in\partial G$.

Let $M:=\max_{z\in\bar{G}}\|z\|^2$. Let $\lambda>0$ be such that $v_\varepsilon(a)+\lambda(\|a\|^2-M)>u(a)$ and let $w(z):=v_\varepsilon(z)+\lambda(\|z\|^2-M)$. Then $w\in\mathcal{PSH}(\Omega_\varepsilon)$, $w(p)\leq u(p)$ for all $p\in\partial G$ and $w(a)>u(a)$ and $\mathscr{L}w(z;X)>0$ for all $z\in G$ and $X\in\mathbb{C}^n\setminus\{0\}$.

Let $\tau\in G$ be the local maximum of $w-u$. Since $w(a)-u(a)>0$ and $w-v\leq 0$ on ∂G such a point exists.

Notice that $\det[\frac{\partial^2u(\tau)}{\partial z_j\partial\bar{z}_k}]=0$ is equivalent to the vector $X\in\mathbb{C}^n\setminus\{0\}$ such that $\mathscr{L}u(\tau;X)=0$. Let $f(\zeta):=(w-u)(\tau+X\zeta)$ for $\zeta\in\mathbb{C}$, $|\zeta|<<1$. Since $\zeta=0$ is a local maximum and f is of class C^2 then $\Delta f(0)\leq 0$. Therefore

$$0\geq\Delta f(0)=4\mathscr{L}(w-u)(\tau;X)=4\mathscr{L}w(\tau;X)>0,$$

a contradiction. Therefore u is maximal.

Conversely, let u be maximal plurisubharmonic function then by above theorem ,we have

$$\det\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right]_{1\le j,k\le n} \equiv 0$$

hence $(dd^c u)^n = 0$ by equation (3.1.1).

□

3.2 The Relative Extremal Functions

An extremal function, which has become known as the *relative extremal function*, was introduced by Siciak[52] in 1962 and Zaharyuta[60] in 1976. Given an open set Ω in $\mathbb{C}^n$ and a compact subset E of Ω he defined a function $h := u^*_{E,\Omega}$, where the star denotes the upper semicontinuous envelope and where

$$u_{E,\Omega}(z) := \sup_u \{v(z) : v \in \mathcal{PSH}(\Omega), v \le -1 \quad \text{on} \quad E, v \le 0 \quad \text{in} \quad \Omega\} \quad z \in \Omega$$

The function $(u_{E,\Omega})^*$ is plurisubharmonic in Ω. Since in one-dimensional case $(u_{E,\Omega})^*$ is closely related to the notion harmonic measure in higher dimensions, it sometimes called the *plurisubharmonic measure* of E relative to Ω[48].

As a direct consequence of definition, we have the following monotonicity property of the relative extremal function.

Proposition 3.2.1. *[24] If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$ then*

$$u_{E_1,\Omega_1} \ge u_{E_2,\Omega_1} \ge u_{E_2,\Omega_2}.$$

Definition 17. A function $u : \Omega \to \mathbb{R}$ is called an *exhaustion function* if for any $t \in \mathbb{R}$ the set $\{z \in \Omega : u(z) \le t\}$ is relatively compact in Ω and an open set Ω in $\mathbb{C}^n$ is called *hyperconvex* (or *pluriregular*) if there exists a function $u \in \mathcal{PSH}(\Omega)$, $u < 0$, such that

$$\{z \in \Omega : u(z) < t\} \subset\subset \Omega, \quad t < 0.$$

Proposition 3.2.2. *[24] If Ω is hyperconvex and E is a relatively compact subset of Ω, then at any point of $w \in \partial\Omega$*

$$\lim_{z \to w} u_{E,\Omega}(z) = 0.$$

Proof. [24] If $\varrho < 0$ is an exhaustion function for Ω, then for some $M > 0$, $M\varrho < -1$ on E. Thus $M\varrho \leq u_{E,\Omega}$ in Ω. Clearly, $\lim_{z\to w} \varrho(z) = 0$, and so we obtained the required result. □

Proposition 3.2.3. *[2] Let $\Omega \subset \mathbb{C}^n$ be a connected open set, and let $E \subset \Omega$. The following conditions are equivalent*

(i) $u^*_{E,\Omega} \equiv 0$;

(ii) there exists a negative function $v \in \mathcal{PSH}(\Omega)$ such that

$$E \subset \{z \in \Omega : v(z) = -\infty\}$$

Proof. [2] If $E \subset \{z \in \Omega : v(z) = -\infty\}$, where $v \leq 0$ and $v \in \mathcal{PSH}(\Omega)$, then $u_{E,\Omega} \geq \sup\{v/j : j = 1, 2, \ldots\}$. So $u_{E,\Omega} = 0$ on the complement of a set of measure zero. Hence $u^*_{E,\Omega} \equiv 0$[1]. Conversely, assume that $u^*_{E,\Omega} \equiv 0$. Then by main approximation theorem there exist a point $a \in \Omega$ such that $u_{E,\Omega}(a) = 0$. Therefore, we may choose $v_j \in \mathcal{PSH}(\Omega)$ such that $v \leq 0$ for all $j \in \mathbb{N}$, $v_j \leq -1$ on E and $v_j(a) \leq -2^{-j}$. Now, define

$$v(z) := \sum_{j=1}^{\infty} v_j(z).$$

Since $v \not\equiv -\infty$ and $v \in \mathcal{PSH}(\Omega)$ with $v|_E = -\infty$, we get the desired.

□

Proposition 3.2.4. *[24] Let Ω be a domain in $\mathbb{C}^n$. Suppose that $E = \bigcup_j E_j$, where $E_j \subset \Omega$ for $j = 1, 2, \ldots$. If $u^*_{E_j,\Omega} \equiv 0$ for each j then $u^*_{E,\Omega} \equiv 0$*

Proof. [24] Choose $v_j \in \mathcal{PSH}(\Omega)$ such that $v_j \leq 0$ and $v|_E \equiv -\infty$. Take a point $a \in \left(\Omega \setminus \bigcup_j v_j^{-1}(\{-\infty\})\right)$. By multiplying all v_j's with a suitable constants, we may assume that $v_j(a) > -2^{-j}$. Define $v(z) := \sum_{j=1}^{\infty} v_j(z)$. Since $v \in \mathcal{PSH}(\Omega)$, $v \leq 0$ and $v|_E \equiv -\infty$, by the above proposition, $u^*_{E,\Omega} \equiv 0$. □

Proposition 3.2.5. *[24] Let Ω be a hyperconvex subset of $\mathbb{C}^n$, and let K be a compact subset of Ω. Suppose that Ω_j is an increasing sequence of open subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ and $K \subset \Omega_1$. Then*

$$\lim_{j\to\infty} u_{K,\Omega_j}(z) = u_{K,\Omega}(z) \qquad z \in \Omega.$$

Proof. [24] Take a point $z_0 \in \Omega$ and assume w.l.o.g. that $K \cup \{z_0\} \subset \Omega_1$. Let $\varrho < 0$ be an exhaustion function for Ω such that $\varrho \leq -1$ on K.
Take $\varepsilon \in (0, 1)$ satisfying $\varrho(z_0) < \varepsilon$. There is $j_0 \in \mathbb{N}$ for which the open set $\omega = \varrho^{-1}((-\infty, \varepsilon))$ is relatively compact in Ω_{j_0}. Now, take $u \in \mathcal{PSH}(\Omega_{j_0})$ such that $u \leq 0$ on Ω_{j_0} and $u \leq -1$ on K. Then

$$v(z) = \begin{cases} \max\{u(z) - \varepsilon, \varrho(z)\} & z \in \omega, \\ \varrho(z), & z \in \Omega \setminus \omega \end{cases}$$

defines a plurisubharmonic function with $v \leq -1$ on K and $v \leq 0$. Hence $v(z_0) \leq u_{K,\Omega}(z_0)$. Since u is an arbitrary function in the family of defining $u_{K,\Omega_{j_0}}$, we have

$$u_{K,\Omega_{j_0}}(z_0) - \varepsilon \leq u_{K,\Omega}(z_0).$$

By monotonicity property of relative extremal functions, for all $j \geq j_0$ we have,

$$u_{K,\Omega_j}(z_0) - \varepsilon \leq u_{K,\Omega}(z_0) \leq u_{K,\Omega_j}(z_0).$$

Since ε arbitrarily small, the result follows. $\square$

Proposition 3.2.6. *If $\Omega \subset \mathbb{C}^n$ is hyperconvex and K is compact, then $u^*_{K,\Omega}$ is maximal in $\Omega \setminus K$, that is,*

$$(dd^c u^*_{K,\Omega})^n = 0 \quad \text{in} \quad \Omega \setminus K.$$

Proof. Consider a ball $B \subset \Omega \setminus K$ and let Φ_B be the solution of the Dirichlet problem in B:

$$(dd^c \Phi)^n = 0, \quad \Phi_B|_{\partial B} = u^*_{K,\Omega}|_{\partial B}.$$

By maximality of Φ_B, for $z \in B$ we have $\Phi_B(z) \geq u^*_{K,\Omega}(z)$. Now, define

$$w(z) = \begin{cases} \Phi_B & z \in B, \\ u^*_{K,\Omega}, & z \in \Omega \setminus B \end{cases}$$

Then w is plurisubharmonic, non-positive and $v \leq -1$ on K. Hence $w \leq u^*_{K,\Omega}$ in Ω. On the other hand, $\Phi_B \geq u^*_{K,\Omega}$ in B. Therefore $u^*_{K,\Omega} = \Phi_B$ in B, which means $u^*_{K,\Omega}$ is maximal in B. Since B was chosen arbitrarily, we arrived the required result. $\square$

3.3 Siciak-Zaharyuta Extremal Functions

Let X be a locally compact metric space, and let $\bar{X}$ be a compactification of X. We write that $f(x) = O(g(x))$ as $x \to a$, to indicate that, for some $M > 0$ and for all $x \in X$ sufficiently close to a we have $|f(x)| \leq M|g(x)|$. If f is real valued, the notation '$f(x) \leq O(g(x))$ as $x \to a$' means that there exists a $h : X \to \mathbb{R}$ such that $h(z) = O(g(x))$ as $x \to a$ and $f(x) \leq h(x)$ for all $x \in X$ sufficiently close to a.
A function $u \in \mathcal{PSH}(\mathbb{C}^n)$ is said to be *minimal growth* if

$$(u(z) - \log \|z\|) \leq O(1) \quad \text{as} \quad \|z\| \to \infty.$$

The family of these functions is called *Lelong class* and will be denoted by $\mathcal{L}(\mathbb{C}^n)$ or, simply, $\mathcal{L}$ if no confusion can arise.

Definition 18. [60][53] Let E be any set in $\mathbb{C}^n$. The function

$$V_E(z) = \sup\{u(z) \in \mathcal{PSH}(\mathbb{C}^n) : u \in \mathcal{L}, u \leq 0 \quad \text{on} \quad E\}$$

is called *Siciak-Zaharyuta extremal function of E.*

Example 5. For any complex norm $\|\cdot\|$ on $\mathbb{C}^n$, let $\bar{B}_{\|\cdot\|}(a,r)$ denote the closed ball with center a and radius r. Then for all $z \in \mathbb{C}^n$,

$$V_{\bar{B}_{\|\cdot\|}(a,r)}(z) = \log^+ \frac{\|z-a\|}{r} \tag{3.3.1}$$

where $\log^+$ is positive part of logarithm function.

Indeed, let $E = \bar{B}_{\|\cdot\|}(a,r)$. Since all norms on $\mathbb{C}^n$ are equivalent (Lemma 5.14 in [40]), there is a constant C such that $\|.\| \leq C|.|$ an $\mathbb{C}^n$. Then the function $\log^+ \frac{\|z-a\|}{r}$ belongs to Lelong class $\mathcal{L}$ and since it is 0 on E, $\log^+ \frac{\|z-a\|}{r} \leq V_E(z)$. Therefore we need to show for any $u \in \mathcal{L}$, such that $u \leq 0$ on E, that $u(z) \leq \log^+ \frac{\|z-a\|}{r}$. Note that this is clearly holds when $z \in E$

Now, for such u and $w \in \mathbb{C}^n \setminus E$ we define a function v on $B(0, \|w-a\|/r) \setminus \{0\} \subset \mathbb{C}$,

$$v(\zeta) = u\left(a + \zeta^{-1}(w-a)\right) - \log^+ \frac{\|w-a\|}{|\zeta|r}.$$

Then v is subharmonic and $v(\zeta)$ is bounded when $\zeta \to 0$ since $u \in \mathcal{L}$. Therefore, by removable singularities theorem, v can be extended over 0 to a subharmonic function $\tilde{v}$ on $B(0, \|w-a\|/r)$. Now, $\lim_{|\zeta| \to \|w-a\|/r} u(\zeta) \leq 0$, so by maximum principle, $\tilde{v} \leq 0$ on $B(0, \|w-a\|/r)$. In particular, $v(1)$ is defined since $\|w-a\|/r \geq 1$ and $v(1) = \tilde{v}(1) = u(w) - \log^+ \frac{\|w-a\|}{r} \leq 0$ as desired.

Proposition 3.3.1. *[2] If K is a compact subset of $\mathbb{C}^n$, then V_K is lower semicontinuous.*

Proof. [2] Note that if $v \in \mathcal{L}$ then also $v * \chi_\varepsilon \in \mathcal{L}$ where χ_ε is the standard smoothing kernel in $\mathbb{C}^n$. Then if $u \in \mathcal{L}$, $u \leq 0$ on K, and $\delta > 0$, then in view of the main approximation theorem for the plurisubharmonic functions and the compactness of K we can find $\varepsilon > 0$ such that $(u * \chi_\varepsilon - \delta) \leq 0$ on K. As a consequence, we see that V_K is the supremum of a family of continuous functions, and hence the result follows. □

Proposition 3.3.2. *[24] If $K_1 \supset K_2 \supset \ldots$ is sequence of compact subsets in $\mathbb{C}^n$ and $K = \bigcap_j K_j$, then*

$$\lim_{j\to\infty} V_{K_j} = V_K$$

Proof. [24] Clearly, $V_{K_1} \leq V_{K_2} \leq \ldots \leq V_K$, and so the limit of left-hand side exists and is not grater than V_K. Take $u \in \mathcal{L}$ such that $u \leq 0$ on K. If $\varepsilon > 0$, then if j is sufficiently large, K_j is not contained in the open set $\{z \in \mathbb{C}^n : u(z) \leq \varepsilon\}$. Thus for such values of j, $u - \varepsilon \leq V_{K_j} \leq \lim_{j\to\infty} V_{K_j}$ in $\mathbb{C}^n$. Consequently, $V_K \leq \lim_{j\to\infty} V_{K_j}$. □

To prove the maximality of envelope function of V_E, V_E^*, we need to propose the following.

Proposition 3.3.3. *[2] Let Ω be open in $\mathbb{C}^n$ and $\psi \in \mathcal{PSH}(\Omega) \cap \mathrm{L}^\infty_{loc}(\Omega)$. If $D \subset\subset \Omega$ is strongly pseudoconvex, then there exists a unique function $\tilde{\psi} \in \mathcal{PSH}(\Omega) \cap \mathrm{L}^\infty_{loc}(\Omega)$ such that*

$$(dd^c\tilde{\psi})^n = 0 \quad on \quad D, \tag{3.3.2}$$

$$\tilde{\psi} = \psi \quad on \quad \Omega \setminus D. \tag{3.3.3}$$

Proof. [2] Let ψ_j be a continuous plurisubharmonic function which decreases to ψ on a neighborhood of $\bar{D}$. By theorem 8.3, p.42 of [1], there exists $\tilde{\psi}_j \geq \psi_j$ continuous and plurisubharmonic on a neighborhood of $\bar{D}$ satisfying $(dd^c\tilde{\psi}_j)^n = 0$ on D and $\tilde{\psi}_j = \psi_j$ on $\Omega \setminus D$. Then since $(dd^c\tilde{\psi}_j)^n \to (dd^c\tilde{\psi})^n$ as $j \to \infty$, $\tilde{\psi}$ satisfies 3.3.2 on D. Now, since $\tilde{\psi}_j \searrow \tilde{\psi}$ on a neighborhood of $\bar{D}$ and $\tilde{\psi} = \psi_j$ on $\Omega \setminus D$, the function $\tilde{\psi} = \psi$ on $\Omega \setminus D$ satisfies 3.3.3 also. □

Now, we consider some envelope function more general than V_E^*.

Corollary 3.3.1. *[2] Let h be a bounded, lower semicontinous function on $\bar{\Omega} \subset\subset \mathbb{C}^n$. If*

$$u_h(z) := \sup\{v(z) \in \mathcal{PSH}(\Omega) \cap \mathrm{L}^\infty_{loc}(\Omega), v \leq h\},$$

then the envelope function u_h^* is maximal on the set $\{u_h^* < h\}$.

Proof. [2] If $z \in \{u_h^* < h\}$, then since u_h^* is upper semicontinuous and h is lower semicontinuos there exists a small ball B centered at z with $\sup\{u_h^*(\zeta) : \zeta \in \bar{B}\} < \inf\{h(\zeta) : \zeta \in \bar{B}\}$. Let function ψ be as in the above proposition with $\psi = u_h^*$. It follows that, we have that $\tilde{\psi} \leq h$. Hence $\tilde{\psi} \leq u_h^*$, so $\tilde{\psi} \leq u_h^*$. Since $(dd^c\tilde{\psi})^n = 0$ on B, we have $(dd^c u_h^*)^n = 0$ on B. Because B is a neighborhood of an arbitrary point $z \in \{u_h^* < h\}$, the corollary is proved. □

Corollary 3.3.2. *[2] Let $E \subset \mathbb{C}^n$. Then $(dd^c V_E^*)^n = 0$ on $\mathbb{C}^n \setminus \bar{E}$.*

Proof. [2] If B is a small ball in $\mathbb{C}^n \setminus \bar{E}$, then $\tilde{V_E^*} = V_E^*$ as in the previous corollary. Hence $(dd^c V_E^*)^n = 0$ on B, therefore on $\mathbb{C}^n \setminus \bar{E}$. □

3.4 Pluricomplex Green Functions with a Logarithmic Pole

Let Ω be an open bounded subset of $\mathbb{C}$ and let a be a point Ω. Suppose that the function $z \mapsto G_\Omega(z, a)$ has the following properties:

(i) $G_\Omega(., a)$ is harmonic on $\Omega \setminus a$, and bounded outside neigborhood of a;

(ii) for each $w \in \partial\Omega$ we have $\lim_{z \to w} G_\Omega(z, a) = 0$;

(iii) $z \mapsto G_\Omega(z, a) + \log|z - a|$ extends a harmonic function on Ω.

Then the function $z \mapsto G_\Omega(z,a)$ is called the *classical Green function for* Ω *with pole* a. In view of the maximum principle, each set Ω can have at most one Green function with a given pole. Furthermore, if u is the solution to the Dirichlet problem

$$\begin{cases} u \in C^2(\Omega) \cap C(\bar{\Omega}), & , \\ \Delta u = 0 \quad \text{in} \quad \Omega, \\ u(z) = \log|z-a| \quad \text{for each} \quad z \in \Omega, & . \end{cases}$$

then $u(z) - \log|z-a|$ is the classical Green function with pole at a. Conversely, if Ω has a classical Green function, the function defined by (iii) solves the Dirichlet problem.

Lempert [38],[39] constructed an analogues function in several variables: his function is plurisubharmonic in Ω, has a logarithmic pole at a given point $a \in \Omega$ and tends to zero at the boundary of Ω. Further, it solves the homogenous Monge-Ampère equation in $\Omega \setminus \{a\}$, in other words, it is maximal on this set. Namely, the function forms as the following

$$u(z) = \begin{cases} u \in C^\infty(\Omega \setminus \{a\}) \cap \mathcal{PSH}(\Omega), \\ (dd^c u)^n = 0 \quad \text{in} \quad \Omega, \\ u(z) - \log\|z-a\| = O(1) \quad \text{as} \quad z \to a, \\ u(z) \to 0 \quad \text{as} \quad z \to w \in \partial\Omega & . \end{cases}$$

For the case $n = 1$ the function $-u$ is just the classical Green function for Ω with pole a. Bearing in mind the analogue between the Laplacian in $\mathbb{C}$ and the Mongére-Ampere in $\mathbb{C}^n$, one can regard u as a $\mathbb{C}^n$-version of classical Green function.

Let Ω be a connected open subset of $\mathbb{C}^n$ and a be a point in Ω. If u is a plurisubharmonic function in a neighborhood of a, we say u has a *logarithmic pole* at a if

$$u(z) - \log\|z-a\| \leq O(1) \quad \text{as} \quad z \to a.$$

In 1985, Klimek, replaced Lempert's construction by a Perron-Bremermann approach in [26] as follows

$$g_\Omega := \sup\{u(z) : u \in \mathcal{PSH}(\Omega), \quad u \le 0, \quad u \quad \text{has a logarithmic pole at} \quad \text{a}\}.$$

It is assumed here $\sup \emptyset := -\infty$. Later and independently, Demailly [10],[11] proved regularity results of the pluricomplex Green's function and its relation to Monge-Ampère equation. Moreover, Guan [15] and Blocki [3] provided finer regularity for strongly pseudoconvex domains D.

In next proposition we give basic properties of the pluricomplex Green function g_Ω.

Proposition 3.4.1. *[5] If Ω and Ω' are domains in $\mathbb{C}^n$ and $w \in \Omega$, then the following statements hold:*

(i) if $z \in \Omega$ and $\Omega \subset \Omega'$, then $g_\Omega(z, w) \ge g_{\Omega'}(z, w)$;

(ii) if $z \in \Omega$, $\Omega \subset \Omega'$ and $\Omega' \setminus \Omega$ is pluripolar, then $g_\Omega(z, w) = g_{\Omega'}(z, w)$;

(iii) if $R > r > 0$ and $\bar{B}(w, r) \subset \Omega \subset B(w, R)$ then

$$\log(\|z - w\|/R) \le g_\Omega(z, w) \le \log(\|z - w\|/r);$$

(iv) if Ω is bounded, then $z \mapsto g_\Omega(z, w)$ is a negative plurisubharmonic function with a logarithmic pole at w;

(v) if $f \in \mathcal{O}(\Omega, \Omega')$, then

$$g_{\Omega'}(f(z), f(w)) \le g_\Omega(z, w) \quad (z \in \Omega);$$

Proof. [5] First property (i) comes from directly definition and (ii) is a consequence of removable singularities theorem for plurisubharmonic functions. First inequality in (iii) comes from definition and second inequality is a special case of Lemma 1.2.1. For (iv), note that, according to (iii), we have

$(z \mapsto g_\Omega(z,w))^*$ belongs to the family that defines the g_Ω.
To see property (v), let $w \in \Omega$ and u be a function from the defining family for $g_{\Omega'}(\cdot, f(w))$. Then $u \circ f \in \mathcal{PSH}(\Omega, [-\infty, 0])$ and

$$\begin{aligned} & u(f(z)) - \log \|z - w\| = \\ & u(f(z)) - \log \|f(z) - f(w)\| + \log \frac{\|z - w\|}{\|f(z) - f(w)\|} \leq O(1) \end{aligned}$$

as $z \to w$ which means $u \circ f$ has a logarithmic pole at w and hence $u \circ f \leq g_\Omega(\cdot, w)$. ☐

Corollary 3.4.1. *[24] If $(\Omega_j)_{j \in \mathbb{N}}$ is an increasing sequence of domains in $\mathbb{C}^n$ and $\Omega = \bigcup \Omega_j$, then*

$$g_\Omega(z,w) = \lim_{j \to \infty} g_{\Omega_j}(z,w) \quad (z, w \in \Omega_1).$$

Proof. [24] Fix $w \in \Omega_1$. If for any j, $g_{\Omega_j}(.,w) \equiv -\infty$, the result is obvious. Suppose that for each j, $g_{\Omega_j}(.,w) \in \mathcal{PSH}(\Omega_j)$. Then $g(z) = \lim_{j \to \infty} g_{\Omega_j}(z,w)$ is a plurisubharmonic function. The former implies the desired convergence, $g \geq g_\Omega(.,w)$ by (i). ☐

Proposition 3.4.2. *[24] If $\Omega \subset \mathbb{C}^n$ is bounded, then $z \mapsto g_\Omega(z,w)$ is maximal in $\Omega \setminus w$, i.e.*

$$(dd^c g_\Omega(z,w))^n \equiv 0 \quad in \quad \Omega \setminus w.$$

Proof. [24] Take a point $w \in \Omega$. Let G be a domain which is relatively compact in $\Omega \setminus \{w\}$, and let $v \in \mathcal{PSH}(\Omega \setminus \{w\})$ be such that $v \leq g_\Omega(\cdot, w)$ on ∂G. Define

$$u(z) = \begin{cases} \max\{v(z), g_\Omega(z,w)\}, & z \in G, \\ g_\Omega(z,w), & z \in \Omega \setminus G. \end{cases}$$

Then u belongs to defining family of $g_\Omega(\cdot, w)$. Hence, $v \leq g_\Omega(\cdot, w)$ in G and this proves the maximality of function $z \mapsto g_\Omega(\cdot, w)$ by Proposition 1.2.1. ☐

Lemma 3.4.1. *[24] Suppose that* $h : \mathbb{C}^n \to [0, \infty)$ *is upper semicontinuous,* $h^{-1}(0) = 0$ *and* $h(\xi z) = |\xi| h(z)$ *for each* $\xi \in \mathbb{C}$ *and* $z \in \mathbb{C}^n$*. If* $\Omega = \{z \in \mathbb{C}^n : h(z) < 1\}$*, then*

$$g_\Omega(z, 0) \leq \log h(z) \quad (z \in \Omega$$

Proof. [24] Take a point $w \in \Omega$. For $\zeta \in B(0, h^{-1}(w))$ consider the function,

$$v(\zeta) = g_\Omega(\zeta w, 0) - \log h(\zeta w).$$

Then $v \in \mathcal{SH}(B(0, h^{-1}(w)) \setminus \{0\})$ and for each $\zeta \in B(0, h^{-1}(w))$ we have $\limsup_{\zeta \to \xi} \leq 0$. Also, since h is homogeneous and upper semicontinuous, for all $z \in \mathbb{C}^n$ we have

$$0 \leq h(z) \leq \|h\|_{B(0,1)} \|z\|.$$

Hence v is bounded in a neighborhood of the origin and, by removable singularities theorem, extends to a subharmonic function in $B(0, h^{-1}(w))$. In view of maximum principle, $v \leq 0$ and $g_\Omega(w, 0) \leq \log h(w)$, as required. □

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