Mathematizing Bayesian situations in school by using multiple representations

Matematizando situaciones Bayesianas en la escuela usando múltiples representaciones

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Abstract

Bayesian problem situations are well known as being difficult for people to judge adequately. Teachers in school are confronted with the question what can be done on the long run to support their students in coping with Bayesian situations. In this regard the paper refers to the learning about mathematizing Bayesian situations in school, especially focused on using the unit square in settings of learning with multiple representations. Firstly, the topic of Bayesian reasoning will be analysed from the different perspectives of mathematical, modelling, and mathematizing structure. Afterwards, potentials of theories of learning with multiple representations will be reflected within the topic of Bayesian reasoning. Bringing both together yields in in a 2x2 matrix which allows for categorizing task types about Bayesian situations.

Keywords: Bayesians situations, learning, multiple representations, tasks categorization

Resumen

Se conocen bien las dificultades de las personas para juzgar adecuadamente las situaciones-problemas Bayesianas. Los profesores se enfrentan en la escuela a la cuestión de qué se puede hacer a largo plazo para apoyar a sus estudiantes en la resolución de situaciones Bayesianas. A este respecto este artículo refiere al aprendizaje de la matematización de situaciones Bayesianas en la escuela, especialmente centrado en el uso del cuadrado unidad en entornos de aprendizaje con múltiples representaciones. En primer lugar, se analizará el tema del razonamiento Bayesiano desde las diferentes perspectivas de matematización y modelización de la estructura matemática. Seguidamente, el potencial de teorías del aprendizaje que tienen en cuenta el uso de múltiples representaciones será reflejado dentro del tópico del razonamiento Bayesiano. Conjuntando ambos campos se elabora una matriz de 2x2 que permite la categorización de tipos de tareas sobre situaciones Bayesianas.

Palabras claves: Situaciones Bayesianas, aprendizaje, representacionesmúltiples, categorización de tareas.

1. Introduction

It is well known in fields of research as well as of practice in classrooms that people have difficulties in adequately probabilistic reasoning within Bayesian reasoning situations. This kind of reasoning requires high demanding mental processes in which intuition plays a key role (cf. Batanero, 2015). However, intuition and probability are not always going well together, which gets Cosmides and Tooby (1996) to speak of "clashes between intuition and probability". Even medicine experts have difficulties in diagnosing a Bayesian situation in fields of medicine like e.g. a Bayesian situation referring to a health test (Fig. 1).

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A person who participates in a health test is expected to have a specific disease by probability of 10%. Given a person has the disease the probability of getting a positive test result is 80%. However, even a person not having the disease is expected to get also a positive test result by probability of 20%. Imagine there is a person who gets a positive test result. By which probability is this person expected to have the disease actually?

Figure 1. A Bayesian situation referring to a health test

For example, in their studies Eddy (1982) or Hoffrage and Gigerenzer (1998) found that only 5% resp. 10% of the participating physicians were able to interpret the given information of comparable tasks in the right way. Because misinterpretations in such situations of decision making could have serious consequences huge efforts in research have been made to identify difficulties in Bayesian reasoning situations and to look for strategies how to overcome them. In the meanwhile, psychological and educational research found out that the question of the Bayesian problems' representation is crucial with regard to the kind of numerical information on the one hand, and to the kind of visualization on the other (McDowell & Jacobs, 2017). This refers not only to the training of experts but also to the training of students in school on the long run. Thus, in the following section the mathematical and the modelling structure of Bayesian problem situations will be analysed in short only regarding the case of two events usually being most relevant in school. Afterwards the mathematising process which is at the very heart of the modelling process will be focussed on and discussed concerning the representation formats of the contextual as well as mathematical information given to describe Bayesian problem situations.

2. Mathematizing Bayesian situations in school

Mathematical aspects: Bayes situations are standing for that kind of problems which could be solved by using the Bayes formula. The Bayes formula in turn is an algebraic expression for the Bayes theorem which describes the probability of an event, based on prior knowledge of conditions that might be related to the event being of interest. Spoken generally, there are two events A and B which are related to each other in sense of that one event (e.g. event A) is conditioning the outcome of the other event (e.g. event B) and the question is about the probability of occurring this conditioned event B given A. More formally in sense of mathematics: Given there are two events A and B (from the sigma-field of a probability space) with the unconditional probability of A which is the probability P(A) of the event A occurring (with P(A) > 0). Then, the conditional probability of B given A is defined as the quotient of the probability of the joint of events A and B, and the probability of B, i.e. $P(B|A) = \frac{P(A \cap B)}{P(A)}$, where $P(A \cap B)$ is the probability of the joint event $A \cap B$. Correspondingly, the conditional probability of A given B equals to the quotient of the probability of the joint of events A and B, and the probability of A, i.e. $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Using some algebra for bringing both equations together one yields the Bayesian formula for $P(B|A)$ with

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A)} \qquad (1)$$

Keeping in mind that the probability of $P(A)$ can be specified regarding the cases of occurring or not occurring of event B by $P(A) = P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})$ (with $\bar{B} = \Omega \backslash B$ being the complementary event and $P(\bar{B}) = 1 - P(B)$) one gets the extended version of the Bayesian formula for $P(B|A)$ with

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})} \qquad (2)$$

which proves often to be useful for inserting numerical information typically given within Bayesian tasks into the Bayesian formula. Given all information would be directly available and correspondingly labelled according to the formula's variables the solution of a Bayesian problem would be a question of pure calculation. However, usually it is not comfortable like this.

Modelling aspects: Keeping a typical medical Bayesian situation (Fig. 1) in mind there arise some questions which must be answered before being able to apply Bayesian formula, like e.g. "what is a conditioning event?", "what is a conditioned event?", "how must the numerical information given in the text be translated in the formula's parameter?", "is it a Bayesian situation at all and thus, is the Bayesian formula applicable at all?", etc. Answering questions like this is at the core of coping with Bayesian situations and decision making under uncertainty and fits the main aspects of modelling in school (cf. Eichler & Vogel, 2015). Modelling is an important competency for the thinking and teaching of statistics and probability (e.g. Wild & Pfannkuch, 1999; Eichler & Vogel, 2014; Eichler & Vogel, 2015; Pfannkuch, Ben-Zvi, & Budgett, 2018) and describes an overarching conception of mathematics education (e.g. OECD, 2003) referring to the four modelling phases which are arranged in the modelling cycle (Blum, Galbraith, Henn, & Niss, 2007). Figure 2 exemplifies the modelling process in the Bayesian situation context.

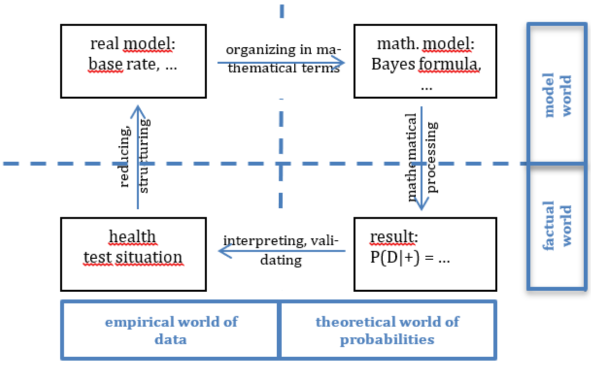

Figure 2. Bayesian reasoning as a modelling process

By analysing the Bayesian problem situation and the coping with it within the terms of the modelling approach it becomes obvious that there are many potential traps on the way from the real situation problem to the result and its interpretation (Fig. 2) including the counterintuition of the result which might be one explanation for the devastating findings of Eddy (1982) or Hoffrage and Gigerenzer (1998) mentioned in the introduction. On the other hand, the modelling cycle is also applicable for locating useful strategies in supporting people by better coping with Bayesian situations, especially in school, where the modelling approach is at the core of different curricula of different countries all over the world since the PISA studies of the OECD.

Bridging real world and mathematical world – mathematizing: The transition processes between the "real" world, i.e. here the empirical world of data, and the mathematical world, i.e. here the theoretical world of probabilities, are at the very heart of the modelling processes. They are estimated to be very demanding and they are always endangered to be shortened inappropriately (Schupp, 2004, p. 3). Especially the sub-process of organizing in mathematical terms is crucial but it is also considered as being the most complex one (Klieme, Neubrand, & Lüdtke, 2001, p. 144). For the reason of taking into account this complexity and of reducing risk of inappropriate shortening the process of mathematizing has to be considered as a unit which consists of the two complementary subprocesses of abstraction one the one hand (model construction: constituting a mathematical model based on the real model) and of contextualizing (model inspection: checking the mathematical model's structural and contextual adequateness in terms of the real model) on the other hand (Pfannkuch et al., 2018; Vogel, 2006). The subprocess of contextualizing has an important meaning in terms of mathematics or sciences learning in school: Students are often confronted with readymade models, like for example mathematical, chemical, biological or physical laws given in graphs or formulas, like e.g. the law of falling bodies with $d = 0{,}5 \cdot g \cdot t^2$ (d for distance, g for gravity, t for time). If students do not reconstruct the structural as well as the contextual meaning of the formulas, they are endangered to operate only on a sub-semantic level. Of course, this general aspect of mathematizing also touches the point in Bayesian reasoning. In general, the interpretation of a representation is an inherently contextualised activity (Roth and Bowen, 2001). Thus, learners must come to understand the relation between the representation, which refers in this case to the Bayes formula, and the domain that it represents, i.e. the Bayesian situation. According to these considerations the process of mathematizing can be specified and displayed as in Figure 3.

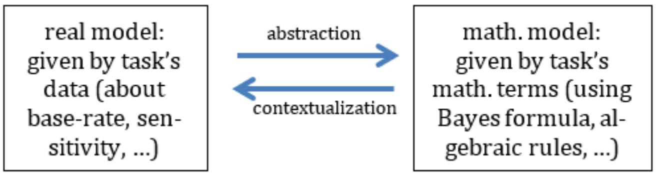

Figure 3. Process of mathematizing (in Bayesian situations)

3. Multiple representations in mathematizing Bayesian situations

Fundamentals about learning with multiple representations

According to Kaput (1987a) the root phenomena of mathematics learning and application are concerned with dealing with multiple representations. Following OECD (2003, p. 41) the representation competence involves "[…] the choosing and switching between different forms of representation of mathematical objects and situations, and translating and distinguishing between different forms of representation." The empirical findings regarding the learners' benefit from using multiple representations is two-fold: On the one hand there is empirical evidence that multiple representations can effectively support learners in their success of problem solving (DeBellis & Goldin, 2006; Kaput, 1987b; van Someren, Reimann, & Boshuzien, 1998).

However, on the other hand, research findings also yielded that sometimes the usage of multiple representations can hinder learning (e.g., Ainsworth, 2006; English & Halford, 1995; Yerushalmy, 1991). By Following Ott, Brünken, Vogel, and Malone (2018) two main factors, i.e. *learner characteristics*, such as their prior knowledge, and *the characteristics of the provided representations*, might be accountable for positive learning effects of using multiple learning in mathematics. Concerning the factor *characteristics of the provided representations* there have to be two theoretical frameworks taken into account which found the discussion of learning with multiple representations: the Cognitive Theory of Multimedia Learning (CTML, Mayer, 2009) and the Integrated Model of Text and Picture Comprehension (ITPC, Schnotz & Bannert, 2003). Instead of discussing specific theoretical details of these psychological models of information processing the common theoretical issues are sufficient for describing the central principles here (cf. Vogel, Girwidz, & Engel, 2007). These principles can be used for deriving didactical implications about the learning of mathematizing Bayesian situations afterwards.

Both the CTML and the ITPC are basing on Paivio's dual coding theory (Paivio, 1986), Wittrock's view of learning as a generative process (Wittrock, 1974), and on Baddeley's assumptions about the dual-channel working memory model (Baddeley, 1992). According to the dual-channel-approach two different kinds of external representations, i.e. *descriptive* and *depictive* representations (Schnotz & Bannert, 2003), are processed in two different channels by each being translated step by step into internal representations (and vice versa). Descriptive representations are composed of symbols like e. g. a certain text or the Bayesian formula. Depictive representations (e.g. pictures or diagrams, like the unit square or a tree diagram) are analogous to the represented phenomenon, as they include iconic signs and often express contextual relations without the use of symbols. By further processing in working memory internal representations, whether of homogeneous or heterogenous kind (cf. Ott et al., 2018), are related to each other on pre-knowledge-based processes of conceptual organization. Schnotz and Bannert (1999) speak of mutual supplementation via the construction of mental models and model inspection.

A necessary prerequisite of benefitting from multiple representations is coherence formation (Seufert, 2003). Gentner (1983) is speaking of processes of structure mapping. Brünken, Seufert, & Zander (2005) distinguish between local coherence formation which refers to each one of the focused multiple representations and global coherence formation which refers to the conceptually mapping of the focused multiple representations and integrating them in a coherent mental representation. Thus, it becomes clear that the use of multiple representations does not per se lead to a benefit of the learners but need to be reflected on a theoretical base.

Using multiple representations in mathematizing Bayesian situations

The students' learning goals of mathematizing Bayesian situations in school can be specified within the process model of Figure 3: On the one hand, it is about learning of abstracting a Bayesian task's data in that way that the Bayesian formula's ratio can be derived which allows for calculating the right (mathematical) solution afterwards. However, school learning about Bayesian situation is not only about Bayesian reasoning, but also about flexible Bayesian reasoning which means by following Borovcnik (2012) the understanding of parameter dependency in Bayesian reasoning situations (Böcherer-Linder, Eichler, & Vogel, 2017). More general, it is also about the conceptual understanding of the mathematical structure in terms of the problem situation beyond pure calculating probabilities, it is also about contextualization and model inspection of mathematical terms representing Bayesian situations.

The basic idea of using multiple representations in mathematizing Bayesian situations is: If a certain representation proves not to be sufficient (like the task's representation of the data of the Bayesian situation of Fig. 1) for different reasons like e.g. missing pre-knowledge or a bad „performance of matching the nature of the problem representation to the nature of the task." (Vessey, 1991, p. 220) then it seems to be promising to provide an alternative representation matching better the prerequisites of the learner or of the task, or, a combination of an alternative representation with the original representation.

People can be effectively supported in a Bayesian reasoning situation when the numerical information is provided via so-called natural frequencies instead of using probabilities given in percentages (e.g. Hoffrage & Gigerenzer, 1998; Binder, K., Krauss, & Bruckmaier, 2015). Gigerenzer & Hoffrage (1995, p. 697) explain the facilitating effect of natural frequencies by using an evolutionary argument: "An evolutionary point of view suggests that the mind is tuned to frequency formats, which is the information format humans encountered long before the advent of probability theory.” This quotation fits that point what Salomon (1979, p. 137) calls “cognitive make-up”. From a mathematical point of view the most important thing for getting the right solution is not necessarily to apply the Bayesian formula in a narrow sense of the symbolic expression but to find the crucial numerical information defining the numerator and denominator of the ratio which represents the result of the Bayesian problem. In concrete, the result of 30,1% for the health task given in Figure 1 can not only derived by calculating a single-event-probability using the probabilities $P(D), P(+|D), P(\overline{D}), P(D|+)$ given by percentages. In case of interpreting probabilities as relative frequencies the same result can be calculated by representing the numerical information of the data via natural numbers instead of percentages, so-called natural frequencies (Johnson & Tubau, 2015, p. 5). In terms of this approach, the problem of Figure 1 could be correspondingly given by those versions of Figure 4.

A person who participates in a health test is expected to have a specific disease by probability of 10%. Given a person has the disease the probability of getting a positive test result is 80%. However, even a person not having the disease is expected to get also a positive test result by probability of 20%.	10 out of 100 persons who participate in a health test are expected to have a specific disease. 8 out of 10 persons having the disease are expected to get a positive test result. However, even 18 out of 90 persons not having the disease also get a positive test result.

Figure 4. Bayesian health test situation represented via single-event-probability and via natural frequencies

In case of using natural frequencies the computation becomes easier (cf. Johnson & Tubau, 2015): When comparing both terms for the same ratio in equation (3) only three absolute numbers instead of six percentages must be processed in the term. The correct solution, which is the so-called positive predictive value of the test, can be calculated by Bayes’ rule either by using probabilities or natural frequencies:

$$P(D|+) = \frac{80\% \cdot 10\%}{80\% \cdot 10\% + 20\% \cdot 90\%} = \frac{8}{8+18} = \text{"}8\ out\ of\ 26\text{"} \qquad (3)$$

However, some issues have to be discussed regarding this problem approach in school because there are several steps to go from the wording and numbering of the original task to the application of the Bayes formula (cf. Johnson & Tubau, 2015). The following aspects teachers should have in mind in teaching this way of abstraction:

- The format of the numerical information must be distinguished from the number of events. The numerical format of percentages can be used as normalized measurement of either a single-event-probability, e.g. given in the task (10% chance of having a disease) or of a proportion of a set (10% of the persons are expected to have the disease). In the same way, whole numbers can be used to express relative frequencies (10 out of 100 persons are expected to have the disease) or single events (in 10 out of 100 chances having the disease is expected).

- Both the format of the numerical information and the number of events have to be distinguished from the sampling structure. The sampling structure could be given by the base-rate D, the test sensitivity $(+|D)$, test specificity $(+|\overline{D})$ or via the joint events $(+\cap D)$, $(+\cap \overline{D})$. The same information could be represented either via the normalized or frequency formats.
- From a pure mathematical point of view the two versions are algebraically not equivalent (probabilities are functions of events) but they are functionally equivalent. Thus, teachers have to decide what is at the forefront of their teaching interest: Using and applying probabilities in the field of Bayesian situations or training of Bayesian reasoning abilities in sense of problem solving or even both. All these learning goals are valuable, but they are different.

Consequently, in the following scenarios of using multiple representations in the two-folded process of mathematizing are discussed: abstraction and contextualization regarding coherence formation. Because of the huge amount of available depictional representations in the field of Bayesian reasoning (cf. Khan, Breslav, Glueck, & Hornbæk, 2015) the focus will be reduced on the unit square. The unit square is comparably unknown in regard to the more common tree diagram. However, there are some issues concerning the psychological models of processing multiple representations which makes the unit square interesting from a theoretical point of view (e.g. Eichler & Vogel, 2010; Eichler & Vogel, 2013, 2015).

Descriptive coherence formation

Given there are these two tasks' representations of Figure 4 to describe the Bayesian situation then the local coherence formation in sense of Seufert and Brünken (2003) would mean to extract the given data of each task's text, label them with regard to their contextual meaning, and relate the data to each other in the right order. This is the most crucial issue in Bayesian situations because there are conditioning information and conditioned information which are known as often being mistaken one for another. With regard to the medical setting of the Bayesian situations the constituting information of the real-model's data are the base-rate, the true-positive-rate and the false-positive rate. With regard to the mathematical structure of the having-to-find ratio which yield into the solution of the Bayesian situation the relevant information is to be separated into the conditioning information (base-rate D) and the conditioned information (true-positive-rate $(+|D)$, false-positive rate $(+|\overline{D})$); Fig. 5).

Via natural frequencies all numerical information are absolutely quantified to a single reference class (i.e. the superordinate set of 100 persons), where categories are naturally classified into (in terms of probability theory) expected values of the compounded events $D\cap +$, $D\cap -, \overline{D}\cap +, \overline{D}\cap -$. In this case, the conditional distribution does not depend on the between-group (having the disease, not having the disease) base rates, but only on the within-group frequencies (true-positive-rate, false-positive rate). Accordingly, the base rates can be ignored, numbers are on the same scale and can be directly compared and (additively) integrated, and the required computations are reduced to a simpler form of Bayes rule (cf. equation (3)). Correspondingly, in the natural frequency version also the structure of the nested sets of the Bayesian situation becomes more transparent.

The next step after having extracted the crucial numerical out of the tasks' wording forward to "the having-to-find ratio" is bridging the gap between the sequential ordering of the numerical information in the tasks' wording (in sense of the natural language) and the relational setting of the numerical information in the tasks' formula structure (in sense of the mathematical language of algebra). In a representational point of view this step from the natural language to what Vollrath (2003) calls "formula language" has to be seen at the core of the abstraction process leading from the real-model world to the mathematical model world. In a theoretical point of view this step should be expected being supportable by using representations of both worlds and bring them together in an artificial notion of what in analogy to Eichler and Vogel (2010) can be called a "word-formula" (Fig. 5; the analogue case of a "picture-formula" will be discussed afterwards).

Base-rate: $P(D) = 10\%$, true-positive-rate: $P(+\|D) = 80\%$ false-positive-rate: $P(+\|\overline{D}) = 20\%$	Base-rate-numbers: $H_{100}(D) = 10$ true-positive number: $H_{100}(D \cap +) = 8$ false-positive number: $H_{100}(\overline{D} \cap +) = 18$
probability of interest $= \frac{\textit{probabilty of having the disease and a positive test}}{\textit{probability of having a positive test}}$ $= \textit{probability of having the disease actually}$	proportion of interest $= \frac{\textit{number of persons with the disease and a positive test}}{\textit{number of persons with a positive test}}$ $= \textit{proportion of having the disease actually}$
$P(D\|+)$ $= \frac{P(D \cap +)}{P(+)} = \frac{P(+\|D) \cdot P(D)}{P(+\|D) \cdot P(D) + P(+\|\overline{D}) \cdot P(\overline{D})}$ $= \frac{80\% \cdot 10\%}{80\% \cdot 10\% + 20\% \cdot 90\%}$	$h(D\|+)$ $= \frac{H_{100}(D \cap +)}{H_{100}(+)}$ $= \frac{8}{8+18}$

Figure 5. Bayesian health test situation represented via single-event-probability and via natural frequencies

The "word-formula"-notion uses the common fraction-notation and combines it with text being inserted as numerator and denominator. There is some empirical evidence given by Ott et al. (2018) on which the "word-formula"-idea can be based on: When looking for effective multimedia support in coping with propositional logic tasks text in sense of natural wording proved to be constantly the reference representation throughout different treatment groups either working within a purely descriptive mode or a mixed mode including depictive representations. Thus, the students who are not feeling familiar enough with the formal notation Bayesian formula can make an intermediate step by thinking only about the ratio but not the formal notation first. Afterwards, when having defined the Bayes' ratio by using wording of natural language both the denominator and numerator can be translated in the mathematical terms given by the task. Going this way, the abstracting process can be displayed schematically like in Figure 5.

The idea of using „word-formulas" is not a new one, it can be already found being implicitly used by e.g. Gigerenzer and Hoffrage (1995), Eichler and Vogel (2010), or Eichler and Vogel (2015). Of course, this kind of formula-notation is formally lacking the syntactical rules of the usual algebraic notation. However, this kind of notation represents an intermediate step of translation between real-world representations and mathematical world representations and thus, it must not be judged in an algebraic but in a pedagogical respect: Students should be supported in their learning about applying mathematical language (and their representations) to real-world affairs like e.g. Bayesian situations.

Further, regarding the two-folded process of mathematizing (Fig. 3) it is not only about abstracting towards the mathematical world. It is also about going backwards from a given Bayesian formula towards the real-world model: In this perspective the ratio being a ready-made mathematical model for the Bayesian situation either given via single-event-probabilities or via natural frequencies must be reflected concerning the mathematical structure and the contextual meaning of the situation. This refers to the contextual meaning of the parameters $(D, (D|+), \dots)$ as well as to the meaning of their position within the Bayesian formula. The "word-formula"-approach is expected as also being useful in this turn-around of mathematizing.

Depictive coherence formation

By mainly focusing on tree diagrams Spiegelhalter and Gage (2014) emphasize the need of visualizing Bayesian situations for facilitating a specific situation's interpretation and allowing people to estimate a risk adequately. Whereas there is a considerable amount of different visualizations for communicating Bayesian situations (Khan, Breslav, Glueck, & Hornbæk, 2015), research on visualizations of Bayesian situations for students' learning is mainly restricted firstly to pure calculating the right solution of the problem situation given in percentages and, secondly, restricted to the tree diagram (as well as to 2x2-tables, cf. Veaux, Velleman, & Bock, 2011) but not to the unit square (Fig. 6).

The unit square allows for a depictional way of studying probability problems because, unlike Venn diagrams, it is semantically consistent with the rules of probability. Referring to the diagnosis task represented either in the probability or the frequency format, the unit square is partitioned into four areas (Fig. 6) concerning the events having the disease (D), not having the disease ($\overline{D}$), getting a positive test result $(+)$ or a negative test result $(-)$.

The vertical partitioning is determined by the event of having the disease which corresponds to the probability P(D) = 20% of having the disease and accordingly to the probability P($\overline{D}$) = 80% of not having the disease. The horizontal partitioning depends on the vertical partitioning and, thus, represent conditional events, which correspond to the probability that a healthy person gets wrongly a positive test result ($P(+|\overline{D})$ (Fig. 6, length of the vertical side right above) and accordingly to the probability that a person having the disease gets correctly a positive test result ($P(+|D)$ (Fig. 6, length of the vertical side left above). The areas represent joint probabilities, i.e. $P(D \cap +)$, $P(D \cap -)$, $P(\overline{D} \cap +)$ and $P(\overline{D} \cap -)$. The natural frequencies shown in Figure 6 on the right represent from the perspective of probability theory the expected values for the compounded events, i.e. 8 (D∩ +), 2 (D∩ −), 18 ($\overline{D}$ ∩ +) and 72 ($\overline{D}$ ∩ −).

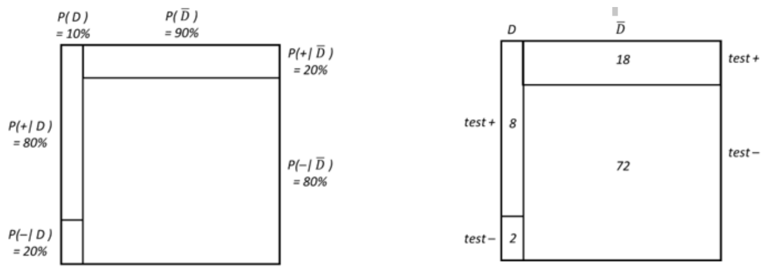

Figure 6. Health test situation represented in the unit square via single-event-probability (left) and via natural frequencies (right)

Because of its characteristics there are theoretical arguments suggesting that the unit square is especially suited to address mathematizing Bayesian situations: The unit square is a statistical graph (Tufte, 2015), which means, that the sizes of the partitioned areas are proportional to the sizes of the represented data. Therefore, because of this characteristic of area-proportionality the proportions of incidences, like e.g. the base-rate, in a population are represented numerically as well as geometrically. Thus, the benefit in a mathematical regard is that the unit square can be used to calculate the numerical value of probabilities and to determine the Bayes ratio (cf. Oldford, 2003, p. 1). From a theoretical point of view of multimedia learning (cf. above) the benefit of using the unit square is that it represents the depictive information principle. Thus, the numerical descriptively represented information gets an analogues depictive counterpart which may lead to a deeper elaboration of the Bayesian situation's mental model via mutual supplementation (cf. above; Schnotz & Bannert, 1999).

Furthermore, the characteristic of area-proportionality allows for using the unit square in the abstracting step forward to "the having-to-find ratio": By bridging the gap between the sequential ordering of the numerical information in the tasks' wording and the relational setting of the numerical information in the tasks' formula structure the so-called "picture-formula of Bayes" (Eichler & Vogel, 2010) can be applied (Fig. 7).

One essential issue of the unit square is that the numerically represented products of conditioning probabilities and conditioned probabilities (e.g. $P(+|D) \cdot P(D)$) which determine the denominator and the numerator of the Bayes formula correspond to the calculation of the rectangular subareas (length multiplied with width) of the unit square. With this kind of calculation, the students are usually very familiar when learning about conditional probabilities and Bayesian situations. Thus, the in the novices' eyes complex looking Bayesian formula gets potentially better accessible for school students because its parts are based on well-known mathematical subroutines.

Because of the area-proportional characteristic of this depictive representation it is even possible to estimate qualitatively the approximate value of the Bayes ratio: relating the subareas of denominator and numerator of the "picture-formula" of Figure 7 to each other with a good eye yield into nearly one third because the white rectangular seems approximately twice as large as the grey one. This fits roughly the exact value of 30,1% (cf. above). Furthermore, it becomes obvious that conjoint probabilities representing the subareas of the unit square derive from multiplying conditioning and conditioned probabilities like $P(+\cap D) = P(+|D) \cdot P(D)$ (of course, also $P(+\cap D) = P(D|+) \cdot P(+)$ which fits contextually another problem situation).

Going beyond, the unit square even suites to support the studying of parameter dependency of flexible Bayesian reasoning (in sense of Borovcnik, 2012 andBöcherer-Linder et al., 2017): If there would be, for example, a base-rate of a probability of 80% of having the disease in this articles working example the "picture-formula" of Figure 7 allows for predicting the resulting probability as becoming nearly 1 without calculating exact values by upsizing mentally the width of the grey rectangular and correspondingly downsizing the width of the white rectangular. Concerning the nested sets hypothesis which is "[…] the general claim that making nested-set relations transparent will increase the coherence of probability judgment." (Sloman, Over, Slovak, & Stibel, 2003, p. 307) it could be stated that unit square fits the transparency criterion by directly neighboring the decisive subareas of the Bayesian situation (fig, 7 in the middle).

These considerations about the representation characteristic of the unit square and the "picture-formula" are not depending on the two-folded direction of the mathematizing process, they can become relevant in the process of abstraction as well as in the process of contextualization. Given for example a Bayesian formula representing a single-event probability is given within a certain problem context it is possible to reconstruct all information via the descriptive way and/or the depictive way of representation. In the psychological point of view of processing multiple representations, the potential of depictional coherence formations must be seen in the combination of depictive and descriptive representations (Fig. 7). This potential should be made useful for teaching about mathematizing Bayesian situations in school via systematic tasks.

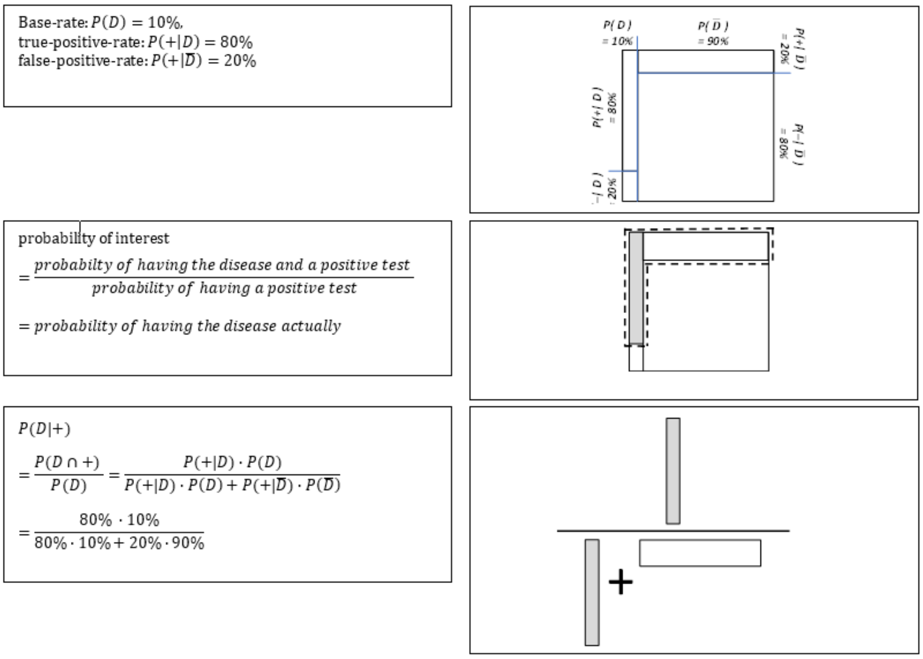

Figure 7. Depictional coherence formation via the “picture-formula of Bayes”

4. Task variations of mathematizing Bayesian situations in school

With regard to the mathematizing of Bayesian situations learning goals in school can be differentiated by the subprocesses of abstracting and contextualization. With regard to the use of multiple descriptional and depictional representations these learning goals can be specified according to their use of reception and of translation which are essential elements of coherence formation (cf. above; Seufert, 2003; Seufert & Brünken, 2003). Thus, task variations addressing these processes can be structurally located in the 2x2-matrix of Figure 8 which exemplifies the case of single-event-probabilities. It should be remarked that the elements of the matrix cells display the typical intended state of each cell. The correctly applied Bayes formula, for example, represents the intended state of the mathematical model which allows for calculating the right solution in the next step of modelling (cf. above) or, with regard to mathematizing, it allows for asking for the structural as well as contextual meaning of the formula’s parameter. Or, the correctly marked unit square allows for deriving the right Bayes formula within the mathematizing step of abstraction.

There are *task variations of the reception type:* These address questions about the interpretation of either a single representation or the combination of at least two combined representations. Potential examples for the interpretation of a single representation could be: “What is the meaning of each subarea of the completely labeled unit square?”, “Why is there the same term being part of the denominator as well as of the numerator?", "Which numerical information stands for base-rate, true-positive-rate, false-positive-rate, …, which information are conditioning, which one are conditioned, which wording in the task indicates what?" Correspondingly, reception type tasks for the case of two combined representations. This could be: "How do the rectangular relate to the numerical information of the Bayes formula?" (given picture-formula and Bayes formula) or "Where can the conditioning information, where the conditioned information be found in the text, where in the unit square?" (given unit square and text of the task). Task variations of the reception type aim for the students' understanding of the different types of representations by which they are confronted when they work on Bayesian situations.

There are also *task variations of the translation type:* These address questions for generating a representation based on a given one. Potential examples could be: "How does the Bayesian formula look like on base of the given numerical information of the unit square?", "How can the numerical information of the task's text be placed within the unit square?" or "Make up a story out of the given Bayesian formula." Task variations of the translation type aim for the students' learning to use flexibly different representations in different situations of Bayesian problem solving. Therefore, the students have to become more and more confident in handling these representations which trigger different mental perspectives on the Bayesian situation (cf. Schnotz & Bannert, 1999; Mayer, 2009). Especially in this point of view, it becomes obvious that it is one thing to support school students on demand via giving them alternative representations and it is another thing to enable them helping themselves on the long run.

Both the reception type and the translation type are exemplified by the single-event-probability version of Bayesian situations. Of course, these types can also be applied in the natural frequency version. Furthermore, they can be applied for the combination of both versions because students should also learn about the different perspectives of modelling of the same Bayesian problem structure. In this point of view, the students could for example be trained in translating the tasks' wording of the single-event-probability-version into the natural frequency-version and vice versa. Thus, they get an additional heuristic to cope with different demands of Bayesian situations.

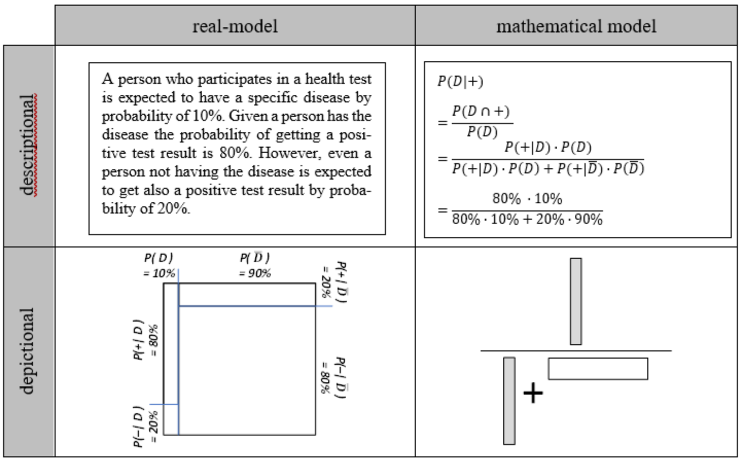

Figure 9. Elements of coherence formation in multimedia based mathematizing process

Realizing the underlying abstract structure is also the learning goal of varying situations (Eichler & Vogel, 2015). Ainsworth (2006, p. 186) defines: "Abstraction is the process by which learners create mental entities that serve as the basis for new procedures and concepts at a higher level of organization." Thus, by confronting learners with contextually different Bayesian situations they should stimulated to construct across-references that then expose the underlying structure of the domain represented as being a common structural element of all situations. The varying situations principle can not only be referred to the contextual side but also to the mathematical side: In general, the Bayesian structure is a certain case of building proportions of quantities which could be measured either via percentages but also via common fractions (percentages correspond to fractions with the denominator 100).

Thus, the students in school should also being confronted with not-Bayesian problems of proportion-building to learn to distinguish between Bayesian and not-Bayesian situations. This corresponds to the fundamentals of learning via examples and counterexamples which is a central principle of acquiring conceptual mathematical knowledge (e.g. Ausubel, 1980). In this point of view learning about Bayesian ratios could be already addressed in in lower secondary or even primary schools when students learn about fractional arithmetic. Of course, in this case the contextual and mathematical setting of the problem's presentation should fit the horizon of children's development status (e.g. Binder & Vogel, 2018; Martignon, & Erickson, 2014).

5. Discussion and conclusion

The question of how to promote insight into Bayesian reasoning situations is a crucial issue for mathematics education and has practical consequences for the teaching and learning of statistics in school. However, this question is also a very complex and multifaceted one. Thus, it needs a tailored focus to come to conclusions which could shed some light on possible theoretical, empirical and practical answers. This paper's approach about treating Bayesian situations in school is based on two columns which were integrated successively in the argumentation: the mathematizing process and the learning with multiple representations. The process of mathematizing is the essential part of the modeling cycle which turns out to be at the very heart of learning statistics and probability. It is argued that this mathematizing process is bridging the real-model world and the mathematical model world. Thereby, the subprocesses of abstraction on the one hand and contextualization on the other hand have to be distinguished. This process of mathematizing is reflected within the fundamentals of theories of processing multiple representations with a special focus on the unit square. As a consequence, a 2x2-matrix of multimedia-based mathematizing of Bayesian situations was derived. This 2x2-matrix can be used as a diagnosis tool for teachers and allows for a systematical developing of task variations in school.

This approach's tailoring brings some limitations which should be discussed. Firstly, the referenced theories of multimedia learning focus on the process of information processing but not on theories about human cognitive dispositions. Thus, the "the standard 'natural frequency vs. nested-sets' debate" (Johnson & Tubau, 2015, p. 5) was left out. However, doing so doesn't mean that these theoretical approaches and their debate would not be of great importance. It is simply a question of space and a question of self-limitation. Those readers who are interested in this regard are recommended to take the references of the introduction part into account. The referenced authors contributed seminal works to and around the mentioned debate.

The same question of reducing refers to theories of semiotics: There is a huge amount of fundamental and important works which reflect among others on the meaning of signs and sign processes, analogy, symbolism, signification, and communication. These aspects are all part of a representation debate in a wide sense. However, the question of representation is connected with many far-reaching questions (cf. Cuoco, 2001). This may explain why Kaput (1987a, p. 19) states that there is „the apparent lack of a comprehensive, systematic theoretical framework of symbol systems and representation systems capable of supporting the kinds of understandings necessary to solve the problems just alluded to.“

However, in the meanwhile there has been some progress and by honoring Carmen Batanero and Juan D. Godino their great works about the *Onto-Semiotic Approach* in fields of research in mathematics education should be explicitly emphasized (e.g. Godino, Batanero, & Font, 2007; Godino, Batanero, & Roa, 2005; Godino & Batanero, 2003).

The theory-based development of task variations of mathematizing Bayesian situations in school asks for empirical evidences, especially concerning the unit square. This is one of the main goals in our ongoing research program that we called BAYES[2]. We research on potentials of the unit-square in supporting people to cope with Bayesian situations.

In the meanwhile, we got some empirical evidence that the unit-square can be a useful representation in mathematizing Bayesian situations referring the visibility of the relevant mathematical structure of the problem situation. Recognizing this structure is crucial when computing set-subset relations in Bayesian situations, applying Bayes' rule and estimating the effect of a changed base rate (e.g. Böcherer-Linder et al., 2017; Böcherer-Linder & Eichler, 2017; Böcherer-Linder, Eichler, & Vogel, 2018).

A limitation of these studies must be seen in the level of the educational background of the participants. They all were students at university which reached university entrance level by having successfully finished the upper secondary school before. Accordingly, after having got these promising results we enlarged our research focus and ask for the possibility of transferring the unit square's supporting effects into the level of secondary school. On this base, we firstly conducted a replication study with students of lower secondary level to research on potential effects of mathematical education (Vogel & Böcherer-Linder, 2018). A further study has been carried out in the upper secondary school, this study will be presented at this CIVEEST conference 2019 by Andreas Eichler.

At the moment, there is a survey in classes of middle secondary level going on which is about the effects of different abstract levels of contents of different Bayesian situations. First results of this study are expected to be presentable at the conference. All these studies can be referenced to the 2x2-matrix which was theoretically derived in this contribution. Thus, this representational model of mathematizing Bayesian situations could not only be used for purposes of diagnosing and task variation developing (cf. above) but also for referencing our empirical studies in school.

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