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ZERO–HOPF BIFURCATIONS IN A HYPERCHAOTIC LORENZ SYSTEM *II*

JAUME LLIBRE AND MURILO RODOLFO CÂNDIDO

Abstract. We characterize the zero–Hopf bifurcation at four singular points of a hyperchaotic Lorenz system. Using averaging theory, we find sufficient conditions so that at the bifurcation points periodic solutions emerge and describe the stability of some orbits.

1. Introduction

The Lorenz system of differential equations in $\mathbb{R}^3$ arose from the work of meteorologist/mathematician Edward N. Lorenz [20], who studied forced dissipative hydrodynamical systems. As he was computing numerical solutions to the system, he notice that initial conditions with small differences eventually produced vastly different solutions, a characteristic of chaos. Since that time, about 1963, the Lorenz system has become one of the most widely studied systems of ODEs because of its wide range of behaviors. Although the origins of this system lies in atmospheric modeling, the Lorenz equations also appear in other areas as in the modeling of lasers see [12], and dynamos see [16].

In recent times a so-called *hyperchaotic Lorenz* system was introduced; see for instance [1, 4, 7, 9, 14, 15, 22, 23, 24, 25, 27, 28, 30] and the references therein. MathSciNet presently lists 32 papers on *hyperchaotic Lorenz* systems. We observe that not all these hyperchaotic Lorenz systems are similar, since they can vary in one or two terms. However these systems can be precisely definite as autonomous differential systems in a phase space of dimension at least four, with a dissipative structure, and at least two unstable directions, such that at least one is due to a nonlinearity. The hyperchaotic systems has a dynamics hard to predict or control, for this reason such systems are as well of use in secure communications systems see, for instance [29].

Our aim in this work is to study, from a dynamical point of view, the 4–*dimensional zero–Hopf equilibria* in the hyperchaotic Lorenz system. Here, a 4–dimensional zero–Hopf equilibrium means an equilibrium point with two zeros and a pair of pure conjugate imaginary numbers as eigenvalues. Using the method of averaging and convenient changes of variables and parameters we can analyse the zero–Hopf bifurcations. More precisely we study zero–Hopf bifurcations of the

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following hyperchaotic Lorenz system (as given in [7, 15])

$$\begin{aligned}\dot{x} &= a(y-x)+w,\\ \dot{y} &= cx-y-xz,\\ \dot{z} &= -bz+xy,\\ \dot{w} &= dw-xz,\end{aligned} \tag{1}$$

for appropriate choices of the parameters a, b, c and d.

There are several works studying zero–Hopf bifurcation see for instance Guckenheimer [10], Guckenheimer and Holmes [11], Han [13], Kuznetsov [17], Llibre [19], Marsden, Scheurle [21].... It has been shown that, under specific conditions, some elaborated invariant sets of the unfolding could be bifurcated from a zero–Hopf equilibrium and hence, in some cases, a zero–Hopf equilibrium could signal a local birth of "chaos" (see [5, 21]). Also, recently Li and Wang [18] published a paper on a Hopf bifurcation in a 3-dimensional Lorenz-type system. Due to the complexity related to the high dimensionality, there is very little work done on the n–dimensional zero–Hopf bifurcation with $n > 3$.

The characterization of the zero-Hopf bifurcation at the origin was recently study by Cid-Montiel, Llibre and Stoica in [6]. In this work we are going to complete the characterization analysing all singulars points for system (1).

2. Statement of the main results

In the first instance we are going to compute the equilibrium points of hyperchaotic Lorenz system (1). One may verify that for any choice of the parameters, the origin of coordinates of $\mathbb{R}^4$ is always an equilibrium point for the system. Moreover if $ad \neq 1$ and $abd(1-c)(c-ad) > 0$, system (1) will have two additional equilibrium points

$$\boldsymbol{p}_\pm = \left(\pm\frac{\sqrt{abd(1-c)}}{\sqrt{c-ad}}, \pm\frac{\sqrt{abd(1-c)(c-ad)}}{1-ad}, \frac{ad(1-c)}{1-ad}, \pm\frac{a(1-c)\sqrt{abd(1-c)}}{(1-ad)\sqrt{c-ad}} \right).$$

Considering $b = 0$ then all the z–axis is filled of equilibria. And if $b = 0$ and $ad(1-c)(1-ad) \neq 0$ we have the additional equilibrium point

$$\boldsymbol{p} = \left(0, 0, \frac{ad(1-c)}{1-ad}, 0\right).$$

We observe that the two equilibria $\boldsymbol{p}_\pm$ tends to the equilibrium point $\boldsymbol{p}$ when $b \to 0$. In short, the equilibrium point of system (1) can be $\boldsymbol{p}_+$, $\boldsymbol{p}_-$, $\boldsymbol{p}$ and the origin.

Note that system (1) is invariant by the symmetry $(x, y, z, w) \to (-x, -y, z, -w)$, i.e. it is invariant under the symmetry with respect to the z-axis. Therefore, we can study $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$ simultaneously using only one of these points. Due to that, in what follows we consider the only equilibrium $\boldsymbol{p}_+$ in order to verify its possibility of being a zero–Hopf equilibrium for some values of the parameter, and clearly the same will occur for the other equilibrium $\boldsymbol{p}_-$.

In the next result we characterize when the equilibria $\boldsymbol{p}$, $\boldsymbol{p}_\pm$ and the origin are zero–Hopf equilibria. To simplify the expressions we define

$$D_a = \sqrt{a^6 + 2a^5 - 3a^4 - 14a^3 - 14a^2 - 4a + 1}.$$

Proposition 1. *The following statements hold.*

(a) *The origin is a zero–Hopf equilibrium if and only if* $a = -1$, $b = d = 0$ *and* $c > 1$.

(b) *Assume that* $ad(1-c)(1-ad) \neq 0$ *and* $b = 0$. *The equilibrium point* $\boldsymbol{p}$ *is a zero–Hopf equilibrium if and only if* $d = a+1$, *and*

($b.1$) *either* $\dfrac{1+d^3-d^4+\left(d^2-1\right)c}{d^2-d-1} > 0$;

($b.2_+$) *or* $c_+ = \dfrac{-1+a(1+a)^2(a^2+2a+3)+(a^2+a-1)D_a}{2a(a^2+3a+3)}$ *and* $\dfrac{-4-9a-10a^2-5a^3-a^4+(2+a)D_a}{3+3a+a^2} > 0$;

($b.2_-$) *or* $c_- = \dfrac{-1+a(1+a)^2(a^2+2a+3)-(a^2+a-1)D_a}{2a(a^2+3a+3)}$ *and* $\dfrac{-4-9a-10a^2-5a^3-a^4-(2+a)D_a}{3+3a+a^2} > 0$;

(c) *The equilibria* $\boldsymbol{p}_\pm$ *never are zero–Hopf equilibria.*

Note that despite z–axis being filled of equilibria when $b = 0$, its only zero–Hopf equilibria are the origin and the equilibrium point $\boldsymbol{p}$. Furthermore, in Proposition 1, corresponding to the condition (a), there is one 1-dimensional parametric family for which the origin is a zero–Hopf equilibrium point and there are three parametric families for which the equilibrium point $\boldsymbol{p}$ is a zero–Hopf equilibrium of the hyperchaotic Lorenz system, one 2–dimensional parametric family corresponding to conditions ($b.1$) and two 1–dimensional parametric families corresponding to conditions ($b.2_+$) and ($b.2_-$).

If (a) holds the eigenvalues at the origin are 0, 0 and

$$\pm\omega i = \pm\sqrt{c-1}\, i.$$

If ($b.1$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_0 i = \pm\sqrt{\frac{1+d^3-d^4+\left(d^2-1\right)c}{d^2-d-1}}\, i. \tag{2}$$

If ($b.2_+$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_+ i = \pm\sqrt{\frac{-4-9a-10a^2-5a^3-a^4+(2+a)D_a}{2(3+3a+a^2)}}\, i. \tag{3}$$

If ($b.2_-$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_- i = \pm\sqrt{\frac{-4-9a-10a^2-5a^3-a^4-(2+a)D_a}{2(3+3a+a^2)}}\, i. \tag{4}$$

The next results characterizes when periodic orbits bifurcate from these zero–Hopf equilibrium points.

Theorem 2. (i) *Consider system* (1) *where*

$$a = -1+\varepsilon a_1, \quad b = \varepsilon b_1, \quad c = 1+c_0^2, \quad \text{and} \quad d = \varepsilon d_1. \tag{5}$$

For the zero–Hopf equilibrium at the origin we have:

(a) *If* $a_1 b_1 \neq 0$, $a_1 \neq d_1$ *and* $c_0 > 0$, *then there exists an* $\varepsilon_1 > 0$ *such that when* $|\varepsilon| < \varepsilon_1$ *the hyperchaotic Lorenz system* (1) *has a periodic solution*
$\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big) =$

$$\varepsilon\left(\sqrt{2a_1b_1}c_0\sin(c_0t),\ \sqrt{2a_1b_1}c_1\big(\sin(c_0t)-c_0\cos(c_0t)\big),\ a_1c_0^2,\ 0\right)+\mathcal{O}(\varepsilon^2), \tag{6}$$

bifurcating from the origin. The periodic solution (6) *is stable if* $b_1 > 0$, $a_1 < 0$ *and* $d_1 < a_1$.

(b) *If* $b_1 d_1 \neq 0$, $a_1 \neq d_1$ *and* $c_0 > 0$, *then there is a convenient choice of* ε *such that for either* $\varepsilon \in (-\varepsilon_1, 0)$ *or* $\varepsilon \in (0, \varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_\pm =$

$$\varepsilon\left(\pm c_0\sqrt{\frac{b_1d_1}{1+c_0^2}},\ \pm c_0\sqrt{b_1d_1(1+c_0^2)},\ c_0^2d_1,\ \pm c_0^3\sqrt{\frac{b_1d_1}{1+c_1^2}}\right)+\mathcal{O}(\varepsilon^2) \tag{7}$$

emerging from the origin. These solutions are stable if $b_1 > 0$, $d_1 > 0$ *and* $d_1 < a_1$.

(ii) *Considering system* (1) *where*

$$a = d-1+\varepsilon a_1 \quad \textit{and} \quad b = \varepsilon^2 b_1. \tag{8}$$

If $a_1 b_1 \neq 0$, $c \neq 1$, $d \notin \{0, \pm 1\}$ *and* $\omega_0 \in \mathbb{R}^*$, *then there exists an* $\varepsilon_1 > 0$ *such that when* $|\varepsilon| < \varepsilon_1$ *the hyperchaotic Lorenz system* (1) *has a periodic solution*
$\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big) =$

$$\begin{aligned}&\Bigg(\varepsilon\sqrt{\frac{2b_1(1-c)(d-1)d(d^2-1)}{d(d-1)-1}}\sin(\omega_0t),\\ &\varepsilon\sqrt{\frac{2b_1(1-c)(d-1)d}{(d(d-1)-1)(d^2-1)}}(\omega_0\cos(\omega_0t)-\sin(\omega_0t)),\\ &\frac{(c-1)(d-1)d}{d^2-d-1}+\varepsilon\frac{a_1\left(-cd^2-cd+c+d^4-d^3+d-1\right)}{(d^2-d-1)^2},\\ &\varepsilon d(d-1)\sqrt{\frac{2b_1(1-c)d(d-1)}{(d(d-1)-1)(d^2-1)}}(d\sin(\omega_0t)+\omega_0\cos(\omega_0t))\Bigg)+\mathcal{O}(\boldsymbol{\varepsilon^2}),\end{aligned} \tag{9}$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$. *The periodic solution* (9) *is unstable if* $a_1 < 0$ *or* $b_1(c-1)(d-1)d < 0$. *Furthermore, there is a convenient choice of* ε *such that for either* $\varepsilon \in (-\varepsilon_1, 0)$ *or* $\varepsilon \in (0, \varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big)_1 =$

$$\begin{aligned}&\left(-\varepsilon\sqrt{\frac{b_1(c-1)(1-d)d}{c+d-d^2}},\ \varepsilon\sqrt{\frac{b_1(c-1)(1-d)d(c+d-d^2)}{d(d-1)-1}},\right.\\ &\left.\frac{(c-1)(d-1)d}{d(d-1)-1}-\varepsilon\frac{a_1(c-1)d}{(1-d-d^2)^2},\ \frac{\varepsilon}{d(d-1)-1}\sqrt{\frac{b_1d(c-1)^3(1-d)^3}{c+d-d^2}}\right)+\mathcal{O}(\varepsilon^2)\end{aligned} \tag{10}$$

and $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_2 =$

(11)
$$\Bigg(\varepsilon\sqrt{\frac{b_1(c-1)(1-d)d}{c+d-d^2}},\ -\varepsilon\sqrt{\frac{b_1(c-1)(1-d)d(c+d-d^2)}{d(d-1)-1}},\\ \frac{(c-1)(d-1)d}{d(d-1)-1}-\varepsilon\frac{a_1(c-1)d}{(1-d-d^2)^2},\ \frac{-\varepsilon}{d(d-1)-1}\sqrt{\frac{b_1d(c-1)^3(1-d)^3}{c+d-d^2}}\Bigg)+\mathcal{O}(\varepsilon^2)$$

emerging from $\boldsymbol{p}$. *These orbits are unstable if* $a_1 < 0$ *or* $b_1(c-1)(d-1)d > 0$.

(iii) *Considering system* (1) *where*

(12)
$$b = \varepsilon^2 b_1, \quad c = c_+ + \varepsilon c_1 \quad \text{and} \quad d = 1 + a + \varepsilon d_1.$$

If $b_1 d_1 \neq 0$, $a \notin \left\{-2,\ -1,\ 0,\ \frac{1\pm\sqrt{5}}{2},\ \frac{-3\pm\sqrt{5}}{2}\right\}$ *and* $\omega_+ \in \mathbb{R}^*$, *then there exists* $\varepsilon_1 > 0$ *such that when* $|\varepsilon| < \varepsilon_1$ *the hyperchaotic Lorenz system* (1) *has a periodic solution* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big) =$

(13)
$$\Bigg(\varepsilon\frac{\sqrt{(2+a)}}{(1+a)\omega_+}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59+2a(20\\ +a(7+a)))))+D_a)\right)^{\frac{1}{2}}\sin(\omega_+ t),\\ \frac{\varepsilon}{a(1+a)\omega_+\sqrt{2+a}}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59\\ +2a(20\ +a(7+a)))))+D_a)\right)^{\frac{1}{2}}\big(\omega_+\cos(\omega_+ t)-\sin(\omega_+ t)\big),\\ \frac{(a+1)\left(a^3+3a^2+4a+D_a+1\right)}{2\left(a^2+3a+3\right)}+\varepsilon\frac{(a+1)}{2\left(a^2+a-1\right)\left(a(a+3)+3\right)}\\ (2a(a(a+3)+3)c_1-a(a(a+5)+8)d_1+d_1D_a-5d_1),\\ \frac{\varepsilon}{\omega_+\sqrt{2+a}}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59\\ +2a(20\ +a(7+a)))))+D_a)\right)^{\frac{1}{2}}\big(\omega_+\cos(\omega_+ t)+(1+a)\sin(\omega_+ t)\big)\Bigg)+\mathcal{O}(\varepsilon^2),$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$. *Furthermore, there is a convenient choice of* ε *such that for either* $\varepsilon \in (-\varepsilon_1,\, 0)$ *or* $\varepsilon \in (0,\, \varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_1 =$

$$\Bigg(-\varepsilon\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\ \frac{\varepsilon(1-a(1+a)(2+a)+Da)}{3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\\ \frac{(1+a)(1+4a+3a^2+a^3+Da)}{2(3+a(3+a))}+\frac{\varepsilon}{2(a^2+a-1)(3+a(3+a))}\\ \Big(2a(1+a)(3+a(3+a))c_0-d_1-a(4+a(3+a))d_1-d_1Da\Big),$$

(14)

$$-\frac{\varepsilon(1+4a+3a^2+a^3+Da)}{2(3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}}\Bigg)+\mathcal{O}(\varepsilon^2)$$

and $\big(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon),w(t,\varepsilon)\big)_2 =$

$$\Bigg(\varepsilon\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\ -\frac{\varepsilon(1-a(1+a)(2+a)+Da)}{3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},$$

$$\frac{(1+a)(1+4a+3a^2+a^3+Da)}{2(3+a(3+a))}+\frac{\varepsilon}{2(a^2+a-1)(3+a(3+a))}$$

$$\Big(2a(1+a)(3+a(3+a))c_0-d_1-a(4+a(3+a))d_1-d_1Da\Big),$$

(15)

$$\frac{\varepsilon(1+4a+3a^2+a^3+Da)}{2(3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}}\Bigg)+\mathcal{O}(\varepsilon^2)$$

emerging from $\boldsymbol{p}$. *These orbits are unstable if* $d_1>0$ *or if the eigenvalues* (34) *are non-zero real numbers.*

(iv) *Consider system* (1) *where*

(16) $$b=\varepsilon^2 b_1,\quad c=c_-+\varepsilon c_1\quad\text{and}\quad d=1+a+\varepsilon d_1.$$

If $b_1d_1\neq 0$, $a\notin\left\{-2,\,-1,\,0,\,\frac{-1\pm\sqrt{5}}{2},\,\frac{-3\pm\sqrt{5}}{2}\right\}$ *and* $\omega_-\in\mathbb{R}^*$, *then there exists* $\varepsilon_1>0$ *such that when* $|\varepsilon|<\varepsilon_1$ *the hyperchaotic Lorenz system* (1) *has a periodic solution* $\big(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon),w(t,\varepsilon)\big)=$

$$\Bigg(\varepsilon\frac{2+a}{a(1+a)(3+a(3+a))\omega_-}\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20$$

$$+a(7+a)))))-D_a)\big)^{\frac{1}{2}}|a|\sin(\omega_-t),\ \frac{\varepsilon}{\sqrt{2+a}(3+a(3+a))|a|(a+a^2)\omega_-}$$

$$+\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20a(7+a)))))-D_a)\big)^{\frac{1}{2}}\big(\omega_-|a|\cos(\omega_-t)$$

$$-a\sin(\omega_-t)\big),\ \frac{(a+1)(a^3+3a^2+4a-D_a+1)}{2(a^2+3a+3)}+\frac{\varepsilon(a+1)}{2(3+a(3+a))(a^2+a-1)}$$

$$\big(2a(3+a(3+a))c_1-(5+a(8+a(5+a))d_1-d_1D_a)\big),\ \frac{\varepsilon}{\sqrt{2+a}(3+a(3+a))|a|\omega_-}$$

$$\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20+a(7+a)))))-D_a)\big)^{\frac{1}{2}}$$

(17)

$$\big(\cos(\omega_-t)+a(1+a)\sin(\omega_-t)\big)\Big)+\mathcal{O}(\varepsilon^2)$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$. *Furthermore, there is a convenient choice of* ε *such that for either* $\varepsilon\in(-\varepsilon_1,\,0)$ *or* $\varepsilon\in(0,\,\varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon),w(t,\varepsilon)\big)_\pm=$

$$\Bigg(\pm\varepsilon\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}},\ \pm\varepsilon\frac{a(1+a)(2+a-1+D_a)}{2a(a^2+3a+3)}\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}}$$

$$\frac{(1+a)(1+a(4+a(3+a))-D_a)}{2(a^2+3a+3)}+\frac{\varepsilon}{2(a^3+a^2-1)(a^2+3a+3)}$$

$$\Big(2a(1+a)(a^2+3a+3)c_0 - d_1 - a(4+a(3+a))d_1 + d_1 D_a\Big),$$

(18)

$$\pm\varepsilon\frac{1+a(4+a(3+a))-D_a}{2(a^2+3a+3)}\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}}\Bigg) + \mathcal{O}(\varepsilon^2)$$

emerging from $\boldsymbol{p}$.

Theorem 2 is proved in Section 4.

In statements (iii) and (iv) Theorems 2 we do not provide the type of linear stability for the solutions (13), (17) and (18) because the expressions of the eigenvalues are huge.

3. The averaging theory for periodic orbits

The averaging theory is a classical and well known tool for studying the behaviour of the dynamics of nonlinear dynamical systems, and in particular their periodic orbits. The method of averaging was originated in the works of Lagrange and Laplace in 18th century, although its formal proof only came out in 1928 by Fatou [8]. Before the Fatou's work there was only intuitive justifications for the method. After the formalization of the theory important contributions were made to averaging by Krylov and Bogoliubov [3] in the 1930s and Bogoliubov [2] in 1945. The first order averaging theory of periodic orbits can be found in [26], see also [11]. The small part of the averaging theory that we shall need for proving our results can be summarized as follows.

We consider the initial Value problem

$$\dot{\mathbf{x}} = \varepsilon\mathbf{F}_1(t,\mathbf{x}) + \varepsilon^2\mathbf{F}_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{19}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0 \tag{20}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\boldsymbol{x_0}$ in some open Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume $\mathbf{F}_1$, $\mathbf{F}_2$ are T–periodic in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T}\int_0^T \mathbf{F}_1(t,\mathbf{y})dt. \tag{21}$$

Theorem 3. *Assume that* $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{D_x F}_1$, $\mathbf{D_{xx} F}_1$ *and* $\mathbf{D_x F}_2$ *are continuous and bounded by a constant* M *independent of* ε *in* $[0,\infty)\times\Omega\times[-\varepsilon_0,\varepsilon_0]$, *and that* $y(t) \in \Omega$ *for* $t \in [0, 1/|\varepsilon|]$. *Then the following statements holds:*

(*a*) *For* $t \in [0, 1/|\varepsilon|]$ *we have* $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ *as* $\varepsilon \to 0$.

(*b*) *If* $\boldsymbol{s}$ *is a singular point of system* (21) *and* $\det D_{\mathbf{y}} g(p) \neq 0$, *then there exists a* T*–periodic solution* $\Phi(t,\varepsilon)$ *for system* (19) *which is close to* $\boldsymbol{s}$ *and such that* $\Phi(0,\varepsilon) - \boldsymbol{s} = \mathcal{O}(\varepsilon)$ *as* $\varepsilon \to 0$.

(*c*) *The stability of the periodic solution* $\Phi(t,\varepsilon)$ *is given by the stability of the singular point.*

For a proof of Theorem 3 and more information on averaging theory see Theorem 11.5 of the book [26], where it is stated on the $\varepsilon \in [0, \varepsilon_0)$ but in fact following the proof the same result works for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ as it is stated here.

Lemma 4. *Let $\Phi(t,\varepsilon)$ the periodic solution given by the statement (b) of Theorem 3, then as $\varepsilon \to 0$ we have*

$$\Phi(t,\varepsilon) = \boldsymbol{s} + \mathcal{O}(\varepsilon) \tag{22}$$

Proof. Consider

$$\Phi(t,\varepsilon) = \Phi(0,\varepsilon) + \varepsilon \int_0^t \Big(\mathbf{F}_1(s,\Phi(s,\varepsilon)) + \varepsilon \mathbf{F}_2(s,\Phi(s,\varepsilon),\varepsilon)\Big) ds \tag{23}$$

the periodic solution given in (b). Assuming $t \in [0,T]$ we have by hypothesis that

$$\left| \int_0^t \Big(\mathbf{F}_1(s,\Phi(s,\varepsilon)) + \varepsilon \mathbf{F}_2(s,\Phi(s,\varepsilon),\varepsilon)\Big) ds \right| \leq (1+|\varepsilon|)MT,$$

so as $\varepsilon \to 0$ we can say that $\Phi(t,\varepsilon) = \Phi(0,\varepsilon) + \mathcal{O}(\varepsilon)$. Thus using also that $\Phi(0,\varepsilon) - \boldsymbol{s} = \mathcal{O}(\varepsilon)$ we obtain the equation (22). □

4. Proofs

In this section we prove our results.

Proof of Proposition 1. First we assume that $b = 0$. The characteristic polynomial $P(\lambda)$ of the linear part of the differential system (1) at the equilibrium point $(0,0,z_0,0)$ is

$$\lambda^4 + (a-d+1)\lambda^3 + (-ca - da + z_0 a + a - d + z_0)\lambda^2 + (-ad + acd - az_0 d + z_0)\lambda.$$

Clearly an equilibrium point is a zero–Hopf equilibrium if and only if $P(\lambda) = \lambda^2\left(\lambda^2+\omega^2\right)$ with $\omega > 0$. Hence solving the equation $P(\lambda) = \lambda^2\left(\lambda^2+\omega^2\right)$, with respect to the parameters a, b, c, d and ω, we get in only two real solutions:

$$S_1: \quad \omega = \sqrt{c-1}, \qquad z_0 = 0, \qquad d = 0, \qquad a = -1;$$

$$S_2: \quad \omega = \sqrt{\frac{(2+a)z_0}{1+a} - (1+a)^2}, \qquad z_0 = \frac{(a^2+a)(c-1)}{a^2+a-1}, \qquad d = a+1.$$

The solution S_1 says when the equilibrium point located at the origin is zero–Hopf, proving statement (a), and it is easy to check that the solution S_2 corresponds to the equilibrium $\boldsymbol{p}$.

Now we shall provide necessary and sufficient conditions under which either $\boldsymbol{p}_+$ if $b \neq 0$, or $\boldsymbol{p}$ if $b = 0$, is a zero–Hopf equilibrium point. The Jacobian matrix of system (1) evaluated at $\boldsymbol{p}_+$ is

$$A = \begin{pmatrix} -a & a & 0 & 1 \\ \dfrac{ad-c}{ad-1} & -1 & \dfrac{\sqrt{a\,b\,d\,(1-c)}}{\sqrt{c-ad}} & 0 \\ \dfrac{\sqrt{a\,b\,d\,(1-c)\,(c-ad)}}{ad-1} & \dfrac{\sqrt{a\,b\,d\,(1-c)}}{-\sqrt{c-ad}} & -b & 0 \\ \dfrac{ad(1-c)}{ad-1} & 0 & \dfrac{\sqrt{a\,b\,d\,(1-c)\,(c-ad)}}{\sqrt{c-ad}} & d \end{pmatrix}$$

and its characteristic polynomial is $P(\lambda) = \lambda^4 + \sigma_3\lambda^3 + \sigma_2\lambda^2 + \sigma_1\lambda + \sigma_0$ with

$$\sigma_0 = -\,2abd(c-1),$$

$$\sigma_1 = \frac{-a^2d^2 - 2a^3d^2 - a^3d^3 + ac - dc + adc + a^2dc + 2ad^2c + 3a^2d^2c + 2a^3d^2c}{(ad-1)(ad-c)}$$

$$
\begin{aligned}
&+\frac{-a^2d^3c-ac^2-2adc^2-a^2dc^2}{(ad-1)(ad-c)},\\
\sigma_2 =&\frac{ad^2-a^2d-a^2bd+abd^2+a^3bd^2-a^2d^3-a^3d^3-a^2bd^3+ac+bc+abc-dc}{(ad-1)(ad-c)}\\
&+\frac{a^2dc-bdc-2abdc-a^2bdc+ad^2c+2a^2d^2c+abd^2c+a^2bd^2c-ac^2-adc^2}{(ad-1)(ad-c)},\\
\sigma_3 =&1+a+b-d.
\end{aligned}
$$

The expressions for the matrix A and for its characteristic polynomial also work for the equilibrium $\boldsymbol{p}$ taking $b=0$.

Forcing that $P(\lambda)=\lambda^2\left(\lambda^2+\omega^2\right)$, i.e. we must solve the following system: $\sigma_3=0$, $\sigma_2=\omega^2$, $\sigma_1=0$ and $\sigma_0=0$. Obtaining the following three real solutions:

$$
\begin{aligned}
S^1 :&\omega=\omega_0,\ b=0 \text{ and } a=d-1;\\
S^2 :&\omega=\omega_-, b=0,\ d=1+a \text{ and } c=c_-;\\
S^3 :&\omega=\omega_+,\ b=0,\ d=1+a \text{ and } c=c_+;
\end{aligned}
$$

where $c_\pm$ are defined in the statement of Proposition 1, ω_0 in (2) and $\omega_\pm$ in (3) and (4).

The solution S^1 says that $\boldsymbol{p}$ is a zero–Hopf equilibrium if condition $(b.1)$ holds. While the solutions S^2 and S^3 correspond to the fact that $\boldsymbol{p}$ is a zero–Hopf equilibrium when conditions $(b.2+)$ and $(b.2-)$ hold. This completes the proof of statement (b).

Since in the three solutions we have $b=0$ it follows that the equilibrium $\boldsymbol{p}_+$ never can be a zero–Hopf equilibrium, proving statement (c) and consequently proving Proposition 1.

Now we are going to proof the results about the bifurcation of periodic orbits from zero–Hopf equilibrium points. All the next proofs will follow some standards steps. First let $\boldsymbol{p}$ be a zero–Hopf equilibrium point of system (1), then we translate it to the origin of coordinates using the change of variables $(x,y,z,w)=(\overline{x},\overline{y},\overline{z},\overline{w})+\boldsymbol{p}$, and we re-scale the system doing $(\overline{x},\overline{y},\overline{z},\overline{w})=\varepsilon(X,\,Y,\,Z,\,W)$. In these new variables $(X,\,Y,\,Z,\,W)$ the linear part of the system has eigenvalues 0, 0, $i\omega$ and $-i\omega$, so in the systems here studied there exist a linear change of variables,

$$
\begin{pmatrix} X\\ Y\\ Z\\ W\end{pmatrix} = \mathrm{A}\begin{pmatrix} \widetilde{x}\\ \widetilde{y}\\ \widetilde{z}\\ \widetilde{w}\end{pmatrix},
$$

that transform the linear part of the differential system into the matrix

$$
\mathrm{J}=\begin{pmatrix} 0 & -\omega & 0 & 0\\ \omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},
$$

in the variables $\mathbf{x}=(\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{w})$ the system becomes $\dot{\mathbf{x}}=\mathrm{J}\,\mathbf{x}+\varepsilon G_1(t,\mathbf{x})+\varepsilon^2G_2(t,\mathbf{x})$. Using generalized cylindrical coordinates $(r,\,\theta,\,z,\,w)$ where $\widetilde{x}=r\cos\theta$, $\widetilde{y}=r\sin\theta$ with $r>0$, $\widetilde{z}=z$ and $\widetilde{w}=w$, we obtain a new system

$$
\dot{r} =\varepsilon\big(H_1(t,\,(r,\,z,\,w),\,\varepsilon)\big),
$$

$$
\begin{aligned}
\dot{\theta} =& \omega + \varepsilon\big(H_2(t,\, (r,\, z,\, w),\, \varepsilon)\big),\\
\dot{z} =& \varepsilon\big(H_3(t,\, (r,\, z,\, w),\, \varepsilon)\big),\\
\dot{w} =& \varepsilon\big(H_4(t,\, (r,\, z,\, w),\, \varepsilon)\big),
\end{aligned} \tag{24}
$$

where the expressions for the functions $H_i(t,\, (r,\, z,\, w),\, \varepsilon)$ with $i = 1, 2, 3, 4,$ are given in (28), (38), (39) and (40). Taking θ as the new independent variable we finally get the system

$$
\begin{aligned}
\frac{dr}{d\theta} =& \varepsilon F_{11}(\theta, \mathbf{y}) + \varepsilon^2 F_{21}(\theta, \mathbf{y}, \varepsilon),\\
\frac{dz}{d\theta} =& \varepsilon F_{12}(\theta, \mathbf{y}) + \varepsilon^2 F_{22}(\theta, \mathbf{y}, \varepsilon),\\
\frac{dw}{d\theta} =& \varepsilon F_{13}(\theta, \mathbf{y}) + \varepsilon^2 F_{23}(\theta, \mathbf{y}, \varepsilon),
\end{aligned} \tag{25}
$$

where

$$
\mathbf{F}_1(\theta, \mathbf{y}) = \big(F_{11}(\theta, \mathbf{y}),\, F_{12}(\theta, \mathbf{y}),\, F_{13}(\theta, \mathbf{y})\big)
$$

and

$$
\mathbf{F}_2(\theta, \mathbf{y}, \varepsilon) = \big(F_{21}(\theta, \mathbf{y}, \varepsilon),\, F_{22}(\theta, \mathbf{y}, \varepsilon),\, F_{23}(\theta, \mathbf{y}, \varepsilon)\big)
$$

are 2π–periodic in θ, with $\mathbf{y} = (r, z, w)$. Then we can apply Theorem 3 to the differential system (25). If $\mathbf{s} = (r_0,\, z_0,\, w_0)$ is a singular point of the averaged system (20) corresponding to system (25) by Theorem 3 and Lemma 4 there is a periodic solution $(r(\theta, \varepsilon),\, z(\theta, \varepsilon),\, w(\theta, \varepsilon)) = (r_0,\, z_0,\, w_0) + \mathcal{O}(\varepsilon)$ for system (25).

Using the expression of $\theta = \omega t + \mathcal{O}(\varepsilon)$ obtained from (24) and going back to the cylindrical coordinates we have the periodic solution

$$
(r(t, \varepsilon),\, \theta(t, \varepsilon),\, z(t, \varepsilon),\, w(t, \varepsilon)) = (r_0,\, \omega t,\, z_0,\, w_0) + \mathcal{O}(\varepsilon)
$$

of system (24). This periodic solution becomes the periodic solution for the system $(\dot{\tilde{x}}, \dot{\tilde{y}}, \dot{\tilde{z}}, \dot{\tilde{w}})$. So for the system $(\dot{X}, \dot{Y}, \dot{Z}, \dot{W})$ we have the periodic solution

$$
(X(t, \varepsilon),\, Y(t, \varepsilon),\, Z(t, \varepsilon),\, W(t, \varepsilon)) = A \begin{pmatrix} r_0 \cos(\omega t) \\ r_0 \sin(\omega t) \\ z_0 \\ w_0 \end{pmatrix} + \mathcal{O}(\varepsilon)
$$

Hence for the system $(\dot{\overline{x}},\, \dot{\overline{y}},\, \dot{\overline{z}},\, \dot{\overline{w}})$ get the periodic solution

$$
(\overline{x}(t, \varepsilon),\, \overline{y}(t, \varepsilon),\, \overline{z}(t, \varepsilon),\, \overline{w}(t, \varepsilon)) = \varepsilon \mathrm{A} \begin{pmatrix} r_0 \cos(\omega t) \\ r_0 \sin(\omega t) \\ z_0 \\ w_0 \end{pmatrix} + \mathcal{O}(\varepsilon^2)
$$

Finally for the system $(\dot{x},\, \dot{y},\, \dot{z},\, \dot{w})$ we obtain the periodic solution

$$
(x(t, \varepsilon),\, y(t, \varepsilon),\, z(t, \varepsilon),\, w(t, \varepsilon)) = p + \varepsilon \mathrm{A} \begin{pmatrix} r_0 \cos(\omega t) \\ r_0 \sin(\omega t) \\ z_0 \\ w_0 \end{pmatrix} + \mathcal{O}(\varepsilon^2). \tag{26}
$$

the equation (26) is used to write the periodic solutions founded in this paper.

Proof. of statement (i) of Theorem 2. First we assume the condition (5) and the new scale $(x, y, z, w) = \varepsilon\,(\overline{x}, \overline{y}, \overline{z}, \overline{w})$ to system (1), obtaining

$$\begin{aligned}
\dot{\overline{x}} &= (-1+\varepsilon a_1)\,(\overline{y}-\overline{x})+\overline{w},\\
\dot{\overline{y}} &= (1+c_0^2)\overline{x}-\overline{y}-\overline{x}\,\overline{z},\\
\dot{\overline{z}} &= -\,\varepsilon b_1\overline{z}+\varepsilon\overline{x}\,\overline{y},\\
\dot{\overline{w}} &= \varepsilon d_1\overline{w}-\varepsilon\overline{x}\,\overline{z}.
\end{aligned} \tag{27}$$

We now proceed the linear change of variables

$$\begin{aligned}
\overline{x} &= \frac{\widetilde{z}}{c_0^2}+\widetilde{y},\\
\overline{y} &= \widetilde{z}\left(\frac{1+c_0^2}{c_0^2}\right)-c_0\widetilde{x}+\widetilde{y},\\
\overline{z} &= \widetilde{w},\\
\overline{w} &= \widetilde{z},
\end{aligned}$$

in order to write the real Jordan normal form for the unperturbed part of system (27) in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -c_0 & 0 & 0\\ c_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

thus, omitting the tilde, we obtain the new system

$$\begin{aligned}
\dot{x} &= -\,c_0 y+\varepsilon\left(\frac{z(a_1+d_1)}{c_0}-a_1 x\right),\\
\dot{y} &= c_0 x+\varepsilon\left(a_1(z-c_0x)+\frac{c_0^2(wy-d_1z)+wz}{c_0^4}\right),\\
\dot{z} &= \varepsilon\left(d_1 z-w\left(\frac{z}{c_0^2}+y\right)\right),\\
\dot{w} &= \varepsilon\left(\left(\frac{z}{c_0^2}+y\right)\left(\frac{z}{c_0^2}-c_0x+y+z\right)-b_1w\right).
\end{aligned}$$

Finally we use the generalized cylindrical coordinates to write the previous system in the form (19),

$$\begin{aligned}
\dot{r} &= \varepsilon\Big(\frac{z}{c_0^4}\big(c_0^3(a_1+d_1)\cos\theta+(a_1c_0^4-c_0^2d_1+w)\sin\theta\big)+\\
&\qquad r\Big(\frac{w\sin\theta^2}{c_0^2}-a_1\cos\theta^2-a_1c_0\cos\theta\sin\theta\Big)\Big),\\
\dot{\theta} &= c_0+\varepsilon\big(\cos\theta\big((a_1c_0^4-c_0^2d_1+w)z+c_0^2r(a_1c_0^2+w)\sin\theta\big)\\
&\qquad -a_1c_0^5r\cos\theta^2-c_0^3(a_1+d_1)z\sin\theta\big),\\
\dot{z} &= \frac{\varepsilon}{c_0^2}\big(c_0^2d_1z-wz-c_0^2rw\sin\theta\big),\\
\dot{w} &= \varepsilon\Big(-b1w+\Big(\frac{z}{c_0^2}+r\sin\theta\Big)\Big(z+\frac{z}{c_0^2}-c_0r\cos(\theta+r\sin\theta)\Big)\Big).
\end{aligned} \tag{28}$$

We also take θ as a new independent variable, obtaining the system

$$\begin{aligned}\frac{dr}{d\theta} =&\varepsilon\left(\frac{a_1c_0^4r\cos^2(\theta)-\sin\theta\left(z\left(a_1c_0^4-c_0^2d_1+w\right)+c_0^2rw\sin(\theta)\right)}{-c_0^5}\right.\\&+\left.\frac{c_0^3\cos(\theta)\left(a_1c_0^2r\sin\theta+z(-a_1-d_1)\right)}{-c_0^5}\right)+\varepsilon^2F_{21}(\theta,\mathbf{y},\varepsilon),\\ \frac{dz}{d\theta} =&\varepsilon\left(-\frac{wz}{c_0^3}+\frac{d_1z}{c_0}-\frac{rw\sin\theta}{c_0}\right)+\varepsilon^2F_{22}(\theta,\mathbf{y},\varepsilon),\\ \frac{dw}{d\theta} =&\varepsilon\left(-\frac{b_1w}{c_0}+\frac{z^2}{c_0^5}+\frac{2rz\sin\theta}{c_0^3}+\frac{z^2}{c_0^3}-\frac{rz\cos\theta}{c_0^2}+\frac{r^2\sin^2(\theta)}{c_0}\right.\\&+\left.\frac{rz\sin\theta}{c_0}-r^2\sin\theta\cos\theta\right)+\varepsilon^2F_{23}(\theta,\mathbf{y},\varepsilon),\end{aligned} \tag{29}$$

where $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon)=\left(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\right)$ is a 2π–periodic function in θ and $\mathbf{y}=(r,z,w)$.

To study the periodic orbits of system (29) we compute the average system (21) of Theorem 3 corresponding to this system

$$g(\mathbf{y})=\left(\frac{r\left(w-a_1c_0^2\right)}{2c_0^3},\ \frac{z\left(c_0^2d_1-w\right)}{c_0^3},\ \frac{r^2-2b_1w}{2c_0}+\frac{\left(c_0^2+1\right)z^2}{c_0^5}\right)$$

solving the non-linear system $g(\mathbf{y})=0$, we have

$$\begin{aligned}\mathbf{s}_0&=\left(c_0\sqrt{2a_1b_1},\ 0,\ a_1c_0^2\right),\\ \mathbf{s}_1^0&=\left(0,\ \frac{\sqrt{b_1d_1}c_0^3}{\sqrt{1+c_0^2}},\ \sqrt{1+c_0^2}\right),\\ \mathbf{s}_2^0&=\left(0,\ -\frac{\sqrt{b_1d_1}c_0^3}{\sqrt{1+c_0^2}},\ \sqrt{1+c_0^2}\right).\end{aligned}$$

The solution $\mathbf{s}_0$ has the Jacobian

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_0)\right)=\frac{a_1b_1(a_1-d_1)}{c_0^3},$$

which is non-zero under conditions (a), then by Theorem 3 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_0$ such that $\mathbf{\Phi}(0,\varepsilon)=\mathbf{s}_0+\mathcal{O}(\varepsilon)$. Which, using Lemma 4, provide the periodic solution (6) of (1).

We also notice that the eigenvalues of $\mathbf{s}_0$ are $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}$ and $\dfrac{d_1-a_1}{c_0}$, we use Theorem 3 (c) to study the stability of the periodic solution (6). Here we divide the analysis in two cases:

(1) When $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}\in\mathbb{R}$: In this case the solution is stable if $d_1<a_1$, $b_1>0$ and $\dfrac{-b_1}{4}\le a_1<0$.

(2) When $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}\in\mathbb{C}$: In this case the solution is stable if $d_1<a_1$, $b_1>0$ and $a_1<\frac{-b_1}{4}$.

In summary this periodic solution is stable if $b_1 > 0$, $a_1 < 0$ and $d_1 < a_1$.

The solutions $\mathbf{s}_1^0$ and $\mathbf{s}_2^0$ are such that $\det\left(\dfrac{\partial g}{\partial y}(\mathbf{s}_i^0)\right) = \dfrac{b_1 d_1(d_1-a_1)}{c_0^3}$, for $i = 1, 2$. By hypothesis (b) they also provide two additional periodic solution for system (1) if $r = \varepsilon r_1 + \mathcal{O}(\varepsilon^2) > 0$, this is possible restricting ε to one of the half intervals $\varepsilon \in (-\varepsilon_1, 0)$ or $\varepsilon \in (0, \varepsilon_1)$.

We notice that $\mathbf{s}_1^0$ and $\mathbf{s}_2^0$ has the same eigenvalues $\dfrac{b_1 \pm \sqrt{b_1(b_1-8d_1)}}{-2c_0}$ and $\dfrac{d_1-a_1}{2c_0}$. Thus, following the previous analysis, we can say by Theorem 3 (c) that the periodic solution (7) is stable if $b_1 > 0$, $d_1 > 0$ and $a_1 > d_1$. □

Proof. of statement (ii) of Theorem 2. Assuming the conditions (8), system (1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0, 0, \frac{(d-1)d(c-1)}{(d-1)d-1}, 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First, we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon\,(X, Y, Z, W)$. With these changes of variables the hyperchaotic Lorenz system (1) becomes

$$\begin{aligned}
\dot{X} &= (1-d)X + (d-1)Y + W + \varepsilon a_1(Y - X),\\
\dot{Y} &= \frac{-d^2+d+c}{-d^2+d+1}X - Y - \varepsilon XZ,\\
\dot{Z} &= \varepsilon\left(\frac{b_1 d(d(-c)+d+c-1)}{(d-1)d-1} + XY\right) - \varepsilon^2 b_1 Z,\\
\dot{W} &= d\left(\frac{(d(-c)+d+c-1)}{(d-1)d-1}X + W\right) - \varepsilon XZ.
\end{aligned} \tag{30}$$

We no proceed the linear change of variables

$$\begin{aligned}
X =& \frac{(d^2-1)}{\omega_0}\widetilde{y} + \widetilde{z},\\
Y =& -\frac{1}{\omega_0}\widetilde{y} - \frac{\left(c-d^2+d\right)}{(d-1)d-1}\widetilde{z} + \widetilde{x},\\
Z =& \widetilde{w},\\
W =& \frac{(d^2(d-1))}{\omega_0}\widetilde{y} + \frac{(c-1)(d-1)}{(d-1)d-1}\widetilde{z} + (d-1)d\widetilde{x},
\end{aligned}$$

in order to write the real Jordan normal form for the unperturbed part of system (30) in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_0 & 0 & 0\\ \omega_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

thus, omitting the tilde, we obtain a system in the form $\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x})$ where $\mathbf{x}=(x, y, z, w)$,

$$G_1(t,\mathbf{x}) = \begin{pmatrix} G_{11}(t,\mathbf{x}) \\ G_{12}(t,\mathbf{x}) \\ G_{13}(t,\mathbf{x}) \\ G_{14}(t,\mathbf{x}) \end{pmatrix} \qquad \text{and} \qquad G_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ -b_1 z \\ 0 \end{pmatrix},$$

with

$$\begin{aligned}
G_{11}(t,\mathbf{x}) =& \frac{\left(a_1(c-1)(d-1)z - a_1\left((d-2)d^2+1\right)x + d\left(-d^2+d+1\right)wz\right)}{((d-1)d-1)\left(d^2-1\right)} \\
&+ \frac{\left(d^4-2d^3+d\right)y(a_1 d-(d+1)w)}{((d-1)d-1)\left(d^2-1\right)\omega_0}, \\
G_{12}(t,\mathbf{x}) =& \frac{\left(-d^2+d+1\right)^2 W\left(\left(d^2-1\right)Y+\omega_0 Z\right)}{\left(d^2-1\right)\left(-cd^2+c+(d-1)d^3-1\right)} \\
&- \frac{a_1}{\left(-d^2+d+1\right)^2\left(d^2-1\right)\left(-cd^2+c+(d-1)d^3-1\right)} \\
&\left((d-2)d^2+1\right)\left(-c(d+1)+(d-1)d^2+1\right)(\omega_0((c-1)Z \\
&+\left(-d^2+d+1\right)X\right)+((d-1)d-1)d^2 Y), \\
G_{13}(t,\mathbf{x}) =& \frac{a_1 d\omega_0\left((c-1)Z+\left(-d^2+d+1\right)X\right)+a_1((d-1)d-1)d^3 Y}{\omega_0\left((d-1)\left(cd+c-d^3\right)+1\right)} \\
&+ \frac{\left(-d^2+d+1\right)^2 W\left(\left(d^2-1\right)Y+\omega_0 Z\right)}{\omega_0\left((d-1)\left(cd+c-d^3\right)+1\right)}, \\
G_{14}(t,\mathbf{x}) =& \frac{-(b_1(c-1)(d-1)d+Z(cZ-(d-1)d(X+Z))+XZ)}{((d-1)d-1)} \\
&+ \frac{Y^2\left(-\left(c\left(d^2-1\right)+d\left(-d\left(\omega_0^2+2\right)+\omega_0^2+1\right)+\omega_0^2+2\right)\right)}{((d-1)d-1)\omega_0^2} \\
&+ \frac{Y\left(\left(d^2-1\right)X-\left(\omega_0^2+2\right)Z\right)}{\omega_0}.
\end{aligned}$$

We now use generalized cylindrical coordinates obtaining system (38). Taking θ as the new independent variable we have

$$\begin{aligned}
\frac{dr}{d\theta} =& \frac{\varepsilon}{(d(d-1)-1)^2\omega_0}\Big(\frac{z}{d^2-1}\big((d(d-1)-1)(a_1(c-1)(d-1)+d(d \\
&-d^2-1)w)\omega_0\cos\theta+(-a_1(c-1)(d-1)(c+cd+d^2-d^3-1) \\
&-(d(d-1)-1)^3 w)\sin\theta\big)+\big((d(d-1)-1)^2 r\omega_0(a_1(d(d \\
&-1)-1)\omega_0^2\cos^2\theta+\varepsilon^2 F_{21}(\theta,\mathbf{y},\varepsilon), \\
\frac{dz}{d\theta} =& \frac{\varepsilon}{(d(d-1)-1)\omega_0}\Big((a_1(c-1)d+(1+d-d^2)w)z\omega_0+(d(d-1)-1)r
\end{aligned}$$

(31)

$$\begin{aligned}
&(-a_1 d\omega_0\cos\theta+(a_1 d^3+w+d(1+(d-2)d(1+d))w)\sin\theta)\Big)+\varepsilon^2 F_{22}(\theta,\mathbf{y},\varepsilon), \\
\frac{dw}{d\theta} =& \frac{\varepsilon}{\omega_0^2}\left(\frac{-b1(c-1)(d-1)d\omega_0}{d(d-1)-1}+\left(z\omega_0+(d^2-1)r\sin\theta\right)\left(\frac{-(c+d-d^2)z}{d(d-1)-1}\right.\right.
\end{aligned}$$

$$+ \ r\cos\theta - \frac{r\sin\theta}{\omega_0}\Big)\Big) + \varepsilon^2 F_{23}(\theta,\mathbf{y},\varepsilon),$$

where $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon) = \big(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ is 2π–periodic in θ and $\mathbf{y} = (r,z,w)$.

We are going to applying Theorem 3, thus we compute the average system $g(\mathbf{y}) = (g_1(\mathbf{y}),\ g_2(\mathbf{y}),\ g_3(\mathbf{y}))$ corresponding to system (31) where

$$\begin{aligned}
g_1(\mathbf{y}) &= -\frac{r(a_1(1-d+d^3-d^4+c(d^2-d-1))+(1+d-d^2)w)}{2(d(d-1)-1)\omega_0^3},\\
g_2(\mathbf{y}) &= \frac{(a_1(c-1)d+(1+d-d^2)^2w)z}{(d(d-1)-1)\omega_0^3},\\
g_3(\mathbf{y}) &= \frac{-2b_1(c-1)(d-1)d-2(c+d-d^2)z^2}{2(d(d-1)-1)\omega_0}\\
&\quad + \frac{(d(d-1)-1)^2(d^2-1)r^2}{2(d(d-1)-1)(d^3(d-1)-cd+c-1)\omega_0},
\end{aligned}$$

solving the non-linear system $g(\mathbf{y}) = 0$, we have

$$\begin{aligned}
\mathbf{s}_1 &= \left(\sqrt{\frac{2b_1d(d-cd+c-1)}{(d^2-1)(d(d-1)-1)}},\ 0,\ \frac{a_1(d-1+(d-1)d^3-c(d^2+d-1))}{(d(d-1)-1)}\right)\\
\mathbf{s}_1^1 &= \left(0,\ \frac{\sqrt{(1-c)(d-1)db_1}}{\sqrt{c+d-d^2}},\ -\frac{a_1(c-1)d}{(1+d-d^2)^2}\right),\\
\mathbf{s}_2^1 &= \left(0,\ -\frac{\sqrt{(1-c)(d-1)db_1}}{\sqrt{c+d-d^2}},\ -\frac{a_1(c-1)d}{(1+d-d^2)^2}\right).
\end{aligned}$$

The solution $\mathbf{s}_1$ has the Jacobian

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_1)\right) = -\frac{a_1b_1(c-1)(d-1)d}{\omega_0^5},$$

which is non-zero, then by Theorem 3 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_1$ such that $\mathbf{\Phi}(0,\varepsilon) = \mathbf{s}_1 + \mathcal{O}(\varepsilon)$. Which, using Lemma 4, provide the periodic solution (9) of (1).

We also notice that the eigenvalues of $\mathbf{s}_1$ are $\pm\dfrac{\sqrt{-b_1(c-1)(d-1)d}}{\omega_0^2}$ and $-\dfrac{a_1}{\omega_0}$. By Theorem 3 (c) the periodic solution (9) is unstable if $a_1 < 0$ or $b_1(c-1)(d-1)d < 0$.

The solutions $\mathbf{s}_1^1$ and $\mathbf{s}_2^1$ are such that $\det\left(\dfrac{\partial g}{\partial y}(\mathbf{s}_i^1)\right) = \dfrac{a_1b_1(c-1)(d-1)d}{\omega_0^5}$, for $i = 1,2$. They also provide two additional periodic solution for system (1) if $r = \varepsilon r_1 + \mathcal{O}(\varepsilon^2) > 0$, this is possible restricting ε to one of the half intervals $\varepsilon \in (-\varepsilon_1, 0)$ or $\varepsilon \in (0, \varepsilon_1)$.

We notice that $\mathbf{s}_1^1$ and $\mathbf{s}_2^1$ has the same eigenvalues $\pm\dfrac{\sqrt{2b_1(c-1)(d-1)d}}{\omega_0^2}$ and $-\dfrac{a_1}{2\omega_0}$. Thus, by Theorem 3 (c), the periodic solutions (10) and (11) are unstable if $a_1 < 0$ or $b_1(c-1)(d-1)d > 0$. □

Proof. of statement (iii) of Theorem 2. Assuming the conditions (12), system (1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0, 0, \frac{(a+1)\left(a^3+3a^2+4a+D_a+1\right)}{2\left(a^2+3a+3\right)}, 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First, we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon\,(X, Y, Z, W)$. With these changes of variables the system (1) becomes

$$\begin{aligned}
\dot{X} =& a(Y-X)+W,\\
\dot{Y} =& \frac{(-1+a(1+a)(2+a)-D_a)X}{2a(3+a(3+a))} - Y + X(c_1 - Z),\\
\dot{Z} =& \varepsilon\left(XY - \frac{(1+a)b_1(1+a(4+a(3+a))+D_a)}{2(3+a(3+a))}\right) - \varepsilon^2 b_1 Z,\\
\dot{W} =& W + aW - \frac{(1+a)(1+a(4+a(3+a))+D_a)X}{2(3+a(3+a))} + \varepsilon\big(d_1 W - XZ\big).
\end{aligned} \tag{32}$$

We now proceed the linear change of variables

$$\begin{aligned}
X =& \widetilde{z} + \frac{(2+a)\widetilde{y}}{(1+a)\omega_+},\\
Y =& \frac{\widetilde{x}}{a+a^2} + \frac{(a(1+a)(2+a)-D_a-1)\widetilde{z}}{2a(3+a(3+a))} - \frac{\widetilde{y}}{(a+a^2)\omega_+},\\
Z =& \widetilde{w},\\
W =& \widetilde{x} + \frac{(1+a(4+a(3+a))+D_a)\widetilde{z}}{2(3+a(3+a))} + \frac{(1+a)\widetilde{y}}{(a+a^2)\omega_+},
\end{aligned}$$

in order to write the real Jordan normal form for the unperturbed part of system (32) in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_+ & 0 & 0\\ \omega_+ & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, omitting the tilde, we obtain a system in the form $\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x})$ with $\mathbf{x}=(x, y, z, w)$. Following the idea of the previous demonstration we can use cylindrical coordinates, obtaining system (39). In order to put this system in the form (19), we take θ as the new independent variable and then we have

$$\begin{aligned}
\frac{dr}{d\theta} =& -\varepsilon\Big((1+a)(3+a(3+a))d_1\left(\sqrt{2}\omega_+\right)^3 r\cos\theta^2\\
&- \frac{\sin\theta}{3+a(3+a)}\Big(-a(1+a)(3+a(3+a))(4+a(9+a(10+a(5+a))))c_1\\
&+ a(16+a(45+a(59+2a(7+a))))d_1 + (a(1+a)(2+a)(3+a(3+a))c_1\\
&+ d_1)D_a + d_1 + (-1+a+a^2)(3+a(3+a))(4+a(9+a(10+a(5\\
&+ a)) - D_a) - 2D_a)w\Big)z + 2(3+a(3+a))^2
\end{aligned}$$

$$
\begin{aligned}
&\sqrt{\frac{-a(9+a(10+a(5+a))-D_a)}{3+a(3+a)}}r\Big((1+a)(a(2+a)c_1-(1+a)d_1)\\
&-(2+a)(-1+a+a^2)w\Big)\sin\theta\Big)+\varepsilon^2F_{21}(\theta,\mathbf{y},\varepsilon),\\
\frac{dz}{d\theta}=&\varepsilon\Big(\sqrt{2}(3+a(3+a))\varepsilon\big((1+a)\sqrt{2}\omega_+(a(6c_1-4d_1)-d_1(1+D_a)+a^2(12c_1\\
&-3d_1-10w)+a^3(8c_1-d_1-8W)+2a^4(c_1-w)+6w)z+2(3\\
&+a(3+a))r(-(1+a)d_1\sqrt{2}\omega_+\cos\theta-\sqrt{2}((1+a)(d_1+a(-(2+a)c_1\\
&+d_1))+(2+a(a^2+a-1)w)\sin\theta)\big)\Big)+\varepsilon^2F_{22}(\theta,\mathbf{y},\varepsilon), \qquad (33)\\
\frac{dw}{d\theta}=&-\varepsilon\Big(2(1+a)^3\big(ab_1(1+a(16+a(45+a(59+2a(20+a(7+a))))))+D_a\big)\\
&+(1-a(1+a)(a^2+a-1)(3+a(3+a))+D_a+a(5+a(4+a))D_a)z^2\big)\\
&+(3+a(3+a))r\big(-2(1+a)(4+a(9+a(10+a(5+a))-D_a)z\cos\theta\\
&+2(2+a)(3+a(3+a))r\cos(2\theta)+\sqrt{2}(1+a)(a(2+a)(3+a)-3-D_a)\\
&-2(4+D_a))\sqrt{2}\omega_+z\sin\theta+(2+a)(3+a(3+a))r\big(\sqrt{2}\sqrt{2}\omega_+\sin(2\theta)\big)\big)\Big)\Big/\\
&\Bigg(a(1+a)^2(3+a(3+a))(4+a(9+a(10+a(5+a))-D_a)\\
&-2D_a)\sqrt{\frac{-2(4+a(9+a(10+a(5+a))))+2(2+a)D_a}{3+a(3+a)}}\Bigg)\\
&+\varepsilon^2F_{23}(\theta,\mathbf{y},\varepsilon),
\end{aligned}
$$

with $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon)=\big(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ a 2π–periodic functions function in θ.

Now we can apply Theorem 3 and calculate the averaging system $g(\mathbf{y})=(\mathbf{g}_1(\mathbf{y}),\mathbf{g}_2(\mathbf{y}),\mathbf{g}_3(\mathbf{y}))$ of (33) where

$$
\begin{aligned}
\mathbf{g}_1(\mathbf{y})=&\frac{1}{4\big(3+a(3+a)\big)\omega_+^3}\Big((1+a)(2a(3+a(3+a))c_1-5d_1\\
&-a\big(8+a(5+a)\big)d_1+d_1D_a)r-2\big(a^2+a-1\big)\big(3+a(3+a)\big)rw\Big),\\
\mathbf{g}_2(\mathbf{y})=&\frac{-1}{2(3+a(3+a))\omega_+}z\Big(2(a(a+3)+3)(a(a+1)(c_1-w)+w)\\
&-(a(a(a+3)+4)+1)d_1-d_1D_a\Big)\\
\mathbf{g}_3(\mathbf{y})=&-\frac{1}{4\big(3+a(3+a)\big)\omega_+}\Bigg(-2(1+a)b_1\big(1+a(4+a(3+a))+D_a\big)\\
&+\frac{(2+a)\big(3+a(3+a)\big)\big(2(2+D_a)+a(9+a(10+a(5+a))+D_a)\big)r^2}{a(1+a)^5\big(1+a(3+a)\big)}\\
&+\frac{2\big(-1+a(1+a)(2+a)-D_a\big)z^2}{a}\Bigg).
\end{aligned}
$$

The non-linear system $g(\mathbf{y}) = 0$ has the solutions

$$\mathbf{s}_2 = \left(\sqrt{\frac{a(1+a)^3 b_1 \big(1 + a(16 + a(45 + a(59 + 2a(20 + a(7+a))))) + D_a\big)}{(2+a)\big(3 + a(3+a)\big)^2}}, \right.$$
$$\left. 0, \frac{(1+a)\big(2a(3+a(3+a))c_1 - (5 + a(8 + a(5+a)))d_1 + d_1 D_a\big)}{2(a^2+a-1)\big(3+a(3+a)\big)} \right),$$
$$\mathbf{s}_1^2 = \left(0, \frac{\sqrt{(a^3+a^2-1+D_a)\, b}}{\sqrt{2}}, \right.$$
$$\left. \frac{-2a(1+a)(3+a(3+a)c_1 + d_1 + a(4+a(3+a))d_1 + D_a d_1)}{2(a^2+a-1)(3+a(3+a))} \right),$$
$$\mathbf{s}_1^2 = \left(0, -\frac{\sqrt{(a^3+a^2-1+D_a)\, b}}{\sqrt{2}}, \right.$$
$$\left. \frac{-2a(1+a)(3+a(3+a)c_1 + d_1 + a(4+a(3+a))d_1 + D_a d_1)}{2(a^2+a-1)(3+a(3+a))} \right).$$

The solution $\mathbf{s}_2$ has the Jacobian

$$\det\left(\frac{\partial \mathbf{g}}{\partial y}(\mathbf{s}_2)\right) = \frac{(1+a)(a^2+a-1)b_1 d_1 (1 + a(4+a(3+a)) - D_a)}{2(3+a(3+a))\omega_+^5}$$

which is non-zero, then by Theorem 3, there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_2$ such that $\mathbf{\Phi}(0,\varepsilon) = \mathbf{s}_2 + \mathcal{O}(\varepsilon)$. Which, using Lemma 4, provide the periodic solution (13) of (1).

The solutions $\mathbf{s}_1^2$ and $\mathbf{s}_2^2$ are such that

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_i^2)\right) = -\frac{(a^2+a-1)b_1 d_1 (a(1+a)(2+a) - D_a - 1)(D_a + a^2 + a^3 - 1)}{4a(3+3a+a^2)\omega_+^5},$$

for $i = 1, 2$. They also provide two additional periodic solutions for system (1) if $r = \varepsilon r_1 + \mathcal{O}(\varepsilon^2) > 0$, this is possible restricting ε to one of the half intervals $\varepsilon \in (-\varepsilon_1, 0)$ or $\varepsilon \in (0, \varepsilon_1)$.

We notice that $\mathbf{s}_1^2$ and $\mathbf{s}_2^2$ has the same eigenvalues $\dfrac{d_1}{2\omega_+}$ and

$$\frac{\pm 1}{a(3+a(3+a))^3\omega_+^5}\Big(-a^2(a+1)\left(a^2+a-1\right)(a(a+3)+3)^2 b_1\Big(a\left(a^{13}+15a^{12}\right.$$
$$+ 103a^{11} + 428a^{10} + 1202a^9 + 2427a^8 + 3699a^7 + 4487a^6 + 4581a^5 + 4038a^4$$
$$+ 2948a^3 - (a+1)^2(a+2)^2\left(a^2+a+1\right)(a(a+3)+3)(a(a+4)+5)D_a$$
$$\left.+1614a^2 + 573a + 100\right) + D_a + 1\Big)\Big)^{\frac{1}{2}}. \tag{34}$$

Thus, by Theorem 3 (c), the periodic solutions (14) and (15) are unstable if $d_1 > 0$ or if the eigenvalues (34) are non-zero real numbers. □

Proof. of statement (iv) of Theorem 2. Assuming the conditions (16), system (1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0,\ \frac{(a+1)(a^3+3a^2+4a-D_a+1)}{2(a^2+3a+3)},\ 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First, we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x,y,z,w) = (\overline{x},\overline{y},\overline{z},\overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x},\overline{y},\overline{z},\overline{w}) = \varepsilon\,(X,Y,Z,W)$. With these changes of variables the system (1) becomes

$$\begin{aligned}
\dot{X} =& W + a(Y-X),\\
\dot{Y} =& \frac{a(1+a)(2+a)+D_a-1}{2a\big(3+a(3+a)\big)}X - Y + \varepsilon(c_1-Z)X,\\
\dot{Z} =& \varepsilon\left(XY - \frac{(1+a)b_1\big(1+a(4+a(3+a))-D_a\big)}{2\big(3+a(3+a)\big)}\right) - \varepsilon^2 b_1 Z,\\
\dot{W} =& (1+a)W - \frac{(1+a)\,\big(1+a(4+a(3+a))-D_a\big)}{2\big(3+a(3+a)\big)}X + \varepsilon(d_1W - XZ).
\end{aligned} \tag{35}$$

We now proceed the linear change of variables

$$\begin{aligned}
X =& \widetilde{w} + \frac{\sqrt{2}(2+a)}{(1+a)\sqrt{2}\omega_-}\widetilde{y} + \widetilde{z},\\
Y =& \frac{1}{2a(1+a)\big(3+a(3+a)\big)\sqrt{2}\omega_-}\Big(-2\sqrt{2}\big(3+a(3+a)\big)\widetilde{y} + \sqrt{2}\omega_-\big((1+a)(-1\\
&+ a(1+a)(2+a)+D_a)\widetilde{w} + 2\big(3+a(3+a)\big)\widetilde{x} + (1+a)\big(-1+a(1\\
&+ a)(2+a)+D_a\big)\widetilde{z}\Big),\\
Z =& \widetilde{w},\\
W =& \widetilde{x} + \frac{\sqrt{2}}{\sqrt{2}\omega_-}\widetilde{y} + \frac{1}{2\big(3+a(3+a)\big)\sqrt{2}\omega_-}\Big(2\sqrt{2}a\big(3+a(3+a)\big)\widetilde{y} + \big(1+a(4\\
&+ a(3+a))-D_a\big)\sqrt{2}\omega_-(\widetilde{w}+\widetilde{z})\Big),
\end{aligned}$$

in order to write the real Jordan normal form for the unperturbed part of system (35) in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_- & 0 & 0\\ \omega_- & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, omitting the tilde, we obtain a system in the form $\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon G_1(t,\mathbf{x}) + \varepsilon^2 G_2(t,\mathbf{x})$ with $\mathbf{x}$=(x,y,z,w). Using cylindrical coordinates we obtain system (40) and taking θ as the new independent variable, we have

$$\begin{aligned}
\frac{dr}{d\theta} =& -\frac{\varepsilon\sqrt{2}}{(2+a)\big(4+a(9+a(10+a(5+a)))-D_a\big)-2D_a\big)^2}\\
&\Big((1+a)\sqrt{2}\omega_-\big(a\big(3+a(3+a)\big)c_1\big(4+a(9+a(10+a(5+a))-D_a\big)-2Da\big)
\end{aligned}$$

$$
\begin{aligned}
&+ d_1\big(1 + a(16 + a(45 + a(59 + 2a(20 + a(7 + a))))) + D_a\big)\\
&- (1 + a)\big(3 + a(3 + a)\big)\big(4 + a(9 + a(10 + a(10 + a(5 + a)) - D_a)\\
&- 2D_a\big)w\big)z\cos\theta - (1 + a)\big(3 + a(3 + a)\big)^2 d_1\sqrt{2}\omega_-^3\, r\cos\theta^2\\
&- \sin\theta\big(\sqrt{2}(1 + a)\big(- a(1 + a)\big(3 + a(3 + a)\big)\big(4 + a(9 + a(10 + a(5\\
&+ a))))\big)c_1 + d_1 + a\big(16 + a(59 + 2a(20 + a(7 + a))))\big)d_1 + \big(a(1 + a)(2 + a)\\
&\big(3 + a(3 + a)\big)c_1 + d_1\big)D_a + (a^2 + a - 1)\big(3 + a(3 + a)\big)\big(4 + a(9 + a(10\\
&+ a(5 + a)) - D_a)w\big)z + 2\big(3 + a(3 + a)\big)^2\sqrt{2}\omega_-\big)r\big((1 + a)\big(a(2 + a)c_1\\
&- (1 + a)d_1\big) - 2(2 + a)\big(a^2 + a - 1\big)w\big)\sin\theta + 2^{\frac{-1}{2}}\big(3 + a(3 + a)\big)\\
&\big(4 + a(9 + a(10 + a(5 + a)) - D_a) - 2D_a\big)r\big(a\big((2 + a)c_1 + d_1 + ad_1\big)
\end{aligned}
$$

(36)

$$
\begin{aligned}
&- (1 + a)(1 + a(2 + a)w)\big)\sin(2\theta)\Big) + \varepsilon^2 F_{21}(\theta, \mathbf{y}, \varepsilon)\\
\frac{dz}{d\theta} =& \frac{-\varepsilon\sqrt{2}\big(3 + a(3 + a)\big)}{(1 + a)\big(4 + a(9 + a(10 + a(5 + a)) - D_a) - 2D_a\big)^2}\Big((1 + a)\sqrt{2}\omega_-\big(a(6c_1\\
&- 4d_1) - d_1(1 + D_a) + a^2(12c_1 - 3d_1 - 10w) + a^3(8c_1 - d_1 8w) + 2a^4(c_1 - w)\\
&+ 6w\big)z2\big(3 + a(3 + a)\big)r\big((1 + a)\sqrt{2}\omega_-\cos\theta + \sqrt{2}\big((1 + a)(d_1
\end{aligned}
$$

(37)

$$
\begin{aligned}
&+ a(-(2 + a)c_1 + d_1)) + (2 + a)\big(a^2 + a - 1\big)w\big)\sin\theta\big)\Big) + \varepsilon^2 F_{22}(\theta, \mathbf{y}, \varepsilon)\\
\frac{dw}{d\theta} =& \frac{-\varepsilon\sqrt{2}\big(3 + a(3 + a)\big)}{(1 + a)\big(4 + a(9 + a(10 + a(5 + a)) - D_a) - D_a\big)^2}\Big((1 + a)\sqrt{2}\omega_-\big(a(6c_1 - 4d_1)\\
&- (1 + D_a) + a^2(12c_1 - 3d_1 - 10w) + a^3(8c_1 - d_1 - 8w) + 2a^4(c_1 - w)\\
&+ 6w\big)z - 2\big(3 + a(3 + a)\big)r\big((1 + a)d_1\sqrt{2}\omega_-\cos\theta + \sqrt{2}\big((1 + a)(d_1\\
&+ a(-(2 + a)c_1 + d_1)) + (2 + a)(a^2 + a - 1)w\big)\sin\theta\big)\Big) + \varepsilon^2 F_{23}(\theta, \mathbf{y}, \varepsilon)
\end{aligned}
$$

with $\mathbf{F}_2(\theta, \mathbf{y}, \varepsilon) = \big(F_{21}(\theta, \mathbf{y}, \varepsilon),\, F_{22}(\theta, \mathbf{y}, \varepsilon),\, F_{23}(\theta, \mathbf{y}, \varepsilon)\big)$ a 2π–periodic functions function in θ. Now we can apply Theorem 3 and calculate the averaging system $g(\mathbf{y}) = (\mathbf{g}_1(\mathbf{y}), \mathbf{g}_2(\mathbf{y}), \mathbf{g}_3(\mathbf{y}))$ corresponding to (36), where

where

$$
\begin{aligned}
\mathbf{g}_1(\mathbf{y}) =& \frac{-1}{2\big(3 + a(3 + a)\big)\omega_-^3}\Big(- (1 + a)\big(2a\big(3 + a(3 + a)\big)c_1\\
&- 5d_1 - a\big(8 + a(5 + a)\big)d_1 + d1D_a\big)r + 2(a^2 + a - 1)\big(3 + a(3 + a)\big)rw\Big),\\
\mathbf{g}_2(\mathbf{y}) =& \frac{-1}{\big(4 + a(9 + a(10 + a(5 + a)) - D_a) - 2D_a\big)^2}\Big(\big(3 + a(3 + a)\big)2\omega_-\\
&\big((-1 - a(4 + a(3 + a)))d_1 - d_1 D_a + 2\big(3 + a(3 + a)\big)\big(a(1 + a)(c_1 - w)\\
&+ w)\big)z\Big),\\
\mathbf{g}_3(\mathbf{y}) =& \frac{1}{4a\big(3 + a(3 + a)\big)\omega_-}\Big(- 2a(1 + a)b_1\big(1 + a(4 + a(3 + a)) + D_a\big)
\end{aligned}
$$

$$+\frac{(2+a)\big(3+a(3+a)\big)\big(2(2+D_a)+a(9+a(10+a(5+a))+D_a)\big)r^2}{(1+a)^5\big(1+a(3+a)\big)} + 2\big(-1+a(1+a)(2+a)-D_a\big)z^2\Big).$$

The non-linear system $g(\mathbf{y})=0$ has the solutions

$$\mathbf{s}_3=\left(\sqrt{\frac{a(1+a)^3b_1\big(1-D_a+a(16+a(45+a(59+2a(20+a(7+a)))))\big)}{(2+a)\big(3+a(3+a)\big)^2}},\right. \left. 0,\frac{(1+a)\big(2a\big(3+a(3+a)\big)c_1-\big(5+a(8+a(5+a))\big)d_1+d_1D_a\big)}{2(a^2+a-1)\big(3+a(3+a)\big)}\right),$$

$$\mathbf{s}_1^3=\left(0,\frac{2a(1+a)(3+a(3+a))c_1-(1+a(4+a(3+a)))d_1+D_ad_1}{2(a^2+a-1)(3+a(3+a))},\right. \left.\frac{\sqrt{(a^3+a^2-1-D_a)\,b_1}}{\sqrt{2}}\right),$$

$$\mathbf{s}_2^3=\left(0,\frac{2a(1+a)(3+a(3+a))c_1-(1+a(4+a(3+a)))d_1+D_ad_1}{2(a^2+a-1)(3+a(3+a))},\right. \left.-\frac{\sqrt{(a^3+a^2-1-D_a)\,b_1}}{\sqrt{2}}\right).$$

The solution $\mathbf{s}_3$ has the Jacobian

$$\det\left(\frac{\partial \mathbf{g}}{\partial y}(\mathbf{s}_3)\right)=\frac{a}{|a|2(3+a(3+a))^2\omega_-^7}\Big((1+a)(a^2+a-1)b_1d_1(1+a(16+a(45 + a(59+2a(20+a(7+a)))))-D_a)\Big)$$

which is non-zero, then by Theorem 3 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_3$ such that $\mathbf{\Phi}(0,\varepsilon)=\mathbf{s}_3+\mathcal{O}(\varepsilon)$. Which, using Lemma 4, provide the periodic solution (17) of (1).

The solutions $\mathbf{s}_1^3$ and $\mathbf{s}_2^3$ can also provide two additional periodic solution for (1) if $r=\varepsilon r_1+\mathcal{O}(\varepsilon^2)>0$, this is possible restricting ε to one of the half intervals $\varepsilon\in(-\varepsilon_1,0)$ or $\varepsilon\in(0,\varepsilon_1)$.

The solutions $\mathbf{s}_1^3$ and $\mathbf{s}_2^3$ are such that

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_i^3)\right)=\frac{-a}{|a|2(3+a(3+a))^2\omega_-^7}\Big((1+a)(a^2+a-1)b_1d_1(1+a(16+a(45 + a(59+2a(20+a(7+a)))))-D_a)\Big)$$

for $i=1,2$. They also provide two additional periodic solution for system (1) if $r=\varepsilon r_1+\mathcal{O}(\varepsilon^2)>0$, this is possible restricting ε to one of the half intervals $\varepsilon\in(-\varepsilon_1,0)$ or $\varepsilon\in(0,\varepsilon_1)$. $\square$

5. Appendix

Equations to functions defined in (24) according to statement (ii) Theorem 2 where $R_a = (d-1)\left(cd+c-d^3\right)+1$ and $R_b = d^2-d-1$ and.

$$
\begin{aligned}
\dot r =& \varepsilon\Big(-\frac{r\left(a_1(d(2R_a+R_b(R_b+2))+R_b(R_a+R_b+1))+R_b^2w(d+R_b)\right)}{2R_a(d+R_b)} \\
&+\frac{z\cos\theta(a_1(-c(R_b+1)+R_b(d+R_b+2)+R_a+1)-dR_bw)}{R_b(d+R_b)} \\
&+\frac{r\cos(2\theta)\left(a_1(R_b(d(R_b+2)+R_b+1)+R_a(R_b+2))+R_b^2w(d+R_b)\right)}{2R_a(d+R_b)} \\
&-\frac{z\sin\theta\left(a_1\left(c(R_a-1)+d\left(R_a+R_b^2+R_b\right)-2R_a+1\right)+R_b^3w\right)}{\sqrt{R_a}R_b^{3/2}(d+R_b)} \\
&+\frac{r\sin(2\theta)(a_1(dR_b(R_b+2)+R_a)-(d+1)R_b(R_b+1)w)}{2\sqrt{R_a}\sqrt{R_b}(d+R_b)}\Big) \\
\dot\theta =& \omega_0+\varepsilon\Big(\frac{a_1\cos^2(\theta)(dR_b+R_a)}{\sqrt{R_a}\sqrt{R_b}(d+R_b)}+\frac{\sqrt{R_b}(R_b+1)\sin^2(\theta)(-a_1d+dw+w)}{\sqrt{R_a}(d+R_b)} \\
&-\frac{z\sin\theta(a_1(-c(R_b+1)+R_b(d+R_b+2)+R_a+1)-dR_bw)}{rR_b(d+R_b)} \\
&\cos\theta\Big(-\frac{z\left(a_1\left(c(R_a-1)+d\left(R_a+R_b^2+R_b\right)-2R_a+1\right)+R_b^3z\right)}{r\sqrt{R_a}R_b^{3/2}(d+R_b)} \\
&\frac{\sin\theta\left(a_1(R_b(d(R_b+2)+R_b+1)+R_a(R_b+2))+R_b^2w(d+R_b)\right)}{R_a(d+R_b)}\Big)\Big) \\
\dot z =& \varepsilon\Big(\frac{z\left(a_1(R_b(-c+d+R_b+2)+R_a)+R_b^2w\right)}{R_a}-\frac{a_1drR_b\cos\theta}{R_a} \\
&+\frac{rR_b^{3/2}\sin\theta(a_1(d(R_b+2)+R_b+1)+R_bw(d+R_b))}{R_a^{3/2}}\Big) \\
\dot w =& \varepsilon\Big(\frac{Z\left(a_1(R_b(-c+d+R_b+2)+R_a)+R_b^2w\right)}{R_a}-\frac{a_1drR_b\cos\theta}{R_a} \\
&\frac{rR_b^{3/2}\sin\theta(a_1(d(R_b+2)+R_b+1)+R_bw(d+R_b))}{R_a^{3/2}}\Big)
\end{aligned}
\tag{38}
$$

Equations to functions defined in (24) according to Theorem **??**.

$$
\begin{aligned}
\dot r =& \varepsilon\Big(-\frac{(a+1)d_1r(a(a(a(a+5)+10)-D_a+9)-2D_a+4)\cos^2(\theta)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2} \\
&+\sin\theta\Big(\frac{\sqrt{2}(a+1)d_1Z(a(a(a(2a(a(a+7)+20)+59)+45)+16)+D_a+1)}{(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^3} \\
&-\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(a(a+1)(c_1-W)+W)}{(a+2)(a(a+3)+3)\omega_+^3}\Big) \\
&+\sin\theta^2\Big(\frac{2a(a+1)c_1r-2\left(a^2+a-1\right)rW}{\sqrt{2}\omega_+^2}-\frac{2(a+1)^2d_1r}{(a+2)\sqrt{2}\omega_+^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\theta\Big(-\frac{(a+1)d_1Z(a(a(a(2a(a(a+7)+20)+59)+45)+16)+D_a+1)}{(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2}\\
&-\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2}\\
&-\sin\theta\Big(\frac{\sqrt{2}a(a+1)d_1r(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\frac{\sqrt{2}r(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a(a+3)+3)\sqrt{2}\omega_+^3}\Big)\Big)\Big)\\
\dot{\theta}=&\omega_+ +\varepsilon\Big(\frac{\sqrt{2}(a+1)d_1(a(a(a(a+5)+10)-D_a+9)-2D_a+4)\cos^2(\theta)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\sin\theta\Big(\frac{(a+1)d_1Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{2(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2r}\\
&\frac{(a(a(a+3)+4)+D_a+1)}{2(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2r}\\
&+\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2r}\Big)\\
&+\sin\theta^2\Big(\frac{\sqrt{2}(a+1)^2d_1(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\frac{(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a(a+3)+3)\omega_+^3}\Big)\\
&+\cos\theta\Big(\frac{1}{4(a+2)(a(a+3)+3)^2\omega_+^3r}\big((a+1)d_1Z(a(a(a(a+5)\\
&+10)-D_a+9)-2D_a+4)(a(a(a+3)+4)+D_a+1)\big)\\
&-\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(a(a+1)(c_1-W)+W)}{(a+2)(a(a+3)+3)\omega_+^3r}\Big)\\
&+\sin\theta\Big(\frac{(a+1)d_1(a(a(a+1)(a+2)-D_a-3)-2(D_a+1))}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2}
\end{aligned}
$$

(39)

$$
\begin{aligned}
&+\frac{2(a(a+1)(c_1-W)+W)}{\sqrt{2}\omega_+^2}\Big)\Big)\\
\dot{z}=&-\frac{\varepsilon}{(a+1)(a(a+3)+3)\sqrt{2}\omega_+^3}\Big((1+a)\sqrt{2}\omega_+\Big(a(6c_1-4d_1)-d_1(1+D_a)\\
&+a^2(12c_1-3d_1-10w)+a^3(8c_1-d_1-8w)+2a^4(c_1-w)+6w\Big)z-2(3+a(3\\
&+a))r\Big((1+a)d_1\sqrt{2}\omega_+\cos\theta+\sqrt{2}\Big((1+a)(d_1+a(-(2+a)c_1+d_1))+(2+a)(a^2\\
&+a-1)w\Big)\sin\theta\Big)\Big)+b_1w\varepsilon^2\\
\dot{w}=&\varepsilon\Big(\frac{1+a}{a(3+a(3+a))(4+a(9+a(10+a(5+a))-D_a)-2D_a)}\Big(-ab_1\Big(1+a\big(16
\end{aligned}
$$

$$+ a(45 + a(59 + 2a(20 + a(7 + a)))) \Big) + D_a \Big) - z^2 + \big(a(1 + a)\big(a^2 + a$$
$$- 1\big)(3 + a(3 + a)) - \big(1 + a(5 + a(4 + a))\big)D_a\big)z^2\Big)$$
$$+ \frac{\sqrt{2}\omega_+ rZ(a(-a(a + 2)(a + 3) + D_a + 3) + 2(D_a + 4)) \sin\theta}{\sqrt{2}a(a + 1)(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)}$$
$$+ \cos\theta \left(\frac{rZ}{a^2 + a} - \frac{\sqrt{2}(a + 2)(a(a + 3) + 3)\sqrt{2}\omega_+ r^2 \sin\theta}{a(a + 1)^2(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)} \right)$$
$$+ \frac{2(a + 2)(a(a + 3) + 3)r^2 \sin^2(\theta)}{a(a + 1)^2(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)}$$

Equations to functions defined in (24) according to Theorem **??**.

$$\dot{r} = \frac{\varepsilon}{2(a(a + 3) + 3)(a + 2)^2 D_a + 2(a(a + 3) + 3)(a(a(a(a + 5) + 10) + 9) + 4)(a + 2)}$$
$$\Big(2(a + 1)(a(a + 3) + 3)(a(a(a(a + 5) + 10) + 9) + 4)d_1 r \cos^2(\theta)$$
$$2 \cos\theta\Big(\big((1 + a)(d_1 - 12w + a((3 + a(3 + a))(4 + a(9 + a(10 + a(5 + a))))c_1$$
$$+ (16 + a(45 + a(59 + 2a(20 + a(7 + a)))))d_1 - (51 + a(100$$
$$+ a(115 + a(82 + a(36 + a(9 + a))))))w))(w + z) - \sqrt{2}(3 + a(3$$
$$+ a))^2\sqrt{2}\omega_- r(a((2 + a)c_1 + d_1 + ad_1) - (1 + a)(2 + a)w) \sin\theta\Big)$$
$$- (3 + a(3 + a)) \sin\theta\big(\sqrt{2}(1 + a)\sqrt{2}\omega_-(6w - d_1 + a(2(1 + a(3 + a(3 + a))c_1$$
$$- (4 + a(3 + a))d_1 - 2a(5 + a(4 + a))w))(w + z) + 4(3 + a(3 + a))r((1 + a)(a(2$$
$$+ a)c_1 - (1 + a)d_1) - (2 + a)(a^2 + a - 1)w) \sin\theta\big) + D_a\big(2(1 + a)(a(2 + a)(3$$
$$+ a(3 + a))c_1 - d_1 - 6w - a(3 + a(5 + a(3 + a))w)(w + z) \cos\theta$$
$$+ (1 + a)(2 + a)(3 + a(3 + a))d_1 r \cos\theta^2$$
$$- \sqrt{2}(1 + a)(3 + a(3 + a))d_1)\sqrt{2}\omega_-(w + z) \sin\theta\big)\Big)$$

$$\dot{\theta} = \omega_- + \frac{\varepsilon}{2(2 + a)(3 + 2a + a^2)^2\sqrt{2}\omega_-^3 r}\Big(2\sqrt{2}(1 + a)(3 + a(3 + a))d_1(2(2 + D_a)$$
$$+ a(9 + a(10 + a(5 + a)) + D_a))r \cos\theta^2 + (1 + a)(2(2 + D_a)$$
$$+ a(9 + a(10 + a(5 + a)) + D_a))\sqrt{2}\omega_-(d_1 - d_1 D_a + a^2(6c_1 + 3d_1 - 8w)$$
$$+ 2a(3c_1 + 2d_1 - 6w) + a^3(2c_1 + d_1 - 2w) - 6w)(w + z) \sin\theta$$
$$+ 2\sqrt{2}(3 + a(3 + a))(2(2 + D_a) + a(9 + a(10 + a(5 + a)) + D_a))r(d_1$$
$$+ a(2 + a)(c_1 + d_1) - 2w - a(3 + a)w) \sin\theta^2 + \cos\theta(-\sqrt{2}(1 + a)(2(2$$
$$+ D_a) + a(9 + a(10 + a(5 + a)) + D_a))(a(6c_1 - 4d_1) + d_1(D_a - 1)$$
$$+ a^2(12c_1 - 3d_1 - 10w) + a^3(8c_1 + d_1 - 8w) + 2a^4(c_1 - w) + 6w)(w + z)$$
$$+ 2(3 + a(3 + a))\sqrt{2}\omega_- r(2d_1(D_a - 1) + a^3(28c_1 + 5d_1 - 26w)$$
$$+ a^2(30c_1 + d_1(D_a - 1) - 20w) + 4a^4(3c_1 + d_1 - 3w) + a^5(2c_1 + d_1 - 2w)$$
$$+ 12w + a(12c_1 - 5d_1 + 3c_1 D_a + 6w)) \sin\theta)\Big)$$

(40)
$$
\begin{aligned}
\dot{z} =& \varepsilon\Big(2\left(-(a+1)(a(a+1)(a(a+3)+3)(a(a(a(a(a(a+5)+7)-4)-17)-12)\right.\\
&-3)-1)Z^2+a(a(a+3)+3)(a(a+1)(a(a+3)+3)(a(a(a(a+5)+10)+9)\\
&+4)c_1-(a(a(a(2a(a(a+7)+20)+59)+45)+16)+1)d_1)Z\\
&+a(a+1)(a(a(a(a(a(a(a(a(a+12)+65)+208)+435)+623)+621)\\
&+424)+183)+40)+1)b_1)\,(a+1)^2+(-2(a(a(a(a(a(a(a(a(2a(a+11)\\
&+107)+296)+495)+470)+141)-219)-312)-180)-46)-1)(a+1)^2\\
&-2(a(a+1)(a+2)(a(a+3)+3)(a(2a(a+3)+5)-1)+1)D_a(a+1)^2\big)\,W^2\\
&+W\left((a+1)D_a\left(2(a+1)(a(a(a+3)+3)(a(a+1)(a+2)(a(a+3)+3)c_1\right.\right.\\
&+d_1-a(a(a+3)(3a(a+3)+14)+17)Z-2Z)-2Z)+(a+2)(a(a+3)\\
&+3)r\left(\sqrt{2}(a+2)(a(a+1)(a+2)-1)\sqrt{2}\omega_{\_}\,\sin\theta-2(a(a(a(a+5)+10)+9)\right.\\
&+4)\cos\theta))+(a+1)\left(2(a+1)(a(a(a+3)+3)(a(a+1)(a(a+3)+3)(a(a(a\right.\\
&+5)+10)+9)+4)c_1-(a(a(a(2a(a(a+7)+20)+59)+45)+16)+1)d_1)\\
&+a(56-a(a(a(a(a(a(a(a(3a+32)+149)+386)+576)+411)-107)\\
&-519)-501)-243))Z+2Z)+(a(a+3)+3)r\left(\sqrt{2}(a(a(a(a(a(a(3a+26)\right.\\
&+93)+168)+134)-26)-131)-84)-14)\sqrt{2}\omega_{\_}\,\sin\theta-2(a(a(a(a(a(a(a\\
&+8)+27)+50)+58)+52)+45)+30)+10)\cos\theta)))+\frac{1}{2}(a(a+3)\\
&+3)r\left(-2(a+2)(a(a+3)+3)(a(a(a(a+5)+10)+9)+4)r+2(a+2)(a(a\right.\\
&+3)+3)(a(a(a(a+5)+10)+9)+4)\cos(2\theta)r-4(a+1)(a(a+1)(a(a+3)\\
&+3)(a(a(a(a+5)+10)+9)+4)d_1+(a(a(a(a(a(a(a(a+8)+27)+50)+58)\\
&+52)+45)+30)+10)Z)\cos\theta+\sqrt{2}\sqrt{2}\omega_{\_}\,((a+2)(a(a+3)+3)(a(a(a(a\\
&+5)+10)+9)+4)r\sin(2\theta)-2(a+1)\left(2a(a+1)(a(a+2)c_1\right.\\
&-(a+1)d_1)(a(a+3)+3)^2-a(a(a(a(a(a(a(a+8)+25)+34)+2)-56)\\
&-77)-48)Z+14Z)\sin\theta))+D_a\left(-2(a(a(a(a(a(a(a+8)+27)+49)\right.\\
&+51)+31)+11)+1)Z^2(a+1)^3+2a\left(a(a+1)^2(a+2)^2(a(a+3)+3)\right.\\
&-1)\,b_1(a+1)^3+2a(a(a+3)+3)(a(a+1)(a+2)(a(a+3)+3)c_1\\
&+d_1)Z(a+1)^2-a(a(a+5)+9)(a(a(a+5)+9)+12)r^2\\
&-36r^2+\frac{1}{2}(a+2)(a(a+3)+3)r\left((a+2)\left(2(a(a+3)+3)r\cos(2\theta)\right.\right.\\
&+\sqrt{2}\sqrt{2}\omega_{\_}(2(a(a(a(a+4)+5)+1)-1)Z\sin\theta+(a(a+3)+3)r\sin(2\theta))\Big)\\
&-4(a+1)(a(a+1)(a(a+3)+3)d_1+(a(a(a(a+5)+10)+9)+4)Z)\cos\theta))\Big)\\
&\Big/\big(2a(1+a)^2(3+a(3+a))(10+8D_a+a(30+22D_a+a(52+20D_a\\
&+a(58+7D_a+a(50+a(27+a(8+a))+Da))))))\big)+b_1w\varepsilon^2\\
\dot{w} =& \frac{\varepsilon}{(2a(1+a)^2(3+a(3+a))(2(2+Da)+a(9+a(10+a(5+a))+Da)))}
\end{aligned}
$$

$$\Big((a+1)^2(a(a(a(a+5)+10)+D_a+9)+2(D_a+2))\left(a^5b_1+4a^4b_1\right.$$
$$+a^3\left(7b_1-(W+Z)^2\right)-a^2\left(b_1(D_a-5)+3(W+Z)^2\right)-a\left(b_1(D_a-1)\right.$$
$$\left.\left.+2(W+Z)^2\right)-(D_a-1)(W+Z)^2\right)+\sqrt{2}(a(a+3)+3)(a+1)\sqrt{2}\omega\_r(a(a(a$$
$$+2)(a+3)+D_a-3)+2(D_a-4))\sin\theta(W+Z)-4(a+2)(a(a+3)$$
$$+3)^2r^2\sin^2(\theta)\Big)-wb_1\varepsilon^2$$

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `candidomr@mat.uab.cat`