Students' attitudes towards two famous paradoxes. Emerged strategies in various school grades.

Las actitudes de los estudiantes hacia dos paradojas famosas. Estrategias emergidas en varios grados escolares

Charalampos Papadatos, Dionysia Bakogianni and Theodossios Zachariades

National and Kapodistrian University of Athens, Greece

Abstract

This study aims at getting insight in the strategies employed by secondary students when they are faced with some famous probabilistic paradoxes. Four different age groups of students participated the study (48 in Grade 8, 63 in Grade 9, 53 in Grade 10 and 49 in Grade 12). All students were given the Bertrand's Box Paradox and the Gardner's Two Children Paradox modified in an understandable way for all age groups. The 213 written responses (answers and explanations) were analysed and categorized according to various heuristics, misconceptions and types of strategy that seemed to guide students' choices. Differences among the various grade levels and also differences between the two problems in students' responses of the same grade level were explored and discussed.

Keywords: secondary education, stochastics learning, paradoxes, students' strategies.

Resumen

Este estudio tiene como objetivo conocer las estrategias empleadas por los estudiantes de secundaria cuando se enfrentan a algunas paradojas probabilísticas famosas. Participaron en el estudio cuatro grupos de estudiantes de diferentes edades (48-Grado 8, 63-Grado 9, 53-Grado 10, 49-Grado 12). Todos los estudiantes resolvieron la paradoja de Bertrand y la paradoja de los dos niños de Gardner, modificadas de manera comprensible para todos los grupos de edad. Las 213 respuestas escritas (respuestas y explicaciones) se analizaron y categorizaron, de acuerdo con diversas heurísticas, conceptos erróneos y tipos de estrategia que parecían guiar las elecciones de los estudiantes. Las diferencias entre los distintos niveles de grado y también las diferencias entre los dos problemas en las respuestas de los estudiantes del mismo nivel de grado fueron exploradas y discutidas

Keywords: educación secundaria, aprendizaje estocástico, paradojas, estrategias de los alumnus.

1. Introduction

Probabilistic reasoning constitutes a high demanding mental process in which intuition plays a central role (Batanero, 2016; Borovcnik & Kapadia, 2014; Leviatan, 2002). The concept of probability was developed at a slow pace through history to reach its current foundation with the various applications. The existence of some problematic situations in the path of probability theory development challenged the common sense and caused cognitive conflict. Additionally, the mathematical solution in such situations has led to the development of fundamental concepts in probability theory, i.e. sample space, independence, distinction between permutations and combinations, expected value, need for simple equiprobable events (Chernoff & Sriraman, 2014). In contrast to the principles of a deterministic world, where mathematical tools and mathematical proofs

Papadatos, C., Bakogianni, D. y Zachariades T. (2019). Students' attitudes towards two famous examples of paradoxes. Explanations and strategies in various school grades. En J. M. Contreras, M. M. Gea, M. M. López-Martín y E. Molina-Portillo (Eds.), *Actas del Tercer Congreso Internacional Virtual de Educación Estadística.* Disponible en www.ugr.es/local/fqm126/civeest.html

have a power to convince, in such stochastic situations the power and logic of mathematics seem not enough to convince a person and influence well established intuitive conceptions. Such situations are found in the literature with the term paradoxes.

Paradoxes are imbedded in situations where the mathematical path contradicts our intuition and there is a resistance to the construction of a new knowledge related to chance and probabilistic notions (Borovcnik & Kapadia, 2014). This resistance is seen more frequent while manipulating probabilistic ideas rather than other subject areas or disciplines of mathematics.

In addition, when people need to make judgements or decisions in situations where chance is inherent, they employ cognitive mechanisms, which are known in the literature as "heuristics". These mechanisms are sometimes based on misleading intuition and established misconceptions, and they constitute an obstacle in thinking with probabilities (Tversky & Kahneman, 1974). The case of paradoxes is a characteristic example of people relying on heuristics. Due to their complex nature, paradoxes have been a point of focus for many researchers. At times, paradoxes have been used as research tools for the exploration of types of people's reasoning or strategies they follow to find a solution (Fischbein & Schnarch, 1997; Taylor & Stacey, 2014; Falk & Konold, 1992). Other times, such problems have been used as didactical tools in undergraduate studies or professional development programs (Batanero, Contreras, Fernándes & Ojeda, 2010; Klymchuk & Kachapova, 2012; Gauvrit & Morsanyi, 2014). Paradoxes, due to the lack of empirical control, constitute a great challenge for students in all levels, but they also may constitute a fruitful terrain for stochastic reasoning to develop (Falk & Konold, 1992; Movshovitz-Hadar & Hadass, 1990). However, despite the didactic potentiality of paradoxes, very little is still known for their role in secondary mathematics education.

This study aims at giving an insight on the way secondary mathematics students of different school Grades deal with some popular paradoxes. Particularly, we studied students in Grades 8, 9, 10 & 12 and we were guided by the following research questions:

1) What strategies emerged while secondary students were confronted with probabilistic paradoxes?
2) Are there differences in students' strategies among the various students' Grades?

2. Theoretical considerations

A paradox characterizes "a situation which reflects a contradiction to the current base of knowledge" and a puzzle is "a situation in which the current concept yields a solution that seems intuitively unacceptable" (Borovcnik & Kapadia, 2014, p.35). Paradoxes and puzzles are often the motivating power for the development of new conjectures and new theories and they contribute to the construction of new knowledge in many scientific areas. The role of paradoxes and puzzles is essential, particularly for the development of probability theory, since contradictions and counterintuitive examples are in abundance in situations where uncertainty and chance dominate (Batanero & Borovcnik, 2016; Chernoff & Sriraman, 2014).

In this paper we focus on two famous paradoxes. The one is known as the Bertrand's Box Paradox (Bertrand, 1889) and the second as the two children problem (Gardner, 1959) (see Table 1 below).

According to Borovcnik & Kapadia (2014), Bertrand's box paradox stems from viewing boxes as equiprobable cases and a resistance to consider the changes in probabilities when new information comes to play. Other researchers (e.g. Gauvrit & Morsanyi, 2014) have also related this problem with the ability to construct correctly the sample space of the underlying situation, namely focusing on the 6 simple cases of coins and not on the 3 boxes. It is worth mentioning that Poincaré in his book *Calcul des probabilités* refers to this problem in the very first chapter for the definition of probability (Poincaré, 1912, p.26). The impact of the given information on the calculation of probabilities as well as the ambiguity of the wording result on various assumptions which opened a lively debate for the problem's setup and further modifications of it (e.g. Nickerson, 2004; Borovcnik & Kapadia, 2014). The two children problem, similarly to the Bertrand's box paradox, revealed the misleading guidance of intuition and the solution is also based on a careful construction of sample space which can be facilitated by a two-way table (Taylor & Stacey, 2014).

Table 1. The two famous paradoxes used in this study

Bertrand's box paradox	Two children problem
Consider 3 boxes. The 1st box contains 2 gold coins, the 2nd box contains 2 silver coins and the 3rd box contains one gold and one silver coin. You choose one box randomly and then the coins in that box are chosen one at a time. Suppose that the first coin is gold. What is the probability that the second coin is also gold?	Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls? Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Although problems like the two discussed above have a mathematical solution, people often rely on their intuitions rather than on their mathematical knowledge to deal with them. The intuitive perceptions and simple cognitive mechanisms employed when judging under uncertainty have been of great interest among researchers in psychology and mathematics education community and many of the paradoxes have been used as research tools for the exploration of misleading intuitions in stochastic situations.

Tversky and Kahneman (1974) highlighted some dominant heuristics, namely principles that reduce the complexities of uncertain tasks to simpler operations for making decisions. One of these heuristics is the *availability* heuristic, when people overestimate the probability of an event by the ease this event can be recalled in memory. For example, one may assess the risk of heart attack among middle-aged people by recalling such occurrences among one's acquaintances (Tversky & Kahneman, 1974, p.1127). In a further study Fischbein & Schnarch (1997) also observed the appearance of this heuristic. Particularly in a research involving five age groups (from Grade 5 to college level) they found that, as subjects are getting older, their skills and knowledge regarding the complementarity and subsequent equality of the two groups increases, and their strategies rely on the ease of the underlying combinations. The second heuristic identified by Tversky and Kahneman (1974) is the representativeness heuristic, when people's assumptions are based on a representative case or pattern that is expected to appear. This heuristic has been also related to various misconceptions such as the gamblers' fallacy (e.g. Falk & Konold, 1992), namely the anticipation that a sequence of same outcomes would be reversed next, or the positive recency effect (e.g. Fischbein & Schnarch, 1997), namely an assumption that the conditions are not fair given a long sequence of same outcomes. The third heuristic identified by Tversky and Kahneman (1974) is anchoring and adjustment according to which people rely on the first available information to make a judgement and then they make necessary adjustments when needed. For example, when a group of high school students estimated the product 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 (= 40320) within 5 seconds, the median estimate was 2250, but when another group of students estimated the same product written as 1 x 2 x 3 x 4 x 5 x 6 x 7 x 8, the median estimate was 512. Estimation was based on the first few steps of multiplication (Tversky & Kahneman,1974).

In addition to heuristics, misconceptions and attitudes that rely on false intuitions when people addressing probabilistic problems have been also a point of attention for researchers. One of the widely discussed such misconceptions is *equiprobability bias* which appears when people assign equal probability to all possible outcomes (e.g. Batanero, Serrano & Garfield, 1996). This misconception about randomness seems to increase with probability teaching especially when the emphasis is on the classical definition of probability (e.g. Gauvrit & Morsanyi, 2014). Another misconception named *sample space miscount* refers to students' construction of the sample set (set of all possible but not equiprobable events) as described by Chernoff and Zazkis (2011) rather than the sample space without recognizing that it cannot be used to estimate probabilities. A further approach to probabilistic problems is what Konold (1989) named *outcome approach*. This approach refers to strategies which focus only on a successful prediction for the outcome of the next trial of a random experiment rather than the probability of it. Lastly, *personal interpretation* refers to judgments based neither on formal tools nor heuristics and misconceptions but on personal opinions or beliefs. Some examples are summarized by Savard (2014).

The various heuristics, misconceptions and informal strategies identified when people are confronted with randomness and uncertainty reveal the dominant and persistent role of intuition as well as the complexities of the cognitive mechanisms that take place in stochastic situations. These complexities have opened a field of inquiry not only with respect to the aspects of probabilistic thinking but also to the teaching and learning of probability (Batanero & Borovcnik, 2016; Pfannkuch, Budgett, Fewster, Fitch, Pattenwise, Wild & Ziedins, 2016). The discussion regarding teaching approaches to facilitate students to grasp the complexities and counterintuitive aspects of probabilistic notions have brought many times to the fore the use of paradoxes as a didactical tool (Falk & Konold, 1992; Movshovitz-Hadar & Hadass, 1990). Some studies reflect on the use of paradoxes on teaching university students (e.g. Klymchuk & Kachapova, 2012) or prospective teachers (e.g. Batanero, Contreras, Fernándes & Ojeda, 2010), thus showing the potentiality of these problems to support the learning of probability. There are also studies that suggest the use of such problems for the teaching of probability even for the secondary education (Batanero, Contreras, Díaz & Cañadas, 2014; Batanero, Godino & Roa, 2004; Taylor & Stacey, 2014). However, despite the recognized potentiality of paradoxes to motivate students' learning in secondary education and to provide a fruitful ground for the construction and reconstruction of meaning around randomness, we still know very little about their role in the teaching practice as well as about students attitudes when dealing with them.

This study aims at giving insight on secondary students' attitudes towards the two paradoxes shown on Table 1. The misconceptions and heuristics mentioned in the literature have been acknowledged and explored while students of different grade levels respond to these problems.

3. Methodology

3.1 The context of the study

According to the Greek official curriculum, in secondary education, students are introduced to some fundamental statistical concepts (population and sample, statistical graphs, frequency and relative frequency distribution, grouping of observations, mean and median value in data sets) in Grade 8 for first time and in Grade 9 with some probabilistic ideas (sets, sample space and events, the classical definition of probability). After these introductory lessons in Grade 8 for statistics and Grade 9 for probability, the next time students learn about probability and statistics content is in Grade 12. In this lesson there are two large chapters dedicated to statistics and probability. One chapter includes descriptive statistics as well as some elements from linear regression and linear correlation. The other chapter includes an extended version of the Grade's 9 content as well as some elements of conditional probabilities and combinatorics. In all Grades the approach to the content of both statistics and probability is formalist, paying attention mainly on formulas, definitions and proofs. Despite the guidelines set by the official curriculum, the teaching of statistics and probability is often omitted in Grades 8 and 9, due to time limitations.

In the study participated 213 students in total. More particularly, 48 students of Grade 8, 63 of Grade 9, 53 of Grade 10 and 49 of Grade 12. Until the time the study took place, the students in Grade 12 were the only participants who had typical knowledge of basic probabilistic concepts. The others hadn't been taught about probability and statistics.

3.2 The paradoxes used and the data of the study

For our study, we used a questionnaire that consisted of two tasks based on the paradoxes seen on Table 1. The wording we used for the problems was due to the students' background with a main consideration to have a common questionnaire for all participants, i.e. to be understood by all no matter in which grade level they are.

The questionnaire of our study is presented in Table 2 below. The original formulation was in Greek. Here we present an English translation of the tasks. Each task was given in a separate page.

Table 2. The tasks that we gave to the students (the same for all grade levels).

TASK 1	TASK 2
Answer the following questions and explain in detail: A. We meet a man who is known to have two kids. We ask him: "Do you have at least one boy?" and he responds "Yes". Which of the following you consider to be more likely: a. He has two boys b. He has a boy and a girl c. Both are equally likely to happen B. We meet a man who is known to have two kids. We ask him: "Is your eldest child a girl?" and he responds "Yes". Which of the following you consider to be more likely: a. He has two girls b. He has a boy and a girl c. Both are equally likely to happen	Answer the following question and explain in detail: A game is played with three cards. One card is black on both sides, another card is red on both sides and the third card has one black side and one red side. We put all cards in a box and we shake the box. Without looking we draw one card and put it on a table. The side we can see is red. What can we say for the other side? The hidden side is more likely to be: a. Red b. Black c. Both colours are equally likely

In Task 1 we consider the two questions as one problem, since one of the parameters we explore is whether the students will face the two questions as separate problems or as the same. Particularly, our aim was to investigate their ability to see the difference in the structure of the sample space in the two different cases. For the second task, our focus was on whether the students would define correctly the sample space of the problem.

The questionnaires were distributed by the teacher of mathematics during the lesson and students had approximately 40 minutes to answer. Responsible for the distribution and the collection of the questionnaire were the teachers of each classroom. The guidelines the teachers in charge were given by the researchers were that they shouldn't give any information or clarification with regard to the content of the questions, and they would also guarantee the anonymity of the participants. The data of our study were 213 written responses of the participants in the two tasks described in Table 2.

3.3 The method of the data analysis

The students' written responses were first grouped according to their grade level and then according to the strategy identified in their justification. For the categorization of the emerged strategies we acknowledged heuristics and probabilistic misconceptions that are discussed in the theoretical section of this paper. In Table 3 we present the categories emerged in students' responses illustrated by some characteristic examples.

Table 3. Categories used for the analysis of students' responses

<table>
<tr><th>Types of students' strategies</th><th>Examples of students' responses (translated from Greek)</th></tr>
<tr><td>No misconception /
other strategy
(mathematical arguments or using reasoning beyond mathematics discipline, in either correct or erroneous way)</td><td>(1st task – Grade 12)
A) He has 2 children:
<table>
<tr><th>1st child</th><th>2nd child</th></tr>
<tr><td>Boy</td><td>Boy</td></tr>
<tr><td>Boy</td><td>Girl</td></tr>
<tr><td>Girl</td><td>Boy</td></tr>
<tr><td>Girl</td><td>Girl</td></tr>
</table>
So it is more likely to be a boy and a girl
Or (1st task – Grade 9)
A) Once we ask someone and use the word "at least", which is negative [in meaning], it means [we ask] if he has at least one boy. Since he answers "yes", one of the two children [only] is a boy otherwise he would have answered "both".</td></tr>
<tr><td>Availability
(students recall similar events or relative cases)</td><td>(1st task – Grade 9)
A) Because I am a boy and I have a sister, so it sounds more normal to be a boy-girl.
B) I know many girls who have only sisters and I also have a bigger sister.</td></tr>
<tr><td>Representativeness
(estimations based on an expected pattern)</td><td>(1st task – Grade 8)
A) It is more likely to have a boy and a girl because one child is a boy for sure, and for the other one is more probable to be of different gender than the first one, that is to be a girl</td></tr>
<tr><td>Equiprobability bias
(assigning equal probability to all possible events)</td><td>(1st task – Grade 10)
A) From the two children we know that one of them is a boy. It is equally likely that the second kid is either a boy or a girl</td></tr>
<tr><td>Outcome approach
(focus only on a successful prediction for the next trial)</td><td>(1st task – Grade 12)
B) The question refers to the older of the two children, so we can't know anything about the gender of the second child</td></tr>
<tr><td>Sample space miscount
(constructing a sample set rather than a sample space)</td><td>(1st task – Grade 12)
A) 2 kids: (i) boy - boy, (ii) girl - boy, (iii) girl - girl. But he has at least one boy, so it is likely to be either case (i) or (ii)</td></tr>
</table>

Personal interpretation (judgement based on personal opinion or beliefs)	(1st task – Grade 9) A) Due to the fact that he has a boy and a girl, he thinks fast that he has a boy and so he answers "yes". B) 2 girls are nicer.
No answer	(the student neither answers nor gives any justification)

4. Results

In Table 4 we summarize the intensity of students participation. More than 92% of the students gave back a justified response (valid cases) while less than 8% responded without justification (missing cases). The missing cases may be due to the volunteer character of the participation in the study.

Table 4. Case Processing summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Strategy in the 1st task	198	93.0	15	7.0	213	100.0
Strategy in the 2nd task	197	92.5	16	7.5	213	100.0

In Figure 1 we present the frequencies of the strategies appearing on Table 3, as emerged from students' responses in the given tasks.

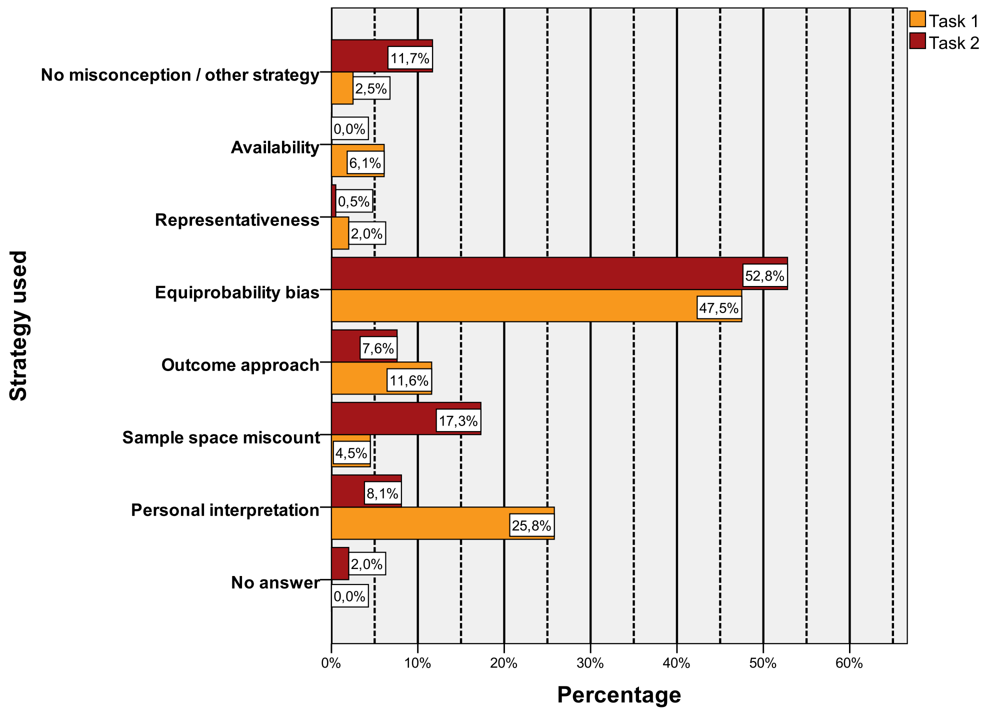

Figure 1. Strategies emerged in students' responses in the two tasks

As we can see, in about half of the students' responses we identified the *equiprobability bias* in both tasks. In the first task, the next more frequent strategies were: *personal interpretation* (25.8%), *outcome approach* (11.6%) and *availability* (6.1%). In the second task, strategies appearing in high frequency were: *sample space miscount* (17.3%), *no misconception / other strategy* (11.7%), *personal interpretation* (8.1%) and *outcome approach* (7.6%).

We believe that the differentiation in the way students addressed the two tasks is due to the problem context in each case. The first task, which has been set in terms of everyday life, causes more personalized interpretations, as well as more detectable desire to determine the gender of the children this particular person we meet has. The second task seems more mathematical as it refers to a gambling game and so it emerges more attempts to record the sample space (both erroneous and correct) and fewer personalized interpretations. Moreover, availability comes up in the first task alone, where one has the ability (and the tendency) to recall information about two-children families.

To determine possible differences among different Grades, we use the contingency table of each strategy employed by the students per each grade level (Table 5).

Table 5. Contingency table for the various strategies per Grade in the two tasks

Strategies used	Grade 8		Grade 9		Grade 10		Grade 12		Total	
	Task1	Task2	Task1	Task2	Task1	Task2	Task1	Task2	Task1	Task2
No mis-conception / other strategy	0 .0%	3 1.5%	2 1.0%	9 4.6%	0 .0%	4 2.0%	3 1.5%	7 3.6%	5 2.5%	23 11.7%
Availability	6 3.0%	0 .0%	1 .5%	0 .0%	2 1.0%	0 .0%	3 1.5%	0 .0%	12 6.1%	0 .0%
Representati-veness	3 1.5%	1 0.5%	0 .0%	0 0.0%	1 .5%	0 0.0%	0 .0%	0 0.0%	4 2.0%	1 0.5%
Equiproba-bility bias	16 8.1%	21 10.7%	31 15.7%	29 14.7%	24 12.1%	28 14.2%	23 11.6%	26 13.2%	94 47.5%	104 52.8%
Outcome approach	4 2.0%	1 0.5%	9 4.5%	10 5.1%	5 2.5%	2 1.0%	5 2.5%	2 1.0%	23 11.6%	15 7.6%
Sample space miscount	0 .0%	8 4.1%	2 1.0%	9 4.6%	1 .5%	3 1.5%	6 3.0%	14 7.1%	9 4.5%	34 17.3%
Personal interpretation	15 7.6%	8 4.1%	18 9.1%	4 2.0%	9 4.5%	4 2.0%	9 4.5%	0 0.0%	51 25.8%	16 8.1%
No answer	0 0.0%	0 0.0%	0 0.0%	1 0.5%	0 0.0%	3 1.5%	0 0.0%	0 0.0%	0 0.0%	4 2.0%
Total	44 22.2%	42 21.3%	63 31.8%	62 31.5%	42 21.2%	44 22.3%	49 24.7%	49 24.9%	198 100%	197 100%

As we can see on Table 5, *equiprobability bias* dominates upon all Grades. We can detect a slight decrease as we move from Grade 9 to Grade 12, from 15.7% to 11.6% for the first task and from 14.7% to 13.2% for the second task. Grade 8 uses more *personal interpretations*, i.e. 7.6% and 4.1% respectively, while upper Grades stay at 4.5% and less than 2.0% respectively. This is to be expected if one considers that students have neither the knowledge nor the maturity to deal with the problems. Grade 9 uses *personal interpretations* in a surprisingly higher extend for the first task than for the second one (9.1% and 2.0% respectively). *Outcome approach* makes a "peak" of 4.5% and 5.1% for the two tasks respectively in Grade 9, but it is consistent in the other Grades (between 2-2.5% and 0.5-1% respectively). *Sample space miscount* seems to be more frequent among Grade 12 students than in the other Grades. Particularly, the frequency in Grade 12 is 3% and 7.1% respectively, while in the other three Grades is less than 1% and less than 4.1% respectively. We interpret that as an effort to use the tools acquired during teaching, albeit not always in a successful manner. Therefore, education students received seems to influence the chosen strategy. Similarly, the use of *personal interpretation* in the upper Grades seems to be diminishing. Overall, *personal interpretation* is used over three times more frequently for the first task than for the second task, which appears to be more mathematical. This finding reinforces the view that problems that differ in their external characteristics (even if they have a similar mathematical structure) are associated with different intuitive misconceptions. *Representativeness* has very little (almost zero) representation in the problems used.

5. Conclusions

With respect to our research questions, we identified six strategies in students' responses to the Bertrand's (1989) box paradox and the two children problem. The emerged strategies are related to heuristics and misleading conceptions widely discussed on the literature (Tversky & Kahneman, 1974; Fischbein & Schnarch, 1997; Falk & Konold, 1992; Konold, 1989; Chernoff and Zazkis, 2011), while there is a small percentage for the first task (2,5%) and a bigger for the second (11,7%) that could not be related to any of those strategies. This result may indicate that the second problem was comprehended more mathematically than intuitively. Another notable result was the high percentage of the *equiprobability bias* in both tasks. This finding agrees with Gauvrit and Morsanyi's (2014) findings that *equiprobability* dominates particularly the two children problem also used there. As we saw, there is some persistence with maturation regarding this bias, so we may assume that school teaching does not seem to have helped to a substantial reduction of this misconception. Furthermore, we identified important differences not only among the different grade levels but also between the tasks in the responses of the same grade level. This indicates the special role of the content of the problem in terms of its wording and the underlying context. Particularly, if it is closer to mathematical or everyday life context, seemed to be significant in employing a mathematical rather than an intuitive approach respectively.

Besides the limitations of this study (sampling based on accessibility, voluntary character, curriculum limitations), we got a deeper insight on how students address paradoxical situations in stochastic contexts and how the employed strategies change and develop through maturation and formal education. The persistence in time of some misleading conceptions which have been also mentioned by other researchers (Fischbein & Schnarch, 1997; Gauvrit & Morsanyi, 2014) indicates a need for alternative approaches in probability teaching, were paradoxes may have a central role (e.g. Leviatan, 2002).

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