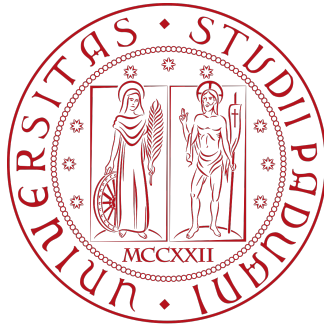


# Università degli studi di Padova



Dipartimento di Ingegneria dell'Informazione  
Laurea magistrale in Ingegneria delle Telecomunicazioni

## Unambiguous State Discrimination for Quantum Communications

Studente:  
Federico Meggetto

Relatore:  
Prof. Roberto Corvaja

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*To my parents,  
Giulia,  
Francesco  
and my advisor*



# Preface

Quantum mechanics represents one of the most successful theories in the history of science. It was born more than a hundred years ago but, for several decades, quantum mechanics was confined to a revolutionary interpretation of physics and related fields, like astronomy.

Only in the last decades, after the discovery of laser with the possibility of producing coherent light, the quantum mechanics receive a strong interest in the area of information, with very innovative and promising applications, like those concerning computer, cryptography, and communications. In particular, the original ideas of quantum communications were developed by Helstrom and by scientists from MIT proving the superiority of quantum systems with respect to classic optical systems. Measurement of the quantum properties of a physical system is fundamental in quantum mechanics and quantum information science; the first one prohibits the perfect discrimination of non-orthogonal states, due to their intrinsic overlap. This seemingly impossible task can, however, be accomplished by performing generalized quantum measurements that allow for inconclusive results as a possibility for the measurement outcomes. Quantum-state discrimination among non-orthogonal states, besides being of fundamental interest, is critical for many realizations of quantum information processing. In particular, the discrimination of non-orthogonal coherent states is essential in quantum key distribution for unconditionally secure communications, coherent state-based quantum repeaters and quantum computing. Moreover, multi-state discrimination can enable: coherent state-assisted entanglement generation with high fidelity, entanglement swapping and photon number state preparation and detection. In addition, unambiguous multi-state comparison allows for quantum digital signatures.

There are two complementary approaches for non-orthogonal state discrimination. The first of these, minimum error discrimination (*MED*), seeks measurements that minimize the probability of erroneously identifying the state.

The second approach, that is unambiguous state discrimination (*USD*), introduces inconclusive results to achieve perfect state discrimination, and aims to maximize the probability of conclusive results. However, imperfections in realistic implementations make ideal (error-free) *USD* impossible. Thus, real-world *USD* becomes an

intermediate measurement strategy between *MED* and ideal *USD*, which retains the conclusive results of ideal *USD*, but contains some errors in those conclusive results.

First, we demonstrate the realization of generalized quantum measurement for *USD* of two non-orthogonal coherent states using coherent displacement and photon counting.

Finally, we demonstrate the both realizations for  $K$  non-orthogonal ( $K > 2$ ).

The performance of each specific system is compared to that of the corresponding classical and quantum optical system.

### Organization of the thesis

The thesis is organized into 6 chapters:

Chapter 1 Collects the mathematical background needed in the formulation and development of Quantum Mechanics: mainly notions of linear vector spaces and Hilbert spaces, with special emphasis on the eigendecomposition of linear operators.

Chapter 2 Introduces the fundamentals of Quantum Mechanics, in four postulates. Postulate 1 is concerned with the environment of Quantum Mechanics: a Hilbert space. Postulate 2 formulates the evolution of a quantum system, according to Schrödinger's and Heisenberg's visions. Postulate 3 is concerned with the quantum measurements, which prescribes the possibility of extracting information from a quantum system. Finally, Postulate 4 deals with the combination of two or more interacting quantum systems. A particular emphasis is given to Postulate 3, because it manages the information in a quantum system and will be the basis of Quantum Communications and Quantum Information consideration.

Chapter 3 Develops the concept of unambiguous state discrimination. Here a non-trivial effort is made to express the results within the language of telecommunications, where the *USD* is applied to the receiver.

Chapter 4 Deals with the general formulation of quantum communication systems, where the transmitter (Alice) prepares and launches the information in a quantum channel and the receiver (Bob) extracts the information by applying the quantum decision rules. Although, in principle, the transmission of analog information would be possible, according to the lines of present-day technology, only digital information (data) is considered. In any case, we will refer to optical communications, in which the information is conveyed through a coherent radiation produced by a laser. The quantum formulation of coherent radiation is expressed according to the universal and celebrated Glauber's theory.

Chapter 5 The basic ideas of chapter 4 are applied to most popular *USD* communication systems, each one characterized by a specific binary modulation format (*OOK*, *BPSK*, *PPM*). The performance of each specific system is compared to that of the corresponding classical and quantum optical system.

Chapter 6 The concepts of chapter 4 are applied to *USD* communication systems, each one characterized by a specific multilevel modulation format, i.e. *QAM*,

*PSK, PPM.* Of course, the performance of each specific system is compared to that of the corresponding classical and quantum optical system.

Padova, 25 February 2019

*Federico Meggetto*





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# Acronyms

Throughout the thesis, notations are explicitly specified at the first use and are frequently recalled.

Matrices and operators are denoted by uppercase letters, i.e.,  $A$ ,  $\psi$ .

For quantum states, Diracs notation *bra* and *ket* is used, as  $\langle x|$  and  $|x\rangle$ .

BPSK	Binary PSK
DFT	Discrete Fourier transform
EID	Eigendecomposition
GUS	Geometrically uniform symmetry
IDP	Ivanovic-Dieks-Peres limit
OOK	Onoff keying
POVM	Positive Operator-Valued Measurements
PPM	Pulse position modulation
PSD	Positive semidefinite
PSK	Phase shift keying
QAM	Quadrature amplitude modulation
QKD	Quantum key distribution
SNR	Signal to noise ratio
SRM	Square root measurement
SVD	Singular-value decomposition
USD	Unambiguous state discrimination





# Chapter 1

## Vector spaces and operative methods

This chapter given an introduction to the important concept of vector spaces, that are thoroughly discussed in the following chapters. For a more detailed discussion please refer to the dedicated literature [1] – [5].

### 1.1 Vector spaces

A vector space [3],  $\mathbf{V}$ , is a collection of vectors,  $\{v_1, \dots, v_n\}$  in which an addition and a multiplication operation by scalar quantities  $\{\alpha, \beta, \gamma, \dots\} \in \mathbf{F}$  are defined such that:

$$\mathbf{v} + \mathbf{w} \in \mathbf{V} \quad (1.1)$$

$$\alpha \mathbf{v} \in \mathbf{V}, \quad \text{if } \mathbf{v} \in \mathbf{V}, \alpha \in \mathbf{F} \quad (1.2)$$

$\mathbf{F}$  is called the scalar field and the addition and product operations  $\alpha + \beta \in \mathbf{F}$ ,  $\alpha\beta \in \mathbf{F}$  are defined on  $\mathbf{F}$ . These operators satisfy some properties:

**Definition 1.1.** Axioms for addition

1.  $v + w = w + v \quad \forall v, w \in V$
2.  $v + (w + z) = (v + w) + z \quad \forall v, w, z \in V$
3.  $\exists$  *null vector*,  $0, \quad v_i + 0 = 0 + v_i = v_i$
4.  $\exists$  *single vector*,  $(-v_i)$ , such that  $v_i + (-v_i) = 0$

**Definition 1.2.** Axioms for multiplication by scalars

1.  $\alpha(v + w) = \alpha v + \alpha w, \quad \alpha \in F \text{ and } v, w \in V$
2.  $(\alpha + \beta)v = \alpha v + \beta v$
3.  $\alpha(\beta v) = (\alpha\beta)v$
4.  $1v = v, \quad 1 \in F$

where  $1$  is the identity in the  $F$  field of the scalars, that is  $1\alpha = \alpha$ .

A set of  $n$  non-null vectors is called **linearly independent** if there are no scalar solutions to the equation

$$\sum_{i=1}^n \alpha_i v_i = 0 \quad (1.3)$$

except for the trivial solution  $\alpha_i = 0 \forall i$ .

This definition implies the fact that if  $n$  vectors are independent, they cannot be written as linear combinations between one and the others.

A vector space is said to be  $n$ -dimensional, if it admits at most  $n$  linearly independent vectors. We will denote a  $n$ -dimensional space on  $F$  with the symbol  $V^n(F)$ .

**Definition 1.3.** Given  $n$  linearly independent vectors  $(v_1, v_2, \dots, v_n)$ , any other vector  $v \in V^n(F)$  can be written as a linear combination of the  $n$  vectors. This is called the **basis**, or a set of **basis vectors**.

## 1.2 Vector spaces with inner product

An inner product in a vector space associates two vectors to a scalar of  $F$ , i.e. it is a bilinear mapping  $V \times V \rightarrow F$  that satisfies the following axioms:

1.  $\langle v|v \rangle \geq 0$
2.  $\langle v_i|v_j \rangle = \langle v_j|v_i \rangle^*$
3.  $\langle v_i|\alpha v_j + \beta v_k \rangle = \alpha \langle v_i|v_j \rangle + \beta \langle v_i|v_k \rangle$

The operation  $*$  is a complex conjugation if  $F = \mathbb{C}$  is the field of the complex numbers.

The second property says that the inner (or scalar) product is called Hermitian in the case of complex.

The third property outlines the linearity of the inner product. It is antilinear when compared to the first.

When an inner product is defined within a vector space, this is called "vector space with an inner product". The norm of a vector induced by the inner product can be defined:

$$|v| = \sqrt{\langle v|v \rangle} \quad (1.4)$$

A vector is called normalized or unitary if it has a norm equal to one. Moreover, two vectors are said to be orthogonal if their scalar product is null:

$$\langle v|w \rangle = 0 \Leftrightarrow v \perp w \quad (1.5)$$

A set of vectors  $(e_1, e_2, \dots, e_n)$  is called orthonormal if

$$\langle e_i|e_j \rangle = \delta_{ij} \quad (1.6)$$

where  $\delta_{ij}$  is the Kronecker delta.

An important property of the inner product is Schwarz's inequality:

$$|\langle v_i | v_j \rangle|^2 \leq |v_i|^2 |v_j|^2 \quad (1.7)$$

Another important inequality in which the inner product is satisfied is the triangular inequality:

$$|v_i + v_j| \leq |v_i| + |v_j| \quad (1.8)$$

### 1.3 The Dirac notation

Dirac introduced a notation for vector space's description that is extremely relevant for discussions of the following topics [5]. As we have already observed, a vector is entirely specified by assigning its components to a base vector introduced.

For example, on an orthonormal basis:

$$\mathbf{v} = \sum_{i=1}^n \mathbf{e}_i v_i \quad (1.9)$$

all operations on vectors are reported to operations on the  $v_i$  components. Therefore there is a two-way correspondence between the vector  $\mathbf{v}$  and the n-ple of its components in a given basis.

$$\mathbf{v} \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad (1.10)$$

In this orthonormal basis, the inner product can be written as:

$$\langle \mathbf{v} | \mathbf{w} \rangle = \sum_{i=1}^n v_i w_i = (v_1 \cdots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \quad (1.11)$$

where we have associated with the vector  $\mathbf{v}$  the n-upla  $(v_1 \cdots v_n)$

We are now able to introduce a more compact way of expressing the inner product between two vectors.

Precisely, the **ket**, that corresponds to the column vector, i.e.

$$|v\rangle \Leftrightarrow \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad (1.12)$$

and the **bra**, that corresponds to the row vector:

$$\langle v| \Leftrightarrow (v_1 \cdots v_n) \quad (1.13)$$

The representation that associates bra to vector  $\mathbf{v}$  is the dual of the one that associates the ket. Note that, in matrix notation, the dual representation is obtained through the operations of transposition and complex conjugation. This operation goes from ket to bra and viceversa.

$$\langle v| = |v\rangle^\dagger \quad (1.14)$$

and obviously it also applies

$$\left(|v\rangle^\dagger\right)^\dagger = |v\rangle \quad (1.15)$$

In the base (1.9), the base vectors have the representation:

$$|e_i\rangle \equiv |i\rangle \Leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow i^{\text{th}} \text{ place} \quad (1.16)$$

So the ket  $|v\rangle$  can be written in this base as

$$|v\rangle = \sum_i v_i |i\rangle \quad (1.17)$$

and the bra as

$$\langle v| = \sum_i v_i^* \langle i| \quad (1.18)$$

We now notice that the inner product of a vector  $\mathbf{v}$  with another  $\mathbf{w}$  is:

$$\langle v|w\rangle = (v_1 \cdots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_i v_i w_i \quad (1.19)$$

The orthonormality of the basic vectors is written in the form

$$\langle i|j\rangle = \delta_{ij} \quad (1.20)$$

If a ket has representation

$$|v\rangle = \sum_i v_i |i\rangle \quad (1.21)$$

its components  $v_i$  are calculated by taking the inner product with the basic vectors:

$$\langle j|v\rangle = \sum_i v_i \langle j|i\rangle = \sum_i v_i \delta_{ij} = v_j \quad (1.22)$$

It follows the expression

$$|v\rangle = \sum_i |i\rangle \langle i|v\rangle \quad (1.23)$$

Similarly,

$$\langle v| = \sum_i \langle i|z_i \quad (1.24)$$

$$\langle v|j\rangle = \sum_i \langle i|j\rangle z_i = z_j \quad (1.25)$$

$$\langle v| = \sum_i \langle v|i\rangle |i\rangle \quad (1.26)$$

$$(1.27)$$

We now would like to show how it is possible, given  $n$  linearly independent vectors, to build a set of orthonormal vectors (Gram-Schmidt process). We begin by defining  $n$  orthogonal vectors from the given set  $(|v_1\rangle, \dots, |v_n\rangle)$ .

$$|1'\rangle = |v_1\rangle \quad (1.28)$$

$$|2'\rangle = |v_2\rangle - \frac{|1'\rangle \langle 1'|v_2\rangle}{\langle 1'|1'\rangle} \quad (1.29)$$

The ket  $|2'\rangle$  is constructed by subtracting its projection along  $|v_2\rangle$  from  $|1'\rangle$ . This makes it orthogonal to  $|1'\rangle$  as we immediately verify. We choose then

$$|3'\rangle = |v_3\rangle - \frac{|1'\rangle \langle 1'|v_3\rangle}{\langle 1'|1'\rangle} - \frac{|2'\rangle \langle 2'|v_3\rangle}{\langle 2'|2'\rangle} \quad (1.30)$$

Given the orthogonality of  $|1'\rangle$  and  $|2'\rangle$  it immediately follows

$$\langle 1'|3'\rangle = \langle 2'|3'\rangle = 0 \quad (1.31)$$

Proceeding in an iterative way, the following result is obtained

$$|k'\rangle = |v_k\rangle - \sum_{i'=1}^{k-1} \frac{|i'\rangle \langle i'|v_k\rangle}{\langle i'|i'\rangle}, \quad k = 2, \dots, n \quad (1.32)$$

The orthonormal base is obtained by normalizing each ket  $|i'\rangle$

$$|i'\rangle = \frac{|i'\rangle}{\langle i'|i'\rangle} = \frac{|i'\rangle}{\sqrt{\langle i'|i'\rangle}} \quad (1.33)$$

The linear independence of the  $n$  initial vectors is implicitly used here. In fact, if the previous construction was linearly dependent, it could have been stopped before reaching the vector  $|n'\rangle$ .

**Definition 1.4.** The maximum number of orthogonal vectors in a vector space is equal to the maximum number of linearly independent vectors.

## 1.4 Linear operators

An operator  $A$  is an operation that transforms a vector into another vector. That is,  $A$  is a mapping of the vector space  $V$  in itself:

$$A : V \rightarrow V \quad (1.34)$$

The action of  $A$  on the vectors is represented as follows

$$|v'\rangle = A|v\rangle \quad (1.35)$$

An operator is **linear** if it satisfies the following property:

$$\boxed{A(\alpha|v\rangle + \beta|w\rangle) = \alpha A|v\rangle + \beta A|w\rangle} \quad (1.36)$$

An operator  $A$  can also act on the bra:

$$(\langle v|\alpha + \langle w|\beta)A = \langle v|A\alpha + \langle w|A\beta \quad (1.37)$$

A simple example of an operator is the identity operator which transforms each vector into itself

$$I|v\rangle = |v\rangle, \quad \forall |v\rangle \in V \quad (1.38)$$

The operators can be combined by the following operation:

Product:  $(A_1A_2)|v\rangle = A_1(A_2|v\rangle)$

Sum:  $(A_1 + A_2)|v\rangle = A_1|v\rangle + A_2|v\rangle$

Scalar multiplication:  $(\alpha A)|v\rangle = \alpha(A|v\rangle)$

It is important to note that the **product** of two operators is generally not commutative, that is,

$$A_1A_2 \neq A_2A_1 \quad (1.39)$$

## 1.5 Matrix representation of a linear operator

We have seen that a vector with given base corresponds to an n-ple of numbers. These numbers are its components in that base. Similarly, given the base, a linear operator is represented by an array of  $n \times n$  numeric values that correspond to its matrix elements. Obviously, the matrix elements depend on the choice of the base. Recall that an operator is completely assigned once the action on the elements of a base has been defined. In particular, the components of the transformed ket can be easily calculated under the action of the operator if this equality is valid:

$$|w\rangle = A |v\rangle \quad (1.40)$$

ensue

$$w_i = \langle i|w\rangle = \langle i|A|v\rangle = \langle i|A \sum_j v_j |j\rangle = \sum_j \langle i|A|j\rangle v_j \quad (1.41)$$

setting as

$$A_{ij} = \langle i|A|j\rangle \quad (1.42)$$

follows that

$$w_i = \sum_j A_{ij} v_j \quad (1.43)$$

Thus, the operator's action can be evaluated by working with its matrix elements. Such action is simply the product of its column vector representable matrix and the usual rows by columns product.

The matrix elements of the **identity operator** are calculated as:

$$\langle i|I|j\rangle = \langle i|j\rangle = \delta_{ij} \quad (1.44)$$

The identity operator has matrix representation given by the identity matrix. Recall that we have demonstrated the relationship (1.23)

$$|v\rangle = \sum_{i=1}^n |i\rangle \langle i|v\rangle \quad (1.45)$$

which can be written in the form

$$|v\rangle = \left( \sum_{i=1}^n |i\rangle \langle i| \right) |v\rangle \quad (1.46)$$

Note that the vector  $|v\rangle$  appears both at the right and left hand-side of the equation. Therefore, it is possible to rewrite it as:

$$I = \left( \sum_{i=1}^n |i\rangle \langle i| \right) \quad (1.47)$$

and with these steps we have arrived at the identity matrix.

### 1.5.1 Projectors

It is interesting to consider the quantities  $|i\rangle \langle i|$  as operators. Remind that  $|i\rangle$  as an orthonormal basis. Their action is given by

$$(|i\rangle \langle i|) |v\rangle = |i\rangle \langle i|v\rangle = v_i |i\rangle \quad (1.48)$$

Also, they are called **projectors**

$$P_i = |i\rangle \langle i| \quad (1.49)$$

We observe that

$$\sum_{i=1}^n P_i = I \quad (1.50)$$

Furthermore the projectors respect the following property <sup>1</sup>

$$P_i P_j = |i\rangle \langle i|j\rangle \langle j| = \delta_{ij} |i\rangle \langle i| = \delta_{ij} P_i \quad (1.51)$$

To calculate the matrix elements of a projector, we can use the representation of the bra and ket

$$P_i = |i\rangle \langle i| \Leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} (0 \cdots 1 \cdots 0) \quad (1.52)$$

and this expression is equal to

---

<sup>1</sup> If  $i = j$ , in particular, we have  $P_i^2 = P_i$ , and is called idempotency. Otherwise  $P_i P_j = 0$  for  $i \neq j$  is called orthogonality



$$i \left\{ \begin{array}{c} \overbrace{\left( \begin{array}{cccc} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{array} \right)}^i \end{array} \right.$$

or by direct calculation

$$(P_i)_{jk} = \langle j|P_i|k\rangle = \langle j|i\rangle = \langle i|k\rangle = \delta_{ij}\delta_{ik} \quad (1.53)$$

which, in fact, tells us that the only nonzero element is at column and row  $i$ . The product  $AB$  of two operator has matrix representation the product of the matrices

$$(AB)_{ij} = \langle i|AB|j\rangle = \langle i|AIB|j\rangle = \sum_{k=1}^n \langle i|A|k\rangle \langle k|B|j\rangle = \sum_{k=1}^n A_{ik}B_{kj} \quad (1.54)$$

### 1.5.2 Hermitian operators

It is important at this time, is to introduce the **Hermitian operators**. Then in the matrix representative

$$A^\dagger = A \quad (1.55)$$

Given the operator  $A$ , it is Hermitian if

$$\langle \langle A|v\rangle, |w\rangle \rangle = \langle |v\rangle, A|w\rangle \rangle \quad (1.56)$$

and the **Anti-Hermitian operators**

$$A^\dagger = -A \quad (1.57)$$

Each operator can always be decoupled into a Hermitian and an anti-Hermitian part

$$A = \underbrace{\frac{1}{2}(A+A^\dagger)}_{\text{hermitian}} + \underbrace{\frac{1}{2}(A-A^\dagger)}_{\text{anti-hermitian}} \quad (1.58)$$

Other important operators are **unitary operators**:

$$UU^\dagger = U^\dagger U = I \quad (1.59)$$

or, in other words,

$$U^\dagger = U^{-1} \quad (1.60)$$

We observe that the product of unitary operators is unitary. In fact

$$(U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1} \quad (1.61)$$

Moreover, the unitary operators have the important property of preserving the inner product.

$$|w_1\rangle = U |v_1\rangle \quad (1.62)$$

$$|w_2\rangle = U |v_2\rangle \quad (1.63)$$

resulting in

$$\langle w_2 | w_1 \rangle = \langle v_2 | U^\dagger U | v_1 \rangle = \langle v_2 | v_1 \rangle \quad (1.64)$$

Leaving unchanged the norm of a vector, they generalize the rotations to the complex case.

It is important to observe that if we think of the columns of a  $n \times n$  unitary matrix as  $n$  vectors, they form an orthonormal set. In fact, starting from  $U^T U = I$  we have

$$\begin{aligned} \delta_{ij} = \langle i | I | j \rangle &= \langle i | U^\dagger U | j \rangle = \sum_{k=1}^n \langle i | U^\dagger | k \rangle \langle k | U | j \rangle = \\ &= \sum_{k=1}^n (U^\dagger)_{ik} U_{kj} = \sum_{k=1}^n U_{ki}^* U_{kj} \end{aligned} \quad (1.65)$$

Also, it is important to mention the fact that the same considerations can be done for the  $n$  rows.

### 1.5.3 Trace

An important operator, which is invariant under unitary transformations is the **trace**. An example in the trace (sum of the diagonal elements):

$$Tr[A] = \sum_{i=1}^n A_{ii} \quad (1.66)$$

An important property in the follow:

$$Tr[AB] = Tr[BA] \quad (1.67)$$

indeed

$$Tr[AB] = \sum_{i,j} A_{ij} B_{ji} = \sum_i B_{ji} A_{ij} = Tr[BA] \quad (1.68)$$

From this, it follows the cyclic property of the trace

$$\text{Tr}[ABC] = \text{Tr}[A(BC)] = \text{Tr}[(BC)A] = \text{Tr}[B(CA)] = \text{Tr}[CAB] \quad (1.69)$$

And by using the latter

$$\text{Tr}[U^\dagger AU] = \text{Tr}[UU^\dagger A] = \text{Tr}[A] \quad (1.70)$$

which shows the important property of invariance to unitary transformation. Another operator which is unaltered under unitary transformations is the **determinant**. In fact

$$\det[U^\dagger AU] = \det[U^\dagger] \det[U] \det[A] = \det[U^\dagger U] \det[A] = \det[A] \quad (1.71)$$

A result of the trace is:

$$\begin{aligned} \text{tr}(A|\psi\rangle\langle\psi|) &= \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle = \\ &= \sum_i \langle\psi|i\rangle\langle i|A|\psi\rangle = \langle\psi|\sum_i |i\rangle\langle i|A|\psi\rangle = \langle\psi|A|\psi\rangle \end{aligned} \quad (1.72)$$

## 1.6 Eigenvalues and eigenvectors

Given an operator  $A$ , we are often interested in determining the vector space's directions that have not changed since action of  $A$ . This is defined by the equation

$$A|v\rangle = \lambda|v\rangle \quad (1.73)$$

where vector  $|v\rangle$  is called an eigenvector of  $A$  with eigenvalue  $\lambda$ . The eigenvalue equation for operator  $A$  can be written in the form:

$$(A - \lambda I)|v\rangle = 0 \quad (1.74)$$

or, in components:

$$\sum_{j=1}^n (A_{ij} - \lambda \delta_{ij}) v_j = 0 \quad (1.75)$$

This is a linear and homogeneous equation in the components of the autovector. Therefore, we do not have identically zero solutions for  $v_j$  if and only if the determinant of the system is null

$$\boxed{\det[A - \lambda I] = 0} \quad (1.76)$$

By expanding the determinant, we find an equation of degree  $n$  which determines the possible eigenvalues. This is also referred to as the **characteristic equation**.

From the fundamental theorem of algebra (every equation of degree  $n$  has  $n$  complex roots) it follows that every operator in  $C^n(C)$  has  $n$  complex non-necessarily distinct eigenvalues. Once the eigenvalue  $\lambda$  is found, one can obtain the correspondent eigenvector solving the eigenvalue equation

$$\sum_j (A_{ij} - \lambda \delta_{ij}) v_j = 0 \quad (1.77)$$

**Theorem 1.1.** *For each Hermitian operator there is at least one orthogonal eigenvector base. In this base the operator matrix is diagonal and its eigenvalues are the diagonal matrix elements.*

*Proof.* First of all, it is shown that if  $A$  is Hermitian, then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Suppose that  $Au = \lambda u$  and  $Av = \mu v$  for  $\lambda \neq \mu$ . Then

$$\lambda \langle u, v \rangle = \langle u, Av \rangle = \langle u, \lambda v \rangle = \lambda \langle u, v \rangle \quad (1.78)$$

since  $\mu \neq \lambda$  it follows that  $\lambda \langle u, v \rangle = 0$ .

Suppose that  $A$  has  $n$  distinct eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$  with corresponding orthogonal eigenvectors  $\{u_1, \dots, u_n\}$ . Let us also agree to scale the eigenvectors so that

$$\langle u_i, u_j \rangle = \delta_{i,j} \quad (1.79)$$

We recall that the eigenvalue equation can be written in matrix form

$$AU = U\Lambda \quad (1.80)$$

where  $U = (u_1 \ u_2 \ \dots \ u_n)$  and  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ . And now, the outcome is

$$A = U\Lambda U^\dagger \quad (1.81)$$

In other words,  $A$  can be diagonalized in a particularly simple way.

Now we can show that if  $A$  is Hermitian then  $A$  has  $n$  orthonormal eigenvectors  $\{u_1, \dots, u_n\}$  and Equation (1.81) is verified where

$$\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\} \quad (1.82)$$

If all the eigenvalues were distinct then the results follows from the forgoing discussion. Here we shall outline a proof this result for the case where  $A$  is real so that the setting is  $\mathbb{R}_n$  rather than  $C_n$ . Define the function

$$f(y) = \langle Ay, y \rangle \quad (1.83)$$

Analysis tells us that this function has a minimum on the set  $\langle y, y \rangle = 1$ . The minimizer can be found with Lagrange multipliers. We have the Lagrangian

$$L(y) = \langle Ay, y \rangle - \lambda (\langle y, y \rangle - 1) \quad (1.84)$$

and the minimizer  $x_1$  is found from the necessary condition

$$\nabla L(y) = 0 \quad (1.85)$$

We compute

$$\frac{\partial L}{\partial y_k} = \langle A_k, y \rangle + \langle A_y, e_k \rangle - 2\lambda y_k = 2(\langle A_k, y \rangle - \lambda y_k) \quad (1.86)$$

so that we have to solve

$$\begin{aligned} \nabla L &\equiv Ay - \lambda y = 0 \\ \langle y, y \rangle &= 1 \end{aligned} \quad (1.87)$$

The solution to this problem is the eigenvector  $x_1$  with eigenvalue  $\lambda_1$ .

If we now apply the Gram-Schmidt method to the set of  $n+1$  dependent vectors  $\{x_1, e_1, \dots, e_n\}$  we end up with  $n$  orthogonal vectors  $\{x_1, y_2, \dots, y_n\}$ .

We now minimize

$$f(y) = \langle Ay, y \rangle \text{ subject to } \|y\| = 1 \quad (1.88)$$

over all  $y \in \text{span}\{y_2, \dots, y_n\}$  and get the next eigenvalue  $\lambda_2$  and eigenvector  $x_2$  which then necessarily satisfies  $\langle x_1, x_2 \rangle = 0$ . Next we find an orthogonal basis of  $\mathbb{R}_n$  of the form  $\{x_1, x_2, y_3, \dots, y_n\}$  and minimize  $f(y)$  over  $\text{span}\{y_3, \dots, y_n\}$  subject to the constraint that  $\|y\| = 1$ . We keep going and find eventually  $n$  eigenvectors, all of which are mutually orthogonal. Afterwards all eigenvectors can be normalized so that we have an orthonormal eigenvector basis of  $\mathbb{R}_n$  which we denote by  $\{u_1, \dots, u_n\}$ .

We say that  $A$  can be diagonalized.

The diagonalization theorem guarantees that for each eigenvalue one can find an eigenvector which is orthogonal to all the other eigenvectors regardless of whether the eigenvalue is distinct or not. This means that if an eigenvalue  $\mu$  is a repeated root of multiplicity  $k$  (meaning that  $\det(A - \lambda I) = (\lambda - \mu)^k g(\lambda)$ , where  $g(\mu) \neq 0$ ), then  $\dim N(A - \mu I) = k$ . We simply find an orthogonal basis of this null space. The process is, as usual, the application of Gaussian elimination to

$$(A - \mu I)x = 0 \quad (1.89)$$

and finding  $k$  linearly independent vectors in the null space which can then be made orthogonal with the Gram-Schmidt process.  $\square$

## 1.7 Hilbert spaces

A Hilbert space  $H$  is a complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. To say that  $H$  is a complex inner product space means that  $H$  is a complex vector space on which there is an inner product  $\langle x, y \rangle$  associating a complex number to each pair of elements  $x, y$  of  $H$  that satisfies the following properties:

1. The inner product of a pair of elements is equal to the complex conjugate of the inner product of the swapped elements:

$$\langle y, x \rangle = \overline{\langle x, y \rangle} \quad (1.90)$$

2. The inner product is linear in its first argument. For all complex numbers  $a$  and  $b$ ,

$$\langle ax_1 + bx_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle \quad (1.91)$$

3. The inner product of an element with itself is positive definite:

$$\langle x, x \rangle \geq 0 \quad (1.92)$$

where the case of equality holds precisely when  $x = 0$ .

It follows from properties 1 and 2 that a complex inner product is antilinear in its second argument, meaning that

$$\langle x, ay_1 + by_2 \rangle = \bar{a} \langle x, y_1 \rangle + \bar{b} \langle x, y_2 \rangle \quad (1.93)$$

A real inner product space is defined in the same way, except that  $H$  is a real vector space and the inner product takes real values. Such an inner product will be bilinear: that is, linear in each argument.

The norm is the real-valued function:

$$\|x\| = \sqrt{\langle x, x \rangle} \quad (1.94)$$

and the distance  $d$  between two points  $x, y$  in  $H$  is defined in terms of the norm by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle} \quad (1.95)$$

That this function is a distance function means firstly that it is symmetric in  $x$  and  $y$ , secondly that the distance between  $x$  and itself is zero, and otherwise the distance between  $x$  and  $y$  must be positive, and lastly that the triangle inequality holds, meaning that the length of one leg of a triangle  $xyz$  cannot exceed the sum of the lengths of the other two legs:

$$d(x, z) \leq d(x, y) + d(y, z) \quad (1.96)$$

This last property is ultimately a consequence of the more fundamental Cauchy-Schwarz inequality, which asserts

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad (1.97)$$

with equality if and only if  $x$  and  $y$  are linearly dependent.

Relative to a distance function defined in this way, any inner product space is a metric space, and sometimes is known as a pre-Hilbert space.

Any pre-Hilbert space that is additionally also a complete space is a Hilbert space. Completeness is expressed using a form of the Cauchy criterion for sequences in  $H$ : a pre-Hilbert space  $H$  is complete if every Cauchy sequence converges with respect to this norm to an element in the space.

Completeness can be characterized by the following equivalent condition: if a series of vectors

$$\sum_{k=0}^{\infty} u_k \quad (1.98)$$

converges absolutely in the sense that

$$\sum_{k=0}^{\infty} \|u_k\| < \infty, \quad (1.99)$$

then the series converges in  $H$ , in the sense that the partial sums converge to an element of  $H$ .

As a complete normed space, Hilbert spaces are by definition also Banach spaces. As such they are topological vector spaces, in which topological notions like the openness and closedness of subsets are well-defined. Of special importance is the notion of a closed linear subspace of a Hilbert space that, with the inner product induced by restriction, is also complete (being a closed set in a complete metric space) and therefore a Hilbert space in its own right.

## 1.8 Tensor products

The tensor product is used to put together vector spaces to create larger ones. Suppose that  $V$  and  $W$  are vector spaces of dimension  $m$  and  $n$ , respectively, and that  $V$  and  $W$  are also Hilbert spaces over a field  $K$ .

The vectors  $V \otimes W$  are then defined to be the equivalence classes of the congruence generated by the following relations:

$$\begin{aligned}
\forall v, v_1, v_2 \in V, \quad \forall w, w_1, w_2 \in W, \quad \forall z \in K : \\
(v_1, w) + (v_2, w) &= (v_1 + v_2, w) \\
(v, w_1) + (v, w_2) &= (v, w_1 + w_2) \\
z(v, w) &= (zv, w) \\
z(v, w) &= (v, zw)
\end{aligned} \tag{1.100}$$

and

$$V \times W = [v_1w_1, v_1w_2, \dots] \tag{1.101}$$

Hence,  $V \otimes W$  is a vector space of  $m \times n$  size. The elements of  $V \otimes W$  are linear combinations of "tensor products"  $|v\rangle \otimes |w\rangle$  of elements of  $|v\rangle$  of  $V$  and  $|w\rangle$  of  $W$ .

In particular, if  $|i\rangle$  and  $|j\rangle$  are orthonormal bases for the spaces  $V$  and  $W$ , then  $|i\rangle \otimes |j\rangle$  is a basis for  $V \otimes W$ .

By definition the tensor product satisfies the following properties:

1. For an arbitrary scalar  $z$ :

$$z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle) \tag{1.102}$$

2.

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle \tag{1.103}$$

3.

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle \tag{1.104}$$

If we now suppose that  $A$  and  $B$  are linear operators on  $V$  and  $W$ , and that  $|v\rangle$  and  $|w\rangle$  are vectors respectively in  $V$  and  $W$ . We can define a linear operator  $A \otimes B$  on  $V \otimes W$  by the equation

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle \tag{1.105}$$

The given definition of  $A \otimes B$  is then extended to all elements of  $V \otimes W$  in a natural way to guarantee the linearity of  $A \otimes B$ , that is,

$$(A \otimes B) \left( \sum_i a_i |v_i\rangle \otimes |w_i\rangle \right) \equiv \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle \tag{1.106}$$



## Chapter 2

# Quantum mechanics

The purpose of this chapter is to introduce some fundamental concepts concerning quantum mechanics. The aim is to support the basics to understand the content of this thesis: for more information, refer to the dedicated literature [1] [5] – [8]. In the following sections, we will give a description of the base mathematical formulation of quantum mechanics. These postulates provide a connection between the physical world and the mathematical formalism of quantum mechanics.

### 2.1 Postulates 1 and 2

#### 2.1.1 Postulate 1: State space

The **first postulate** of quantum mechanics states that a Hilbert space is associated with any isolated physical system. This space is also known as the state space of the system.

This system is entirely described by its state vector. The latter is a unit vector. Given a physical system, quantum mechanics does not give us hints on whether the system is found or about its state.

#### 2.1.2 Postulate 2: Evolution

The **second postulate** of quantum mechanics states that the evolution of a closed quantum system is described by a unitary transformation. The state  $|\psi(t_1)\rangle$  of the system at time  $t_1$  is related to the state  $|\psi(t_2)\rangle$  of the system at time  $t_2$  by a unitary operator  $U(t_1, t_2)$  which depends only on the times  $t_1$  and  $t_2$ ,

$$|\psi(t_2)\rangle = U|\psi\rangle \quad (2.1)$$

In quantum mechanics, the space and quantum state of a system are not specified. The description of a real world quantum dynamic, operator  $U$ , is also not specified. This postulate requires a system to be closed or that does not have interactions with others. In reality, this scenario never happens because there will be always be at least a minimal interactions between systems. However, systems that are approximately closed and described by a good approximated unit evolution are still considerably interesting.

As we have just seen, this postulate describes how well correlated are states of a closed system in two different periods of time. At this point, it is now possible to reformulate such postulate to describe an evolution in continuous time.

### 2.1.2.1 Another formulation

The **other formulation of the postulate 2** of quantum mechanics says that the time evolution of the state of a closed quantum system is described by the Schrödinger equation,

$$\boxed{i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle} \quad (2.2)$$

where  $\hbar$  is Planck's constant.

$H$  is a fixed Hermitian operator, known as the Hamiltonian of the closed system.

For many systems, it is possible to write a time-varying Hamiltonian where  $H$  is not actually a constant.

## 2.2 Postulate 3: Quantum measurement

We said that close quantum systems evolve unitary. The third postulate provides a means of describing the effect of measurements on quantum systems.

The quantum measurement postulate essentially:

1. provides a rule that describes the measurement statistics, the probabilities of different possible measurements
2. provides a rule that describes the post-measurement status of the system

The **postulate 3** states that quantum measurements are described by a collection  $\prod_m$  of measurement operators. These are operators acting on the state space of the system being measured. The measurement outcomes that may can be associate to the index  $m$  of the operator. If the state of the quantum system is  $|\psi\rangle$  immediately before the measurement, then the probability of having  $m$  is given by

$$\boxed{p(m) = \langle \psi | \prod_m | \psi \rangle} \quad (2.3)$$

and the state of the system after measurement is

$$\boxed{|\psi_{post}^{(m)}\rangle = \frac{\Pi_m |\psi\rangle}{\sqrt{\langle \psi | \Pi_m | \psi \rangle}} = \frac{\Pi_m |\psi\rangle}{\sqrt{p(m)}}} \quad (2.4)$$

The operators satisfy the completeness equation,

$$\sum_m \Pi_m = I \quad (2.5)$$

Which expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m \langle \psi | \Pi_m | \psi \rangle \quad (2.6)$$

### 2.2.1 Projective measurements

The model of a quantum measurement is formulated on the basis of appropriate Hermitian operators. The standard formulation is based on projectors, and was introduced by von Neumann (this is why they are referred to as **projective** or **von Neumann measurements**), but other equivalent formulations are found in the literature, and also various generalizations.

For many applications of quantum computation and information, we will mainly be concerned about projective measurements.

To this measurement we can associate the observable Hermitian operator  $M$ . Such operator has a spectral decomposition,

$$M = \sum_m m P_m \quad (2.7)$$

where  $P_m$  is the projector of eigenvalue  $m$  on the eigenspace of  $M$ . The possible measurement's outcome correspond to the eigenvalues  $m$  of the observable. Upon measuring the state  $|\psi\rangle$ , the probability of getting result  $m$  is given by

$$p(m) = \langle \psi | P_m | \psi \rangle \quad (2.8)$$

That the measurement operators, in addition to satisfying the completeness relation (2.5), also satisfy the conditions that  $\Pi_m$  are orthogonal projectors, i.e. the  $\Pi_n$  are Hermitian and

$$\Pi_m \Pi_n = \delta_{m,n} \quad (2.9)$$

The projective measures have different properties; in particular, it is fairly straightforward to calculate the average values for measurements. By definition, the average value of the measurement is

$$\begin{aligned}
E(M) &= \sum_m m p(m) = \sum_m m \langle \psi | P_m | \psi \rangle = \\
&= \langle \psi | \left( \sum_m m P_m \right) | \psi \rangle = \text{Tr}(M | \psi \rangle \langle \psi |)
\end{aligned} \tag{2.10}$$

### 2.2.2 POVM measurements

The generalized quantum measurements are carried out through a set of Hermitian operators, which are not necessarily projectors, and are called **POVM** (positive operator-valued measurements).

A system of general measurement operators (POVM)  $Q_m$  is defined imposing the following conditions to the operators  $Q_m$ :

1. They are Hermitian operators,  $Q_m^* = Q_m$
2. They are positive semidefinite:  $Q_m \geq 0$
3. They resolve the identity:  $\sum_m Q_m = I$

The above properties on POVMs ensure that the probabilities calculated

$$\boxed{p(m) = \langle \psi | Q_m | \psi \rangle} \tag{2.11}$$

respect the conditions relative to a probability distribution, that is,

$$p(m) \geq 0, \quad \sum_m p(m) = 1 \tag{2.12}$$

We note that, with the POVMs, Postulate 3 cannot be fully applied to know the systems state immediately after the measurement. On the other hand, this knowledge is irrelevant in many applications and in particular in quantum communications.

According to many authors, for example Helstrom [65], and Eldar and Forney [66], through the POVM measurements from a quantum system more useful results can be obtained with respect to von Neumanns projective measurements, based on projectors. As will be seen in the next chapter, to improve generality and simplicity, it is convenient to work out the formulation considering POVMs, but often, due to contextual constraints, we will arrive at the conclusion that such operators turn out to be projectors.

## 2.3 Postulate 4: Composite systems

The fourth postulate of quantum mechanics is used to describe a composite quantum system consisting of several distinct physical systems.

The **postulate 4** of quantum mechanics states that the state space of a composite

physical system is the tensor product of the component state spaces. Moreover, if we have systems numbered 1 to  $n$ , and system number  $i$  is in the state  $|\psi_i\rangle$ , then the joint state of the total system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle \quad (2.13)$$

## 2.4 The density operator

To study quantum mechanics, we have used state vectors; another equivalent tool is the density operator [9] or density matrix. It provides a much more convenient language to reflect about some commonly encountered scenarios in quantum mechanics.

### 2.4.1 Ensembles of quantum states

The density operator provides a means of describing quantum systems whose state where we have an additional uncertainty on the system state.

More precisely, suppose that a quantum system is in one of the numerous states  $|\psi_i\rangle$ , where  $i$  is an index, with respective probabilities  $p_i$ .

The density operator for the system is defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle \psi_i| \quad (2.14)$$

Two new important concepts related to the density operator can now be introduced, i.e. the **pure state** and the **mixed state**.

1. A quantum state is said to be pure, when the state  $|\psi\rangle$  is known. In this case, the density operator is  $\rho = |\psi\rangle \langle \psi|$  and the state satisfies the equation  $tr(\rho^2) = 1$ .
2. Otherwise,  $\rho$  is in a mixed state. The state satisfies  $tr(\rho^2) < 1$  and it is a mixture of the pure states of  $\rho$ .

Through this mathematical notation, the postulates of quantum mechanics can be completely reformulated in an equivalent way.

If through the unitary operator  $U$  we can describe the evolution of a closed quantum system and if this system was initially in state  $|\psi_i\rangle$  with probability  $p_i$ , the system is going to be in state  $U|\psi_i\rangle$  with probability  $p_i$ . Of course, this only after the evolution has occurred. The evolution of the density operator is described by the equation

$$\rho(t_0) = \sum_i p_i |\psi_i\rangle \langle \psi_i| \rightarrow \sum_i p_i U |\psi_i\rangle \langle \psi_i| U^\dagger = U \rho U^\dagger = \rho(t_1) \quad (2.15)$$

With the density operator it is possible to describe measurements. If we try to perform a measurement, described by the operators  $\Pi_m$ , we can say that, if the initial state was  $|\psi_i\rangle$ , then the probability of obtaining result  $m$  is

$$p(m|i) = \langle \psi_i | \Pi_m | \psi_i \rangle = \text{tr}(\Pi_m |\psi_i\rangle \langle \psi_i|) \quad (2.16)$$

with equation (1.72) for the last equality.

According to the laws of total probability, the probability of obtaining result  $m$  is

$$p(m) = \sum_i p(m|i) p_i = \sum_i p_i \text{tr}(\Pi_m |\psi_i\rangle \langle \psi_i|) = \text{tr}(\Pi_m \rho) \quad (2.17)$$

### 2.4.2 General properties of the density operator

The density operator was introduced as a means of describing sets of quantum states. The operators that are of type density, are characterized by the following theorem:

**Theorem 2.1.** *An operator  $\rho$  is the density operator associated with some ensemble  $p_i, |\psi_i\rangle$  if and only if it satisfies the conditions:*

1. **Trace condition**, that is,  $\rho$  has a trace equal to one.
2. **Positive condition**, that is,  $\rho$  is a positive operator.

*Proof.* Suppose  $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$  is a density operator. Then

$$\text{tr}(\rho) = \sum_i p_i \text{tr}(|\psi_i\rangle \langle \psi_i|) = \sum_i p_i = 1 \quad (2.18)$$

therefore the trace condition  $\text{tr}(\rho) = 1$  is satisfied.

Suppose, now, that  $|\varphi\rangle$  is an arbitrary vector in the state space.

$$\langle \varphi | \rho | \varphi \rangle = \sum_i p_i \langle \varphi | \psi_i \rangle \langle \psi_i | \varphi \rangle = \sum_i p_i |\langle \varphi | \psi_i \rangle|^2 \geq 0 \quad (2.19)$$

The positivity condition is satisfied.

Vice versa, suppose that  $\rho$  is any operator that satisfies the conditions of trace and positivity conditions.

Since  $\rho$  is positive, it must have a spectral composition

$$\rho = \sum_j \lambda_j |j\rangle \langle j| \quad (2.20)$$

where the vectors  $|j\rangle$  are orthogonal, and  $\lambda_j$  are real, non-negative eigenvalues of  $\rho$ . From the trace condition, we see that  $\sum_j \lambda_j = 1$ . We can infer that a system in the state  $|j\rangle$  and with probability  $\lambda_j$ , has the density operator  $\rho$ . That is,  $\lambda_j, |j\rangle$  is an ensemble of states that creates the density operator  $\rho$ .  $\square$

With this theorem we can define a density operator as  $\rho$  and with a trace equal to one. The postulates of quantum mechanics can be reformulated as:

**Postulate 1:** Associated with any isolated physical system, there is a complex vector space with inner product (Hilbert space) known as the state space of the system. The system is entirely described by its density operator. It is a positive operator  $\rho$ , with trace one and acting on the state space of the system. If a quantum system is in state  $\rho_i$  and with probability  $p_i$ , the density operator for such system is  $\sum_i p_i \rho_i$ .

**Postulate 2:** The evolution of a closed quantum system is described by a unitary transformation. That is, the state  $\rho$  at time  $t_1$  is related to the state  $\rho'$  at time  $t_2$  by a unitary operator  $U$  which depends only on  $t_1$  and  $t_2$ ,

$$\rho' = U \rho U^\dagger \quad (2.21)$$

**Postulate 3:** Quantum measurements are described by a collection  $M_m$  of measurement operators. These act on the state space of the system being measured. The index  $m$  refers to the outcomes that may occur during the experiment. If the state is  $\rho$  immediately before the measurement, then the probability of having result  $m$  is

$$p(m) = \text{tr} (M_m^\dagger M_m \rho), \quad (2.22)$$

and the state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\text{tr} (M_m^\dagger M_m \rho)} \quad (2.23)$$

Lastly, the measurement operators also satisfy the completeness equation,

$$\sum_m M_m^\dagger M_m = I \quad (2.24)$$

**Postulate 4:** The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. If we have systems numbered 1 to  $n$ , and the system number  $i$  is in state  $\rho_i$ , then the joint state is

$$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n \quad (2.25)$$

Naturally, these reformulations of the fundamental postulates of quantum mechanics in terms of density operator are mathematically equivalent to the description in terms of the state vector.





## Chapter 3

# Discrimination of quantum states

In quantum information processing and quantum computing protocols, information is encoded in a state. It is fundamental to identify the state of the system. We can talk about quantum state discrimination if the possible states are non-orthogonal.

### 3.1 Introduction

The system sends information encoded in a state belonging to a set of target states. [6]. Determining the state of the system, after information processing, it still an issue. If we are in the situation where the set of possible target states is known and they are mutually orthogonal, then the state of the system can be unambiguously detected.

It is important to remark that whenever target states are non-orthogonal, target states cannot be perfectly identified.

The problem of discrimination between non-orthogonal states is ubiquitous in quantum information and quantum computing. In many communication schemes such as cryptography and probability, these states are the basis and therefore very much in use. Generally speaking, it is a measurement optimization problem. For more information on this matter we recommend the readings of reviews [6] [10] [11].

Non-orthogonal quantum states have a considerable importance in the communication of quantum cryptographic secure protocols, especially in the scheme of quantum key distribution (QKD), for example in the technique based on the procedure developed by Bennett [15] in two states, i.e. B92 protocol.

### 3.2 Unambiguous discrimination

For measure optimization, we will consider two schemes:

1. Unambiguous discrimination
2. Minimal error discrimination

By waking through the next section, we will see that the optimal measure for the first strategy is a POVM whereas for the second is a standard von Neumann.

Unambiguous discrimination is relatively simple to generalize for more than two states but it gets rather difficult to deal with in the eventuality of mixed states.

The error minimization approach, initially developed for two mixed states, is rather difficult to generalize to more than two states.

Before starting the treatment of the optimal unambiguous discrimination for two pure states, a physical representation of the phenomenon must be presented. To this end, we describe the optimal unambiguous discrimination procedure between two non-orthogonal polarization states of the single photon, in terms of classical optics.

Consider weak impulses, each containing on average one photon.

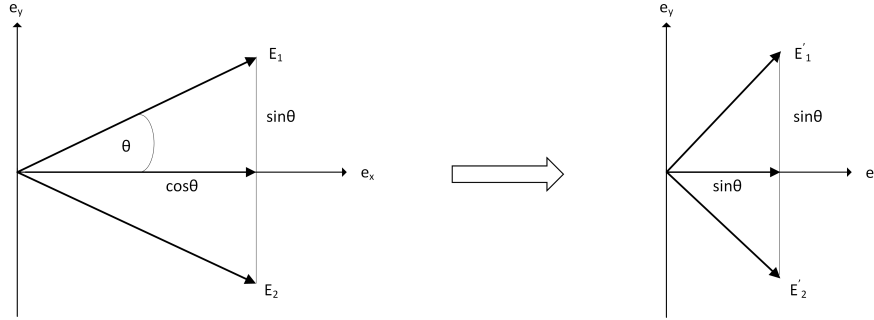
Then the electric field has amplitude  $E_0$  with  $|E_0|^2 = 1$ .

Each pulse is linearly polarized, with equal probability in the direction  $e_{1,2}$

$$e_{1,2} = \cos\theta e_x \pm \sin\theta e_y \quad (3.1)$$

where  $e_x$  and  $e_y$  denote the reference polarization base vectors and we assume  $\cos\theta \geq \sin\theta$ .

In Figure 3.1 shows the graphical representation of the polarization vectors.



**Fig. 3.1** Linearly polarized pulse

If, at this point, the pulses go through a linear optical device and undergo a polarization selective attenuation in direction by a factor of  $\tan\theta$ , their polarization vectors may be orthogonal.

For this purpose the attenuation process has to be designed in such a way that the amplitude of the x-component is reduced.

The electric field vectors of the respective pulses are then transformed into vectors

$$E'_{1,2} = \frac{E_0 \sqrt{2} \sin\theta (e_x \pm e_y)}{\sqrt{2}} \quad (3.2)$$

After leaving the linear optical device, the polarization directions of the two pulses are orthogonal. It can be said that pulses can be discriminated unambiguously even **when the pulses contain only a single photon.**

Due to the reduction of the total initial intensity by a factor  $2 \sin^2 \theta$ , the intensity that can be used for the unambiguous polarization state discrimination is equally reduced.

Since in the classical detection the detection probability depends on the intensity, on each final orthogonal probability equal to the probability of having at least one photon

$$\begin{aligned} P_D &= P[N_{1,2} > 0] = 1 - e^{-|E'_{1,2}|^2} = \\ &= 2 \sin^2 \theta = 1 - \cos(2\theta) = 1 - |\langle e_1 e_2 \rangle| \end{aligned} \quad (3.3)$$

The module of the scalar product was introduced in equation (3.3) because, when the direction of one of the vectors ( $e_1$  or  $e_2$ ) is inverted, the linear polarization state remains the same.

The probability of a discrimination failure, i.e. the polarization state of the photon cannot be determined in a unambiguously way, is

$$Q_F = 1 - P_D = |\langle e_1 e_2 \rangle| \quad (3.4)$$

As it can be seen, the procedure described above provides the maximum  $P_D$  or the minimum  $Q_F = 1 - P_D$  given the unit vector and the fact that the only possible operation on the vector is a selective attenuation.

In general, if two states,  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , occur with equal probability, the optimal probability of obtaining an unambiguous result is given by equation (3.3) with the scalar product  $e_1 e_2$  replaced by the  $\langle \psi_1 | \psi_2 \rangle$  overlap.

### 3.2.1 Unambiguous discrimination of two pure states

Unambiguous discrimination treats the problem in which a quantum system is prepared in either one of the two known states,  $|\psi_1\rangle$  or  $|\psi_2\rangle$  with probability  $\eta_1$  or  $\eta_2$  (such that  $\eta_1 + \eta_2 = 1$ ) respectively. These preparation probabilities are called a priori probabilities.

The states are, in general, non-orthogonal,  $\langle \psi_1 | \psi_2 \rangle \neq 0$  but linearly independent. Ivanovic [16] has come to the conclusion that if you allow inconclusive detection results, Bob can, in all the cases, permanently determine the status of the individual system. To carry out this task it is useful to use the von Neumann measurement.

We indicate the Hilbert space of the two given states by  $H$  and introduce the projector  $P_1$  for  $|\psi_1\rangle$  and  $\overline{P_1}$  for the orthogonal subspace, such that  $P_1 + \overline{P_1} = I$ , the identity in  $H$ .

So we know with certainty that  $|\psi_2\rangle$  has been prepared if in the measurement a click

in the  $\overline{P}_1$  detector occurs. A similar conclusion for  $|\psi_1\rangle$  can be reached with the roles of  $|\psi_1\rangle$  and  $|\psi_2\rangle$  reversed.

Naturally, when a click in  $P_1$  occurs, nothing is learned about what state was prepared and therefore corresponding to inconclusive results.

The probability of failure is

$$Q_F = |\langle\psi_1|\psi_2\rangle| \quad (3.5)$$

whereas success is

$$P_D = 1 - Q_F = 1 - |\langle\psi_1|\psi_2\rangle| \quad (3.6)$$

These probabilities (3.5) – (3.6) are optimal and are called the Ivanovic-Dieks-Peres limit (IDP) [16] – [18].

This result can be generalized for the case when the preparation probabilities of the states,  $\eta_1$  and  $\eta_2$ , are different

$$\eta_1 \neq \eta_2 \quad (3.7)$$

IDP thus corresponds to the case of equal a priori probability and the generalization for a priori arbitrary probabilities is due to Jaeger and Shimony [19].

### 3.2.2 Optimal POVM and the complete solution

The von Neumann projective measurement has two results: it can correctly identify one of the two states but lose information of the other, or information of the identifiable state is occasionally lost.

If we want to have a better result we would need a measurement with three results,  $|\psi_1\rangle$ ,  $|\psi_2\rangle$  and failure.

However, in the two-dimensional Hilbert space  $H$ , the number of possible results for a von Neumann measurement cannot exceed two, since it is always limited by the dimensionality of the Hilbert space.

Therefore, we must move to generalized measurements that allow a greater flexibility [44]. In particular, the number of distinguishable results may exceed the dimensionality of the corresponding Hilbert space.

In our case, this means that we replace the projector  $\overline{P}_2$  with the quantum detection operator  $\Pi_1$ ,  $\overline{P}_1$  by  $\Pi_2$  and introduce  $\Pi_0$  for the inconclusive results so that

$$p_1 = \langle\psi_1|\Pi_1|\psi_1\rangle \quad (3.8)$$

is the probability of successfully identifying  $|\psi_1\rangle$  and

$$q_1 = \langle\psi_1|\Pi_0|\psi_1\rangle \quad (3.9)$$

is the failing probability to identify  $|\psi_1\rangle$ . Analogously for  $|\psi_2\rangle$ .

For an unambiguous discrimination, it is also required that

$$\langle \psi_2 | \Pi_1 | \psi_2 \rangle = \langle \psi_1 | \Pi_2 | \psi_1 \rangle = 0 \quad (3.10)$$

These possibilities must respect the constraint,

$$\Pi_1 + \Pi_2 + \Pi_0 = I \quad (3.11)$$

where  $I$  is the unit operator in  $H$ . The probabilities are always real and non-negative, which implies that quantum detection operators are semi-definite positive.

It should be noted that equation (3.11) does not correspond to orthogonal measurements when all detection operators are different from zero. This equation describes a POVM or simply a generalized measurement with the detection operators as its elements.

Decomposing  $\Pi_k$  as

$$\Pi_k = A_k^\dagger A_k \quad (3.12)$$

the detection probability can be expressed as

$$\langle \psi_i | A_k^\dagger A_k | \psi_i \rangle = \|A_k \psi_i\|^2 \geq 0 \quad (3.13)$$

This expression can also help us to identify the operator  $A_k$ .

**The expression  $A_k | \psi_i \rangle$  corresponds to the post-detection state.**

The condition of unambiguous discrimination is equivalent to the condition

$$A_1 | \psi_2 \rangle = A_2 | \psi_1 \rangle = 0 \quad (3.14)$$

$| \psi_1^\perp \rangle$  is an orthogonal vector to  $| \psi_2 \rangle$  and  $| \psi_2^\perp \rangle$  to  $| \psi_1 \rangle$  and  $| \bar{\psi}_1 \rangle$   $| \bar{\psi}_2 \rangle$  the post-detection states applying  $A_1, A_2$ , we have

$$A_1 = c_1 | \bar{\psi}_1 \rangle \langle \psi_1^\perp | \quad (3.15)$$

and

$$A_2 = c_2 | \bar{\psi}_2 \rangle \langle \psi_2^\perp | \quad (3.16)$$

It must also be noted that the  $c_i$  coefficients belong to the complex field and they are determined from the optimal condition.

For perfect distinguishability of post-detection states, corresponding to optimal discrimination, we must verify their orthogonality, i.e.  $\langle \bar{\psi}_1 | \bar{\psi}_2 \rangle = 0$ .

Then we can write the detection operators as

$$\Pi_1 = A_1^\dagger A_1 = |c_1|^2 | \psi_1^\perp \rangle \langle \psi_1^\perp | \quad (3.17)$$

end

$$\Pi_2 = A_2^\dagger A_2 = |c_2|^2 | \psi_2^\perp \rangle \langle \psi_2^\perp | \quad (3.18)$$

Now, by inserting these expressions in the definition of  $p_1$  and  $p_2$  we get

$$|c_1|^2 = \frac{p_1}{|\langle \psi_1 | \psi_1^\perp \rangle|^2} \quad (3.19)$$

A similar result holds for  $|c_2|^2$ .

Finally, by introducing  $\cos \Theta = |\langle \psi_1 | \psi_2 \rangle|$  and  $\sin \Theta = |\langle \psi_1 | \psi_1^\perp \rangle|$ , we can write the detection operators as

$$\Pi_1 = \frac{p_1}{\sin^2 \Theta} |\psi_1^\perp\rangle \langle \psi_1^\perp| \quad (3.20)$$

end

$$\Pi_2 = \frac{p_2}{\sin^2 \Theta} |\psi_2^\perp\rangle \langle \psi_2^\perp| \quad (3.21)$$

We have just obtained operators  $\Pi_1$  and  $\Pi_2$ .

However, we need to have a further condition for the existence of the POVM. This condition is the positivity of the inconclusive detection operator, i.e.

$$\Pi_0 = I - \Pi_1 - \Pi_2 \quad (3.22)$$

This  $2 \times 2$  matrix problem in  $H$  and the corresponding eigenvalue problem can be solved analytically.

The non-negativity of the eigenvalues leads to the condition

$$q_1 q_2 \geq |\langle \psi_1 | \psi_2 \rangle|^2 \quad (3.23)$$

where  $q_1 = 1 - p_1$  and  $q_2 = 1 - p_2$  are the failure probabilities for the corresponding input states.

Equation (3.23) represents the constraint imposed by the positivity requirement on the optimal detection operators. Now, let

$$Q = \eta_1 q_1 + \eta_2 q_2 \quad (3.24)$$

be the average probability of failure for unambiguous discrimination.

We want to minimize this probability of failure through the constraint of equation (3.23). Considering the relation

$$P = \eta_1 p_1 + \eta_2 p_2 = 1 - Q \quad (3.25)$$

then the minimum of  $Q$  is also the maximum probability of success.

Through equation (3.23), to optimize the product  $q_1 q_2$  we can express  $q_2$  as

$$q_2 = \frac{\cos^2 \Theta}{q_1} \quad (3.26)$$

By inserting this expression into (3.24) we obtain

$$Q = \eta_1 q_1 + \eta_2 \frac{\cos^2 \Theta}{q_1} \quad (3.27)$$

where  $q_1$  can now be considered as an independent parameter of the problem. The optimization of  $Q$  with respect to  $q_1$  gives the following result:

$$q_1^{POVM} = \sqrt{\frac{\eta_2}{\eta_1}} \cos \Theta \quad (3.28)$$

and

$$q_2^{POVM} = \sqrt{\frac{\eta_1}{\eta_2}} \cos \Theta \quad (3.29)$$

Finally, by substituting these optimal values just found in equation (3.24), the probability of optimal failure is obtained by

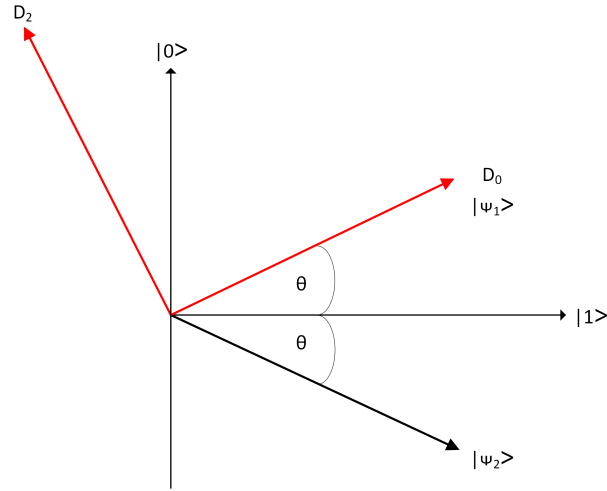
$$Q^{POVM} = 2\sqrt{\eta_1 \eta_2} \cos \Theta \quad (3.30)$$

It is to be noted that for

$$\eta_1 = \eta_2 = \frac{1}{2} \quad (3.31)$$

we obtain the IDP result of equation (3.5).

Such result is extremely useful for the average probability of failure of the two von Neumann measurements.



**Fig. 3.2** A von Neumann measurement that discriminates  $|\psi_2\rangle$  unambiguously

This probability for the first measurement of von Neumann, with its error direction in  $|\psi_1\rangle$ , can be written with a simple inspection as

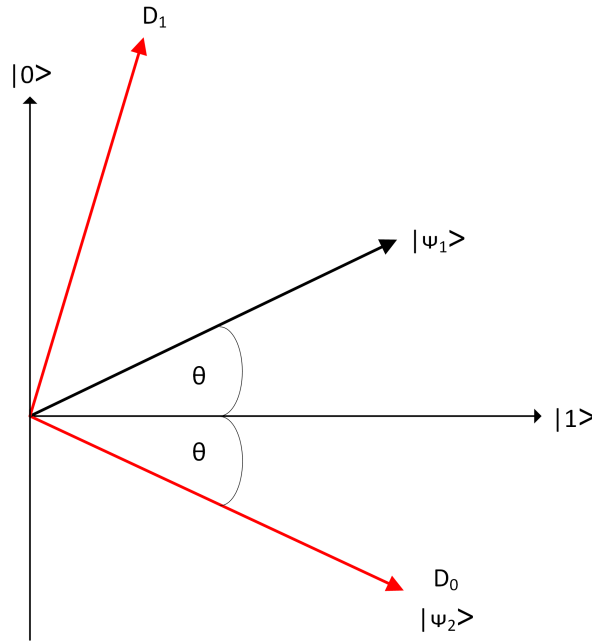
$$Q_1 = \eta_1 + \eta_2 |\langle \psi_1 | \psi_2 \rangle|^2 \quad (3.32)$$

because, in this direction and prepared with probability  $\eta_1$ ,  $|\psi_1\rangle$  gives a click with probability 1, and, prepared with probability  $\eta_2$ ,  $|\psi_2\rangle$  gives a click with probability  $|\langle\psi_1|\psi_2\rangle|^2$ . The corresponding configuration of the detector, which provides  $Q_1$  for the probability of failure, is shown in Figure 3.2.

With a similar reasoning, the probability of von Neumann second measurement's average failure, with error direction along  $|\psi_2\rangle$  is

$$Q_2 = \eta_1 |\langle\psi_1|\psi_2\rangle|^2 + \eta_2 \quad (3.33)$$

The corresponding detector set-up, which provides  $Q_2$  for the failure probability, is shown in Figure 3.3.



**Fig. 3.3** A von Neumann measurement that discriminates  $|\psi_1\rangle$  unambiguously

It can be observed that  $Q_1$  and  $Q_2$  are the arithmetic mean of two terms and  $Q^{POVM}$  is the geometric mean of the same terms in both cases.

For the existence of the POVM solution, we have

$$\begin{aligned} q_1^{POVM} &\leq 1 \\ q_2^{POVM} &\leq 1 \end{aligned} \quad (3.34)$$

Using  $\eta_2 = 1 - \eta_1$ , then the POVM exists in the interval



$$\frac{\cos^2 \Theta}{1 + \cos^2 \Theta} \leq \eta_1 \leq \frac{1}{1 + \cos^2 \Theta} \quad (3.35)$$

We now have two cases:

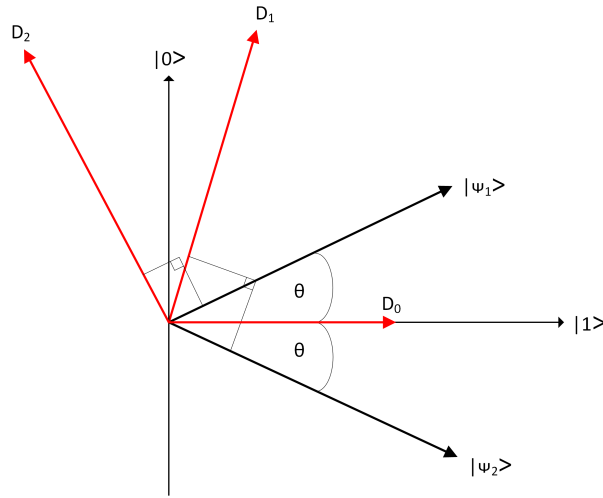
1. if  $\eta_1$  is smaller than the lower limit, the POVM is the first von Neumann measurement.
2. if  $\eta_1$  is higher than the upper limit, the POVM is the second von Neumann measurement

This can be easily seen from equations (3.20), (3.21) and (3.22) since

$$p_1 = 1 - q_1 = 0 \quad (3.36)$$

$\Pi_0$  becomes a projection along  $|\psi_1\rangle$  for  $q_1 = 1$ .

The configuration of the detection operators, which provides  $Q^{POVM}$  for the failure probability, is shown in Figure 3.4.



**Fig. 3.4** Optimal POVM that discriminates  $|\psi_1\rangle$  and  $|\psi_2\rangle$  unambiguously

The results we have just obtained are summarized as follows. The probability of optimal failure,  $Q^{opt}$ , is

$$Q^{opt} = \begin{cases} Q^{POVM}, & \text{if } \frac{\cos^2 \Theta}{1 + \cos^2 \Theta} \leq \eta_1 \leq \frac{1}{1 + \cos^2 \Theta} \\ Q_1, & \text{if } \eta_1 < \frac{\cos^2 \Theta}{1 + \cos^2 \Theta} \\ Q_2, & \text{if } \eta_1 < \frac{1}{1 + \cos^2 \Theta} \end{cases} \quad (3.37)$$

### 3.2.3 Neumark theorem and the realization of the POVM

In reality, detection operators are non-orthogonal and therefore POVM is difficult to achieve in a Hilbert space. To overcome this problem, we can make use of Neumark theorem [45].

It states that a POVM can be realized according to the following constructive procedure, also known as a generalized measurement.

The system is embedded in a larger Hilbert space, where extra degrees of freedom are called *ancilla*. A unitary transformation commits the degrees of freedom of the system with those of the *ancilla*. Von Neumann's projective measurements can still be performed on this larger system. You can choose the unitary transformation and the subsequent von Neumann measurement in such a way that if the result  $k$  is found on the larger system and using von Neumann measurement, the resulting transformation on the original system state is  $A_k |\psi\rangle$ . This corresponds to an element of the POVM in the original Hilbert space of the system.

We illustrate the power of this theorem on an alternative derivation of the condition on the individual failure probabilities (3.23). The joint Hilbert space  $K$  of the *system plus ancilla* is a tensor product of the two Hilbert spaces,  $H$  of the system and  $A$  of the *ancilla*,  $K = H \otimes A$ . This means that a state in  $K$  is a superposition of product states where, in each product, the first member is from  $H$  and the second is from  $A$ . Specifically, the two inputs now correspond to  $|\psi_1\rangle |\phi_0\rangle$  and  $|\psi_2\rangle |\phi_0\rangle$ , where  $|\phi_0\rangle$  describes the initial state of the *ancilla*. We choose the unitary transformation as

$$U(|\psi_1\rangle |\phi_0\rangle) = \sqrt{p_1} |\psi'_1\rangle |\phi_0\rangle + \sqrt{q_1} |\psi_0\rangle |\phi_1\rangle \quad (3.38)$$

and

$$U(|\psi_2\rangle |\phi_0\rangle) = \sqrt{p_2} |\psi'_2\rangle |\phi_0\rangle + \sqrt{q_2} e^{i\theta} |\psi_0\rangle |\phi_1\rangle \quad (3.39)$$

where  $|\phi_1\rangle$  is chosen to be orthogonal to  $|\phi_0\rangle$ , and  $|\psi'_1\rangle$  and  $|\psi'_2\rangle$  correspond to orthogonal vectors in the original Hilbert space.

If we now perform a von Neumann measurement on the *ancilla*, then a click along direction  $|\phi_1\rangle$  collapses both inputs onto the same output,  $|\psi_0\rangle$ . All information about the inputs is lost. The probability for this to happen, and with input  $i$ , is  $q_i$  ( $i = 1, 2$ ). Obviously, this outcome corresponds to the inconclusive result, so  $q_i$  are the failure probabilities of the corresponding input states  $i$ . On the other hand, a click along direction  $|\phi_0\rangle$  transforms the original inputs into orthogonal outputs in the Hilbert space system. The probability for this to happen, and with input  $i$ , is  $p_i$  ( $i = 1, 2$ ). Obviously, this outcome corresponds to full distinguishability in the original Hilbert space system, so  $p_i$  are the probabilities of success for discriminating the corresponding input states  $i$ . From unitarity we obtain  $p_i + q_i = 1$  for  $i = 1, 2$  and by taking the inner product of (3.38) and (3.39) we obtain (3.42).

### 3.2.4 More than Two Pure States

So far we have only considered a discrimination between two states, a case where a complete solution can be found. This is not true for more than two states. There are only a few general results and only for special case explicit solutions exist.

Let's analyze a general result that apply to the case of unambiguous discrimination. Only linearly independent states can be discriminated without ambiguity, according to Chefles [46]. Let the POVM for discriminating the  $N$  states  $|\psi_1\rangle, \dots, |\psi_N\rangle$ .

Consider the POVM  $\Pi_i = A_i^\dagger A_i$ ,  $i = 1, \dots, N, I$  with

$$A_I^\dagger A_I + \sum_{j=1}^N A_j^\dagger A_j = I \tag{3.40}$$

is an obvious generalization of equation (3.11) to  $N$  states.

We note that

$$\begin{aligned} \Pi_1 &= A_1^\dagger A_1 \\ &\vdots \\ &\vdots \\ \Pi_N &= A_N^\dagger A_N \end{aligned} \tag{3.41}$$

The operator  $A_I$  again corresponds to the inconclusive outcome, whereas the operator  $A_j$  to the identification of the state as  $|\psi_j\rangle$ . Since it has to be errorless, we must have

$$\langle \psi_k | A_j^\dagger A_j | \psi_k \rangle = \langle \psi_k | \Pi_j | \psi_k \rangle = p_j \delta_{jk} \tag{3.42}$$

where  $0 \leq p_j \leq 1$  is the probability of successfully identifying  $|\psi_j\rangle$ .

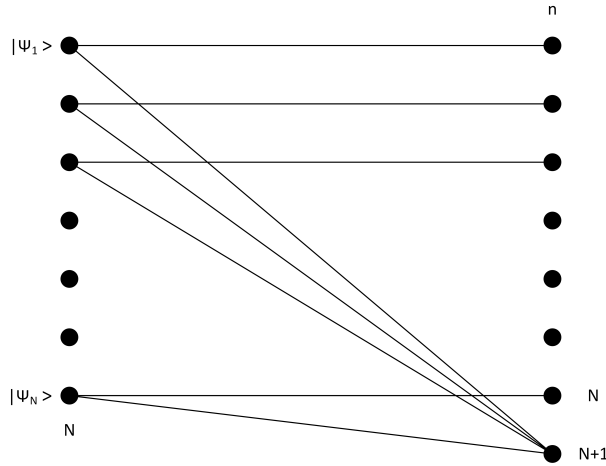


Fig. 3.5 Graphic representation of Unambiguous State Discrimination.

Note that (3.42) applies to the case of independent states. This can be proven as follow.

*Proof.* If we now suppose that the states are linearly dependent so that they can be expressed in terms of each other

$$|\psi_j\rangle = \sum_{k=1}^N c_{jk} |\psi_k\rangle \quad (3.43)$$

We can replace this relationship in equation (3.42) and we obtain that

$$\sum_{m,n=1}^N c_{km}^* c_{kn} \langle \psi_m | A_j^\dagger A_j | \psi_n \rangle = p_j \delta_{jk} \quad (3.44)$$

This result can be simplified by noting that

$$|\langle \psi_m | A_j^\dagger A_j | \psi_n \rangle|^2 \leq \langle \psi_m | A_j^\dagger A_j | \psi_m \rangle \langle \psi_n | A_j^\dagger A_j | \psi_n \rangle \quad (3.45)$$

which gives

$$\langle \psi_m | A_j^\dagger A_j | \psi_n \rangle = p_j \delta_{mn} \delta_{jm} \quad (3.46)$$

Finally, replacing this in equation (3.44) we find

$$|c_{kj}|^2 = \delta_{jk} \quad (3.47)$$

implying that states are non linear combinations and therefore necessarily linearly independent.  $\square$

At this point, consideration on the form of the operator  $A_j$  can be made, keeping in mind that we have

$$A_j |\psi_k\rangle = 0 \quad \forall j \neq k \quad (3.48)$$

Let  $|\psi_k^\perp\rangle$  the unit orthogonal vector to all the vectors  $|\psi_i\rangle \neq |\psi_k\rangle$ . We can therefore choose

$$A_j = \frac{\sqrt{p_j}}{\langle \psi_j^\perp | \psi_j \rangle} |\psi_j'\rangle \langle \psi_j^\perp| \quad (3.49)$$

where  $|\psi_j'\rangle$ ,  $j = 1, \dots, N$  are arbitrary orthogonal unit vectors.

The problem now is to find the values of  $p_j$ , given the a priori probability of the  $|\psi_j\rangle$  state  $\eta_j$ .

The values of  $p_j$  should be chosen to maximize the average probability of success,  $P$ , where

$$P = \sum_{j=1}^N \eta_j p_j \quad (3.50)$$

and in such a way that the operator

$$A_I^\dagger A_I = I - \sum_{j=1}^N A_j^\dagger A_j = I - \sum_{j=1}^N \frac{p_j |\psi_j^\perp\rangle\langle\psi_j^\perp|}{|\langle\psi_j|\psi_j^\perp\rangle|^2} \quad (3.51)$$

is positive.

The solution to the general  $N > 2$  problem must be obtained via numerical procedures and it is not trivial.

### 3.2.4.1 Upper and Lower bounds

The second general result states that there exists upper and lower bounds on the success probability. An **upper bound** is given by [76]

$$P \leq 1 - \frac{1}{N-1} \sum_{j=1}^N \sum_{\substack{k=1 \\ k \neq j}}^N \sqrt{\eta_j \eta_k} |\langle\psi_j|\psi_k\rangle| \quad (3.52)$$

Starting from the work of Duan and Guo [77], X. Sun, et al. derived a **lower bound** [78]. Consider the  $N \times N$  matrix whose elements are  $\langle\psi_j|\psi_k\rangle$ , and let  $\lambda_N$  be the smallest eigenvalue of this matrix. They showed that

$$P \geq \lambda_N \quad (3.53)$$

The problem of discriminating among three non-orthogonal states was first considered by Peres and Terno [79].

They developed a geometric approach and numerically applied it to several examples. A different method was considered by Duan and Guo [77] and Y. Sun and ourselves [80].

### 3.2.4.2 Case of three states

We considered the three vectors to be discriminated,  $|\psi_j\rangle$ ,  $j = 1, 2, 3$ , and lied in the space  $H$ . To this, a failure space,  $A$ , is appended so that the whole problem takes place in the space obtained by the direct sum extension

$$K = H \otimes A \quad (3.54)$$

If the procedure fails, the vector  $|\psi_j\rangle$  is mapped into a vector in the failure space,  $|\phi_j\rangle$ , and if it succeeds it is mapped onto a vector in the original space,  $\sqrt{p_j}|\psi'_j\rangle$ , where  $\|\psi'_j\| = 1$ , and  $0 \geq p_j \geq 1$ .

The vectors  $|\psi'_j\rangle$  are mutually orthogonal, so that they can be perfectly distin-

guished. Chefles showed that the set of failure vectors must be linearly dependent for the optimal procedure [46], so that the dimension of  $A$  is one or two.

Making this more explicit, we assume that there is a unitary operator,  $U$ , acting on  $K$ , such that

$$U|\psi_j\rangle = \sqrt{p_j}|\psi'_j\rangle + |\phi_j\rangle \quad (3.55)$$

It should be noted that, unlike in (3.39), the vector  $|\phi_j\rangle$  is not-normalized to unity. Instead, we use  $\langle\phi_j|\phi_j\rangle = q_j$  here. After  $U$  has been performed, we measure the projection operator onto the space  $H$ . If we obtain 1, the procedure succeeded, and we know what the input state was. If the input was  $|\psi_j\rangle$ , the procedure succeeds with probability  $p_j$ . If we obtain 0, the procedure failed, and this happens with probability  $q_j = 1 - p_j = \|\phi_j\|^2$ , if the input state was  $|\psi_j\rangle$ . The above equation implies that

$$\langle\phi_j|\phi_k\rangle = \langle\psi_j|\psi_k\rangle - \delta_{jk}p_j \quad (3.56)$$

Defining the matrix

$$C_{jk} = \langle\phi_j|\phi_k\rangle \quad (3.57)$$

we see by its definition that it must be positive definite. Therefore, the problem of finding the optimal unambiguous state discrimination procedure is reduced to finding the values of  $p_j$  that optimize the success probability

$$P = \sum_{j=1}^3 \eta_j p_j \quad (3.58)$$

subject to the constraint that the  $3 \times 3$  matrix, whose elements are  $\langle\psi_j|\psi_k\rangle - \delta_{jk}p_j$  is positive.

This can be solved in some special cases. We shall assume that all the a priori probabilities are the same, so that they are all  $1/3$ . If all overlaps are the same, i.e.

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\psi_3\rangle = \langle\psi_2|\psi_3\rangle = s \quad (3.59)$$

where  $s$  is real and positive, then  $q_j = s$ , for  $j = 1, 2, 3$ , and  $Q = 1 - P = s$  as well. There is also an explicit solution if

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \langle\psi_1|\psi_3\rangle = s_1 \\ \langle\psi_2|\psi_3\rangle &= s_2 \end{aligned} \quad (3.60)$$

where both  $s_1$  and  $s_2$  are real and positive. We first note that for a fixed value of  $s_1$  there is a restriction on how large  $s_2$  can be.

The largest angle between  $|\psi_2\rangle$  and  $|\psi_3\rangle$  is twice the angle between  $|\psi_1\rangle$  and  $|\psi_2\rangle$  (this maximum is achieved when the vectors are coplanar). This implies that

$$s_2 \geq 2s_1^2 - 1 \quad (3.61)$$

The solution to the state discrimination problem depends on whether  $s_1/s_2 < 2$  or not.

If  $s_1/s_2 < 2$ , we have

$$\begin{aligned} q_1 &= \frac{s_1^2}{s_2}, & q_2 &= q_3 = s_2 \\ Q &= \frac{1}{3} \left[ \frac{s_1^2}{s_2} + 2s_2 \right] \end{aligned} \quad (3.62)$$

If  $s_1/s_2 > 2$ , we have

$$\begin{aligned} q_1 &= 2s_1, & q_2 &= q_3 = s_1 - s_2 \\ Q &= \frac{2}{3} (2s_1 - s_2) \end{aligned} \quad (3.63)$$

### 3.2.4.3 Case of Symmetry (GUS)

There is another case where the exact solution to the unambiguous discrimination problem is known, and to which we now put our focus on. First we note that a set of  $N$  states is called symmetric [7] [11] [60] if there exists a unitary operator,  $V$ , such that, for  $j = 1, \dots, N-1$ ,

$$\begin{aligned} V |\psi_j\rangle &= |\psi_{j+1}\rangle \\ V |\psi_N\rangle &= |\psi_1\rangle \end{aligned} \quad (3.64)$$

This, in turn, means that  $V$  can be expressed as

$$V = \sum_{k=0}^{N-1} e^{2\pi i k/N} |\gamma_k\rangle \langle \gamma_k| \quad (3.65)$$

where  $|\gamma_k\rangle$  is the eigenstate of  $V$  with eigenvalue  $e^{2\pi i k/N}$ .

This implies that  $|\psi_j\rangle = V^{j-1} |\psi_1\rangle$ .

The case of unambiguous state discrimination for  $N$  symmetric states was analyzed by Chefles and Barnett [82]. They found an analytical expression for the optimal success probabilities when the a priori probabilities of the states are the same. The vectors  $|\psi_j\rangle$ ,  $j = 1, \dots, N$  are now assumed to span the entire space. Because the states  $|\psi_j\rangle$  form a basis for the space, (3.64) now implies that  $V^N = I$ .

The states can now be expanded as

$$|\psi_j\rangle = \sum_{k=0}^{N-1} e^{2\pi i k(j-1)/N} c_k |\gamma_k\rangle \quad (3.66)$$

The optimal success probability is found to be

$$P = N \min |c_k|^2 \quad (3.67)$$

where the minimum is taken over  $k$  [82].

### 3.2.5 Mixed States

Unambiguous discrimination can also be extended to the case of mixed states [25], where it may be applied to problems such as quantum state comparison [13] [25], subset discrimination [52], and determining whether a given state is pure or mixed [53].

Consider the problem of discriminating between two mixed states  $\rho_0$ ,  $\rho_1$ , which may be written in terms of their eigenvalues and eigenvectors as follows

$$\rho_0 = \sum_i \lambda_i^{(0)} |\lambda_i^{(0)}\rangle \langle \lambda_i^{(0)}| \quad (3.68)$$

and

$$\rho_1 = \sum_i \lambda_i^{(1)} |\lambda_i^{(1)}\rangle \langle \lambda_i^{(1)}| \quad (3.69)$$

where  $0 < \lambda_i^{(j)} \leq 1$ . Define the projectors

$$\Lambda_{ker}^{(0)} = I - \sum_i |\lambda_i^{(0)}\rangle \langle \lambda_i^{(0)}| \quad (3.70)$$

and

$$\Lambda_{ker}^{(1)} = I - \sum_i |\lambda_i^{(1)}\rangle \langle \lambda_i^{(1)}| \quad (3.71)$$

such that

$$\Lambda_{ker}^{(0)} \rho_0 = \Lambda_{ker}^{(1)} \rho_1 = 0 \quad (3.72)$$

These are the projectors onto the kernels of  $\rho_0$  and  $\rho_1$  respectively <sup>1</sup>. If we now define  $\Pi_1$  to lie in the kernel of  $\rho_0$  then

$$\Pi_1 = \Lambda_{ker}^{(0)} \Pi_1 \Lambda_{ker}^{(0)} \quad (3.73)$$

and clearly

$$Tr(\rho_0 \Pi_1) = Tr(\rho_0 \Lambda_{ker}^{(0)} \Pi_1 \Lambda_{ker}^{(0)}) = 0 \quad (3.74)$$

Thus, if there exists a positive operator  $\Pi_1$  in the kernel of  $\rho_0$  for which  $Tr(\rho_1 \Pi_1) > 0$ , then  $\rho_1$  may be unambiguously discriminated from  $\rho_0$ .

---

<sup>1</sup> The support of a mixed state  $\rho$  is the subspace spanned by its eigenvectors with non-zero eigenvalues. The kernel of a mixed state is the subspace orthogonal to its support.



Similarly  $\rho_0$  should lie in the kernel of  $\rho_1$ . A necessary and sufficient condition for unambiguous discrimination between two mixed states is that they have non-identical kernels, and therefore non-identical supports [25].

Unless the states are orthogonal, we get an inconclusive outcome, as before

$$\Pi_I = 1 - \Pi_0 - \Pi_1 \quad (3.75)$$

The solution that minimizes the probability of occurrence of the inconclusive result is difficult.

Some solutions are presented and known for some special cases. For example: one-dimensional kernel states [25], unambiguous discrimination between a pure and a mixed state, initially in two dimensions [54], and later extended to  $N$  dimensions [51]. Other examples may be found in [30],[39] and [40].

Reduction theorems given in [26] show that it is always possible to reduce a general problem to be in the form of discriminating two states, each of rank  $r$ , with them spanning across a  $2r$ -dimensional space.

Thus, the simplest case, which is non-reducible to pure state discrimination, is the problem of two rank-2 density operators in a 4-dimensional space. This was indeed analyzed in detail by Kleinmann et al [55]. Upper and lower bounds for the general case are given in [25], [28] and [31], a further reduction theorem in [40], and numerical algorithms are discussed in [29].

### 3.3 State discrimination with minimum error

As we have seen so far, whenever a definitive answer is returned after a state measurement, the result should be unambiguous, even with the possibility of obtaining inconclusive results.

However, we want to have only conclusive results for applications in quantum communication. This constraint, however, can generate errors that are inevitable when states are non-orthogonal.

Based on the measurement result, in each individual case a hypothesis must be made about the state of the quantum system. This procedure is known as quantum hypothesis testing.

The problem is to find the optimal measurement strategy that minimizes the probability of errors. In the most general case, we want to distinguish, with minimum probability of error, between  $N$  states of a quantum system ( $N > 2$ ), being characterized by the density operators  $\rho_j$ , with  $j = 1, \dots, N$ , and occurring with given a priori probabilities  $\eta_j$  which sum up to unity.

The measurement can be formally described with the help of a series of measurement operators  $\Pi_j$  that refer to the possible measurement results [7] and [11].

They are defined in such a way that  $Tr(\rho \Pi_j)$  is the probability of infer that the system is in the state  $\rho_j$  if it has been prepared in the state  $\rho$ .

In particular for pure states (third postulate) the transition probability are

$$p(i|j) = \text{Tr}(\Pi_i |\psi_j\rangle\langle\psi_j|) \quad (3.76)$$

Since the probability is a non-negative real number, detection operators must be Hermitean and positive semidefinite.

All of this leads to the following relationship:

$$\sum_{j=1}^N \Pi_j = I \quad (3.77)$$

where  $I$  denotes the unit operator in the Hilbert space of the quantum system.

Thus the overall probability  $P_{err}$  of making an incorrect hypothesis for one of the incoming states is then given by

$$P_{err} = 1 - P_{corr} = 1 - \sum_{j=1}^N \eta_j \text{Tr}(\rho_j \Pi_j) \quad (3.78)$$

with  $\sum_{j=1}^N \eta_j = 1$ .

Here, we have introduced the probability  $P_{corr}$  that the hypothesis is correct. To find the minimum error measurement strategy, it is necessary to determine the specific set of detection operators that minimizes the value of  $P_{err}$  under the constraint given by equation (3.77). By inserting these optimal detection operators into equation (3.78), the minimum probability of error

$$P_{err}^{min} \equiv P_E \quad (3.79)$$

is determined.

### 3.3.1 Distinguishing two quantum states with minimum error

If we consider the case of only two states, we can calculate the minimum probability of error  $P_E$  using the **Helstrom formula** [7].

We now analyze the measurement of the two-state minimum error with the help of an alternative method [47] [48] which allows us to obtain an immediate view of the structure of the optimal detection operators.

Starting from equation (3.78) and using the relations

$$\eta_1 + \eta_2 = 1 \quad (3.80)$$

and

$$\Pi_1 + \Pi_2 = I \quad (3.81)$$

that must be satisfied by the a priori probabilities and by the detection operators, respectively, we see that the total probability of obtaining an incorrect result in the measurement is given by

$$P_{err} = 1 - \sum_{j=1}^2 \eta_j Tr(\rho_j \Pi_j) = \eta_1 Tr(\rho_1 \Pi_2) + \eta_2 Tr(\rho_2 \Pi_1) \quad (3.82)$$

This formula can be alternatively written as follows

$$P_{err} = \eta_1 + Tr(\Lambda \Pi_1) = \eta_2 - Tr(\Lambda \Pi_2) \quad (3.83)$$

where we introduced the Hermitean operator

$$\Lambda = \eta_2 \rho_2 - \eta_1 \rho_1 = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k| \quad (3.84)$$

It should be noted that the states  $|\phi_k\rangle$  indicate the orthonormal eigenstates which belong to the eigenvalues  $\lambda_k$  of the operator  $\Lambda$  and, moreover, the eigenvalues are real.

Using the spectral decomposition of  $\Lambda$ , we obtain the following relation

$$P_{err} = \eta_1 + \sum_k \lambda_k \langle \phi_k | \Pi_1 | \phi_k \rangle = \eta_2 - \sum_k \lambda_k \langle \phi_k | \Pi_2 | \phi_k \rangle \quad (3.85)$$

Now, for the optimization, we need to determine the specific operators  $\Pi_1$ , or  $\Pi_2$ , that minimize the equation (3.85) under the constraint that

$$0 \leq \langle \phi_k | \Pi_j | \phi_k \rangle \leq 1 \quad (3.86)$$

with  $j = 1, 2$  and for all eigenstates  $|\phi_k\rangle$ .

This last constraint is due to the fact that  $Tr(\rho \Pi_j)$  denotes a probability for any  $\rho$ . Taking into account this constraint and the equation (3.85) it immediately follows that the smallest possible error probability is achieved when the detection operators are chosen in such a way that the equations

$$\langle \phi_k | \Pi_1 | \phi_k \rangle = 1 \quad (3.87)$$

to the positive eigenvalues and

$$\langle \phi_k | \Pi_1 | \phi_k \rangle = 0 \quad (3.88)$$

for all the negative eigenvalues.

For all the values of  $k$  corresponding to the optimum detection operators can be

written as

$$\Pi_1 = \sum_{k \text{ "positive"}} |\phi_k\rangle\langle\phi_k| \quad (3.89)$$

and

$$\Pi_2 = I - \Pi_1 \quad (3.90)$$

Obviously, provided that there are positive and negative eigenvalues in the spectral decomposition of  $\Lambda$ , the minimum error measurement for discrimination two quantum states is a von Neumann measurement that consists in carrying out projections on the two orthogonal subspaces of states  $\{|\phi_1\rangle, \dots, |\phi_{k_0-1}\rangle\}$ , on the one hand, and  $\{|\phi_{k_0}\rangle, \dots, |\phi_{D_S}\rangle\}$  on the other side.

Now, by inserting the optimum detection operators in the equation (3.83) it turns out that the minimum error probability [48] is

$$P_E = \eta_1 - \sum_{k \text{ "positive"}} \lambda_k \quad (3.91)$$

Now, using the condition  $\eta_1 + \eta_2 = 1$ , we obtain

$$P_E = \frac{1}{2} \left( 1 - \sum_k |\lambda_k| \right) = \frac{1}{2} (1 - \text{Tr}|\Lambda|) \quad (3.92)$$

where  $|\Lambda| = \sqrt{\Lambda^\dagger \Lambda}$ .

Using now, also equation (3.78), this brings us to the Helstrom formula [3] for the minimum error probability in discriminating  $\rho_1$  and  $\rho_2$ ,

$$P_E = \frac{1}{2} (1 - \text{Tr}|\eta_2 \rho_2 - \eta_1 \rho_1|) \quad (3.93)$$

### 3.3.1.1 Pure state

In the special case in which the states to be distinguished are **pure states**  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , this expression is reduced to [7]

$$P_E = \frac{1}{2} \left( 1 - \sqrt{1 - 4\eta_1 \eta_2 |\langle\psi_1|\psi_2\rangle|^2} \right) \quad (3.94)$$

The expression (3.94) can be cast to the equivalent form,

$$P_E = \eta_{\min} \left( 1 - \frac{2\eta_{\max} (1 - |\langle\psi_1|\psi_2\rangle|^2)}{\eta_{\max} - \eta_{\min} + \sqrt{1 - 4\eta_{\min}\eta_{\max} |\langle\psi_1|\psi_2\rangle|^2}} \right) \quad (3.95)$$

where  $\eta_{min}$  is the smallest of the prior probabilities,  $\eta_1$  and  $\eta_2$  and  $\eta_{max}$  that is greatest of the prior probabilities,  $\eta_1$  and  $\eta_2$ .

The first factor on the right-hand-side is what we would get if we always guessed the state that is prepared more often, without any measurement at all. Thus, the factor multiplying  $\eta_{min}$  is the result of the optimized measurement.

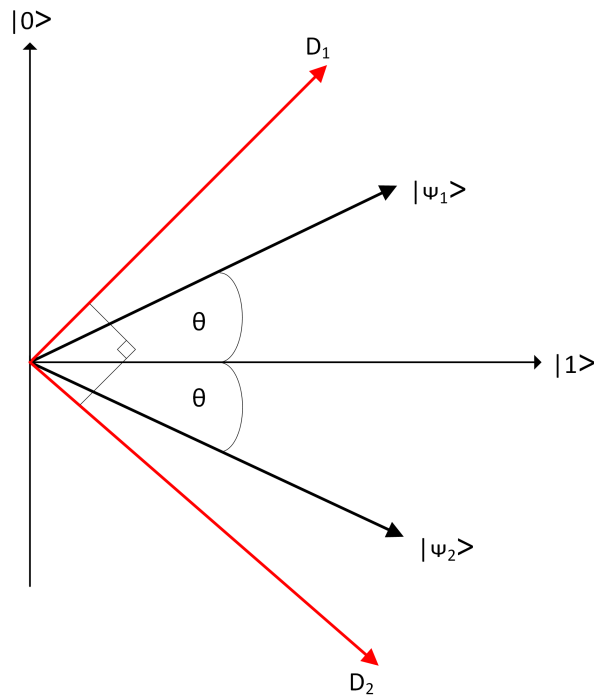
The setting of the detectors that obtain the optimum error probabilities is particularly simple in the case of equal probability a priori.

Two orthogonal detectors, symmetrically positioned around the two pure states, will perform the task, [7] with measurement vector  $|D_1\rangle$  and  $|D_2\rangle$  as shown in Figure 3.6 with

$$\Pi_1 = |D_1\rangle\langle D_1| \quad (3.96)$$

and

$$\Pi_2 = |D_2\rangle\langle D_2| \quad (3.97)$$



**Fig. 3.6** Detector configuration for the optimum minimum-error discrimination of two pure states with equal a priori probabilities. A von Neumann measurement with two orthogonal detectors placed symmetrically around  $|\psi_1\rangle$  and  $|\psi_2\rangle$  will achieve the optimum.

The set-up of the detectors that achieve the optimum error probabilities is particu-

larly simple for the case of equal a priori probabilities.

The result of equation (3.93) was obtained independently by Helstrom [7] and Holevo [14] and is commonly referred to as the Helstrom bound.

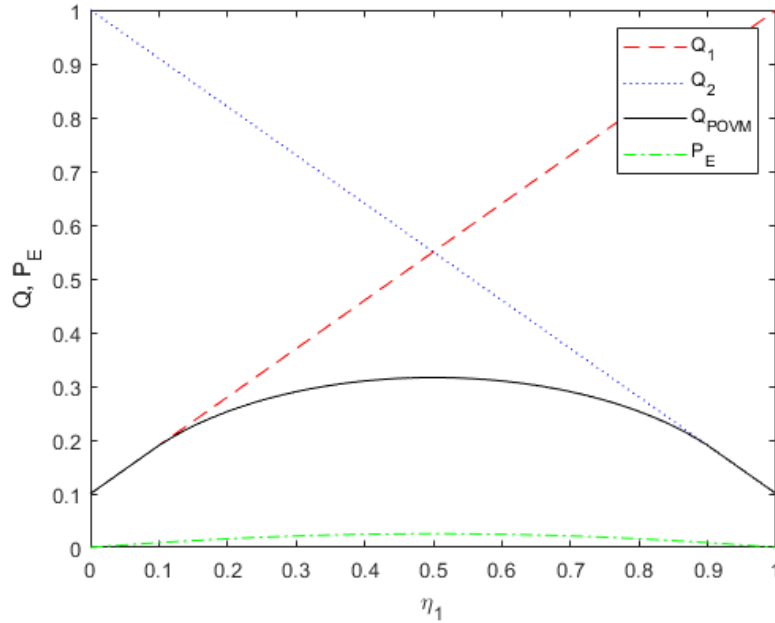
Finally, it should be noted that the relation

$$P_E \leq \frac{1}{2} Q^{opt} \quad (3.98)$$

is always satisfied between the minimum-error probability of the minimum-error detection and the optimal failure probability of unambiguous detection [22].

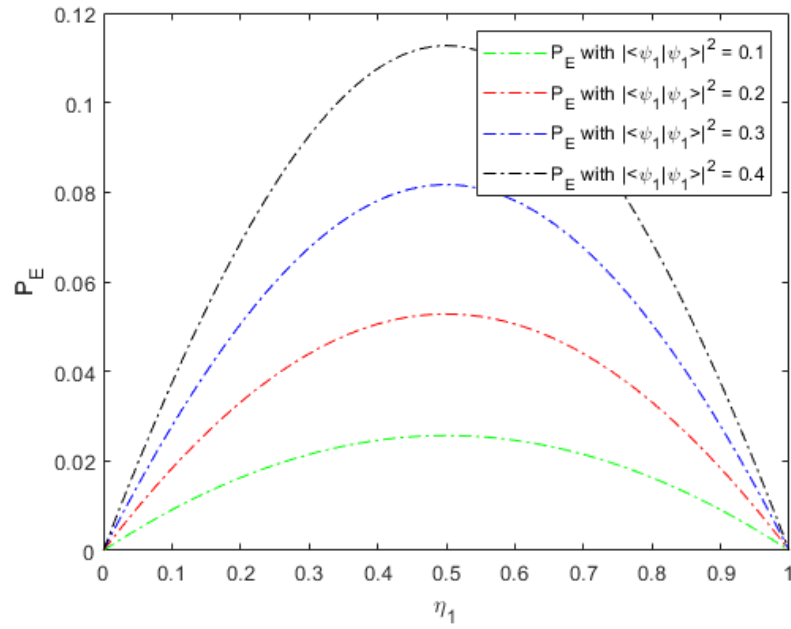
This result holds for pure as well as mixed states. It means that for two arbitrary states (mixed or pure), prepared with arbitrary a priori probabilities, the smallest possible failure probability in unambiguous discrimination is at least twice as large as the smallest probability of errors in minimum-error discrimination of the same states.

In Figure 3.7 we display the failure probabilities,  $Q_1$ ,  $Q_2$  and  $Q^{POVM}$  from equation (3.37) as well as the minimum error probability  $P_E$  from equation (3.95) vs  $\eta_1$  for a fixed value of the overlap,  $\cos^2 \Theta$ .



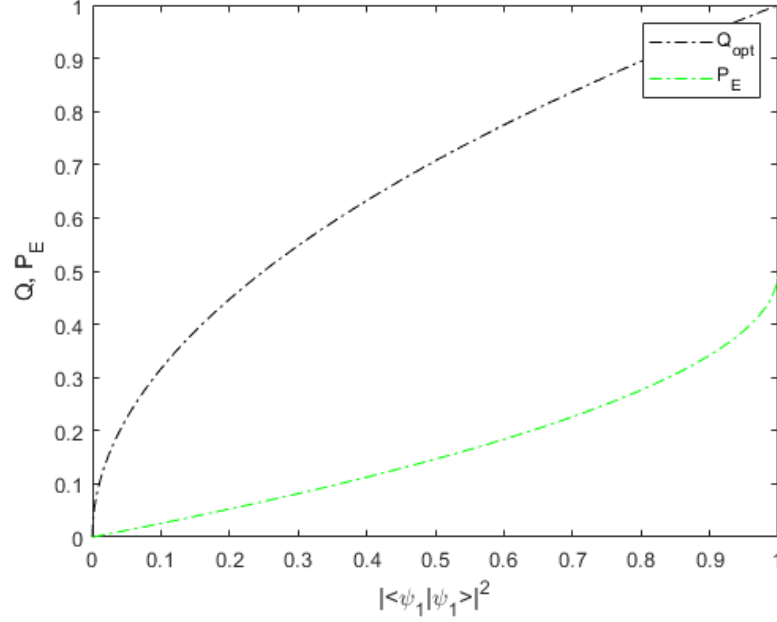
**Fig. 3.7** Failure probability,  $Q$ , and minimum error probability,  $P_E$ , vs the prior probability,  $\eta_1$ . Dashed line:  $Q_1$ , dotted line:  $Q_2$ , solid line:  $Q^{POVM}$ . For the figure we used the following representative value:  $|\langle \psi_1 | \psi_2 \rangle|^2 = 0.1$ . For this the optimal failure probability,  $Q_{opt}$  is given by  $Q_1$  for  $0 < \eta_1 < 0.09$ , by  $Q^{POVM}$  for  $0.09 \leq \eta_1 \leq 0.91$ , and by  $Q_2$  for  $0.91 < \eta_1$ .

In Figure 3.8 we display the minimum error probability,  $P_E$ , vs the prior probability,  $\eta_1$ .



**Fig. 3.8** Minimum error probability,  $P_E$ , vs the prior probability,  $\eta_1$

In Figure 3.9 we display the failure probability,  $Q^{POVM}$ , vs minimum error probability,  $P_E$ .



**Fig. 3.9** Failure probability,  $Q^{POVM}$ , vs minimum error probability,  $P_E$ . For the figure we used the following representative value:  $\eta_1 = \eta_2 = 1/2$

### 3.3.2 Distinguishing $N$ Symmetric Pure States

We conclude our review of minimum-error discrimination of a pure-state discrimination with a problem that is solvable and has found wide application in quantum communication. It consists in the so called square-root measurement that discriminates with minimum error between  $N$  equally probable symmetric states. Symmetric pure states are defined in such a way that each state derives from its predecessor by applying a unitary operator  $V$  in a cyclic way [86]

$$\begin{aligned} |\psi_j\rangle &= V |\psi_{j-1}\rangle = V^{j-1} |\psi_1\rangle \\ |\psi_1\rangle &= V |\psi_N\rangle \end{aligned} \quad (3.99)$$

for  $j = 1, \dots, N$  and implying that  $V^N = I$ .

For the case that the states occur with equal a priori probability, i. e. that  $\eta_j = 1/N$  for each of the states, Ban et al. found that the optimum detection operators for minimum-error discrimination are given by [86]

$$\Pi_j = A_j^\dagger A_j = B^{-1/2} |\psi_j\rangle \langle \psi_j| B^{-1/2} \equiv |\mu_j\rangle \langle \mu_j| \quad (3.100)$$



where

$$B = \sum_{j=1}^N |\psi_j\rangle\langle\psi_j| \quad (3.101)$$

$|\mu_j\rangle = B^{-1/2}|\psi_j\rangle$  are in general non-normalized and are called detection states. It is obvious that the special structure of the detection operators, or of the detection states, respectively, suggest the name *square-root-measurement*. The minimum error probability  $P_E$  for this measurement is [86]

$$P_E = 1 - \frac{1}{N} \sum_{j=1}^N |\langle\mu_j|\psi_j\rangle|^2 \quad (3.102)$$

in accordance with the fact that in the corresponding optimized measurement scheme the quantum system is inferred to have been prepared in the state  $|\psi_j\rangle$  provided that the state  $|\mu_j\rangle$  is detected. When the detection states  $|\mu_j\rangle$  are orthonormal, the detection operators are projection operators and the minimum-error measurement is a von-Neumann measurement. If not, it is a generalized measurement. The latter always holds true when the number of states exceeds the dimensionality of the physical state space of the quantum system, as can be immediately seen from the fact that the detection operators have to sum up to the unit operator in that space. In this case the given states are linearly dependent and form an overcomplete set in the Hilbert space of the system.

Let us apply the general solution in order to investigate minimum-error discrimination for the set of the  $N$  symmetric states

$$|\psi_j\rangle = \sum_{k=1}^D c_k e^{i\frac{2\pi}{N}j(k-1)} |\gamma_k\rangle, \quad (N \geq D) \quad (3.103)$$

where the coefficients  $c_k$  are arbitrary non-zero complex numbers with  $\sum_k |c_k|^2 = 1$ , and the states  $|\gamma_k\rangle$  ( $k = 1, \dots, D$ ) form a  $D$ -dimensional orthonormal basis. The given symmetric states are non-orthogonal except for the case that both the conditions  $N = D$  and  $|c_k|^2 = 1/N$  are fulfilled. To distinguish them with minimum error, provided that they occur with equal a priori probability, we obtain the optimum detection states

$$|\mu_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^D \frac{c_k}{|c_k|} e^{i\frac{2\pi}{N}jk} |\gamma_j\rangle \quad (3.104)$$

yielding the minimum error probability [87]

$$P_E = 1 - \frac{1}{N} \left( \sum_{k=1}^D |c_k| \right)^2 \quad (3.105)$$

In this case the detection states  $|\mu_j\rangle$  are non-orthogonal and non-normalized, with

$$\langle \mu_j | \mu_j \rangle = \frac{D}{N} \quad (3.106)$$

When  $N = D$ , however, the states  $|\psi_j\rangle$  are linearly independent and therefore can be discriminated also unambiguously.

Assuming equal a priori probabilities, the minimum failure probability,  $Q_F$ , for unambiguous discrimination of **symmetric states** has been derived to be

$$Q_F = 1 - N \min |c_k|^2 \quad (3.107)$$

according with [82].

Comparing this with the expression for  $P_E$ , the minimum error probability is found to be smaller than  $Q_F$ . It is worth mentioning that the minimum error probability,  $P_E$ , on the one hand, and the failure probability,  $Q_F$ , on the other hand, have been considered as distinguishability measures for ordering different ensembles of  $N$  equally probable symmetric pure states, and it has been found that these two measures impose different orderings [88].

### 3.4 SRM Approach

Now, we introduce the SRM approach that will be very useful for the study of *PPM* modulation performance in the next chapters.

We summarize the main steps of the SRM theory, and for a more detailed discussion is recommended reading the dedicated literature [1].

Starting from the constellation of  $K$  coherent states  $C = \{|\gamma_0\rangle, \dots, |\gamma_{K-1}\rangle\}$ , we evaluate in sequence

1. Grams matrix of the inner products  $G = [\langle \gamma_i | \gamma_j \rangle]$ , where  $i, j = 0, 1, \dots, K-1$ , calculated according to (4.8). In the cases of interest, the matrix  $G$ , which is  $K \times K$ , has rank  $K$ .
2. The spectral decomposition (EID) of  $G$

$$G = V \Lambda_G V^* = \sum_{i=0}^{K-1} \sigma_i^2 |v_i\rangle \langle v_i| \quad (3.108)$$

From this EID we find the eigenvalues  $\sigma_i^2$  and the orthonormal basis  $\{|v_i\rangle\}$ .

3. The square roots of  $G$

$$G^{\pm \frac{1}{2}} = V \Lambda_G^{\pm \frac{1}{2}} V^* \quad (3.109)$$

4. The transition probabilities according to [1]

$$p_c(i|j) = \left| \left( G^{\frac{1}{2}} \right)_{ij} \right|^2 \quad (3.110)$$

and the error probabilities (with equiprobable symbols)

$$P_e = 1 - \frac{1}{K} \sum_{i=0}^{K-1} \left| \left( G^{\frac{1}{2}} \right)_{ij} \right|^2 \quad (3.111)$$

5. The measurement vectors as linear combination of the states according to

$$M = \Gamma G^{\frac{1}{2}} \rightarrow |\mu_i\rangle = \sum_{j=0}^{K-1} \left( G^{\frac{1}{2}} \right)_{ij} |\gamma_j\rangle \quad (3.112)$$

With geometrically uniform symmetry (GUS). If the states  $|\gamma_i\rangle$  have the GUS, Grams matrix becomes circulant and its EID is given by

$$G = W_{[K]} \Lambda_G W_{[K]}^* = \sum_{i=0}^{K-1} \sigma_i^2 |w_i\rangle \langle w_i| \quad (3.113)$$

where the vectors  $|w_i\rangle$  are the columns of the DFT matrix  $W_{[K]}$

$$|w_i\rangle = \frac{1}{\sqrt{K}} \left[ W_K^{-i}, W_K^{-2i}, \dots, W_K^{-i(K-1)} \right]^T \quad i = 0, 1, \dots, K-1 \quad (3.114)$$

and the eigenvalues are given by the DFT of the first row  $[r_0, r_1, \dots, r_{K-1}]$  of the matrix  $G$

$$\lambda_i = \sigma_i^2 = \sum_{k=0}^{K-1} r_k W_K^{-ki}, \quad r_k = \langle \gamma_0 | \gamma_k \rangle \quad (3.115)$$

The square roots of  $G$  have elements

$$\left( G^{\pm \frac{1}{2}} \right)_{ij} = \frac{1}{K} \sum_{p=0}^{K-1} \lambda_p^{\pm \frac{1}{2}} W_K^{-p(i-j)} \quad (3.116)$$

and in particular the diagonal elements are all equal. Therefore, the error probability is given by

$$P_e = 1 - \left| \left( G^{\frac{1}{2}} \right)_{00} \right|^2 \quad (3.117)$$



## Chapter 4

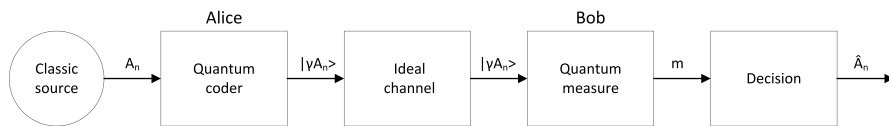
# Concepts of Communications

As previously discussed, the quantum decision theory can be applied to quantum communications systems.

In such systems, the nature of the states that carry the information is specified. A constellation of  $K$  quantum states, to which to commit a symbol belonging to a  $K$ -ary alphabet, corresponds, in the classical version, to a  $K$ -ary modulation format. We are still only considering states that operate at optical frequencies (optical quantum systems), because at radio frequencies quantum phenomena are not very relevant. In practice, the quantum states are usually treated as coherent states of a coherent monochromatic radiation emitted by a laser.

In the chapter 5, we are going to discuss binary systems, with the differences among classical, quantum and USD versions of the *OOK* (on-off keying), *2-PSK* (phase-shift keying) and *2-PPM* (Pulse position) modulations.

In chapter 6 we discuss a multilevel system, and examine differences for classical, quantum and USD versions of the  $K$ -*QAM* (quadrature amplitude),  $K$ -*PSK*, and  $K$ -*PPM* modulations. All of the above mentioned systems are examined in the absence of thermal noise. Thus, in this chapter, the scheme of Figure 4.1 is going to be followed where the channel is ideal and the received state is directly given by the transmitted state.



**Fig. 4.1** Quantum communications system for digital transmission.  $\{A_n\}$  is a sequence of classical symbols of information that Alice carries into a sequence of quantum states  $\{|\gamma A_n\rangle\}$ . For each symbol period, Bob performs a quantum measurement to identify, from the result  $m$  of the measurement, which symbol was transmitted.

Quantum measurements are affected by an intrinsic randomness (corresponding to shot noise in the classical model). Therefore, even if we ignore thermal noise, the

following analysis can still be done in the presence of noise. To learn more, readings of reviews [1] [74] are strongly encouraged.

## 4.1 Coherent States

In physics, and specifically in quantum mechanics, a coherent state is the specific quantum state of the quantum harmonic oscillator. It is often described as being the state with dynamics to be closely similar to the oscillatory behavior of a classical harmonic oscillator. When Erwin Schrödinger derived it in 1926, it was the first example of quantum dynamics. He derived it while searching for solutions of his own equation that satisfy the correspondence principle. The quantum harmonic oscillator and the coherent states arise in the quantum theory for a wide range of physical systems. For instance, a coherent state well describes the oscillating motion of a particle confined in a quadratic potential. Also, it describes a state in a system for which the ground-state wave packet is displaced from the origin of the system. This state can be related to classical solutions by a particle oscillating with an amplitude equivalent to the displacement. These states, expressed as eigenvectors of the lowering operator and forming an overcomplete family, were introduced in the early papers of John R. Klauder.

In the quantum theory of light (quantum electrodynamics) and other bosonic quantum field theories, coherent states were introduced by Roy J. Glauber in 1963.

The concept of coherent states has always been considerably abstract; it has become a major topic in mathematical physics and in applied mathematics, with applications ranging from quantization to signal processing and image processing. For this reason, the coherent states associated to the quantum harmonic oscillator are sometimes referred to as canonical coherent states (CCS), standard coherent states, Gaussian states, or oscillator states.

In quantum optics, the coherent state refers to a state of the quantized electromagnetic field, that describes a maximal kind of coherence and a classical kind of behavior. Erwin Schrödinger derived it as a "minimum uncertainty" Gaussian wave packet in 1926, searching for solutions of the Schrödinger equation that satisfy the correspondence principle. It is a minimum uncertainty state, with the single free parameter chosen to make the relative dispersion (standard deviation in natural dimensionless units) equal for position and momentum, each being equally small at high energy.

Further, in contrast to the energy eigenstates of the system, the time evolution of a coherent state is concentrated along the classical trajectories. The quantum linear harmonic oscillator, and hence coherent states, arise in the quantum theory of a wide range of physical systems. They occur in the quantum theory of light (quantum electrodynamics) and other bosonic quantum field theories. While minimum uncertainty Gaussian wave packets were well-known, they did not attract full attention until when Roy J. Glauber, in 1963, provided a complete quantum-theoretic

description of coherence in the electromagnetic field.

He wanted to describe the Hanbury-Brown and Twiss' experiment, which generated very wide baseline (hundreds or thousands of miles) interference patterns that could be used to determine stellar diameters. This opened the door to a much more comprehensive understanding of coherence.

In classical optics, light is thought of as electromagnetic waves radiating from a source. Often, coherent laser light is thought of as light that is emitted by many sources that are in phase. Actually, the picture of one photon being in-phase with another is not valid in quantum theory. Laser radiation is produced in a resonant cavity where the resonant frequency of the cavity is the same as the frequency associated with the atomic electron transition providing energy flow into the field. As energy in the resonant mode builds up, the probability for stimulated emission, in that mode only, increases. That is a positive feedback loop in which the amplitude in the resonant mode increases exponentially until some non-linear effects limit it. As a counter-example, a light bulb radiates light into a continuum of modes, and there is nothing that selects one over the other. The emission process is highly random in space and time. In a laser, however, light is emitted into a resonant mode, and that mode is highly coherent. Thus, laser light is idealized as a coherent state.

The energy eigenstates of the linear harmonic oscillator (i.e. masses on springs, lattice vibrations in a solid, vibrational motions of nuclei in molecules, or oscillations in the electromagnetic field) are fixed-number quantum states. The Fock state (i.e. a single photon) is the most particle-like state; it has a fixed number of particles, and phase is indeterminate. A coherent state distributes its quantum-mechanical uncertainty equally between the canonically conjugate coordinates, position and momentum, and the relative uncertainty in phase (defined heuristically), and amplitude are roughly equal and small at high amplitude.

That said, let's consider a generic model of the quantum state, created by an electromagnetic field at a certain frequency (optical). This model is given by a coherent quantum state according to Glauber's theory which will now be formulated in detail, with, of course, an emphasis on USD.

### 4.1.1 Glauber's Representation

The coherent radiation emitted by a laser is modeled as a coherent state. It has been demonstrated [67] – [69] that the coherent states of a single mode can be represented in a Hilbert space of infinite dimensions. This space, has an orthonormal basis  $\{|n\rangle, n = 0, 1, 2, \dots\}$ , called number states, because  $|n\rangle$  contains exactly  $n$  photons.

Considering this base, the number operator is associated, which is defined by

$$N = \sum_{n=0}^{\infty} n |n\rangle \langle n| \quad (4.1)$$

Then  $N$  has eigenvectors  $|n\rangle$  with eigenvalues  $n$ .

In this mathematical context, a generic coherent state (or Glauber state) is expressed as follows:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (4.2)$$

where  $\alpha$  is a complex amplitude whose meaning is

$$|\alpha|^2 = \text{average number of photons in the state } |\alpha\rangle \quad (4.3)$$

According to equation (4.2), to each point  $\alpha$  of the complex plane  $\mathbb{C}$ , a coherent state is associated with a physical meaning given by equation (4.3).

Thus, the more  $\alpha$  gets further from the origin of  $\mathbb{C}$ , the higher the photonic intensity associated to the state  $|\alpha\rangle$  becomes.

The set of coherent states will be indicated by

$$\mathbb{G} = \{|\alpha\rangle, \alpha \in \mathbb{C}\} \quad : \quad \text{coherent states} \quad (4.4)$$

### 4.1.2 Poisson regime in the coherent states

Photon statistics are the theoretical and experimental study of the statistical distributions produced in photon counting experiments that use Photodetectors to analyze the intrinsic statistical nature of photons in a light source. In these experiments, light's incident on the photodetector generates photoelectrons and a counter registers electrical pulses, generating a statistical distribution of photon counts. Low intensity disparate light sources can be differentiated by the corresponding statistical distributions produced in the detection process.

We now set up a quantum measurement with the number operator  $N$  (interpreted as an observable), in such a way to find a relationship between a representation of a coherent state  $|\alpha\rangle$  and Poisson's regime.

The outcome  $m$  of the measurement gives the number of photons of the quantum system in the state  $|\alpha\rangle$ .

The probability of having an outcome of the measurement  $m = i$  is

$$\begin{aligned} p(m = i | |\alpha\rangle) &= |\langle i | |\alpha\rangle|^2 = \left| \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \langle i | n \rangle \right|^2 = \\ &= \left| e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^i}{\sqrt{i!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2i}}{i!} \end{aligned} \quad (4.5)$$

Therefore,

$$p(m = i | |\alpha\rangle) = e^{-N\alpha} \frac{(N\alpha)^i}{i!} \quad (4.6)$$



with  $N_\alpha = |\alpha|^2$ .

It can also be proved that the average of  $m$  is

$$E[m|\alpha] = \langle \alpha | N | \alpha \rangle = |\alpha|^2 = N_\alpha \quad (4.7)$$

In conclusion, the outcome of the measurement  $m$  is a Poisson random variable with average  $N_\alpha = |\alpha|^2$ .

### 4.1.3 Degree of Superposition of Coherent States

It is important to evaluate the degree of superposition of two distinct coherent states  $|\alpha\rangle$  and  $|\beta\rangle$ , within the geometry given by the inner product. We have

**Definition 4.1.** The inner product of two coherent states is given by

$$\langle \alpha | \beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta)} \quad (4.8)$$

Hence two distinct coherent states are never orthogonal

The quadratic degree of superposition of two states is expressed by

$$|X|^2 := |\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2} \quad (4.9)$$

where  $X = \langle \alpha | \beta \rangle$ .

## 4.2 Constellations of Coherent States

In a quantum communication system, the target is the transmission of a sequence of classical symbols  $\{A_n\}$  through a sequence of quantum states  $\{|\gamma_{A_n}\rangle\}$ . Thus, in general, with a  $K$ -ary alphabet  $A = \{0, 1, \dots, K-1\}$ , Alice must be able to prepare a constellation of  $K$  coherent states

$$S = \{|\gamma_0\rangle, |\gamma_1\rangle, \dots, |\gamma_{K-1}\rangle\} \quad (4.10)$$

to realize the  $c \rightarrow q$  mapping

$$A_n \in A \rightarrow |\gamma_{A_n}\rangle \in S \quad (4.11)$$

which must be bijective. This operation is also called quantum encoding.

Our goal is to choose the constellation, and to do so, we will use the modulations

that are used for optical transmission systems, also called classical systems. This approach also has the advantage of allowing us to compare the performances of classical, quantum and USD systems.

A classical  $K$ -ary modulation, in general non-linear, is specified by  $K$  complex waveforms

$$\gamma_0(t), \gamma_1(t), \dots, \gamma_{K-1}(t) \quad (4.12)$$

of duration limited to the signaling interval  $[0, T]$ , with the rule that if  $A_n \in A$  is the  $n$ -th source symbol, the modulator forms a signal with complex envelope [70]

$$c(t) = \gamma_{A_n}(t) \quad 0 \leq t < T \quad (4.13)$$

With a sequence of symbols  $\{A_n\}$ , the complete expression of the complex envelope becomes

$$c(t) = \sum_{n=-\infty}^{+\infty} \gamma_{A_n}(t - nT) \quad (4.14)$$

from which a real modulated signal is obtained as

$$v(t) = \Re [c(t)e^{i2\pi\nu t}] \quad (4.15)$$

where  $\nu$  is the carrier optical frequency.

To proceed from the classical system, characterized by the waveforms  $\gamma_i(t)$ ,  $i \in A$ , to the quantum system and USD with coherent state constellation  $|\gamma_i\rangle$  with  $i \in A$ , we must somehow "remove" the dependence on time, which is not present in the coherent states.

### 4.2.1 State Constellations from Scalar Modulations

In *PSK* and *QAM* modulations, the waveforms (4.12) are of the form

$$\gamma_i(t) = \gamma_i h(t), \quad i \in A = \{0, 1, \dots, K-1\} \quad (4.16)$$

where  $h(t)$  is a real pulse and  $\gamma_i$  are complex numbers.

The complex envelope  $c(t)$  of the modulated signal is then produced by an encoder, mapping the symbols  $i \in A$  into the complex symbols  $\gamma_i$ , and by an interpolator with impulse response  $h(t)$ . The resulting complex envelope becomes

$$c(t) = \sum_{n=-\infty}^{+\infty} C_n h(t - nT) \quad (4.17)$$

where  $\{C_n\}$  is the sequence of complex symbols obtained by the mapping

$$A_n = i \rightarrow C_n = \gamma_i \quad (4.18)$$

In this way, a constellation of complex symbols is identified

$$C = \{\gamma_0, \gamma_1, \dots, \gamma_{K-1}\}, \quad \gamma_i \in \mathbb{C} \quad (4.19)$$

from which one can form the constellation of coherent states

$$S = \{|\gamma_0\rangle, |\gamma_1\rangle, \dots, |\gamma_{K-1}\rangle\}, \quad |\gamma_i\rangle \in \mathbb{G} \quad (4.20)$$

that are in a one-to-one relationship with the constellation of complex symbols  $C$ .

### 4.2.2 State Constellations from Vector Modulations

Everything that has been previously seen consisted of directly creating the constellation of coherent states from the constellation of symbols.

This system is not always possible, because in general the  $K$  waveforms (4.12) cannot be expressed in the form (4.16).

To remove the dependence on time, we can proceed as follows [70]. Let's take a function base,  $h_1(t), \dots, h_N(t)$ , orthonormal in the interval  $[0, T)$ , with  $N \leq K$ , and we expand the waveforms (4.12) on this basis, i.e.

$$\gamma_i(t) = \sum_{j=1}^N \gamma_{ij} h_j(t), \quad i = 0, 1, \dots, K-1 \quad (4.21)$$

where the coefficients are given by

$$\gamma_{ij} = \int_0^T \gamma_i(t) h_j^*(t) dt, \quad j = 1, \dots, N \quad (4.22)$$

The vectors of the complex coefficients

$$\gamma_i = (\gamma_{i1}, \dots, \gamma_{iN}), \quad i = 0, 1, \dots, K-1 \quad (4.23)$$

uniquely identify the waveform  $\gamma_i(t)$ .

The classic modulator is implemented in such a way that the encoder makes the map

$$A_n = i \in A \rightarrow C_n = \gamma_i \in \mathbb{C}^N \quad (4.24)$$

with

$$C_n = [C_{n1}, \dots, C_{nN}], \quad \gamma_i = [C_{i1}, \dots, C_{iN}] \quad (4.25)$$

Then from the vector  $C_n$ , a sequence of interpolators forms the complex envelope  $c(t)$  of the modulated signal, as

$$c(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^N C_{ni} h_i(t - nT) \quad (4.26)$$

This generalizes the scalar modulation, which is obtained with  $N = 1$ .

What we have already described, allows us to identify a constellation of complex vectors  $\{\gamma_i, i = 0, \dots, K-1\}$  with  $\gamma_i \in \mathbb{C}^N$ .

Now, to introduce the coherent states, we must consider a composite Hilbert space, given by the tensor product

$$H = H_0 \otimes H_0 \otimes \dots \otimes H_0 \quad (4.27)$$

of  $N$  equal Hilbert spaces  $H_0$ .

In this composite space, the states become the tensor product of coherent states and, through equation (4.23), to each symbol  $i \in A$  the tensor product of coherent states is associated

$$|\gamma_i\rangle = |\gamma_{i1}\rangle \otimes |\gamma_{i2}\rangle \otimes \dots \otimes |\gamma_{iN}\rangle \quad (4.28)$$

that, with  $i$  varying in  $A$ , forms the desired constellation of coherent states.

This composite constellation of coherent states will be used in *PPM* modulation.

### 4.3 Parameters in a Constellation of Coherent States

Note that the constellation of coherent states  $S$  given by equation (4.20) can be structured in a matrix form as

$$\Gamma = [|\gamma_0\rangle, |\gamma_1\rangle, \dots, |\gamma_{K-1}\rangle] \quad (4.29)$$

and becomes the state matrix.

In modulation formats, the states of  $S$  are always independent (in the sense of vector spaces), and therefore the state matrix always has the complete rank, that is,  $\text{rank}(\Gamma) = K$ . From the state matrix, we obtain the Gram's matrix, a  $K \times K$  matrix formed by the inner products between the pairs of states

$$G = \Gamma^* \Gamma = [\langle \gamma_i | \gamma_j \rangle], \quad |\gamma_i\rangle, |\gamma_j\rangle \in \mathbb{G} \quad (4.30)$$

that can be calculated using equation (4.8).

Also  $G$  has always the full rank and since the states are non-orthogonal, all the entries  $G$  are non-zero.

Also in the  $N$ -dimensional case, when the states are given by the tensor product of the  $N$  component states (4.28), to calculate the state overlap, we evaluate the inner

products

$$\langle \gamma_i | \gamma_j \rangle = \langle \gamma_{i1} | \gamma_{j1} \rangle \langle \gamma_{i2} | \gamma_{j2} \rangle \dots \langle \gamma_{iN} | \gamma_{jN} \rangle \quad (4.31)$$

In this relation we have kept in mind that the inner product of the states (given by a tensor product) is obtained as a product of the inner products of the component states.

Each of the inner products of the component states is evaluated by equation (4.8).

### 4.3.1 Number of Signal Photons in a Constellation

From equation (4.7) we have that the average number of photons associated to the coherent state  $|\gamma\rangle \in \mathbb{G}$  is given by the squared norm of the complex amplitude  $\gamma$

$$N_\gamma = |\gamma|^2 \quad (4.32)$$

In a constellation of coherent states, we introduce the signal photons per symbol. We can observe that the generic symbol of the constellation,  $C \in \mathbb{C}$ , must be considered as a random variable with probability  $p(C = \gamma)$ ,  $\gamma \in \mathbb{C}$ , and also the average number of photons  $N_C$  associated to  $C$  becomes a random variable; the statistical average of  $N_C$ ,

$$N_S = E[N_C] = \sum_{\gamma \in \mathbb{C}} p(C = \gamma) N_\gamma = \sum_{\gamma \in \mathbb{C}} p(C = \gamma) |\gamma|^2 \quad (4.33)$$

defines the average number of photons per symbol, or in other words, the number of signal photons per symbol.

Now, given the one-to-one correspondence  $A = i \Leftrightarrow C = \gamma_i$ , the probability of these two events turns out to be equal to the prior probability  $\eta_i$ .

Therefore, we have

$$N_S = \sum_{i \in A} \eta_i |\gamma_i|^2 \quad \text{photons/symbol} \quad (4.34)$$

In particular, with equally likely symbols, the number of signal photons per symbol becomes

$$N_S = \frac{1}{K} \sum_{i \in A} |\gamma_i|^2 = \frac{1}{K} \sum_{\gamma \in \mathbb{C}} |\gamma|^2 \quad (4.35)$$

Finally, knowing that, with equiprobable symbols, there are  $\log_2 K$  bit/symbol, the number of photons of the signal per bit is given by

$$N_R = \frac{N_S}{\log_2 K} \quad \text{photons/bit} \quad (4.36)$$

We have seen above that in a  $N$ -dimensional constellation  $\mathbb{A}$ , whose states are  $N$ -mode coherent states,  $|\gamma\rangle = |\gamma_1\rangle \otimes |\gamma_2\rangle \otimes \dots \otimes |\gamma_N\rangle$ , the average number of photons associated to the composite state  $|\gamma\rangle$  is

$$N_\gamma = |\gamma_1|^2 + |\gamma_2|^2 + \dots + |\gamma_N|^2 \quad (4.37)$$

where  $\gamma = [\gamma_1, \gamma_2, \dots, \gamma_N]$ . Consequently, the number of signal photons per symbol must be evaluated according to

$$N_S = \sum_{\gamma \in \mathbb{C}} p(C = \gamma) N_\gamma \quad (4.38)$$

with  $N_\gamma$  given by equation (4.37), and the sum extended to the  $N$ -dimensional constellation.

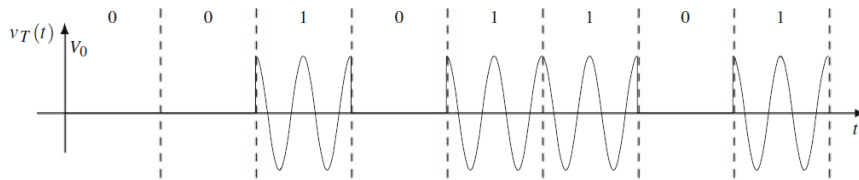
## Chapter 5

### Communications with binary systems

#### 5.1 OOK Modulation

##### 5.1.1 Classical Systems with OOK Modulation

Classical signal with *OOK* as shown in Figure 5.1.



**Fig. 5.1** A realization of a binary sequence and corresponding *OOK* signal

At reception ideal photon counter receiver uses the decision criterion

$$\hat{A}_0 = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n \geq 1 \end{cases} \quad (5.1)$$

where  $n$  is the number of photons counted in a symbol period. Then, with the transmission of the symbol  $A_0 = 0$ , we always have a correct decision

$$P_e(0) = 0 \quad (5.2)$$

When  $A_0 = 1$  the number of arrivals  $n$  is a Poisson variable with average  $N_R(1)$ , and therefore with (conditioned) distribution

$$p_n(k|1) = e^{-N_R(1)} \frac{N_R(1)^k}{k!}, \quad k = 0, 1, \dots \quad (5.3)$$

and we have an error when  $n = 0$ , which occurs with probability

$$P_e(1) = p_n(0|1) = e^{-N_R(1)} = e^{-2N_R} \quad (5.4)$$

The average error probability in the classical system is therefore

$$P_{e,classical} = \frac{1}{2} e^{-2N_R} \quad (5.5)$$

where equally likely symbols are assumed.

In optical communications this probability is called the **quantum limit** [71] or **shot noise limit**, and it is the optimum for any detection that does not exploit the coherence property of the optical beam. Notice, in fact, that in this classical context the decision criterion (5.1) is optimal.

The receiver scheme is called direct detection of the incident light pulses. The main advantage of this approach is its simplicity. In particular, phase and frequency instabilities of the laser source are well tolerated. Moreover, at the receiver direct detection is used and phase sensitive devices are avoided.

### 5.1.2 Quantum Interpretation of Photon Counting in OOK

The OOK modulation, has a simple quantum equivalent, employing the coherent states  $|0\rangle$  and  $|\alpha\rangle$  where the photon counting can be treated as a quantum measurement with not optimal measurement operators.

The quantum measurement realized by the photon counter is obtained with the elementary projectors  $|n\rangle\langle n|$ , where  $|\alpha\rangle$  is the number state, and the outcome of the measurement is given by the number of photons  $n$ . The transition probabilities in the measurement are

$$\begin{aligned} p(i|\alpha) &= p(n = i|\alpha) = e^{-|\alpha|^2} \frac{|\alpha|^{2i}}{i!}, \\ p(i|0) &= p(n = i|0) = \delta_{i0} \end{aligned} \quad (5.6)$$

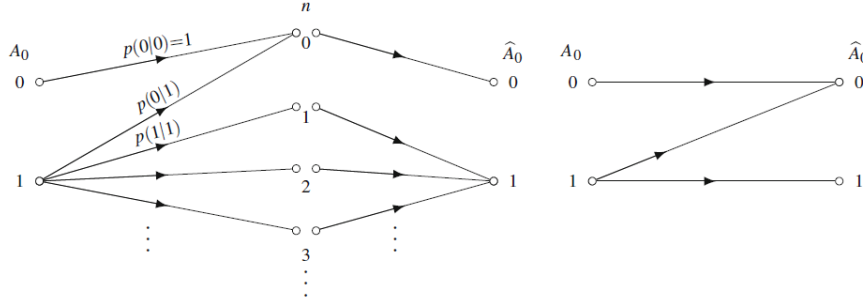
The alphabet of the measurement is then  $M = \{0, 1, 2, \dots\}$ , and it is different from the alphabet  $A = \{0, 1\}$  of the source.

To find the global performance, we must introduce a decision criterion consisting in the partitioning of  $M$  into two decision regions  $M_0$  and  $M_1$  to obtain two global measurement operators. The optimal partition is  $M_0 = \{0\}$  and  $M_1 = \{1, 2, \dots\}$  and



so we have the global projectors

$$\begin{aligned} Q_0 &= |0\rangle\langle 0| \\ Q_1 &= \sum_{n=1}^{\infty} |n\rangle\langle n| \end{aligned} \tag{5.7}$$



**Fig. 5.2** Quantum interpretation of the decision made via a photon counter in an *OOK* system. The outcome of quantum measurement is given by the number of photons  $n$  present in the state  $|\alpha\rangle$  or  $|0\rangle$ . The decision converts the measurement alphabet  $M = \{0, 1, 2, \dots\}$  into the binary alphabet  $A = \{0, 1\}$ , thus realizing a binary channel.

The global transition probabilities, from [1], are  $p_c(0|0) = Tr[\rho_0 Q_0]$  and  $p_c(0|1) = Tr[\rho_1 Q_0]$ , where  $\rho_1 = |\alpha\rangle\langle\alpha|$  and  $\rho_0 = |0\rangle\langle 0|$ . Then

$$\begin{aligned} p_c(0|0) &= \langle 0|0\rangle\langle 0|0\rangle = 1 \\ p_c(0|1) &= \langle\alpha|Q_0|\alpha\rangle = |\langle\alpha|0\rangle|^2 = e^{-|\alpha|^2} = e^{-2N_R} \end{aligned} \tag{5.8}$$

### 5.1.3 Quantum Systems with OOK Modulation

Now let's analyze the constellation of the *OOK* modulation, shown in Figure 5.3

$$|\gamma_0\rangle = |0\rangle, |\gamma_1\rangle = |\Delta\rangle \in G \tag{5.9}$$

where  $|0\rangle$  is the ground state and the state  $|\Delta\rangle$  is by the number  $\Delta$  which is not restrictive to assume real and positive. The quadratic superposition of the two states is  $|\langle 0|\Delta\rangle|^2 = e^{-\Delta^2}$ .

The number of signal photons associated with the symbol  $A_0 = 0$  is  $N_R(0) = 0$ , while the one associated to the symbol  $A_0 = 1$  is  $N_R(1) = \Delta^2$ .

The number of signal photons for bit is then

$$N_R = \frac{1}{2}N_R(0) + \frac{1}{2}N_R(1) = \frac{1}{2}N_R(1) \quad (5.10)$$

and the quadratic superposition of the two states can be written in the meaningful form

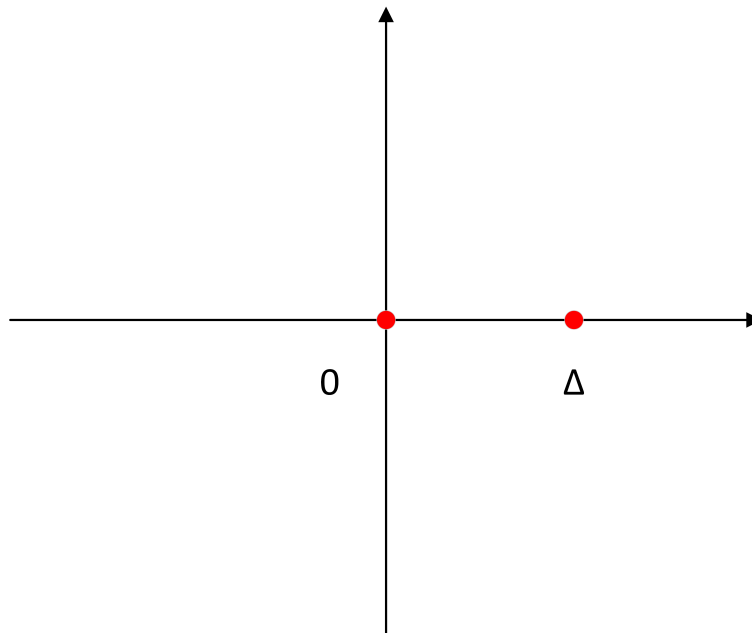
$$|X|^2 = e^{-2N_R} \quad (5.11)$$

From the formula of Helstrom [7], we obtain that the error probability of the *OOK* quantum system with equiprobable symbols becomes

$$P_{e,quantum} = \frac{1}{2} \left[ 1 - \sqrt{1 - |\langle \psi_1 | \psi_2 \rangle|^2} \right] \quad (5.12)$$

and finally becomes

$$P_{e,quantum} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-2N_R}} \right] \quad (5.13)$$



**Fig. 5.3** Constellation of symbols in *OOK* modulation

### 5.1.4 USD Systems with OOK Modulation

Now let's look at how the *OOK* system is detected in the USD system.

Starting from the equation (3.5) we consider the equiprobable symbols, as was done in the case of the quantum system.

So we make  $\eta_1 = \eta_2 = \frac{1}{2}$ . From this we get that

$$P_{e,USD} = Q = |\langle \psi_1 | \psi_2 \rangle| \quad (5.14)$$

becomes

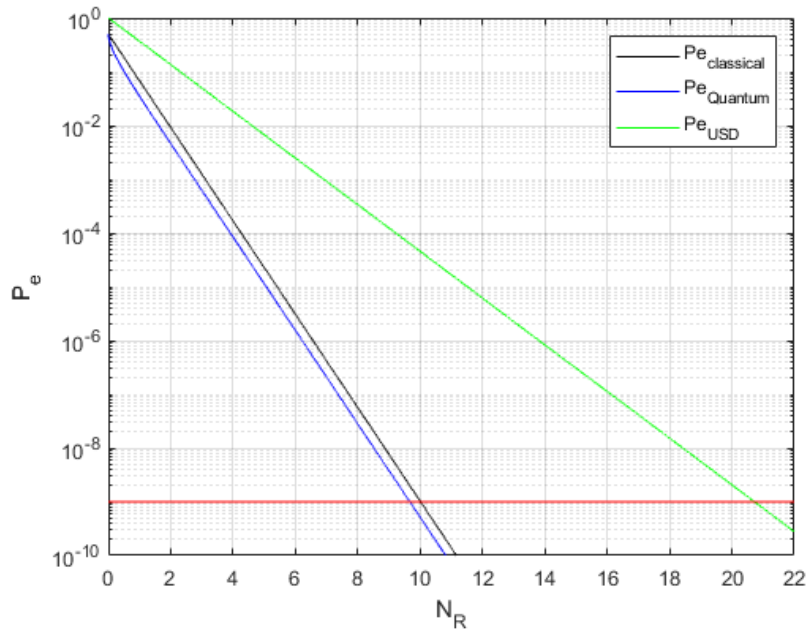
$$P_{e,USD} = e^{-\frac{1}{2}|\alpha-\beta|^2} \quad (5.15)$$

and follows

$$P_{e,USD} = e^{-\frac{1}{2}|\alpha|^2} \quad (5.16)$$

that can be rewritten as

$$P_{e,USD} = e^{-N_R} \quad (5.17)$$



**Fig. 5.4** Comparison of classical, quantum and USD *OOK* modulation

The comparison between the  $P_{e, \text{classical}}$  of the classical receiver, given by (5.5), the

$P_{e,quantum}$  of the quantum receive, given by (5.13) and  $P_{e,USD}$  of USD receive, given by (5.17) is shown in Figure 5.4 as a function of the average number of photons per bit  $N_R$ .

Thus we not have a great improvement in quantum *OOK* with respect to classical *OOK* (in error probability a relevant improvement is expressed in decades).

While, as far as *OOK* is concerned, in the case of USD, we have a deterioration in performance compared to both the classic and the quantum (in the probability of error).

We can see how the ambiguous state of the USD substantially influences the performance of this modulation.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity thresholds  $10^{-9}$ :

$$\begin{aligned} N_{R,quantum} &= 9.668 \text{ photons/bit} \\ N_{R,classical} &= 10.015 \text{ photons/bit} \\ N_{R,USD} &= 20.723 \text{ photons/bit} \end{aligned}$$

Finally, it should be noted that the relation (3.98) is valid because the value of  $N_{R,USD}$  is more than double of  $N_{R,quantum}$ .

## 5.2 BPSK Modulation

Phase-shift keying (*PSK*) is a digital modulation process which conveys data by changing (modulating) the phase of a constant frequency reference signal (the carrier wave).

### 5.2.1 Classical Systems with BPSK Modulation

The *BPSK* modulator encoding mapping

$$A_0 \rightarrow C_0 = e^{iA_0\pi} = \begin{cases} +1, & A_0 = 0 \\ -1, & A_0 = 1 \end{cases} \quad (5.18)$$

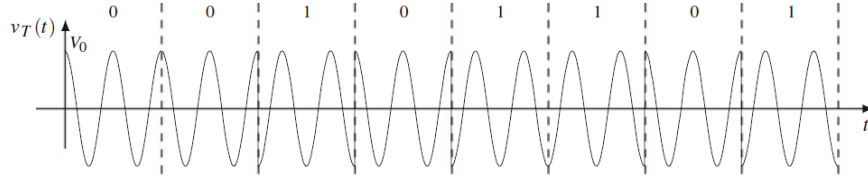
which gives the modulated signal

$$v_T(t) = \Re [C_0 V_0 e^{i2\pi\nu t}] = V_0 \cos(2\pi\nu t + A_0\pi), \quad 0 < t < T \quad (5.19)$$

where  $V_0$  is the amplitude of the carrier  $v_0(t) = V_0 \cos(2\pi\nu t)$ .

Figure 5.5 shows a sequence of binary symbols and the corresponding *BPSK* signal,

which in the interval  $(nT, nT + T)$  is given by  $V_0 \cos(2\pi vt + A_n\pi)$ .



**Fig. 5.5** A realization of a binary sequence and corresponding BPSK signal

Since different symbols have the same optical energy over the bit period, and hence the same photon counting, direct detection cannot discriminate between them.

Then the receiver adds to the incoming signal  $v_R(t) = V_R \cos(2\pi vt + A_n\pi)$  the signal  $V_L \cos(2\pi vt)$ , generated by a local laser tuned at the same frequency as  $v_0(t)$ , to get the signal

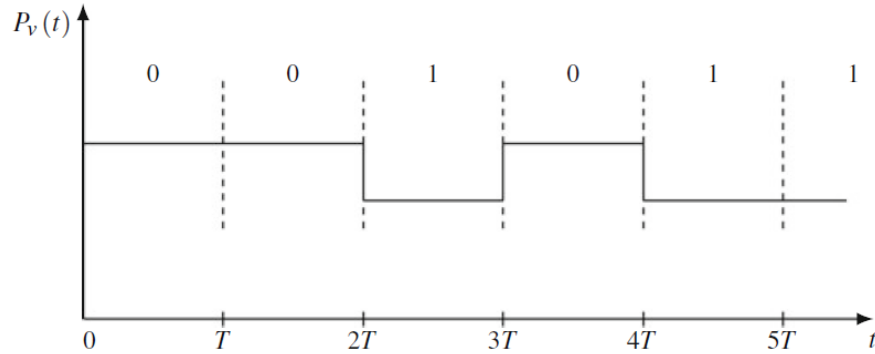
$$v(t) = V_R \cos(2\pi vt + A_0\pi) + V_L \cos(2\pi vt) \tag{5.20}$$

We assume that the local carrier has an amplitude  $V_L$  much greater than that of the received signal,  $V_L \gg V_0$  [1].

Since  $\cos(2\pi vt + A_0\pi) = \cos(A_0\pi)\cos(2\pi vt)$ , the power becomes

$$P_v(t) = (V_R \cos(A_0\pi) + V_L)^2 = V_R^2 + V_L^2 + 2V_R V_L \cos(A_0\pi) \tag{5.21}$$

which is illustrated in Figure 5.6 for a sequence of source symbols.



**Fig. 5.6** Example of the optical power  $P_R(t)$  after the introduction of the local carrier in a homodyne receiver

Applying this power to a photon counter, we obtain a number of arrivals  $n$  in a symbol period, which can be decomposed in the form

$$n = \bar{n}(A_0) + u \quad (5.22)$$

where  $\bar{n}(A_0) = E[n|A_0]$  is the useful signal and the fluctuation  $u$  is the shot noise. Now, from the theory of semiclassical detection developed in [1], the number of signal photons is given by the photonic intensity  $P_v(t)/h\nu$  integrated over  $(0, T)$ , and therefore it results in

$$\bar{n}(A_0) = H (V_L^2 + V_R^2 + 2V_R V_L \cos(\pi A_0)) = N_L + N_R + U_0 \cos(\pi A_0) \quad (5.23)$$

where  $H = T/h\nu$ ,  $N_L + N_R = H (V_L^2 + V_R^2)$  is a bias term,  $U_0 = 2\sqrt{N_L N_R}$ , and  $U_0 \cos(\pi A_0)$  is the symbol-dependent part. The variance, coinciding with the average, is

$$\sigma_n^2(A_0) = N_L + N_R + U_0 \cos(\pi A_0) \cong N_L \quad (5.24)$$

where the approximation follows from the hypothesis  $V_L \gg V_0$ . In conclusion, the decision on the transmitted symbol  $A_0$  is made on the value

$$n = N_L + N_R + U_0 \cos(\pi A_0) + u \quad (5.25)$$

At this point we introduce the Gaussian approximation, where it is assumed that the photon number  $n$  is a Gaussian random variable and hence specified by the mean  $\bar{n}(A_0)$  and by the variance  $\sigma^2(A_0) = N_L$ , which is independent of the symbol  $A_0$ . So, using [1], the error probability in the classical *BPSK* with homodyne receiver is given by

$$P_{e,classical} = Q\left(\sqrt{4N_R}\right) \quad (5.26)$$

This error probability is known as the **standard quantum limit**.

In the ideal counter the error probability in the classical *BPSK* is given by

$$P_{e,classical} = \frac{1}{2} e^{-4N_R} \quad (5.27)$$

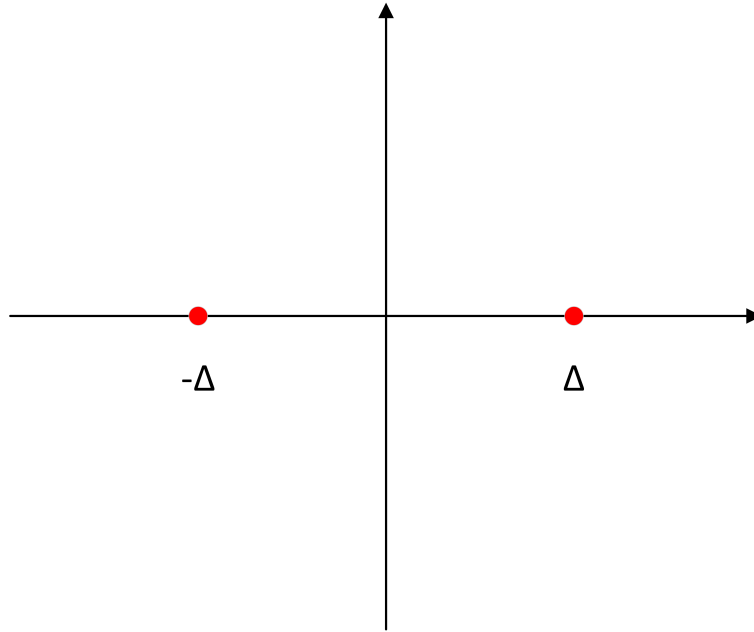
which represents the **super quantum limit**.

### 5.2.2 Quantum Systems with BPSK Modulation

In the *BPSK* quantum system, the symbol  $A_0 = 0$  (phase  $\varphi = 0$ ) is encoded into a coherent state  $|\Delta\rangle$  with a given amplitude  $\Delta$  and the symbol  $A_0 = 1$  (phase  $\varphi = \pi$ ) into the coherent state  $|-\Delta\rangle$ . It is show in Figure 5.7

$$|\gamma_0\rangle = |\Delta\rangle, \quad |\gamma_1\rangle = |-\Delta\rangle \in G \quad (5.28)$$

Obviously, the number of signal photons associated to the two states is equal



**Fig. 5.7** Constellation of symbols and states in 2-PSK modulation

$$N_R(0) = N_R(1) = |\Delta|^2 = N_R \quad (5.29)$$

and the (quadratic) superposition degree of the two states becomes

$$|X|^2 = e^{-\Delta - (-\Delta)^2} = e^{-4\Delta^2} = e^{-4N_R} \quad (5.30)$$

which yields the error probability

$$P_{e,quantum} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N_R}} \right] \quad (5.31)$$

Compared to the quantum BPSK modulation, we have an improvement, because the term at the exponent  $4N_R$  in place of  $2N_R$ .

### 5.2.3 USD Systems with BPSK Modulation

Now let's look at how the BPSK system is detected in the USD system. Starting from the equation (3.5) we consider the equiprobable symbols, as was done in the case of the quantum system.

So we make  $\eta_1 = \eta_2 = \frac{1}{2}$ . From this we get that

$$P_{e,USD} = Q = |\langle \psi_1 | \psi_2 \rangle| \quad (5.32)$$

becomes

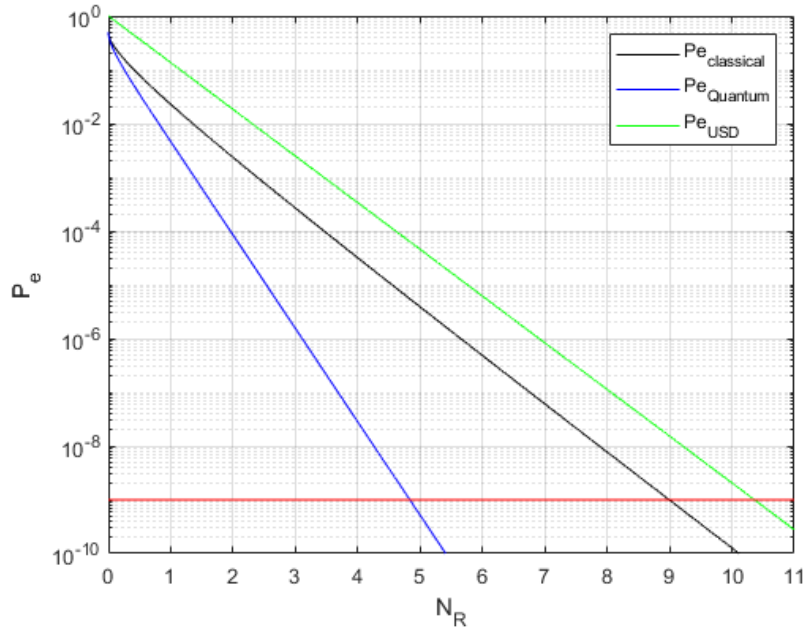
$$P_{e,USD} = e^{-\frac{1}{2}|\alpha - (-\alpha)|^2} \quad (5.33)$$

And we rewrite

$$P_{e,USD} = e^{-\frac{1}{2}4|\alpha|^2} \quad (5.34)$$

And finally becomes

$$P_{e,USD} = e^{-2N_R} \quad (5.35)$$



**Fig. 5.8** Comparison of classical, quantum and USD *BPSK* modulation

The comparison between the  $P_{e, \text{classical}}$  of the classical receiver, given by (5.26), the  $P_{e, \text{quantum}}$  of the quantum receiver, given by (5.31) and  $P_{e, \text{USD}}$  of USD receiver, given by (5.35) is shown in Figure 5.8 as a function of the average number of photons per bit  $N_R$ .

Analyzing the results we can see a worsening of the *BPSK* in USD compared to the classic *BPSK* (in the probability of error a significant improvement is expressed in decades).



Even if we compare the USD system with the quantum we have a deterioration in performance (in the probability of error).

So we can conclude by saying that with a binary modulation we can have unsatisfactory performance with the USD channel.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity thresholds  $10^{-9}$ :

$$\begin{aligned} N_{R,quantum} &= 4.837 \text{ photons/bit} \\ N_{R,classical} &= 8.913 \text{ photons/bit} \\ N_{R,USD} &= 10.361 \text{ photons/bit} \end{aligned}$$

Finally, it should be noted that the relation (3.98) is valid because the value of  $N_{R,USD}$  is more than double of  $N_{R,quantum}$ .

## 5.3 2-PPM Modulation

### 5.3.1 Classical System with 2-PPM Modulation

In the classical version, the symbol period  $T$  is subdivided into  $K = 2$  parts with spacing  $T_0 = T/K$ , obtaining  $K$  positions. Then, to the symbol  $i \in A = \{0, 1, \dots, K-1\}$  the waveform is associated consisting of a rectangle in the  $i$ th position  $iT_0$  of the symbol period

$$\gamma_i(t) = \begin{cases} \Delta, & iT_0 < t < (i+1)T_0 \\ 0, & elsewhere \end{cases} \quad i = 0, 1, \dots, K-1 \quad (5.36)$$

where  $\Delta > 0$  is the scale factor. But, we will adopt the specular format

$$\gamma_i(t) = \begin{cases} \Delta, & (K-1-i)T_0 < t < (K-i)T_0 \\ 0, & elsewhere \end{cases} \quad i = 0, 1, \dots, K-1 \quad (5.37)$$

where the  $i$ th position becomes  $(K-1-i)T_0$  instead of  $iT_0$ .

The reason of this choice is due to the fact that it simplifies the formulation of the symmetry operator in the quantum version. To waveforms (5.37),  $K$  binary words can be associated of length  $K$

$$\gamma_i = [\gamma_{i,K-1}, \dots, \gamma_{i,1}, \gamma_{i,0}] \quad i = 0, 1, \dots, K-1 \quad (5.38)$$

where  $\gamma_{ij} = \Delta \delta_{ij}$ . For example, for  $K = 2$  the words are

$$\begin{aligned}\gamma_0 &= [0 \ \Delta] \\ \gamma_1 &= [\Delta \ 0]\end{aligned}\tag{5.39}$$

### 5.3.2 Quantum System with 2-PPM Modulation

In the case of the *PPM* modulation, the states are represented by the formulation given by the paragraph 4.2.2 where the coherent states take the form:

$$|\gamma_i\rangle = |\gamma_{i,K-1}\rangle \otimes \dots \otimes |\gamma_{i,1}\rangle \otimes |\gamma_{i,0}\rangle, \quad i = 0, 1, \dots, K-1 \tag{5.40}$$

with

$$|\gamma_{ij}\rangle = \begin{cases} |\Delta\rangle, & i = j \\ |0\rangle, & i \neq j \end{cases} \tag{5.41}$$

where  $|\Delta\rangle$  is a coherent state with parameter  $\Delta$ , and  $|0\rangle$  is the ground state. In our case, i.e. with  $K = 2$ , we have two states:

$$\begin{aligned}|\gamma_0\rangle &= |0\rangle \otimes |\Delta\rangle \\ |\gamma_1\rangle &= |\Delta\rangle \otimes |0\rangle\end{aligned}\tag{5.42}$$

### 5.3.3 Performance of Quantum 2-PPM Systems

For binary quantum *PPM* ( $K = 2$ ), from (6.82) we get from Helstrom

$$P_{e,quantum} = \frac{1}{2} \left( 1 - \sqrt{1 - |X|^2} \right) \tag{5.43}$$

where  $|X|^2$  as the superposition of the states  $|\gamma_0\rangle$  and  $|\gamma_1\rangle$  which is equal to  $e^{-N_S}$ . Then since  $N_S = N_R$  the error probability is

$$\boxed{P_{e,quantum} = \frac{1}{2} \left( 1 - \sqrt{1 - e^{-2N_R}} \right)} \tag{5.44}$$

### 5.3.4 Performance of Classical 2-PPM Systems

For the study of performance of Classic *PPM* systems we consider  $A_0$  the transmitted word,  $B_0$  received word and  $\hat{A}_0$  decided word.

Let us consider the case  $K = 2$  with the standard format and with a unitary scale factor ( $\Delta = 1$ ), in which the possible transmitted words  $A_0$  are

$$\begin{aligned}\gamma_0 &= [1 \ 0] \\ \gamma_1 &= [0 \ 1]\end{aligned}\tag{5.45}$$

With a photon counter, the symbol 0 is always received correctly, whereas the symbol 1 may be received as 0, with an error probability  $e^{-N_S}$ . Then we have three possible received words  $B_0$ : the two correct words (5.45) and the wrong word  $[0 \ 0]$ , (see Figure 6.11) and we have to decide to which correct word the wrong word should be associated.

The optimum criterion (with equiprobable symbols) is to associate the wrong word  $[0 \ 0]$  to whatever correct word.

Then the decision criterion becomes

$$\begin{aligned}\hat{A}_0 &= \gamma_0 \quad \text{if } B_0 = [1 \ 0] \text{ or } B_0 = [0 \ 0] \\ \hat{A}_0 &= \gamma_1 \quad \text{if } B_0 = [0 \ 1]\end{aligned}\tag{5.46}$$

and we can get an error only in the last case, each with probability  $e^{-N_S}$ . Thus

$$P_{e,classical} = \frac{1}{2} [P_e(\gamma_1) + P_e(\gamma_2)] = \frac{1}{2} [0 + e^{-N_S}] = \frac{1}{2} e^{-N_S}\tag{5.47}$$

The general result is

$$P_{e,classical} = \frac{K-1}{K} e^{-N_S}\tag{5.48}$$

and since  $N_S = N_R$ , for binary classical PPM, from (5.48) we have

$$\boxed{P_{e,classical} = \frac{1}{2} e^{-N_R}}\tag{5.49}$$

### 5.3.5 Performance of USD 2-PPM Systems

Now let's look at how the 2-PPM system is detected in the USD system.

Starting from the equation (3.5) we consider the equiprobable symbols, as was done in the case of the quantum system.

So we make  $\eta_1 = \eta_2 = \frac{1}{2}$ . From this we get that

$$P_{e,USD} = Q = |\langle \psi_1 | \psi_2 \rangle|\tag{5.50}$$

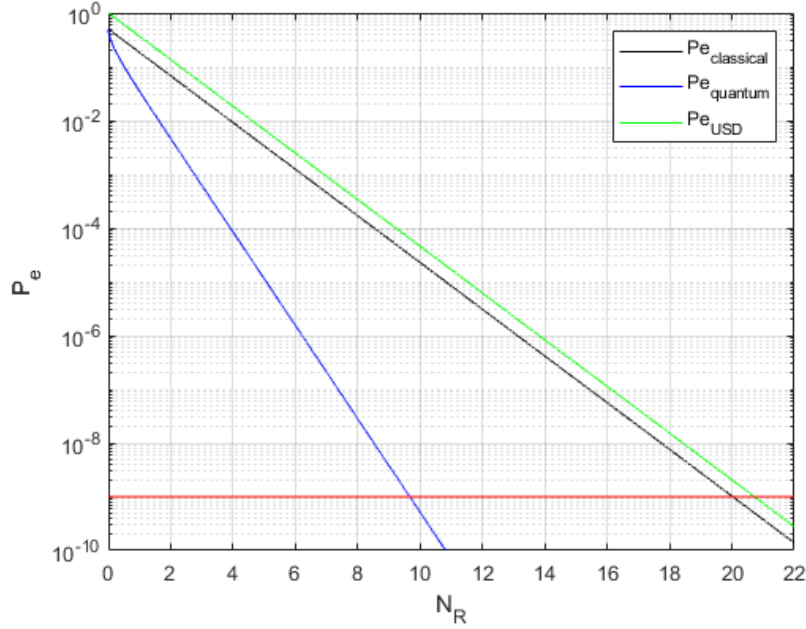
The inner product  $|\langle \psi_1 | \psi_2 \rangle|$  can be written in the following form

$$|\langle \psi_1 | \psi_2 \rangle| = |\langle \Delta | 0 \rangle| |\langle 0 | \Delta \rangle| = e^{-\frac{1}{2}N_S} e^{-\frac{1}{2}N_S}\tag{5.51}$$

and it follows that

$$P_{e,USD} = e^{-N_R}\tag{5.52}$$

The comparison between the  $P_{e, \text{classical}}$  of the classical receiver, given by (5.49), the  $P_{e, \text{quantum}}$  of the quantum receiver, given by (5.44) and  $P_{e, \text{USD}}$  of USD receiver, given by (5.52) is shown in Figure 5.9 as a function of the average number of photons per bit  $N_R$ .



**Fig. 5.9** Comparison of classical, quantum and USD 2-PPM modulation

Analyzing the results we can notice a nearness of the performances of the 2-PPM with USD compared to the classic 2-PPM.

While, if we compare the USD system with the quantum one, we have a deterioration in performance.

So we can conclude by saying that with a low-complexity PPM modulation we can have similar performance for USD and classic channel.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity thresholds  $10^{-9}$ :

$$N_{R, \text{quantum}} = 9.668 \text{ photons/bit}$$

$$N_{R, \text{classical}} = 20.031 \text{ photons/bit}$$

$$N_{R, \text{USD}} = 20.723 \text{ photons/bit}$$

Finally, it should be noted that the relation (3.98) is valid because the value of  $N_{R, \text{USD}}$  is more than double of  $N_{R, \text{quantum}}$ .

## Chapter 6

# Communications with multilevel systems

After dealing with the case of binary modulation in the last chapter, we now go to study multilevel modulations. In particular, *QAM*, *PSK* and *PPM* modulations.

### 6.1 K-QAM Modulation

#### 6.1.1 QAM Formats

The alphabet of *QAM* modulation consists of a constellation of  $K = L^2$  equally spaced points on a square grid of the complex plane, which can be defined starting from the  $L$ -ary balanced alphabet

$$A_L = \{-(L-1) + 2(i-1) \mid i = 1, 2, \dots, L\} \quad \text{with } L = 2, 3, \dots \quad (6.1)$$

The  $K$ -ary *QAM* constellation is then formed by the complex numbers

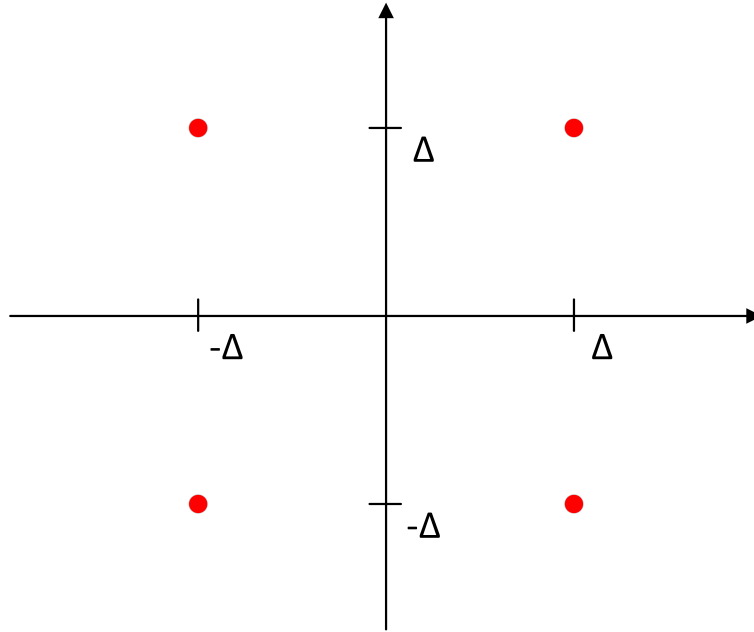
$$C = \{\Delta(u + iv) \mid u, v \in A_L\} \quad (6.2)$$

where  $\Delta$  is the scale factor and  $2\Delta$  gives the spacing of symbols in the constellation, with  $\Delta$  real and positive.

Figure 6.1 illustrates the constellation for  $L = 2$  i.e. 4-*QAM*.

To obtain the constellation of the coherent states in quantum *QAM* it suffices to assign to each symbol  $\gamma$  of the constellation  $C$  the corresponding coherent state  $|\gamma\rangle$ . Then the constellation of the coherent states becomes

$$S = \{|\gamma_{uv}\rangle = |\Delta(u + iv)\rangle \mid u, v \in A_L\} \quad (6.3)$$



**Fig. 6.1** Constellation of 4-*QAM* with scale factor  $\Delta$

For example, for the 4-*QAM*, where  $L = 2$  and  $A_2 = \{-1, 1\}$ , we have the following coherent states listed in lexicographic order

$$\begin{array}{lll}
 u = -1 & v = -1 & |\gamma_0\rangle = |\gamma_{-1,-1}\rangle = |\Delta(-1 - i)\rangle \\
 u = -1 & v = +1 & |\gamma_1\rangle = |\gamma_{-1,+1}\rangle = |\Delta(-1 + i)\rangle \\
 u = +1 & v = -1 & |\gamma_2\rangle = |\gamma_{+1,-1}\rangle = |\Delta(+1 - i)\rangle \\
 u = +1 & v = +1 & |\gamma_3\rangle = |\gamma_{+1,+1}\rangle = |\Delta(+1 + i)\rangle
 \end{array}$$

### 6.1.2 Performance of Quantum *QAM* Systems

We consider the decision based on the SRM method, described in paragraph 3.4. Let's start with the construction of the Gram matrix  $G$ , whose elements are inner products

$$\langle \gamma_{uv} | \gamma_{u'v'} \rangle = \langle \Delta(u + iv) | \Delta(u' + iv') \rangle \quad (6.4)$$

Remembering equation (4.8), we get

$$\langle \gamma_{uv} | \gamma_{u'v'} \rangle = e^{-\frac{1}{2}\Delta^2[(u'-u)^2 + (v'-v)^2 - 2i(u'v - v'u)]}, \quad u, v, u', v' \in A_L \quad (6.5)$$

The only problem in the construction of the Gram matrix  $G$  is the ordering of the four-index elements in a standard (bidimensional) matrix. To this end, we can use the lexicographical order indicated above. The main point of the SRM is the spectral decomposition of  $G$ , according to (3.108), that is,

$$G = V \Lambda_G V^* = \sum_{i=0}^{K-1} \sigma_i^2 |v_i\rangle \langle v_i| \quad (6.6)$$

which identifies the eigenvalues  $\sigma_i^2$  and the orthonormal basis  $|v_i\rangle$ ,  $i = 1, 2, \dots, K$ , and also the square roots

$$G^{\pm \frac{1}{2}} = V \Lambda_G^{\pm \frac{1}{2}} V^* \quad (6.7)$$

We can then compute the transition probabilities from (3.110) and the error probability from (3.111), that is,

$$p(j|i) = \left| \left( G^{\frac{1}{2}} \right)_{ij} \right|^2 \quad (6.8)$$

and

$$P_{e,quantum} = 1 - \frac{1}{K} \sum_{i=0}^{K-1} \left[ \left( G^{\frac{1}{2}} \right)_{ij} \right]^2 \quad (6.9)$$

The performance is evaluated according to the number of photons per symbol,  $N_S$ , given in general by (4.35).

We find for the *QAM*

$$N_S = \frac{2}{3} (L^2 - 1) \Delta^2 = \frac{2}{3} (K - 1) \Delta^2 \quad (6.10)$$

so that the shape factor of the *QAM* constellation is given by

$$\mu_K = \frac{2}{3} (K - 1) \quad (6.11)$$

### 6.1.3 Performance of classical Optical QAM Systems

The signal at the decision point is

$$z = C_0 U_0 + u_a + i u_b \quad (6.12)$$

where  $C_0 = A_0 + iB_0$  is the transmitted symbol,  $U_0$  is the amplitude,  $u_a$  and  $u_b$  are statistically independent Gaussian noises with null average and the same variance  $\sigma_u^2$ .

So the probability of error [1] turns out to be

$$P_{e,classical} = 1 - \left[ 1 - 2 \left( 1 - \frac{1}{L} \right) Q \left( \frac{U_0}{\sigma_u} \right) \right]^2 \quad (6.13)$$

where  $Q(x)$  is the normalized complementary Gaussian distribution.

The result depends on the cardinality  $K$  and on the SNR  $\Lambda = U_0^2/\sigma_u^2$ , which can be expressed as a function of the average number of photons per symbol  $N_S$  that is,

$$\Lambda = \frac{4N_S}{\mu_K} \quad (6.14)$$

with

$$\mu_K = \frac{2}{3}(K-1) \quad (6.15)$$

### 6.1.3.1 Case 4-QAM

In the specific case of 4-QAM, we have that

$$P_{e,classical} = 1 - \left[ 1 - Q \left( \sqrt{4N_R} \right) \right]^2 \quad (6.16)$$

where  $N_R = \frac{N_S}{2}$ .

### 6.1.3.2 Case 16-QAM

In the case of 16-QAM, we have that  $N_R = \frac{N_S}{4}$

$$P_{e,classical} = 1 - \left[ 1 - \frac{3}{2} Q \left( \sqrt{\frac{8}{5}N_R} \right) \right]^2 \quad (6.17)$$

## 6.1.4 Performance of USD QAM Systems

Let's see how the QAM system is detected in the USD system.

As we saw in Chapter 3, with a number of symbols greater than two, we can find upper and lower bounds for the USD system.



### 6.1.4.1 Lower bound

Regarding the lower bound, let's start with the formula (3.52). Knowing that the symbols are equiprobable then the  $\eta_i$  are equal to  $1/K$ .

So the error probability can be rewritten as

$$P_{e,USD_{Lower}} \geq \frac{1}{K-1} \sum_{i=1}^K \sum_{\substack{j=1 \\ j \neq i}}^K \sqrt{\frac{1}{K} \frac{1}{K}} |\langle \psi_i | \psi_j \rangle| \quad (6.18)$$

and follows

$$P_{e,USD_{Lower}} \geq \frac{1}{K-1} \sum_{i=1}^K \sum_{\substack{j=1 \\ j \neq i}}^K \frac{1}{K} |\langle \psi_i | \psi_j \rangle| \quad (6.19)$$

### 6.1.4.2 Upper bound

Now let's analyze the upper bound. Considering the equation (3.53) we can write the upper bound as

$$P_{e,USD_{Upper}} \leq 1 - \lambda_K \quad (6.20)$$

where  $\lambda_K$  is the minimum eigenvalue of the matrix

$$\begin{bmatrix} \langle \psi_1 | \psi_1 \rangle & \cdots & \langle \psi_1 | \psi_K \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_K | \psi_1 \rangle & \cdots & \langle \psi_K | \psi_K \rangle \end{bmatrix} \quad (6.21)$$

$\langle \psi_i | \psi_j \rangle$  can be written in the form

$$\langle \psi_i | \psi_j \rangle = e^{-\frac{1}{2}(|\psi|^2 + |\psi|^2 - 2\psi^* \psi)} \quad (6.22)$$

### 6.1.4.3 Case 4-QAM

In the case of 4-QAM the inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewritten as

$$|\langle \psi_i | \psi_j \rangle| = \begin{cases} e^{-2N_R}, & \text{if } |i-j| = 1 \text{ and } |i-j| = 3 \\ e^{-4N_R}, & \text{if } |i-j| = 2 \end{cases} \quad (6.23)$$

Therefore, the general equation given in (6.19) can be rewritten explicitly in the case of 4-QAM.

$$P_{e,USD_{Lower}} \geq \frac{1}{3} (2e^{-2N_R} + e^{-4N_R}) \quad (6.24)$$

For the upper bound we have  $|\psi_i|^2 = 2N_R \forall i$

$$\langle \psi_i | \psi_j \rangle = e^{-\frac{1}{2} \left( N_S + N_S - 2N_S e^{i\frac{2\pi}{K}(j-i)} \right)} \quad (6.25)$$

where  $N_S = 2N_R$ .

Therefore, we can write in extended form as follows

$$\langle \psi_i | \psi_j \rangle = \begin{bmatrix} 1 & e^{-N_S(1+i)} & e^{-2N_S} & e^{-N_S(1-i)} \\ e^{-N_S(1-i)} & 1 & e^{-N_S(1+i)} & e^{-2N_S} \\ e^{-2N_S} & e^{-N_S(1-i)} & 1 & e^{-N_S(1+i)} \\ e^{-N_S(1+i)} & e^{-2N_S} & e^{-N_S(1-i)} & 1 \end{bmatrix} \quad (6.26)$$

#### 6.1.4.4 Upper bound with GUS in the case of 4-QAM

Assuming equal a priori probabilities, the minimum failure probability for unambiguous discrimination of symmetric states is given by the formula (3.107) which is shown below

$$P_{e,USD} \leq 1 - K \min_k |c_k|^2 \quad (6.27)$$

The value of  $|c_k|^2$  is given by [82] and is equal to

$$|c_k|^2 = \frac{1}{K^2} \sum_{i=1}^K \sum_{j=1}^K e^{-\frac{2\pi i k(i-j)}{K}} \langle \psi_j | \psi_i \rangle \quad (6.28)$$

where  $\langle \psi_j | \psi_i \rangle$  is equal to

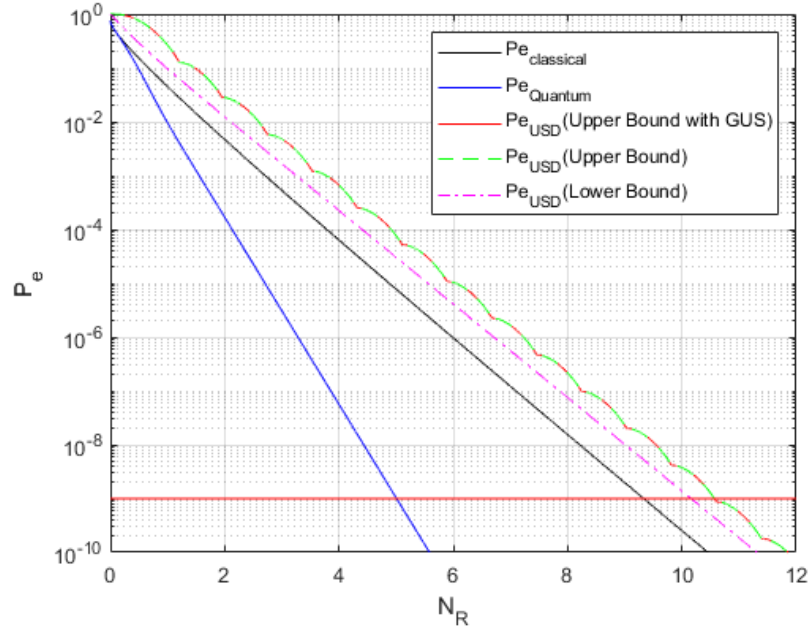
$$\langle \psi_j | \psi_i \rangle = e^{-\frac{1}{2} \left( N_S + N_S - 2N_S e^{i\frac{2\pi}{K}(j-i)} \right)} \quad (6.29)$$

So, the failure probability is as follows

$$P_{e,USD} \leq 1 - K \frac{1}{K^2} \min_k \left[ \sum_{i=1}^K \sum_{j=1}^K e^{-\frac{2\pi i k(i-j)}{K}} e^{-\frac{1}{2} \left( N_S + N_S - 2N_S e^{i\frac{2\pi}{K}(j-i)} \right)} \right] \quad (6.30)$$

and finally (for  $K = 4$ )

$$P_{e,USD} \leq 1 - \frac{1}{4} \min_k \left[ \sum_{i=1}^4 \sum_{j=1}^4 e^{-\frac{i\pi k(i-j)}{2}} e^{-\left( N_S - N_S e^{i\frac{\pi}{2}(j-i)} \right)} \right] \quad (6.31)$$



**Fig. 6.2** Comparison of classical, quantum and USD (Lower bound, Upper bound and GUS) 4-*QAM* modulation

The comparison between the  $P_{e,classical}$  of the classical receiver, given by (6.16), the  $P_{e,quantum}$  of the quantum receiver, given by (6.9) and  $P_{e,USD}$  of USD receiver, given by upper bound (6.20), lower bound (6.19) and GUS (6.31) is shown in Figure 6.2 as a function of the  $N_R$ .

Analyzing the results we can see a worsening of the *QAM* in USD compared to the classic *QAM*.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity  $10^{-9}$ :

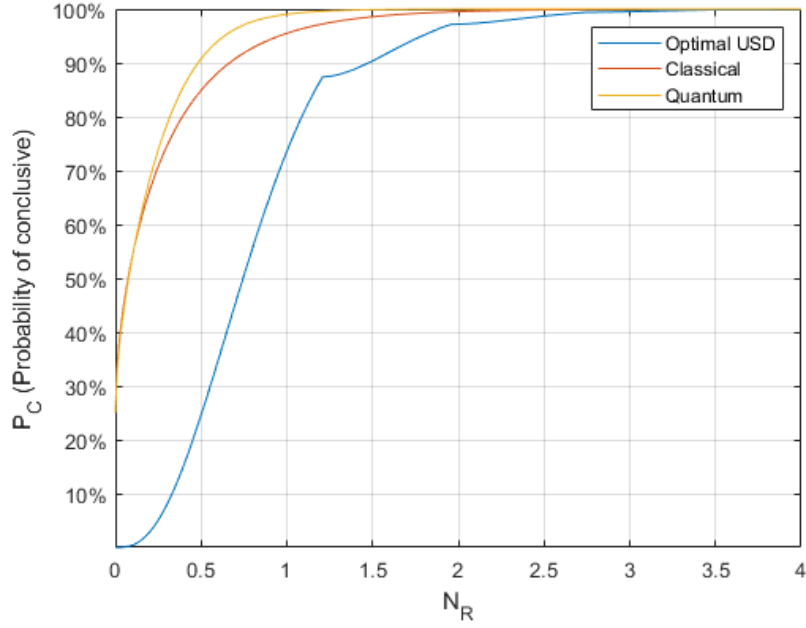
$$N_{R,quantum} = 5.007 \text{ photons/bit}$$

$$N_{R,classical} = 9.331 \text{ photons/bit}$$

$$N_{R,USD_{upper,GUS}} = 10.5677 \text{ photons/bit}$$

$$N_{R,USD_{upper}} = 10.5677 \text{ photons/bit}$$

$$N_{R,USD_{lower}} = 10.1589 \text{ photons/bit}$$



**Fig. 6.3** Probability of conclusive results as a function of mean photon number  $N_R$  for the USD scheme with 4- $QAM$  modulation in comparison with classical, quantum and Optimal USD

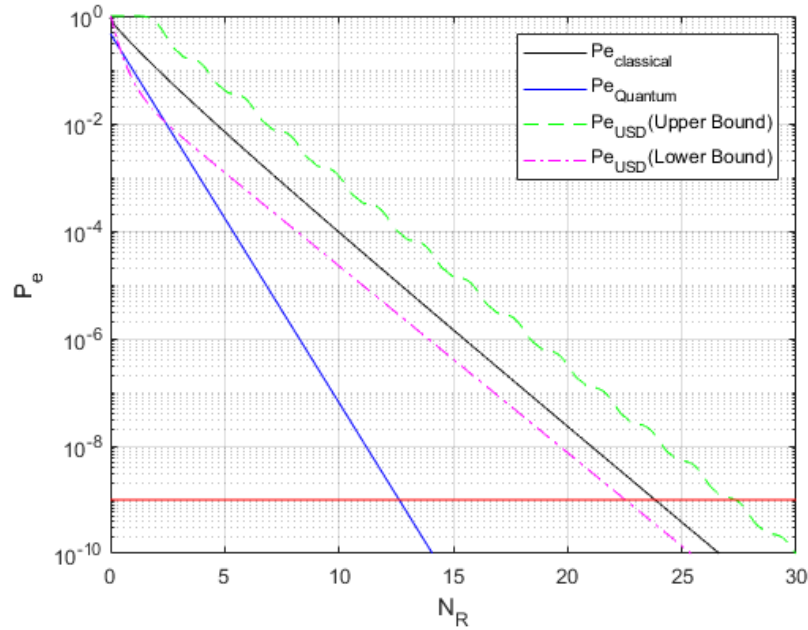
Figure 6.3 shows the probability of conclusive results as a function of mean photon number  $N_R$  of the input state for the USD scheme. The maximum probability of conclusive results for optimum USD is included in Figure 6.3 as a result of equation

$$P_{Optimum,USD} = K \min_k |c_k|^2 \quad (6.32)$$

#### 6.1.4.5 Case 16-QAM

In the case of 16- $QAM$  we have  $N_S = 10\Delta^2$  that is equal to  $N_R = \frac{5}{2}\Delta^2$  and  $\mu_K = 10$ . The inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewritten as

$$|\langle \psi_i | \psi_j \rangle| = \begin{cases} e^{-\frac{4}{5}N_R}, & \text{if } |i-j| = 1 \text{ and } |i-j| = 4 \\ e^{-\frac{16}{5}N_R}, & \text{if } |i-j| = 2 \text{ and } |i-j| = 8 \\ e^{-\frac{36}{5}N_R}, & \text{if } |i-j| = 3 \text{ and } |i-j| = 12 \\ e^{-\frac{8}{5}N_R}, & \text{if } |i-j| = 5 \\ e^{-4N_R}, & \text{if } |i-j| = 6 \text{ and } |i-j| = 9 \\ e^{-8N_R}, & \text{if } |i-j| = 7 \text{ and } |i-j| = 13 \\ e^{-\frac{32}{5}N_R}, & \text{if } |i-j| = 10 \\ e^{-\frac{52}{5}N_R}, & \text{if } |i-j| = 11 \text{ and } |i-j| = 14 \\ e^{-\frac{72}{5}N_R}, & \text{if } |i-j| = 15 \end{cases} \quad (6.33)$$



**Fig. 6.4** Comparison of classical, quantum and USD (Lower and Upper bound) 16-QAM modulation

Therefore, the general equation given in (6.19) can be rewritten explicitly in the case of 16-QAM.

$$P_{e,USD_{Lower}} \geq \frac{1}{15} \left( 2e^{-\frac{4}{5}N_R} + 2e^{-\frac{16}{5}N_R} + 2e^{-\frac{36}{5}N_R} + e^{-\frac{8}{5}N_R} + 2e^{-4N_R} \right) + \frac{1}{15} \left( 2e^{-8N_R} + e^{-\frac{32}{5}N_R} + 2e^{-\frac{52}{5}N_R} + e^{-\frac{72}{5}N_R} \right) \quad (6.34)$$

Instead, for the upper bound can be solved by considering the the general equation (6.20).

In the case of 16-*QAM* we report in Figure 6.4 the comparison between quantum, classic and USD (lower and upper bound).

The values of  $N_R$  are shown below for the various channels analyzed at the sensitivity  $10^{-9}$ :

$$\begin{aligned} N_{R,quantum} &= 12.48 \text{ photons/bit} \\ N_{R,classical} &= 23.83 \text{ photons/bit} \\ N_{R,USD_{Upper}} &= 27.34 \text{ photons/bit} \\ N_{R,USD_{Lower}} &= 22.53 \text{ photons/bit} \end{aligned}$$

## 6.2 K-PSK Modulation

Since the 4-*PSK* and 4-*QAM* overlap, as mentioned above, the evaluation of the performance of the USD will have the same result but with different resolution approach.

### 6.2.1 K-PSK Format

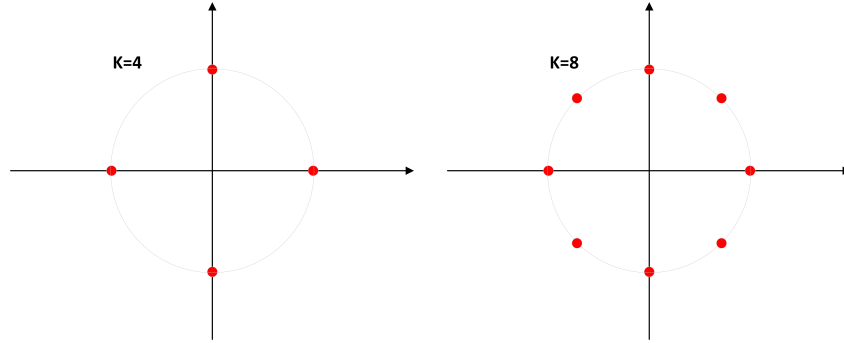
The constellation of the *PSK* modulation consists of  $K$  points uniformly distributed along a circle of the complex plane

$$C = \{ \Delta W_K^m \mid m = 0, 1, \dots, K-1 \} \quad (6.35)$$

where the scale factor  $\Delta$  is given by the radius of the circle and  $W_K = e^{i2\pi/K}$ . The constellation is illustrated in Figure 6.5 for some values of  $K$ .

In the quantum version, the states are obtained by simply associating to every complex symbol  $\psi$  of the constellation (6.35) the corresponding coherent state, which is given by

$$|\psi_m\rangle = |\Delta W_K^m\rangle = e^{-\frac{1}{2}\Delta^2} \sum_{n=0}^{\infty} \frac{(\Delta W_K^m)^n}{\sqrt{n!}} |n\rangle, \quad m = 0, 1, \dots, K-1 \quad (6.36)$$



**Fig. 6.5** Constellations of *PSK* modulation

In this constellation, all the coherent states have the same number of signal photons given by

$$N_S = \Delta^2 \quad (6.37)$$

Constellation (6.36) enjoys GUS [1]. The unitary operation  $S$  is obtained from the rotation operator, which is defined as

$$R(\phi) := e^{i\phi N} \quad (6.38)$$

where  $N$  is the number operator given by (4.1). Specifically we have

$$S = R\left(\frac{2\pi}{K}\right) = e^{\left(\frac{i2\pi}{K}N\right)} = W_K^N \quad (6.39)$$

### 6.2.2 Performance of Quantum K-PSK Systems

For the decision we apply the SRM, which gives an optimal result. Then, for the performance evaluation, we follow the procedure described in paragraph 3.4, taking into account that the *PSK* constellation satisfies the GUS. The generic element  $i, j$  of Gram's matrix  $G = [G_{ij}]$  is the inner product  $G_{ij} = \langle \psi_i | \psi_j \rangle$  obtained from (4.8) with  $\alpha = \Delta W_K^i$  and  $\beta = \Delta W_K^j$ , namely,

$$G_{ij} = e^{-\Delta^2(1 - W_K^{j-i})}, \quad i, j = 0, 1, \dots, K-1 \quad (6.40)$$

The element  $i, j$  depends only on the difference  $j-i$ ; and therefore Gram's matrix becomes circulant. The eigenvalues are obtained computing the DFT of the first row of the Gram's matrix, that is,

$$\lambda_i = \sum_{k=0}^{K-1} G_{0k} W_K^{-ki} \quad (6.41)$$

and the corresponding eigenvectors are given by the columns of the DFT matrix

$$|w_i\rangle = \frac{1}{\sqrt{K}} \left[ 1, W_K^{-i}, W_K^{-2i}, \dots, W_K^{-(K-1)i} \right]^T \quad (6.42)$$

Thus the matrices  $G^{\pm\frac{1}{2}}$  are obtained from (3.116), where the element  $i, j$  is given by

$$\left( G^{\pm\frac{1}{2}} \right)_{ij} = \frac{1}{K} \sum_{p=0}^{K-1} \lambda_p^{\pm\frac{1}{2}} W_K^{p(j-i)} \quad (6.43)$$

Finally, the error probability with equiprobable symbols is simply

$$P_{e,quantum} = 1 - \frac{1}{K} \left( \sum_{i=0}^{K-1} \sqrt{\lambda_i} \right)^2 \quad (6.44)$$

Therefore, to calculate  $P_{e,quantum}$  it suffices to evaluate the eigenvalues according to (6.41) and to apply (6.44). As usual,  $P_{e,quantum}$  can be expressed as a function of the number of signal photons per symbol  $N_S$ , given by (6.37).

### 6.2.3 Performance of Classical K-PSK Systems

The classical optical PSK system falls into the general model of quadrature modulations (with homodyne receiver). The signal at the decision point becomes (based on [1])

$$z_0 = C_0 U_0 + u_a + i u_b \quad (6.45)$$

where  $C_0$  is the transmitted symbol,  $C_0 \in \mathbb{C}_0 = \{W_K^i | i = 1, \dots, K\}$ ,  $u_a$  and  $u_b$  are independent zero-mean Gaussian components with the same variance  $\sigma_u^2$ . At this point, we introduce the count parameters

$$\begin{aligned} U_0 &= (2V_R V_L) H \\ \sigma_u^2 &= H V_L^2 \end{aligned} \quad (6.46)$$

and that in this case the number of signal photons contained in the received power is

$$N_S = H V_L^2 \quad (6.47)$$

So the SNR becomes



$$\Lambda = \frac{U_0^2}{\sigma_u^2} = 4N_S \quad (6.48)$$

the shape factor  $\mu_K$  being unitary.

For  $K > 4$  the exact computation is not known, and we resort to the inequality

$$P_{e,classical} < P'_e = e^{-\frac{1}{2} \frac{U_0^2}{\sigma_u^2} \sin^2(\frac{\pi}{K})} = e^{-2N_S \sin^2(\frac{\pi}{K})} \quad (6.49)$$

### 6.2.3.1 Case 8-PSK

In that case  $K=8$ ,  $N_S = 3N_R$ , and the outcome is follows

$$P_{e,classical} < e^{-2N_S \sin^2(\frac{\pi}{8})} = e^{-6N_R \sin^2(\frac{\pi}{8})} \quad (6.50)$$

### 6.2.3.2 Case 16-PSK

In this case  $K=16$ ,  $N_S = 4N_R$ , the result is as follows

$$P_{e,classical} < e^{-2N_S \sin^2(\frac{\pi}{16})} = e^{-8N_R \sin^2(\frac{\pi}{16})} \quad (6.51)$$

## 6.2.4 Performance of USD K-PSK Systems

In the case of USD we have the lower bound and the upper bound.

### 6.2.4.1 Lower bound in the case of 8-PSK

The error probability is as follows

$$P_{e,USD,Lower} \geq \frac{1}{7} \sum_{i=1}^8 \sum_{\substack{j=1 \\ j \neq i}}^8 \frac{1}{8} |\langle \psi_i | \psi_j \rangle| \quad (6.52)$$

where the inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewrite as

$$|\langle \psi_i | \psi_j \rangle| = \begin{cases} e^{-3N_R(1-1/\sqrt{2}-i/\sqrt{2})}, & \text{if } |i-j| = 1 \\ e^{-3N_R(1-i)}, & \text{if } |i-j| = 2 \\ e^{-3N_R(1+1/\sqrt{2}-i/\sqrt{2})}, & \text{if } |i-j| = 3 \\ e^{-6N_R}, & \text{if } |i-j| = 4 \\ e^{-3N_R(1+1/\sqrt{2}+i/\sqrt{2})}, & \text{if } |i-j| = 5 \\ e^{-3N_R(1+i)}, & \text{if } |i-j| = 6 \\ e^{-3N_R(1-1/\sqrt{2}+i/\sqrt{2})}, & \text{if } |i-j| = 7 \end{cases} \quad (6.53)$$

Therefore, the general equation can be rewritten explicitly in the case of 8-PSK.

$$P_{e,USD_{Lower}} \geq \frac{8}{56}e^{-3N_R(1-1/\sqrt{2}-i/\sqrt{2})} + \frac{8}{56}e^{-3N_R(1-i)} + \frac{8}{56}e^{-3N_R(1+1/\sqrt{2}-i/\sqrt{2})} + \frac{8}{56}e^{-6N_R} + \frac{8}{56}e^{-3N_R(1+1/\sqrt{2}+i/\sqrt{2})} + \frac{8}{56}e^{-3N_R(1+i)} + \frac{8}{56}e^{-3N_R(1-1/\sqrt{2}+i/\sqrt{2})} \quad (6.54)$$

#### 6.2.4.2 Upper bound in the case of 8-PSK

The upper bound is as follows

$$P_{e,USD_{Upper}} \leq 1 - \lambda_K \quad (6.55)$$

where  $\lambda_K$  is the minimum eigenvalue in the matrix thus formed

$$\begin{bmatrix} \langle \psi_1 | \psi_1 \rangle & \cdots & \langle \psi_1 | \psi_8 \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_8 | \psi_1 \rangle & \cdots & \langle \psi_8 | \psi_8 \rangle \end{bmatrix} \quad (6.56)$$

Thus, we have  $|\psi_i|^2 = 3N_R \forall i$

$$\langle \psi_i | \psi_j \rangle = e^{-\frac{1}{2}(2N_S - 2N_S e^{i\frac{\pi}{4}(j-i)})} = e^{-(3N_R - 3N_R e^{i\frac{\pi}{4}(j-i)})} \quad (6.57)$$

#### 6.2.4.3 Upper bound with GUS in the case of 8-PSK

The minimum failure probability for unambiguous discrimination of symmetric states is given by the formula (3.107) which is shown below

$$P_{e,USD} \leq 1 - K \min_k |c_k|^2 \quad (6.58)$$

The value of  $|c_k|^2$  is given by [82] and is equal to (in the case of 8-PSK)

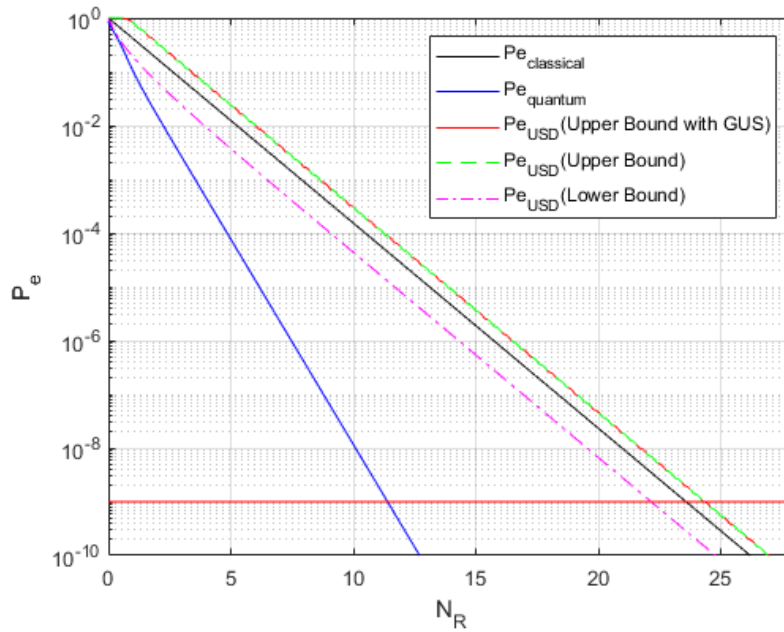
$$|c_k|^2 = \frac{1}{64} \sum_{i=1}^8 \sum_{j=1}^8 e^{-\frac{\pi k(i-j)}{4}} \langle \psi_j | \psi_i \rangle \quad (6.59)$$

where  $\langle \psi_j | \psi_i \rangle$  is equal to

$$\langle \psi_j | \psi_i \rangle = e^{-\frac{1}{2} (2N_S - 2N_S e^{i\frac{\pi}{4}(j-i)})} \quad (6.60)$$

So, the failure probability for  $K=8$  is as follows

$$P_{e,USD} \leq 1 - \frac{1}{8} \min_k \left[ \sum_{i=1}^8 \sum_{j=1}^8 e^{-\frac{i\pi k(i-j)}{4}} e^{(-N_S + N_S e^{i\frac{\pi}{4}(j-i)})} \right] \quad (6.61)$$



**Fig. 6.6** Comparison of classical, quantum and USD (Upper bound, Lower bound and GUS) 8-PSK modulation

The comparison between the  $P_{e,classical}$  of the classical receiver, given by (6.50), the  $P_{e,quantum}$  of the quantum receiver, given by the general equation (6.44) and  $P_{e,USD}$  of USD receiver, given by upper bound (6.55), lower bound (6.54) and GUS (6.61) is shown in Figure 6.6 as a function of the  $N_R$ .

Analyzing the results we can see like the 8-PSK in USD is better in the lower bound and worse in the upper bound than the classic 8-PSK.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity  $10^{-9}$ :

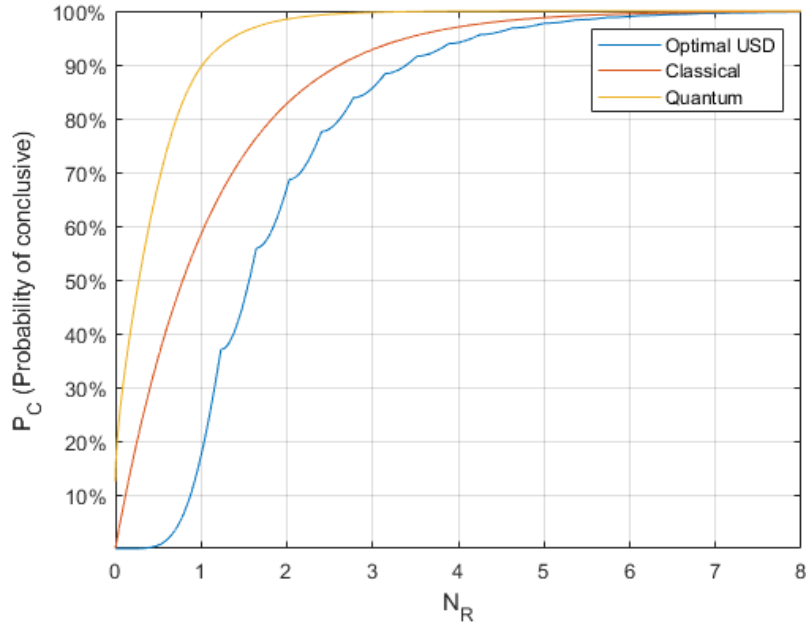
$$N_{R,quantum} = 11.3978 \text{ photons/bit}$$

$$N_{R,classical} = 23.584 \text{ photons/bit}$$

$$N_{R,USDUpper,GUS} = 24.3575 \text{ photons/bit}$$

$$N_{R,USDUpper} = 24.3575 \text{ photons/bit}$$

$$N_{R,USDLower} = 22.1588 \text{ photons/bit}$$



**Fig. 6.7** Probability of conclusive results as a function of the mean photon number per bit  $N_R$  for the USD scheme with 8-PSK modulation in comparison with classical, quantum and Optimal USD

In Figure 6.7 shows the probability of conclusive results as a function of mean pho-

ton number  $N_R$  of the input state for the USD scheme. The maximum probability of conclusive results for optimum USD is included in Figure 6.7 as a result of equation

$$P_{Optimum,USD} = K \min_k |c_k|^2 \quad (6.62)$$

#### 6.2.4.4 Lower bound in the case of 16-PSK

The error probability is as follows

$$P_{e,USD,Lower} \geq \frac{1}{15} \sum_{i=1}^{16} \sum_{\substack{j=1 \\ j \neq i}}^{16} \frac{1}{16} |\langle \psi_i | \psi_j \rangle| \quad (6.63)$$

where the inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewrite as

$$|\langle \psi_i | \psi_j \rangle| = \begin{cases} e^{-4N_R(1-e^{i\pi/8})}, & \text{if } |i-j| = 1 \\ e^{-4N_R(1-e^{i\pi/4})}, & \text{if } |i-j| = 2 \\ e^{-4N_R(1-e^{i3\pi/8})}, & \text{if } |i-j| = 3 \\ e^{-4N_R(1-e^{i\pi/2})}, & \text{if } |i-j| = 4 \\ e^{-4N_R(1-e^{i5\pi/8})}, & \text{if } |i-j| = 5 \\ e^{-4N_R(1-e^{i3\pi/4})}, & \text{if } |i-j| = 6 \\ e^{-4N_R(1-e^{i7\pi/8})}, & \text{if } |i-j| = 7 \\ e^{-8N_R}, & \text{if } |i-j| = 8 \\ e^{-4N_R(1-e^{-i7\pi/8})}, & \text{if } |i-j| = 9 \\ e^{-4N_R(1-e^{-i3\pi/4})}, & \text{if } |i-j| = 10 \\ e^{-4N_R(1-e^{-i5\pi/8})}, & \text{if } |i-j| = 11 \\ e^{-4N_R(1-e^{-i\pi/2})}, & \text{if } |i-j| = 12 \\ e^{-4N_R(1-e^{-i3\pi/8})}, & \text{if } |i-j| = 13 \\ e^{-4N_R(1-e^{-i\pi/4})}, & \text{if } |i-j| = 14 \\ e^{-4N_R(1-e^{-i\pi/8})}, & \text{if } |i-j| = 15 \end{cases} \quad (6.64)$$

Therefore, the general equation can be rewritten explicitly in the case of 16-PSK.

$$\begin{aligned}
P_{e,USD_{Lower}} \geq & \frac{16}{240} e^{-4N_R \left(1 - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i}{2} \sqrt{2-\sqrt{2}}\right)} + \frac{16}{240} e^{-4N_R \left(1 - \frac{1+i}{\sqrt{2}}\right)} + \\
& \frac{16}{240} e^{-4N_R (1 - i \cos(\pi/8) + \sin(\pi/8))} + \frac{16}{240} e^{-4N_R (1-i)} + \\
& \frac{16}{240} e^{-4N_R (1 - i \cos(\pi/8) - \sin(\pi/8))} + \frac{16}{240} e^{-4N_R \left(1 + \frac{1-i}{\sqrt{2}}\right)} + \\
& \frac{16}{240} e^{-4N_R \left(1 + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i}{2} \sqrt{2-\sqrt{2}}\right)} + \frac{16}{240} e^{-8N_R} + \\
& \frac{16}{240} e^{-4N_R \left(1 + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i}{2} \sqrt{2-\sqrt{2}}\right)} + \frac{16}{240} e^{-4N_R \left(1 + \frac{1+i}{\sqrt{2}}\right)} + \\
& \frac{16}{240} e^{-4N_R (1 + i \cos(\pi/8) + \sin(\pi/8))} + \frac{16}{240} e^{-4N_R (1+i)} \\
& \frac{16}{240} e^{-4N_R (1 + i \cos(\pi/8) - \sin(\pi/8))} + \frac{16}{240} e^{-4N_R \left(1 - \frac{1-i}{\sqrt{2}}\right)} + \\
& \frac{16}{240} e^{-4N_R \left(1 - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i}{2} \sqrt{2-\sqrt{2}}\right)}
\end{aligned} \tag{6.65}$$

#### 6.2.4.5 Upper bound in the case of 16-PSK

The upper bound is as follows

$$P_{e,USD_{Upper}} \leq 1 - \lambda_K \tag{6.66}$$

where  $\lambda_K$  is the minimum eigenvalue in the matrix thus formed

$$\begin{bmatrix}
\langle \psi_1 | \psi_1 \rangle & \cdots & \langle \psi_1 | \psi_{16} \rangle \\
\vdots & \ddots & \vdots \\
\langle \psi_{16} | \psi_1 \rangle & \cdots & \langle \psi_{16} | \psi_{16} \rangle
\end{bmatrix} \tag{6.67}$$

Thus, we have  $|\psi_i|^2 = 4N_R \forall i$

$$\langle \psi_i | \psi_j \rangle = e^{-\left(N_S - N_S e^{i\frac{\pi}{8}(j-i)}\right)} = e^{-\left(4N_R - 4N_R e^{i\frac{\pi}{8}(j-i)}\right)} \tag{6.68}$$

#### 6.2.4.6 Upper bound with GUS in the case of 16-PSK

The minimum failure probability for unambiguous discrimination of symmetric states is given by the formula (3.107) which is shown below

$$P_{e,USD} \leq 1 - K \min_k |c_k|^2 \quad (6.69)$$

The value of  $|c_k|^2$  is given by [82] and is equal to (in the case of 16-PSK)

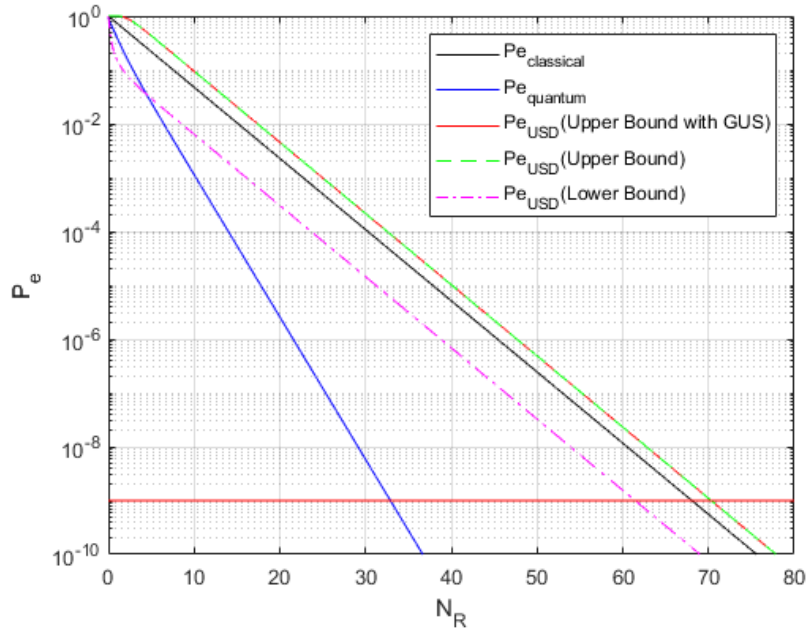
$$|c_k|^2 = \frac{1}{256} \sum_{i=1}^{16} \sum_{j=1}^{16} e^{-\frac{\pi i k (i-j)}{8}} \langle \psi_j | \psi_i \rangle \quad (6.70)$$

where  $\langle \psi_j | \psi_i \rangle$  is equal to

$$\langle \psi_j | \psi_i \rangle = e^{-\frac{1}{2} (2N_S - 2N_S e^{i\frac{\pi}{8}(j-i)})} \quad (6.71)$$

So, the failure probability for  $K=16$  is as follows

$$P_{e,USD} \leq 1 - \frac{1}{16} \min_k \left[ \sum_{i=1}^{16} \sum_{j=1}^{16} e^{-\frac{\pi i k (i-j)}{8}} e^{-(N_S - N_S e^{i\frac{\pi}{8}(j-i)})} \right] \quad (6.72)$$



**Fig. 6.8** Comparison of classical, quantum and USD (Upper bound, Lower bound and GUS) 16-PSK modulation

The comparison between the  $P_{e,classical}$  of the classical receiver, given by (6.51), the  $P_{e,quantum}$  of the quantum receiver, given by the general equation (6.44) and  $P_{e,USD}$  of USD receiver, given by upper bound (6.66), lower bound (6.65) and GUS (6.72) is shown in Figure 6.8 as a function of the  $N_R$ .

Analyzing the results we can see like the 16-PSK in USD is much better in the lower bound and worse in the upper bound than the classic 16-PSK.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity  $10^{-9}$ :

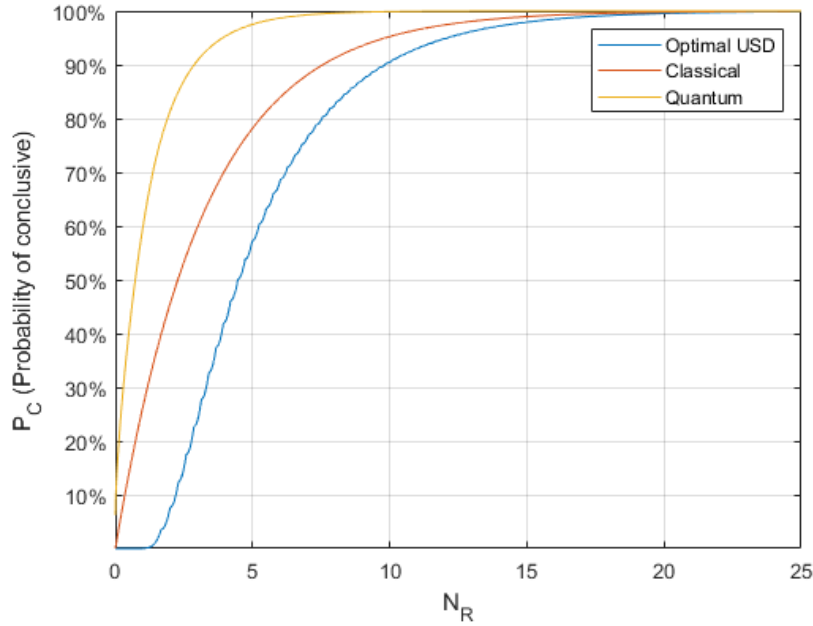
$$N_{R,quantum} = 32.8921 \text{ photons/bit}$$

$$N_{R,classical} = 68.06 \text{ photons/bit}$$

$$N_{R,USDUpper,GUS} = 70.3315 \text{ photons/bit}$$

$$N_{R,USDUpper} = 70.3315 \text{ photons/bit}$$

$$N_{R,USDLower} = 61.443 \text{ photons/bit}$$



**Fig. 6.9** Probability of conclusive results as a function of mean photon number  $N_R$  for the USD scheme with 16-PSK modulation in comparison with classical, quantum and Optimal USD

In Figure 6.9 shows the probability of conclusive results as a function of mean pho-



ton number  $N_R$  of the input state for the USD scheme. The maximum probability of conclusive results for optimum USD is included in Figure 6.9 as a result of equation

$$P_{Optimum,USD} = K \min_k |c_k|^2 \quad (6.73)$$

## 6.3 K-PPM Modulation

In paragraph 5.3 the modulation *2-PPM* was treated. Now, in this section, we will see the performance of *K-PPM* for the classic, quantum and USD channels.

One of the key difficulties of implementing this technique is that the receiver must be properly synchronized to align the local clock with the beginning of each symbol. Therefore, it is often implemented differentially as differential pulse-position modulation, whereby each pulse position is encoded relative to the previous, such that the receiver must only measure the difference in the arrival time of successive pulses. It is possible to limit the propagation of errors to adjacent symbols, so that an error in measuring the differential delay of one pulse will affect only two symbols, instead of affecting all successive measurements.

Aside from the issues regarding receiver synchronization, the key disadvantage of *PPM* is that it is inherently sensitive to multipath interference that arises in channels with frequency-selective fading, whereby the receiver's signal contains one or more echoes of each transmitted pulse. Since the information is encoded in the time of arrival, the presence of one or more echoes can make it extremely difficult to accurately determine the correct pulse. Multipath in the Pulse Position Modulation systems can be easily supported by using the same techniques that are used in Radar systems.

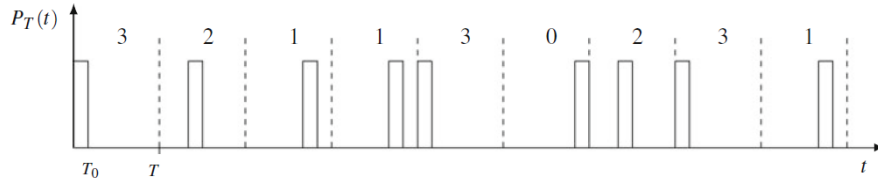
One of the principal advantages of *PPM* is that it is an *M-ary* modulation technique that can be implemented non-coherently, such that the receiver does not need to use a phase-locked loop (PLL) to track the phase of the carrier. This makes it a suitable candidate for optical communications systems, where coherent phase modulation and detection are difficult and extremely expensive. The only other common *M-ary* non-coherent modulation technique is *M-ary* frequency-shift keying (*M-FSK*), which is the frequency-domain dual to *PPM*.

In Figure 6.10 is presented an example of *PPM* with  $K = 4$ .

### 6.3.1 Performance of Quantum K-PPM Systems

Considering that the *PPM* states have the GUS and that the SRM gives an optimal detection, we can then apply the method summarized in paragraph 3.4, on the SRM detection in the presence of GUS.

The performance evaluation, then, is structured as follows.



**Fig. 6.10** Realization of transmitted symbols and corresponding optical power in classic 4-PPM modulation

$G$  has as element  $i, j$  the inner product  $G_{ij} = \langle \psi_i | \psi_j \rangle$ , where now the states  $|\psi_i\rangle$  are composite. Recalling that

$$\langle \psi_i | \psi_j \rangle = \langle \psi_{i0} | \psi_{j0} \rangle \langle \psi_{i1} | \psi_{j1} \rangle \dots \langle \psi_{iK-1} | \psi_{jK-1} \rangle \quad (6.74)$$

Then, using equation (4.8), we get

$$\langle \psi_i | \psi_j \rangle = \begin{cases} 1, & i = j \\ e^{-|\Delta|^2}, & i \neq j \end{cases} \quad (6.75)$$

For example, in our case where  $K = 2$  the inner product  $\langle \psi_0 | \psi_1 \rangle$  results in

$$\langle \psi_0 | \psi_1 \rangle = \langle \Delta | 0 \rangle \langle 0 | \Delta \rangle = e^{-\frac{1}{2}|\Delta|^2} e^{-\frac{1}{2}|\Delta|^2} = e^{-|\Delta|^2} \quad (6.76)$$

We observe that the same energy  $E$  is associated to all symbols, and, according to equation (5.40), to each composite state the same signal photons are associated, given by

$$N_S = |\Delta|^2 = \text{number of signal photons/symbol} \quad (6.77)$$

Therefore Gram's matrix becomes

$$G = \begin{bmatrix} 1 & |X|^2 & \dots & |X|^2 \\ |X|^2 & 1 & \dots & |X|^2 \\ \vdots & & \ddots & \vdots \\ |X|^2 & |X|^2 & \dots & 1 \end{bmatrix} \quad (6.78)$$

Notice that  $G$  is a circulant matrix, as a consequence of the GUS.

Considering the GUS, the eigenvalues of  $G$  are given by the DFT of the first row  $[1, |X|^2, \dots, |X|^2]$  and therefore

$$\lambda_i = \sum_{k=0}^{K-1} G_{0k} W_K^{-ki} = 1 + |X|^2 \sum_{k=1}^{K-1} W_K^{-ki} \quad (6.79)$$

we have

$$\lambda_i = \begin{cases} 1 + (K-1)|X|^2, & i = 0 \\ 1 - |X|^2, & i = 1, \dots, K-1 \end{cases} \quad (6.80)$$

where

$$\begin{aligned} \lambda_0 &= 1 + (K-1)|X|^2 \\ \lambda_1 &= 1 - |X|^2 \end{aligned} \quad (6.81)$$

The error probability is computed from (3.111) and becomes

$$P_{e,quantum} = 1 - \frac{1}{K^2} \left( \sqrt{1 + (K-1)|X|^2} + (K-1)\sqrt{1 - |X|^2} \right)^2 \quad (6.82)$$

In the quantum case, the error probability is given by (6.82), which can be rewritten in the form

$$P_{e,quantum} = \frac{K-1}{K^2} \left[ K - (K-2)(1 - |X|^2) - 2\sqrt{(1 - |X|^2)(1 + (K-1)|X|^2)} \right] \quad (6.83)$$

where the superposition degree  $|X|^2$  can be expressed as a function of the number of signal photons per symbol  $N_S$ , or of the number of signal photons per bit  $N_R$

$$|X|^2 = e^{-N_S} = e^{-N_R \log_2 K} \quad (6.84)$$

### 6.3.1.1 Case 4-PPM

So, in our case, where  $K = 4$ , the equation (6.84) becomes

$$|X|^2 = e^{-N_S} = e^{-2N_R} \quad (6.85)$$

and then the equation (6.83) can be rewritten as

$$P_{e,quantum} = \frac{3}{8} \left[ 4 - 2(1 - |X|^2) - 2\sqrt{(1 - |X|^2)(1 + 3|X|^2)} \right] \quad (6.86)$$

Thus the equation (6.86) can be rewritten with the following

$$P_{e,quantum} = \frac{3}{2} - \frac{3}{4}(1 - |X|^2) - \frac{3}{4}\sqrt{(1 - |X|^2)(1 + 3|X|^2)} \quad (6.87)$$

and finally

$$P_{e,quantum} = \frac{3}{4} + \frac{3}{4}e^{-2N_R} - \frac{3}{4}\sqrt{1 + 2e^{-2N_R} - 3e^{-4N_R}} \quad (6.88)$$

### 6.3.1.2 Case 16-PPM

So, in our case, where  $K = 16$ , the equation (6.84) becomes

$$|X|^2 = e^{-N_S} = e^{-4N_R} \quad (6.89)$$

and then the equation (6.83) can be rewritten as

$$P_{e,quantum} = \frac{15}{256} \left[ 16 - 14(1 - |X|^2) - 2\sqrt{(1 - |X|^2)(1 + 15|X|^2)} \right] \quad (6.90)$$

and then

$$P_{e,quantum} = \frac{15}{16} - \frac{210}{256}(1 - e^{-4N_R}) - \frac{15}{128}\sqrt{(1 - e^{-4N_R})(1 + 15e^{-4N_R})} \quad (6.91)$$

## 6.3.2 Performance of Classical $K$ -PPM Systems

Now, recall that the equation of  $K$ -PPM in the classical channel (according to (5.48)) is as follows

$$P_{e,classical} = \frac{K-1}{K}e^{-N_S} \quad (6.92)$$

### 6.3.2.1 Case 4-PPM

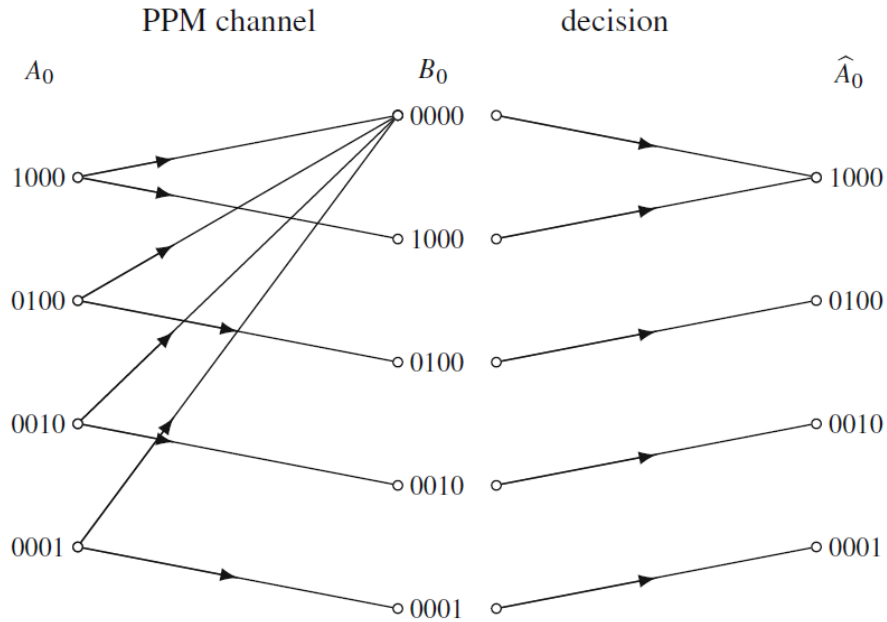
So, in our case where  $K = 4$  this equation becomes

$$P_{e,classical} = \frac{3}{4}e^{-N_S} \quad (6.93)$$

and then

$$P_{e,classical} = \frac{3}{4}e^{-2N_R} \quad (6.94)$$

In the classic 4-PPM system the channel and decision criterion is made as follows in the Figure 6.11



**Fig. 6.11** Channel and decision criterion of a classical 4-PPM.  $A_0$  is the transmitted word,  $B_0$  the received word, and  $\hat{A}_0$  the decided word

### 6.3.2.2 Case 16-PPM

So, in our case where  $K = 16$  this equation becomes

$$P_{e, \text{classical}} = \frac{15}{16} e^{-4N_R} \quad (6.95)$$

### 6.3.3 Performance of USD K-PPM Systems

Now let's see at how the 4-PPM system is detected in the USD system. As we saw in Chapter 3, with a modulation order number of symbols greater than two, we can find upper and lower bounds for the USD system.

### 6.3.3.1 Lower bound in the case of 4-PPM

Regarding the lower bounds, let's start with the formula (3.52). Knowing that the symbols are equiprobable then the  $\eta_i$  are equal to  $1/K$  with  $K = 4$ .

So the error probability can be rewritten as

$$P_{e,USD_{Lower}} \geq \frac{1}{3} \sum_{i=1}^4 \sum_{\substack{j=1 \\ j \neq i}}^4 \frac{1}{4} |\langle \psi_i | \psi_j \rangle| \quad (6.96)$$

The inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewritten as

$$|\langle \psi_i | \psi_j \rangle| = e^{-N_s} = e^{-2N_R} \quad (6.97)$$

And finally

$$\boxed{P_{e,USD_{Lower}} \geq e^{-2N_R}} \quad (6.98)$$

### 6.3.3.2 Lower bound in the case of 16-PPM

The inner products  $|\langle \psi_i | \psi_j \rangle|$  can be rewritten as

$$|\langle \psi_i | \psi_j \rangle| = e^{-N_s} = e^{-4N_R} \quad (6.99)$$

And then

$$\boxed{P_{e,USD_{Lower}} \geq e^{-4N_R}} \quad (6.100)$$

### 6.3.3.3 Upper bound

Now let's analyze the upper bound. Considering the equation (3.53) we can write the upper bound as

$$\boxed{P_{e,USD_{Upper}} \leq 1 - \lambda_K} \quad (6.101)$$

where  $\lambda_K$  is the minimum eigenvalue in the matrix thus formed

$$\begin{bmatrix} \langle \psi_1 | \psi_1 \rangle & \cdots & \langle \psi_1 | \psi_K \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi_K | \psi_1 \rangle & \cdots & \langle \psi_K | \psi_K \rangle \end{bmatrix} \quad (6.102)$$

## 6.3.3.3.1 Case 4-PPM

$\langle \psi_i | \psi_j \rangle$  can be written in the form

$$\langle \psi_i | \psi_j \rangle = \begin{cases} 1, & \text{if } i = j \\ e^{-N_s} = e^{-2N_R}, & \text{if } i \neq j \end{cases} \quad (6.103)$$

Therefore, we can write in extended form as follows

$$\langle \psi_i | \psi_j \rangle = \begin{bmatrix} 1 & e^{-2N_R} & e^{-2N_R} & e^{-2N_R} \\ e^{-2N_R} & 1 & e^{-2N_R} & e^{-2N_R} \\ e^{-2N_R} & e^{-2N_R} & 1 & e^{-2N_R} \\ e^{-2N_R} & e^{-2N_R} & e^{-2N_R} & 1 \end{bmatrix} \quad (6.104)$$

## 6.3.3.3.2 Case 16-PPM

In the case of 16-PPM the inner product  $\langle \alpha_i | \alpha_j \rangle$  can be written in the form

$$\langle \psi_i | \psi_j \rangle = \begin{cases} 1, & \text{if } i = j \\ e^{-N_s} = e^{-4N_R}, & \text{if } i \neq j \end{cases} \quad (6.105)$$

## 6.3.3.4 Upper bound with GUS in the case of 4-PPM

Assuming equal a priori probabilities, the minimum failure probability for unambiguous discrimination of symmetric states is given by the formula (3.107) which is shown below

$$P_{e,USD} \leq 1 - K \min_k |c_k|^2 \quad (6.106)$$

The value of  $|c_k|^2$  is given by [82] and is equal to

$$|c_k|^2 = \frac{1}{K^2} \sum_{i=1}^K \sum_{j=1}^K e^{\frac{-2\pi i k(i-j)}{K}} \langle \psi_j | \psi_i \rangle \quad (6.107)$$

where  $\langle \psi_j | \psi_i \rangle$  is equal to

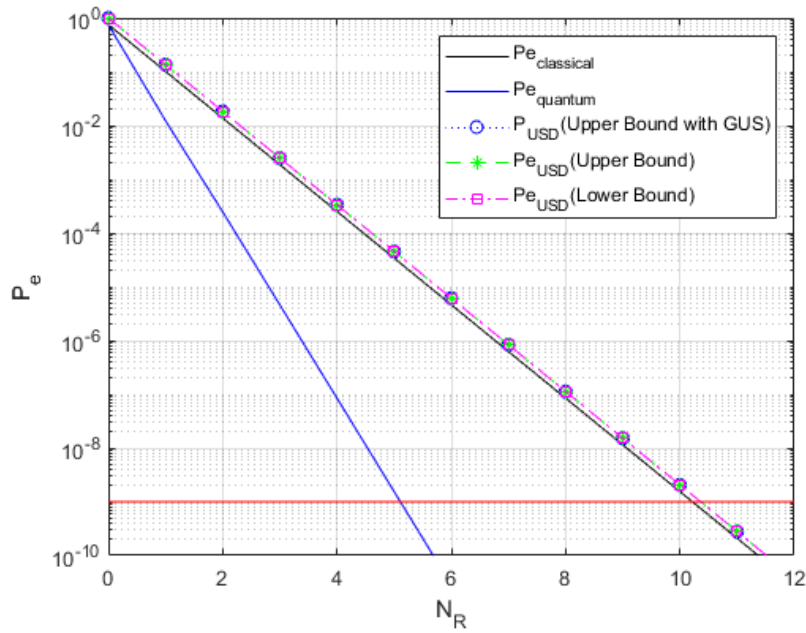
$$\langle \psi_j | \psi_i \rangle = \begin{cases} 1, & \text{if } i = j \\ e^{-N_s} = e^{-2N_R}, & \text{if } i \neq j \end{cases} \quad (6.108)$$

So, the failure probability is as follows

$$P_{e,USD} \leq 1 - K \frac{1}{K^2} \min_k \left[ \sum_{\substack{i,j \\ i \neq j}} e^{-\frac{2\pi k(i-j)}{K}} e^{-2N_R} + K \right] \quad (6.109)$$

and finally (for  $K = 4$ )

$$P_{e,USD} \leq 1 - \frac{1}{4} \min_k \left[ \sum_{\substack{i,j \\ i \neq j}} e^{-\frac{i\pi k(i-j)}{2}} e^{-2N_R} + 4 \right] \quad (6.110)$$



**Fig. 6.12** Comparison of classical, quantum and USD (Upper bound, Lower bound and GUS) 4-PPM modulation

The comparison between the  $P_{e,classical}$  of the classical receiver, given by (6.94), the  $P_{e,quantum}$  of the quantum receiver, given by (6.88) and  $P_{e,USD}$  of USD receiver, given by upper bound (6.101), lower bound (6.96) and GUS (6.110) is shown in Figure 6.12 as a function of the average number of photons per bit  $N_R$ .

Analyzing the results we can see an improvement of 4-PPM in USD.

Interestingly, the bounds of the USD overlap. Finally, the USD compared to the classic 4-PPM has very close performances. So we can conclude by saying that



with a modulation of this type we can have satisfactory performances with the USD channel.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity  $10^{-9}$ :

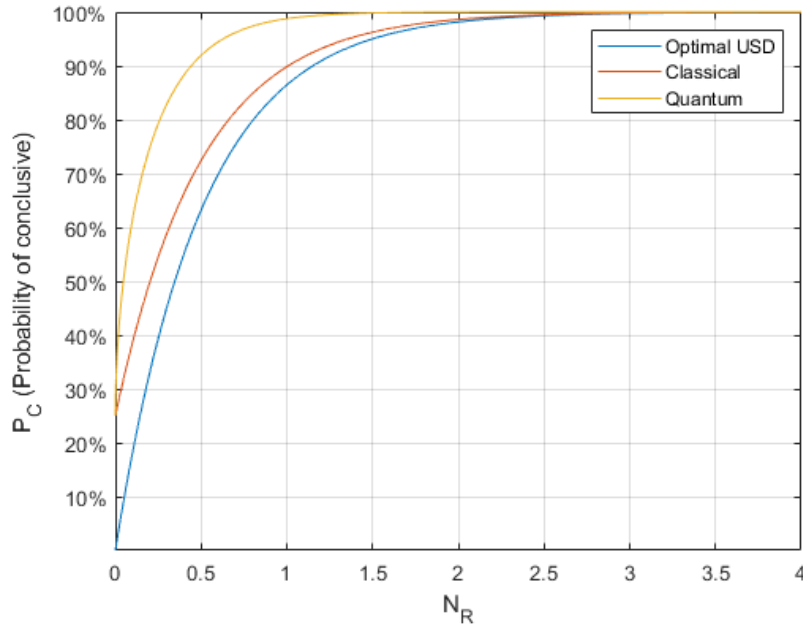
$$N_{R,quantum} = 5.109 \text{ photons/bit}$$

$$N_{R,classical} = 10.217 \text{ photons/bit}$$

$$N_{R,USD_{Upper,GUS}} = 10.361 \text{ photons/bit}$$

$$N_{R,USD_{Lower}} = 10.361 \text{ photons/bit}$$

$$N_{R,USD_{Upper}} = 10.361 \text{ photons/bit}$$



**Fig. 6.13** Probability of conclusive results as a function of mean photon number  $N_R$  for the USD scheme with 4-PPM modulation in comparison with classical, quantum and Optimal USD

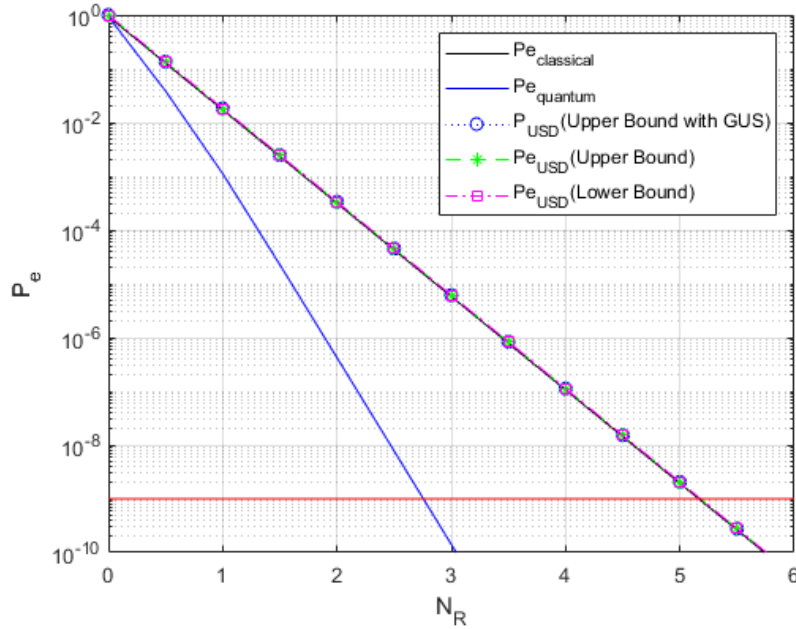
In Figure 6.13 shows the probability of conclusive results as a function of mean photon number  $N_R$  of the input state for the USD scheme. The maximum probability of conclusive results for optimum USD is included in Figure 6.13 as a result of equation

$$P_{Optimum,USD} = K \min_k |c_k|^2 \quad (6.111)$$

### 6.3.3.5 Upper bound with GUS in the case of 16-PPM

The failure probability in the case of  $K = 16$  is the follow.

$$P_{e,USD} \leq 1 - \frac{1}{16} \min_k \left[ \sum_{\substack{i,j \\ i \neq j}} e^{\frac{-i\pi k(i-j)}{8}} e^{-4N_R} + \sum_{\substack{i,j \\ i=j}} e^{\frac{-i\pi k(i-j)}{8}} \right] \quad (6.112)$$



**Fig. 6.14** Comparison of classical, quantum and USD (Upper bound, Lower bound and GUS) 16-PPM modulation

The comparison between the  $P_{e,classical}$  of the classical receiver, given by (6.95), the  $P_{e,quantum}$  of the quantum receiver, given by (6.91) and  $P_{e,USD}$  of USD receiver, given by upper bound (6.101), lower bound (6.100) and GUS (6.112) is shown in Figure 6.14 as a function of the average number of photons per bit  $N_R$ .

Analyzing the results we can see an improvement of 16-PPM in USD.

It is noteworthy that bounds of the USD compared to the classic 16-PPM have overlapping performances.

So we can conclude by saying that with a modulation of this type we can have satisfactory performances with the USD channel.

Now, we report the values of  $N_R$  for the various channels analyzed at the sensitivity

$10^{-9}$ :

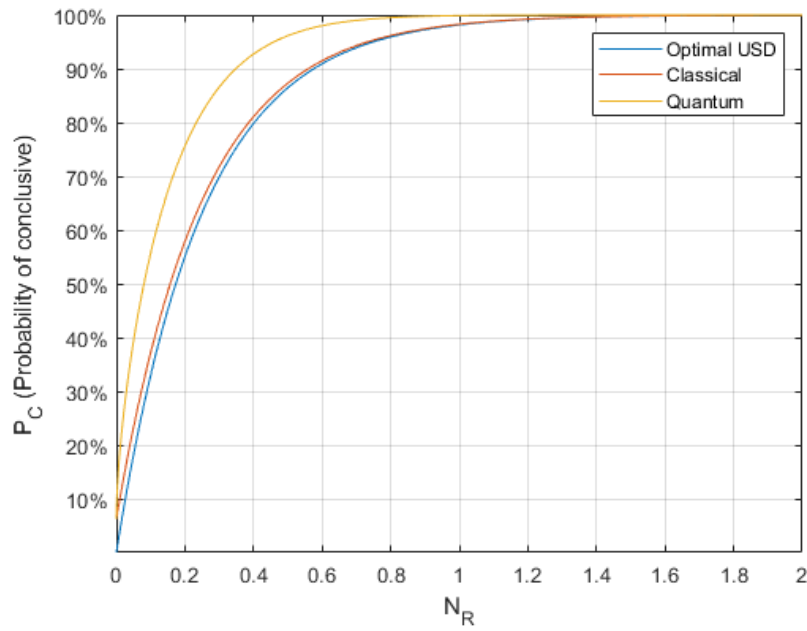
$$N_{R,quantum} = 2.775 \text{ photons/bit}$$

$$N_{R,classical} = 5.164 \text{ photons/bit}$$

$$N_{R,USD_{Upper,GUS}} = 5.18 \text{ photons/bit}$$

$$N_{R,USD_{Lower}} = 5.18 \text{ photons/bit}$$

$$N_{R,USD_{Upper}} = 5.18 \text{ photons/bit}$$



**Fig. 6.15** Probability of conclusive results as a function of mean photon number  $N_R$  for the USD scheme with 16-PPM modulation in comparison with classical, quantum and Optimal USD

In Figure 6.15 shows the probability of conclusive results as a function of mean photon number  $N_R$  of the input state for the USD scheme. The maximum probability of conclusive results for optimum USD is included in Figure 6.15 as a result of equation

$$P_{Optimum,USD} = K \min_k |c_k|^2 \quad (6.113)$$

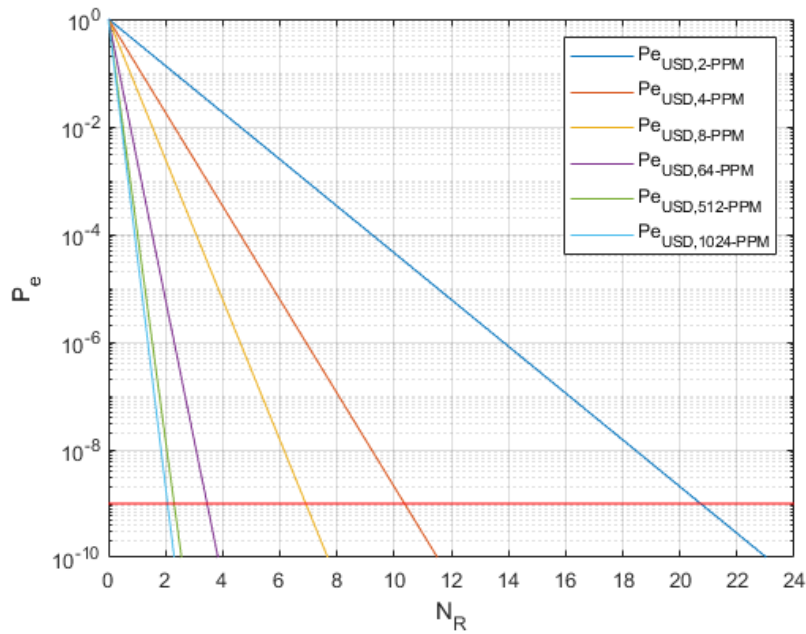
### 6.3.3.6 Comparison of K-PPM USD systems

In Figure 6.16 the error probability of the USD *PPM* is plotted as a function of the number of signal photons per bit  $N_R$  for various values of  $K$  given by the formula (6.106).

We realize that USD *PPM* receivers have an extraordinary sensitivity.

In particular, we report the values of  $N_{R,USD}$  for the USD channel, given by the formula (3.107), analyzed at the sensitivity  $10^{-9}$ :

2 – PPM	$N_{R,USD} = 20.723 \text{ photons/bit}$
4 – PPM	$N_{R,USD} = 10.217 \text{ photons/bit}$
8 – PPM	$N_{R,USD} = 6.907 \text{ photons/bit}$
64 – PPM	$N_{R,USD} = 3.453 \text{ photons/bit}$
512 – PPM	$N_{R,USD} = 2.305 \text{ photons/bit}$
1024 – PPM	$N_{R,USD} = 2.072 \text{ photons/bit}$



**Fig. 6.16** Error probability of USD  $K$ -PPM as a function of number of signal photons per bit  $N_R$

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