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PAPER

Explicit tight bounds on the stably recoverable information for the inverse source problem

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Abstract

For the inverse source problem with the two-dimensional Helmholtz equation, the singular values of the source-to-near-field operator reveal a sharp frequency cut-off in the stably recoverable information on the source. We prove and numerically validate an explicit, tight lower bound $\mathcal{B}_-$ for the spectral location $\mathcal{B}$ of this cut-off. We also conjecture, justify and support numerically a tight upper bound $\mathcal{B}_+$ for the cut-off. The bounds are expressed in terms of zeros of Bessel functions of the first and second kind. Finally, we propose our near-field upper bound $\mathcal{B}_+$ as an improvement of a commonly used upper bound on the spectral cutoff for the source-to-far-field operator.

1. Introduction

We treat the single-frequency inverse source problem for the Helmholtz equation in the plane, illustrated in figure 1. Fix a positive constant wavenumber $k = 2\pi/\lambda$, where λ is the operating wavelength, and let D_0 and D be open disks in $\mathbb{R}^2$ centered at the origin and with radii R_0 and $R \geqslant R_0$, respectively. Write $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ for the Laplacian, and consider the Helmholtz problem

$$
\begin{cases}
(\Delta + k^2)u = s & \text{in } \mathbb{R}^2, \\
\lim_{|x|\to\infty} \sqrt{|x|}\,(\partial_{|x|} - \mathrm{i}k)u(x) = 0, & \text{uniformly for } x/|x| \in S^1,
\end{cases}
\tag{1}
$$

for some source $s \in L^2(D_0)$ extended by zero to the whole plane. The second condition in (1) is the outgoing Sommerfeld radiation condition in the plane. The inverse source problem, ISP, is now
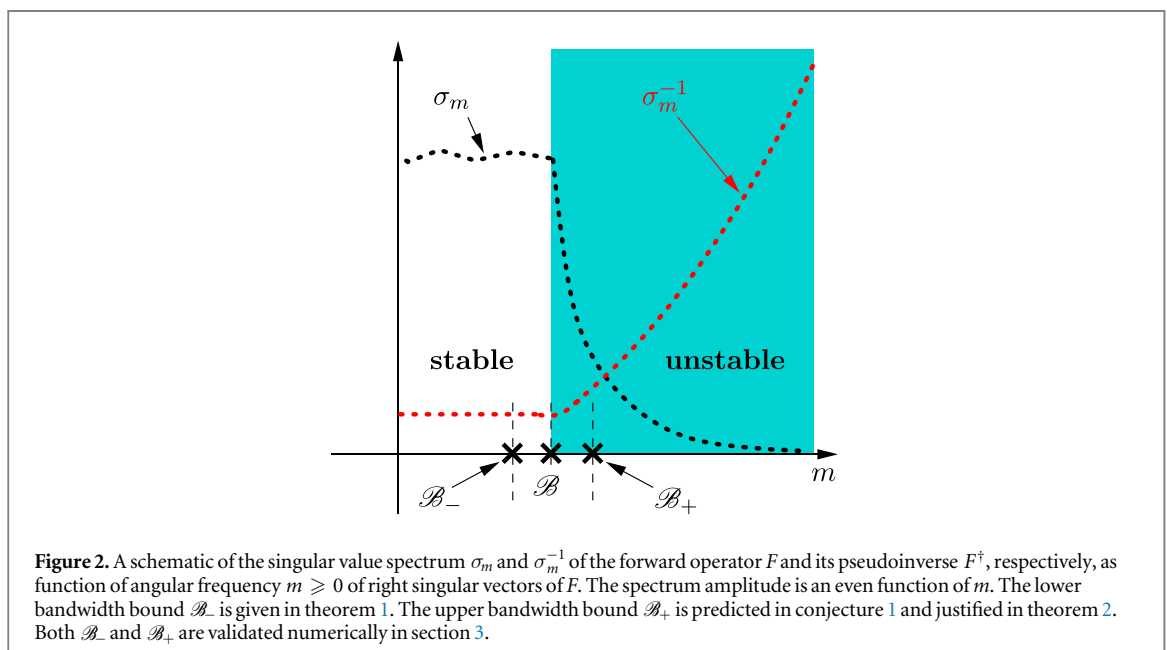
given a single measurement $U \in L^2(\partial D)$, *find a source* $s \in L^2(D_0)$ *such that there is a function ('radiated field')* u *satisfying* $u|_{\partial D} = U$ *and satisfying the system* (1).

The ISP arises naturally in inverse acoustic and electromagnetic scattering, and has been devoted a substantial body of literature. The ISP is treated, e.g., in the multi-frequency regime by Bao *et al* (2010), and with far-field measurement data by Griesmaier *et al* (2012, 2014), Griesmaier and Sylvester (2016, 2017); see also El Badia and Nara (2011). It occurs in antenna synthesis and diagnostics (Persson and Gustafsson 2005, Jørgensen *et al* 2010), the analytic continuation of solutions of exterior scattering problems (Sternin and Shatalov 1994, Zaridze *et al* 1998, Bliznyuk *et al* 2005, Karamehmedović 2015), and in linearized inverse obstacle scattering problems.

In terms of the source-to-near-field forward operator $F : s \mapsto U$, described in detail in section 2, solving the ISP amounts to solving

$$
Fs = U \quad \text{for} \quad s \in L^2(D_0).
\tag{2}
$$

This problem is ill-posed, since $\ker F = (\Delta + k^2)H^2(D_0)$, where $H^2(D_0)$ is the Sobolev space $\{\partial^\alpha w \in L^2(D_0) \text{ for } \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \leqslant 2\}$. Also, measurements U are typically noisy and sampled over an only finite set of points. A common regularizing measure is to look for the minimum-L^2-norm, or minimum-energy, solution of (2), which is given by $s^\dagger = F^\dagger U$; here, $F^\dagger = (F^*F)^{-1}F^* = F^*(FF^*)^{-1}$ is the Moore-Penrose pseudoinverse of F. Another regularization scheme uses a truncated singular value decomposition (TSVD) of the

Figure 1. Problem geometry and coordinate systems.

Figure 2. A schematic of the singular value spectrum σ_m and σ_m^{-1} of the forward operator F and its pseudoinverse $F^\dagger$, respectively, as function of angular frequency $m \geqslant 0$ of right singular vectors of F. The spectrum amplitude is an even function of m. The lower bandwidth bound $\mathscr{B}_-$ is given in theorem 1. The upper bandwidth bound $\mathscr{B}_+$ is predicted in conjecture 1 and justified in theorem 2. Both $\mathscr{B}_-$ and $\mathscr{B}_+$ are validated numerically in section 3.

forward operator F. Here, s is approximated by a finite sum of the form $\sum_1^N \sigma_m^{-1}(U, \phi_m)\psi_m$, with $(\sigma_m, \psi_m, \phi_m)$ a singular system of F such as that shown in Bao *et al* (2010). Our aim is to estimate the maximal amount of information about *any* source $s \in L^2(D_0)$ that can be stably recovered *in principle*, that is, *regardless* of the sampling frequency in the measurement and of the choice of the regularisation scheme. By 'stably recoverable information' we mean 'information recoverable robustly to noise', and we refer to figures 2 and 3 for further clarification.

Namely, plotting the singular values σ_m of the forward operator F reveals a seemingly low-pass filter behavior with well-defined 'passband' and 'stopband'. A low-pass filter characteristic was earlier proved for the singular spectrum of the source-to-*far*-field operator (Griesmaier and Sylvester 2017, Griesmaier *et al* 2014), also called the restricted Fourier transform there. However, since we are here looking for the limits of the stably recoverable information under even the ideal conditions, we consider the source-to-near-field operator F, as some information might be lost in the transition to the far field. Also, as will become clear in section 2, the singular values σ_m of the source-to-near-field operator have a more involved dependence on the frequency index m than the singular values of the source-to-far-field operator, and we find that this necessitates a separate analysis.

Numerical investigation indicates that the supposed low-pass filter behavior of the singular spectrum of F persists also when the singular values σ_m are ordered according to increasing angular frequency of the right singular vectors of F, that is, of the singular vectors defined at the measurement boundary ∂D. In this case, the singular values within the 'passband' generally do not increase or decrease monotonically (they are uniformly large there), and the singular values in the 'stopband', still ordered according to angular frequency m, are observed to be monotonically decreasing with m. It is not *a priori* clear that the monotonicity properties of the
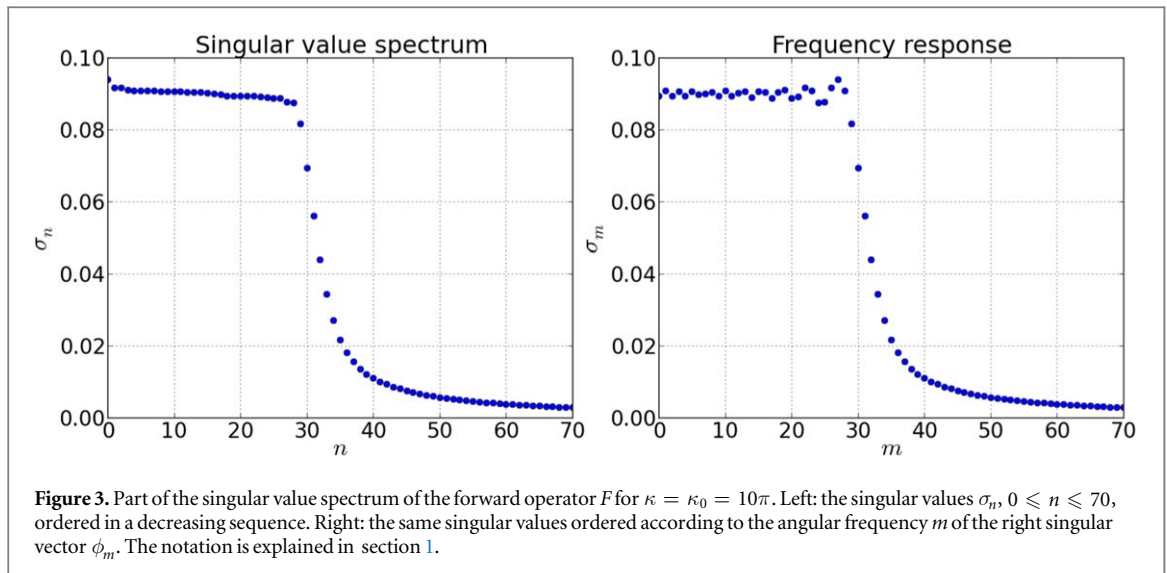
Figure 3. Part of the singular value spectrum of the forward operator *F* for $\kappa = \kappa_0 = 10\pi$. Left: the singular values σ_n, $0 \leqslant n \leqslant 70$, ordered in a decreasing sequence. Right: the same singular values ordered according to the angular frequency *m* of the right singular vector ϕ_m. The notation is explained in section 1.

singular spectrum $\{\sigma_m\}$ should be directly connected with significantly different regimes of decay of the singular values. However, motivated by the numerical studies, and by our theorem 2 below, we embark on a non-asymptotic analysis of the singular spectrum of the forward operator *F* and call the *bandwidth* $\mathscr{B}$ of *F* the singular value index (angular frequency *m* of a right singular vector of *F*) at which the singular value spectrum of *F* becomes strictly decreasing as function of nonnegative *m*:

$$\mathscr{B} = \text{argmin}_{m \in \mathbb{N}_0} \{\sigma_{m+n} > \sigma_{m+n+1} \text{ for all } n \in \mathbb{N}_0\}.$$

We now define the stably recoverable information on a source *s* to be the projection of *s* onto the singular subspace of *F* defined by $|m| \leqslant \mathscr{B}$. Then, finding the maximal amount of stably recoverable information about any source *s*, regardless of measurement sampling quality and of regularization scheme, amounts to estimating the bandwidth $\mathscr{B}$ of the forward operator *F*.

To simplify the notation, write $\kappa_0 = kR_0$ and $\kappa = kR$ for the size parameters of the source support and of the measurement boundary, respectively. Also, for real *m* and positive integer ν, write $j_{m,\nu}$ and $y_{m,\nu}$ for the ν'th positive zero of the Bessel function J_m of the first kind, respectively Bessel function Y_m of the second kind, and order *m*. It is well-known that, for all $m \in \mathbb{N}_0$, we have $j_{m,1} > 0$ (Watson 1945, p 479), as well as $y_{m,1} > \sqrt{m(m+2)} \geqslant 0$ (Watson 1945, p 487). Our main result, proved in section 2.2, is

Theorem 1. *The bandwidth $\mathscr{B}$ of the forward operator F: $s \mapsto U$ associated with the Helmholtz problem* (1) *and measurement at ∂D is bounded from below by*

$$\mathscr{B}_- = \text{argmin}_{m \in \mathbb{N}_0} \{j_{m,1} \geqslant \kappa_0\}.$$

The general form of the result in theorem 1, as well as numerical experimentation, lead us to conjecture a tight upper bound on the bandwidth $\mathscr{B}$:

Conjecture 1.

$$\mathscr{B}_+ = \text{argmin}_{m \in \mathbb{N}_0} \{y_{m,1} \geqslant \kappa_0\}.$$

To justify both this conjecture and our notion of the bandwidth of the forward operator, we also prove that the singular spectrum $\{\sigma_m\}$ is majored by a quickly decaying sequence when $m \geqslant \mathscr{B}_+$:

Theorem 2. *There is a positive constant* C *that depends only on* κ *and* κ_0 *and such that, for every* $m \in \mathbb{N}$ *with* $y_{m,1} \geqslant \kappa_0$, *we have* $\sigma_m \leqslant C/m$.

After the proof of theorem 2, we indicate one way to estimate more precisely a majorant for σ_m when $R = R_0$, $y_{m,1} \geq \kappa_0$ and $m \geq 2$; we leave a more detailed treatment to future work. For convenience, in section 2.2 we also show that the bandwidth bounds of theorem 1 and conjecture 1 can be expressed explicitly in the source size parameter κ_0:

Corollary 1. *Let* $a_- = 1.855757$ *and* $a_+ = 0.931577$. *For sufficiently large* κ_0, *we have*

$$\mathscr{B}_\pm \approx \widetilde{\mathscr{B}}_\pm = \left\lceil \left(\frac{1}{6}(108\kappa_0 + 12\sqrt{12a_\pm^3 + 81\kappa_0^2})^{1/3} - \frac{2a_\pm}{(108\kappa_0 + 12\sqrt{12a_\pm^3 + 81\kappa_0^2})^{1/3}} \right)^3 \right\rceil$$

as well as $\mathscr{B}_+ \lesssim \lceil \kappa_0 \rceil$.

Finally, as announced, we can use our analysis in the near field to improve a commonly used bound for the far-field spectral cutoff. The singular values of the source-to-far-field operator are given by Griesmaier *et al.* (2014)

$$\sigma_{far,m} = 2\pi \sqrt{\int_{r=0}^{R_0} J_m(kr)^2 r}.$$

Griesmaier *et al* 2014 and Griesmaier and Sylvester (2017) show that these singular values decay rapidly when the index *m* satisfies $|m| \geqslant kR_0 = \kappa_0$. The last inequality in corollary 1 aligns well with the upper bound $|m| \geqslant \kappa_0$ of the far-field spectral cut-off. A tighter upper bound for this cut-off is given in the following:

Theorem 3. *The singular values* $\sigma_{far,m}$ *of the source-to-far-field operator are bounded from above by*

$$\pi\sqrt{2}\,R_0 \frac{(\kappa_0/2)^m}{m!} \sqrt{e^{-\kappa_0^2/2(m+1)} + \frac{(\kappa_0/2)^2}{(m+1)^2} e^{-\kappa_0^2/2(m+2)}} \text{ when } m \geqslant \mathscr{B}_+ = \operatorname{argmin}_{\mu \in \mathbb{N}_0} \{ y_{\mu,1} \geqslant \kappa_0 \}.$$

Since for any $m \in \mathbb{N}_0$ we have $y_{m,1} > m$ (Pálmai and Apagyi, 2011), an index *m* satisfying $m \geqslant \kappa_0$ will also satisfy $m \geqslant \mathscr{B}_+$, so proving theorem 3 may indeed improve the upper bound on the spectral cutoff in the far field. We demonstrate numerically in section 3 that $\mathscr{B}_+$ is an improvement of the upper bound $|m| \geqslant \kappa_0$.

In section 2 we analyze the singular value spectrum of the forward operator *F*. In particular, we prove theorem 1, theorem 2, corollary 1 and theorem 3 in section 2.2. We validate the bounds $\mathscr{B}_-$ and $\mathscr{B}_+$ on the bandwidth $\mathscr{B}$ numerically in Section 3, and discuss some implications of theorem 1 in section 4. A conclusion and suggestions for further work are given in section 5.

2. Spectral analysis of the forward operator

The function $(i/4)H_0^{(1)}(k|x|)$, $x \in \mathbb{R}^2$, is the radial outgoing fundamental solution of the Helmholtz operator in the plane, with singularity at the origin. Recall that $H_0^{(1)} = J_0 + iY_0$ is the Hankel function of zero order and of the first kind. As in Bao *et al* (2010), introduce the forward operator

$$Fs(x) = \int_{y \in D_0} H_0^{(1)}(k_0|x-y|)s(y), \quad x \in \partial D, \ s \in L^2(D_0),$$

that maps sources *s* to the traces at ∂D of the corresponding radiated fields. It is well-known (Bao *et al* 2010) that $F: L^2(D_0) \to L^2(\partial D)$ is compact. The adjoint F^* is defined by

$$F^*U(y) = \int_{x \in \partial D} H_0^{(2)}(k|x-y|)U(y), \quad y \in D_0, \ U \in L^2(\partial D),$$

where $H_0^{(2)} = J_0 - iY_0$ is the Hankel function of zero order and of the second kind.

2.1. A singular system of *F*
Bao *et al* (2010) derived a singular system of the forward operator *F*. We here slightly improve a part of their proposition 2.1:

Lemma 1. *The forward operator F admits the singular value decomposition*

$$F = \sigma_0(\cdot, \psi_0)_{L^2(D)}\phi_0 + \sum_{m \in \mathbb{N}} \sigma_m[(\cdot, \psi_m)_{L^2(D)}\phi_m + (\cdot, \psi_{-m})_{L^2(D)}\phi_{-m}],$$

where

$$\sigma_m = \sqrt{2R}\,\pi R_0 |H_m^{(1)}(\kappa)| A_m(\kappa_0), \quad m \in \mathbb{N}_0, \tag{3}$$

and

$$\psi_m(y) = (\sqrt{\pi}\, R_0 A_m(\kappa_0))^{-1} J_m(k|y|) \mathrm{e}^{\mathrm{i}m \arg y},$$
$$\phi_m(x) = (2\pi R)^{-1/2} \mathrm{e}^{\mathrm{i} \arg H_m^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}m \arg x},$$

for $m \in \mathbb{Z}$, $x \in \partial D$ *and* $y \in D_0$. *Here*

$$A_m(\kappa_0) = \sqrt{J_m(\kappa_0)^2 - J_{m-1}(\kappa_0)J_{m+1}(\kappa_0)} = \sqrt{J_m(\kappa_0)^2 + J_{m+1}(\kappa_0)^2 - \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0)}$$

for $m \in \mathbb{Z}$.

Our slight improvement of proposition 2.1 of Bao *et al* (2010) consists in explicitly evaluating the integral $\int_{\varrho=0}^{R_0} \varrho J_m^2(k\varrho)$, occurring in σ_m and ψ_m, in terms of $A_m(\kappa_0)$. This explicit evaluation is crucial to our proof of theorem 1. We also note that our expressions for the singular vectors ϕ_m, as well as the singular values σ_m, differ from Bao *et al* (2010) in that they are only proportional to those given in that reference.

Proof of lemma 1. For $s \in L^2(D_0)$ and $y \in D_0$ we have

$$F^*Fs(y) = \int_{z \in D_0} s(z) \int_{x \in \partial D} H_0^{(1)}(k|x-z|)H_0^{(2)}(k|x-y|). \tag{4}$$

A special case of the Graf addition theorem (Abramowitz and Stegun 1972, equation (9)) reads

$$H_0^{(1)}(k|x-y|) = \sum_{m \in \mathbb{Z}} H_m^{(1)}(\kappa)J_m(k|y|)\mathrm{e}^{\mathrm{i}m(\arg x - \arg y)}, \quad x \in \partial D, \, y \in D_0.$$

Similar to Bao *et al* (2010), inserting this in (4) we get

$$F^*Fs(y) = \sum_{m,n \in \mathbb{Z}} H_m^{(1)}(\kappa)H_n^{(2)}(\kappa)J_n(k|y|)\mathrm{e}^{-\mathrm{i}n \arg y} \times \int_{z \in D_0} s(z)J_m(k|z|)\mathrm{e}^{-\mathrm{i}m \arg z} \int_{x \in \partial D} \mathrm{e}^{\mathrm{i}(m+n)\arg x}$$
$$= 2\pi R \sum_{m \in \mathbb{Z}} |H_m^{(1)}(\kappa)|^2 J_m(k|y|)\mathrm{e}^{\mathrm{i}m \arg y} \int_{z \in D_0} s(z)J_m(k|z|)\mathrm{e}^{-\mathrm{i}m \arg z},$$

since $J_{-m} = (-1)^m J_m$ and $Y_{-m} = (-1)^m Y_m$ for all integer m. This gives an eigendecomposition of the operator F^*F; to normalize the eigenvectors, we note that Gradsteyn and Ryzhik (2007, equation (5).54.2, p 629) gives

$$\int \varrho J_m(k\varrho)^2 = \frac{\varrho^2}{2}(J_m(k\varrho)^2 - J_{m-1}(k\varrho)J_{m+1}(k\varrho)), \quad m \in \mathbb{Z},$$

and the recursion formula for cylinder functions (Gradsteyn and Ryzhik 2007, equation (8).471.1, p 926) implies

$$J_{m-1}(\kappa) + J_{m+1}(\kappa) = \frac{2m}{\kappa}J_m(\kappa), \quad m \in \mathbb{Z}. \tag{5}$$

Thus,

$$\int_{\varrho=0}^{R_0} \varrho J_m(k\varrho)^2 = R_0^2 \int_{\varrho=0}^1 \varrho J_m(\kappa_0\varrho)^2 = \frac{R_0^2}{2}(J_m(\kappa_0)^2 - J_{m-1}(\kappa_0)J_{m+1}(\kappa_0))$$
$$= \frac{R_0^2}{2}\left(J_m(\kappa_0)^2 + J_{m+1}(\kappa_0)^2 - \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0)\right) = \frac{R_0^2 A_m(\kappa_0)^2}{2},$$

and F^*F admits the spectral decomposition

$$F^*F = \sigma_0^2(\cdot, \psi_0)_{L^2(D_0)} + \sum_{m \in \mathbb{N}} \sigma_m^2[(\cdot, \psi_m)_{L^2(D_0)}\psi_m + (\cdot, \psi_{-m})_{L^2(D_0)}\psi_{-m}].$$

Evidently, σ_0^2 has multiplicity one and all the other eigenvalues σ_m^2, $m \in \mathbb{N}$, have multiplicity two. The lemma now follows from theorem 4.7 on p 100 of Colton and Kress (2013); it here just remains to compute

$$\phi_m(x) = \sigma_m^{-1}F\psi_m(x) = \frac{\int_{y \in D_0} H_0^{(1)}(k|x-y|)J_m(k|y|)\mathrm{e}^{\mathrm{i}m \arg y}}{\sqrt{2R}\,\pi^{3/2}R_0^2|H_m^{(1)}(\kappa)|A_m(\kappa_0)^2}$$
$$= \frac{\sum_{\nu \in \mathbb{Z}} H_\nu^{(1)}(\kappa)\mathrm{e}^{\mathrm{i}\nu \arg x}\int_{\varrho=0}^{R_0}\varrho J_\nu(k\varrho)J_m(k\varrho)\int_{\theta=0}^{2\pi}\mathrm{e}^{\mathrm{i}\theta(m-\nu)}}{\sqrt{2R}\,\pi^{3/2}R_0^2|H_m^{(1)}(\kappa)|A_m(\kappa_0)^2}$$
$$= (2\pi R)^{-1/2}\mathrm{e}^{\mathrm{i} \arg H_m^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}m \arg x}, \quad x \in \partial D, \, m \in \mathbb{N}_0.$$
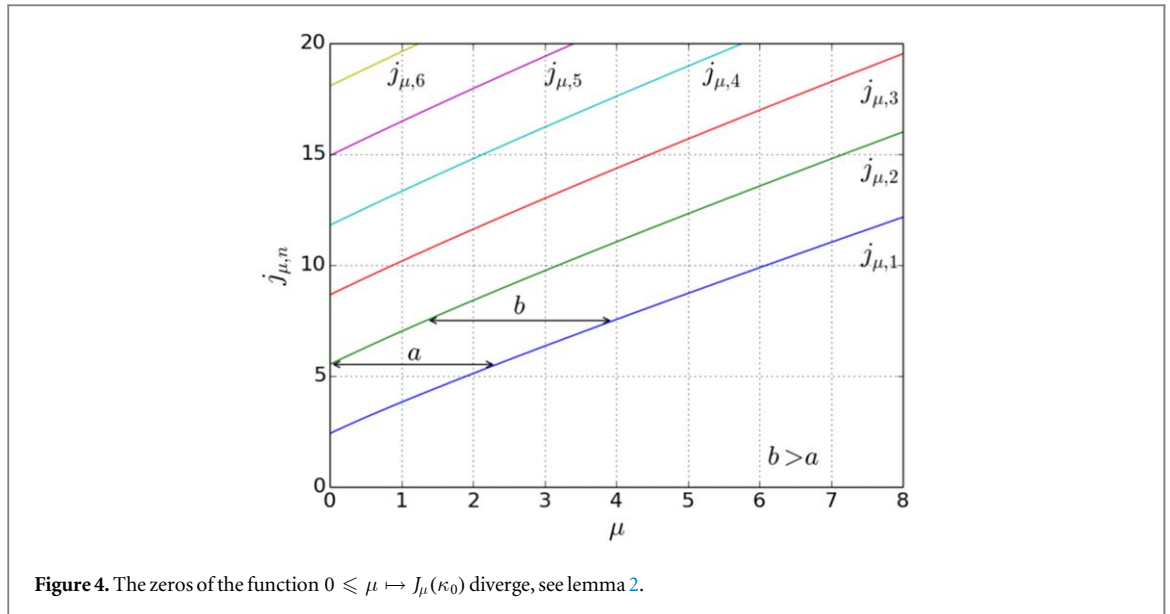
□

Figure 4. The zeros of the function $0 \leqslant \mu \mapsto J_\mu(\kappa_0)$ diverge, see lemma 2.

Figure 3 shows the first 71 nonnegative-index singular values of the forward operator F with size parameters $\kappa = \kappa_0 = 10\pi$. It seems the forward operator is a low-pass filter with respect to the singular values σ_m, with bandwidth $\mathscr{B} = 27$. We quantify the frequency response of this filter in section 2.2.

2.2. Proof of theorem 1, theorem 2, corollary 1 and theorem 3

To arrive at the lower bound $\mathscr{B}_-$ of the bandwidth $\mathscr{B}$, we first prove that the distance between the nonnegative zeros of the function $\mu \mapsto J_\mu(\kappa_0)$ is greater than 1. To this end, we use three fundamental and well-known properties of the positive zeros $j_{\mu,\nu}$ of the Bessel function $J_\mu(z)$:

1. **P1** For any positive integer n, the function $\mu \mapsto j_{\mu,n}$ is strictly increasing over real μ.

2. **P2** For any real μ and any positive integer ν, we have $j_{\mu,\nu} < j_{\mu,\nu+1}$.

3. **P3** For any positive integer n, we have $j_{1,n} < j_{0,n+1}$.

Property P1 is shown, e.g., in Watson (1945, p 508) while properties P2 and P3 follow from Pálmai and Apagyi (2011, theorem 1.1, interlacing property (2) of zeros of Bessel functions).

Lemma 2. *If $\mu_2 > \mu_1 \geqslant 0$ and $J_{\mu_1}(\kappa_0) = J_{\mu_2}(\kappa_0) = 0$ then $\mu_2 - \mu_1 > 1$.*

Proof. First let $\mu_1 = 0$. Under the assumption of the Lemma, there are numbers n_1, $n_2 \in \mathbb{N}$ and $\mu_2 > 0$ such that $n_1 > n_2$ and $\kappa_0 = j_{0,n_1} = j_{\mu_2,n_2}$. The number n_1 is necessarily greater that 1: due to P2 and P1 in conjunction with the assumption $\mu_2 > 0$, we have $j_{\mu_2,1} \geqslant j_{\mu_2,1} > j_{0,1}$, and hence $j_{0,1} \neq j_{\mu_2,\nu}$ for any positive integer ν. Also, n_2 is necessarily smaller than n_1: by P1 and the assumption $\mu_2 > 0$, it holds that $j_{0,n_1} < j_{\mu_2,n_1}$, and by P2 we have $j_{\mu_2,n_1} \leqslant j_{\mu_2,\nu}$ when $\nu \geqslant n_1$, so $j_{0,n_1} \neq j_{\mu_2,\nu}$ for any integer $\nu \geqslant n_1$. Having established the conditions on n_1 and n_2 in the assumption $j_{0,n_1} = j_{\mu_2,n_2}$ of the Lemma, we now use P2 and P3 to see that $j_{0,n_1} > j_{1,n_1-1} \geqslant j_{1,n_2}$, so necessarily $j_{\mu_2,n_2} > j_{1,n_2}$. This implies, due to P1, that $\mu_2 > 1$, that is, $\mu_2 - \mu_1 > 1$. The situation corresponds to the numerical value of κ_0 being equal to, e.g., the height (along the second axis) of the linear piece a in figure 4. To generalize this argument to the case $\mu_1 > 0$, that is, to the case of the numerical value of κ_0 being equal to, e.g., the height of the linear piece b in figure 4, it now suffices to show that

$$\frac{dj_{\mu,n}}{d\mu} < \frac{dj_{\mu,n+1}}{d\mu} \quad \text{for all } \mu \geqslant 0, \, n \in \mathbb{N}. \tag{6}$$

To see why (6) suffices, assume $\kappa_0 = j_{\mu_1,n+1} = j_{\mu_2,n}$ with $\mu_2 > \mu_1 > 0$ and $n \in \mathbb{N}$, and consider the inverse functions $f_n \colon j_{\mu,n} \mapsto \mu$ and $f_{n+1} \colon j_{\mu,n+1} \mapsto \mu$. These functions are well-defined due to the abovementioned property P1. Also, let $\widetilde{\mu}$ be a real satisfying $j_{0,n+1} = j_{\widetilde{\mu},n}$. By the first part of this proof, addressing the case $\mu_2 > \mu_1 = 0$, we know that necessarily $\widetilde{\mu} > 1$. Now if (6) holds, then $(df_n/dx)(x) > (df_{n+1}/dx)(x)$ for all positive x where both f_n and f_{n+1} are defined, and we have

$$\mu_2 = \widetilde{\mu} + \int_{x=j_{\widetilde{\mu},n}}^{j_{\mu_2,n}} \frac{df_n}{dx}(x) = \widetilde{\mu} + \int_{x=j_{0,n+1}}^{j_{\mu_1,n+1}} \frac{df_n}{dx}(x) > \widetilde{\mu} + \int_{x=j_{0,n+1}}^{j_{\mu_1,n+1}} \frac{df_{n+1}}{dx}(x)$$

$$= \widetilde{\mu} + f_{n+1}(j_{\mu_1,n+1}) - f_{n+1}(j_{0,n+1}) = \widetilde{\mu} + \mu_1,$$

that is, $\mu_2 - \mu_1 > \widetilde{\mu} > 1$. We now turn to proving the inequality (6). For nonnegative order μ, the n'th zero $j_{\mu,n}$ of the Bessel function J_μ satisfies (Watson 1945, pp 508–510)

$$\frac{dj_{\mu,n}}{d\mu} = 2j_{\mu,n} \int_{t=0}^{\infty} K_0(2j_{\mu,n} \sinh t) e^{-2\mu t}.$$

Substituting $q = 2j_{\mu,n} \sinh t$ and using that $j_{\mu,n} > 0$, $\exp(\operatorname{arcsinh} \tau) = \tau + \sqrt{1 + \tau^2}$, as well as that $\cosh\operatorname{arcsinh} \tau = \sqrt{1 + \tau^2}$, we get

$$\frac{dj_{\mu,n}}{d\mu} = \int_{q=0}^{\infty} K_0(q)(q/2j_{\mu,n} + \sqrt{1 + q^2/4j_{\mu,n}^2})^{-2\mu}(1 + q^2/4j_{\mu,n}^2)^{-1/2}. \tag{7}$$

Let us consider the right-hand side of (7). Setting

$$g(x) = \int_{q=0}^{\infty} K_0(q)\left(q/2x + \sqrt{1 + q^2/4x^2}\right)^{-2\mu}(1 + q^2/4x^2)^{-1/2}, \quad x > 0,$$

we find

$$\frac{dg}{dx}(x) = 2^{2\mu+1}x^{2\mu} \int_{q=0}^{\infty} \frac{qK_0(q)(2\mu\sqrt{4x^2 + q^2} + q)}{(4x^2 + q^2)^{3/2}(\sqrt{4x^2 + q^2} + q)^{2\mu}} > 0 \quad \text{for } x > 0$$

since the modified Bessel function K_0 of the second kind is positive-valued for positive-valued arguments. Thus, the right-hand side of (7) is monotonically increasing with increasing value of $j_{\mu,n}$. Finally, recall the property P2: for any $\mu \geqslant 0$ and $n \in \mathbb{N}$, we have $j_{\mu,n} < j_{\mu,n+1}$. Thus, (6) indeed holds. □

We can now link the variation of the function $m \mapsto A_m(\kappa_0)$ with that of the Bessel function of the first kind. Fix $m \in \mathbb{N}_0$.

Lemma 3. *If* $J_\xi(\kappa_0) = 0$ *for some* $\xi \in [m, m+1]$ *then* $A_m(\kappa_0) \leqslant A_{m+1}(\kappa_0)$.

Proof. The recursion formula (5) implies

$$A_m(\kappa_0)^2 - A_{m+1}(\kappa_0)^2 = J_m(\kappa_0)^2 + J_{m+1}(\kappa_0)^2 - \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0)$$

$$- J_{m+1}(\kappa_0)^2 - J_{m+2}(\kappa_0)^2 + \frac{2(m+1)}{\kappa_0}J_{m+1}(\kappa_0)J_{m+2}(\kappa_0)$$

$$= \frac{2(m+1)}{\kappa_0}J_{m+1}(\kappa_0)(J_m(\kappa_0) - J_{m+2}(\kappa_0))$$

$$- \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0) + \frac{2(m+1)}{\kappa_0}J_{m+1}(\kappa_0)J_{m+2}(\kappa_0)$$

$$= \frac{2}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0).$$

For any fixed positive argument x, the function $\mathbb{R} \ni \mu \mapsto J_\mu(x)$ is differentiable and not identically zero. Thus, by the assumption of the current Lemma, and by lemma 2, this function has the value zero at $\mu = m$ or at $\mu = m + 1$, or the function changes sign precisely once in the interval $[m, m+1]$, so

$$A_m(\kappa_0)^2 - A_{m+1}(\kappa_0)^2 = \frac{2}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0) \leqslant 0.$$

□

Remark 1. Clearly, the function $\mathbb{N}_0 \ni m \mapsto |H_m^{(1)}(\kappa)|^2$ is positive-valued. It is also strictly increasing, as can be seen from Nicholson's integral for $|H_m^{(1)}(\kappa)|^2$ (Watson 1945, pp 441–444)

$$|H_m^{(1)}(\kappa)|^2 = \frac{8}{\pi^2} \int_{t=0}^{\infty} K_0(2\kappa \sinh t) \cosh 2mt, \quad m \in \mathbb{Z}, \kappa > 0,$$

where K_0 is the modified Bessel function of the second kind. This namely implies

$$\partial_m(|H_m^{(1)}(\kappa)|^2) = \frac{16m}{\pi^2} \int_{t=0}^{\infty} K_0(2\kappa \sinh t) \sinh 2mt > 0, \quad m > 0,$$

since both K_0 and the hyperbolic sine are positive over positive reals.

The above discussion suffices for a proof of the lower bound $\mathscr{B}_-$.

Proof of theorem 1. Let $m \in \mathbb{N}_0$. If $j_{m,1} < \kappa_0$ then, due to the continuity and the strictly increasing nature of the function $\mu \mapsto j_{\mu,1}$, the value κ_0 is the first positive zero of some Bessel function J_ξ with $\xi > m$; that is, there are $n \in \mathbb{N}_0$ and $\xi \in [m + n, m + n + 1]$ satisfying $J_\xi(\kappa_0) = 0$, and, by lemma 3, $A_{m+n}(\kappa_0) \leqslant A_{m+n+1}(\kappa_0)$. Since $\mathbb{N}_0 \ni \mu \mapsto |H_\mu^{(1)}(\kappa)|$ is strictly increasing, and the singular value σ_μ from (3) is proportional to $|H_\mu^{(1)}(\kappa)|A_\mu(\kappa_0)$ for $\mu \in \mathbb{N}_0$, we have $\sigma_{m+n} \leqslant \sigma_{m+n+1}$, hence $m < \mathscr{B}$. In conclusion,
$\mathscr{B} \geqslant \mathrm{argmax}_{m \in \mathbb{N}_0}\{j_{m,1} < \kappa_0\} + 1 = \mathrm{argmin}_{m \in \mathbb{N}_0}\{j_{m,1} \geqslant \kappa_0\}$. $\qquad\square$

We can also justify the conjectured upper bound $\mathscr{B}_+$:

Proof of theorem 2. Let $m \in \mathbb{N}$. Using the standard series representation (Watson 1945, p 40) of the Bessel function of the first kind, we have, for fixed positive z, that

$$|J_m(z)| = \left| \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{\mu!(m+\mu)!}\left(\frac{z}{2}\right)^{m+2\mu} \right| \leqslant \frac{(z/2)^m}{m!}\sum_{\mu=0}^{\infty}\frac{1}{\mu!}\left(\frac{z}{2}\right)^{2\mu} = \frac{(z/2)^m}{m!}e^{z^2/4}$$

and thus

$$J_m^2(z) \leqslant C_1 \frac{(z/2)^{2m}}{m!^2} \tag{8}$$

for some constant C_1 dependent only on z. Since m is a positive integer, we also have, for fixed positive z, that (NIST 2017)

$$\begin{aligned}
|Y_m(z)| &= \left| \frac{(2/z)^m}{\pi}\sum_{\mu=0}^{m-1}\frac{(m-\mu-1)!}{\mu!}\left(\frac{z^2}{4}\right)^m - \frac{2}{\pi}J_m(z)\ln\frac{z}{2} \right. \\
&\quad \left. + \frac{(z/2)^m}{\pi}\sum_{\mu=0}^{\infty}(\psi(\mu+1) + \psi(m+\mu+1))\frac{(-z^2/4)^\mu}{\mu!(m+\mu)!} \right| \\
&\leqslant \left(\frac{2}{z}\right)^m\frac{(m-1)!}{\pi}\sum_{\mu=0}^{m-1}\frac{1}{\mu!}\left(\frac{z^2}{4}\right)^\mu + \frac{2}{\pi}J_m(z)\left| \ln\frac{z}{2} \right| \\
&\quad + \frac{(z/2)^m}{\pi}\sum_{\mu=0}^{\infty}(\psi(\mu+1) + \psi(m+\mu+1))\frac{(z^2/4)^\mu}{\mu!(m+\mu)!}
\end{aligned}$$

The digamma function ψ satisfies[1] (Abramowitz and Stegun 1972, p 258)

$$\psi(\mu) = -\gamma + \sum_{k=1}^{\mu-1}1/k,$$

so

$$\psi(\mu+1) + \psi(m+\mu+1) = -2\gamma + 2\sum_{k=1}^{\mu}1/k + \sum_{k=\mu+1}^{m}1/k \leqslant 2\mu + (m-\mu) = \mu + m.$$

Hence

$$\begin{aligned}
|Y_m(z)| &\leqslant \left(\frac{2}{z}\right)^m\frac{(m-1)!}{\pi}e^{z^2/4} + \frac{2}{\pi}\left| \ln\frac{z}{2} \right|\frac{(z/2)^m}{m!}e^{z^2/4} + \frac{(z/2)^m}{\pi}\frac{1}{(m-1)!}e^{z^2/4} \\
&\leqslant C_2 + \left(\frac{2}{z}\right)^m\frac{(m-1)!}{\pi}e^{z^2/4}
\end{aligned}$$

and

$$Y_m^2(z) \leqslant C_3 + C_4\left(\frac{2}{z}\right)^{2m}(m-1)!^2 \tag{9}$$

for some constants C_2, C_3 and C_4 dependent only on z. Recall from (3) that $\sigma_m^2 \propto |H_m^{(1)}(\kappa)|^2 A_m^2(\kappa_0) = (J_m^2(\kappa) + Y_m^2(\kappa))A_m^2(\kappa_0)$. By assumption, $y_{m,1} \geqslant \kappa_0$, so, again using the inequality $y_{m,1} < j_{m,1}$ from Watson (1945, p 487) we have $\kappa_0 < j_{m,1} < j_{m+1,1}$. Therefore $J_m(\kappa_0)J_{m+1}(\kappa_0) > 0$, and hence

---

[1] Here $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

$$A_m^2(\kappa_0) = J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0) - \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0) \leqslant J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0),$$

so

$$\sigma_m^2 \leqslant \mathrm{const.} \times (J_m^2(\kappa) + Y_m^2(\kappa))(J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0)). \tag{10}$$

Multiplying out the right-hand side of (10), and using the derived estimates (8) and (9) for $J_m^2(z)$ and $Y_m^2(z)$, yields terms that are each bounded from above by const. $\times (\kappa_0/\kappa)^{2m}/m^2$. Finally, since by assumption $\kappa_0 \leqslant \kappa$, we have $\sigma_m \leqslant \mathrm{const.}/m$.   □

We next briefly describe one way to obtain a tighter majorant of σ_m than what is presented in theorem 2, for the case $R = R_0$, that is, $\kappa = \kappa_0$. Using the standard series representation of $J_m(z)$ we readily find that, for $t \in]0, 1[$, we have $|t^{-m}J_m(t\kappa_0)| \leq (\kappa_0/2)^m \exp((t\kappa_0/2)^2)/m!$. Furthermore, since $j_{m,1} > y_{m,1}$, we have by the inequality (2.4) on p. 77 of Laforgia (1986) that $y_{m,1} \geq \kappa_0$ implies $|J_m(\kappa_0)| < t^{-m}|J_m(t\kappa_0)|\exp(-\kappa_0^2(1 - t^2)/4(m + 1))$ for any $t \in]0, 1[$, hence it implies $|J_m(\kappa_0)| \leq (\kappa_0/2)^m \exp(-\kappa_0^2/4(m + 1))/m!$. To find an improved upper bound on $|Y_m(\kappa_0)|$, we can use the well-known estimates $(m/e)^m e \leq m! \leq ((m + 1)/e)^{m+1}e$. We get

$$\sum_{\mu=0}^{m-1} \frac{(m - \mu - 1)!}{\mu!}\left(\frac{\kappa_0^2}{4}\right)^\mu \leq \sum_{\mu=0}^{m-1} \frac{(\kappa_0^2/4)^\mu}{\mu!}\left(\frac{m - \mu}{e}\right)^{m-\mu}e \leq \sum_{\mu=0}^{m-1} \frac{(\kappa_0^2/4)^\mu}{\mu!}e^{1-m+\mu}m^{m-\mu}$$

$$\leq m!\sum_{\mu=0}^{m-1} \frac{(\kappa_0^2 e/4m)^\mu}{\mu!} \leq m!e^{\kappa_0^2 e/4m},$$

as well as, for $m \geq 2$,

$$\sum_{\mu=0}^{\infty} (\psi(\mu + 1) + \psi(m + \mu + 1))\frac{(\kappa_0^2/4)^\mu}{\mu!(m + \mu)!} \leq \sum_{\mu=0}^{\infty} \frac{(\kappa_0^2/4)^\mu}{\mu!(m + \mu - 1)!}$$

$$\leq \sum_{\mu=0}^{\infty} \frac{(\kappa_0^2/4)^\mu}{\mu!e}\left(\frac{e}{m + \mu - 1}\right)^{m+\mu-1} \leq \frac{\exp(e\kappa_0^2/4(m - 1))}{(m - 2)!}.$$

Thus, assuming

$$y_{m,1} \geq \kappa_0$$

and

$$m \geq 2,$$

and using the above estimates, we find the dominant term (with respect to the order m) in the bound to be

$$\sigma_m^2 = 2R_0^3\pi^2|H_m^{(1)}(\kappa_0)|^2 A_m(\kappa_0)^2 \leq 2R_0^3\pi^2(J_m(\kappa_0)^2 + Y_m(\kappa_0)^2)(J_m(\kappa_0)^2 + J_{m+1}(\kappa_0)^2)$$

$$2R_0^3\exp\left(\frac{\kappa_0^2}{2}\left(\frac{e}{m} - \frac{1}{m + 1}\right)\right) + 4R_0^3\frac{(\kappa_0/2)^{2m}}{(m - 1)!^2}\exp\left(\frac{\kappa_0^2}{4}\frac{(2m^2 + m - 1)e - 2m^2 + 2m}{m(m^2 - 1)}\right).$$

Proof of corollary 1. To get estimates on the bandwidth $\mathscr{B}$ that are explicit in κ_0, we use that (Watson 1945) $j_{m,1} = m + a_-m^{1/3} + O(m^{-1/3})$ and $y_{m,1} = m + a_+m^{1/3} + O(m^{-1/3})$, with $a_- = 1.855757$ and $a_+ = 0.931577$. Assuming $j_{m_-,1} = \kappa_0 \gg 1$, $y_{m_+,1} = \kappa_0 \gg 1$, and setting $n_\pm = m_\pm^{1/3}$ leads us to solve the equation $n^3 + a_\pm n - \kappa_0 = 0$ for $n_\pm$. We find

$$n_\pm = \frac{1}{6}(108\kappa_0 + 12\sqrt{12a_\pm^3 + 81\kappa_0^2})^{1/3} - \frac{2a_\pm}{(108\kappa_0 + 12\sqrt{12a_\pm^3 + 81\kappa_0^2})^{1/3}}.$$

In particular, $\mathscr{B}_\pm \approx \lceil n_\pm^3\rceil$ for sufficiently large κ_0. Finally, we have $m < (m - 2) + a_+(m - 2)^{1/3}$ for integer $m \geqslant 12$, so, for sufficiently large κ_0, $m \geqslant \kappa_0$ implies $y_{m-2,1} \approx (m - 2) + a_+(m - 2)^{1/3} > m \geqslant \kappa_0$ and hence $\mathscr{B}_+ \lesssim \lceil\kappa_0\rceil$.   □
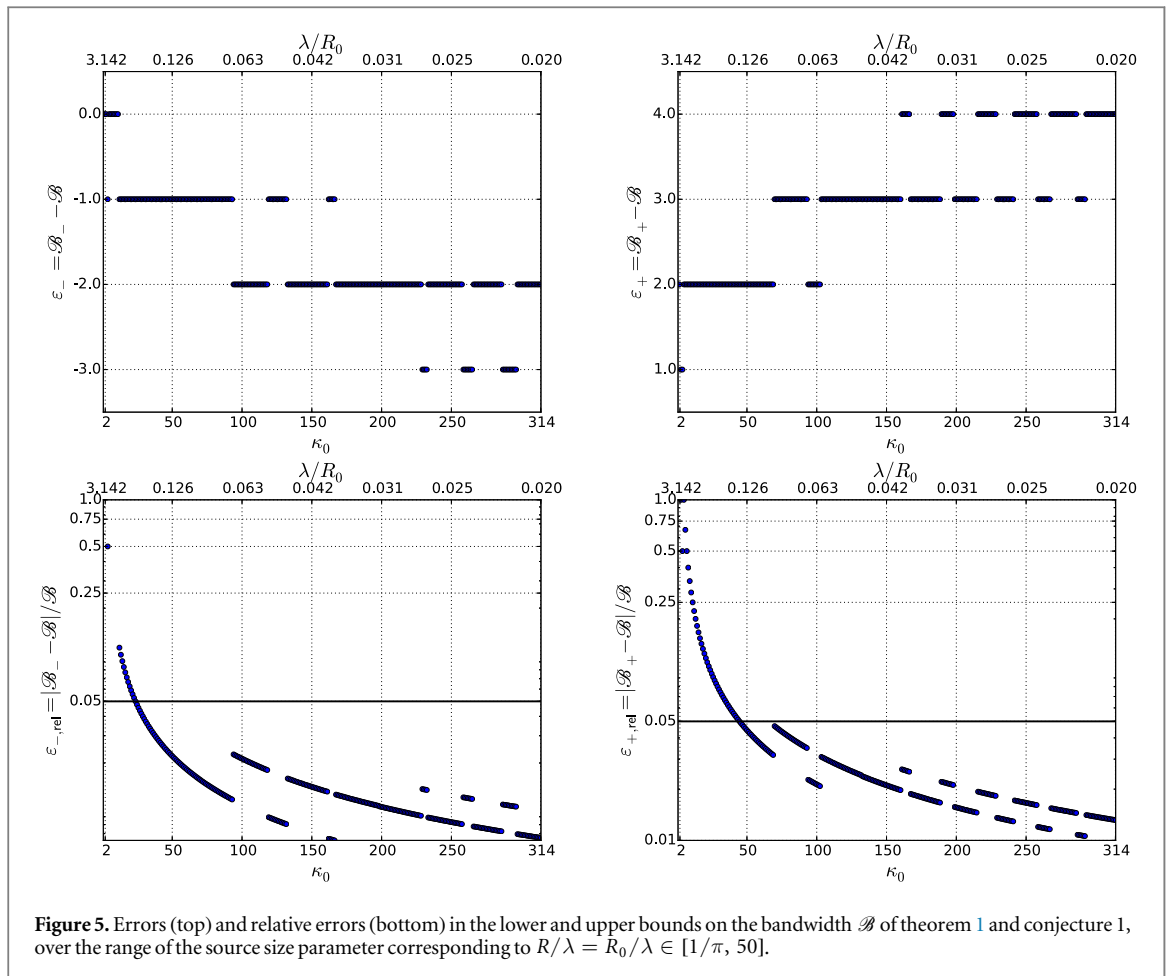
Proof of theorem 3. We can here start with the same argument as seen in our proof of theorem 2: since $\kappa_0 \leq y_{m,1}$, we have $\kappa_0 < j_{m,1} < j_{m+1,1}$ so $J_m(\kappa_0)J_{m+1}(\kappa_0) > 0$ and

$$A_m^2(\kappa_0) = J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0) - \frac{2m}{\kappa_0}J_m(\kappa_0)J_{m+1}(\kappa_0) \leqslant J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0).$$

It is now readily checked that the mth singular value $\sigma_{far,m}$ of the source-to-far-field operator satisfies

$$\sigma_{far,m}^2 = (2\pi)^2 \int_{r=0}^{R_0} rJ_m^2(kr) = 2(\pi R_0)^2 A_m(\kappa_0)^2 \leqslant 2(\pi R_0)^2(J_m^2(\kappa_0) + J_{m+1}^2(\kappa_0)).$$

Using the estimate $|J_m(\kappa_0)| \leq (\kappa_0/2)^m \exp(-\kappa_0^2/4(m + 1))/m!$, obtained just after the proof of theorem 2,

Figure 5. Errors (top) and relative errors (bottom) in the lower and upper bounds on the bandwidth $\mathscr{B}$ of theorem 1 and conjecture 1, over the range of the source size parameter corresponding to $R/\lambda = R_0/\lambda \in [1/\pi, 50]$.

we find that

$$\sigma_{far,m}^2 \leqslant 2(\pi R_0)^2 \frac{(\kappa_0/2)^{2m}}{m!^2}\left(e^{-\kappa_0^2/2(m+1)} + \frac{(\kappa_0/2)^2}{(m+1)^2}e^{-\kappa_0^2/2(m+2)}\right). \qquad \square$$

The following technical result is useful in conjunction with theorem 3.

Lemma 4. *If $j_{m,1} \geq \kappa_0$ and $m \geq 1$ then $m > \kappa_0/2 - 1$.*

Proof. Since m⩾1, the result follows immediately for $\kappa_0 < 4$. Assume in the following that $\kappa_0 \geq 4$. In Watson (1945, p 487) it is shown that $j_{m,1} < \sqrt{4(m+1)(m+5)/3}$. Thus, if $\kappa_0 \leq j_{m,1}$ then

$$\kappa_0^2 < 4(m+1)(m+5)/3,$$

and hence

$$m > \frac{1}{2}\sqrt{3\kappa_0^2 + 16} - 3.$$

It is now a straightforward matter to verify that

$$\frac{1}{2}\sqrt{3\kappa_0^2 + 16} - 3 \geq \frac{\kappa_0}{2} - 1. \qquad \square$$

The sequence $((\kappa_0/2)^m/m!)$, occurring in the estimate of theorem 3, is decaying when $m > \kappa_0/2 - 1$. By lemma 4, this last inequality is satisfied under the assumption $y_{m,1} \geq \kappa_0$ of theorem 3, since $j_{m,1} > y_{m,1}$.

3. Numerical validation

We here compute the bandwidth $\mathscr{B}$, as well as the bandwidth bounds $\mathscr{B}_-$ and $\mathscr{B}_+$ of theorem 1 and conjecture 1, respectively, for 300 values of the size parameters $\kappa = \kappa_0$ uniformly distributed over the interval $\kappa \in [2, 100\pi]$. Recall that $\kappa = kR = 2\pi R/\lambda$ and $\kappa_0 = kR_0 = 2\pi R_0/\lambda$, where R is the radius of the sampling circle ∂D, R_0 is the radius of the source domain, and λ is the operating wavelength. Thus, we consider 300 values of the relative
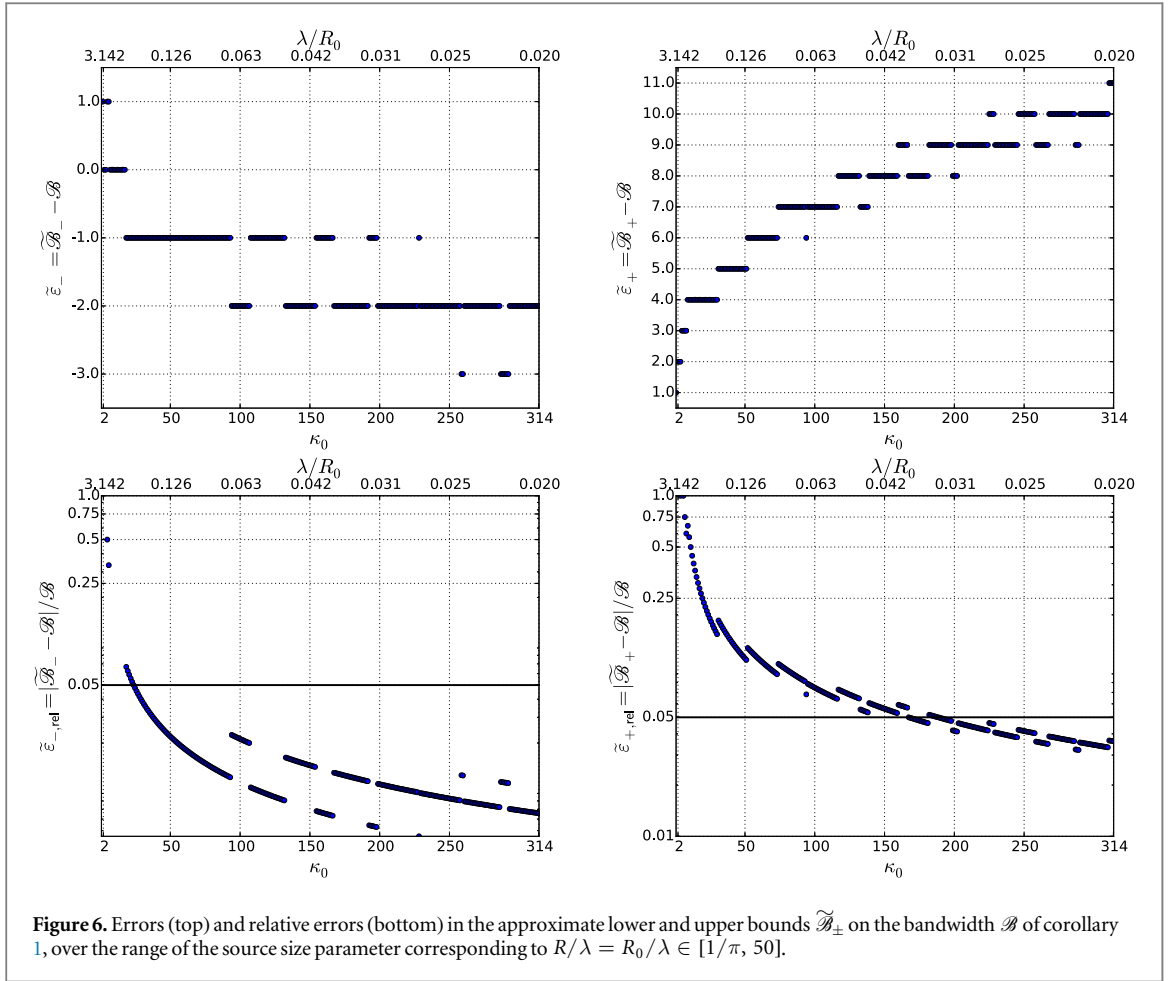
Figure 6. Errors (top) and relative errors (bottom) in the approximate lower and upper bounds $\widetilde{\mathscr{B}}_\pm$ on the bandwidth $\mathscr{B}$ of corollary 1, over the range of the source size parameter corresponding to $R/\lambda = R_0/\lambda \in [1/\pi, 50]$.

Table 1. Linear regression of the computed bandwidth $\mathscr{B}$, and the lower ($\mathscr{B}_-$) and upper ($\mathscr{B}_+$) bandwidth bounds. We have here held equal the size parameters κ and κ_0.

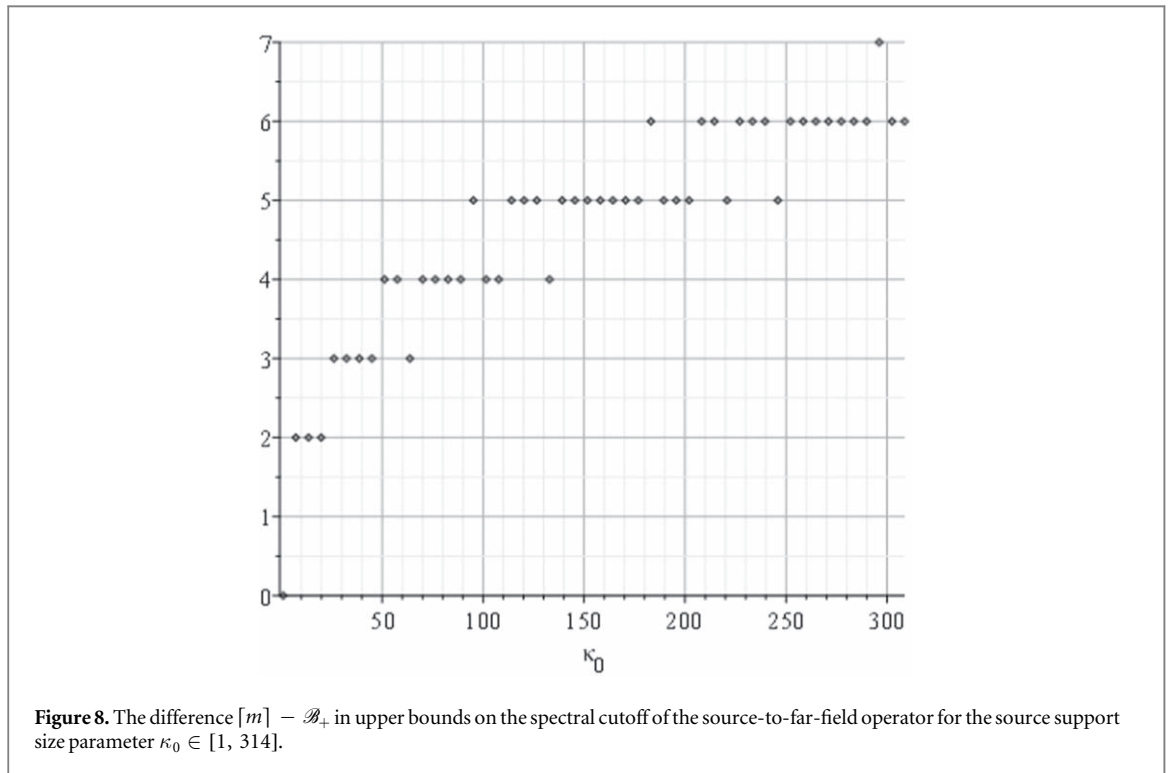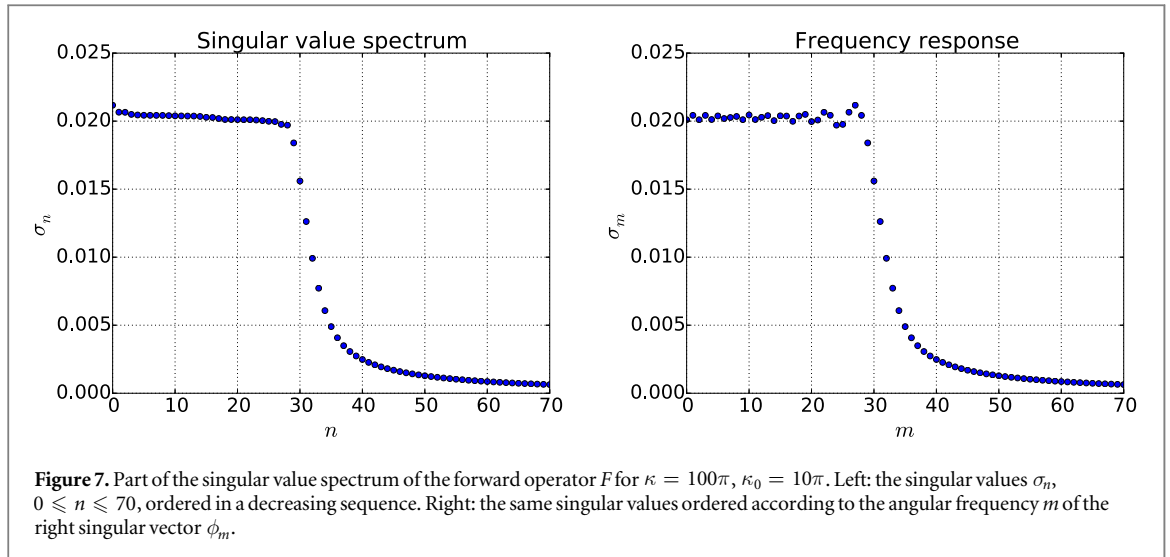|	linear interpolant	Mean absolute error	Standard deviation
$\mathscr{B}$	$0.9793\kappa - 3.9569$	0.4813	$3.9 \cdot 10^{-4}$
$\mathscr{B}_-$	$0.9736\kappa - 4.7394$	0.5715	$4.8 \cdot 10^{-4}$
$\mathscr{B}_+$	$0.9861\kappa - 2.0083$	0.4052	$3.4 \cdot 10^{-4}$

wavelength $\lambda/R = \lambda/R_0$ distributed nonuniformly over the interval $\lambda/R \in [1/50, \pi]$. Figure 5 shows the errors $\varepsilon_\pm = \mathscr{B}_\pm - \mathscr{B}$ and the relative errors $\varepsilon_{\mathrm{rel},\pm} = |\mathscr{B}_\pm - \mathscr{B}|/\mathscr{B}$ in the estimated bandwidth as function of the problem size parameter κ.

For the two lowest considered values of κ, we find that $\mathscr{B} = 0$ and $\mathscr{B}_- = 0$; there, we set $\varepsilon_{\mathrm{rel},-} = 0$. Both $\mathscr{B}$ and $\mathscr{B}_-$ are positive for higher considered values of κ. In particular, there is zero bandwidth for κ smaller than some threshold value between approx. 1.7 and approx. 2.7, and for such size parameters κ the inverse source problem is, from the viewpoint of the bandwidth of the singular values, similar to the inverse heat conduction problem. Over the considered interval for κ, the mean errors are $\overline{\varepsilon}_- = -1.68$, $\overline{\varepsilon}_+ = 3.02$, and the maximum absolute errors are $\max|\varepsilon_-| = 3$, $\max|\varepsilon_+| = 4$. The relative error in $\mathscr{B}_-$ is below 5% for $\kappa \geqslant 24.7461$, i.e., for $R/\lambda \geqslant 3.94$, and $\mathscr{B}_+$ is below 5% for $\kappa \geqslant 45.7181$, i.e., for $\lambda/R \geqslant 7.28$.

We find both $\mathscr{B}$, $\mathscr{B}_-$ and $\mathscr{B}_+$ to be approximately linear functions of κ in the given interval, with least-squares fits summarized in table 1.

Figure 6 shows errors in the approximations $\widetilde{\mathscr{B}}_\pm$ of corollary 1, where we choose the simpler form $\widetilde{\mathscr{B}}_+ \approx \lceil \kappa_0 \rceil$ for the estimate of the upper bound. The approximate expression for the lower bound shows almost the same small error as the lower bound itself, and the approximate expression $\widetilde{\mathscr{B}}_+ \approx \lceil \kappa_0 \rceil$ for the upper bound, while simple, has error below 5% only for problem size parameters of approx. 175 or higher when $\kappa = \kappa_0$ is maintained.

Our bounds $\mathscr{B}_\pm$ are independent of the radius of the measurement surface, and we next validate this property numerically. Figure 7 shows the first 71 nonnegative-index singular values of the forward operator F

Figure 7. Part of the singular value spectrum of the forward operator *F* for $\kappa = 100\pi$, $\kappa_0 = 10\pi$. Left: the singular values σ_n, $0 \leqslant n \leqslant 70$, ordered in a decreasing sequence. Right: the same singular values ordered according to the angular frequency *m* of the right singular vector ϕ_m.

Figure 8. The difference $\lceil m \rceil - \mathcal{B}_+$ in upper bounds on the spectral cutoff of the source-to-far-field operator for the source support size parameter $\kappa_0 \in [1, 314]$.

with size parameters $\kappa = 100\pi$, $\kappa_0 = 10\pi$. The bandwidth is unchanged at $\mathcal{B} = 27$ (compare with figure 3), as predicted by our bounds. The decrease in the numerical stability of the ISP due to the measurement boundary being farther away from the source is instead expressed in terms of the overall lower level of the singular values.

Let us finally compare numerically our proposed upper bound $\mathcal{B}_+ = \mathrm{argmin}_{m \in \mathbb{N}_0} \{ y_{m,1} \geqslant \kappa_0 \}$ for the spectral cutoff of the source-to-far-field-operator (theorem 3) with the bound $|m| \geqslant \kappa_0$ given in, e.g., Griesmaier *et al* (2014) and Griesmaier and Sylvester (2017). Figure 8 shows, for $\kappa_0 \in [1, 314]$ the difference $\lceil \kappa_0 \rceil - \mathcal{B}_+$. Over the considered range of source sizes κ_0, our bound gets progressively tighter, and estimates the spectral cutoff at the order *m* up to 7 lower than suggested by the alternative bound.

4. Discussion

The bandwidth estimates $\mathcal{B}_\pm$ are directly applicable as optimal filter estimates in the numerical solution of the inverse source problem in terms of a truncated singular value decomposition (TSVD) of the forward operator. Next, it has been amply observed in the literature concerning the single-frequency inverse source problem that

the numerical stability of the solution increases with the operating frequency. Theorem 1 confirms and explicitly quantifies this increase in numerical stability, also for non-asymptotic frequencies.

Theorem 1 of course has direct implications for the maximum achievable stable resolution of the reconstruction in the inverse source problem. Detailed analysis of this resolution requires an investigation of the pointwise behavior of the left singular vectors of the forward operator. While we here do not perform such analysis, we do note that the left singular vectors tend to be supported near the origin for low values of index m, and near the measurement boundary for high index values. This means the amplified noise produces is a 'wall of non-information' near the measurement boundary and blocks faithful reconstruction of the source inside D.

As shown in Bao *et al* (2010) and in section 2 here, the right singular vectors (defined over the measurement boundary) of the forward operator are proportional to $\exp(im\theta)$, $m \in \mathbb{Z}$. This means that the bandwidth index $\mathscr{B}$ is approximately the angular frequency of the highest-frequency data component that can be stably inverted. Thus, the sampling theorem (Shannon 1949) is directly applicable with theorem 1 to give the following: in case the radiated field u is sampled equidistantly at the boundary ∂D, any angular sampling rate greater than approximately $\Delta\theta \approx \pi/\mathscr{B} \leqslant \pi/\mathscr{B}_- = \pi/\operatorname{argmin}_{m\in\mathbb{N}_0}\{j_{m,1} \geqslant \kappa_0\}$ is excessive due to the limited bandwidth of the forward operator.

Bandwidth bounds in theorem 1 and conjecture 1 involve the size parameter of only the source support, and in light of the successful numerical validation of these bounds, we find it justified to propose that the bandwidth may generally be independent of the radius R of the measurement boundary relative to the radius R_0 of the source support (as long as $R \geqslant R_0$). As illustrated in section 3, the decrease in the robustness of the inversion (in the presence of noise) as R_0/R decreases seems instead to be expressed by a lower overall level of the singular values. We therefore briefly analyze the asymptotic behavior of the singular spectrum (3) as $m \to 0$, and as $m \to \infty$. The standard large-argument approximation of the Bessel functions of the first and second kind, valid for $\kappa_0 \gg m^2 - 1/4$, yields

$$J_m(\kappa_0) \sim \sqrt{\frac{2}{\pi\kappa_0}}\cos\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4}\right), \quad Y_m(\kappa_0) \sim \sqrt{\frac{2}{\pi\kappa_0}}\sin\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4}\right),$$

so

$$
\begin{aligned}
A_m(\kappa_0)^2 &\sim \frac{2}{\pi\kappa_0}\Bigg(\cos\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4}\right)^2 \\
&\quad - \cos\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4} + \frac{\pi}{2}\right)\cos\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4} - \frac{\pi}{2}\right)\Bigg) \\
&= \frac{2}{\pi\kappa_0}\left(\cos\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4}\right)^2 + \sin\left(\kappa_0 - \frac{m\pi}{2} - \frac{\pi}{4}\right)^2\right) = \frac{2}{\pi\kappa_0}
\end{aligned}
$$

and, since $\kappa \geqslant \kappa_0$, we also have $H_m(\kappa)^2 \sim 2/\pi\kappa$. Thus

$$\sigma_m \sim \frac{\sqrt{2}}{\pi}\lambda\sqrt{R_0}$$

for $R_0/\lambda \gg (m^2 - 1/4)/2\pi$. Forward operators mapping from source spaces with larger supports thus have higher-valued singular values in the bandpass region, regardless of the size R of the measurement boundary relative to the size R_0 of source support. However, we also see that the height of the bandpass decreases when the operating wavelength lambda decreases (equivalently, when the operating frequency increases), which may counteract the increase in stably recoverable information gained due to the increase in bandwidth. In the small-argument limit ($0 < \kappa^2 \ll m + 1$) the standard approximation is

$$J_m(\kappa_0) \approx \frac{1}{m!}\left(\frac{\kappa_0}{2}\right)^m, \quad Y_m(\kappa_0) \approx -\frac{(m-1)!}{\pi}\left(\frac{2}{\kappa_0}\right)^m,$$

so (since $\kappa_0 \leqslant \kappa$) $A_m(\kappa_0)^2 \sim (\kappa_0/2)^{2m}m!^{-2}(m+1)^{-1}$ and $H_m(\kappa)^2 \sim (\kappa/2)^{2m}m!^{-2} + m!^2m^{-2}\pi^{-2}(\kappa/2)^{-2m}$, resulting in

$$A_m(\kappa_0)^2 H_m(\kappa)^2 \sim \left(\frac{R_0}{R}\right)^{2m}\frac{1}{\pi^2 m^2(m+1)},$$

and thus

$$\sigma_m \sim \frac{1}{m}\sqrt{\frac{2}{m+1}}\left(\frac{R_0}{R}\right)^{m-1/2}R_0^{3/2}.$$

Evidently, the ratio R_0/R of the source support radius to the measurement boundary radius strongly affects the rate of decay of the singular values, the robustness of the inversion to noise generally improving as the source support approaches the measurement boundary.

5. Conclusion and further work

We analyzed the singular values of the source-to-near-field operator associated with the single-frequency inverse source problem for the Helmholtz equation in the plane. In particular, we considered bounds on the information content that is preserved by the forward operator, proving a tight lower bound and conjecturing and rigorously justifying a tight upper bound on the singular value index of the highest-frequency data component that is stably recoverable. The bounds were expressed in terms of the zeros of Bessel functions of the first and the second kind. We validated both bounds numerically, establishing concrete estimates on the stably recoverable information in the inverse source problem regardless of the data sampling rate and the choice of regularization. The result can be used directly, e.g., to estimate optimal TSVD filters and data sampling rates. We also showed that our upper bound constitutes an improvement of a widely used upper bound on the spectral cut-off of the source-to-far-field operator.

Proving the statement in conjecture 1 is a natural next step. Also, it would complete the picture to supplement the results on the bandwidth with a more precise description of the general levels and decay rates of the singular values as function of the size parameters of the source support and of the measurement boundary, individually or in relation to one another. Finally, a spectral analysis of the forward operator in dimension greater than 2 will be interesting.

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