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A NOTE ON PERIODIC POINT FREE SELF-MAPS

JAUME LLIBRE[1] AND VÍCTOR F. SIRVENT[2]

Abstract. We study the periodic point free maps on connected retract of a finite simplicial complex using the Lefschetz numbers. We put special emphasis in the self-maps on the product of spheres and of the wedge sums of spheres.

1. Introduction and statement of the main results

Let X be a topological space and f a continuous self-map on X. We say that a point x in X is a *periodic point of f of period m* if $f^m(x) = x$ and $f^j(x) \neq x$ for $1 \leq j \leq m-1$; if $m = 1$, x is called a *fixed point*. We say that f is a *periodic point free map*, if it does not have periodic points.

Let X be a retract of a finite simplicial complex, see [7] for a precise definition. The compact manifolds, the CW complexes are spaces of this type. Let n be the topological dimension of X. If $f : X \to X$ is a continuous map on X, it induces a homomorphism on the k–th rational homology group of X for $0 \leq k \leq n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and f_{*k} is a linear map whose matrix has integer entries, then *Lefschetz number* of f is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}).$$

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then f has a fixed point, see for more details [3, 11].

The Lefschetz fixed point theorem and its improvements is one of the most used tools for studying the existence of fixed points and periodic

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points, for continuous self maps on compact manifolds, among other references see for instance [1, 2, 3, 4, 8, 13].

The map f is called *Lefschetz periodic point free* if $L(f^m) = 0$, for $m \geq 1$. Clearly the periodic point free maps are Lefschetz periodic point free.

Note that if f is periodic point free, then f is Lefschetz periodic point free, but in general the converse does not hold. Periodic point free and Lefschetz periodic point free maps have been studied previously, see for instance [5, 6, 12, 13].

In the present note we give necessary and sufficient conditions for a continuous self-map in a connected retract of a finite simplicial complex to be a Lefschetz periodic free (Theorem 1 and Corollay 2). These conditions are given in term of the eigenvalues of the induced maps on homology. In Theorems 3 and 4 we give criteria for a map on the product of spheres to be periodic point free and Lefschetz periodic point free. In Theorem 5 we give a criteria for a map on the wedge sums of spheres to be periodic point free and Lefschetz periodic point free.

Let Λ_k be the set of eigenvalues of f_{*k}, and $\Lambda := \cup_{k=1}^{n} \Lambda_k$, i.e. the set of all eigenvalues of the induced maps on the homology by f.

Let $\lambda \in \Lambda$, we define

$$e(\lambda) := \sum_{k=0}^{n} (-1)^k \text{mult}_k(\lambda),$$

where $\text{mult}_k(\lambda)$ is the multiplicity of λ as eigenvalue of f_{*k}, if λ is not an eigenvalue of f_{*k}, then $\text{mult}_k(\lambda) = 0$. The quantity $e(\lambda)$ counts the multiplicities of λ as eigenvalue of all maps f_{*k} with k even, minus the multiplicities of λ as eigenvalue of all maps f_{*k} with k odd.

Theorem 1. *Let X be a connected retract of a finite simplicial complex, and f be a continuous self-map on X. The map f is Lefschetz periodic point free if and only the next two conditions hold.*

(a) *If λ is a non zero-eigenvalue of f_{*k} for some k, then $e(\lambda) = 0$.*
(b) *The number 1 is an eigenvalue of f_{*k} for some $k > 0$ and $\sum_{k \geq 1} \text{mult}_k(1) = 1$.*

Theorem 1 is proved in section 2. From Theorem 1 follows immediately Corollary 2.

Corollary 2. *Let X be a connected retract of a finite simplicial complex, and f be a continuous self-map on X. If the map f is periodic point free then the next two conditions hold.*

(a) *If λ is an eigenvalue, different from 0, of f_{*k}, for some k, then $e(\lambda) = 0$.*
(b) *The number 1 is an eigenvalue of f_{*k}, for some $k > 0$ and $\sum_{k\geq 1}(-1)^k mult_k(1) = 1$.*

We consider the case of periodic point and Lefschetz periodic point free maps on the product of spheres. It is well known that if X is the n–dimensional torus and $f : X \to X$ is a continuous map, then f is Lefschetz periodic point free if and only if 1 is an eigenvalue of f_{*1}, see for instance [1, 8, 14]. A similar result is known for maps on the product of spheres of the same dimension. Let $X = \mathbb{S}^n \times \cdots \times \mathbb{S}^n$ and $f : X \to X$ is a continuous map. If n is odd then f is Lefschetz periodic point free if and only if 1 is an eigenvalue of f_{*1}. If n is even then f is never periodic point free, see [13].

Theorem 3. *Let $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$ and let $f : X \to X$ be a continuous map. If the numbers n_i are even for all $1 \leq i \leq l$, then f is not Lefschetz periodic point free.*

The combinatorics of the periodic structure of self-maps on X could be very complicated in general. In the following particular case we give a criteria when a self-map on a product of spheres of different dimensions are Lefschetz free periodic.

Let

$$M = M(n_1, \ldots, n_l) := \bigcup_{s=1}^{l} \{n_{i_1} + \cdots + n_{i_s} \, : \, i_1 < \cdots < i_s\}.$$

We suppose that $n_1 < \cdots < n_l$. By elementary combinatorics we have that the cardinality of the set is at most $2^l - 1$. We shall assume that the cardinality of M is exactly $2^l - 1$, i.e. the numbers $n_1, \ldots, n_l$ are such that all the sums defined in the set M are different.

Theorem 4. *Let $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$ with $n_1 < \cdots < n_l$ and let $f : X \to X$ be a continuous map. Assume that the cardinality of M is $2^l - 1$. Then the following statements hold.*

(a) *If 0 is not an eigenvalue of f_{*k} for $k \in M$, and*

$$\sum_{k\in M} (-1)^{k+1} \neq 1,$$

then f in not periodic point free.

(b) *If there exists $j \in M$ with j odd and $f_{*j} = (1)$, and for $i \in M \setminus \{j\}$ there is i' such that $f_{*i} = f_{*i'}$ and $i \not\equiv i' \pmod 2$, then f is Lefschetz periodic point free.*

Theorems 3 and 4 are proved in section 3.

Given topological spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the *wedge sum* $X \vee Y$ is the quotient of the disjoint union X and Y obtained by identifying x_0 and y_0 to a single point (see for more details pp. 10 of [7]). The wedge sum is also known as "one point union". For example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure "8," two circles touching at a point. We can think the wedge sums of spheres as generalization of graphs in higher dimensions. In the following theorem we consider conditions for a continuous self-map on the wedge sum of spheres to be periodic point free.

Theorem 5. *Let $X = X_1 \vee \cdots \vee X_l$ where $X_i = \mathbb{S}^{n_i} \vee \overset{s_i - times}{\cdots} \vee \mathbb{S}^{n_i}$, $l > 0$ and $1 \leq n_1 < \cdots < n_l$, i.e. X is a wedge sum of spheres, and Let $f : X \to X$ be a continuous map.*

(a) *If 0 is not an eigenvalue of f_{*k} for $1 \leq k \leq n_l$ and*

$$-1 + (-1)^{n_1+1} s_1 + \cdots + (-1)^{n_l+1} s_l \neq 0, \tag{1}$$

then f is not periodic point free.

(b) *If $n_i \equiv 0 \pmod 2$, for $1 \leq i \leq l$ then f is not periodic point free.*

(c) *If $n_i \equiv 1 \pmod 2$, then f is Lefschetz periodic point free if and only if there exists a unique $1 \leq j \leq l$ such that 1 is a simple eigenvalue for f_{*j} and all the other eigenvalues f_{*i} for $1 \leq i \leq l$ are equal to 0.*

Theorem 5 is proved in section 4.

If $l = 1$, and $n_1 = 1$, then X is the circle, a particular case of a graph. Periodic point free maps on graphs were studied in [12].

If $l = 1$ and $n_1 > 1$, then X is the n_1–dimensional sphere, and a continuous self–map f of X is not periodic point free, unless n_1 is odd, 1 is a simple eigenvalue of f_{*n_1}, and 0 is eigenvalue of f_{*n_1} with multiplicity $s_1 - 1$. This fact follows from expression of the Lefschetz zeta function of f which in this case is $\zeta_f(t) = (1-t)^{-1} q_1(t)^{(-1)^{n_1+1}}$.

2. Proof of Theorem 1

Let X be a connected retract of a finite simplicial complex, and f be a continuous self-map on X. Then the *Lefschetz zeta function* of f is defined as

$$\zeta_f(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m} t^m\right).$$

This function keeps the information of the Lefschetz number for all the iterates of f, so this function gives information about the set of periods of f.

Note that if f is Lefschetz periodic point free if and only if $\zeta_f(t) = 1$.

There is an alternative way to compute the Lefschetz zeta function, namely

$$\zeta_f(t) = \prod_{k=0}^{n} \det(Id_{*k} - tf_{*k})^{(-1)^{k+1}}, \tag{2}$$

where $n = \dim X$, $n_k = \dim H_k(X, \mathbb{Q})$, $Id := Id_{*k}$ is the identity map on $H_k(X, \mathbb{Q})$, and by convention $\det(Id_{*k} - tf_{*k}) = 1$ if $n_k = 0$, for a proof see [4].

Lemma 6. *The Lefschetz zeta function of X can be expressed in the form*

$$\zeta_f(t) = \prod_{\lambda\in\Lambda} (1 - t\lambda)^{-e(\lambda)}. \tag{3}$$

Proof. We consider

$$\begin{aligned} \zeta_f(t) &= \prod_{k=0}^{n} \det(Id_{*k} - tf_{*k})^{(-1)^{k+1}} = \prod_{k=0}^{n} \prod_{\lambda\in\Lambda_k} ((1 - t\lambda)^{\mathrm{mult}_k \lambda})^{(-1)^{k+1}} \\ &= \prod_{\lambda\in\Lambda} (1 - t\lambda)^{\sum_{k=0}^{n} (-1)^{k+1} \mathrm{mult}_k \lambda} = \prod_{\lambda\in\Lambda} (1 - t\lambda)^{-e(\lambda)}. \end{aligned}$$

This completes the proof of the lemma. □

From Lemma 6 it follows that the zero eigenvalues of f_{*k} for $1 \leq k \leq n$ do not contribute to any non-trivial factor in the expression (3) of the Lefschetz zeta function.

*Proof of Theorem*1. If f is Lefschetz periodic point free then $\zeta_f(t) = 1$. If $\lambda \in \Lambda$ is different from zero, then the factor $(1 - t\lambda)$ cannot be in (3), i.e. $e(\lambda) = 0$.

Since X is connected then 1 is a simple eigenvalue of f_{*0}, then the term $(1-t)^{-1}$ is a factor of (3). Hence 1 is an eigenvalue of f_{*k}, for some $k \neq 0$, and it is required that $\sum_{k\geq 1} \text{mult}_k(1) = 1$ in order to have $e(1) = 0$.

From the previous arguments it is clear that if conditions (a) and (b) of Theorem 1 hold, then f is Lefschetz periodic point free. This completes the proof of the theorem. □

3. Proofs of Theorems 3 and 4

Let $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$. Using the Künneth Theorem (see for instance [7]), we compute the homology groups of X over the rational numbers $\mathbb{Q}$. They are given by

$$H_k(X, \mathbb{Q}) = \begin{cases} \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{b_k} & \text{if } b \neq 0, \\ \{0\} & \text{if } b_k = 0, \end{cases}$$

with $0 \leq k \leq n_1 + \cdots + n_k$; where b_k is the number of ways that k can be obtained as by summing up subsets of $(n_1, \ldots, n_l)$, i.e. b_k is the cardinality of the set

$$\left\{ S \subset \{1, \ldots, l\} \mid \sum_{j \in S} n_j = k \right\}.$$

The numbers b_k are called the *Betti numbers of* X.

Proof of Theorem 3. We remark that $b_k \neq 0$ if and only if $k \in M$. So the formula (2) of the Lefschetz zeta function can be expressed in the following manner:

$$\zeta_f(t) = (1-t)^{-1} \prod_{k=1}^{l} q_k(t)^{(-1)^{k+1}} = (1-t)^{-1} \prod_{k \in M} q_k(t)^{(-1)^{k+1}},$$

where $q_k(t) = \det(Id_{*k} - tf_{*k})$. Since the numbers n_i are even, clearly all the elements of the set M are even numbers. Therefore

$$\zeta_f(t) = \frac{1}{(1-t) \prod_{k \in M} q_k(t)},$$

the map f is not Lefschetz periodic point free. □

Let $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$ with $n_1 < \cdots < n_l$ and we assume that the cardinal of M is $2^l - 1$. In this case the Betti numbers of X are $b_k = 1$

if $k \in M$ and $b_k = 0$ otherwise, i.e.

$$H_k(X, \mathbb{Q}) = \mathbb{Q} \text{ if } k \in \{0\} \bigcup_{s=1}^{l} \{n_{i_1} + \cdots + n_{i_s} \,|\, i_1 < \cdots < i_s\},$$

and trivial otherwise.

If $f : X \to X$ is a continuous map, the formula (2) can be written as

$$\zeta_f(t) = (1-t)^{-1} \prod_{k \in M} (1 - a_k t)^{(-1)^{k+1}}, \tag{4}$$

where $f_{*k} = (a_k)$.

Proof of statement (a) *of Theorem* 4. The sum of the degree of the polynomials of numerator of (4) minus the degree of the polynomials in the denominator of (4), is

$$-1 + \sum_{k \in M} (-1)^{k+1}.$$

So, if this sum is not zero then $\zeta_f(t) \neq 1$. Hence f is not periodic point free. □

Proof of statement (b) *of Theorem* 4. If f has the property described, then its Lefschetz zeta function is

$$\begin{aligned} \zeta_f(t) &= (1-t)^{-1} \prod_{k \in M} (1 - a_k t)^{(-1)^k+1} \\ &= (1-t)^{-1} (1-t)^{(-1)^j+1} \prod_{k \in M \setminus \{j\}} (1 - a_k t)^{(-1)^k+1} \\ &= \prod_{\substack{i \in M \setminus \{j\} \\ i \equiv 1 \pmod 2}} (1 - a_i t) \prod_{\substack{i \in M \setminus \{j\} \\ i \equiv 0 \pmod 2}} (1 - a_i t)^{-1}. \end{aligned}$$

Since for each $i \in M$, with $i \neq j$ and $i \equiv 1 \pmod 2$, there exists i' satisfying $i' \equiv 0 \pmod 2$, and $a_i = a_{i'}$, we have $\zeta_f(t) = 1$. □

The particular case of a product of spheres with $l = 2$ was studied in [6]. If $l = 3$ then

$$M = \{n_1, n_2, n_3, n_1 + n_2, n_1 + n_3, n_2 + n_3, n_1 + n_2 + n_3\}.$$

In this case the possible situations are:

(1) If n_1, n_2 and n_3 are even, then f in not periodic point free. As it is a particular case of Theorem 3.

(2) If n_1, n_2 and n_3 are odd, then

$$\zeta_f(t) = \frac{(1-a_{n_1}t)(1-a_{n_2}t)(1-a_{n_3}t)(1-a_{n_1+n_2+n_3}t)}{(1-t)(1-a_{n_1+n_2}t)(1-a_{n_1+n_3}t)(1-a_{n_2+n_3}t)}.$$

(3) In the case that two n_i are odd and the other even. We can suppose that n_1 and n_2 are odd and n_3 even. Then

$$\zeta_f(t) = \frac{(1-a_{n_1}t)(1-a_{n_2}t)(1-a_{n_1+n_3}t)(1-a_{n_2+n_3}t)}{(1-t)(1-a_{n_3}t)(1-a_{n_1+n_2}t)(1-a_{n_1+n_2+n_3}t)}.$$

(4) In the case that two n_i are even and the other odd. We can suppose that n_1 and n_2 are even and n_3 odd. Then

$$\zeta_f(t) = \frac{(1-a_{n_3}t)(1-a_{n_1+n_3}t)(1-a_{n_2+n_3}t)(1-a_{n_1+n_2+n_3}t)}{(1-t)(1-a_{n_1}t)(1-a_{n_2}t)(1-a_{n_1+n_2}t)}.$$

In the cases (2), (3) and (4) the map f is not periodic point free, unless the condition of statement (c) of Theorem 4 is satisfied.

In the following lines, we consider for a product of three spheres a situation when n_1, n_2, n_3 do not satisfy that the cardinal of the set M is $2^3 - 1 = 7$, but it satisfies that this cardinal is 6, being $n_3 = n_1 + n_2$. By the Künneth formula we have that the homology groups of X are

$$H_k(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = n_1, n_2, n_1+n_3, n_2+n_3, n_1+n_2+n_3; \\ \mathbb{Q}\oplus\mathbb{Q} & \text{if } k = n_3; \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence

$$\zeta_f(t) = \frac{q_{n_3}(t)^{(-1)^{n_3+1}} \displaystyle\prod_{i=1}^{2} (1-a_{n_i}t)^{(-1)^{n_i+1}} (1-a_{n_i+n_3}t)^{(-1)^{n_i+n_3+1}}}{(1-t)(1-a_{n_1+n_2+n_3}t)},$$

where $q_{n_3}(t) = \det(Id_{*n_3} - tf_{*n_3})$. The degree of the polynomial $q_{n_3}(t)$ is at most 2, and it is equal to 2 if and only if 0 is not an eigenvalue of f_{n_3}.

In this case the possible situations are:

(1) If n_1 and n_2 are even, then the numerator of the Lefschetz zeta function is 1 and the denominator is a polynomial of degree greater than 1. Hence $\zeta_f(t) \neq 1$. Therefore f it is not periodic point free. This is a particular case of Theorem 3.
(2) If n_1 and n_2 are odd. Then

$$\zeta_f(t) = \frac{(1-a_{n_1}t)(1-a_{n_2}t)(1-a_{n_1+n_3}t)(1-a_{n_2+n_3}t)}{(1-t)(1-a_{n_1+n_2+n_3}t)q_{n_3}(t)}.$$

(3) If n_1 and n_2 have different parities, say n_1 even and n_2 odd, then

$$\zeta_f(t) = \frac{(1-a_{n_2}t)(1-a_{n_1+n_3}t)q_{n_3}(t)}{(1-t)(1-a_{n_1+n_2+n_3}t)(1-a_{n_1}t)(1-a_{n_2+n_3}t)}.$$

In the cases (2) and (3), $\zeta_f(t) \neq 1$ if the eigenvalues of the induced maps on homology are different from 0 and 1. However in (2) and (3) it is possible that $\zeta_f(t) = 1$ and consequently the map f be Lefschetz periodic point free.

As we can see the combinatorial analysis of the Lefschetz zeta function becomes quite complicated when the numbers $n_1, \ldots, n_l$ do not satisfy that the cardinal of the set M is $2^l - 1$.

4. Proof of Theorem 5

Let $X = X_1 \vee \cdots \vee X_l$ where $X_i = \mathbb{S}^{n_i} \vee \overset{s_i-\text{times}}{\cdots} \vee \mathbb{S}^{n_i}$, $l > 0$ and $1 \le n_1 < \cdots < n_l$. The homology groups of X over the rational numbers $\mathbb{Q}$ are

$$H_k(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} \oplus \overset{m_k}{\cdots} \oplus \mathbb{Q} & \text{for } k = 0, m_0 = 1, \text{ and for } k = n_i, m_{n_i} = s_i, \\ \{0\} & \text{otherwise.} \end{cases}$$

The computation of these homology groups follows from the facts $H_k(X,\mathbb{Q}) = \oplus_{i=1}^{l} H_k(X_i,\mathbb{Q})$, $H_k(X_i,\mathbb{Q}) = \oplus_{j=1}^{s_i} H_k(\mathbb{S}^{n_i},\mathbb{Q})$, and $H_k(\mathbb{S}^{n_i},\mathbb{Q}) = \mathbb{Q}$ for $k = 0, n_i$ and trivial otherwise, (*cf.* [7]).

Let $f : X \to X$ be a continuous self-map on X, its Lefschetz zeta function has the form:

$$\zeta_f(t) = (1-t)^{-1} q_1(t)^{(-1)^{n_1+1}} \cdots q_l(t)^{(-1)^{n_l+1}}, \tag{5}$$

where $q_k(t) = \det(Id_{*n_k} - tf_{*n_k})$, clearly it is a polynomial of degree smaller than or equal to s_k, the degree is s_k if and only if 0 is not an eigenvalue of f_{*n_k}. So if $p_k(t)$ is the characteristic polynomial of f_{*k}, we have $p_k(t) = (-1)^{s_k} t^{s_k} q_k(1/t)$. Therefore

$$\zeta_f(t) = (-1)^{s_1+\cdots+s_l}(1-t)^{-1}(t^{s_1}p_1(1/t))^{(-1)^{n_1+1}} \cdots (t^{s_l}p_l(1/t))^{(-1)^{n_l+1}}. \tag{6}$$

Proof of statement (*a*) *of Theorem* 5. From (1) and (6) if follows that $\zeta_f(t) \neq 1$, so the map f is not Lefschetz periodic point free, and consequently f is not periodic point free. □

Proof of statement (*b*) *of Theorem* 5. Since $n_i \equiv 0 \pmod 2$ we have that $\zeta_f(t) = ((1-t)q_1(t)\cdots q_l(t))^{-1}$. Then $\zeta_f(t) \neq 1$. Therefore statement (b) follows. □

Proof of statement (c) *of Theorem* 5. If n_i are odd, for $1 \leq i \leq l$, then $\zeta_f(t) = q_1(t) \cdots q_l(t)(1-t)^{-1}$. Hence $\zeta_f(t) = 1$, if and only if there exists only one $1 \leq j \leq l$, such that $q_j(t) = (1-t)$ and for $i \neq j$, $q_i(t) = 1$. We remind that $q_i(t) = 1$ if and only if the eigenvalues of f_{*n_i} are equal to zero. And $q_j(t) = 1 - t$ if and only if 1 is the only non-zero eigenvalue of f_{*n_j} and it is a simple eigenvalue. This proves statement (c). □

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References

[1] Ll. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, *Torus maps and Nielsen numbers*, Contemporary Math. **152** (1993), 1–7.

[2] R.B. Brooks, R.F. Brown, J. Pak and D. Taylor, Nielsen numbers of maps of tori, *Proc. Amer. Math. Soc.* **52** (1975), 398–400.

[3] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[4] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.

[5] G. Graff, A. Kaczkowska, P. Nowak-Przygodzki, and J. Signerska Lefschetz periodic point free self-maps of compact manifolds, *Topology Appl.* **159** (2012), 2728–2735.

[6] J.L.G. Guirao and J. Llibre, On the Lefschetz periodic point free continuous self–maps on connected compact manifolds, *Topology Appl.* **158** (2011), 2165–2169.

[7] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[8] B. Jiang and J. Llibre, Minimal sets of periods for torus maps, *Discrete Contin. Dyn. Syst.* **4** (1998), 301–320.

[9] J. Jezierski, A sufficient condition for the realizability of the least number of periodic points of a smooth map, *J. Fixed Point Theory Appl.* **18** (2016), 609–626.

[10] J. Jezierski and W. Marzantowicz, *Homotopy methods in topological fixed and periodic points theory*, Springer Verlag, Berlin, 2006.

[11] S. Lefschetz, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.

[12] J. Llibre, Periodic point free continuous self–maps on graphs and surfaces, *Topology Appl.* **159** (2012), 2228–2231.

[13] J. Llibre and V.F. Sirvent, Partially periodic point free self-maps on surfaces, graphs, wedge sums and product of spheres, *J. Differ. Equ. Appl.* **19** (2013), 1654–1662.

[14] J.W. Vick, *Homology theory. An introduction to algebraic topology*, Springer–Verlag, New York, 1994. Academic Press, New York, 1994.

[1] Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Catalunya, Spain

E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matemáticas, Universidad Simón Bolívar, Apartado 89000, Caracas 1086-A, Venezuela

E-mail address: vsirvent@usb.ve