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Aspects of String Perturbation Theory

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A me stesso.

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Contents

"Nature seems to enjoy running on the verge of inconsistency"
(D. Friedan, E. Martinec, S. Shenker)

Introduction

The union of gravity with quantum field theory leads to non renormalization issues so, despite of its impressive success, the Standard Model is surely not the ultimate theory describing the reality at its very fundamental level. Other hints in this direction also come from the fact that the Standard Model appears to be, in a certain sense, too "arbitrary" and "unnatural". It looks like arbitrary because it is not able to explain the reason why its particular pattern of gauge fields and multiplets exists and because it cannot determine the values of the many parameters entering its Lagrangian. The latter have to be fixed by comparison with experiments and some of them turn out to be unexpectedly much smaller than the values that one would guess a priori: hence the unnaturalness of the model, which arises as a fine-tuning problem. For instance, the Higgs boson is much lighter than the Planck mass $M_P \sim 10^{19}$ GeV, whereas one would expect that the large quantum contributions to its mass would inevitably make the latter very huge (i.e. comparable to the scale at which new physics should emerge) unless there is an incredible fine-tuning cancellation between the quadratic radiative corrections and the bare mass.

All the attempts at unification of the interactions of nature have been essentially based on enlarging the symmetry group of short distance physics. This has brought us to theories characterized by large non-abelian gauge groups, supersymmetry (which helps with the naturalness issue, by solving the fine-tuning problem for the Higgs mass), higher dimensions (that allow to describe several and different four-dimensional fields in term of the same higher-dimensional object), etc...
Out of the theories that have been put forward to go beyond the SM, the most promising one seems to be String Theory, according to which fundamental particles are nothing but different excitations of a one-dimensional object. This idea leads - in a very natural way - to a theory where the characteristic length of the string is the only arbitrarily adjustable parameter and where all interactions (gravity inluded) are unified in a truly elegant formalism which is free from those UV divergences which typically affect any QFT of pointlike particles. From an abstract point of view, string theory represents a radical step in enlarging the symmetry group of fundamental physics, because it brings the infinite-dimensional algebra of two-dimensional conformal transformations into the game. Such vast extension in symmetry is correspondingly followed by tight restrictions on the structure of the theory. Indeed, because of the presence of several potential anomalies, the consistency of string theory is a non-trivial issue. It requires stringent constraints on the framework, which lead to the critical dimension (we can say that the superstring is allowed to live only in a ten-dimensional spacetime), to precise restrictions on the possible gauge groups (for example, the heterotic superstring is consistent only with $E_8 \times E_8$ or $SO(32)$ gauge vectors) and to the spacetime supersymmetry.

So far, no evidence of supersymmetric partners of the SM particles has been found. Thus, there must be a mechanism that breaks it at a certain energy scale Λ_b considerably higher than the typical scale Λ_{SM} of the SM, namely the electroweak one: $\Lambda_b > \Lambda_{SM} \sim 10^2$ GeV. If supersymmetry is really the solution to the hierarchy problem, the cancellation of the quantum corrections to the Higgs mass requires Λ_b not to be too much above Λ_{SM} and it's thus believed that the breaking of supersymmetry should occur around an energy scale like $10^2 \text{ GeV} \lesssim \Lambda_b \lesssim 10^3$ GeV.
From the superstring point of view, the fact that $\Lambda_b \lesssim 10^3$ GeV means that the compactification of the six extra-dimensions should not be responsible for the breaking of the supersymmetry; otherwise, we would have $\Lambda_b \sim 1/R_c$ where R_c is the characterisitic length of the internal space, which has to be taken very small in order for the additional dimensions to be penetrable only at very high energy, that is $1/R_c \gg 10^3$ GeV. In other words, the six extra dimensions are curled up into special manifolds which don't break supersymmetry (i.e. they must be Calabi-Yau, if the metric is the only background field) and Λ_b cannot be tied to the compactification scale. In the context of the superstring, the link between Λ_b and $1/R_c$ appears to be a generic[1] problem with tree-level supersymmetry breaking (see [1]) and we are left with the possibility of breaking supersymmetry by means of loop or non-perturbative effects.

[1]At least in the traditional approach to phenomenology based on the heterotic superstring.

In the final chapter of this thesis, we will analyze the *only known*[2] example in which supersymmetry can be spontaneously broken in superstring perturbation theory, despite being unbroken at the tree-level. We will be able to detect the breaking of the supersymmetry by looking at the mass-splitting that arises as a one-loop effect affecting a chiral supermultiplet of the low-energy limit of the $SO(32)$ heterotic superstring compactified in a Calabi-Yau manifold. To be more precise, we will determine - at one loop level - the $\rho_i \rho_i^*$ correlation function for a particular complex scalar field ρ_i and we will find that a possible non-vanishing mass term is developed. We will perform the calculation around a vacuum which is supersymmetric at the tree-level and in which ρ_i and its superpartner appear massless. Given that the latter will continue to be massless also perturbatively, the non-vanishing mass term for ρ_i will precisely coincide with the craved mass-splitting, which is due to the presence of a non-vanishing D-term.
This model is not interesting from a phenomenological point of view. Nevertheless, it's worth of being studied, because it lets us have a look at the differences between the old literature and the current state of the art of on-shell string perturbation theory. Indeed, this calculation has been done in various ways in the old literature (see [2], [3]) where, in order to get a non-vanishing mass term for ρ_i, they needed to impose the momentum conservation condition only at the very end of the calculation. Clearly, this sounds like a trick, because nothing prevents us from imposing the momentum conservation in a previous step of the computation. We will determine the mass-splitting by following the strategy outlined by A. Sen in 2015 (see [4]), a strategy that doesn't require the momentum conservation condition to be imposed necessarily at the end. His approach was inspired by some recent advancements that he and his collaborators carried out in the context of the closed superstring field theory/off-shell superstring scattering amplitudes. We will not present the heavy formalism underlying such advanced topics; rather, we will help the reader to understand the structure of that calculation by following a very humble and elementary path, partially based on the old-fashioned approach to string perturbation theory (indeed we will cite articles like [5], [6] and [7] several times throughout all the course of the thesis); when big theorems and long proofs will be required, we will refer to the proper reference.

We will not take any background in string theory for granted. After an informal introduction, we will devote the second and the third chapter to presenting the quantization of the string; we will do it for the bosonic string, a theory that we consider as an unrealistic but simple toy model that lets us get acquaintance with the main ideas of string theory. In the fourth chapter, the most important ingredients of CFT borrowed by string theory will be given and we will also have our first approach to the BRST formalism, an essential tool that will help us to build up physically meaningful correlation functions. Then, we will be ready to analyze how the global aspects of the worldsheet enter string perturbation theory, something that we will describe in the fifth chapter, where particular attention will be paid to the toroidal worldsheet, given that it will be precisely the worldsheet on which the final calculation will be performed. At this point, the reader should have acquired enough familiarity with string theory, so, in the sixth chapter we will be more sketchy with the superstring and we will describe in detail only those features of it that have no analogy in the bosonic theory; here, for example, the basics of the technology involving the picture changing operators will be presented[3]. Finally, in the seventh chapter we will study the spontaneous supersymmetry breaking mechanism for the heterotic $SO(32)$ superstring and we will explicitly show the difference between the way the calculation used to be done in the 80's and the strategy proposed by Sen.

[2] At least, the only known example according to the knowledge of E. Witten (see [2]).

[3] We like to mention that, essentially, there are two possible ways to do superstring perturbation theory. As E. Witten likes, one could describe the superstring perturbation theory by resorting to the mathematically involved machinery of super-Riemann surfaces, otherwise one could decide to follow the favourite approach by A. Sen, which consists of integrating out the odd moduli of the worldsheet and dealing with the picture changing operators in a more intuitive and manifest way since the beginning of the calculation. We prefer the second option, because it requires less mathematical background and because it is more direct for computations.

1 String Theory: an informal introduction

In this section, we are going to introduce the reader to the technical language that will be used in the following chapters, by presenting some of the fundamental ideas underlying string theory. We will not delve into computations; rather, we will explain basic concepts on which this fascinating theory is based. Particular emphasis will be laid on the features of the interactions allowed among extended objects and on the formalisms used to deal with them.

1.1 $p-$branes and worldvolume formalisms

Let's consider a $p-brane$, that is a classical object (relativistic or not) of p space dimensions propagating in a flat $D-$dimensional spacetime $\mathcal{M}$.
Such an object is defined to be *fundamental* if it is not an assembly of lower-dimensional branes which are bound together by some forces and if it appears to be $p-$dimensional at any scales. In particular, this means that a fundamental $p-$brane has no internal structure and it cannot be described either as a subset of a higher-dimensional object by forgetting the not-observable dimensions of the latter. As we will see, $p-$branes can interact and split into other branes; if the initial $p-$brane is fundamental, then also the final ones are fundamental. Given that a fundamental object has no internal structure, we can require it to be homogeneous; this stronger assumption lets us characterize the $p-$brane only by specifying a scalar number T, called *tension*, which can be thought of as the homogeneous energy density of the object.
To completely characterize the dynamics of a fundamental $p-$brane, we have to describe its history - also called *worldvolume* - as a subset of $\mathcal{M}$ - also called *target space*; using a standard language, we refer to the worldvolume of a $0-$brane (a pointlike particle) as *worldline* and to that one of a *string* (a $0-$brane) as *worldsheet*.
If we are searching for a relativistic theory, time and spatial coordinates must be considered on the same footing and there are only two formalisms that achieve this within the framework of a lagrangian formulation. To be more precise, the dynamics of the $p-$brane can be described in two fashions, which differ for the choice of the dynamical variables and for the way they relate the $p-$brane to the spacetime $\mathcal{M}$.

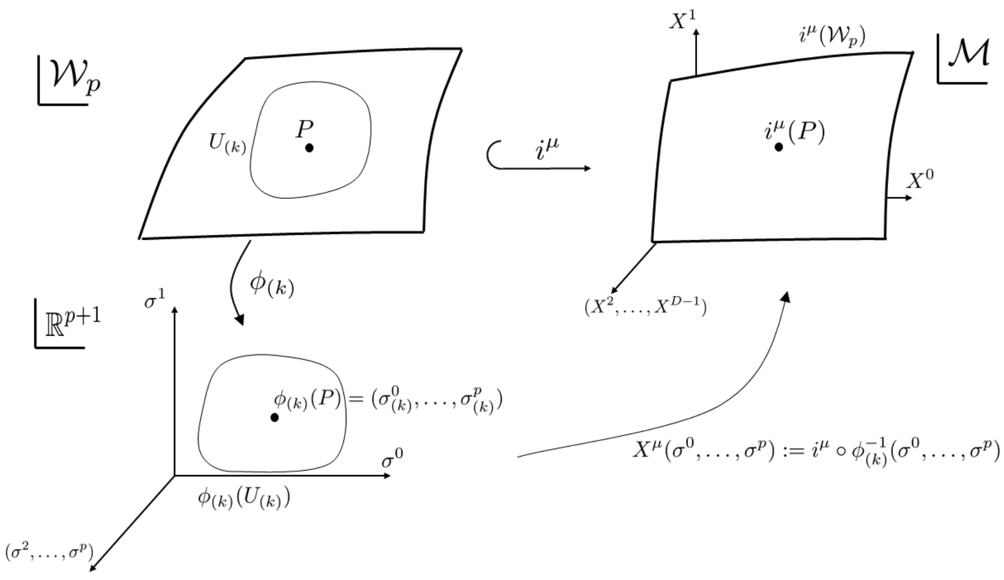

Figure 1
The abstract and the embedded worldvolume.

- According to the *worldvolume* formalism, both space and time positions of the object are dynamical variables. They are given as functions $X^\mu(\sigma^a)$ on the worldvolume and the latter is parametrized in terms of coordinates σ^a ($a \in \{0,1,...,p\}$). Usually, $X^\mu(\sigma^a)$ are sloppily called embeddings of the worldvolume into the spacetime $\mathcal{M}$. In the language of differential geometry, the worldvolume of a $p-$brane is a manifold $\mathcal{W}_p$ of real dimension $p+1$ defined by the atlas of its local charts $\{(U_{(k)},\phi_{(k)})\}_{k\in\{1,2,...\}}$ and which is embedded into the spacetime $\mathcal{M}$ by means of some functions i^μ (see Figure 1). Then, if $P \in U_{(k)}$ is a point of $\mathcal{W}_p$, we define X^μ as

$$X^\mu(\sigma^\alpha) = X^\mu(\sigma^\alpha_{(k)}) \equiv i^\mu \circ \phi^{-1}_{(k)}(\sigma^\alpha_{(k)})\,, \tag{1.1}$$

where $\sigma^\alpha_{(k)} = \phi_{(k)}(P)$ are the coordinates of P in the k^{th} local chart $(U_{(k)},\phi_{(k)})$; for the sake of simplicity, we will always drop, as we have just done, the label of the chart and we will write σ^α instead of $\sigma^\alpha_{(k)}$. We will continue to refer to the abstract manifold $\mathcal{W}_p$ as the worldvolume of the $p-$brane, whereas its embedded version $i^\mu(\mathcal{W}_p)$ will be called *history* of the $p-$brane. Actually, at the classical level, there is no essential difference between the worldvolume and the history of the fundamental object: the functions $X^\mu(\sigma^\alpha)$ will be the solutions of the equations of motion and, thus, we expect i^μ to be particularly nice functions which are able to identify the structure of $\mathcal{W}_p$ with that one of $i^\mu(\mathcal{W}_p)$. But, at the quantum level, we have to "sum" all possible histories of the $p-$brane, as suggested by the path-integral approach to quantization; a way to do this is to fix the topology of the abstract worldsheet and to "sum" over *all* the possible i^μ (not only over the embeddings!) and then, as last step, to let the topology of the worldsheet vary.
Anyway, given a geometric object it is natural to consider its shape and its deformations; hence, there is a natural action S_p for a fundamental $p-$brane of tension T which is simply given by the integration over its worldvolume, that is

$$S_p \equiv -T\int_{\mathcal{W}_p} dVol = -T\int d^{p+1}\sigma\sqrt{-\det\gamma_{\alpha\beta}} = -T\int d^{p+1}\sigma\sqrt{-\det\left(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\right)} \tag{1.2}$$

where $\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}$ denotes the induced metric on the worldvolume by the Minkowskian structure of $\mathcal{M}$.
Not all $X^\mu(\sigma^a)$ correspond to degrees of freedom of the $p-$brane, because this description is apparently redundant by contruction. Indeed, a manifold is defined up to diffeomorphisms and this means, in particular, that its description must be invariant under reparametrizations (after all, σ^a are merely unphysical labels introduced to have a practical description of the abstract $\mathcal{W}_p$ and to perform computations). Local symmetries must be fixed, so it is understood that, in a worldvolume description, the action S_p must go with constraints, whose physical meaning become immediately transparent in the easy case of a $0-$brane. Indeed, for a pointlike particle, the tension coincides with its mass m, so the action (1.2) is nothing but the familiar integration of the proper time s over the worldline $\mathcal{W}_0$:

$$S_0 \equiv -m\int_{\mathcal{W}_0} ds = -m\int d\tau\sqrt{-\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau}} \quad . \tag{1.3}$$

From the definition of the canonical momentum p^μ associated to the position variable X^μ

$$p_\mu = \frac{\partial L}{\partial\dot{X}^\mu} = m\frac{\dot{X}_\mu}{\sqrt{-(\dot{X})^2}}\ ,$$

one can see that the dynamics of the action S_0 has to be constrained by the on-shell condition

$$p^\mu p_\mu + m^2 = 0 \quad . \tag{1.4}$$

So, in the case of the $0-$brane, reparametrization redundancy of the worldline has let us consider the time position X^0 of the particle as a dynamical variable on the same footing of the positions X^i, without quitting the natural requirement that the particle cannot freely move into Minkowski space: at the very least, it has to follow a timelike direction with $(p^0)^2 \geq m^2$.
Turning back to the case of a general $p-$brane, it is important to know that the action S_p is

classically equivalent[4] to the action $\hat{S}_p$ defined by

$$\hat{S}_p \equiv -\frac{T}{2}\int d^{p+1}\sigma\sqrt{-\det h_{\alpha\beta}}\eta_{\mu\nu}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + \frac{p-1}{2}T\int d^{p+1}\sigma\sqrt{-\det h_{\alpha\beta}} \quad , \tag{1.5}$$

where $h_{\alpha\beta}$ is a new metric on the worldvolume, which a priori is not dependent on $\gamma_{\alpha\beta}$; nevertheless, the equations of motion for $h_{\alpha\beta}$ states that, classically, these two metrics are proportional.
The importance of the action $\hat{S}_p$ can be appreciated when one is interested in quantizing the $p-$brane, a procedure which is very difficult to perform with the action S_p because of the square root appearing in (1.2). This is the reason why in string theory - which is a worldsheet formulation of fundamental homogeneous strings - one defines $\hat{S}_p$ to be the action of the system; this action is called *Polyakov action* and in the following will be denoted as S_{Poly}, that is

$$S_{Poly} \equiv \hat{S}_1 = -\frac{T}{2}\int d^2\sigma\sqrt{-\det h_{\alpha\beta}}\eta_{\mu\nu}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu \quad . \tag{1.6}$$

This action is invariant under *local Weyl transformations* of the metric $h_{\alpha\beta}$ which act on the latter as

$$h_{\alpha\beta}(\sigma^0,\sigma^1) \mapsto e^{2f(\sigma^0,\sigma^1)}h_{\alpha\beta}(\sigma^0,\sigma^1), \tag{1.7}$$

where f is a general real scalar function on the worldsheet. Geometrically speaking, this means that string theory is not sensitive to a local change of scale which preserves the angles between all lines; for example, the two worldsheet metrics shown in the Figure 2 are surprisingly viewed, by the Polyakov string, as equivalent (at the quantum level, these two wordsheets will define the same physical state). It is not difficult to imagine that a theory that enjoys such a gauge symmetry is

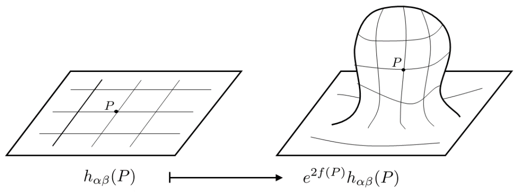

Figure 2
A Weyl transformation on the worldsheet.

extremely rare and has to go with very stringent requirements in its structure. In particular, the kind of interactions that we can add to S_{Poly} are strongly limited and familiar terms like

$$\int d\sigma^0 d\sigma^1\sqrt{-\det h_{\alpha\beta}}V(X^\mu) \quad , \quad \text{V polynomial potential} \tag{1.8}$$

are not allowed because they (explicitly) break the Weyl invariance of the theory; note that, in particular, we cannot admit a cosmological constant μ on the worldsheet, because it corresponds to introduce a term like (1.8) with a constant potential $V(X^\mu)$.
At the quantum level, Weyl invariance will become even a more stringent requirement, to such a point that it will lead us to introduce the concept of the *critical dimension*: the quantum theory of a string moving in Minkowski spacetime $\mathcal{M}$ is consistent only if the latter has a particular dimension.
As we have already mentioned, in the worldvolume approach, to quantize a system one has to sum over all possible trajectories (equivalently, over all possible histories) of the string: this is the so-called *first quantization* of the string, where the basic object of the theory is the trajectory of the string rather than a function(al) of strings.

[4]Actually, this is true only for $p = 0, 1$; for $p > 1$ the action (1.5) has not enough symmetry to fix all the entries of the (unphysical) metric $h_{\alpha\beta}$ so in this case the worldvolume of the brane will contain inner unphysical degrees of freedom. So, for $p > 1$ we have to use more sophisticated actions.

- According to the *spacetime* or *field* formalism, the dynamical variables are chosen to be some fields $\Psi[X^\mu]$, which are functions (better, functionals) of the spacetime coordinates of the $p-$brane for every given $p-$dimensional shape of the latter. This is the well-known approach to classical and quantum field theory, where both the time coordinate and the spatial coordinates of pointlike particles are nothing but labels.
  By analogy with QFT, we can forecast that the field description is particularly suitable when we have to work with a huge number of $p-$branes or when the number of the fundametal objects is not fixed.
  When the canonical quantization scheme is performed within the field formalism, one obtains the so-called *second quantization* of the $p-$brane. In the case of fundamental strings, the second quantized theory is called "string field theory", whose consistency has been proven in the last ten years. A lot of work has still to be done in this subject, but string field theory has already obtained its first important success, among which there is the explanation of tachyon condensation.

Sharing the description of the same extended object, these two formalisms are conceptually equivalent. Instead, they differ as computational tools, insofar as some calculations are simpler (if at all possible) in one of them. Sometimes, it is impossible even to address a particular question in one formalism and we are forced to select the other one. This happens, for example, when we are interested[5] in going off-shell. If this is the case, it is difficult to deal with the wordvolume approach, because – as we explained for the pointlike particle case – the redundant structure of the theory requires us to work on-shell (something that holds also within string theory, as we will see).

1.2 Why strings?

Among all other fundamental $p-$brane theories, the theory of fundamental 1-branes occupies a distinguished place. Roughly speaking:

- Strings are nicer than pointlike particles. Being extended objects, strings do not suffer from UV divergences. In a QFT of pointlike particles, UV divergences arise because interactions are arbitrarily localized at a point of $\mathcal{M}$ (in the language of Feynman diagrams, UV divergences appear because two vertices come together or, equivalently, because the momentum flowing into a loop becomes infinite). The *point of interaction* – defined as the locus in spacetime where the number and/or the nature of the objects change – of two pointlike particles is *geometrical*, perfectly localized in spacetime, independent of the Lorentz frame of observation. The geometrical nature of this point is apparent in Figure 3, where it is clear that both observers (one boosted with respect to the other one) always agree on which parts of the worldline (there, the Feynman diagram of a cubic interaction) correspond to one or two particle states. Instead, the spatial extension of the strings makes their interactions non-local in spacetime; for instance, the two observers of the previous example, this time will recognize the point of interaction in two different points of spacetime. In other words, the interactions of strings appear "smeared out" and so, intuitively, there is hope for getting a theory free of UV divergences.
  The action (1.6) defines a local field theory on the worldsheet: there are no non-local objects at our disposal that can be attached "around" the "smeared out" region of interaction to specify the nature of the interaction on the worldsheet. This means that the interaction is determined by the "shape" of the worldsheet, namely by its topology[6]. String interactions result from non-trivial topology of the surface and, as such, they are "maximally smeared out", because the topology of the worldsheet appears as a global concept. As we will see, it will turn out that there is only one possible topology for the worldsheet at a given loop level[7] and this is in apparent contrast with the perturbation theory of pointlike particles, where the number of possible Feynman diagrams grows at each loop level.
  To sum up, the extended nature of strings makes string theory more appealing than a theory of

[5]It is important to learn how to go off-shell in a theory, because it is needed for the renormalization.
[6]We will be interested in compact oriented Riemann surfaces; in this case, it is legitimate to think of "topology" as the number of handles of the surface.
[7]This is true only if we are considering closed oriented strings.

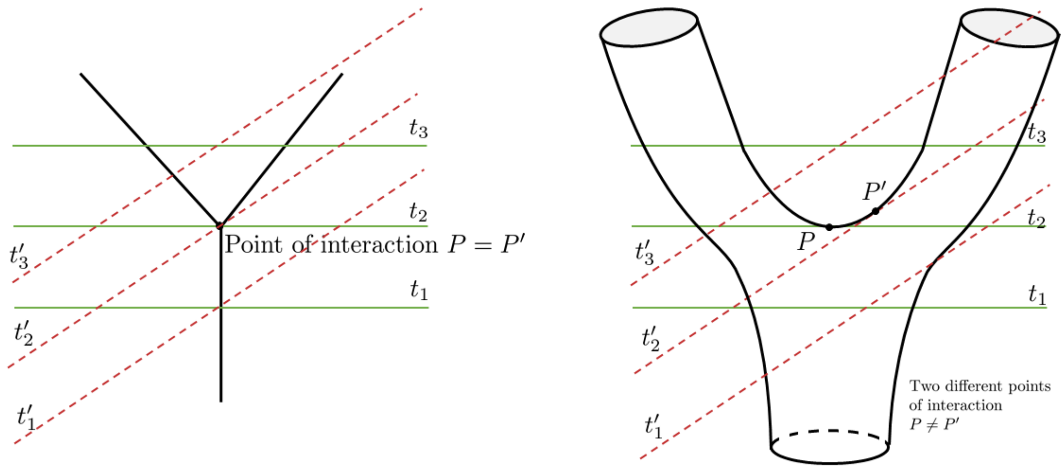

Figure 3
Interactions between pointlike particles and strings. The red and green inertial observers are the same in both cases. In the pointlike case (on the left), the interaction has a geometrical nature. On the right, instead, the interaction is non-geometrical.

pointlike particles, both from conceptual (UV finite behaviour) and practical (drastic reduction of the possible kind of interactions) points of view.

- Strings are nicer than $p-$branes with $p > 1$. In fact, there appear to be both technical and conceptual obstacles when ones tries to quantize higher dimensional objects.
  For $p > 1$ the worldvolume action is not Weyl invariant anymore and this makes it hard to quantize. Indeed, one would like to use the action $\hat{S}_p$, but it hasn't enough local symmetries to gauge fix all the independent entries of the metric $h_{\alpha\beta}$; given that we cannot let the abstract worldvolume (something unphysical) to host inner degrees of freedom, we have to search for a more complex action and the quantization procedure will end up with getting complicated.
  The conceptual issue lays on the deep fact that the quantization of a higher dimensional object leads to a continuum spectrum, but we need a discrete one, otherwise we can not interpret its excitations as single particle states.

1.3 Quantization and interactions with the worldsheet: Polyakov's formulation

A fundamental $0-$brane (a pointlike particle like the W^- boson, the electron or the antineutrino $\bar{\nu}_e$ of the Standard Model) is ontologically one and can not be divided; it can decay into other particles but these can not be considered as components of the first one. For example, in the framework of the Standard Model, it is completely nonsense to say that the electron and the antineutrino coming from the decay of a W^- boson used to be constituents of the W^- . It doesn't make sense, because the interaction of pointlike particles is totally localized in spacetime; this also implies that for each observer there is a glaring notion of particle, which can be geometrically defined as the intersection point between the equal-time surface of the observer and the worldline of the particle.
This is not the case for the blurred world of strings, because their interactions are not-localized. This is the most striking physical difference between fundamental particles and fundamental strings. A string can be cut into pieces, but these are nothing but pieces of the original string: they can carry different quantum numbers like the mass and the spin (and so they appear, in our laboratory, as different particles) but they cannot aquire new ontological attributes (their tension is the same as that of the original string), otherwise the string would not be fundamental; at the same time, we can not say that the original string

used to consist of these two pieces melded together, because it could have been separated at any of its points.
The upshot is that only the full worldsheet[8] Σ carries an ontological attribute and that interactions among strings can be understood only by looking globally at it. We are then lead to two basilar ideas:

- the information about strings interactions is encoded in the topology of the worldsheet, as we have already mentioned. In the following chapters, only the oriented closed string will be studied; its worldsheet is an oriented Riemann surface (a complex manifold of real dimension 2) that, as we will soon see, can be considered compact. The classification of compact oriented surfaces is well-known: their topology can be distinguished by specifying the *Euler number* Ξ, an integer number that describes their topological structure, regardless of the way they are bent (it is a topological invariant quantity). For oriented and compact 2−dimensional surfaces, Ξ is given by

$$\Xi = 2(1-g) \tag{1.9}$$

  where g is the *genus* of the surface, which simply counts the number of handles of Σ. For example, a torus has one handle, so its Euler number is zero.

- the local dynamic of the string does not depend on whether there are interactions or not. In a Lorentz-covariant theory like that one defined by S_{Poly}, non-local interaction terms are not allowed to explicitly appear in the action. So, the action of the free string must already contain, somehow, interactions. This is in apparent contrast with the case of the pointlike particle, whose free action is

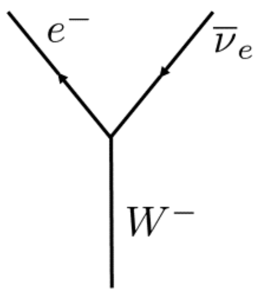

Figure 4
A representation of the decay of the weak boson.

  different from the interacting one. Let's consider the decay of the W^- boson depicted in Figure 4: the smooth worldline of the free propagating boson experiences a singular joining at the interaction point, meaning that "something different from the free propagation" happens there and the nature of this "something different" must be specified, by adding - to the action of the free particle - the action of the interaction. Analogously, in (Q)FT, one has to point out the Feynman rule for the vertex of the corresponding diagram; for the pointlike particle, at every singularity of the worldline one has to add a covariant object, such as a gamma matrix or the momentum of a particle. In string theory, instead, worldsheets don't experience any singularities, a symptom of the fact that string interactions don't need another action[9] to be specified.

Both of these features have to be taken into account to perform a meaningful quantization, because they are direct consequences of the extended nature of the string.
A natural way to incorporate them in the quantized theory is achieved by resorting to the path integral. Indeed, this is a method which is clearly suitable for describing interactions in string theory, because it lets us get physical quantities by dealing directly with the worldsheet, which is the unique object which has an ontological existence.
To be more precise, in this approach, amplitudes are given by summing over all histories (over all

[8]Given that strings are nicer than any other $p-$brane, we refer to its worldvolume $\mathcal{W}_{p=1}$ with a particular symbol: $\Sigma \equiv \mathcal{W}_{p=1}$.
[9]Actually, this is true at the classical level and "almost" true at the quantum level (see later the introduction of S_{HE}).

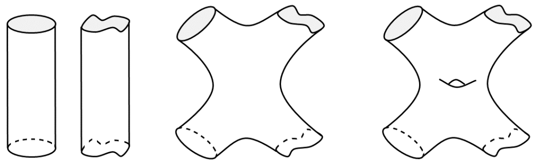

Figure 5
From the left: non-interacting strings, interacting strings at tree level and the one-loop interaction. The external states are the same for all the pictures.

worldsheets and over all fields defined on them) interpolating between the initial and final states; in the case of the string, the external states are identified with the boundary curves of Σ, as illustrated in Figure 5, where a free and an interacting worldsheet with the same external states are depicted. In the path integral approach[10], each history Σ is weighted by ($\hbar = c = 1$)

$$e^{-S_{cl}[\Sigma]} \tag{1.10}$$

where $S_{cl}[\Sigma]$ is the classical action for the given worldsheet. So, at this point, we need to specify the most general action S_{cl} for the classical string. This task is largely simplified by the strict structure imposed by the gauge redundancies of the classical string, namely reparametrization and Weyl transformations, which restrict the possible actions to be of the following form:

$$S_{cl} = S_{Poly} + \lambda S_{HE} = S_{Poly} + \lambda \frac{1}{4\pi} \int d\sigma^0 d\sigma^1 \sqrt{h} R, \tag{1.11}$$

where λ is a real dimensionless parameter and S_{HE} is the usual Hilbert-Einstein action for the metric $h_{\alpha\beta}$ of the worldsheet.
In two dimensions, the Hilbert-Einstein action doesn't carry any dynamical information, essentially because the metric has three independent entries which locally can be fixed by gauging Weyl invariance and reparametrization invariance of σ^0 and σ^1. Indeed, S_{HE} turns out to be a topological term, because, in the two dimensional case, Gauss-Bonet theorem states that

$$S_{HE} = \Xi = 2(1-g) \quad , \tag{1.12}$$

where Ξ and g are the topological invariant quantities defined above: for a given worldsheet, S_{HE} is a constant integer.
At the classical level, we don't have to sum over all the possible worldsheets and only the $(g = 0)-$topology (a sphere) contributes; thus, at the classical level, S_{HE} - being a constant- can be forgotten and this is the reason why the action of string theory is usually defined to be S_{Poly} alone. At the quantum level, all possible $g-$topologies have to be considered and S_{HE} implies that the worldsheet with g holes must be weighted by a factor of

$$e^{-\lambda\Xi} = e^{-2\lambda(1-g)} \quad .$$

We get that *perturbation theory* in string theory is a sum of all the contributions coming from all possible worldsheets Σ_g of genus g ordered by the *string coupling constant*

$$g_s \equiv e^{\lambda} \quad . \tag{1.13}$$

So, for $g_s << 1$, we have a good *perturbative expansion* in which the sum over all histories reads as

$$\begin{aligned} \sum_{\Sigma_g} \int D[X^\mu] D[h_{\alpha\beta}] \exp(-S_{cl}[\Sigma_g]) &= \sum_{g\geq 0} \int D[X^\mu] D[h_{\alpha\beta}] \exp\Big(- S_{Poly}[\Sigma_g] - 2\lambda(1-g) \Big) = \\ &= \sum_{g\geq 0} g_s^{2(g-1)} \int D[X^\mu] D[h_{\alpha\beta}] \exp(-S_{Poly}[\Sigma_g]) \quad . \end{aligned} \tag{1.14}$$

[10] We will always consider a path integral over the Euclidean version of the worldsheet, obtained by the lorentzian one through a Wick rotation, because it leads to a better defined sum over the metrics of the worldsheet.

This is an asymptotic expansion, just as in QFT.
We can also make contact with QFT by taking the *pointlike limit* of the string; every worldsheet becomes a worldline and, pictorially, all the Riemann surfaces that appear in the perturbative expansion can be interpreted as Feynman diagrams; in particular, this tells us that the number of handles of a Riemann surface is the string analogue of the number of loops. Whereas in QFT the number of Feyman diagrams grows factorially with the loop level, in string theory there is only one topological distinct Riemann surface contributing. In this sense, perturbation theory with strings is considered to be "cheaper" than that one of QFT.
It is not only a cheaper description, but it also seems more fundamental, because:

- the string coupling $g_s = e^{\lambda}$ is not an independent parameter of string theory, because λ can be determined by the dynamics of the string moving in a curved background. To be more precise, λ turns out to be the spacetime expectation value of a massless field - the dilaton field - whose quanta can be described as particular excitations of the string. String theory doesn't admit free parameters (except for α') and it doesn't leave room for adjusting any dimensionless constant that enters the theory. This is a remarkable property of string theory which is not shared by any QFT, whose coupling constants usually cannot be fixed by any inner mechanisms.

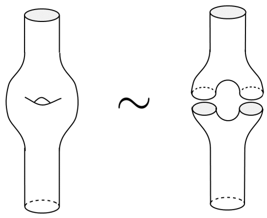

Figure 6
Intuitive decomposition of 2-point correlation function at one loop.

- all the possible topologies of the closed worldsheet can be decomposed into various copies of a particular worldsheet (see Figure 6, for example). This particular worldsheet represents the basic interaction of closed strings, a process in which a closed string splits into two, or - reversing time direction- a process in which two closed strings join into one. We can now appreciate the unifying language of string theory: in closed string theory, not only all particles (graviton, gauge bosons, fermions,...) are obtained as various states of excitation of the string, but also all interactions (gravity, gauge, Yukawa, ...) arise from the single process of Figure 7, which - in the pointlike limit of the string - can be interpreted as a Feynmann diagram of a cubic interaction. Perturbation theory in closed (bosonic) string theory can thus be interpreted as a perturbation theory of a (scalar) two-dimensional quantum field theory with cubic interaction on the worldsheet[11]; the latter is renormalizable, so we guess that string theory is a good candidate for describing a microscopic (i.e. fundamental) theory of all interactions (gravity included).

1.4 Vertex operators and S-matrix

In order to get this cheap description, compactness of the worldsheet has revealed itself to be an essential ingredient because it has allowed us to use the classification theorem of the 2−dimensional surfaces mentioned above.
In string theory, this compactness can be reached by exploiting the *state-operator correspondence map*, that will be discussed in section 4.8. Roughly speaking, it consists of replacing the external state of a string with a *vertex operator*, namely a local operator on the worldsheet which carries information about all the quantum numbers of the replaced state. This correspondence can be schematically represented

[11]Please note that this is true only if the string is propagating on a flat target space. Indeed, if the target space is characterized by a metric $G_{\mu\nu}(X)$, then the action that generalizes the Polyakov one should be something like $\int d^2\sigma G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta}$; the metric $G_{\mu\nu}$ can depend in a complicated way on X^μ and the worldsheet theory is not cubic in general.

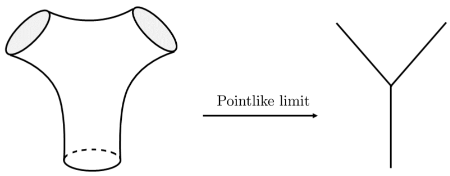

Figure 7
The basic process of closed strings as a cubic interaction.

as done in Figure 8, where the cross stands for the point where the vertex operator is inserted. The state-operator correspondence is a tool which naturally appears in string theory, thanks to the gauge redundancy of the worldsheet description. Indeed, within the framework of a theory which is Weyl and reparametrization invariant, this correspondence can be understood as a *conformal transformation*, where the latter is nothing but a residual gauge transformation that the theory still admits after fixing the gauge redundancies.

With the introduction of vertex operators, we are outlining another analogy between string theory

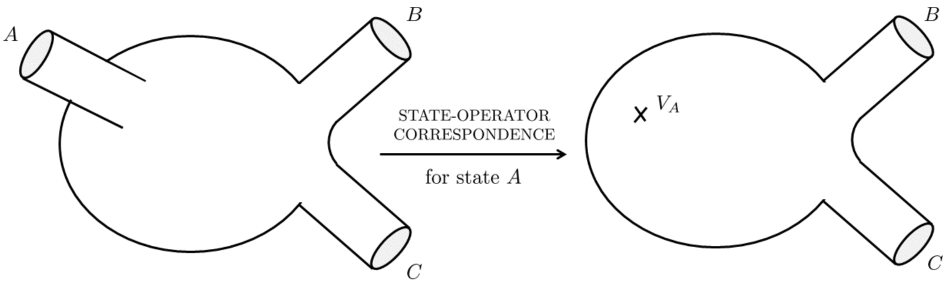

Figure 8
We replace the state A of the string with the vertex operator $V_{(A)}$ on the worldsheet.

and QFT, because external states (that is, on-shell particles) are represented by operators acting on the classical vacuum $|0\rangle$ of the theory. In light of this similarity, it appears natural to define the scattering amplitude $S_{j_1,...,j_n}(p_1,...,p_n)$ among n states (labeled by quantum numbers $\{j_i\}$ and spacetime momenta $\{p_i\}$, $i \in \{1,...,m\}$) as the Polyakov path integral over the worldsheet with the insertions of the vertex operators $V_{j_i}(p_i)$ corresponding to the external states, that is

$$S_{j_1,...,j_n}(p_1,...,p_n) \equiv \sum_{g\geq 0} g_s^{2(g-1)} \int D[X^\mu]D[h_{\alpha\beta}]e^{-S_{Poly}[\Sigma_g]} \prod_{i=i}^{m} V_{j_i}(p_i) \quad . \tag{1.15}$$

As in QFT, this formula defines[12] the $S-matrix$ element for m external states, once the latter are taken to sit at the infinity of the spacetime $\mathcal{M}$.

1.5 Quantum gravity and compactifications

As we have just mentioned, the Polyakov string admits, among its excitations, the graviton, namely the hypothetical massless boson of spin 2 that should mediate the force of gravity. Thus, it is common sense to say that string theory is a theory of "quantum gravity", because it allows us to explain, in terms of

[12]The complete formula for the S-matrix will be derived in section 5.4.

the same fundamental object, the nature of all elementary particles and of all interactions between them, gravity included.
But, with a theory of quantum gravity, one would like to determine, dynamically, the geometry of the spacetime, given that the latter is expected to be a classical object consisting of interrelated fundamental quanta. Obviously, this cannot be achieved in the framework of string theory, because it is a first quantized theory and in the worldvolume formalism such a question cannot even be addressed. In order to describe the metric of spacetime as an emergent property coming from collaborating strings, one should be able to do "statistical mechanics with strings", which is one of the aims of string field theory (the second quantized version of string theory). Instead, in string theory the (finite) number of strings is fixed (according to the scattering amplitude that we need to compute) and the geometry of $\mathcal{M}$ is specified a priori; in this context, the string simply represents a fluctuation (namely: a graviton, an electon, etc.) propagating with negligible backreaction on this background.
Actually, this is not completely true, because the gauge symmetries of string theory are very demanding; they are so stringent that, to get a consistent dynamics of a single string in a curved[13] background, the latter must satisfy particular constraints: for example, a void spacetime can host a string only if it is $10-$dimensional[14] and Ricci-flat. There are a lot of backgrounds that satisfy these requirements. Among them, we find solutions which admit the not-observed six dimensions to be curled up in very tiny compact manifolds, in such a way that they are penetrable only at very high energy; moreover, these manifolds have to be very special if we want for important properties of our ten dimensional theory to survive also in the four dimensional bulk[15].
It is a remarkable and significant fact that string theory, albeit being "only" a first quantized description, it is able to give - in the attempt of recovering the daily phenomenology of our $4-$dimensional spacetime - additional and highly non-trivial information about the structure of spacetime and, as such, it is not only a respectable theory of quantum gravity, but it is a more than respectabe one. Demanding string theory to uniquely and dynamically determine the background of spacetime is simply asking too much a theory which has already yielded enough. From this point of view, string theory shoud not be wickedly criticized; instead, as a scientific theory, we should understand to what extent we can trust it. So let's have a closer look at it!

[13] The action S_σ of a string moving in a curved background $G_{\mu\nu}(X)$ is obtained, as one would expect, by substituting the flat metric $\eta_{\mu\nu}$ appearing in S_{Poly} with the curved one, as dictated by the minimal-coupling principle:

$$S_\sigma \equiv -\frac{T}{2}\int d^2\sigma\sqrt{-\det h_{\alpha\beta}}G_{\mu\nu}(X)h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu \quad . \tag{1.16}$$

We specified the subscript σ to denote this action only because S_σ defines a model that, for historical reasons, is called *non-linear sigma model.*
[14] To be more precise, it has to be 10 dimesnional in the case of the superstring, 26 dimensional if we are considering the bosonic string.
[15] We are talking about supersymmetry, a feature of the superstring.

2 Bosonic String Theory

After discussing the fundamental ideas underlying string theory, in the following chapters we are going to concisely present the structure of classical and quantum closed string theory. We will use the bosonic version of string theory as a (non-realistic) toy model that allows us to get acquaintance with the standard tools/techniques developed to study string theories.

The main references that we used for this chapter are [8] and [9].

2.1 The classical theory

The classical bosonic string theory is the theory on the 2−dimensional worldsheet $\Sigma_{g=0}$ defined by S_{Poly}

$$\begin{aligned} S_{Poly}[X^\mu, h_{\alpha\beta}; \Sigma_0] = &-\frac{T}{2}\int_{\Sigma_0} d^2\sigma\sqrt{-h}h^{\alpha\beta}\gamma_{\alpha\beta} = \\ = &-\frac{T}{2}\int_{\Sigma_0} d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad , \end{aligned} \tag{2.1}$$

where h stands for the determinant of the metric $h_{\alpha\beta}$ and the tension T is usually expressed as

$$\begin{aligned} T &= \frac{1}{2\pi\alpha'} \\ \alpha' &\approx l_s^2 \end{aligned} \tag{2.2}$$

with l_s the characteristic length of the string. We specify that l_s is an invariant quantity of the theory (*any* observer agrees on its value) that is expected to approximately coincide with the characteristic length of quantum gravity[16], namely the Planck length l_p ($l_s \approx l_p \approx 10^{-35}m$).
This action appears to be the appropriate setting for a fundamental string model of elementary particles, since it involves only the intrinsic geometry of the string, with no reference to the extrinsic curvature experienced by the string.
It describes relativistic (homogeneous fundamental) strings: the Poincaré invariance

$$\begin{aligned} X^\mu &\mapsto \Lambda^\mu_\nu X^\nu + c^\mu \\ h_{\alpha\beta} &\mapsto h_{\alpha\beta} \end{aligned}$$

appears as a global symmetry on the worldsheet (the index μ is seen, from the worldsheet, as an inner one) that gives the usual associated conserved Noether currents in the target space. The gauge structure of S_{Poly} consists of:

- Reparameterization invariance. If we redefine the worldsheet coordinates as $\sigma^\alpha \mapsto \tilde{\sigma}^\alpha(\sigma)$, the fields X^μ transform as worldsheet scalars, whereas $h_{\alpha\beta}$ transform as a metric should do. At the infinitesimal level, this means

$$\begin{aligned} \delta X^\mu(\sigma) &= -\delta\sigma^\alpha\partial_\alpha X^\mu(\sigma) \\ \delta h_{\alpha\beta}(\sigma) &= -\nabla_\alpha\delta\sigma_\beta - \nabla_\beta\delta\sigma_\alpha \quad , \end{aligned} \tag{2.3}$$

  where $\delta\sigma^\alpha$ are defined by the linear term of the transformation ($\tilde{\sigma}^\alpha \approx \sigma^\alpha + \delta\sigma^\alpha$).

[16] Actually, in this thesis we are interested in the case in which it is possible to get a well-defined perturbation expansion in g_s. It can be shown that $g_s \ll 1$ implies that $l_s \gg l_p$. This means that the perturbative aspects of string theory are better understood when it is possible to disentangle stringy physics from strong coupling effects of gravity and it maybe means that quantum gravity phenomena could appear even before reaching l_p. It is worth to mention that compactifications could explain the reason why $l_s \gg l_p$, by means of the presence, in our spacetime, of the (relatively big) volume of the compactified extra-dimensions. See [8] for details.

- Weyl invariance. As we have already seen in (1.7), Weyl transformations rescale the metric of the worldsheet by a local factor that we like to express as $e^{2f(\sigma)}$ (f is a function on the worldsheet). At the infinitesimal level, this transformation reads as

$$\begin{aligned} \delta X^\mu(\sigma) &= 0 \\ \delta h_{\alpha\beta}(\sigma) &= 2f(\sigma)h_{\alpha\beta}(\sigma) \quad . \end{aligned} \tag{2.4}$$

The local invariances allow for a convenient gauge choice for the worldsheet metric $h_{\alpha\beta}$, called *conformal gauge*. For any point, we can consider the two null-vectors that exist because the metric has Minkowskian signature; their integral curves give the light-cone coordinates σ^+ and σ^- and with respect to them we must have

$$ds^2 = -\Omega^2(\sigma^+, \sigma^-)d\sigma^+ d\sigma^- \quad , \tag{2.5}$$

for some real function Ω. Now we can use reparametrization invariance to define the coordinates σ^0 and σ^1 as

$$\sigma^\pm \equiv \sigma^0 \pm \sigma^1 \quad . \tag{2.6}$$

With respect to these new coordinates, the metric takes its conformal gauge form, namely

$$ds^2 = \Omega^2(\sigma)(-d^2\sigma^0 + d^2\sigma^1) \quad . \tag{2.7}$$

At this point, one could also use Weyl invariance to bring $\Omega(\sigma)$ to 1 and we obtain that, *locally*, we can always suppose[17] $h_{\alpha\beta} = \eta_{\alpha\beta}$, that is

$$ds^2 = (-d^2\sigma^0 + d^2\sigma^1) \quad . \tag{2.8}$$

One should be aware of an essential fact: the choice of the flat metric for $h_{\alpha\beta}$ doesn't fix completely the gauge redundancies. The residual gauge consists of *conformal transformations* of the coordinates, particular diffeomorphisms that can be undone by a Weyl rescaling. From formula (2.5), we understand that these peculiar transformations are given by the redefinitions of new coordinates ($f_\pm$ real functions of only one variable)

$$\tilde{\sigma}^\pm \equiv f_\pm(\sigma^\pm) \quad . \tag{2.9}$$

Indeed, the only effect of (2.9) would have been changing the coordinates $\sigma^\pm \mapsto \tilde{\sigma}^\pm$ and changing Ω: up to a Weyl transformation, we would have ended up again with the flat metric

$$ds^2 = -d\tilde{\sigma}^+ d\tilde{\sigma}^- = -d^2\tilde{\sigma}^0 + d^2\tilde{\sigma}^1 \quad ,$$

where, as before, $\tilde{\sigma}^0$, $\tilde{\sigma}^1$ are defined by $\tilde{\sigma}^\pm = \tilde{\sigma}^0 \pm \tilde{\sigma}^1$. This means that when we work with the flat metric (2.8), we have still the freedom to specify what me mean with the coordinates σ^0 and σ^1, the latter being defined up to a conformal tranformation.
Anyway, with the choice of the flat metric (2.8) (or, more in general, with the conformal metric (2.7)), the Polyakov action simplifies tremendously and becomes the theory of D free scalar fields:

$$S_{Poly}[X^\mu, \eta_{\alpha\beta}; \Sigma_0] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial^\alpha X^\mu \partial_\alpha X_\mu \quad , \tag{2.10}$$

whose equations of motion are nothing but the free wave ones for the worldsheet scalars X^μ, that is

$$\Box X^\mu = 0 \quad . \tag{2.11}$$

At this point, two comments must follow.

1. As always, to obtain the equations of motion for a field one has to stationarize the action against all the possible synchronic variations of that field. When one does it with the action (2.10), it has to be kept in mind that we are considering only closed strings, so all the total derivatives with respect to σ^1 vanish upon integration on Σ_0; indeed, Σ_0 has the topology of a cylinder and there are no possible contributions from the spatial boundary term of Σ, because all the fields respect the periodicity of σ^1 (to be clear, let's define the latter by $\sigma^1 \approx \sigma^1 + 2\pi$). We like to stress this elementary point, because it is precisely at this level that we are introducing in our model the information regarding the topology of the string.

[17] Please note that this argument can be used for showing that gravity is not dynamical in two dimensions, given that, in two dimensions, S_{HE} and S_{Poly} enjoy the same symmetries. Now it should be clear the reason why S_{HE} contributes as a topological term when it is added to S_{Poly} in the framework of perturbative string theory.

2. The equations of motions (2.11) are not enough to specify the dynamics of the Polyakov string. Indeed, by looking at the original action $S_{Poly}[X^\mu, h_{\alpha\beta}; \Sigma_0]$ (see (2.1)) it is apparent that we have to take into account also the equations of motion for $h_{\alpha\beta}$, that are given by

$$0 = \frac{\delta S_{Poly}}{\delta h_{\alpha\beta}}[X^\mu, h_{\alpha\beta}; \Sigma_0] \propto T_{\alpha\beta} \quad .$$

The metric of the worldsheet appears only algebraically in (2.1) (consistently with the fact that it doesn't carry any dynamical degrees of freedom), so the equations of motion of $h_{\alpha\beta}$ are constraints that have to be imposed on the solutions of the free wave equation (2.11). In the flat gauge, the stress-energy tensor is

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X \quad , \tag{2.12}$$

and the constraints read as

$$\begin{aligned} T_{00} = T_{11} &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \\ T_{01} &= \dot{X} \cdot X' = 0 \quad , \end{aligned} \tag{2.13}$$

where we have introduced the standard notation for $\dot{X} \equiv \partial_{\sigma^0} X^\mu$ and $X' \equiv \partial_{\sigma^1} X^\mu$.
The physical meaning of these constraints becomes transparent if we exploit the residual gauge that has survived after imposing $h_{\alpha\beta} = \eta_{\alpha\beta}$ to relate σ^0 to the time coordinate X^0 in the Lorentz frame of our laboratory[18], namely

$$\sigma^0 \equiv \frac{X^0}{R} \quad , \tag{2.14}$$

where R is a constant that is needed on dimensional grounds (it has the dimension of a length[19]). This choice of parameterization of σ^0 is called the *static gauge* because the hypersurface of the worldsheet defined by $\sigma^0 = \tau \in \mathbb{R}$ becomes, once embedded into the target-space, the closed string that we see in our laboratory at "fixed" time $X^0 = R\tau$. In the static gauge, the equation of motion (2.11) naturally becomes

$$\Box X^i = 0, \tag{2.15}$$

and the constraints (2.13) can be nicely written in vector notation as

$$\begin{aligned} (\dot{\vec{X}})^2 + (\vec{X}')^2 &= R^2 \\ \dot{\vec{X}} \cdot \vec{X}' &= 0 \quad . \end{aligned} \tag{2.16}$$

So, in our laboratory, we see the string oscillating according to the well-known wave equation...but with a peculiarity: the physical oscillations must be perpendicular to the string itself, namely they must be *transverse*, otherwise the last constraint would be violated.

To find the solution to (2.11), it is convenient to rewrite it in terms of the lightcone coordinates $\sigma^\pm$ of the worldsheets:

$$\partial_+\partial_- X^\mu(\sigma^+, \sigma^-) = 0 \quad . \tag{2.17}$$

The most general smooth solution to the wave equations (2.11) is *locally*[20]

$$X^\mu(\sigma^0, \sigma^1) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \quad , \tag{2.18}$$

[18] Later we will explain the reason why this is possible. Anyway, we are not entering the details of the static gauge, because - for us - it is only a nice way to get an idea about the physical meaning of the constraints. We will never use the static gauge again.

[19] In our conventions, the scalar fields X^μ have the dimension of a length (they are the spacetime positions of the string in our spacetime), whereas σ^α and $h_{\alpha\beta}$ are dimensionless (they respectively are unphysical labels and an auxiliary metric field on the worldsheet).

[20] A very important detail. As we've already known from classical mechanics, the solution (2.18) is the right one only if $X^\mu(\sigma^0, \sigma^1)$ is defined on a simply connected region. If the worldsheet has a non-trivial topology, the decomposition into right/left movers has to be performed in each local patch. If we choose the conformal gauge in each patch, the transition functions that let us go from a patch to the other one are conformal transformations. If $X^\mu_{L/R}$ were tensors under these transformations, the decomposition of X^μ into right/left movers would be globally defined. It will turn out that this is the case and the notion of right/left movers for X^μ is, thus, global. Note that if $X^\mu_{L/R}$ are globally defined then also X^μ is and this is consistent with the fact that X^μ is an embedding of all the worldsheet.

where $X^\mu_{L,R}$ are arbitrary functions on the worldsheet that depend only on one of the lightcone variable $\sigma^\pm$; we use the subscripts L, R because we will call them - for intuitive reasons - *left/right movers*.
By requiring $X^\mu(\sigma^0, \sigma^1)$ to respect the periodicity of $\sigma^1 \approx \sigma^1 + 2\pi$, we can express the most general solution in terms of the Fourier modes and we end up with

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+} \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^-} \quad , \end{aligned} \tag{2.19}$$

where we have introduced:

- the real constants x^μ and p^μ; to see that they can be interpreted as the initial positon and the momentum of the center of mass of the string, it is enough to get rid of the Fourier modes of the string (that can be thought of as mechanical oscillations of the string that determine how it appears in spacetime) by integrating over the periodicity of σ^1, that is

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma^1 X^\mu(\sigma^0, \sigma^1) = x^\mu + \alpha' p^\mu \sigma^0 \tag{2.20}$$

  and to notice that p^μ is precisely the Noether charge associate to the spacetime translation symmetry of S_{Poly}.

- the complex constants $\tilde{\alpha}^\mu_n$ and α^μ_n; they are the coefficients of the Fourier modes of the right and left movers that, because of the reality of X^μ, have to satisfy:

$$\begin{aligned} \tilde{\alpha}^\mu_n &= (\tilde{\alpha}^\mu_{-n})^* \\ \alpha^\mu_n &= (\alpha^\mu_{-n})^* \quad . \end{aligned} \tag{2.21}$$

  According to the conformal structure of the Polyakov string, the fields $\partial_\pm X^\mu = \partial_\pm X^\mu_{LR}$ are[21], in a certain sense, more important than the fields X^μ themselves; so, it is convenient to interpret the momentum of the centre of mass as the coefficient of the zero Fourier mode and to define

$$\begin{aligned} \tilde{\alpha}^\mu_0 &\equiv \sqrt{\frac{\alpha'}{2}} p^\mu \\ \alpha^\mu_0 &\equiv \sqrt{\frac{\alpha'}{2}} p^\mu \quad , \end{aligned} \tag{2.22}$$

  so as to obtain compact formulae for $\partial_\pm X^\mu$, which are

$$\partial_\pm X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha^\mu_n e^{-in\sigma^\pm} \quad . \tag{2.23}$$

As we explained with one of the comments above, to get the Polyakov dynamics we have to impose, on the solution (2.19), the constraints (2.13) which, in terms of the $\sigma^\pm$ coordinates, are simply

$$(\partial_- X)^2 = (\partial_+ X)^2 = 0 \quad . \tag{2.24}$$

These can be easily rewritten as

$$\begin{aligned} 0 = (\partial_- X)^2 &= \alpha' \sum_{n\in\mathbb{Z}} L_n e^{-in\sigma^-} \\ 0 = (\partial_+ X)^2 &= \alpha' \sum_{n\in\mathbb{Z}} L_n e^{-in\sigma^+}, \end{aligned} \tag{2.25}$$

[21] With obvious notation, we define $\partial_\pm$ as the partial derivatives with respect to $\sigma^\pm$.

where we have introduced the Fourier modes of the constraints L_n and $\tilde{L}_n$

$$\begin{aligned} L_n \equiv& \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m \\ \tilde{L}_n \equiv& \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m \quad . \end{aligned} \tag{2.26}$$

What we have just found is a very important lesson: any classical solution of the Polyakov string must obey an infinite number of constraints, namely

$$\tilde{L}_n = L_n = 0 \qquad \forall \quad n\in\mathbb{Z}, \tag{2.27}$$

which implies that α_n and α_m are somehow able to talk to each other, something that is not at all true at the level of the solution (2.19) (the same is valid for $\tilde{\alpha}_n$ and $\tilde{\alpha}_m$). They are able to talk to each other in a "Polyakov way", because they organize themselves in order to make the worldsheet energy-stress tensor vanish, which means, in particular, that the physical oscillations of the string are transverse.
By looking at formulae (2.26), one could expect something particular arising from L_0 and $\tilde{L}_0$ because, in this case, formulae (2.26) appear to be "symmetric"; indeed, L_0 are $\tilde{L}_0$ are very special, because they include the square of the spacetime momentum p^μ of the centre of mass, which is interpreted, in the Minkowskian target spacetime, essentially as the square mass of a particle:

$$p^\mu p_\mu = -M^2. \tag{2.28}$$

More explicitly, we have:

$$\begin{aligned} 0 = L_0 =& \frac{1}{2}\sum_{m\in\mathbb{Z}}\alpha_{-m}\cdot\alpha_m = \frac{1}{2}\bigg(\frac{\alpha'}{2}p^\mu p_\mu + \sum_{m\in\mathbb{Z}\setminus\{0\}}\alpha_{-m}\cdot\alpha_m\bigg) = \\ =& \frac{1}{2}\bigg(-\frac{\alpha'}{2}M^2 + 2\sum_{m>0}\alpha_{-m}\cdot\alpha_m\bigg) \quad , \\ 0 = \tilde{L}_0 =& \frac{1}{2}\sum_{m\in\mathbb{Z}}\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m = \frac{1}{2}\bigg(\frac{\alpha'}{2}p^\mu p_\mu + \sum_{m\in\mathbb{Z}\setminus\{0\}}\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\bigg) = \\ =& \frac{1}{2}\bigg(-\frac{\alpha'}{2}M^2 + 2\sum_{m>0}\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\bigg) \quad , \end{aligned} \tag{2.29}$$

from which we can deduce that the *effective mass* of the string[22] can be expressed in terms of the excited oscillator modes as

$$M^2 = \frac{4}{\alpha'}\sum_{n>0}\alpha_n\cdot\alpha_{-n} = \frac{4}{\alpha'}\sum_{n>0}\tilde{\alpha}_n\cdot\tilde{\alpha}_{-n} \quad . \tag{2.30}$$

This formula defines the *spectrum* of the Polyakov theory and it forecasts for the masses of the string to be either zero or, otherwise, of the order of

$$M \approx \sqrt{T} \approx \frac{1}{l_s} \quad . \tag{2.31}$$

Usually, one (sometimes naively) takes l_s to be approximately the Planck length. The Planck mass is incredibly high above the rest masses of all the elementary particles that we know, so it is believed that the latter should be massless excitations of the string that acquire their masses by means of a lower-energy mechanism (Higgs) of quantum field theory. This is the reason why we will focus on the massless spectrum of the Polyakov string. Obviously, it doesn't mean that the massive string spectrum is not important; it is essential to remove the UV divergences from loop integrals and its existence is thus necessary for the

[22] Here we are deliberately talking about masses but the reader should be aware that this is a sloppy language, borrowed from field theory. At the classical level, strictly speaking, M is a frequency which can be translated into a mass only at the quantum level, by means of a multplication with $\hbar$.

consistency of the theory: in string theory, every bit of its rigid structure is needed.
Formula (2.30) states that, at the classical level, the string mass M is a real[23] continuous quantity. At the quantum level, α_n^μ and $\tilde{\alpha}_n^\mu$ will become operators which are able to create/dissolve a quantum of excitation of the n^{th} harmonic and, thus, the mass spectrum (2.30) will be discrete, letting us interpret the quantum string state as a single particle state. Another important feature of the quantum theory is that the formula (2.30) will get a quantum correction given by a constant shift and, thus, at the quantum level there is the possibility to obtain particles with imaginary mass, called *tachyons*, which prevents the theory from having a time-dependent stable solution. As we will see, these instabilities will be cured by the superstring, by introducing fermions on the worldsheet. But, as first step, we have to understand the reason why the spectrum receives these quantum corrections.

2.2 The quantum theory

Here we are going to sketch the basics about the quantization of the Polyakov string with the canonical formalism, only to explain the new features that enter the theory at the quantum level and to get an idea about the massless spectrum of the bosonic closed string, so as to persuade the reader that string theory does include gravity.

The most natural (and old) approach with canonical quantization is the *covariant quantization*, which is reminiscent of QED's Gupta-Bleuler procedure: in a manifestly Lorentz invariant fashion, we promote all the fields X^μ to operators and then we impose the constraints (2.13) on the states of the Fock space. Tracing the Gubta-Bleuler steps, we promote X^μ's and their conjugate momenta $\Pi_\mu \equiv 1/(2\pi\alpha')\dot{X}_\mu$ to operator-valued fields obeying the canonical equal-time commutation relations

$$\begin{aligned} &[X^\mu(\sigma^0,\sigma^1)\,,\Pi_\nu(\sigma^{0\prime},\sigma^{1\prime})] = i\delta(\sigma^1-\sigma^{1\prime})\delta^\mu_\nu \\ &[\Pi^\mu(\sigma^0,\sigma^1)\,,\Pi_\nu(\sigma^{0\prime},\sigma^{1\prime})] = 0 \\ &[X^\mu(\sigma^0,\sigma^1)\,,X^\nu(\sigma^{0\prime},\sigma^{1\prime})] = 0 \quad . \end{aligned} \tag{2.32}$$

By standard computations we obtain the induced commutation relations for the Fourier modes among which the non-zero ones are

$$\begin{aligned} [\hat{x}^\mu\,,\,\hat{p}_\nu] &= i\delta^\mu_\nu \\ [\alpha_n^\mu\,,\,\alpha_m^\nu] = [\tilde{\alpha}_n^\mu\,,\,\tilde{\alpha}_m^\nu] &= n\eta^{\mu\nu}\delta_{n+m,0} \quad ; \end{aligned} \tag{2.33}$$

We introduced the hat-label above x^μ and p^μ, to stress the fact that they are operators. We then see that:

- the initial position $\hat{x}^\mu$ and the momentum $\hat{p}^\mu$ of the centre of mass of the string satisfy the Heisenberg uncertainty principle of a pointlike particle when $\sigma^0 = 0$ ($\hat{p}^\mu$ is a conserved quantity and $\hat{x}^\mu$ doesn't evolve in time: we are in the Schrödinger picture). This means that:
  1. we can interpret the first-quantized bosonic string as a first-quantized pointlike particle "surrounded" by the oscillations of the string;
  2. the quantum state of the center of mass is encoded in a wavefunction with support in the target-space, which can be decomposed in terms of the eigenfunctions $|p^\mu\rangle$ of the operator $\hat{p}^\mu$:
  $$\hat{p}^\mu\,|p^\mu\rangle = p^\mu\,|p^\mu\rangle \quad .$$
- up to the redefinition of the Fourier modes

$$\begin{aligned} a_n^\mu &\equiv \frac{\alpha_n^\mu}{\sqrt{n}} \quad \forall\, n>0 \\ (a_n^\mu)^\dagger &\equiv \frac{\alpha_{-n}^\mu}{\sqrt{n}} \quad \forall\, n>0 \quad , \end{aligned} \tag{2.34}$$

[23] This is true, thanks to the relations (2.21), only for the excitations in the spatial directions. The excitations in the time direction have imaginary mass and, of course, this is not acceptable. At the quantum level, things are even worse, because the quantum state corresponding to this kind of excitations will become states with negative norm and the unitarity of the theory is at risk. As we will see, we can get rid of these excitation by fixing the gauge.

we can restore the familiar algebra for the harmonic oscillators, namely

$$[a_n^\mu\,,\,(a_m^\nu)^\dagger]=\delta_{mn}\eta^{\mu\nu}\quad . \tag{2.35}$$

We discover that for each n^{th} harmonic (obviously, to talk about harmonic, it must be $n>0$) of the right sector we have a creation ($(a_n^\mu)^\dagger\propto\alpha_{-n}^\mu$) and an annihilation ($a_n^\mu\propto\alpha_n^\mu$) operator; these operators carry a spacetime vector index which points out the target space direction in which the quanta of that harmonic are created/annihilated; so we have an infinite tower[24] of harmonic oscillators for each spacetime directions. Actually, we are dealing with two of these towers, because the same considerations are valid also for the left sector.

Thanks to the commutations relations (2.33), we can build the Fock space of the quantum string. The vacuum state $|0\rangle$ is defined by

$$\alpha_n^\mu\,|0\rangle=\tilde\alpha_n^\mu\,|0\rangle=0\quad\forall\;n>0\quad ,$$

and a generic state is obtained by acting with any number of creation operators on the vacuum (an example: $(\tilde\alpha_{-1}^\nu)^3\alpha_{-3}^\mu\,|0\rangle$). Being a first-quantized theory, the vacuum $|0\rangle$ is the vacuum of a single string and as such, it must carry information about the center of mass of the string; so, to obtain the complete vacuum of the theory, we have to tensor $|0\rangle$ with a wavefunction describing the quantum state of the center of mass. Given that the latter can always be decomposed in terms of the eigenfunctions $|p^\mu\rangle$, the simplest choice is to work in momentum representation and define the vacuum $|0;p^\mu\rangle$ of the single string as

$$|0;p^\mu\rangle\equiv|0\rangle\otimes|p^\mu\rangle\quad . \tag{2.36}$$

From the commutation relations (2.33), we can eventually understand the reason why the quantum mass spectrum is shifted with respect to the classical one. Indeed, the classical constraints $L_n=\tilde L_n=0$ have to be imposed as operator equations on the Hilbert space of the physical states $|\text{phys}\rangle$, namely we have to require the vanishing of all of their matrix elements

$$\langle\text{phys'}|\,L_n\,|\text{phys}\rangle=\langle\text{phys'}|\,L_n\,|\text{phys}\rangle=0\quad ,$$

which are conditions that can be achieved by requiring ($L_n^\dagger=L_{-n}$)

$$L_n\,|\text{phys}\rangle=\tilde L_n\,|\text{phys}\rangle=0\quad\forall\;n>0\quad . \tag{2.37}$$

These are well-defined equations because, by looking at (2.26) and (2.33), we note that, for $n\neq0$, L_n ($\tilde L_n$) is a composite operator of commuting α's ($\tilde\alpha$'s). But, when $n=0$, there is an order ambiguity affecting L_0 ($\tilde L_0$), because this is a composite operator built in terms of the non-commuting modes $\alpha_{-n}^\mu\alpha_n^\mu$ ($\tilde\alpha_{-n}^\mu\tilde\alpha_n^\mu$). This ambiguity must be taken into account, because different prescriptions will lead to a different mass spectrum (different L_0, $\tilde L_0$), namely different quantum theories. We require the quantum operators L_0 and $\tilde L_0$ to be normal ordered in the sense of QFT, with the annihilation operators $\alpha_{n>0}^i$ moved to the right, that is

$$L_0\equiv\sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m+\frac{1}{2}\alpha_0^2\qquad\tilde L_0\equiv\sum_{m=1}^{\infty}\tilde\alpha_{-m}\cdot\tilde\alpha_m+\frac{1}{2}\tilde\alpha_0^2\quad , \tag{2.38}$$

and we take into account other possible prescriptions by introducing the real constants $\tilde a$ and a into the constraints

$$(L_0+a)\,|\text{phys}\rangle=(\tilde L_0+\tilde a)\,|\text{phys}\rangle=0\quad . \tag{2.39}$$

These constants will shift the mass spectrum of the string, which is now given by

$$M^2=\frac{4}{\alpha'}\left(a+\sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m\right)=\frac{4}{\alpha'}\left(\tilde a+\sum_{m=1}^{\infty}\tilde\alpha_{-m}\cdot\tilde\alpha_m\right)\quad . \tag{2.40}$$

[24] The harmonic defines a frequency, so an energy. We like to think about the n^{th} harmonic "above" the m^{th} one if $n>m$, because this means that $(a_n^\mu)^\dagger$ will create states more massive than those created with $(a_m^\mu)^\dagger$.

As we will see later, $L_0 - \tilde{L}_0$ can be thought of as the generator ∂_σ of the rotation of the closed string. Any point of the closed string is indistinguishable from another one, so we must require *every* state of the string to satisfy

$$(L_0 - \tilde{L}_0)\,|\mathrm{phys}\rangle = 0 \quad .$$

But, according to

- formula (2.38), the condition $(L_0 - \tilde{L}_0)\,|\mathrm{phys}\rangle = 0$ means that the number $N \equiv \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m$ of right-moving modes must be equal to the number $\tilde{N} = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m$ of the left-moving ones; the condition $(N - \tilde{N})\,|phys\rangle = 0$ is called *level matching* condition;
- formula (2.39), the condition $(L_0 - \tilde{L}_0)\,|\mathrm{phys}\rangle = 0$ means that $a = \tilde{a}$.

In particular, the quantum theory will require $a = \tilde{a} = -1$, which implies an imaginary mass for the vacuum $|0; p^\mu\rangle$. In the covariant quantisation, these values for a and $\tilde{a}$ are determined by requiring the *unitarity* of the theory. To be more precise, one can show that the ghosts (quantum states of negative norm arising from the quantization of oscillations in the time direction) decouple from any S-matrix only if $a = \tilde{a} = -1$ and if the dimension D of the target-space is 26; this result is called *no-ghost-theorem*: if $a = \tilde{a} = -1$, $D = 26$, then every physical state is of the form

$$|\mathrm{phys}\rangle = |\mathrm{phys}_T\rangle + |s\rangle \quad ,$$

such that $\langle \mathrm{phys}_T | \mathrm{phys}_T \rangle > 0$ and $|s\rangle$ decouples from all physical process; in words, the state $|\mathrm{phys}\rangle$ decomposes into the physical *transverse* state $|\mathrm{phys}_T\rangle$ plus a pure gauge state $|s\rangle$. The proof of this theorem is not straightforward, because it involves an analysis of the unitarity not only at the tree-level, but also at the one-loop level; we refer the reader to [10] for further details.

Instead, we are going to find the values for a, $\tilde{a}$ and D in the *light-cone quantization*, which is the analogue of the Coulomb fixing procedure of QED. The constraints (2.13) are implemented classically, before the quantisation, which is now performed only on the space of physically distinct classical solutions. So, by construction, at the quantum level we will not have any ghosts and we don't have to worry about unitarity. Instead, the critical values for a, $\tilde{a}$ and D will be uniquely determined by requiring that Lorentz invariance will still hold at the quantum level, a requirement that is not trivial, given that, in order to explicitly solve the constraints, some particular directions in spacetime have to be singled out.
We have already mentioned that, after fixing $h_{\alpha\beta}$ to be flat, we still have the freedom of specifying which coordinates we really mean with the lightcone coordinates $\sigma^\pm$ of the worldsheet (see (2.9)), because they are defined up to a conformal transformation, namely:

$$\begin{aligned} \tilde{\sigma}^0 &= \frac{1}{2}\Big(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)\Big) \\ \tilde{\sigma}^1 &= \frac{1}{2}\Big(\tilde{\sigma}^+(\sigma^+) - \tilde{\sigma}^-(\sigma^-)\Big) \quad . \end{aligned} \tag{2.41}$$

We note that $\tilde{\sigma}^0$ satisfies the free wave equation $\partial_-\partial_+\tilde{\sigma}^0 = 0$, which is the same equation governing the dynamics of $X^\mu(\sigma^0, \sigma^1)$ (see (2.17)). We can therefore use the residual symmetry to identify $\tilde{\sigma}^0$ with one of the[25] X^μ. For example, we can choose $\tilde{\sigma}^0 \propto X^0$ and this would lead to the static gauge, which we saw to be very useful to get an idea about the physical meaning of the constraints; but, to explicitly solve the latter, it is more conveninet to require (look at Figure 9)

$$\tilde{\sigma}^0 \propto X^+ \propto (X^0 + X^{(D-1)}) \quad .$$

So let's introduce, in the target-space, the light-cone coordinates

$$X^\pm \equiv \frac{1}{2}(X^0 \pm X^{(D-1)}) \quad \text{and} \quad X^i \quad i \in 1, ..., D-2 \quad , \tag{2.42}$$

[25] We have presented this "quick and dirty" argument only to give an immediate idea about the lightcone scheme. The reader should be aware that, in general, imposing the equation of motion before quantisation can hide subtle problems. For a better understanding of this point, we refer to [11], where light-cone quantisation is introduced without relying on the equation of motion and just by exploiting local Weyl and reparametrisation invariance.

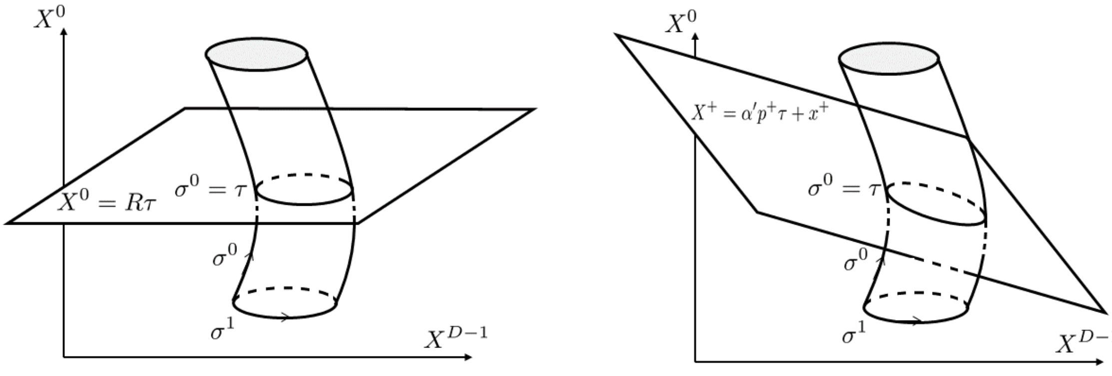

Figure 9
An intuitive sketch for the static (on the left) and lightcone (on the right) gauge.

and let's define the so-called *lightcone gauge*:

$$\tilde{\sigma}^0 = \frac{1}{\alpha' p^+}(X^+(\sigma^0, \sigma^1) - x^+) \quad , \tag{2.43}$$

where x^+ is a (real) constant of integration. In this way we have completely specified what we really mean with $\tilde{\sigma}^\pm$, because (2.43) implies

$$X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \tilde{\sigma}^+ \quad , \quad X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \tilde{\sigma}^- \quad . \tag{2.44}$$

Given that we have fixed the residual gauge, from now on we will forget about the "tilde" above the coordinates $\tilde{\sigma}^\pm$, by relabing them as $\sigma^\pm$. We note that the lightcone gauge is valid as long as $p^+ \neq 0$, namely as long as the string is not in a massless excitation travelling in the X^{D-1} direction ($0 = p^+ = -p_- = (1/\sqrt{2})(p_0 - p_{D-1})$). This is nothing strange, because we are defining σ^0 by specifying the timelike curves of $\sigma^0 = \tau \in \mathbb{R}$ as the intersection of the embedded worldsheet with the null hypersurface $X^+ = \alpha' p^+ \tau + x^+$ in the spacetime (see Figure 9) and, clearly, this cannot always be done.
The advantage of working with the lightcone gauge is that

$$\partial_+ X^+ = \partial_- X^+ = \frac{1}{2}\alpha' p^+ \quad ,$$

which is a condition that allows us to readily solve the constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$. Indeed, the latter can be written, in the lightcone coordinates, as

$$\begin{aligned} 2\partial_+ X^- \partial_+ X^+ &= \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \\ 2\partial_- X^- \partial_- X^+ &= \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \quad , \end{aligned} \tag{2.45}$$

and so we immediately get that $X^- = X_R^- + X_L^-$ can be defined, up to integration constants, uniquely in terms of the other fields X^i, by solving:

$$\begin{aligned} \partial_+ X_L^- &= \partial_+ X^- = \frac{1}{2\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \\ \partial_- X_R^- &= \partial_- X^- = \frac{1}{2\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \quad . \end{aligned} \tag{2.46}$$

This means that the Fourier modes p^-, $\tilde{\alpha}_n^-$ and $\tilde{\alpha}_n^-$ of the usual decomposition of X^-

$$
\begin{aligned}
X_L^-(\sigma^+) =& \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+} \\
X_L^-(\sigma^-) =& \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^-}
\end{aligned}
\tag{2.47}
$$

are functions of the Fourier modes of X^i (x^- is the undetermined integration constant). For example, one can express α_0^- and $\tilde{\alpha}_0^-$ in terms of the other oscillations and, by using $\alpha_0^- = \tilde{\alpha}_0^- = \sqrt{\alpha'/2}p^-$ one reaches the following two expressions

$$
\begin{aligned}
\frac{\alpha' p^-}{2} =& \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\in\mathbb{Z}\backslash\{0\}} \alpha_n^i \alpha_{-n}^i \right) \\
\frac{\alpha' p^-}{2} =& \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\in\mathbb{Z}\backslash\{0\}} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \right) \quad ,
\end{aligned}
\tag{2.48}
$$

which can be used to find the effective mass only in terms of the physical oscillations:

$$
\begin{aligned}
M^2 = -p^\mu p_\mu =& 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \\
=& \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} N_T \\
=& \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i = \frac{4}{\alpha'} \tilde{N}_T \quad ,
\end{aligned}
\tag{2.49}
$$

where we have introduced the quantities N_T and $\tilde{N}_T$ that, in the quantised theory, will become the number operators for the transverse harmonic oscillators. In analogy to electromagnetism, it's custom to call the physical modes *transverse* oscillators[26]: here they are α_n^i and $\tilde{\alpha}_n^i$ for $i \in \{1, ..., D-2\}$ and $n \in \mathbb{Z}\backslash\{0\}$ and they determine the $2(D-2)$ *internal*[27] degrees of freedom. To be more precise: on-shell, the string can propagate (at maximum) $2(D-2)$ degrees of freedom. In fact, the equation of motion states that the string is described by the $2D$ functions X_L^μ and X_R^μ, but: 2 of them (X_L^+ and X_R^+) are killed by fixing the residual gauge (2.44) and other two of them (X_L^- and X_R^-) are fixed by the constraints (4.3). Actually, $2(D-2)$ are only those on-shell degrees of freedom coming from the oscillations of the string, we need to add those coming from the dynamics of the centre of mass. The dynamics of the centre of mass is described in terms of x^i, p^i, p^+, x^-, p^- and x^+, but: x^+ can be absorbed in (2.43) by shifting σ^0 and p^- (which is the canonical momentum associate to x^+) is constrained by (2.48). In other words, the centre of mass of the string cannot freely move in all the D directions of the target-space (as in the case of the pointlike particle, the on-shell condition states that it cannot freely travel in the time direction).

In order to quantize the string in the lightcone gauge, we have to impose the commutation relations as in (2.33), but this time only on the physical degrees of freedom α_n^i, $\tilde{\alpha}_n^i$, x^i, p^i, p^+, x^-. Of course, the vacuum state is now defined by

$$
\hat{p}^\mu |0; p^\mu\rangle = p^\mu |0; p^\mu\rangle \qquad \alpha_n^i |0; p^\mu\rangle = \tilde{\alpha}_n^i |0; p^\mu\rangle = 0 \quad \forall\ n > 0 \quad ,
$$

[26] Technically, they are not number operators, because of the $\sqrt{n}$ that appears in (2.34). For this reason, N_T and $\tilde{N}_T$ are called *level operators*.

[27] We like to stress that this is true - with a little abuse of language - both at the classical and at the quantum level. We know from classical mechanics that a stationary oscillation of a rope is locally described by inner forces that, on average, cancel which each others: stationary waves of a rope give information about these inner forces. At the quantum level, the oscillations α_n^i and $\tilde{\alpha}_n^i$ will give information about the inner quantum degrees of freedom, namely the spin (and the mass). From this point of view, the mass and the spin of the single string can not change because of the background in which the string is moving, because external forces affect only the motion of the centre of mass, without interfering with the inner mechanics of the string.

and the Hilbert space built on this vacuum with the creation operators of the physical modes α^i_{-n} and $\tilde{\alpha}^i_{-n}$ is, by construction, positive definite. As in the covariant approach, we have to impose the non-trivial constraints descending from the equation of motion of $h_{\alpha\beta}$ (which now read as in (2.48)) as operator equations on the physical states. When imposing them, we have to introduce constants a and $\tilde{a}$ ($a = \tilde{a}$, as before) in order to take into account the possibility of different prescriptions for solving the order ambiguity appearing on the right side of (2.48); so, again, the formula for the square mass will differ from the classical one ((2.49)) only by the quantum shift of and we get

$$M^2 = \frac{4}{\alpha'}(N_T + a) = \frac{4}{\alpha'}(\tilde{N_T} + \tilde{a}) \quad . \tag{2.50}$$

Later, with the path integral quantisation, we will be able to derive the value of $a = -1$ in a rigorous way, but here we are going to determine it by means of a heuristic approach: first, we "show" that a must be equal to $a = (2 - D)/24$ and, then, we guess the dimension of the target-space must be $D = 26$.
Let's suppose that the string sits in its quantum vacuum $|0; p^\mu\rangle$ for a well specified momentum p^μ. The centre of mass of the string behaves like a first-quantized pointlike particle (see commutation relations (2.33)) so, if it is required to have a sharp momentum p^μ, then it must have an infinite undeterminacy in its location. But, delocalizing the centre of mass means also delocalizing the fluctuations that naturally characterize the vacuum of the string, because it is not possible to change the centre of mass of the string without moving the string itself. So, we expect that, in this limit, there should be an analogy between the vacuum of the string (usually thought of as quantum noise on the string) and the vacuum of QFT (a quantum background noise in the whole spacetime). If we were in the framework of QFT, to quantize a formula affected by ordering ambiguity, we would simply use the normal ordering (annihilation operators $\hat{a}$ on the right and creation operators $\hat{a}^+$ on the left), but let's have a closer look at it. For example, let's consider the energy E of a scalar in a $4-$dimensional spacetime; classically, we would have ($w_k = \sqrt{m^2 + \vec{k}^2}$)

$$E = \int d^3k w_k (a^+(k) a(k)) \quad .$$

We have to impose the commutation rule $[\hat{a}(k), \hat{a}^+(p)] = \delta^3(\vec{k} - \vec{p})$; first of all we symmetrize the expression in the variables, namely

$$E = \frac{1}{2} \int d^3k w_k (a^+(k) a(k) + a(k) a^+(k)) \quad ,$$

and, then, we turn $a(k)$ and $a^+(k)$ into operators, that is

$$\begin{aligned} E &= \int d^3k w_k \hat{a}^+(k) \hat{a}(k) + \frac{1}{2} \int d^3k w_k [\hat{a}(k), \hat{a}^+(k)] = \\ &= \int d^3k w_k \hat{a}^+(k) \hat{a}(k) + \delta^3(\vec{0}) \frac{1}{2} \int d^3k w_k. \end{aligned} \tag{2.51}$$

The last term contributes as a constant and, given that only differences of energies matter, we can neglect it and we end up with the well-known normal ordered energy; we stress the fact that we can neglect this constant term regardless of its finite or infinite value (actually it diverges, because $w_k \sim_\infty |\vec{k}|$), as long as gravity is not in the game (as in QFT). In string theory, instead, gravity is present and it is not possible to arbitrarily shift the energy anymore: energy is coupled to the metric and the equations of motion of the latter are, in general, non-linear. Let's see what happens if we try to quantize formula (2.50) by following the same steps that we have done for the energy of the scalar field.
We start from the expression in (2.50), namely

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n=1} \alpha^i_{-n} \alpha^i_n \quad ,$$

and, as first thing, we have to symmetrize in α^i_{-n} and in α^i_n:

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n=1} \frac{1}{2} (\alpha^i_{-n} \alpha^i_n + \alpha^i_n \alpha^i_{-n}) \quad .$$

Then, we turn α^i_{-n} and α^i_n into operators satisfying the proper commutation relations (as we saw in (2.33), this is $[\hat{\alpha}^i_n, \hat{\alpha}^i_{-n}] = n$) and we obtain

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n=1} \left(\hat{\alpha}^i_{-n} \hat{\alpha}^i_n + \frac{n}{2} \right) = \frac{4}{\alpha'} \left(N_T + \frac{D-2}{2} \sum_{n=1} n \right) \quad . \tag{2.52}$$

We are tempted to throw away the constant shift given by the contribution $\sum_{n>0} n$, but gravity asks us to take it into account. We guess that the mass of the vacuum of the string cannot be infinite, so we have to isolate the finite contribution that is *naturally* hiding inside $\sum_{n>0} n$. As always, if we want to isolate a finite term inside an infinite-valued one, we have to come up with a regularization, that here we introduce by means of the parameter ϵ:

$$\begin{aligned} \sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} \lim_{\epsilon \to 0^+} n e^{-\epsilon n} = -\sum_{n=1}^{\infty} \lim_{\epsilon \to 0^+} \partial_\epsilon e^{-\epsilon n} \overset{*}{=} -\lim_{\epsilon \to 0^+} \partial_\epsilon \sum_{n=1}^{\infty} e^{-\epsilon n} = \\ = -\lim_{\epsilon \to 0^+} \partial_\epsilon (1 - e^{-\epsilon})^{-1} = \lim_{\epsilon \to 0^+} \left[\frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon) \right] \quad . \end{aligned} \tag{2.53}$$

We stress that, so far, we haven't performed any dirty tricks, because the steps that we have just done are all mathematically rigorous [28]. This, in particular, means that $-1/12$ is the finite contribution that *canonically*[29] can be associated to $\sum_{n>0} n$. Now we can get rid, by hand[30], of the unphysical divergent part $\sim \epsilon^{-2}$ and we can decide to keep only the finite contribution of (2.53), because the vacuum of a single string cannot have an infinite mass. Thus, we are allowed to use

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \quad .$$

If we plug this result back into equation (2.52), we find that the possible mass of the quantum string can be written as

$$M^2 = \frac{4}{\alpha'} \left(N_T - \frac{D-2}{24} \right) \quad , \tag{2.54}$$

from which we can read (by comparison with (2.50)) that

$$a = -\frac{D-2}{24} \quad .$$

[28]The careful reader could be worried by the fact that we let the limit-operation and the partial derivative commute with the series (see step labeled by $*$ in (2.53)).
But it is not difficult to show that we can actually write (here $f(\epsilon, n) \equiv -\partial_\epsilon e^{-\epsilon n} = n e^{-\epsilon n}$)

$$\sum_{n=1}^{\infty} \lim_{\epsilon \to 0^+} f(\epsilon, n) = \lim_{\epsilon \to 0^+} \sum_{n=1}^{\infty} f(\epsilon, n)$$

thanks to the Beppo-Levi theorem.
Moreover, we can also commute ∂_ϵ and $\sum_{n=1}^{\infty}$ because ($q \equiv e^{-\epsilon n}$)

$$\sum_{n=1}^{\infty} \partial_\epsilon e^{-\epsilon n} = \frac{\partial q}{\partial \epsilon} \sum_{n=1}^{\infty} \partial_q q^n = \frac{\partial q}{\partial \epsilon} \partial_q \sum_{n=1}^{\infty} q^n = \partial_\epsilon \sum_{n=1}^{\infty} e^{-\epsilon n} \quad ,$$

where we have used the well-known fact that we can bring the differentiation ∂_q outside the series if the latter is uniformly convergent; indeed, the geometric series $\sum_{n=1}^{\infty} q^n$ uniformly converges in the open interval $-1 < q < 1$ and in our case we have $0 < q < 1$ because the $\lim_{\epsilon \to 0^+}$ is obviously on the left of ∂_ϵ, meaning that we can assume $\epsilon > 0$ on the right of ∂_ϵ.

[29]Canonically from the point of view of a mathematician. A physicist would guess that we still have the freedom to regularize the theory in several ways, because we could get rid of only a part of the infinite value of $1/\epsilon^2$, by leaving in the game a finite contribution different from $-1/12$; but it is very nice to see that this cannot be done because of conformal invariance: look at [12], pag. 46.

[30]We mention that - with the path integral approach to quantization - we could also get rid of it in a more rigorous way, see [12]. In general, Weyl invariance is broken upon quantization and we must then include Weyl-noninvariant counterterms as well. For example, to the classical action $S_{Poly} + \lambda S_{HE}$ we can add a term like $\mu^2 \int d^2\sigma \sqrt{-h}$, where μ plays the role of the cosmological constant of the worldsheet. This counterterm in the bare action then cancels off the divergence $\sim 1/\epsilon^2$ arising in the quantum computation of the vacuum energy.

With formula (2.54), we can now shortly analyse the spectrum of a single free string. We remember that the states of the closed string have to respect - as we have already mentioned- the level-matching condition, which means, in light-cone quantisation, that we have to impose $N_T = \tilde{N}_T$. For:

- $N_T = \tilde{N}_T = 0$, we obviously get the ground state $|0; p^\mu\rangle$, the state with no oscillators excited. The mass formula (2.54) gives

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \quad . \tag{2.55}$$

  Clearly, if we want to embed the 2−dimensional worldsheet in the target-space, the dimension D of the latter must be $D \geq 2$. We have already seen that the string has $2(D-2)$ on-shell oscillating degrees of freedom, so we have to suppose $D > 2$ if we want to gain non-trivial information coming from the extended nature of the string; indeed, for $D = 2$, there is only one spatial direction and the string hasn't enough room to show its extended nature: it appears like a massless pointlike particle travelling in the future of the light-cone (the dynamical variables in $D = 2$ are only p^+ and x^-). But, by assuming $D > 2$, we immediately see that the vacuum of the string is tachyonic! We remark that the unstable nature of the ground state in bosonic string theory shouldn't worry the reader, given that it will be eliminated by the superstring.

- $N_T = \tilde{N}_T = 1$, we get the first excited level of the string; at this level, we find the $(D-2)^2$ states

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0; p^\mu\rangle \quad ,$$

  where i and j are manifestly vector indices of $SO(D-2) \subset SO(1, D-1)$.
  String theory wants to recognise the nature of the fundamental particles as particular "sounds" (i.e. vibrational modes) of the strings; for example, here we are facing the problem of giving a particle interpretation to the states $\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0; p^\mu\rangle$. The target space has been taken flat, so a single particle is identified as an irreducible representation of the little group, namely the subgroup of $SO(1, D-1)$ that leaves the momentum of the particle invariant ($SO(D-1)$ for a massive one, $SO(D-2)$ for a massless one) and that encodes how the internal degrees of freedom of the particle (spin/helicity) transform. In string theory, the internal degrees of freedom are represented by the harmonic oscillators $\tilde{\alpha}^i_n$ and α^i_n, so we should find the connection between the bunch of oscillators characterizing a particular string level and the little group of the particles present at that level. The level specifies the mass of the states so at each level we expect more particles of the same mass, which can be distinguished by the different transformation properties of their internal degrees of freedom. For example, at the first level, the states $\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0; p^\mu\rangle$ are in the $(D-2)^2$−dimensional representation of $SO(D-2) \times SO(D-2)$ which can be decomposed as the direct sum of three irreducible representations of $SO(D-2)$: so the little group of the particles of the first level is $SO(D-2)$ and they must be massless. In formulae,

$$\begin{aligned}\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0; p^\mu\rangle =& \left(\tilde{\alpha}^{(i}_{-1}\alpha^{j)}_{-1} - \frac{1}{D-2}\delta^{ij}\tilde{\alpha}^k_{-1}\alpha^k_{-1}\right)|0; p^\mu\rangle + \\ &+ \tilde{\alpha}^{[i}_{-1}\alpha^{j]}_{-1}|0; p^\mu\rangle + \\ &+ \frac{1}{D-2}\delta^{ij}\tilde{\alpha}^k_{-1}\alpha^k_{-1}|0; p^\mu\rangle \quad .\end{aligned} \tag{2.56}$$

  Respectively, these three representations correspond to

  - a massless, transversely polarised spin 2 particle. Obviously, the first guess is that this particle can be identified with the on-shell graviton, given that these characteristics coincide with those expected from the quantization of the gravitational waves. Actually, to be sure about this identification one should check that also the interactions of this particle are those expected from a graviton, but it is not necessary to check it, because, on general ground[31], it is possible to show that any theory of interacting massless spin two particles must be equivalent to general relativity (plus higher derivative corrections). Thus, we can think of the first line on the right side of (2.56) as an on-shell quantum of the target-space metric.

[31] See [13] for details.

– a particle deriving from the quantization of an antisymmetric $2-$tensor field B_{ij} that must be present in the target-space. This tensor is called *Kalb-Ramond* and it has to be thought of as a “generalized” gauge potential, meaning that the string is “electrically charged” under it. It was possible to forecast the appearance of this generalized gauge potential only by looking at the extended nature of the string. We know that the coupling of the electromagnetism with the pointlike particle is governed by the Lorentz action, which is nothing but the integration over the worldline of the pullback of the spacetime electromagnetic potential A^{μ} (which is a $1-$form); if we want to build the close analog for the string, we need a field which is a $2-$form in the target-space, so its pull-back on the $2-$dimensional worldsheet can be integrated over the worldsheet itself.

– a particle deriving from the quantization of a scalar field ϕ that must be present in the target-space. This is called *dilaton* and, as we have already mentioned, it is possible to show that its constant mode ϕ_0 (namely, its expectation value on the target-space vacuum) determines the string coupling constant $g_s = e^{\lambda} = e^{\phi_0}$: we can only trust perturbation theory if the strings involved in the process are localized in regions of the target-space where the zero modes of the dilaton assumes a negative value. An important question is if string theory is able to dynamically determine the value of ϕ_0: the bosonic string doesn't, but there do exist backgrounds (particular compactifications) of the superstring in which a potential for the dilaton fixes its expectation value. Thus we have to thank the dilaton if, in (super)string theory, we don't need to introduce g_s as a ”god-given“ parameter. On the other hand, having a massless particle in the game means that long-range forces arise and we have to check if the latter interfere with gravity. Indeed, in the framework of the non-linear sigma model it can be shown that the dilaton field does interfere with gravity; in this context, if we want to restore the (strong) equivalence principle of gravity, we have to find a way to make the dilaton massive and, again, such a mechanism does arise in particular compactifications of superstring theory.

- $N_T = \tilde{N}_T > 1$, we obviously get, according to formula (2.50) states that for sure have a mass bigger than that of the first level. So, all the levels above the first one describe massive particles, in which we are not interested.

What we have learnt from the string spectrum is that the quantum string forecasts an unstable vacuum, three massless particles (the gravion, the dilaton and the “photon” of the Kalb-Ramond field) and an infinite number of extremely massive particles. Now that we know that the first level is massless, we can impose the massless condition for $N = \tilde{N} = 1$ in (2.50) and we discover that D must be fixed to be

$$D = 26 \quad . \tag{2.57}$$

This is the famous value of the critical dimension for the target-space in bosonic string theory. It can be rigorously determined, in the context of the light-cone quantisation, by imposing the Lorentz invariance of the quantum theory, namely by requiring that the quantized version of the Noether charges associated to Poincare' invariance must satisfy the Poincare' algebra (have a look at [10] for it).

With the canonical quantisation of the string, we hope that we have given the reader a direct and “mechanical” intuition about the physics of the string. In the next chapter, we are going to use the more abstract language of the path integral. On the one hand, this approach could appear too formal and far from the physical intuition developed so far. On the other hand, the path integral - letting us taste a bit of the deep mathematical structure of string theory - will reveal itself as an essential tool in string theory, because it will allow us to study scattering process in string theory.

3 The path-integral quantization

The modern covariant approach to quantisation of the string uses the Faddeev-Popov gauge fixing procedure to properly deal with the redundant structure of the theory. The key idea is that there is a prize to pay if ones wants to fix the diffeomorphism and Weyl redundancie, which is the introduction of the *b-c ghost system* on the worldsheet. The presence of these two new fields will explain, in an elementary way, the reason why the order ambiguity constant $a = \tilde{a}$ has to be $a = -1$. Moreover, the b-c system is essential to build the BRST algebra and to study the loop interactions of strings. Once we will have the BRST algebra at our disposal, we will discover that the value of the critical dimension $D = 26$ is fixed by the requirement for the BRST charge to be nilpotent also at the quantum level.
It will be easier to understand all the implications coming from the presence of the Faddeev-Popov's ghosts in the game once we will have developed the CFT's tools. For the moment, we are mainly concerned in persuading the reader about the necessity of introducing the b and c fields.

The main references that we used for this chapter are [8] and [11].

3.1 Faddeev-Popov gauge fixing: a first approach

As usual, the fundamental object in the path-integral approach is given by the partition function Z, here corresponding to the path integral with no vertex operator insertions

$$Z = \frac{1}{V_{diff \times Weyl}} \int D[X^\mu] D[h_{\alpha\beta}] e^{-S_{Poly}[X^\mu, h_{\alpha\beta}; \Sigma_g]} \quad , \tag{3.1}$$

where we have divided by the (infinite) volume of the gauge group so as to take into account the overcounting due to diffeomorphisms and Weyl redundancies. The combined infinitesimal version of these two transformations reads as (see (2.3) and (2.4))

$$\begin{aligned} \delta X^\mu &= -\delta\sigma^\alpha \partial_\alpha X^\mu \\ \delta h_{\alpha\beta} &= 2f h_{\alpha\beta} - \nabla_\alpha \delta\sigma_\beta - \nabla_\beta \delta\sigma_\alpha \quad = \\ &= (2f - \nabla_\gamma \delta\sigma^\gamma) h_{\alpha\beta} - 2(P_1 \delta\sigma)_{\alpha\beta} \quad , \end{aligned} \tag{3.2}$$

where we defined a differential operator P_1 that takes vectors into traceless symmetric 2−tensors,

$$(P_1 \delta\sigma)_{\alpha\beta} \equiv \frac{1}{2} (\nabla_\alpha \delta\sigma_\beta + \nabla_\beta \delta\sigma_\alpha - h_{\alpha\beta} \nabla_\gamma \delta\sigma^\gamma) \quad . \tag{3.3}$$

Following a standard route, we define the Faddeed-Popov measure Δ_{FP} by

$$1 = \Delta_{FP}(h_{\alpha\beta}) \int [d\zeta] \delta(h_{\alpha\beta} - \hat{h}^\zeta_{\alpha\beta}) \quad , \tag{3.4}$$

where

- $\hat{h}_{\alpha\beta}$ is a fiducial metric, that is a fixed metric whose form any other metric can assume after a proper gauge transfomation. We have already seen that in a given patch it is always possible to make any $h_{\alpha\beta}$ conformally flat, so we can take[32] $\hat{h}_{\alpha\beta} = \Omega^2(\sigma)\delta_{\alpha\beta}$; obviously, this choice for the fiducial metric works (at least) locally, but here we are not interested in complications due to non-trivial topologies of the worldsheet that will be instead analyzed in chapter 5 (to keep things simple, in this section, one can assume $\Sigma_g = \Sigma_0$);
- ζ is a shorthand for the infinitesimal version of a combined coordinate-Weyl transformation that brings $h_{\alpha\beta}(\sigma)$ to $h_{\alpha\beta}(\sigma) + \delta h_{\alpha\beta}(\sigma)$; here, $\delta h_{\alpha\beta}(\sigma)$ is given by (5.18) and, following the notation of (3.4), we will indicate an infinitesimal gauge transformation ζ as $\zeta = (f, \delta\sigma^\alpha)$;
- the delta function is, to be more precise, a delta functional, because it requires $h_{\alpha\beta}(\sigma) = \hat{h}^\zeta_{\alpha\beta}(\sigma)$ for every point;

[32] We are considering the wick rotated version of the worldsheet: $\sigma^0 \mapsto \sigma^2 \equiv i\sigma^0$ and $\sigma^1 \mapsto \sigma^1$; with the wick rotation, the Polyakov action gets an overall minus sign.

- $\Delta_{FP}(h_{\alpha\beta})$ is gauge invariant, that is $\Delta_{FP}(h_{\alpha\beta}) = \Delta_{FP}(h^{\zeta}_{\alpha\beta})$. In fact:

$$\begin{aligned}\Delta^{-1}_{FP}(h^{\zeta}_{\alpha\beta}) &= \int [D\zeta']\delta(h^{\zeta}_{\alpha\beta} - \hat{h}^{\zeta'}_{\alpha\beta}) = \\ &= \int [D\zeta']\delta(h_{\alpha\beta} - \hat{h}^{\zeta^{-1}\zeta'}_{\alpha\beta}) \overset{\zeta''\equiv\zeta^{-1}\zeta'}{=} \\ &= \int [D\zeta'']\delta(h_{\alpha\beta} - \hat{h}^{\zeta''}_{\alpha\beta}) = \\ &= \Delta^{-1}_{FP}(h_{\alpha\beta}) \quad .\end{aligned} \tag{3.5}$$

Note that we used the obvious gauge invariance of the delta function in the second equality and the hypothetical[33] gauge invariance of the measure $D[\zeta]$ in the third step ($D[\zeta'] = D[\zeta\zeta''] = D[\zeta'']$).

Inserting (3.4) into the functional integral (3.1), the latter becomes[34]

$$Z_{\hat{h}} = \frac{1}{V_{diff\times Weyl}} \int D[\zeta]D[X^{\mu}]D[h_{\alpha\beta}]\Delta_{FP}(h_{\alpha\beta})\delta(h_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta})e^{-S_{Poly}[X^{\mu},h_{\alpha\beta};\Sigma_0]} \quad . \tag{3.6}$$

We can carry out the integration over $h_{\alpha\beta}$ and rename the dummy variable $X \mapsto X^{\zeta}$ to obtain

$$Z_{\hat{h}} = \frac{1}{V_{diff\times Weyl}} \int D[\zeta]D[(X^{\mu})^{\zeta}]\Delta_{FP}(\hat{h}^{\zeta}_{\alpha\beta})e^{-S_{Poly}[(X^{\mu})^{\zeta},\hat{h}^{\zeta}_{\alpha\beta};\Sigma_0]} \quad . \tag{3.7}$$

Now we can use the gauge invariance of Δ^{-1}_{FP}, $D[(X^{\mu})]$ and of the action to write

$$Z_{\hat{h}} = \frac{1}{V_{diff\times Weyl}} \int D[\zeta]D[(X^{\mu})]\Delta_{FP}(\hat{h}_{\alpha\beta})e^{-S_{Poly}[X^{\mu},\hat{h}_{\alpha\beta};\Sigma_0]} \quad . \tag{3.8}$$

Now there is nothing, in the integrand, that depends on the gauge parameter ζ so the integral over ζ gives the volume of the gauge group that perfectly cancels[35] the $V_{diff\times Weyl}$ term in the denominator. So we finally get

$$Z_{\hat{h}} = \int D[(X^{\mu})]\Delta_{FP}(\hat{h}_{\alpha\beta})e^{-S_{Poly}[X^{\mu},\hat{h}_{\alpha\beta};\Sigma_0]} \quad . \tag{3.9}$$

We are thus left with computing the Faddeev-Popov measure $\Delta_{FP}(\hat{h}_{\alpha\beta})$, determined by

$$\left(\Delta_{FP}(\hat{h}_{\alpha\beta})\right)^{-1} = \int [d\zeta]\delta(\hat{h}_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta}) \quad . \tag{3.10}$$

Again, here we want to keep things simple, so we pretend that exactly for one value of ζ the delta functional $\delta(\hat{h}_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta})$ is nonzero. This means that all the nonzero contribution in (3.10) arises when ζ is the identity and, thus, to compute $\Delta_{FP}(\hat{h}_{\alpha\beta})$ it is enough to consider infinitesimal gauge transformations. In other words, the integral

$$\int D[\zeta]$$

over the gauge group can be equivalently substituted with the integral over the tranformations $\zeta = (f, \delta\sigma^{\alpha})$ near the identity, namely with

$$\int D[f]D[\delta\sigma^{\alpha}] \quad .$$

[33] One can suppose that this measure is gauge invariant by an analogy with the Haar measure of finite-dimensional Lie groups, but there are no reasons why this analogy has to hold a priori, given that, here, the gauge group is infinite dimensional. In fact, it turns out that the measure is in general anomalous and, to remove the anomaly, we are forced to fix $D = 26$ (and $a = -1$). This is the reason why, strictly speaking, all the Fadeev-Popov procedure that we are presenting has to be performed under the assumption of the criticality condition.

[34] We add a subscript to Z to explicitly denote the dependece of Z on the choice of the fiducial metric.

[35] This is not true. As we have already seen in the last chapter, the gauge choice $h_{\alpha\beta} = \hat{h}_{\alpha\beta} = \delta_{\alpha\beta}$ (more in general, the conformal gauge choice) doesn't completely fix the gauge, because, locally, we still have the freedom to specify the meaning of the coordinates by a confromal transformation. We will turn back to this problem in chapter 5, where we will see that not the whole of $V_{diff\times Weyl}$ will cancel with the integral over ζ: the volume of the conformal transformations globally defined in Σ_g will survive in the denominator.

So, by introducing formula (5.18) into the Dirac functional of (3.10), we have

$$\left(\Delta_{FP}(\hat{h}_{\alpha\beta})\right)^{-1} = \int D[f]D[\delta\sigma^\alpha]\delta\Big(- (2f - \hat{\nabla}_\gamma \delta\sigma^\gamma)\hat{h}_{\alpha\beta} + 2(\hat{P}_1\delta\sigma)_{\alpha\beta}\Big) \quad , \tag{3.11}$$

where we put a hat on the differential operators ∇ and P_1 because their definition depends on a metric that here is the fiducial one.
To perform the calculation, we go to the analogue of the Fourier space for the Dirac functional, because the latter admits the following integral representation in terms of a symmetric tensor field $B^{\alpha\beta}$:

$$\begin{aligned} &\delta\Big(-(2f - \hat{\nabla}_\gamma \delta\sigma^\gamma)h_{\alpha\beta} + 2(\hat{P}_1\delta\sigma)_{\alpha\beta}\Big) = \\ &\quad = \int D[B^{\alpha\beta}]\mathrm{exp}\Big[2\pi i \int d^2\sigma \sqrt{\hat{h}} B^{\alpha\beta}\Big(- (2f - \hat{\nabla}_\gamma \delta\sigma^\gamma)\hat{h}_{\alpha\beta} + 2(\hat{P}_1\delta\sigma)_{\alpha\beta}\Big)\Big] \quad . \end{aligned} \tag{3.12}$$

After plugging this formula into (3.11), it is immediate to see that the infinitesimal Weyl transformation f acts as a Lagrange multiplier. In fact, the integration over $D[f]$ produces a Dirac functional that forces $B^{\alpha\beta}$ to be traceless, namely

$$B^{\alpha\beta}\hat{h}_{\alpha\beta} = 0 \quad .$$

From a practical point of view: in (3.11) we can set $f = 0$, drop the integral over $D[f]$ whereas, in (3.12), we can put a label $'$ on $B^{\alpha\beta}$ to remember that $B'^{\alpha\beta}$ is traceless. Thus we obtain

$$\begin{aligned} &\left(\Delta_{FP}(\hat{h}_{\alpha\beta})\right)^{-1} = \\ &\quad = \int D[B'^{\alpha\beta}]D[\delta\sigma^\alpha]\mathrm{exp}\Big[2\pi i \int d^2\sigma \sqrt{\hat{h}} B'^{\alpha\beta}\Big(\hat{\nabla}_\gamma \delta\sigma^\gamma \hat{h}_{\alpha\beta} + 2(\hat{P}_1\delta\sigma)_{\alpha\beta}\Big)\Big] \quad , \end{aligned} \tag{3.13}$$

and, using the tracelessness condition of $B^{\alpha\beta}$, we end up with

$$\begin{aligned} &\left(\Delta_{FP}(\hat{h}_{\alpha\beta})\right)^{-1} = \\ &\quad = \int D[B'^{\alpha\beta}]D[\delta\sigma^\alpha]\mathrm{exp}\Big[4\pi i \int d^2\sigma \sqrt{\hat{h}} B'^{\alpha\beta}(\hat{P}_1\delta\sigma)_{\alpha\beta}\Big] = \\ &\quad = \int D[B'^{\alpha\beta}]D[\delta\sigma^\alpha]\mathrm{exp}\Big[4\pi i \int d^2\sigma \sqrt{\hat{h}} B'^{\alpha\beta}\hat{\nabla}_\alpha \delta\sigma_\beta\Big] \quad . \end{aligned} \tag{3.14}$$

The previous manipulations have given us an expression for $\left(\Delta_{FP}(\hat{h}_{\alpha\beta})\right)^{-1}$, but, for (3.9), we need $\Delta_{FP}(\hat{h}_{\alpha\beta})$. Given that the exponent is quadratic, the integral computes the inverse determinant of the operator $\hat{\nabla}_\alpha$. Both $B'^{\alpha\beta}$ and $(\hat{P}_1\delta\sigma)_{\alpha\beta}$ are symmetric traceless tensors, so, to be more precise, we are computing the inverse determinant of the projection of ∇_α onto symmetric, traceless tensors: in this sense the operator is a "square matrix" and we can talk about its determinant. But, in order to find the determinant from the path integral expression of its inverse, we can simply replace, in the latter, each bosonic field with a corresponding Grassmann *ghost field* which inherits the same transformation properties. So, in (3.14), we perform the substitutions

$$\begin{aligned} \delta\sigma^\alpha &\mapsto c^\alpha \\ B'_{\alpha\beta} &\mapsto b_{\alpha\beta} \quad , \end{aligned} \tag{3.15}$$

where c^α is a vector field and $b_{\alpha\beta}$ is a symmetric traceless tensor field on the worldsheet. We obtain

$$\Delta_{FP}(\hat{h}_{\alpha\beta}) = \int D[b_{\alpha\beta}]D[c^\alpha]e^{-S_g} \quad , \tag{3.16}$$

where the ghost action S_g, with a convenient normalization for the fields, can be written as

$$S_g \equiv \frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{h}} b_{\alpha\beta}(\hat{P}_1 c)^{\alpha\beta} = \frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{h}} b_{\alpha\beta}\hat{\nabla}^\alpha c^\beta \quad . \tag{3.17}$$

It is not difficult to see that this action, in the conformal gauge ($\hat{h}_{\alpha\beta} = \Omega^2 \delta_{\alpha\beta}$), reads as

$$S_g \equiv \frac{1}{2\pi}\int d^2z(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}) = \frac{1}{2\pi}\int d^2z(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}) \quad , \tag{3.18}$$

where we introduced the following notation

$$\begin{aligned} z =& \sigma^1 + i\sigma^2 \qquad \bar{z} = \sigma^1 - i\sigma^2 \\ d^2z =& 2d\sigma^1 d\sigma^2 \quad , \end{aligned} \tag{3.19}$$

which appears to be useful, given that the complex coordinates z and $\bar{z}$ are directly related to the lightcone coordinates $\sigma^{\pm}$ of the Minkowskian version of the worldsheet by

$$\begin{aligned} z =& \sigma^1 + i\sigma^2 = \sigma^1 + i(i\sigma^0) = -\sigma^+ \\ \bar{z} =& \sigma^1 - i\sigma^2 = \sigma^1 - i(i\sigma^0) = \sigma^- \quad . \end{aligned} \tag{3.20}$$

The action (3.18) is Weyl invariant, because the conformal factor Ω of the conformal gauge doesn't show up and this means that both $b_{\alpha\beta}$ and c^{α} are neutral under Weyl transformation (in contrast to $b^{\alpha\beta}$ and to c_{α}).

To sum up: the action of bosonic string theory is the diffeomorphism and Weyl invariant action defined by the sum of the Polyakov action and of the $b-c$ system. These two actions describe the free fields X^{μ}, b and c. Locally, there are no interaction terms in these actions that mix the $b-c$ system with the X^{μ}'s so, at the classical level, in a given patch, the ghost fields can be forgotten because they don't influence the dynamics of X^{μ}. At the quantum level, instead, the $b-c$ ghost system assumes a key role both for local and global reasons.
Gobally, the gauge fixing does not fix the metric completely and, for topologies with genus $g \geq 1$, we are left with residual modes of the metric, known as *moduli*, which presence has to be taken into account by a proper insertion of the b, c ghosts in the worldsheet (see chapter 5). Given that $g = 0$ is the tree level worldsheet, this fact finds a close analogy in QFT, where one has to worry about Faddeev-Popov ghosts only in loops. In fact, the Faddeev-Popov fields are Grassmann fields with integer spin and, so, they cannot appear as external fields because of their unphysical spin-statistics; being mathematical objects that are necessary for consistency, their presence is then evanescent and indetectable and this is the reason why these fields are usually called "ghosts". The reader should not confuse the Faddeev Popov ghosts with the ghosts met during the covariant canonical quantisation of the string, the latter being quantum states of the string that cannot have definite positive energy and norm. Of course, there should be a connection between these two kinds of ghosts, because both of them arise from a covariant approach to the quantisation of a gauge redundant system. Sloppily speaking, we can say that the contribution coming from the Faddeev-Popov ghosts cancels the contribution coming from the ghosts which would spoil unitarity[36]. The reason why it is not so wrong to think about this kind of cancellations among different ghosts can be found in the *BRST-symmetry*, the remnant of the local gauge symmetry that survives after the gauge fixing. Being nilpotent, the Noether charge Q_B associated to this symmetry introduces a cohomology on the set of the quantum states of the theory that can be used to distinguish which states are physical and which are not. With respect to this definition of "physical condition", it is possible to recover the no-ghost theorem of the old covariant approach to quantisation, as well as the

[36]For example, if we compute the total order ambiguity constant a^{tot} of the gauge-fixed theory, we would find that it consists of three pieces $a^{tot} = a_T + a_U + a_{FP}$, where a_T is the order ambiguity constant $a_T = -(D-2)/24$ coming from the $2*(D-2)$ transverse degrees of freedom, $a_U = -2/24 = -1/12$ comes from the $2*2$ unphysical degrees of freedom and $a_{FP} = +2/24$ comes from the Faddeev-Popov ghosts; so we would have

$$a^{tot} = a_T + a_U + a_{FP} = a_T \quad ,$$

and we would recover the mass-formula (2.50) of the lightcone quantisation. To understand the reason why $a_{FP} = 2/24 = 1/12$ one should note that the equations of motion of S_g imply that both the b and c fields can be decomposed, locally, into left and right movers so we have $2*2$ degrees of freedom from the Faddeev-Popov ghosts, which is the same number of degrees of freedom coming from the unphysical modes of X^0 and X^{D-1}; but the latter are bosonic variables, whereas the Faddeev-Popov ghosts are Grassmann ones so they contributes with opposite sign to a^{tot}. Following this example, we like to think that at each loop level the running of the Faddeev-Popov ghosts cancels the running of the other ghosts (the unphysical oscillations of X^{μ}).

same Hilbert space of the lightcone quantisation (see [11]). But, as we will see, the nilpotency of Q_B holds at the quantum level only if $D = 26$; only if we are working under the hypothesis of the critical condition we can show that ghosts decouple from physical processess: for $D \neq 26$, unitarity is lost.
Locally, the $b - c$ ghost system is fundamental to preserve the gauge invariance of the gauge fixed theory at the quantum level.
In this section, we have assumed that the measures $D[\zeta]$ and $D[X^\mu]$ are separately gauge invariant. The truth is that they are invariant only under diffeomorphisms, but not under Weyl rescalings (see [6], for example). We have seen that the gauge invariance of $D[\zeta]$ alone is equivalent to the gauge invariance of $\Delta_{FP}(h_{\alpha\beta})$, namely to the gauge invariance of the $b - c$ system; thus, we can expect that, at the quantum level, both the $b - c$ and the matter sectors separately suffer from a Weyl anomaly (the traces of their energy-momentum tensors are nonzero). Later we will find that these two Weyl anomalies cancel if $D = 26$ (and $a = -1$) and this is equivalent to the memorable result due to Polyakov: the combined measure $D[h_{\alpha\beta}]D[X^\mu]$ (equivalently, the combined measure $D[\zeta]D[X^\mu]$) is not anomalous only for the critical string. If we want to obtain a gauge-fixed worldsheet that doesn't depend on the gauge chosen to fix Weyl invariance, we necessarily have to take into account the Faddeev-Popov ghosts *and* to impose the criticality condition.

Given the importance of the criticality condition, in the following chapter we are going to analyze the local aspects of the gauge-fixed worldsheet and its $BRST$ remnant symmetry. The discussion of the global aspects of the worldsheet will follow immediately after.

4 Local aspects

As we have done so far, we are going to pay attention only to one patch of the worldsheet. Nevertheless, it is important to keep in mind that, in general, the latter will be a manifold that can be described by several local coordinate systems and the non-trivial information about the nature of the manifold will be encoded in the transitions functions. The geometry of the 2−dimensional gauge-fixed worldsheet is very rich thanks to the gauge redundancies of the theory and this translates into very nice transition functions. In fact, we have seen that we can fix reparametrization invariance to bring, in each local patch, the worldsheet metric $h_{\alpha\beta}$ to the conformal form $h_{\alpha\beta} = \Omega^2\delta_{\alpha\beta}$. If we denote two local patches that intersect each other with the subscripts A and B, we have

$$\Omega^2_A(\sigma^1_A,\sigma^2_A)\Big((d\sigma^1_A)^2+(d\sigma^2_A)^2\Big) = ds^2\Big|_{A\bigcap B} = \Omega^2_B(\sigma^1_B,\sigma^2_B)\Big((d\sigma^1_B)^2+(d\sigma^2_B)^2\Big) \quad ,$$

and, in the overlapping region $A\bigcap B$, this means

$$(d\sigma^1_A)^2+(d\sigma^2_A)^2 \propto (d\sigma^1_B)^2+(d\sigma^2_B)^2 \quad .$$

In other words, the transition function between the two patches must be a[37] *conformal transformation* and this is a peculiar fact characterizing all 2−dimensional Riemannian manifolds. Observe that, if we denote with A and B the same patch, the transition function between the two patches is nothing but a reparametrization of the coordinates of the same patch so we get that in each patch we still have the freedom to perform a conformal transformation. Obviously, if we think (take the A patch, for example) the conformal factor $\Omega^2_A(\sigma)$ to be fixed by the conformal gauge, then the freedom of reparametrizing the single A patch by a conformal transformation is lost. But it is likewise obvious that we can recover this freedom by adding into the game the Weyl invariance of the worldsheet! This is the reason why some authors like to say - with a misleading abuse of language - that conformal transformations (on a given patch) are diffeomorphisms followed by a compensating Weyl rescaling (see [8], for example); what they really mean is that we have to thank Weyl invariance if we still have conformal invariance in a patch even after fixing the metric to a precise conformal gauge form by the Faddeev-Popov procedure.
Weyl redundancy makes the worldsheet of string theory very special among all the 2−dimensional Riemannian manifolds[38], because a Weyl transformation really changes the Riemannian structure of the worldsheet. It lets us bring the metric from the conformal to the flat gauge form in, at least, one patch. We precise that the Weyl trasformations consist of rescalings of the metric ds^2 by a factor e^{2f} where f is a function which is continuous and globally defined on the worldsheet, otherwise the rescaled worldsheet wouldn't be a Riemannian manifold. In general, $\Omega^2_A(\sigma_A) \neq \Omega^2_B(\sigma_B)$ and, in this case, by a Weyl rescaling it is possible to bring the metric into the flat form only in one of the two patches; only in the case of $\Omega^2_A(\sigma_A) = \Omega^2_B(\sigma_B)$ it is possible to bring the metric to the flat form in both patches by a Weyl transformation.
We also stress that Weyl rescalings don't touch the coordinates so, even after performing a Weyl transformation in each patch, the transition functions will be the conformal transition functions of before.
To sum up:

- One starts with a Riemannian manifold (Σ_g, ds^2) and fixes reparametrization invariance by choosing the conformal gauge in each patch. This choice fixes the transition functions among patches to be conformal transformations.
- Then we remember that in the theory we have also the Weyl redundancy $(\Sigma_g, ds^2) \sim (\Sigma_g, e^{2f}ds^2)$ and we fix it by bringing the conformal gauge choice to the flat one at least in one patch.
- For the same reasoning explained around formula (2.9), we could naively guess that we still have the freedom to specify the meaning of the local coordinates by performing in each local patch a different conformal transformation. But again we have to remember that the Weyl rescalings are continuous and globally defined, so, after fixing reparametrization and Weyl redundancies, we still

[37] This holds if we choose coordinate in A and B such that the metric is in conformal form in both patches.
[38] So special to such a point that, as we will see later, the Euclidean version of the worldsheet is not a Riemannian manifold, but a complex manifold. A complex manifold doesn't need a metric to be defined and, in this sense, the metric $h_{\alpha\beta}$ is completely an auxiliary field, not only at the dynamical level, but also from a geometrical point of view.

have the freedom to perform only those conformal transformations that are globally defined on the whole worldsheet. These transformations form the group of the residual gauge transformations, which is called *conformal killing group (CKG)* of the worldsheet.

The main references that we used for this chapter are [8, 11, 12, 14, 15, 7].

4.1 2−dimensional conformal transformations

A conformal transformation is a diffeomorphism $\sigma^\alpha \mapsto \tilde{\sigma}^\alpha$ under which the metric changes only by an overall factor. If the original metric is the flat Euclidean one, then the transformed metric should be

$$\delta_{\alpha\beta} \mapsto \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}\delta_{\gamma\delta} = \Lambda(\sigma)\delta_{\alpha\beta} \quad .$$

Obviously we are interested in the 2−dimensional case. Let ϵ^α be such that $\sigma^\alpha \mapsto \tilde{\sigma}^\alpha = \sigma^\alpha + \epsilon^\alpha + O(\epsilon^2)$. Then, it is not difficult to show that this definition reduces to

$$\partial_\alpha\epsilon_\beta + \partial_\beta\epsilon_\alpha = \partial^\gamma\epsilon_\gamma\delta_{\alpha\beta} \quad . \tag{4.1}$$

After introducing the usual complex notation

$$\begin{aligned} z &= \sigma^1 + i\sigma^2 \qquad \bar{z} = \sigma^1 - i\sigma^2 \\ \partial_z &= \frac{1}{2}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \\ \epsilon^z &= \epsilon^1 + i\epsilon^2 \qquad \epsilon^{\bar{z}} = \epsilon^1 - i\epsilon^2 \\ dzd\bar{z} &= 2d\sigma^1 d\sigma^2 \\ g_{z\bar{z}} = g_{\bar{z}z} &= \frac{1}{2} \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 = g^{zz} = g^{\bar{z}\bar{z}} = 0 \quad g^{z\bar{z}} = g^{\bar{z}z} = 2 \\ \epsilon_z &= \frac{1}{2}(\epsilon_1 - i\epsilon_2) \qquad \epsilon_{\bar{z}} = \frac{1}{2}(\epsilon_1 + i\epsilon_2) \quad , \end{aligned} \tag{4.2}$$

we find that, locally, the solution to (4.1) is given by

$$\partial_{\bar{z}}\epsilon^z = 0 = \partial_z\epsilon^{\bar{z}} \quad . \tag{4.3}$$

Thus, we have simply

$$\epsilon \equiv \epsilon^z = \epsilon^z(z) \qquad \bar{\epsilon} \equiv \epsilon^{\bar{z}} = \epsilon^{\bar{z}}(\bar{z}) \tag{4.4}$$

and the infinitesimal conformal transformations on the 2−dimensional Euclidean worldsheet are generated, locally, by all the meromorphic[39] functions $\epsilon(z)$ and anti-meromorphic functions $\bar{\epsilon}(\bar{z})$. As such, the group of the local conformal transformations is infinite dimensional, and this is a peculiarity of the 2-dimensional case. This means that the gauge-fixed worldsheet enjoys - locally - a huge number of symmetries and, so, its mathematical structure is very rigid.
The (anti-)meromorphic generators can be expanded in a Laurent series as

$$\begin{aligned} z \mapsto z' &= z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \\ \bar{z} \mapsto \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \end{aligned} \tag{4.5}$$

and we see that the algebra of the infinitesimal conformal transformations is generated by the *Witt generators* l_n, $\bar{l}_n$, namely by

$$l_n = -z^{n+1}\partial_z \quad , \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \quad , \tag{4.6}$$

[39] Not holomorphic, because the vector field ϵ can have singularities outside the local patch of the worldsheet where we are solving (4.3).

which satisfy the *Witt Algebra*

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0 \quad . \end{aligned} \tag{4.7}$$

This is the infinite dimensional algebra of the 2−dimensional conformal transformations. Locally, it characterizes the worldsheet[40] Σ_g, regardless of its genus but most of these transformations fail to be globally defined on Σ_g.
For example, at the tree level ($g = 0$), the worldsheet is a Riemann-sphere which can be covered, as we know, by two patches with complex coordinates z, u with transition function $uz = -1$ in the overlapping region

$$\mathbb{S}^2 \backslash \{\text{North pole } z = 0\} \backslash \{\text{South pole } u = 0\} \quad .$$

In the z patch, the Witt generators are given by (4.6), and they are globally defined only for $n \geq -1$ (for $n < -1$, l_n and $\bar{l}_n$ are not defined at the North pole). In the u patch, instead, the Witt generators are given by

$$\begin{aligned} l_n &= -z^{n+1}\partial_z = -\left(\frac{-1}{u}\right)^{n+1} \frac{\partial u}{\partial z}\partial_u = -(-u)^{-(n+1)}u^2\partial_u = (-1)^n u^{1-n}\partial_u \\ \bar{l}_n &= -\bar{z}^{n+1}\partial_{\bar{z}} = (-1)^n \bar{u}^{1-n}\partial_{\bar{u}} \quad , \end{aligned} \tag{4.8}$$

and only for $n \leq 1$ they are defined also at the South pole. Thus, the only Witt generators that are globally defined on the Riemann sphere are:

- $l_{-1} = -\partial_z$: it generates rigid translations $z \mapsto z + b$, $b \in \mathbb{C}$;
- $l_0 = -z\partial_z$: it generates complex dilatations $z \mapsto az$, $a \in \mathbb{C}$ which consist of real dilatations (for $a \in \mathbb{R}$) and real rotations (for $a \in i\mathbb{R}$);
- $l_1 = z^2\partial_z$: it generates the socalled *special* conformal transformations $z \mapsto z/(cz + 1)$, $c \in \mathbb{C}$.

The combination of these three transformations gives the most general global conformal diffeomorphism on the Riemann sphere:

$$z \mapsto \frac{az + b}{cz + d} \quad , \quad d \in \mathbb{C}. \tag{4.9}$$

Being a diffemorphisms, it has to be invertible, so we have to require $ad-bc \neq 0$; we can rescale a, b, c and d to obtain $ad - bc = 1$ without changing the transformation, and we recognize that the set of the diffeomorphisms (4.9) is isomorphic to $SL(2, \mathbb{C})$. Actually, after imposing the condition $ad-bc = 1$, we still have the freedom to represent the same transformation with both (a, b, c, d) and $(-a, -b, -c, -d)$ so we have to quotient by a $\mathbb{Z}_2$ factor and we obtain $PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C}) \backslash \mathbb{Z}_2$. Given that we have also the antiholomorphic set of transformations, we get that the CKG of the Riemann sphere is $PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$.
The CKG of the sphere is useful to guess the CKG of surfaces with genus $g > 0$. In fact, any closed oriented two-dimensional surface can be obtained by adding g handles to the sphere. Higher genus surfaces are - topologically speaking - more complicated than the sphere and, so, their CKG should be a proper subgroup of $PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$. For instance, the only conformal transformations that preserve the periodicity condition $z \approx z + 2\pi\tau$ of a torus ($\tau \in \mathbb{C}$) are given by the Witt generators which act homogeneously on the plane, namely by the translations l_1 and $\bar{l_1}$ and the CKG is now $U(1) \times U(1) \subset PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$. In the same way, we expect that the CKG(Σ_g) of a worldsheet with genus $g > 1$ should be a proper subgroup of the CKG of the torus, but the latter consists of only translations, so we guess that there are no conformal transformations globally defined on $\Sigma_{g>1}$. This reasoning, though heuristic, gives the right CKG's for all Σ_g; the reader who is not satisfied with it can refer to [6].

[40] Up to conformal anomalies. We will discover that, at the quantum level, the conformal transformations are generated by the Virasoro generators L_n, $\bar{L}_n$, which are a "generalization" of the l_n, $\bar{l}_n$.

4.2 From the cylinder to the plane

We understood a very important lesson: to study the conformal properties of the worldsheet it is convenient to work with complex[41] coordinates z and $\bar{z}$, so, then, we can exploit the results coming from complex analysis/geometry. From this point of view it is thus natural, as first step, to map the worldsheet Σ_0 of the freely moving string to the complex plane and the importance of such a map will become even clearer later, as being the building block of the state-operator correspondence.
On the Euclidean cylinder parametrized[42] by $\sigma^1 \in [0, 2\pi[$ and $\sigma^2 \in]-\infty, +\infty[$ we can define the complex coordinates

$$w \equiv \sigma^2 - i\sigma^1 \quad , \quad \bar{w} \equiv \sigma^2 + i\sigma^1 \tag{4.10}$$

and the complex coordinates z, $\bar{z}$, related to w, $\bar{w}$ by means of the conformal transformation

$$z \equiv e^w \quad \bar{z} \equiv e^{\bar{w}} \quad , \tag{4.11}$$

which identifies, being a biholomorphism, the cylinder $[0, 2\pi[\times]-\infty, +\infty[$ with $\mathbb{C}\backslash\{0\}$. One would like to describe also the string sitting at $\sigma^2 = \pm\infty$; according to the map (5.37), we should add the corresponding points $z = 0$ and $z = \infty$ to the complex plane and we end up with the compactified complex plane $\mathbb{C} \cup \{\infty\}$, namely with the Riemann sphere $\mathbb{S}^2$. One should be aware of the fact that the "closed" cylinder $[0, 2\pi[\times[-\infty, +\infty]$ is not biholomorphic to $\mathbb{S}^2$, so they are different worldsheets (actually, they cannot even be homeomorphic, given that the Riemann sphere is simply connected and the cylinder is not). The problem is that the Riemann sphere has "less" information than what the cylinder has, because all the points of the string sitting at $\sigma^2 = -\infty$ are indiscriminately mapped to the same point, namely to the origin of the complex plane, so we don't have an injective mapping. The right way to keep track of this information is to introduce a vertex operator at the North Pole $z = 0$ of the sphere; in this way the not-simply-connectedness of the cylinder will be somehow recovered on the sphere: indeed, it will not anymore be possible to shrink any contour integral around the origin to a point, because of the presence of the vertex operator at $z = 0$! Obviously, the same reasoning works also for the South pole and, if we want to describe the string at $\sigma^2 = +\infty$, we have to add the same vertex operator at $u = 0$. In other words, one should think of Σ_0 as the "closed" cylinder, or equivalently, as the Riemann sphere with the vertex operator insertions at the poles.
Clearly, according to (5.37), lines of equal time σ^2 are mapped into circles around the origin, σ^1-translations become rotations and time σ^2-translations become dilatations. In the quantized theory, this means that:

- the generator of dilatations will take the role of the Hamiltonian,
- time ordering will be replaced by radial ordering and
- equal time commutators will be substituted by equal radius commutators.

This is the core of the so-called *radial quantization*, according to which products of fields are only defined if we put them in *radial order* $R[...]$; in analogy to the time ordering of QFT, the latter is defined as

$$R[\phi_1(z_1)\phi_2(z_2)] = \begin{cases} \phi_1(z_1)\phi_2(z_2) & \text{for} |z_1| > |z_2| \\ \phi_2(z_2)\phi_1(z_1) & \text{for} |z_2| > |z_1| \end{cases} \quad , \tag{4.12}$$

where, as an example, we have taken two commuting fields; in the case of anti-commuting fields, there will be a minus sign in the second line of the definition of $R[...]$.

[41] A note for the careful reader. In order to use the results coming from complex analysis/geometry, we deal with z and $\bar{z}$ as they were independent coordinates. Strictly speaking, this means that we're really extending the worldsheet from $\mathbb{R}^2$ to $\mathbb{C}^2$. However, after computations one should remember that we're really sitting on the real slice $\mathbb{R}^2 \subset \mathbb{C}^2$ defined by $\bar{z} = z^*$. To streamline our discussion, sometimes we will focus only on z, leaving understood that we should do the analogue construction for $\bar{z}$.

[42] Because of the periodicity condition, the cylinder needs at least two charts to be covered. These charts can be chosen such that their transition functions is only an innocent translation, so we can pretend to have only one "global" chart, as long as the functions defined on it respect the periodicity of the cylinder. Note, for example, that the functions $z(w)$ and $\bar{z}(\bar{w})$ that we are going to introduce respect the periodicity of $w \approx w + 2\pi$.

4.3 Conformal Field Theory

A *conformal field theory* (CFT) is a theory invariant under the group of infinitesimal conformal transformations, as the gauge-fixed Polyakov theory defined on a local patch of the worldsheet.
One can define a CFT even without an action, but let's introduce it by means of the lagrangian perspective, whose logic is reminescent of that one of QFT:

- the classical theory is given by an action $S[\phi_i]$ invariant under infinitesimal conformal transformations;
- the basic objects of the theory are the *fields* $\mathcal{O}_i(x)$. We specify that, in the context of conformal field theories, we call "fields" any local expression written in terms of the $\phi_i(x)$ appearing in the action and of their derivatives;
- the quantum theory is defined by the correlation functions

$$\langle R\big[\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\big]\rangle = \frac{1}{Z}\int\prod_i D[\phi_i]e^{-S[\phi_i]}\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) \quad ;$$

- the equations involving the local fields will always be thought of as *operator equations* in the quantum theory, namely as equations which are valid only if inserted into the path integral, where also other operators can be present, as long as the latter are "far" from the operators of the equations. To be more precise, let's consider a formula like (g is an operator-valued function)

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = g\big(\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\big) \quad ;$$

  this will be a shorthand for

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\rangle = \langle g\big(\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\big)...\rangle \quad ,$$

  where we denoted with ... the hypothetical presence of other operators $\mathcal{O}_i(x_i)$ which have to be inserted at distances bigger than $|x_1 - x_2|$; in the radial quantization, these operators cannot be inserted in the annulus whose boundary are given by the circles of radii given by the radial positions of $\mathcal{O}_1$ and $\mathcal{O}_2$.

In the case a classical action was missing, one can define a CFT as a "complete set" of local fields $\mathcal{O}_i$ with correlation functions given as maps from the space of operators to $\mathbb{C}$ whose forms are constrained by conformal invariance. It turns out that the requirement of conformal invariance is so stringent that, in principle, all correlation functions can be computed in terms of a finite amount of input data[43]; this can be done because, in a CFT, there is a natural notion of "complete set" of operators - the notion of quasi-primary fields, which doesn't exist in a general QFT - and because the "product" (the OPEs) of two such quasi primary fields has remarkable properties. We are going to introduce these concepts directly in the setting of a $2-$dimensional CFT defined on the Riemann sphere $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$, given that it is the case of our interest.

4.4 Primary and quasi-primary fields

We have already seen that, on $\mathbb{S}^2$, the complex dilatations $z \mapsto \lambda z$ ($\bar{z} \mapsto \bar{\lambda}\bar{z}$) have an important role, because they are related to the σ^1 and σ^2 translations on the cylinder, namely with rotations and time translations. It is thus natural to label[44] the fields $\Phi(z,\bar{z})$ defined on $\mathbb{S}^2$ according to their transformation properties under dilatations. If a field transforms as

$$\Phi(z,\bar{z}) \mapsto \Phi'(z',\bar{z}') = \lambda^{-h}\bar{\lambda}^{-\bar{h}}\Phi(z,\bar{z}) \quad ,$$

then we say that it has *conformal dimensions* $(h,\bar{h})$. It is intuitive that $h+\bar{h}$ and $h-\bar{h}$ are, respectively, the eigenvalues of Φ under real dilatations and real rotations on $\mathbb{S}^2$ (see [8]) and, so, $\Delta \equiv h+\bar{h}$ and

[43] See later: the input needed is what we will call structure constants of the theory.
[44] This is reminiscent of our approach to QFT, where one labels fields according to the mass and spin.

$s \equiv h - \bar{h}$ play the role of the scaling dimension and spin of the field Φ. The reader should also keep in mind that a unitary CFT is characterized by $h, \bar{h} \geq 0$ (the conformal dimensions vanish only in the case of Φ proportional to the identity operator).
A *primary field* $\Phi(z, \bar{z})$ is a field that transforms as a tensor under conformal transformations, namely:

$$\begin{aligned} z &\mapsto z' \equiv f(z) \\ \bar{z} &\mapsto \bar{z}' \equiv \bar{f}(\bar{z}) \\ \Phi(z,\bar{z}) &\mapsto \Phi'(z', \bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \Phi(z, \bar{z}) \quad ; \end{aligned} \tag{4.13}$$

at the infinitesimal level, this definition reduces to

$$\begin{aligned} z &\mapsto z' \equiv f(z) = z + \epsilon(z) + O(\epsilon^2) \\ \bar{z} &\mapsto \bar{z}' \equiv \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) + O(\epsilon^2) \\ \delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) &= -(h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\Phi(z,\bar{z}) \quad . \end{aligned} \tag{4.14}$$

A *quasi-primary field* satisfies (4.13) only when the conformal transformation is globally defined on $\mathbb{S}^2$, that is when $(f, \bar{f}) \in PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$.
We have seen, in the last chapters, that the closed string theory consists of the right sector (fields depending only on the σ^- coordinate), and the left one (fields depending only on σ^+); the coordinates w, $\bar{w}$ coincide with $w = i\sigma^-$ and $\bar{w} = i\sigma^+$ so the conformal map to the cylinder preserves the notion of "right" and "left" movers, because z ($\bar{z}$) depends only on w ($\bar{w}$). It makes then sense to introduce proper names for fields that, on the plane, depend only on z or only on $\bar{z}$; these are respectively called *chiral* and *antichiral* fields and correspond to left and right moving fields on the cylinder. On the cylinder, a right-moving field $\Phi(\sigma^-)$ can always be expanded as

$$\Phi(\sigma^-) = \sum_{n\in\mathbb{Z}} \phi_n e^{-in\sigma^-} \quad , \tag{4.15}$$

namely as

$$\Phi(w) = \sum_{n\in\mathbb{Z}} \phi_n e^{-nw} \quad ; \tag{4.16}$$

if it is primary, then, after the conformal map to the complex plane, it becomes the chiral field

$$\Phi_{\text{plane}}(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \quad , \tag{4.17}$$

where we have added the subscript "plane" instead of putting the $'$ as we did in (4.13). So we have learnt that the modes ϕ_n for a chiral primary field $\Phi(z)$ of conformal dimension h can be simply obtained as

$$\phi_n = \frac{1}{2\pi i} \oint \Phi(z) z^{n+h-1} \quad . \tag{4.18}$$

4.5 OPEs

In a QFT, the *operator product expansion* (OPE) is defined as an approximative expansion of two operators $\mathcal{O}_i(x_i)$ and $\mathcal{O}_j(x_j)$ valid in the limit $x_i - x_j \to 0$ and in a certain neighbourhood of the locations of the operators:

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k C_{ij}^k(|x_i - x_j|)\mathcal{O}_k(x_k) \quad . \tag{4.19}$$

In a CFT, the structure of the OPE is constrained by the requirement of conformal invariance to such a point that the functional dependence of $C_{ij}^k(|x_i - x_j|)$ is completely fixed. Moreover, the OPE is an exact expression[45] and the OPE of two quasi-primary fields involves only other quasi-primary fields and

[45]Namely, a convergent series whose radius of convergence is given by the distance to the next field insertion.

their derivatives (the so-called *descendant fields*).
For example, in the case of the 2−dimensional CFT it is possible to show[46] that the OPE of two chiral quasi-primary fields $\phi_i(z)$, $\phi_j(w)$ can be written as

$$\begin{aligned} \phi_i(z)\phi_j(w) &= \sum_{k,n\geq 0} S^k_{ij} \frac{a^n_{ijk}}{n!} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \partial^n \phi_k(w) \\ a^n_{ijk} &= \binom{2h_k+n-1}{n}^{-1} \binom{h_k+h_i-h_j+n-1}{n} \quad , \end{aligned} \tag{4.20}$$

where the sum over k runs only over the quasi-primary fields and the complex constants S^k_{ij} are called *structure constants*. Even though it is not important for what we are going to study later, the reader should be aware about an essential fact: it is possible, by successive applications of the OPEs, to reduce all higher order correlations functions to the correlators (4.20), so the structure constants is all what we need to solve the theory.
Now that we understood the advantages of working with the OPEs in a CFT, we show how to compute them. We will start by focusing on the OPEs between the energy momentum tensor and any other conformal field.

4.6 Conformal Ward-Takahashi identities

We need to borrow[47] a general result from QFT, namely the *Ward-Takahashi indentity*; this computes the variation $\delta\mathcal{O}_i$ of an operator under a tranformation of the fields $\phi \mapsto \phi + \epsilon\delta\phi$ in terms of the integral of the divergence of the Noether current that one would obtain at the classical level by taking ϵ constant. So let's supose that, at the classical level, the action $S[\phi]$ is invariant under a global tranformation $\phi \mapsto \phi + \epsilon\delta\phi$ and let the associated Noether current be J_α. Then, at the quantum level, the following equation is valid as an operator equation:

$$-\frac{1}{2\pi} \int_{B(x_1)} dx\, \partial_\alpha J^\alpha(x) \mathcal{O}_1(x_1)... = \delta\mathcal{O}_1(x_1)... \quad , \tag{4.21}$$

where we denoted with $B(x_1)$ a region that contains the location x_1 of the operator $\mathcal{O}_1$, but that doesn't include any other hypothetical operators present in "...".
In the case of a 2−dimensional QFT, we can use Stoke's theorem and introduce the complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ to rewrite this equation as

$$\delta\mathcal{O}(w,\bar{w}) = -\frac{1}{2\pi i} \oint_{\partial B(x_1)} \Big(dz\, J_z(z,\bar{z}) - d\bar{z}\, J_{\bar{z}}(z,\bar{z}) \Big) \mathcal{O}(w,\bar{w}) \quad , \tag{4.22}$$

where

$$J_z = \frac{1}{2}(J_1 - iJ_2) \quad , \quad J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2) \quad , \tag{4.23}$$

because indices are raised and lowered as in (4.2).
Now we want to apply the Ward-Takahashi identity to the conformal symmetry of a 2−dimensional CFT, so we need to find the Noether current associated to conformal invariance.
We start by applying Noether's theorem to translations and dilatations and we obtain that the energy-momentum tensor has to be conserved and it has to be traceless, that is

$$\partial_\alpha T^{\alpha\beta} = 0 \quad , \quad T^\alpha_\alpha = 0 \quad .$$

Let's rewrite these equations with the complex coordinates. The tracelessness condition becomes

$$T_{z\bar{z}} = 0 \tag{4.24}$$

[46] See [16] for the proof.
[47] See [8] for a derivation and for details.

and the conservation condition now reads as

$$\partial_{\bar{z}} T_{zz} = 0 \quad , \quad \partial_z T_{\bar{z}\bar{z}} = 0 \quad . \tag{4.25}$$

In other words, the energy-momentum tensor splits into a chiral and into an anti-chiral part:

$$T_{zz} \equiv T(z) \quad , \quad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) \quad , \tag{4.26}$$

and now it is not difficult to guess that the Noether currents for the conformal transformations

$$\begin{aligned} z &\mapsto z + \epsilon v(z) = z + \epsilon(z) \\ \bar{z} &\mapsto \bar{z} + \bar{\epsilon}\bar{v}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) \end{aligned} \tag{4.27}$$

are given by the couple

$$J_z = \epsilon(z)T(z) \qquad J_{\bar{z}} = \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}) \quad . \tag{4.28}$$

After pluggig these currents back into (4.22), we get the *conformal Ward-Takahashi identity*:

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O}(w,\bar{w}) = -\frac{1}{2\pi i}\oint_{C_w}\Big(dz\,\epsilon(z)T(z) + d\bar{z}\,\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\Big)\mathcal{O}(w,\bar{w}) \quad , \tag{4.29}$$

where both the contour integrals around w and $\bar{w}$ are counter-clockwise in z and in $\bar{z}$; remember that the Ward-Takahashi identities are operator equations, so, here, both $T(z)\mathcal{O}(w,\bar{w})$ and $\bar{T}(\bar{z})\mathcal{O}(w,\bar{w})$ are taken radially ordered, namely $|z| > |w|$ and $|\bar{z}| > |\bar{w}|$.
So we have learnt that the way an operator $\mathcal{O}$ changes under an infinitesimal conformal transformation is encoded in it OPE with the energy-momentum tensors $T(z)$, $\bar{T}(\bar{z})$.
If we specialize this formula to a primary field $\phi(w,\bar{w})$ of dimension $(h,\bar{h})$, we can be more precise; indeed we can substitute the left side of (4.29) with the last line of (4.14) to obtain

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) &= -(h\partial_w\epsilon(w) + \epsilon(w)\partial_w + \bar{h}\partial_{\bar{w}}\bar{\epsilon}(\bar{w}) + \bar{\epsilon}(\bar{w})\partial_{\bar{w}})\Phi(w,\bar{w}) = \\ &= -\frac{1}{2\pi i}\oint_{C_w}\Big(dz\,\epsilon(z)T(z) + d\bar{z}\,\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\Big)\phi(w,\bar{w}) \quad , \end{aligned} \tag{4.30}$$

and by using the standard formulae

$$\begin{aligned} \epsilon(w)\partial_w\phi(w,\bar{w}) &= \oint_{C_w} dz\,\frac{1}{2\pi i}\frac{\epsilon(z)}{z-w}\partial_w\phi(w,\bar{w}) \\ \partial_w\epsilon(w)\phi(w,\bar{w}) &= \oint_{C_w} dz\,\frac{1}{2\pi i}\frac{\epsilon(z)}{(z-w)^2}\phi(w,\bar{w}) \quad , \end{aligned} \tag{4.31}$$

we end up with the very important formulae for the OPEs between a primary field and the energy-momentum tensor, namely

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + O(1) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + O(1) \quad . \end{aligned} \tag{4.32}$$

4.7 The Virasoro generators

Given the importance of the energy-momentum tensor in a CFT, the most natural question is to ask what happens if we take $\phi(w,\bar{w})$ to be $T(w)$ or $\bar{T}(\bar{w})$ in formulae (4.32). The energy-momentum tensor $T_{\alpha\beta}$ has for sure scaling dimension $\Delta[T_{\alpha\beta}] = h + \bar{h} = 2$ because after integrating it over the space direction we obtain the conserved energy of the system. $T(w)$ is chiral and $\bar{T}(\bar{w})$ anti-chiral, so we understand that the first one has conformal dimensions $(h, \bar{h}) = (2, 0)$ and the latter has $(h, \bar{h}) = (0, 2)$. So we must have

$$\begin{aligned} T(z)T(w) &= ... + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + O(1) \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= ... + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\bar{T}(\bar{w}) + O(1) \quad , \end{aligned} \tag{4.33}$$

where we denoted with the dots the hypothetical presence of terms of higher singular behaviour. Obviously, this hypothetical term must have the same scaling dimension of the product of two energy-momentum tensors, so any operators that appear on the right-hand-side must be of the form

$$\frac{\mathcal{O}_n}{(z-w)^n} \quad ,$$

with $\Delta[\mathcal{O}_n] = 4 - n$. But, there are no operators with negative conformal dimensions in a unitary theory so the most singular term has to be a constant that multiplies $(z-w)^{-4}$ and we write

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{h}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + O(1) \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\bar{T}(\bar{w}) + O(1) \quad , \end{aligned} \tag{4.34}$$

where we didn't include terms proportional to $(z-w)^{-3}$ because the stress energy momentum tensor is bosonic and, so, in a radial ordered equation we must have $T(z)T(w) = T(w)T(z)$. The constants c and $\bar{c}$ are called *central charges* and, roughly speaking, they somehow count the number of degrees of freedom in the CFT[48]; so, if we want to have a theory that describes the same number of degrees of freedom both in the chiral and anti-chiral sector[49], we have to require $c = \bar{c}$.
By comparing (4.34) with (4.32), we discover that the the energy-momentum tensor is a primary field only if $c = \bar{c} = 0$. Here we introduced the charges by hand; one way to compute them is to determine the commutator relations among the modes L_n of the energy-momentum tensor, because the information contained in the OPE of two operators is equivalent to the commutator relations among their modes.
Indeed, let $a(z)$ and $b(z)$ be two fields of our CFT, and let

$$A = \oint_{C_0} dz\, a(z) \quad , \quad B = \oint_{C_0} dz\, b(z)$$

be their contours integrals around $0 \in \mathbb{C}$, with C_0 oriented counterclockwisely. Then, if we remember about the omnipresent radial ordering, it is not difficult to show the validity of the following two equations as operatorial ones:

$$\begin{aligned} [A, b(z)]_\pm &= \oint_{C_z} dw\, a(w)b(z) \\ [A, B]_\pm &= \oint_{C_0} dz \oint_{C_z} dw\, a(w)b(z) \quad , \end{aligned} \tag{4.35}$$

where we denoted with $[\cdot,\cdot]_+$ the anticommutator, that has to be used when both fields are Grassmann odd and with $[\cdot,\cdot]_-$ the anticommutator (that has to be used in all other cases). These formulae are extremely useful in CFT, because they relate OPEs to commutation relations and allow us to translate into operator language the dynamical or symmetry information contained in the OPEs.
If we define the modes L_n of $T(z)$ as in (4.18), namely as

$$L_n = \frac{1}{2\pi i}\oint T(z) z^{n+2-1} = \frac{1}{2\pi i}\oint T(z) z^{n+1} \quad , \tag{4.36}$$

[48]For example, the central charge of a system consisting of D free-bosons is precisely D. We will see that the central charge represents an anomalous behaviour of the conformal theory, so, to be more precise, it counts the anomalous degrees of freedom of the quantum theory.

[49]Which is the case for closed string theory, because of the level matching condition.

then we can insert the OPE (4.34) into (4.35) to get that the *Virasoro Generators* L_n satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad . \tag{4.37}$$

The set of these commutations relations define the so-called *Virasoro algebra*, which is precisely the unique central extension of the Witt algebra.
We can expand $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ appearing in (4.29) into their Laurent series as we have done in (4.5); then, it is not difficult to read from equation (4.29) (one has only to apply (4.35)) that L_n, $\bar{L}_n$ are - at the quantum level - the generators for the conformal transformation $\delta z \sim z^{n+1}$, $\delta\bar{z} \sim \bar{z}^{n+1}$. In other words, the Virasoro generators are the "quantum version" of the Witt generators and, by comparing their algebras, we can understand that the central charges are due to a pure quantum effect: if $c, \bar{c} \neq 0$, then the conformal algebra is anomalous.
To be more precise, it is possible to show that for a $2-$dimensional CFT, the conformal anomaly is given by the expectation value of the trace of the energy-momentum tensor (that, in complex coordinates, is $T_{z\bar{z}}$). In fact, whereas at the classical level we have, thanks to dilatation invariance, that

$$T_{z\bar{z}} = 0 \quad ,$$

at the quantum level we instead have (see [11] for a proof)

$$\langle T_{z\bar{z}} \rangle = -\frac{c}{12}R \quad , \tag{4.38}$$

where we denoted with R the Ricci scalar of the worldsheet. This formula is saying that the curvature and the charges are somehow connected to the breaking of the dilatation invariance at the quantum level. This is intuitive for the curvature, because a non-vanishing one implies the notion of a typical length scale in the theory. Instead, it is not intuitive for the central charge; but also c does define a length scale in the theory, because it turns out (see again [11]) that $c \neq 0$ is equivalent to the presence of a vacuum-energy on the cylinder.
The formula (4.38) is interesting, because on the right side there is no dependence on the states that we used to sandwich $T_{z\bar{z}}$ on the left. According to this formula, it is not important to know the (finite) energies of these states: the conformal anomaly is the same at all energies. Indeed, from a practical point of view, the conformal anomaly is completely[50] due to the *normal ordering* prescription that we have to specify at the quantum level to remove short distance divergences; obvioulsy, at very short distances all the finite energy states look basically the same, so it makes sense that there is no dependence on the state on the right side of (4.38).
In QFT, the normal order $:\phi_i\phi_j:$ is usually defined as moving all creation operators to the left. In the setting of a $2-$dimensional CFT, one can rigorously prove (see [16]) that this notion of normal ordering is equivalent to picking out the non-singular term in the radially ordered OPE, that is

$$:\phi_i(z)\phi_j(w) :\equiv R[\phi_i(z)\phi_j(w)] - \text{singular terms} \quad . \tag{4.39}$$

Clearly, $\langle: \phi_i(z)\phi_j(w) :\rangle = 0$, and this is essential to prove Wick's theorem for two fields, which is

$$R[\phi_i(z)\phi_j(w)] \equiv \langle\phi_i(z)\phi_j(w)\rangle + :\phi_i(z)\phi_j(w): \quad . \tag{4.40}$$

As in a general QFT, one can inductively use the Wick theorem for two fields to relate radial-ordered and normal-ordered products of more than two fields by replacing any pair of them by their two-point correlator.
In the literature it is common not to write the normal order symbol, by giving it as understood. The reader should be aware that every composite operator that involves fields whose OPEs are singular has always to be taken in its normal ordered form, otherwise it is not well-defined[51].

[50] As we have seen for the string, the quantum anomaly $a, \tilde{a}$ affecting $L_0, \tilde{L}_0$ arose because of the ordering prescription that we introduced.

[51] For instance, see the $T(z) \sim \partial X \partial X$ and the $e^{ikX(z)} \sim \sum_n X^n$ operators that will appear later in the CFT of the single boson; the OPEs $\partial X \partial X$ and $X(z)X(z)$ contain singular terms, so $T(z)$ and $e^{ikX(z)}$ have to be considered in their normal ordered version.

4.8 State-operator correspondence

As always, the first step to buid the Hilbert space of the theory is to define the vacuum state. For a 2−dimensional CFT on $\mathbb{S}^2$, we distinguish between the *in-vacuum* $|0\rangle$ and the *out-vacuum* $\langle 0|$ respectively, corresponding to the vacuum at $\sigma^2 = -\infty$ and at $\sigma^2 = +\infty$.
Given that we have already subtracted the infinities that naturally arise in the quantum theory by taking all fields in their normal ordered form, we require the vacuum to give well-defined states under the action of the energy-momentum tensor operator.
Regularity of $T(z) = \sum_n z^{-n-2} L_n$ at $\sigma^2 = -\infty$, i.e. at $z = 0$, implies that

$$L_n |0\rangle = 0 \qquad \forall n \geq -1 \quad , \tag{4.41}$$

and regularity at $\sigma^2 = +\infty$, i.e. at $z = \infty$ (better, at $u = -1/z = 0$) imposes

$$\langle 0| L_n = 0 \qquad \forall n \leq 1 \quad . \tag{4.42}$$

By looking at these two conditions we understand that the only Virasoro generators that annihilate both $|0\rangle$ and $\langle 0|$ are L_{-1}, L_0 and L_1. Obviously, analogous relations hold for the anti-chiral sector and we get that the vacuum of a 2−dimensional CFT is invariant only under $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$.
Then, we can associate a state to every quasi-primary[52] field $\Phi(z,\bar{z})$ by postulating that also the action of $\Phi(z,\bar{z})$ on the vacuum is regular at $\sigma^2 = \pm\infty$. For a chiral field $\phi(z) = \sum_n \phi_n z^{-n-h}$ of conformal dimension h, this implies

$$\begin{aligned} \phi_n |0\rangle = 0 \quad \forall n \geq 1-h \\ \langle 0| \phi_n = 0 \quad \forall n \leq h-1 \quad , \end{aligned} \tag{4.43}$$

where we used the intuitive[53] definition of the hermitian conjugate for a mode, that is $(\phi_n)^\dagger = \phi_{-n}$.
We can now define the *in-state* and *out-state* as

$$\begin{aligned} |\phi_{in}\rangle =& \phi_{-h} |0\rangle = \lim_{z\to 0} \phi(z) |0\rangle = \phi(0) |0\rangle \\ \langle \phi_{out}| =& \langle 0| \phi_h \quad . \end{aligned} \tag{4.44}$$

In these two formulae is encoded the *operator-state correspondence*: thanks to the conformal map between the cylinder and the complex plane, we can bring the entire spatial slice $\sigma^2 = -\infty$ to the point $z = 0$ and, thus, in the path integral[54] the information about the state corresponding to a field configuration in the remote past is represented by a local operator inserted at the origin of the plane. This is a peculiarity of CFT that doesn't happen in a general QFT.

4.9 Highest weight states

As usual, we are interested in the transformation properties of the states of our theory. Thanks to the operator correspondence map, we can focus on the transformation properties of the fields.
With the help of the formulae (4.35), the OPE between the energy-momentum tensor and a chiral primary field $\phi(w)$ becomes

$$[L_m, \phi(z)] = z^m (z\partial_z + (m+1)h)\phi(z) \quad , \quad [L_m, \phi_n] = ((h-1)m - n)\phi_{m+n} \quad . \tag{4.45}$$

These results can be used to determine the action of the Virasoro generators on the primary state $|\phi\rangle = \phi(0) |0\rangle$:

$$\begin{aligned} L_0 |\phi\rangle &= h |\phi\rangle \\ L_n |\phi\rangle &= 0 \quad \forall n > 0 \\ L_0(L_{-n} |\phi\rangle) &= (n+h)(L_{-n} |\phi\rangle) \quad \forall n > 0 \quad ; \end{aligned} \tag{4.46}$$

[52] In order to talk about the values of $\phi(z)$ at both $z = 0$ and $u = 0$, we need a globally defined field on the sphere.
[53] To define the Hilbert space it is not enough to specify its states; also a scalar product is needed. Defining the scalar product is equivalent to define the hermitian conjugation. It turns out that the right definition for the hermitian conjugation is $\Phi^\dagger(z,\bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z)$, see [15] for details. For the modes, this means $\Phi^\dagger_{m,n} = \Phi_{-m,-n}$, as one would intuitively guess.
[54] A clear reference for understanding this is [8].

therefore we recognize L_n and L_{-n} as lowering and raising operator with respect to the eigenstates of L_0. The states corresponding to the primary fields, namely the states of the theory satisfying (4.46), are called *highest weight states.* These states are very important, because they are the building block of the Hilbert space of the 2−dimensional CFT. Indeed, the complete Hilbert space is obtained by acting with L_{-n} on all highest weight states ϕ_j, where the $j-$subscript labels the primary fields. One usually calls *Verma module* V_{h_j} the subspace of the Hilbert space spanned[55] by the set of all the states of the form

$$|\phi_j^{k_1...k_m}\rangle = L_{-k_1}...L_{-k_m}|\phi_j\rangle\,, \qquad k_i > 0 \tag{4.47}$$

of conformal weight $h = h_j + \sum_i^m k_i$.
By means of the operator-state correspondence one can define a conformal field associated to a state in the Verma module V_{h_j} and it turns out that it is not primary; instead, the field that creates the state $|\phi_j^{k_1...k_m}\rangle$ from the $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$ invariant vacuum is called *descendant field* $\phi_j^{k_1...k_m}(z)$.

With these concepts we consider our introduction to CFT as concluded. Now we are going to show a short application of these abstract ideas to bosonic string theory. We have already seen that the gauge-fixed Poyakov action consists of the matter sector (D free bosonic fields with a second-order Lagrangian) and of the ghost sector (a couple of anticommuting fields defining a first-order Lagrangian). So, we are going to focus on the CFT of a single free boson and on the CFT defined by a first-order Lagrangian.

4.10 A single free boson on the sphere

The action of a single free boson on the sphere is

$$S = \frac{1}{2\pi\alpha'}\int dzd\bar{z}\partial X(z,\bar{z})\bar{\partial}X(z,\bar{z}) \quad , \tag{4.48}$$

which gives immediately the classical equation of motion $\partial\bar{\partial}X(z,\bar{z}) = 0$. It is important to check that this equation holds also at the quantum level as an operator equation inside the path integral, because, in that case, we can still use the essential decomposition $X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$. But we simply have

$$0 = \frac{1}{Z}\int D[X]\frac{\delta}{\delta X}e^{-S} = -\frac{1}{Z}\int D[X]\frac{\delta S}{\delta X}e^{-S} = \frac{1}{\pi\alpha'}\langle\partial\bar{\partial}X(z,\bar{z})\rangle \quad , \tag{4.49}$$

and the classical equation of motion does hold as an operator equation.
With a similar trick, we can compute

$$\begin{aligned} 0 &= \int D[X]\frac{\delta}{\delta X}\left(e^{-S}X(z',\bar{z}')\right) = \\ &= \int D[X]e^{-S}\left(\delta^{(2)}(z-z',\bar{z}-\bar{z}') + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}X(z,\bar{z})X(z',\bar{z}')\right) \quad , \end{aligned} \tag{4.50}$$

namely

$$\langle\partial_z\partial_{\bar{z}}X(z,\bar{z})X(z',\bar{z}')\rangle = -\pi\alpha'\langle\delta^{(2)}(z-z',\bar{z}-\bar{z}')\rangle \quad , \tag{4.51}$$

which can be integrated[56] to

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\log(|z-w|^2) \quad . \tag{4.53}$$

Correspondigly, the chiral correlators are

$$\langle X(z)X(w)\rangle = -\frac{\alpha'}{2}\log(z-w) \quad , \quad \langle\bar{X}(\bar{z})\bar{X}(\bar{w})\rangle = -\frac{\alpha'}{2}\log(\bar{z}-\bar{w}) \quad . \tag{4.54}$$

[55] It is possible to show that for $k_1 \geq k_2 \geq ... \geq k_m$ the states $|\phi_j^{k_1...k_m}\rangle$ are linearly independent.
[56] Remember that

$$\partial_z\frac{1}{\bar{z}} = \partial_{\bar{z}}\frac{1}{z} = 2\pi\delta^{(2)}(z,\bar{z}) \tag{4.52}$$

.

Taking the derivatives of equations (4.54), we get the 2−point functions for the fields ∂X and $\bar{\partial}\bar{X}$, which are

$$\langle \partial X(z)\partial X(w)\rangle = -\frac{\alpha'}{2}\frac{1}{(z-w)^2} \quad , \quad \langle \bar{\partial}\bar{X}(\bar{z})\bar{\partial}\bar{X}(\bar{w})\rangle = -\frac{\alpha'}{2}\frac{1}{(\bar{z}-\bar{w})^2} \quad . \tag{4.55}$$

The fields ∂X, $\bar{\partial}\bar{X}$ are important, because the energy-momentum tensor can be written in terms of them, i.e.

$$T(z) = -\frac{1}{\alpha'} : \partial X(z)\partial X(z) : \qquad \bar{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial}\bar{X}(\bar{z})\bar{\partial}\bar{X}(\bar{z}) : \quad . \tag{4.56}$$

Now we can compute the OPE between the energy-momentum tensor and all the fields of our theory. We discover that

- X and $\bar{X}$ have dimensions $(0,0)$. In fact

$$\begin{aligned}
T(z)X(w) =& R\Big[-\frac{1}{\alpha'} : \partial X(z)\partial X(z) : X(w)\Big] = \\
=& -\frac{2}{\alpha'}\partial X(z)\partial X(z)X(w) + \ldots = \partial X(z)\frac{1}{z-w} + \ldots = \\
=& \frac{\partial X(w)}{z-w} + \ldots
\end{aligned} \tag{4.57}$$

- the fields ∂X and $\bar{\partial}\bar{X}$ are primary fields of conformal dimensions $(1,0)$ and $(0,1)$, because

$$T(z)\partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial\partial X(w)}{(z-w)} + \ldots \tag{4.58}$$

- the descendant fields $(n>1)$ $\partial^n X$ and $\bar{\partial}^n\bar{X}$ are fields of conformal dimensions $(n,0)$ and $(0,n)$. They are not primaries, as it is not difficult to check for $\partial^2 X$, by taking the derivative ∂_w of (4.58);
- the energy-momentum tensor is indeed quasi-primary of conformal dimension $(2,0)$:

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \tag{4.59}$$

So we have learnt that:

- the fields ∂X and $\bar{\partial}\bar{X}$ are the fundamental[57] fields in the game: being primaries, they are the fields from which the Hilbert space of theory can be built. Note that, on the complex plane, these fields take the following form

$$\begin{aligned}
\partial X(z) &= -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{z^{n+1}} \\
\bar{\partial}\bar{X}(\bar{z}) &= -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{\bar{z}^{n+1}} \quad ,
\end{aligned} \tag{4.60}$$

  with $\alpha_0 = \tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}\hat{p}$ and

$$\begin{aligned}
\alpha_n &= i\sqrt{\frac{2}{\alpha'}}\oint_{C_0}\frac{dz}{2\pi i}z^n\partial X(z) \\
\tilde{\alpha}_n &= i\sqrt{\frac{2}{\alpha'}}\oint_{C_0}\frac{d\bar{z}}{2\pi i}\bar{z}^n\bar{\partial}\bar{X}(\bar{z}) \quad ,
\end{aligned} \tag{4.61}$$

  where the last two expressions are valid for all $n\in\mathbb{Z}$.

[57] The fields X and $\bar{X}$ are not fundamental, because their correlation functions scale in a logaritmic way, see (4.54); conformal field theories are scale invariant, and this requires correlation functions of conformal fields to behave as power laws.

- the central charges for one free integer-moded real boson are $c = \bar{c} = 1$.

We conclude this section by saying that there is another very important primary field in this CFT, which is given by the (normal ordered, of course) exponential of the X, $\bar{X}$ fields; here its useful OPEs are:

$$
\begin{aligned}
\partial X(z) : e^{ikX(w)} : &= -i\frac{\alpha' k}{2} : e^{ikX(w)} : \frac{1}{z-w} + ... \\
T(z) : e^{ikX(w)} : &= \frac{\alpha' k^2/4}{(z-w)^2} : e^{ikX(w)} : + \frac{\partial_w}{(z-w)} : e^{ikX(w)} : + ... \\
: e^{ik_1 X(z)} :: e^{ik_2 X(w)} : &= (z-w)^{\alpha'(k_1 k_2)} : e^{i(k_1+k_2)X(w)}(1 + O(z-w)) : \quad ,
\end{aligned}
\tag{4.62}
$$

and, in particular, from the second line we get that $: e^{ikX(z,\bar{z})} :$ is a primary field of conformal dimensions $(h, \bar{h}) = (\alpha' k^2/4, \alpha' k^2/4)$.

4.11 First-order Lagrangians

In complex coordinates, the $b - c$ ghost action reads as

$$
S_g = \frac{1}{2\pi} \int d^2 z (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}}) \quad , \tag{4.63}
$$

where $c^z(z)$ is a worldsheet vector so it has $h = -1$ and, correspondingly, $b_{zz}(z)$ has dimension $h = 2$. As we have done for the single boson, we can show that the equations of motion are valid as operator equations; given that the equations of motion impose on the chiral and anti-chiral sectors to be independent, it is enough to analyze only one of them. Thus, by following the approach of [7], we decide to study the chiral part of a theory that is sligthly more general:

$$
S = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c \quad , \tag{4.64}
$$

where $c(z)$ and $b(z)$ have conformal dimensions $h = 1 - \lambda$ and $h = \lambda$. The statistics of b and c is parametrized by ϵ: $\epsilon = 1$ if they are Grassmann odd and $\epsilon = -1$ if they are Grassmann even.
The importance of this first-order lagrangian relies on the fact that for $\lambda = 2$, $\epsilon = 1$ it describes the $b - c$ theory of (4.63) (the conformal ghost system) whereas, for $\lambda = 3/2$, $\epsilon = -1$ it describes the "superconformal ghost system". The latter will appear in the context of the superstring, where we will have more gauge redundancie to fix and this will lead us to introduce another couple of conjugate fields, which are bosonic fields with half-integer spin and which form the so-called $\beta - \gamma$ system.
The equations of motion are easily found to be

$$
\begin{aligned}
\bar{\partial} b &= 0 \quad \text{i.e.} \quad b = b(z) \\
\bar{\partial} c &= 0 \quad \text{i.e.} \quad c = c(z) \quad ,
\end{aligned}
\tag{4.65}
$$

and the propagator is

$$
\langle c(z) b(w) \rangle = \frac{1}{z-w} \quad . \tag{4.66}
$$

The basic OPEs are

$$
c(z)b(w) = \frac{1}{z-w} + ... \quad , \quad b(z)c(w) = \frac{\epsilon}{z-w} + ... \quad , \tag{4.67}
$$

where, as usual, the dots denote the presence of regular terms; the $b(z)b(w)$ and $c(z)c(w)$ products are non-singular.
Following (4.17), we decompose the b and c fields into the mode expansions

$$
\begin{aligned}
b(z) &= \sum_{n \in a - \lambda + \mathbb{Z}} z^{-n-\lambda} b_n \quad , \quad b_n^\dagger = \epsilon b_{-n} \\
c(z) &= \sum_{n \in a + \lambda + \mathbb{Z}} z^{-n-1+\lambda} c_n \quad , \quad c_n^\dagger = c_{-n} \quad ,
\end{aligned}
\tag{4.68}
$$

where we introduced the constant a for reasons that will become clear in the superstring. Indeed, for the case of half-integer λ, there are two sectors: the *Ramond* one (R), which is specified by $a = 0$, and the *Neveu-Schwarz* (NS) characterized by $a = 1/2$. Anyway, the OPEs (4.67) are equivalent to the (anti)commutator relations

$$[c_m, b_n]_\epsilon = \delta_{m+n,0} \quad . \tag{4.69}$$

As we explained in (4.43), the modes have to act in the following way on the $PSL(2,\mathbb{C})$ invariant vacuum:

$$\begin{aligned} b_n \left|0\right\rangle &= 0 \quad \forall n \geq 1-\lambda \\ c_n \left|0\right\rangle &= 0 \quad \forall n \geq \lambda \quad . \end{aligned} \tag{4.70}$$

The energy-momentum tensor is

$$\begin{aligned} T =& -\lambda : b\partial c : + (1-\lambda) : (\partial b) c := \\ =& \frac{1}{2}\Big(: (\partial b) c : - : b\partial c : \Big) + \frac{1}{2}\epsilon Q \partial (: bc :) \quad , \end{aligned} \tag{4.71}$$

where we introduced the quantity

$$Q \equiv \epsilon(1-2\lambda) \quad , \tag{4.72}$$

which will aquire the meaning of a *background charge* soon.
From the energy-momentum tensor, we can find the Virasoro generators as in (4.36):

$$L_n = \sum_m (m - (1-\lambda)n) : b_{n-m} c_m : \qquad . \tag{4.73}$$

The OPEs of the fields in the game with the energy momentum tensor are:

$$\begin{aligned} T(z)b(w) &= \frac{\lambda b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)} + ... \\ T(z)c(w) &= \frac{(1-\lambda)c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)} + ... \\ T(z)T(w) &= \frac{\epsilon(1-3Q^2)/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + ... \qquad , \end{aligned} \tag{4.74}$$

and we find that the b, c fields are primaries of the expected dimensions $h = \lambda$, $h = 1-\lambda$ and that the central charge of the system is

$$c = \epsilon(1-3Q^2) \quad . \tag{4.75}$$

Actually, there is another important field in the game, the Noether current $j(z)$ associated to the following $U(1)$ symmetry of the action (4.64):

$$c(z) \mapsto e^{i\alpha} c(z) \qquad b(z) \mapsto e^{-i\alpha} b(z) \quad \alpha \in \mathbb{R}. \tag{4.76}$$

By applying the Noether theorem we find

$$j(z) = - : b(z) c(z) : \quad , \tag{4.77}$$

whose OPE with the energy-momentum tensor is

$$T(z)j(w) = \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} + ... \qquad . \tag{4.78}$$

This means that the *number current* $j(z)$ is a primary field of conformal dimension $h = 1$ only in the case of vanishing Q, i.e. when b and c have the same conformal dimension $h = 1/2$.
We can decompose j as usual as

$$j(z) = \sum_n z^{-n-1} j_n \quad , \tag{4.79}$$

with

$$j_n = \sum_m \epsilon : c_{n-m} b_m : \quad , \quad [j_m, j_n] = \epsilon m \delta_{m+n} \quad , \tag{4.80}$$

where the commutation relations come from the OPE

$$j(z)j(w) = \frac{\epsilon}{(z-w)^2} + ... \quad . \tag{4.81}$$

The other OPEs which involve the number current are

$$j(z)b(w) = \frac{-b(w)}{z-w} + ... \quad , \quad j(z)c(w) = \frac{c(w)}{z-w} + ... \quad , \tag{4.82}$$

which reflect the fact that b and c have $U(1)$ charges -1 and $+1$, as we see from (4.76), because the conserved Noether charge associated[58] to the current $j(z)$ is nothing but $N_j \equiv \oint_{C_0} \frac{dz}{2\pi i} j(z)$.
To understand the meaning of the quantity Q, we begin with rewriting the OPE (4.78) in the form of the corresponding commutators

$$[L_n, j_m] = \frac{1}{2} Q n(n+1) \delta_{n+m,0} - m j_{m+n} \quad , \tag{4.83}$$

and we see that $j(z)$ transforms covariantly under translations (L_{-1}) and under dilatations (L_0), but not under special conformal transformations (L_{+1}), i.e. the number current is not a quasi-primary field. Then, we note that the Hermiticity conditions (4.68) easily give $j_n^\dagger = -j_{-n}$ for $n \neq 0$. Instead, the case $n = 0$ is delicate because of normal ordering ambiguities and, to find $j_0^\dagger$, we can alternatively exploit the anomalous commutators (4.83):

$$\begin{aligned} (j_0)^\dagger =& (-[L_{-1}, j_1])^\dagger = [L_{-1}^\dagger, j_1^\dagger] = [L_1, -j_{-1}] = -[L_1, j_{-1}] = \\ =& -(Q + j_0) \quad . \end{aligned} \tag{4.84}$$

The relation $j_0^\dagger = -j_0 - Q$ has striking consequences on the operator expectation values of the theory. Indeed, if O_p is an operator with $U(1)$ charge p, i.e. $[j_0, O_p] = pO_p$ and $|q\rangle$ is a state with $U(1)$ charge q, we find that

$$\begin{aligned} p\langle q'|O_p|q\rangle =& \langle q'|[j_0, O_p]|q\rangle = \langle j_0^\dagger q'|O_p|q\rangle - \langle q'|O_p q|q\rangle = \\ =& (-q' - Q - q)\langle q'|O_p|q\rangle \quad , \end{aligned} \tag{4.85}$$

which means that

$$p + q + q' + Q \neq 0 \Longrightarrow \langle q'|O_p|q\rangle = 0 \quad ; \tag{4.86}$$

we then normalize the states such that

$$\langle -q - Q|q\rangle = 1 \tag{4.87}$$

and this makes the meaning of Q as a background charge apparent. By passing from the classical to the quantum level, the background charge Q appears because of the normal ordering prescription (which affects j_0). This is the same mechanism that makes the central charge c emerge; we know that the central charge is an anomaly in the theory (it signals the breaking of the dilatation invariance) and we wonder

[58] As always, the conserved quantity is simply given by the integration of the conserved current over the spatial directions of the theory. In the radial quantization, the spatial slices are given by the circles centered in $z = 0$.

whether the background charge Q is also an anomaly. We can already guess that the answer is yes, by looking at the relation (4.75). The precise statement is that Q is an anomaly which affects the classical conservation of the number current, because it is possible to show[59] that

$$\nabla^z j_z(z) = \frac{1}{4} Q R \quad , \tag{4.88}$$

where R, as usual, is the Ricci scalar of the worldsheet.

Bosonization of the first order Lagrangian systems The number current is extremely useful not only because its anomalous nature constrains the theory to satisfy (4.86), but also because it lets us *bosonize* the system, giving us the opportunity to equivalently describe the latter in terms of a bosonic conformal field theory. We are going to present the bosonization here[60], even though it will become useful only in the context of the superstring; at a first reading, this paragraph can be omitted.
The key idea is that the number current characterized by the OPE (4.81), namely by

$$j(z)j(w) = \frac{\epsilon}{(z-w)^2} + \dots \quad , \tag{4.89}$$

is all what we need to define a new energy-momentum tensor $T_j(z)$ which satisfies the expected OPE (4.78) with j, which is

$$T_j(z)j(w) = \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} + \dots \quad . \tag{4.90}$$

In fact, if we define

$$T_j(z) \equiv \epsilon\Big(-\frac{1}{2}(j(z))^2 - \frac{1}{2} Q \partial_z j_z(z)\Big) \quad , \tag{4.91}$$

then it is not difficult to recover (4.90) and this means that Q still assumes the meaning of a background charge. Note that in this approach we are considering j as a general field satisfying (4.89), in our mind there is not necessarily the field j given in terms of the b and c fields as in (4.77). This is the crux: we want to find another set of fields that equivalently describe the same CFT as the b and c fields.
If we compute the OPE of the new energy-momentum tensor T_j with itself, we find

$$T_j(z)T_j(w) = \frac{(1-3\epsilon Q^2)/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \quad , \tag{4.92}$$

and, thus, the new central charge c_j is now $c_j = (1-3\epsilon Q^2)$.
If we continue to call $c = \epsilon(1-3Q^2)$ the central charge characterizing the theory of the b, c fields, we immediately note that

$$\begin{aligned} c =& c_j \qquad \text{if } \epsilon = +1 \quad \text{(Fermi statistics for } b, c \text{ fields)} \\ c =& c_j - 2 \quad \text{if } \epsilon = -1 \quad \text{(Bose statistics for } b, c \text{ fields)} \quad . \end{aligned} \tag{4.93}$$

The current j and its stress energy momentum tensor T_j completely characterize the Fermi theory, because, in this case, we have $c = c_j$ and, so, we can identify T and T_j.
Instead, in the case of Bose statistics, there is a "residual" central charge c_{-2} given by

$$c_{-2} \equiv c - c_j = -2 \quad ;$$

it is natural to think about it as the effect of a third energy-momentum tensor T_{-2} which commutes with j and T_j and which satisfies

$$T = T_j + T_{-2} \quad . \tag{4.94}$$

[59] See [17].
[60] We are going to closely follow the approach of [7], where the usufulness of the bosonization of the b, c system was discussed for the first time.

By looking at formula (4.75), we see that we can take T_{-2} to be the energy-momentum tensor associated to an auxiliary linear Fermi system with $\lambda_{-2} = 1$ and $Q_{-2} = -1$, consisting of a field $\eta(z)$ of conformal dimension $h = 1$ and of a field $\xi(z)$ of conformal dimension $h = 0$. In other words, to bosonize the b, c system, only the field j is needeed in the case of Fermi statistics, but we have to add the fields η, ξ in the case b, c were commuting fields.
In any case, to bosonize the b, c system, we recognize that if we impose

$$\begin{aligned} j(z) &= \epsilon \partial_z \phi(z) \\ \langle \phi(z)\phi(w) \rangle &\sim \epsilon \log(z-w) \quad , \end{aligned} \tag{4.95}$$

namely if we impose for j to be the conserved current associated to the translation invariance of a free boson ϕ, then we do recover (4.89). This suggests that we should be able to express the b, c CFT in terms of the CFT of a single boson ϕ[61]. The action that describes the bosonized current (4.95) is

$$S_\phi = -\frac{1}{4\pi} \int d^2z \sqrt{h} \Big(\epsilon \partial_z \phi \partial_{\bar{z}} \phi + \frac{1}{2} Q R \phi \Big) \quad , \tag{4.96}$$

where we added also the term with Q in order to reproduce the anomalous behaviour of $j = \epsilon \partial_z \phi$; indeed, formula (4.88) is recovered simply by the equation of motion of the action S_Q.
The last step needed to complete the bosonization of the b, c system is to find, within the theory determined by this new action, primary fields that can correspond to the old b, c fields.
We see that the primary field $: e^{q\phi(z)} :$ satisfies

$$: e^{k_1 \phi(z,\bar{z})} :: e^{k_2 \phi(w,\bar{w})} : = (z-w)^{\epsilon(k_1 k_2)} : e^{(k_1+k_2)\phi(w,\bar{w})} (1 + O(z-w)) : \quad , \tag{4.97}$$

and, so, in the case of Fermi statistics for b, c, namely for $\epsilon = 1$, we get

$$\begin{aligned} : e^{\phi(z,\bar{z})} :: e^{-\phi(w,\bar{w})} : &\sim \frac{1}{z-w} \sim c(z) b(w) \\ : e^{-\phi(z,\bar{z})} :: e^{\phi(w,\bar{w})} : &\sim \frac{1}{w-z} \sim b(w) c(z) \quad , \end{aligned} \tag{4.98}$$

which are precisely the OPE of (4.67). Thus, for the anticommuting b, c fields we guess the correspondence

$$c(z) \longleftrightarrow : e^{\phi(z)} : \qquad\qquad b(z) \longleftrightarrow : e^{-\phi(z)} : \quad .$$

In the case of commuting b, c fields, this guess is clearly wrong, because the OPE among $: e^{\phi(z,\bar{z})} :$ and $: e^{-\phi(w,\bar{w})} :$ has odd powers of $(z - w)$, meaning that these fields behave as Grassmann odd fields; to cook up commuting b, c fields, the fields $e^{\pm\phi(z)}$ have to be combined with other fermionic fields. We guess that the fermionic fields needed are precisely those coming from the T_{-2}'s theory, namely $\eta(z)$ and $\xi(z)$. Indeed, from

$$c(z) \longleftrightarrow : e^{\phi(z)} : \eta(z) \qquad\qquad b(z) \longleftrightarrow : e^{-\phi(z)} : \partial\xi(z) \quad ,$$

we recover the OPEs of (4.67) in the case of Bose statistics ($\epsilon = -1$).
In order to have a proper identification among conformal fields, it is not enough to recover the right OPEs among fields; one has also to check that their conformal properties do match.
The conformal dimensions and the charge of $: e^{q\phi(z)} :$ are determined by

$$\begin{aligned} T(z) : e^{q\phi(w)} : &= \frac{\epsilon q (q+Q)/2}{(z-w)^2} : e^{q\phi(w)} : + \frac{\partial_w : e^{q\phi(w)} :}{z-w} + \ldots \quad . \\ j(z) : e^{q\phi(w)} : &= \frac{q}{z-w} : e^{q\phi(z)} : + \ldots \quad ; \end{aligned} \tag{4.99}$$

and the charges of the η, ξ fields are clearly vanishing, because they do not depend on the field ϕ. With this information it is not difficult to check that our guesses (both of them, for both statistics) identify the b, c fields with conformal fields of the same conformal dimension and charge.
To sum up:

[61] This guess turns out to be completely true only in case b, c are anti-commuting. See later.

1. in case of Fermi statistics ($\epsilon = 1$), the fields b, c themselves can be bosonized simply as the exponential of $\phi(z)$ where the bosonic field $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ is described by the action (4.96), namely

$$\begin{aligned} b(z) =: e^{-\phi(z)} : \quad , \quad c(z) =: e^{\phi(z)} : \\ \phi(z)\phi(w) \sim \log(z-w) \quad ; \end{aligned} \tag{4.100}$$

2. in case of Bose statistics ($\epsilon = -1$), the "bosonization" of the b, c fields needs also the introduction of T_{-2} - namely the introduction of the η and ξ fields - in the system and we have

$$\begin{aligned} & b(z) =: e^{-\phi(z)} : \partial\xi(z) \quad , \quad c(z) =: e^{\phi(z)} : \eta(z) \\ & \phi(z)\phi(w) \sim -\log(z-w) \quad ; \\ & \eta(z)\xi(w) \sim \xi(z)\eta(w) \sim \frac{1}{z-w} \quad . \end{aligned} \tag{4.101}$$

4.12 The critical dimension from the CFT perspective

The starting point of string theory is the Polyakov action $S_{Poly}[X^\mu, h_{\alpha\beta}]$, which enjoys both local Weyl and diffeomorphism invariance on the worldsheet. The Faddeev-Popov procedure allows us to gauge fix the theory; in particular, we can do it locally so as to obtain a CFT on each patch.
The gauge fixed action consists of D copies of the free-boson-CFT (one for each X^μ) and of the $b, c-$ghost CFT; these theories don't talk with each other, so their central charges sum together and we get that the total central charge c^{tot} of the system is given by

$$c^{tot} \equiv c_{(\text{D free bosons})} + c_{(bc)} = Dc_{(1 \text{ free boson})} + \epsilon(1 - 3Q^2) = D - 26 \quad , \tag{4.102}$$

where we used the fact that the $bc-$ghost system is a first-order Lagrangian system characterized by $\epsilon = 1$, $\lambda = 2$ (so $Q = -3$).
From the perspective of the conformal field theory defined on each patch, a non-vanishing central charge is not a problem, given that the conformal transformations are here seen as global symmetries. From the perspective of the Polyakov action, a non-vanishing central charge really is a big problem, because it means that the Weyl invariance is anomalous. Indeed, from the relation

$$\langle T_{z\bar{z}} \rangle = -\frac{c^{tot}}{12} R \quad , \tag{4.103}$$

we see that, if $c^{tot} \neq 0$, then we are able to distinguish which gauge choice has been selected for fixing the Weyl invariance, because the Ricci scalar of the worldsheet does transform under a general Weyl transformation. For $c^{tot} \neq 0$ we have an observable (the expectation value of the energy-momentum tensor) that depends on which conformal gauge choice we used and our quantum theory is not Weyl invariant. This is not acceptable, because Weyl transformations are not symmetries, but gauge redundancies of the theory; so, if we want to obtain a quantum theory that has the same degrees as freedom of the classical one, gauge transformations cannot be anomalous. This leads us to impose the vanishing of the central charge

$$c^{tot} = D - 26 \overset{!}{=} 0 \quad , \tag{4.104}$$

and we then recover that the quantum bosonic string defines a consistent theory only if it is moving in a $26-$dimensional spacetime.
Here we can appreciate the importance of the presence of the $b, c-$ghost system because it is the latter that precisely fixes the central charge of the matter sector. On the other hand, the presence of fields with the wrong spin-statistics (like b and c, which are anticommuting fields with integer spin) raises the question of the physical state condition. But from the path integral quantization of gauge theories in QFT, we already know how to deal with this problem: the physical state condition is implemented by analyzing the BRST symmetry of the system.

4.13 BRST quantization

Here we are going to briefly introduce the basics about the BRST quantization in the Polyakov string. By following the approach of [17] we want to give the reader an operative introduction to the BRST quantization; for further details, [11] is a good reference.

We observe that, after fixing the flat gauge by means of the Faddeev Popov method, the gauge fixed action S_f

$$S_f \equiv S_{Poly}[X^\mu, \hat{h}_{\alpha\beta}; \Sigma_0] + S_g = \frac{1}{2\pi\alpha'}\int d^2z\Big(\partial X\cdot\bar{\partial}X + \alpha' b\bar{\partial}c + \alpha'\bar{b}\partial\bar{c}\Big) \tag{4.105}$$

enjoys a global, fermionic, residual symmetry. Let ϵ be a constant Grassmann odd parameter; then this symmetry is generated by the transformation (let's focus on the chiral part, the antichiral one is analogue)

$$\begin{aligned} \delta_\epsilon X^\mu(z) &= i\epsilon\partial X^\mu(z)c(z) \\ \delta_\epsilon c(z) &= i\epsilon c(z)\partial c(z) \\ \delta_\epsilon b(z) &= i\epsilon T^{tot}(z) \quad , \end{aligned} \tag{4.106}$$

where we denoted with T^{tot} the total energy-momentum tensor, $T^{tot} = T^X + T^{bc}$. This transformation is called *BRST symmetry*; its associated Noether charge is called *BRST charge* Q_B, so we can also rewrite (4.106) as

$$\begin{aligned} \delta_\epsilon X^\mu(z) &= \epsilon[Q_B, \partial X^\mu(z)] \\ \delta_\epsilon c(z) &= \epsilon\{Q_B, c(z)\} \\ \delta_\epsilon b(z) &= \epsilon\{Q_B, b(z)\} \quad . \end{aligned} \tag{4.107}$$

One can compute the Noether current j_B associated to the BRST symmetry to find that

$$j_B(z) =: c(z)\Big[T^X(z) + \frac{1}{2}T^{bc}(z)\Big] : \tag{4.108}$$

and this means that the Q_B satisfying (4.107) is given by (here $T^{tot} = T^X + T^{bc}$)

$$\begin{aligned} Q_B &= \oint \frac{dz}{2\pi i} c(z)\Big[T^{tot}(z) - :\partial(c(z)b(z)):\Big] \quad = \\ &\oint \frac{dz}{2\pi i} : c(z)\Big[T^X(z) + \frac{1}{2}T^{bc}(z)\Big] : \quad = \\ &= \sum_{m\in\mathbb{Z}} : c_{-m}\Big(L^X_m + \frac{1}{2}L^{bc}_m\Big) : \\ L^X_n &= \sum_{m=-\infty}^{+\infty}\Big(\frac{1}{2} : \alpha_{n-m}\cdot\alpha_m :\Big) \\ L^{bc}_n &= \sum_{m=-\infty}^{+\infty}\Big((n+m) : b_{n-m}c_m :\Big) \quad , \end{aligned} \tag{4.109}$$

from which we can see that

$$Q_B^\dagger = Q_B \quad . \tag{4.110}$$

Another important feature of the BRST charge is that it must be nilpotent, i.e. it has to satisfy

$$Q_B^2 = 0 \quad . \tag{4.111}$$

But, in the quantum theory, the evaluation of $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\}$ is complicated by the normal ordering and one would find

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = \frac{1}{2}\sum_{m,n=-\infty}^{+\infty}\big([L^{tot}_m, L^{tot}_n] - (m-n)L^{tot}_{m+n}\big)c_{-m}c_{-n} \quad , \tag{4.112}$$

which implies that $Q_B^2 = 0$ if and only if the total anomaly of the matter-plus-ghost system vanishes (compare (4.112) with (4.37)). In other words, consistency of the BRST symmetry is equivalent to the abscence of the total Weyl anomaly and, so, to the critical dimension condition.
Actually, now we can gain another interpretation of the reason why we need the critical dimension in string theory. As always, we can add a total derivative to the Noether current of a symmetry without affecting the (physical) value of the associated charge. To the definition (4.108) of j_B, we add, by hand, the total derivative $\frac{3}{2}\partial^2 c$ to obtain[62]

$$j_B(z) \equiv: c(z)\left[T^X(z) + \frac{1}{2}T^{bc}(z)\right]: + \frac{3}{2}\partial^2 c(z) \quad ; \tag{4.113}$$

with this definition, the OPE of the BRST-current with the total energy-momentum tensor of the system is

$$T^{tot}(z) j_B(w) = \frac{c^{tot}/2}{(z-w)^4}c(w) + \frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{z-w} + \ldots \quad , \tag{4.114}$$

from which we can see that j_B is a tensor if and only if the critical dimension condition $D = 26$ is satisfied. But we must require for j_B to be a tensor under conformal transformations, because the latter are the transition functions from one patch to the other one. In fact, if j_B didn't transform well under such transformations, it would imply that the definition of Q_B depends on the patch and, with it, also the meaning of the physical state condition defined by Q_B! Said another way: if Q_B defines the notion of physical state and if we want for this notion not to depend on the local features of the worldsheet, then we have to impose the vanishing of the total central charge of the system.
The reason why the BRST symmetry can be used to define the notion of physical state is implicit in the definition of the BRST transformation: it is a remnant of the local gauge symmetry[63]. Physical states must be gauge invariant, so it is natural to ask for them to be also Q_B invariant. Then, a necessary condition for a state to be physical is

$$Q_B \left|\text{phys}\right\rangle = 0 \quad , \tag{4.115}$$

namely it has to be *Q-closed*.
Classically, the Virasoro constraints ($L_n^X = 0$) have to be imposed by hand as the equation of motion of the worldsheet's metric $h_{\alpha\beta}$; in the modern approach to quantization, locally we fix the gauge by fixing the metric and the remnant of the gauge symmetry gives at the quantum level the constraint (4.115), which has to be thought of as the gauge-fixed analogue of the Virasoro constraints.
Obviously, if a state automatically satisfies (4.115), then it cannot be physical, because, in that case, formula (4.115) wouldn't impose any constraints. Because of the nilpotency of Q_B, there exist a lot of states which trivially satisfy (4.115); they are given by

$$\left|\chi\right\rangle = Q_B \left|\psi\right\rangle \qquad \text{for } \left|\psi\right\rangle \text{ arbitrary} \tag{4.116}$$

and they are called *null* states, because they are orthogonal to all physical states and to themselves[64]; indeed

$$\begin{aligned} \langle\text{phys}|\chi\rangle =& \langle\text{phys}|Q_B|\psi\rangle = \langle Q_B^\dagger\text{phys}|\psi\rangle = \langle Q_B\text{phys}|\psi\rangle = 0 \\ ||\left|\chi\right\rangle||^2 = \langle\chi|\chi\rangle =& \langle\psi|Q_B^2|\psi\rangle = 0 \quad . \end{aligned} \tag{4.117}$$

This means that we can add a null state to a physical state $|\text{phys}\rangle$, without changing the (physical) values of $\langle\text{phys'}|\text{phys}\rangle$. In other words, to define the Hilbert space containing the physical states of our theory,

[62] The formula (4.108) defines j_B only up to a total derivative which must be of dimension one and ghost number one. The most general form is then

$$j_B(z) =: c(z)\left[T^X(z) + \frac{1}{2}T^{bc}(z)\right]: + k\partial^2 c(z) \quad ,$$

and requiring j_B to be a conformal field of conformal dimension $h = 1$ we get $k = 3/2$. One can show that the total derivative that we have just added has no effect on flat correlation functions, but it ensures the conservation of the BRST-current on curved worldsheets. So, $j_B(z)$ has to be defined as in (4.113).
[63] As it can be understood by looking at (4.106): the BRST transformation of X^μ is just the conformal transformation with (real bosonic) parameter $i\epsilon c$.
[64] In particular, they are zero norm states.

we have to divide the set of the Q-closed states by the set of the null states - which are also called, for reasons that are apparent in (4.116), *Q-exact* states. We denote this Hilbert space as $\mathcal{H}_B$:

$$\mathcal{H}_B \equiv \frac{\text{closed states}}{\text{exact states}} \quad . \tag{4.118}$$

The physical states of the theory are given by the equivalence classes just defined; one can show (see [11]) that every non-trivial equivalence class has a representative which essentially is given by a highest weight state of the Virasoro algebra with L_0 eigenvalue $+1$, so we recover the correct constraints defining the Hilbert space of the old covariant approach to quantization. To be more precise, every equivalence class has a representative $|\mathcal{V}\rangle$ of the form

$$|\mathcal{V}\rangle = \phi(0)\,|0\rangle_X \otimes c_1\,|0\rangle_{bc} = \phi(0)\,|0\rangle_X \otimes c(0)\,|0\rangle_{bc} = |\phi\rangle \otimes |c\rangle \quad , \tag{4.119}$$

where $|0\rangle_X$ and $|0\rangle_{bc}$ are respectively the $PSL(2,\mathbb{C})$ invariant vacua of the matter and of the ghost CFTs and $|\phi\rangle$ is a highest weight state of the Virasoro algebra of the matter sector with L_0^X eigenvalue $+1$ (or eigenvalue 0, in the only case $|\phi\rangle = |0\rangle_X$). By requiring the state of the form (4.119) to be Q_B invariant, we find that

$$0 = Q_B\,|\mathcal{V}\rangle = Q_B\big(\,|\phi\rangle \otimes c_1\,|0\rangle_{bc}\big) = \sum_{n=0}^{+\infty} c_{-n}(L_n^X - \delta_{n,0})\big(\,|\phi\rangle \otimes c_1\,|0\rangle_{bc}\big) \quad , \tag{4.120}$$

where the term $-\delta_{n,0}$ arises because of the c ghost appearing in (4.119): in the language of the BRST quantization, the normal ordering constant a for L_0^X is fixed to be $a = -1$ by the presence of the ghost c in (4.119).
Formula (4.119) suggests to take, as the vacuum for the ghost sector, the state $|c\rangle$ instead of $|0\rangle_{bc}$; to understand the reason why this is the case, we have to turn back to the definition of the $PSL(2,\mathbb{C})$ vacuum of the ghost CFT.
As explained in (4.43), the $PSL(2,\mathbb{C})$ vacuum $|0\rangle_{bc}$ is defined by

$$\begin{aligned} b_n\,|0\rangle_{bc} &= 0 \quad \text{for } n \geq -1 \\ c_n\,|0\rangle_{bc} &= 0 \quad \text{for } n \geq 2 \quad . \end{aligned} \tag{4.121}$$

This means that $|0\rangle_{bc}$, although a highest weight state of the Virasoro algebra, is not a highest weight state for the b, c algebra, i.e. it is not annihilated by all the negative frequency modes of b and c since

$$\begin{aligned} c(0)\,|0\rangle_{bc} =& c_1\,|0\rangle_{bc} \neq 0 \\ c(0)\partial c(0)\,|0\rangle_{bc} =& c_1 c_0\,|0\rangle_{bc} \neq 0 \quad . \end{aligned} \tag{4.122}$$

These two states are degenerate with respect to the ghost energy

$$L_0^{bc} = \sum_{m\geq 1}^{+\infty} m\big(b_{-m}c_m + c_{-m}b_m\big) - 1 \quad , \tag{4.123}$$

and later we will see that there are no other states with the same energy of these two. It is not $|0\rangle_{bc}$ the lowest energy state of the Hilbert space of $b - c$'s CFT; the two states of (4.122) are the lowest energy states, because they are such that only positive frequency states propagate forward in time (outward from the origin):

$$\begin{aligned} b_n\,|\psi\rangle &= 0 \quad \text{for } n \geq 1 \\ c_n\,|\psi\rangle &= 0 \quad \text{for } n \geq 1 \quad , \end{aligned} \tag{4.124}$$

where here we denoted with $|\psi\rangle$ anyone of the states (4.122). In other words, it is only with respect to the state $|\psi\rangle$ that we recover the interpretation of annihilation operators for the b_n's and c_n's with $n > 0$ and, thus, the ground state of the ghost sector is $|\psi\rangle$. For $|\psi\rangle = |c\rangle$ we have

$$\begin{aligned} b_n\,|c\rangle = b_n(c_1\,|0\rangle_{bc}) &= 0 \quad \text{for } n \geq 0 \\ c_n\,|c\rangle = c_n(c_1\,|0\rangle_{bc}) &= 0 \quad \text{for } n \geq 1 \quad , \end{aligned} \tag{4.125}$$

whereas, for $|\psi\rangle = |c\partial c\rangle$ we get

$$\begin{aligned} b_n\,|c\partial c\rangle &= b_n(c_1 c_0\,|0\rangle_{bc}) = 0 \quad \text{for } n \geq 1 \\ c_n\,|c\partial c\rangle &= c_n(c_1 c_0\,|0\rangle_{bc}) = 0 \quad \text{for } n \geq 0 \quad , \end{aligned} \tag{4.126}$$

and we notice that we can distinguish the states $|c\rangle$ and $|c\partial c\rangle$ by looking at the zero modes of the b and c fields: the first one is annihilated by b_0 and the second one by c_0.
This is nothing strange. The zero modes b_0 and c_0 don't appear in the expression (4.123) and so they commute with the Hamiltonian of the $b-c$ ghost system. This means that the algebra

$$b_0^2 = 0 \quad c_0^2 = 0 \quad \{b_0, c_0\} = 1 \tag{4.127}$$

can be used to define degenerate states for every energy level of the $b-c$ CFT. Indeed, it is not difficult so show that we cannot represent this algebra with only one state, and we need at least two of them. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the states defining the representation of the $b_0 - c_0$ algebra, i.e.

$$c_0\,|\uparrow\rangle = \alpha\,|\uparrow\rangle \quad , \quad b_0\,|\downarrow\rangle = \beta\,|\downarrow\rangle \quad \text{for some } \alpha, \beta \in \mathbb{C} \quad ;$$

because of the first two relations of (4.127), we get $\alpha = \beta = 0$ and we have to check whether the third relation of (4.127) is consistent or not with these conditions. With $\{b_0, c_0\} = 1$ we immediately find that $\langle\downarrow\,|\,\downarrow\rangle = \langle\uparrow\,|\,\uparrow\rangle = 0$. But if the algebra is represented with only one state, namely if $|\downarrow\rangle = |\uparrow\rangle$, then

$$|\downarrow\rangle = |\uparrow\rangle = (c_0 b_0 + b_0 c_0)\,|\downarrow\rangle = 0 \tag{4.128}$$

and thus with only one state it is not possible to implement the $b_0 - c_0$ algebra (4.127).
Instead, if $|\downarrow\rangle \neq |\uparrow\rangle$, then we obtain

$$\begin{aligned} &c_0\,|\uparrow\rangle = 0 \quad\quad , \quad b_0\,|\downarrow\rangle = 0 \\ &b_0\,|\uparrow\rangle = \gamma\,|\downarrow\rangle \quad , \quad c_0\,|\downarrow\rangle = \delta\,|\uparrow\rangle \quad \text{for } \gamma, \delta \in \mathbb{C} \quad , \end{aligned} \tag{4.129}$$

where the equations of the last line come from the relations $b_0^2 = c_0^2 = 0$. Now we can apply the third property of (4.127) and we simply obtain that $\gamma\delta = 1$, because

$$|\downarrow\rangle = (b_0 c_0 + c_0 b_0)\,|\downarrow\rangle = b_0 c_0\,|\downarrow\rangle = \gamma\delta\,|\downarrow\rangle \quad .$$

For example, we can choose

$$\begin{aligned} |\uparrow\rangle &\equiv c_0 c_1\,|0\rangle_{bc} = -\,|c\partial c\rangle \\ |\downarrow\rangle &\equiv c_1\,|0\rangle_{bc} = |c\rangle \quad , \end{aligned} \tag{4.130}$$

and the algebra (4.129) is satisfied with $\gamma = \delta = 1$. We are left with the possibility of rescaling both $c_0 c_1\,|0\rangle_{bc}$ and $c_1\,|0\rangle_{bc}$ with the same constant, but we can fix this freedom by normalizing the non-vanishing value of $\langle\uparrow\,|\,\downarrow\rangle$ to be 1 and we will always work with

$$\langle\uparrow\,|\,\downarrow\rangle = \langle 0|_{bc}\, c_{-1} c_0 c_1\,|0\rangle_{bc} \equiv 1 \quad . \tag{4.131}$$

We have thus found that the algebra of the zero modes of the b, c ghost system is responsible for the double degeneracy of each energy level of the system. In particular, the ground state of the b, c CFT can be described both with $|\uparrow\rangle = -\,|c\partial c\rangle$ and with $|\downarrow\rangle = |c\rangle$.
Instead, when the $b-c$ CFT is thought of as part of string theory, this degeneracy of states doesn't survive because there is the Q_B charge in the game. If we require the ground state of the string

$$|0\rangle_{tot} = |0\rangle_X \otimes |\psi\rangle \quad , \quad |\psi\rangle \in \{|c\rangle\,, |c\partial c\rangle\}$$

to be BRST-closed, we should find the right Virasoro constraints; as we saw in (4.120), this can be done for $|\psi\rangle = |c\rangle$. Instead, for $|\psi\rangle = |c\partial c\rangle$, we wouldn't find the Virasoro constraints; indeed, by looking at the BRST transformations of the various fields:

$$\begin{aligned}
[Q_B, X^\mu(z)] &= c(z)\partial X^\mu(z) \\
[Q_B, T^{tot}(z)] &= \frac{1}{12}(D-26)\partial^3 c(z) \\
\{Q_B, c(z)\} &= c(z)\partial c(z) \\
\{Q_B, b(z)\} &= T^{tot}(z) \\
[Q_B, j(z)] &= -j_B(z)
\end{aligned} \tag{4.132}$$

we can see that T^{tot}, j_B and $c\partial c$ are BRST-exact operators. In particular, this means that $|c\partial c\rangle$ is a null state and, thus, it cannot be the representative state defining the equivalence class of a non-trivial physical state: requiring for it to be BRST-closed wouldn't give us any physical information.

The upshot is that the physical states of the closed bosonic string theory are defined - up to BRST-exact states - by the conditions

$$\begin{aligned}
Q_B\,|\mathrm{phys}\rangle &= 0 \\
b_n\,|\mathrm{phys}\rangle &= 0 \qquad \text{for } n \geq 0 \\
\bar{b}_n\,|\mathrm{phys}\rangle &= 0 \qquad \text{for } n \geq 0 \\
c_n\,|\mathrm{phys}\rangle &= 0 \qquad \text{for } n \geq 1 \\
\bar{c}_n\,|\mathrm{phys}\rangle &= 0 \qquad \text{for } n \geq 1 \quad .
\end{aligned} \tag{4.133}$$

We saw that these requirements give the right Virasoro constraints on the matter sectors; we now note that they also give the level matching condition. In fact, $\{Q_B, b(z)\} = T^{tot}(z)$ is equivalent to

$$\{Q_B, b_n\} = L_n^{tot} \quad \text{for } n \in \mathbb{Z} \tag{4.134}$$

and this implies that

$$\left(L_0^X - \bar{L}_0^X\right)|\mathrm{phys}\rangle = \left(L_0^{tot} - \bar{L}_0^{tot}\right)|\mathrm{phys}\rangle = \{Q_B, b_0 - \bar{b}_0\}\,|\mathrm{phys}\rangle = 0 \quad , \tag{4.135}$$

because of the relations (4.133). We like to mention that the Hilbert space defined by (4.133) and (4.118) doesn't contain the unphysical oscillators of X^μ and the unphysical b, c oscillations. Essentially, the condition of Q_B invariance removes one set of unphysical X^μ oscillators and one set of ghost oscillators, whereas the equivalence relation removes the other set of unphysical X^μ oscillators and the other set of ghost oscillators; for more details, we refer to [11], where it is also shown that the Hilbert space defined by (4.133) contains only positive norm states and that the conditions (4.133) allow us to immediately recover the spectrum of the string that we have already studied in the light-cone quantization. Hence the equivalence of the Hilbert space given by the BRST quantization to that one coming from the canonical quantization.

4.14 Vertex operators

The operators $\mathcal{V}(z,\bar{z})$ that create the states (4.133) from the $PSL(2,\mathbb{C})$ vacuum $|0\rangle_{tot} = |0\rangle_X \otimes |0\rangle_{bc}$ of the theory can be always chosen, modulo a BRST-exact operator, to be of the form (see (4.119))

$$\mathcal{V}(z,\bar{z}) = c(z)\bar{c}(\bar{z})\phi(z,\bar{z}) \quad , \tag{4.136}$$

where $\phi(z,\bar{z})$ is a primary operator of the matter CFT of conformal dimension $h = \bar{h} = 1$ that is called a *vertex operator*; the operator $\mathcal{V}(z,\bar{z})$ has vanishing conformal dimensions and it is called a *fixed vertex operator*.
An important example of a vertex operator is the primary field $:e^{ik\cdot X(z,\bar{z})}:$ which acts on the vacuum to create the state

$$|k\rangle = \lim_{z,\bar{z}\to 0} :e^{ik\cdot X(z,\bar{z})}:|0\rangle_X\,, \quad h = \bar{h} = \frac{\alpha'}{4}k^2 \quad . \tag{4.137}$$

In order for it to be a vertex operator, we have to impose $h = \bar{h} = 1$ and this gives the mass-shell condition for the lowest-lying state (the tachyon). This is nothing strange, because we saw in (2.36) that the vacuum for the oscillations of the string has to be tensored with the eigenvector $|p^\mu\rangle$ of the momentum operator of the centre of mass, and we can show that the above defined $|k\rangle$ is precisely the momentum eigenstate with momentum k^μ of the centre of mass operator:

$$
\begin{aligned}
\sqrt{\frac{\alpha'}{2}}\hat{p}^\mu\,|k\rangle =&\alpha_0^\mu\,|k\rangle = \lim_{z,\bar{z}\to 0}\alpha_0^\mu : e^{ik\cdot X(z,\bar{z})} : |0\rangle_X = \\
=& \lim_{z,\bar{z}\to 0}\left[\alpha_0^\mu\, , : e^{ik\cdot X(z,\bar{z})} : \right]|0\rangle_X = \\
=& \lim_{z,\bar{z}\to 0}\sqrt{\frac{2}{\alpha'}}i\oint_{C_z}\frac{dw}{2\pi i}\partial X^\mu(w) : e^{ik\cdot X(z,\bar{z})} : |0\rangle_X = \\
=& \lim_{z,\bar{z}\to 0}\sqrt{\frac{2}{\alpha'}}i\oint_{C_z}\frac{dw}{2\pi i}\Big(-\frac{i}{2}\alpha' k^\mu\frac{1}{w-z}\Big) : e^{ik\cdot X(z,\bar{z})} : |0\rangle_X = \\
=&\sqrt{\frac{\alpha'}{2}}k^\mu\,|k\rangle \quad ,
\end{aligned}
\tag{4.138}
$$

where we used formula (4.61) and the OPE appearing in the first line of (4.62). Thus, if we want to describe a string (its quantum vacuum) propagating with momentum p^μ, we have to insert $: e^{ik\cdot X(z,\bar{z})} :$ at the origin $z = \bar{z} = 0$ of the complex plane, with $k^\mu = p^\mu$.
If we want to describe also the quantum excitations of the string, $: e^{ik\cdot X(z,\bar{z})} :$ alone is not enough. For instance, the states of the first level are created with momentum k^μ and polarization tensor $\xi_{\mu\nu}$ by the following vertex operator:

$$
|k,\xi\rangle = \lim_{z,\bar{z}\to 0}\xi_{\mu\nu} : \partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z})e^{ik\cdot X(z,\bar{z})} : |0\rangle_X \quad . \tag{4.139}
$$

One can easily compute the OPE of this operator with the energy-momentum tensor of the matter sector to find that it is primary if and only if $k^\mu\xi_{\mu\nu} = 0$, and so we have just recovered the transversality constraint of momentum and polarization for massless particles. The masslessness condition arises by demanding the conformal dimensions $h = \bar{h} = 1 + \frac{\alpha'}{4}k^2$ of this vertex operator to be 1.
The reader should notice again that in the discussion of these two examples, all the physical information (e.g. the transversality constraint and the on-shellness condition) come from the fact that we have to require the vertex operator ϕ of formula (4.136) to be a primary operator of dimensions $h = \bar{h} = 1$; we didn't need to work with the ghost insertions $c\bar{c}$ and, thus, we wonder whether it is possible to rewrite the operator defining the physical state without involving them. As we have already mentioned, a physical state must be gauge invariant so the corresponding operator must have vanishing conformal dimensions. Obviously, this is the case for $\mathcal{V}$, because the $c\bar{c}$ contribution has conformal dimensions $h = \bar{h} = -1$ and if we want to remove the latter, we have to replace it with an object of the same conformal dimensions. But this object cannot be a conformal operator of the matter sector, because the X-theory is a unitary CFT (it consists of operators of non negative conformal dimensions); thus, the only object that can replace the ghosts $c\bar{c}$ is the measure $dzd\bar{z}$ and we end up with

$$
V = \int_\Sigma dzd\bar{z}\phi(z,\bar{z}) \quad , \tag{4.140}
$$

which is the gauge fixed version of the *integrated vertex operator*:

$$
V = \int_\Sigma \sqrt{h}2d\sigma^1 d\sigma^2\phi(\sigma^1,\sigma^2) \quad . \tag{4.141}
$$

One of the advantages of rewriting the vertex operators in their integrated form is that we can immediately see that the vertex operator $\phi(z,\bar{z})$ is defined up to total derivatives:

$$
\phi(z,\bar{z}) \sim \phi(z,\bar{z}) + d\phi'(z,\bar{z}) \quad , \tag{4.142}
$$

where $\phi'(z,\bar{z})$ is another operator.
Obviously, this is due to the fact that the state $|\mathcal{V}\rangle$ of formula (4.119) is only a representative of its equivalence class

$$
|\mathcal{V}\rangle \sim |\mathcal{V}\rangle + Q_B\,|\mathcal{V}'\rangle \quad . \tag{4.143}
$$

By looking at formulae (6.61) and (6.62), we can appreciate the close similarity between our BRST-charge and the differential d, seen as cohomological operators.
Given that derivatives of fields are examples of descendant fields, we guess that defining physical states up to BRST-exact states is translated, by the operator-state correspondence, into the following equivalence relation of operators

$$\phi(z,\bar{z}) \sim \phi(z,\bar{z}) + \text{descendant fields} \quad , \quad \phi(z,\bar{z}) \text{ primary field of } h = \bar{h} = 1 \quad .$$

For further details, we refer to [17]; adding a Q_B exact state to a physical one doesn't change the amplitudes involving the latter and, correspondingly, we can show that descendant fields don't contribute to the amplitudes.

We like to conclude this section by stressing the fact that the fundamental version of the vertex operator is the fixed one, because it derives from the state-operator correspondence. Indeed, the $c\bar{c}$ insertions appearing in (4.136) cannot be neglected because they give the right information about the ghost charge of the vacuum, which has to be taken into account, if we want to obtain non-vanishing amplitudes. For example, the presence of $c\bar{c}$ has led us to formula (4.131), from which we can read off that the physical vacuum of closed string theory at the tree level has ghost charge 6:

$$\langle 0^{tot}| \bar{c}_{-1}\bar{c}_0\bar{c}_1 c_{-1}c_0c_1 \, |0^{tot}\rangle \neq 0 \quad . \tag{4.144}$$

In the next chapter, we will see the geometric interpretation of this result and we will see in which sense the integrated version of the vertex operator is useful.

5 Global aspects

The Faddeev-Popov gauge fixing procedure presented in chapter 3 didn't take into account the complications due to the global aspects of the worldsheet. There are gauge transformations (those ones which belong to the CKG) that are not fixed by the choice of the metric and there are metrics that are not gauge (i.e. conformally) equivalent to one another (because there are the moduli in the game, parameters that specify the complex structure of the worldsheet).
Later on, we will be interested in calculating an amplitude at one loop, so now we are going to introduce the concept of the moduli parameters by focusing our attention to the case of the toroidal worldsheet Σ_1.

The main references used for this chapter are [11, 6, 18].

5.1 The moduli space of the torus

As a topological space, the torus is constructed by introducing the following equivalence relations in the $z-$complex plane ($z = \sigma^1 + i\sigma^2$):

$$\begin{aligned} z \cong& z + 2\pi \\ z \cong& z + 2\pi\tau \quad , \end{aligned} \tag{5.1}$$

so that one can think of the torus as the cylinder of circumference 2π and length $2\pi\tau_2$ with the ends rotated by an angle of $2\pi\tau_1$ and then sewn together. The region inside the parallelogram of Figure 10 is called *fundamental domain* of the torus and the complex number $\tau = \tau_1 + i\tau_2$ is called *Teichmüller parameter*.
By means of this construction, it is clear that one can always turn this topolological space into a flat Rie-

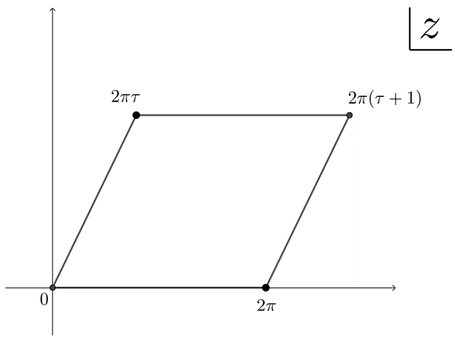

Figure 10
A fundamental region for the torus parametrized by z.

mannian manifold, because the flat Euclidean metric defined on the $z-$complex plane naturally induces the flat metric on the quotient space defined by the relations (5.1), the latter becoming the transition functions on the overlap regions. It is common, in the physics literature (see [11] for example), to cover the torus only with a single coordinate patch which is a little larger than the fundamental region; then, the periodicity conditions (5.1) are precisely the transition functions on the overlap between the opposite edges of the patch. Clearly, such a patch cannot be a chart in the mathematical sense of the word, because it is not injective; the torus is a compact topological space, it cannot be covered with only one chart. For the moment[65], the important thing is that the transition functions of the torus are nothing but translations, as in (5.1): the latter are isometries of the flat Euclidean metric, so it is really possible to equip the torus with the flat Euclidean metric.

[65] At the end of this chapter, the torus will be covered with two coordinate patches (z and u): they will be two cylinders with transition functions corresponding to the periodicity condition in the second line of (5.1), namely $z \cong u$ and $z \cong u + 2\pi\tau$.

In the previous chapters, we showed that it is always possible - locally - to bring the metric of the worldsheet into the flat form by means of the combined action of diffeomorphisms and Weyl transformations. Now we have just seen that, on the torus, this can be achieved globally...but we have to be careful, here. To be more precise, we have just shown that it is always possible to globally reach the flat metric on a torus of a given Teichmüller parameter τ (equivalently, on a given torus defined by the periodicity conditions/transition functions (5.1)). But, in the path integral approach to quantization, we have to sum over all the possible metrics that we can introduce on the *same* worldsheet[66]. Then it is intuitive that, in order to perform the path integral over the metrics of the torus, we have to find a way to remove the τ dependence of the periodicity conditions (5.1), so as to obtain similar transition functions for all the tori. This can be easily done by defining a new coordinate z' by

$$z = \operatorname{Re} z' + \tau \operatorname{Im} z' \quad , \tag{5.2}$$

so that the periodicity conditions now read as

$$\begin{aligned} z' &\cong z' + 2\pi \\ z' &\cong z' + 2\pi i \quad . \end{aligned} \tag{5.3}$$

Please note that, with this choice of coordinates, the flat metric $ds^2 = dzd\bar{z}$ becomes $\tau-$dependent, namely

$$ds^2 = dzd\bar{z} = |d \operatorname{Re} z' + \tau d \operatorname{Im} z'|^2 \quad . \tag{5.4}$$

This is very interesting. We can say that, by means of the combined action of Weyl and diffeomorphism transformations, we can always reach one of the following two *equivalent* descriptions of the *globally flat* torus.

- One can work with the metric $ds^2 = dzd\bar{z}$ and hide the $\tau-$dependence in the transition functions/periodicity conditions. The prize that we have to pay if we want to work with the simple $ds^2 = dzd\bar{z}$ is to admit that we are working with a family of tori whose fundamental regions are parametrized by τ. We can assume $\tau_2 > 0$, because in (5.1) we can take τ and $-\tau$ without changing the torus[67]. The upper complex plane $\operatorname{Im}\tau > 0$ is called *Teichmüller* space and it represents the family of our tori.
- One can fix the fundamental region of the torus to $[0, 2\pi] \times [0, 2\pi i]$ (as illustrated in Figure 11) and work with a $\tau-$dependent metric $ds^2 = |d \operatorname{Re} z' + \tau d \operatorname{Im} z'|^2$.

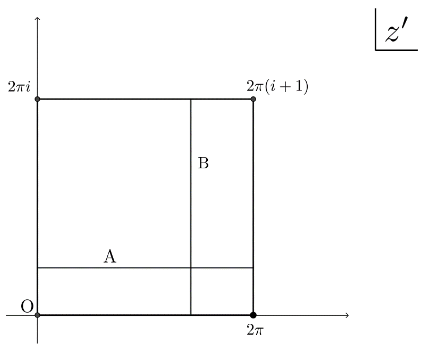

Figure 11
A fundamental region for the torus parametrized by z'.

In the path integral approach it is convenient to adopt the second description, according to which the integration over the metrics should reduce - after locally fixing the flat metric (5.4) - to two ordinary

[66] Soon it will be clear in which sense two worldsheets can be considered equivalent.
[67] One can also see from (5.4) that τ and $\bar{\tau}$ define the same metric.

integrals, over the real and imaginary part (τ_1 and τ_2) of the Teichmüller parameter. On the other hand, we will exploit the other description to understand over which integration domain we have to integrate τ_1 and τ_2 and now we explain how we are able to get this information.
To perform the integration over the metrics, we would like to identify a *gauge slice* on the space $\mathcal{G}_g$ of all the possible metrics on a given worldsheet of genus g, namely a choice of one configuration from each set of (diffeomorphism×Weyl)-inequivalent metrics (see Figure 12). Clearly, as first step, we have to understand

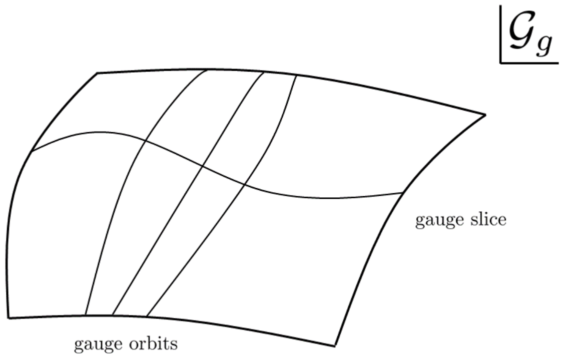

Figure 12
A schematic representation of the space of all the possible metrics of Σ_g.
You can move along the gauge orbits by performing a diff×Weyl transformation.
You can move along the gauge slice by changing the moduli of the metric.

when two metrics on the same worldsheets can be considered (diffeomorphism×Weyl)-inequivalent. As we know, from a local point of view, this question is trivial. From a global perspective, instead, it is difficult to answer. Fortunately, we can resort to a very natural 1 : 1 correspondence between a *Riemann surface* (namely a complex manifolds of real dimension 2) and Riemannian manifolds of the same (real) dimension defined up to (diffeomorphism×Weyl) transformations. Of course, a Riemannian manifold is already defined up to diffeomorphisms, so we reformulate the correspondence as

$$\text{Riemannian manifolds mod Weyl} \overset{1:1}{\longleftrightarrow} \text{Riemann surfaces} \quad , \tag{5.5}$$

where we left as understood that the Weyl transformations over the Riemannian manifold are globally defined and that both the Riemannian manifold and the Riemann surface are oriented 2−real dimensional manifolds.

Proof. Let's start with a (2−dimensional, real and oriented) Riemannian manifold (M, ds^2) and let's suppose for it to be covered with N coordinate patches. From our discussion of conformal gauge, we know that we can find in each coordinate patch a coordinate z_m ($m \in \{1, ..., N\}$) such that $ds^2 \propto dz_m d\bar{z}_m$. In the overlapping region between the m^{th} and n^{th} patches, we then have $dz_m d\bar{z}_m \propto dz_n d\bar{z}_n$ and the transition function must be holomorphic or antiholomorphic. But an antiholomorphic transition function would change the orientation of the manifold and this is not possible, because the latter is, by hypothesis, oriented. Now we could perform a Weyl transformation on ds^2: given that the Weyl rescaling doesn't touch the coordinates, we would recover the same set of holomorphic transition functions and thus the same Riemann surface.
Let's start with a 2−real dimensional, oriented complex manifold M and let's suppose for it to be covered with N coordinate patches. The transition functions are, by definition of a complex manifold, holomorphc functions on the overlapping regions of neighbourhood patches[68]. To turn

[68] A complex manifold is defined by functions which must be holomorphic on the overlapping regions, not necessarily everywhere. Take, for example, the Riemann sphere $\mathbb{S}^2$. It can be covered with the u and z patches that we have already mentioned and the transition function can be taken to be $u = -1/z$. Naively, one would say that this is not a holomorphic function, because it consists of a negative power of the coordinate z. But we have to remember that this transition function is defined only in the intersection of the z and u patches, which is the sphere without the South pole $z = \infty$ and the North pole $z = 0$; in this region it is indeed holomorphic, because $\partial_{\bar{z}}(1/z) = 0$.

M into a Riemannian manifold, we have to introduce a metric on it. But we can always define the flat metric $dz_m d\bar{z}_m$ on each patch and then, to obtain a globally defined and continuous metric, we can use the partition of unity. If we call ds^2 the metric that we have just built, we have to show that both (M, ds) and $(M, e^{2f}ds)$ come from the same complex manifold M. But this is clear, since a Weyl transformation doesn't touch the transition functions among patches. □

We want to persuade the reader about the identification (5.5), by sketching another proof, which is more intuitive; on the other hand, it requires a little of familiarity with the notion of complex manifold, so we will be more sloppy[69]. In the two dimensional case, saying that a manifold has holomorphic transition functions is equivalent to say that it can be equipped with a complex structure J, namely with a (1,1)-tensor that squares to the identity $\mathbb{I}$ and that, in particular, can be locally thought[70] of as a "rotation of 90°" on each tangent space of the manifold, in the precise sense that it acts on the tangent space basis vectors ∂_{σ^1} and ∂_{σ^2} according to

$$\begin{aligned} J\Big(\frac{\partial}{\partial\sigma^1}\Big) =& \frac{\partial}{\partial\sigma^2} \\ J\Big(\frac{\partial}{\partial\sigma^2}\Big) =& -\frac{\partial}{\partial\sigma^1} \quad . \end{aligned} \tag{5.6}$$

These equations imply that, up to a constant, one can write $J^\alpha_\beta \sim \sqrt{h}\epsilon_{\gamma\beta}h^{\alpha\gamma}$, where we have introduced the square root of the determinant of the metric in order to transform the $\epsilon-$object into a tensor and we have then raised one index so as to obtain a tensor that maps vectors into vectors. The constant is fixed by the $J^2 = -\mathbb{I}$ condition and we get

$$J^\alpha_\beta = \sqrt{h}h^{\alpha\gamma}\epsilon_{\gamma\beta} \quad . \tag{5.7}$$

We immediately note that the complex structure is automatically Weyl-invariant (in two dimensions), so we are essentially done, the rest of the proof being obvious.

Now that we have the identification (5.5) at our disposal, we immediately understand that, for our purpose, two tori have to be considered the *same* if and only if they are the *same complex manifold*, namely if they are biholomorphic. We could tackle this problem in a rigorous mathematical way and we would discover that two Teichmüller parameters τ and $\hat{\tau}$ define biholomorphic tori if and only if they are related by a $PSL(2;\mathbb{C})$ transformation, namely by the transformation

$$\tau = \frac{a\tau + b}{c\tau + d} \tag{5.8}$$

for some integer numbers $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$. In fact, if we start with the torus defined by formulae (5.4) and (5.3), that is with the torus

$$\begin{aligned} z' =& \sigma'^1 + i\sigma'^2 \\ z' \cong& z' + 2\pi \\ z' \cong& z' + 2\pi i \\ ds^2 =& |d\,\mathrm{Re}\, z' + \tau d\,\mathrm{Im}\, z'|^2 = |d\sigma'^1 + \tau d\sigma'^2|^2 \quad , \end{aligned} \tag{5.9}$$

then we can define the coordinates

$$\begin{pmatrix} \hat{\sigma}^1 \\ \hat{\sigma}^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \sigma'^1 \\ \sigma'^2 \end{pmatrix} \tag{5.10}$$

and we will obtain the torus

$$\begin{aligned} \hat{z} =& \hat{\sigma}^1 + i\hat{\sigma}^2 \\ \hat{z} \cong& \hat{z} + 2\pi \\ \hat{z} \cong& \hat{z} + 2\pi i \\ ds^2 =& |d\,\mathrm{Re}\,\hat{z} + \hat{\tau} d\,\mathrm{Im}\,\hat{z}|^2 = |\hat{\sigma}^1 + \hat{\tau}\hat{\sigma}^2|^2 \quad , \end{aligned} \tag{5.11}$$

[69] We refer to [19] for all the mathematical details regarding complex manifolds.

[70] We like to depict the complex structure as a "rotation of 90°" because in this way it is more natural to think about a similarity between the notions of complex structure and metric. But it should be clear that a complex manifold doesn't need a metric to be defined, so it makes no sense to talk about length and angles.

which is clearly the same complex torus, only written in different coordinates.
The transformation (5.10) is a *large* coordinate transformation, namely a diffeomorphism of the torus that cannot be obtained from the identity by successive infinitesimal transformations; indeed, the curve A in the coordinates z' (see Figure 11) maps to a curve in the $\hat{z}$ coordinate that runs a times in the $\hat{A}$ direction and $-c$ times in the $\hat{B}$ direction: in order to turn the torus characterized by τ into the torus defined by $\hat{\tau}$, we have to cut the torus to a cylinder, to properly twist it and to sew together again its ends. These large coordinate transformations are called *modular group* and they are generated by the repeated application of the following two transformations:

$$T: \quad \tau \mapsto \tau + 1 \qquad S: \quad \tau \mapsto -1/\tau \quad . \tag{5.12}$$

In order not to overcount the same complex structure, in the path integral we must integrate τ over the *moduli space* of the torus, which is the Teichmüller space mod the action of the modular group $PSL(2,\mathbb{C})$. We usually represent the moduli space as a particular subset of the Teichmüller space, the so-called *fundamental region* of the moduli space. For example, in Figure 13 we depict two possible choice (F_0 and F_1) for the fundamental region, where the lines I and I' are identified, as the arcs II and II' (or the arcs III and III') are. In this thesis, we like to work with the fundamental region F_0, and this means that the integral over the metrics of the torus will reduce, after locally fixing the gauge, to an integral over the τ in the region

$$-\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2} \quad |\tau| \geq 1 \quad , \tag{5.13}$$

which has to be thought of as open only at $\mathrm{Im}\,\tau \to +\infty$, because it is rolled up according to the identification I-I' and II-II'.

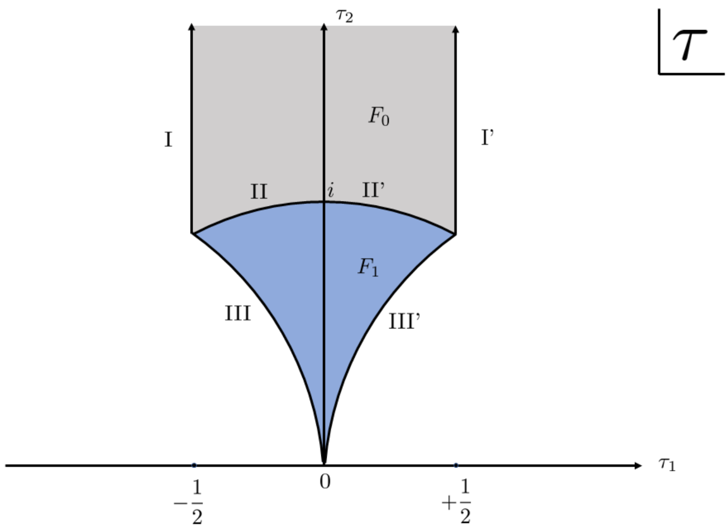

Figure 13
The regions F_0 and F_1 are two possible representations of the moduli space of the torus.
In this thesis we will always use F_0, whose unique boundary is at $\tau_2 = +\infty$.

5.2 Moduli and conformal killing vectors

After discussing the concrete example for the torus, we want to discuss the appearence of the moduli space of the worldsheet in scattering amplitudes from a more general and abstract way, which requires a mathematical language that will be useful in the next section.
The path integral wants us to sum over the space $\mathcal{G}_g$ of all the possible metrics which can be introduced

on the worldsheet of genus g. After taking into account the diff×Weyl redundancie, we are left with the moduli space $\mathcal{M}_g$

$$\mathcal{M}_g \equiv \frac{\mathcal{G}_g}{\text{diff}\times\text{ Weyl}} \quad , \tag{5.14}$$

where we have to be careful and remember that in the game there are both global and infinitesimal diffeomorphisms. Indeed, as we have seen for the torus, the redundancy group consisting of diffeomorphisms in general is not connected and we will refer to its connected component containing the identity as diff_0. Then, the modular group MG is nothing but the quotient

$$MG \equiv \frac{\text{diff}}{\text{diff}_0} \tag{5.15}$$

and we can rewrite the moduli space as

$$\mathcal{M}_g = \frac{\mathcal{G}_g}{\text{diff}_0\times\text{ Weyl}}\frac{\text{diff}_0}{\text{diff}} = \frac{\mathcal{T}_g}{MG} \quad , \tag{5.16}$$

where we denoted with

$$\mathcal{T}_g \equiv \frac{\mathcal{G}_g}{\text{diff}_0\times\text{ Weyl}} \tag{5.17}$$

the Teichmüller space of the worldsheet of genus g.
We like to stress again that in the Faddeev-Popov procedure presented in section 3, we considered only the infinitesimal diffeomorphisms, namely the diff_0 transformations. There we wrote the infinitesimal variation of the metric under the combined action of $\text{diff}_0\times$Weyl as

$$\delta h_{\alpha\beta} = -2(P_1\delta\sigma)_{\alpha\beta} + (2f - \nabla^\gamma\delta\sigma_\gamma)h_{\alpha\beta} \quad , \tag{5.18}$$

where the operator P_1 is given by

$$(P_1\delta\sigma)_{\alpha\beta} \equiv \frac{1}{2}(\nabla_\alpha\delta\sigma_\beta + \nabla_\beta\delta\sigma_\alpha - h_{\alpha\beta}\nabla_\gamma\delta\sigma^\gamma) \quad . \tag{5.19}$$

At the infinitesimal level, changes in the moduli correspond to variations $\delta' h_{\alpha\beta}$ of the metric that cannot be reached by $diff_0 \times Weyl$; equivalently, by changing the moduli we obtain those variations $\delta' h_{\alpha\beta}$ that are orthogonal[71] to all variations given by formula (5.18):

$$\begin{aligned} 0 &= \int_{\Sigma_g} d\sigma^1 d\sigma^2 \sqrt{h}\delta' h_{\alpha\beta}\Big[-2(P_1\delta\sigma)^{\alpha\beta} + (2f - \nabla^\gamma\delta\sigma_\gamma)h^{\alpha\beta}\Big] = \\ &= \int_{\Sigma_g} d\sigma^1 d\sigma^2 \sqrt{h}\Big[-2(P_1^T\delta' h)_\alpha\delta\sigma^\alpha + \delta' h_{\alpha\beta}h^{\alpha\beta}(2f - \nabla^\gamma\delta\sigma_\gamma)\Big] \quad , \end{aligned} \tag{5.20}$$

where we introduced the transpose operator P_1^T which[72] maps traceless symmetric tensors $t_{\alpha\beta}$ to vectors via $(P_T^1 t)_\alpha = -\nabla^\beta t_{\alpha\beta}$.
In order for (5.20) to vanish for every f and $\delta\sigma$, we need

$$\begin{aligned} h^{\alpha\beta}\delta' h_{\alpha\beta} &= 0 \\ (P_1^T\delta' h)_\alpha &= 0 \quad ; \end{aligned} \tag{5.21}$$

the first condition says that $\delta' h_{\alpha\beta}$ is traceless so the second equation is well-defined (P_1^T acts on traceless tensors) and for every solution of the second equation there will be a modulus. Obviously, we are interested

[71] Orthogonal with respect to the scalar product ($\delta h_{\alpha\beta}$ and $\delta' h_{\gamma\delta}$ are infinitesimal variations of the same metric)

$$(\delta h_{\alpha\beta}\,,\,\delta' h_{\gamma\delta}) = \int_{\Sigma_g} d\sigma^1 d\sigma^2 \sqrt{h} h^{\alpha\gamma}h^{\beta\delta}\delta h_{\alpha\beta}\delta' h_{\gamma\delta} \quad ,$$

which induces the norm on $\mathcal{G}_g$ used to define the integration measure $D[h_{\alpha\beta}]$; refer to [6] for the details regarding the geometrical aspects of string perturbation theory.

[72] See [17] for details.

in understanding how many moduli exist for a given worldsheet and we have to find the number of independent globally defined zero modes for the operator P_1^T. This is a highly non-trivial mathematical question, and the best that we can do is, on general ground, to appeal to an index theorem[73], which gives the difference between the number of the globally defined zero modes of the operator and its adjoint (here the transpose) in terms of a topological invariant of the manifold (like the Euler number Ξ, for example). To be more precise, we can resort to the Riemann-Roch theorem which, in our case, reduces to

$$\dim_{\mathbb{R}} \mathrm{Ker} P_1 - \dim_{\mathbb{R}} \mathrm{Ker} P_1^T = 3\Xi = 6(1-g) \quad . \tag{5.22}$$

So, if we are interested in finding $\dim_{\mathbb{R}} \mathrm{Ker} P_1^T$, we can equivalently determine the number of independent globally defined zero modes for the operator P_1...which are something that we have already met because, as we are now going to show, they are nothing but the transformations of the CKG! Indeed, we have already explained that, after locally fixing the gauge in each patch (take the conformal one, for example), we still have the residual freedom of reparametrizing the worldsheet by conformal transformations that are globally defined on it. The CKG's transformations are the globally defined diffeomorphisms that can be "undone" by a global Weyl rescaling. They embody the residual freedom that we have of parametrizing the worldsheet after fixing the gauge so, in mathematical terms, they correspond to the transformations that correspond to a vanishing combined variation of the metric (5.18):

$$0 = \delta h_{\alpha\beta} = -2(P_1 \delta\sigma)_{\alpha\beta} + (2f - \nabla^\gamma \delta\sigma_\gamma) h_{\alpha\beta} \quad . \tag{5.23}$$

The trace of this equation uniquely specifies f (remember: P_1 maps to traceless tensors) and we learn which is the precise Weyl rescaling that is able to undo the transformation. The term $(f - \nabla^\gamma \delta\sigma_\gamma)$ must vanish and (5.23) then states that the transformations of the CKG are precisely those satisfying[74]

$$0 = (P_1 \delta\sigma)_{\alpha\beta} \quad . \tag{5.24}$$

With the description of the CKGs for different worldsheets that we gave at the beginning of the last chapter, we can write

$$\dim_{\mathbb{R}} \mathrm{Ker} P_1 = \begin{cases} 6, & \text{for } g=0 \quad \text{Riemann sphere} \\ 2, & \text{for } g=1 \quad \text{Riemann torus} \\ 0, & \text{for } g>1 \quad \text{Higher genus Riemann surfaces} \end{cases} \tag{5.25}$$

and, by using the Riemann-Roch theorem we finally obtain that the number of moduli is

$$\dim_{\mathbb{R}} \mathrm{Ker} P_1^T = \begin{cases} 0, & \text{for } g=0 \quad \text{Riemann sphere} \\ 2, & \text{for } g=1 \quad \text{Riemann torus} \\ 6(g-1), & \text{for } g>1 \quad \text{Higher genus Riemann surfaces} \end{cases} \quad . \tag{5.26}$$

Clearly, the last formula should also give, according to (5.16), the real dimension of the Teichmüller space; indeed, it is possible to show that, for $g>1$, the Teichmüller space is a complex manifold of real dimension $6(g-1)$ topologically equivalent to $(\mathbb{R}_+ \times \mathbb{R})^{3(g-1)}$. The moduli space is obtained by taking the quotient of the Teichmüller space as explained by (5.16); in general, the modular group MG acts holomorphically on the Teichmüller space, but with fixed points and, thus, the moduli space will have the structure of an orbifold. See [6] for details in this directions; for us it is enough to know that, locally, the moduli space has the structure of a manifold of the dimension given by formula (5.26).
So far, we have discussed only the moduli associated to the metric. When there are vertex operators appearing in the path integral it is useful to treat their positions on the same footing as the moduli from the metric, as we will show later. In general, a vertex operator can be inserted at any point of the worldsheet Σ_g, so the Teichmüller space and the moduli space at topology g with n vertex operators are

[73] Refer always to [6], for more details about the mathematical aspects of what we are presenting.
[74] For the distrustful reader: in the conformal gauge and with the complex notation, we have that $0 = (P_1\delta\sigma)_{\alpha\beta}$ becomes $\bar{\partial}\delta z = \partial\delta\bar{z} = 0$ and the identification among the (globally defined) zero modes of P_1 and CKG's transformations reduces to a tautology.

then

$$
\begin{aligned}
\mathcal{T}_{g,n} \equiv & \frac{\mathcal{G}_g \times \Sigma_g^n}{\text{diff}_0 \times \text{ Weyl}} \\
\mathcal{M}_{g,n} = & \frac{\mathcal{T}_{g,n}}{MG} \quad .
\end{aligned}
\tag{5.27}
$$

We are now ready to discuss the complete Faddeev-Popov procedure for string theory. We will follow the same steps done in section 3 but, this time, we will not neglect the complications coming from the presence of the CKG and the moduli in the game. In particular, the $b-c$ ghost system will reveal all its importance, because it will let us define the proper measure on the moduli space.

5.3 Faddeev-Popov gauge fixing: the complete approach

The Polyakov path integral for the S-matrix with n external states is

$$
S_{j_1 \dots j_n}(k_1, ..., k_n) = \sum_{g \geq 0}^{+\infty} g_s^{2(g-1)} \int \frac{D[X^\mu] D[h_{\alpha\beta}]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_{Poly}[X^\mu, h_{\alpha\beta}; \Sigma_g]} \prod_{i=1}^n \int d^2\sigma_i \sqrt{h(\sigma_i)} \phi_{j_i, k_i}(\sigma_i) \tag{5.28}
$$

where we introduced the vertex operators in their integrated form because we cannot let a physical quantity like the S-matrix depend on unphysical parameters like the positions of the vertex operators on the worldsheet.

In gauge-fixing, the integral over the metrics becomes an integral over the gauge group and over the moduli

$$
D[h_{\alpha\beta}] \to D[\zeta] d^\rho t \quad ,
$$

where we denoted with ζ the combined action of diff$\times$Weyl, with t the real moduli of the worldsheet and with $\rho \equiv \dim_{\mathbb{R}} \text{Ker} P_1^T$ the number of them.

We have to remember that we have still to fix the transformations of the CKG, and we do it by fixing the positions of $\kappa \equiv \dim_{\mathbb{C}} \text{Ker} P_1$ vertex operators:

$$
d^{2n} \to d^{2n-\kappa} \quad ;
$$

for example, on the torus we can fix the position of one vertex operator. Clearly, here we are assuming that in the S-matrix there are at least κ vertex operators. If there are not enough vertex operators to fix the whole CKG, we can divide the S-matrix by the volume of the unfixed subgroup. Please note that if the volume of this subgroup is infinite, then the S-matrix vanishes. For example, the CKG of the Riemann sphere is $PSL(2, \mathbb{C})$, which has an infinite volume, given that $SL(2, \mathbb{C})$ is a non compact Lie group. Thus, the oriented closed string 0-point, 1-point and 2-point functions vanish at tree-level, and this means that there is no vacuum energy, no tadpole and, respectively, no mass renormalisation at tree-level.

In the last chapter we will be interested in computing a mass term at one loop-level, so here we study the case in which we have enough vertex operators to fix the whole of the CKG. We will denote with the hat the κ positions of the vertex operators that we are able to fix ($\sigma_i^\alpha \to \hat{\sigma}_i^\alpha$) in analogy with the hat by which we denote the gauge choice for the metric ($h_{\alpha\beta} \to \hat{h}_{\alpha\beta}$).

The Fadeev-Popov measure is defined by

$$
1 = \Delta_{FP}(h_{\alpha\beta}, \sigma) \int_F d^\rho t \int_{\text{Diff}\times\text{Weyl}} D[\zeta] \delta(h_{\alpha\beta} - \hat{h}_{\alpha\beta}(t)^\zeta) \prod_{(\alpha,i)\in\Omega} \delta(\sigma_i^\alpha - \hat{\sigma}_i^{\zeta\alpha}) \quad , \tag{5.29}
$$

where Ω denotes the set of the fixed coordinates and F is a fundamental region of the moduli space. Note that in this formula we stressed the fact that we are considering the worldsheet as the manifold without moduli dependence inside the transition functions and with the Teichmüller parameters explicitly appearing in the metric (at one loop, for example, the torus that we have in our mind is that one define by (5.9)).

We can insert the expression (5.29) into the S-matrix and, after following the steps that we did in chapter 3, the latter will become

$$S_{j_1\dots j_n}(k_1,...,k_n)=\sum_{g\geq 0}^{+\infty} g_s^{2(g-1)}\int_F d^\rho t\Delta_{FP}(\hat{h}_{\alpha\beta},\hat{\sigma})\int D[X^\mu]\prod_{(a,i)\notin\Omega}\int d\sigma_i^a\times e^{-S_{Poly}[X^\mu,\hat{h}_{\alpha\beta};\Sigma_g]}\times \\ \times\prod_{i=1}^n\sqrt{\hat{h}(\sigma_i)}\phi_{j_i,k_i}(\sigma_i) \quad , \tag{5.30}$$

and we are left with evaluating the Faddeev-Popov measure. As we did in chapter 3, we expand $\Delta_{FP}(h_{\alpha\beta},\sigma)$ around the particular pair (ζ,t) that makes the delta functional nonzero.
The general metric variation consists of a local symmetry variation and of a change in the moduli,

$$\delta h_{\alpha\beta}=\sum_{k=1}^{\rho}\delta t^k\partial_{t^k}\hat{h}_{\alpha\beta}-2(\hat{P}_1\delta\sigma)_{\alpha\beta}+(2f-\hat{\nabla}^\gamma\delta\sigma_\gamma)\hat{h}_{\alpha\beta} \tag{5.31}$$

and we compute the inverse of the Faddeev-Popov determinant as

$$\begin{aligned}\Delta_{FP}(\hat{h}_{\alpha\beta},\hat{\sigma})^{-1}=&n_R\int d^\rho\delta tD[f]D[\delta\sigma^\alpha]\delta(\delta h_{\alpha\beta})\prod_{(\alpha,i)\in\Omega}\delta(\delta\sigma^\alpha(\hat{\sigma}_i))=\\ =&n_R\int d^\rho\delta td^\kappa\chi D[\beta'_{\alpha\beta}]D[\delta\sigma^\alpha]\exp\left(2\pi i(\beta',2\hat{P}_1\delta\sigma-\delta t^k\partial_{t^k}\hat{g})+2\pi i\sum_{(\alpha,i)\in\Omega}\chi_{\alpha i}\delta\sigma^\alpha(\hat{\sigma}_i)\right) \quad ,\end{aligned} \tag{5.32}$$

where we have written the delta functions and functionals respectively as integrals over $\chi_{\alpha i}$ and $\beta_{\alpha\beta}$ and we have also integrated out $D[f]$ to obtain the traceless constraint on $\beta'_{\alpha\beta}$; the inner product among β' and $2\hat{P}_1\delta\sigma-\delta t^k\partial_{t^k}\hat{g}$ is the natural one for traceless symmetric tensors $(T^1,\,T^2)$ of rank 2: $(T^1,T^2)\equiv\int d^2\sigma\sqrt{h}(T^1)^{\alpha\beta}T^2_{\alpha\beta}$. We have also taken into account the possibility that the Dirac deltas could be nonzero at n_R different points; these must be related by a residual discrete[75] group symmetry so we consider only one of these points and we multiply by n_R.
Now we invert the integral (5.32) by replacing all bosonic variables with Grassmann odd fields:

$$\begin{aligned}\delta\sigma^\alpha&\to c^\alpha\\ \beta'_{\alpha\beta}&\to b_{\alpha\beta}\\ \chi_{\alpha i}&\to\eta_{\alpha i}\\ \delta t^k&\to\xi^k \quad ,\end{aligned} \tag{5.33}$$

and, with convenient normalization for the fields, we can arrive at

$$\begin{aligned}\Delta_{FP}(\hat{h}_{\alpha\beta},\hat{\sigma})=&\frac{1}{n_R}\int D[b_{\alpha\beta}]D[c^\alpha]d^\rho\xi d^\kappa\eta\exp\left[-\frac{1}{4\pi}(b,2\hat{P}_1c-\xi^k\partial_k\hat{h})+\sum_{(\alpha,i)\in\Omega}\eta_{\alpha i}c^\alpha(\hat{\sigma}_i)\right]=\\ =&\frac{1}{n_R}\int D[b_{\alpha\beta}]D[c^\alpha]e^{-S_g}\prod_{k=1}^{\rho}\frac{1}{4\pi}(b,\partial_k\hat{h})\prod_{(\alpha,i)\in\Omega}c^\alpha(\hat{\sigma}_i^\alpha) \quad ,\end{aligned} \tag{5.34}$$

[75]For example, in the case of the torus we can fix the metric to the form (5.9). We still have the freedom to perform a CKG's transformation, because a rigid $U(1)\times U(1)$ translation leaves the metric and the periodicity conditions of (5.9) invariant. We can fix this freedom by fixing the position of a vertex operator on the worldsheet. At this point we are left only with the freedom of changing z' of formula (5.9) by $z'\to -z'$. Thus, for the torus, the residual discrete group symmetry is $\mathbb{Z}_2$ and $n_R=2$.

which is the proper integration measure for integration on moduli space.
Finally, we get the full expression for the S-matrix:

$$\begin{aligned} S_{j_1\dots j_n}(k_1,\dots,k_n) = &\sum_{g\geq 0}^{+\infty} g_s^{2(g-1)} \int_F \frac{d^\rho t}{n_R} \int D[X^\mu]D[b_{\alpha\beta}]D[c^\alpha] e^{-S_{Poly}[X^\mu,\hat{h}_{\alpha\beta};\Sigma_g]-S_g} \times \\ &\times \prod_{(\alpha,i)\notin\Omega} \int d\sigma_i^\alpha \prod_{k=1}^{\rho} \frac{1}{4\pi}(b,\partial_k \hat{h}) \prod_{(\alpha,i)\in\Omega} c^\alpha(\hat{\sigma}_i) \prod_{i=1}^{n} \sqrt{\hat{h}}\phi_{j_i,k_i}(\sigma_i) \quad . \end{aligned} \tag{5.35}$$

Even though it would be an instructive thing to do (see [11] and [6]), we don't check the BRST-invariance and the independence of the gauge choice of this formula, because the Faddeev-Popov procedure guarantees these properties for the gauge-fixed amplitude. What we want to stress is that the formula (5.35) is very fundamental, because it depends essentially only on the geometry of the worldsheet. It can be easily extended to all bosonic string theories (closed, open, oriented, not oriented) and S_{Poly} can be replaced by a general $c = \tilde{c} = 26$ matter theory; the measure on moduli space is always given by formula (5.34), so we can use formula (5.35) also for different bosonic string theories, which are characterized by different vertex operators, namely by a different spectrum.
We summarize what we have learnt in this chapter by stating that we can always work locally and the complications caused by the moduli and the CKG are taken into account by the ghost insertions:

- for each fixed coordinate, $\int d\sigma_i^\alpha$ is replaced by $c^\alpha(\hat{\sigma}_i)$; in particular, if we are able to completely fix the position of a vertex operator ϕ_{j_i,k_i}, then this replacement, in the flat gauge and with complex coordinates, reads as

$$\int dz d\bar{z} \phi_{j_i,k_i}(z_i,\bar{z}_i) \rightarrow c(\hat{z}_i)\bar{c}(\hat{\bar{z}}_i)\phi_{j_i,k_i}(\hat{z}_i,\hat{\bar{z}}_i) \quad ,$$

  and we see that the replacement precisely consists of substituting the integrated vertex operator with its fixed form.

- for each modulus t^k we have to introdue a b ghost by means of $\frac{1}{4\pi}(b,\partial_k \hat{h})$.

We mention that these ghost insertions are precisely those that we need to avoid a vanishing result. Indeed, we have to remember that there is the $U(1)$ anomaly in the $b-c$ sector which, locally, is given by (see (4.88) and remember that $Q = \epsilon(1-2\lambda) = -3$)

$$\nabla^z j_z(z) = \frac{1}{4}QR = -\frac{3}{4}R \quad ; \tag{5.36}$$

it is possible to integrate this expression to obtain its global version, namely

$$N_c - N_b = -\frac{Q}{2}\Xi = 3(1-g) \quad , \tag{5.37}$$

where we denoted with N_c, N_b the number of zero[76] modes of c and b which are globally defined on the worldsheet of genus g.
All the dependence of (5.35) on the zero modes of b and c is hidden in Δ_{FP} (better, in the $c^\alpha(\hat{\sigma}_i)$ and in the $(b,\partial_k\hat{h})$ insertions) because $e^{-S_g} = 1$ when the action S_g eats a ghost's zero mode[77]. So, in the formula (5.35) we have $N_c = \kappa$ and $N_b = \rho$ zero modes: Riemann-Roch theorem guarantees that the constraint (5.37) is respected and the $U(1)$ anomaly does not force our S-matrix to necessarily vanish.

[76] Note that the situation is very similar to the $U(1)$ anomaly in gauge theory, where the integral of the anomalous divergence of the chiral $U(1)$ current gives the difference between the number of massless left- and right-handed fermions, which are the zero modes of the chiral Dirac operator. The role of the Dirac operator is played by P_1, P_1^T, since the equation of motions of b and c are $P_1 c = 0$ and $P_1^T b = 0$.

[77] Look at (3.17): the action $S_g[b,c]$ can be written as

$$S_g[b,c] = \frac{1}{2\pi}(b,P_1 c) = \frac{1}{2\pi}(P_1^T b, c) \quad .$$

5.4 S-matrix

In the last chapter we derived the form of the S-matrix by considering the worldsheet with all its moduli dependence encoded in the metric. As we know, we can equivalently describe the worldsheet with a moduli independent metric and with moduli dependent transition functions and now we are going to recast the result (5.35) from this point of view, which is more convenient for practical purposes. Let's consider coordinates z_m in each patch such that the moduli dependent metric $\hat{h}(t)$ is Weyl equivalent to $dz_m d\bar{z}_m$; our goal is to rewrite the terms $(b, \partial_k \hat{h}(t))$ in such a way that they depend only on the transition functions of the worldsheet.
We start by defining the so-called *Beltrami differential* as

$$\mu^{\beta}_{k\alpha} \equiv \frac{1}{2}\hat{h}^{\beta\gamma}(t)\partial_k \hat{h}_{\alpha\gamma}(t) \tag{5.38}$$

so that the b insertions read as

$$\frac{1}{4\pi}(b, \partial_k \hat{h}) = \frac{1}{2\pi}\int d^2z(b_{zz}\mu^{z}_{k\bar{z}} + b_{\bar{z}\bar{z}}\mu^{\bar{z}}_{kz}) = \frac{1}{2\pi}(b, \mu_k) \quad . \tag{5.39}$$

Note that if the metric $\hat{h}(t)$ is Weyl-equivalent to $dz_m d\bar{z}_m$, then the metric $\hat{h}(t+\delta t)$ is equivalent to

$$dz_m d\bar{z}_m + \delta t^k \left(\mu^{\bar{z}_m}_{kz_m} dz_m dz_m + \mu^{z_m}_{k\bar{z}_m} d\bar{z}_m d\bar{z}_m\right) \quad . \tag{5.40}$$

We could also say that after a change δt in the moduli there will be new transition functions and, thus, new coordinates in the m^{th} patch:

$$\begin{aligned} z'_m &= z_m + \delta t^k v^{z_m}_k(z_m, \bar{z}_m) \\ v^{z_m}_k &\equiv \frac{dz'_m}{dt^k} \quad , \end{aligned} \tag{5.41}$$

where we denoted with $v^{z_m}_k$ a vector that is defined only in the m^{th} patch. The metric $dz'_m d\bar{z}'_m$ must be Weyl equivalent to the metric (5.40) because they correspond to metrics with the same value for the moduli; from

$$dz'_m d\bar{z}'_m \propto dz_m d\bar{z}_m + \delta t^k \left(\mu^{\bar{z}_m}_{kz_m} dz_m dz_m + \mu^{z_m}_{k\bar{z}_m} d\bar{z}_m d\bar{z}_m\right) \quad , \tag{5.42}$$

we thus arrive at the *infinitesimal version of Beltrami's equation*, which is the crux:

$$\mu^{\bar{z}_m}_{kz_m} = \partial_{z_m} v^{\bar{z}_m}_k \qquad \mu^{z_m}_{k\bar{z}_m} = \partial_{\bar{z}_m} v^{z_m}_k \quad . \tag{5.43}$$

The generators $v^{z_m}_k$ ($v^{\bar{z}_m}_k$) contain a holomorphic (antiholomorphic) part which is not determined by the equation (5.43) and which correspond - as we already know - to the freedom to make holomorpic (anti-holomorphic) reparametrizations within each patch. If we want to change the moduli of the worldsheet, the Beltrami differentials must be non-vanishing and this means, according to Beltrami's equations, that $v^{z_m}_k$ ($v^{\bar{z}_m}_k$) must not be holomorphic (antiholomorphic).
We can now integrate by parts the $b-$insertions of formula (5.39) to obtain

$$\frac{1}{2\pi}(b, \mu_k) = \frac{1}{2\pi i}\sum_m \oint_{C_m} \left(dz_m v^{z_m}_k b_{z_m z_m} - d\bar{z}_m v^{\bar{z}_m}_k b_{\bar{z}_m \bar{z}_m}\right) \quad , \tag{5.44}$$

with the contour C_m counterclockwisely oriented in the m^{th} patch.
If φ_{mn} was the transition function between the m^{th} and n^{th} patches ($z_m = \varphi_{mn}(z_n)$) then, after the variation δt of the moduli we will have

$$\begin{aligned} z'_m &= z_m + \delta t^k v^{z_m}_k \\ z'_n &= z_n + \delta t^k v^{z_n}_k \\ z'_m &= \varphi'_{mn}(z'_n) \quad ; \end{aligned} \tag{5.45}$$

which, in the overlapping region, can be combined together to get

$$\begin{aligned}\varphi'_{mn}(z'_n) =& z'_m = z_m + \delta t^k v_k^{z_m} = \varphi_{mn}(z_n) + \delta t^k v_k^{z_m} = \\ =& \varphi_{mn}(z'_n - \delta t^k v_k^{z_n}) + \delta t^k v_k^{z_m} = \\ =& \varphi_{mn}(z'_n) - \frac{\partial \varphi_{mn}}{\partial z_n} v_k^{z_n} \delta t^k + \delta t^k v_k^{z_m} \quad ;\end{aligned} \tag{5.46}$$

so, we have

$$\frac{\partial \varphi_{mn}}{\partial t^k} = v_k^{z_m} - \frac{\partial \varphi_{mn}}{\partial z_n} v_k^{z_n} \quad , \tag{5.47}$$

which can be introduced into (5.44) to finally get the expression of the $b-$insertions in terms of the transition functions of the complex manifold:

$$\frac{1}{2\pi}(b, \mu_k) = \frac{1}{2\pi i} \sum_{(mn)} \oint_{C_{mn}} \left(dz_m \frac{\partial \varphi_{mn}}{\partial t^k} b_{z_m z_m} - d\bar{z}_m \frac{\partial \bar{\varphi}_{mn}}{\partial t^k} b_{\bar{z}_m \bar{z}_m} \right) \quad , \tag{5.48}$$

where the C_{mn} contour is running in the overlapping region between the m^{th} and the n^{th} patches, counterclockwise from the point of view of the m^{th} patch.

The formula (5.48) is very useful for computations. Here, we want to take advantage of it in order to reach a more elegant formulation of our S-matrix, where we treat the moduli coming from the location of a vertex operator on the same footing as the moduli coming from the metric. Actually, it is not only a matter of reaching a more elegant formulation; the reader should note that the insertion of a vertex operator modifies the complex structure of the worldsheet, because it brings new moduli into the game. Let z_v be the position of the vertex operator in a coordinate frame z. We introduce a new coordinate system z' around the vertex operator such that the latter sits at $z' = 0$. For example, we can imagine to cut, in the $z-$patch, a small disk of radius $\epsilon > 0$ around $z = z_v$ and then we cover the hole with a disk of radius ϵ' (just a little bigger than ϵ) where we use the coordinate z'; then we sew together the disk with the rest of the worldsheet by using the transition function

$$z = z' + z_v \quad , \tag{5.49}$$

which is obviously holomorphic in the overlapping region, namely in the annular region between the circles of radii ϵ and ϵ'. The transition function (5.49) suggests to treat the position of the vertex operator as a modulus and, according to formula (5.48) applied to the 2 moduli z_v and $\bar{z}_v$, we have to add, in the S-matrix, the following two ghost insertions:

$$\int_C \frac{dz'}{2\pi i} b_{z'z'} \int_C \frac{d\bar{z}'}{-2\pi i} b_{\bar{z}'\bar{z}'} = b_{-1}\bar{b}_{-1} \quad , \tag{5.50}$$

where C is any not shrinkable contour running counterclockwise in the annulus (in the overlapping region between the coordinates z and z').

Now we can rewrite the full expression (5.35) for the S-matrix in a very compact form:

$$S_{j_1 \dots j_n}(k_1, \dots, k_n) = \sum_{g \geq 0}^{+\infty} g_s^{2(g-1)} \int_{F \times \Sigma_g^{2n-\kappa}} \frac{d^m t}{n_R} \left\langle \prod_{k=1}^{m} B_k \prod_{i=1}^{n} \mathcal{V}_{j_i, k_i} \right\rangle \quad , \tag{5.51}$$

where we introduced B_k which is the shorthand for the $b-$insertions of formula (5.48): $B_k \equiv \frac{1}{2\pi}(b, \mu_k)$. The upshot is that we can insert the vertex operators in their fixed versions $\mathcal{V}_{j_i,k_i}(z,\bar{z}) = c(z)\bar{c}(\bar{z})\phi_{j_i,k_i}(z,\bar{z})$ and treat the coordinates of the vertex operators as moduli of the surface: this is the reason why in (5.51) we have considered the integration over the positions of the vertex operators which are not fixed by the CKG as an integration over the moduli space, which is now $m-$dimensional, with m given by

$$m \equiv \mu + 2n - \kappa = 6(g-1) + 2n = 2(3g + n - 3) \quad . \tag{5.52}$$

Clearly, if we have to integrate over the positions of $2n-\kappa$ vertex operators, then the latter have to appear in (5.51) in their integrated form, even though we introduced them in their fixed version. Indeed, the $b-$insertions that we add according to (5.50) remove the $c\bar{c}$ contributions from the fixed version:

$$b_{-1}\bar{b}_{-1}\mathcal{V}_{j_i,k_i}(z_i,\bar{z}_i)=\oint_{C(z_i)}\frac{dw}{2\pi i}b(w)\oint_{C(\bar{z}_i)}\frac{d\bar{w}}{2\pi i}\bar{b}(\bar{w})c(z_i)\bar{c}(\bar{z}_i)\phi_{j_i,k_i}(z_i,\bar{z}_i)=\phi_{j_i,k_i}(z_i,\bar{z}_i)\quad . \tag{5.53}$$

The reader should appreciate again the elegance of formula (5.51), where all vertex operators appear in their fundamental versions, namely in their fixed forms $c(z_i)\bar{c}(\bar{z}_i)\phi_{j_i,k_i}(z_i,\bar{z}_i)$, which are BRST-invariant because they directly come from the state-operator correspondence. The integrated form $\phi_{j_i,k_i}(z_i,\bar{z}_i)$ is not BRST-invariant and it shouldn't be considered fundamental; nevertheless, it is not difficult to show that, under a BRST transformation, we have

$$\begin{aligned}
[Q_B,\phi_{j_i,k_i}(z_i,\bar{z}_i)] =&\frac{\partial}{\partial z_i}\left(c(z_i)\phi_{j_i,k_i}(z_i,\bar{z}_i)\right)\\
[\bar{Q}_B,\phi_{j_i,k_i}(z_i,\bar{z}_i)] =&\frac{\partial}{\partial \bar{z}_i}\left(\bar{c}(\bar{z}_i)\phi_{j_i,k_i}(z_i,\bar{z}_i)\right)\quad ,
\end{aligned} \tag{5.54}$$

and, thus, we can still have integrated vertex operators in the S-matrix, provided that we integrate their positions over the (compact) worldsheet.

5.5 The torus again: an exercise

In the last paragraph, we have used formula (5.48) to understand the kind of b-insertions that are associated to the moduli coming from the positions of the vertex operators. Here, we exploit it to find the ghost insertions corresponding to the metric moduli and, to be more concrete, we will do this exercise in the case of the torus.

First of all, we have to cover the torus with τ-depending charts, so as to apply formula (5.48) in a straightforward way. We use the minimal number of charts, namely two (one chart is not enough, because of compactness) and we decide for them to cover cylindrical open subsets of the torus, as we are going to explain.

As usual, let z be a complex coordinate in which the torus is described by the identification

$$\begin{aligned}
z \cong& z+2\pi\\
z \cong& z+2\pi\tau\quad .
\end{aligned} \tag{5.55}$$

Let b be a positive constant such that $0<b<2\pi\tau_2$; then, as a fundamental region for the torus, we can choose the following portion of the complex plane (see Figure 14)

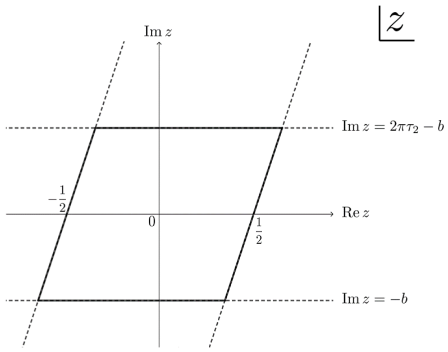

Figure 14
This is the fundamental region for the torus that will be used in this thesis.

$$-b\leq \mathrm{Im}\,z<2\pi\tau_2-b\qquad -\frac{1}{2}+\frac{\tau_1}{\tau_2}\,\mathrm{Im}\,z\leq \mathrm{Re}\,z<+\frac{1}{2}+\frac{\tau_1}{\tau_2}\,\mathrm{Im}\,z\quad . \tag{5.56}$$

Let a be a positive constant such that $0 < a < 2\pi\tau_2 - b$ and let's call C_a the circle on the torus that, on the fundamental region, corresponds to $\operatorname{Im} z = a$. We also call C_{-b} the circle on the torus that, on the fundamental region, corresponds to $\operatorname{Im} z = -b$ (or, equivalently, to $\operatorname{Im} z = 2\pi\tau_2 - b$). Clearly, C_a and C_{-b} are the ends of two cylinders, that we like to name as D_- and D_+ and that are respectively defined as those parts of the fundamental region satisfying $\operatorname{Im} z < a$ and $\operatorname{Im} z > a$. If we slightly extend both D_- and D_+ such that their intersections contain both the circles C_a and C_{-b} then we can consider, as charts, the coordinates u_- (u_+) on D_- (D_+) defined by

$$
\begin{aligned}
u_- &= z + b \qquad && \text{on } D_- \\
u_+ &= 2\pi\tau - b - z \qquad && \text{on } D_+ \quad ;
\end{aligned}
\tag{5.57}
$$

in few words (look at the arrows in Figure 15): u_- parametrizes the points of D_- by starting from the bottom of the fundamental region (the circle $\operatorname{Im} z = -b$) and by going up towards C_a, whereas u_+ parametrizes the points of D_+ by starting from the top of the fundamental region (the circle $\operatorname{Im} z = 2\pi\tau_2 - b$) and by going down towards C_a. Clearly, the transition functions between these two charts are

$$
\begin{aligned}
u_- &= u_+ \qquad && \text{along the intersection containing } C_{-b} \\
u_- &= 2\pi\tau - u_+ \qquad && \text{along the intersection containing } C_a \quad .
\end{aligned}
\tag{5.58}
$$

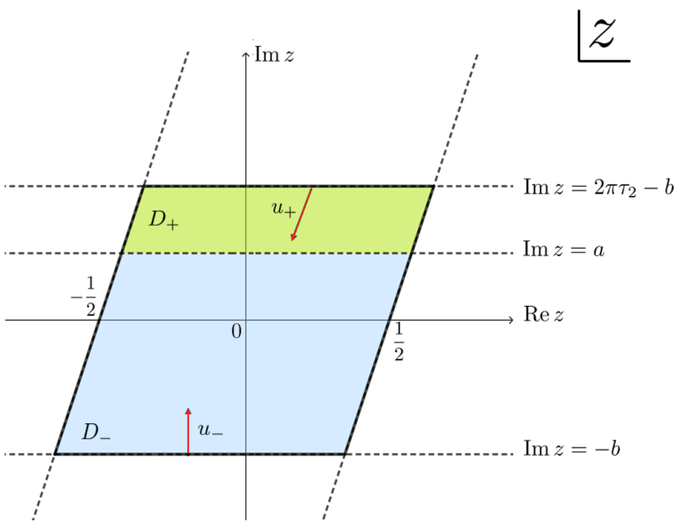

Figure 15
Illustration of the $u_\pm$ charts.

We see that the only transition function that depends on the modulus is that one defined on C_a, to which the following ghost insertion is associated

$$
\begin{aligned}
b_\tau =& \frac{1}{2\pi i}\oint_{C_a} du_- \frac{\partial u_-}{\partial \tau} b_{u_-,u_-} = \frac{1}{2\pi i}\oint_{C_a} du_- 2\pi b_{u_-,u_-} = \\
=& \frac{1}{2\pi i}\oint_{C_a} dz 2\pi b(z) \quad ,
\end{aligned}
\tag{5.59}
$$

where the integration contour C_a is oriented so that the region D_+ lies to its right. Note that in the last step we exploited the facts that the b-ghost is a conformal tensor (of dimension (2,0)) and that $\frac{du_-}{dz} = 1$. Analogously, the modulus $\bar\tau$ appears in the transition function $\bar u_- = 2\pi\bar\tau - \bar u_+$ and we have the insertion (look again at formula (5.48))

$$
\begin{aligned}
\bar b_{\bar\tau} =& -\frac{1}{2\pi i}\oint_{C_a} d\bar u_- \frac{\partial \bar u_-}{\partial \bar\tau} b_{\bar u_-,\bar u_-} = -\frac{1}{2\pi i}\oint_{C_a} d\bar u_- 2\pi b_{\bar u_-,\bar u_-} = \\
=& \frac{1}{2\pi i}\oint_{\bar C_a} d\bar z 2\pi \bar b(\bar z) \quad ,
\end{aligned}
\tag{5.60}
$$

where, in the last step, we absorbed the overall sign by changing the orientation of the contour integral ($C_a \to \bar{C}_a$).

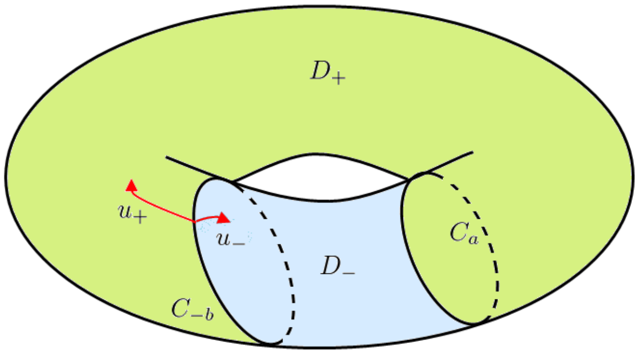

Figure 16
A representation of the $u_\pm$ charts on the torus.

A little heads-up Clearly, in order to get the right expressions for b_τ and $\bar{b}_{\bar{\tau}}$, we must use two charts, as we have seen; but, after obtaining b_τ and $\bar{b}_{\bar{\tau}}$, one would like to work with only one chart, so as to make life easier. This can be achieved by taking the formal limit in which $a \to 2\pi\tau_2 - b$: C_a approaches C_{-b}, the area of the cylinder D_+ becomes "negligible" and we can use the coordinate u_- in a wider region. In this limit, D_- corresponds to the torus deprived[78] of C_{-b}, that we call D^l_-; C_{-b} is a zero-measure set of the torus, so it is completely irrelevant to the S-matrix elements. Once we have taken into account the information coming from the moduli by means of the corresponding b-insertions, we can equivalently work with this limit-case cylinder D^l_- instead of working on the torus. This is what we will do in the last chapter, where we will work with the fundamental region

$$-b < \operatorname{Im} z < 2\pi\tau_2 - b \qquad -\frac{1}{2} + \frac{\tau_1}{\tau_2}\operatorname{Im} z \leq \operatorname{Re} z < +\frac{1}{2} + \frac{\tau_1}{\tau_2}\operatorname{Im} z \quad , \tag{5.61}$$

which differs from that one defined in (5.56) only because now C_{-b} is missing. In this fundamental region we can equivalently use z or u_-; for example, in the last chapter we will use the first one. We want to stress the fact that a Lebesgue-zero-measure set can be neglected from an integration only if there are no distributions (i.e. Dirac delta and its derivatives) with support on it. We will discover, in the computation of the last chapter, that a Dirac delta will be produced precisely where a particular fixed vertex operator is placed. Fortunately, in the case of the torus we have a CKG that allows us to locate this vertex operator where we prefer so we can put it far from the zero-measure-set, and the latter can be consistently neglected. To be more precise, usually one follows the reverse argument: we locate the vertex operator at $z = 0$ (so as to have handier OPEs) and, then, we use the CKG to take a fundamental region like (5.56), such that the point $z = 0$ is in the middle, far from the zero-measure-set.
We very like to stress that neglecting a zero-measure-set is not very important for the torus, where a flat metric can be reached globally, but can be very important for other surfaces, for which the flat metric can be reached only locally. Indeed, one can always take a very big chart that covers the surface up to zero-measure-sets[79] and fix the gauge so as to reach the flat metric on it; then all the information about the moduli is migrated to the boundaries of this chart, namely to the transition functions between this "global" chart and the zero-measure-sets. In these zero-measure-sets there could be a very complicated metric that makes life harder, but it doesn't matter: after considering the proper b-insertions, we can simply neglect these portions of the worldsheet. Obviously, if a Dirac delta pops up in the calculations,

[78]The cylinders $D_\pm$ don't contain their boundaries (which are two circles), because $u_\pm$ are charts in the mathematical sense of the word: they maps *open* subsets of the torus into proper oben subsets of $\mathbb{C}$. Thus, in the $C_a \to C_{-b}$ limit, D_- covers all the torus except for the circle $C_{-b} = C_a$.
[79]For example, for the sphere, we can take the sphere deprived of the two poles, where we can introduce the spherical coordinates.

we must carefully fix the CKG, if we still want to neglect the zero-measure-set. Unfortunately, there is no CKG for surfaces with $g > 1$ and this is one of the reasons why higher-loop computations are more troublesome: nothing can be neglected and one must use several patches.

6 Superstring Theory

We used the bosonic string theory as a toy model that has allowed us to get acquaitance with the tools of string theory. Now we move to a more realistic model, the superstring, which includes fermionic fields in the game. The fermionic excitations are added in a supersymmetric way, so as to cure the tachyonic behaviour of the bosonic string. To be more precise, if we want to remove the tachyon, we have to aim at models with target space supersymmetry. A way to do it is to introduce the fermionic fields in a supersymmetric way on the worldsheet and, then, to perform a projection on the theory so as to single out a spectrum which appears supersymmetric from the point of view of the target space.
At that time this thesis is written, five consistent superstring theory are known and the underlying presence of an intriguing web of connections relating all of them suggests that they should descend from a more fundamental theory. In the next chapters we will be interested only in the heterotic superstring and we will completely neglect all the other superstrings; actually, we will mention also Type IIA/B, but only because studying them will help us to introduce the heterotic theory in a more natural way.
We will be sketchy in this chapter, because we need only the very basics and because a lot of the features of the superstring works as in the bosonic string. The only object that has no comparison with the bosonic string is the picture number operator, something that we will study at the end of the chapter.

We refer to [17] for a systematic and detailed introduction to the superstring.

6.1 The classical fermionic (field) theory

As first thing, it is convenient to have a look at a purely fermionic 2−dimensional free field theory. The action for D real fermionic fields ψ^μ defined on the Euclidean[80] worldsheet $\Sigma_0 = [0, 2\pi] \times \mathbb{R}$ (or in a local patch of the Euclidean worldsheet Σ_g) is

$$S_\psi[\psi^\mu; \Sigma_0] \equiv -\frac{1}{4\pi} \int d\sigma^1 d\sigma^0 \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \quad , \tag{6.1}$$

where:

- each field ψ^μ consists of two real components $\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$ and we defined $\bar{\psi}^\mu$ as $\bar{\psi}^\mu \equiv (\psi^\mu)^t \rho^0$;
- we denoted with ρ^0 and ρ^1 the 2−dimensional Dirac matrices, satisfyng the Dirac algebra $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$.

In the case of the bosonic string it was necesary to assume that X^μ was periodic on σ^1, because X^μ was the embedding of Σ_0 into the target space. The fermionic field ψ^μ is not an embedding of the worldsheet into spacetime, so more general boundary conditions are now allowed. As usual, the consistent boundary conditions are singled out with the equations of motion when we vary the action. The latter can be rewritten, by using the coordinates $\sigma^\pm = \sigma^0 \pm \sigma^1$, as

$$\begin{aligned} S_\psi[\psi^\mu; \Sigma_0] =& \frac{1}{4\pi} \int d\sigma^- d\sigma^+ \big(\psi^\mu_- \partial_+ \psi^\nu_- + \psi^\mu_+ \partial_- \psi^\nu_+\big)\eta_{\mu\nu} = \\ =& \frac{1}{2\pi} \int d\sigma^0 d\sigma^1 \big(\psi^\mu_- \partial_+ \psi^\nu_- + \psi^\mu_+ \partial_- \psi^\nu_+\big)\eta_{\mu\nu} \end{aligned} \tag{6.2}$$

and its variation reads as

$$\begin{aligned} \delta S_\psi[\psi^\mu; \Sigma_0] =& -\frac{1}{\pi} \int d\sigma^0 d\sigma^1 \eta_{\mu\nu} \big(\partial_+ \psi^\mu_- \delta\psi^\nu_- + \partial_- \psi^\mu_+ \delta\psi^\nu_+\big) + \\ &+ \frac{1}{4\pi} \int d\sigma^0 \eta_{\mu\nu} \big(\psi^\mu_- \delta\psi^\nu_- + \psi^\mu_+ \delta\psi^\nu_+\big)\Big|_0^{2\pi} \quad ; \end{aligned} \tag{6.3}$$

from this variation we learn that the fields $\psi^\mu_\pm$

[80] Till this moment we have denoted with σ^2 the Euclidean time. We are changing conventions: from now on, the Euclidean time will be σ^0.

- are chiral, in the sense that they depend only on a single light-cone coordinate $\psi^\mu_\pm = \psi^\mu_\pm(\sigma^\pm)$ because of the equations of motion $\partial_\mp \psi^\mu_\pm = 0$;
- must satisfy one of the following boundary conditions

$$\psi^\mu_\pm(\sigma^1 + 2\pi) = \begin{cases} \psi^\mu_\pm & \text{R-sector} \quad \text{(periodic on } \Sigma_0) \\ -\psi^\mu_\pm & \text{NS-sector} \quad \text{(anti-periodic on } \Sigma_0) \end{cases} \quad , \tag{6.4}$$

where NS and R are shorthands for *Neveu-Schwarz* and *Ramond.*

The boundary conditions can be picked up for the left and right movers independently so we have four sectors: RR, RNS, NSR and NSNS.

6.2 The quantum fermionic (field) theory

The field ψ^μ_- has weight 1/2 so, on the complex plane, it takes the following form

$$\psi^\mu_-(z) = \sqrt{-i}e^{-\frac{i}{2}\sigma^-}\psi^\mu_-(\sigma^-) = \begin{cases} \sum_{n\in\mathbb{Z}} b^\mu_n z^{-n-\frac{1}{2}} & \text{R-sector} \quad \text{(anti-periodic on } \mathbb{C}) \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r z^{-r-\frac{1}{2}} & \text{NS-sector} \quad \text{(periodic on } \mathbb{C}) \end{cases} \quad , \tag{6.5}$$

and we note that, because of the factor $\left(\frac{\partial z}{\partial\sigma^-}\right)^{-\frac{1}{2}} = \sqrt{-i}e^{-\frac{i}{2}\sigma^-}$, now, on the plane, it is the R-sector that respects the anti-periodic boundary condition $\psi^\mu_-(e^{2\pi i}z) = -\psi^\mu_-(z)$. Clearly, the same holds for the fields ψ^μ_+, whose Fourier modes we denote with $\bar{b}^\mu_n$ and $\bar{b}^\mu_r$. It is not difficult to show that the modes satisfy

$$\begin{aligned} \{b^\mu_n, b^\nu_m\} &= \eta^{\mu\nu}\delta_{m+n,0} \quad \text{R-sector} \\ \{b^\mu_r, b^\nu_s\} &= \eta^{\mu\nu}\delta_{r+s,0} \quad \text{NS-sector} \quad . \end{aligned} \tag{6.6}$$

In the case of the NS-sector, it is straighforward to build the Hilbert space of the theory, because in this case there is a unique ground state $|0\rangle_{NS}$ such that

$$b^\mu_r |0\rangle_{NS} = 0 \quad \forall\, r > 0 \quad , \tag{6.7}$$

so that the Hilbert space $\mathcal{H}_{NS}$ is spanned by the states of the form

$$b^{\mu_1}_{r_1}...b^{\mu_i}_{r_i}|0\rangle_{NS} \quad \text{where } r_1 < ... < r_i \leq -\frac{1}{2} \quad , \tag{6.8}$$

where the restriction $r_1 < ... < r_i$ comes from the fact that each creation operator can be applied at most once ($(b^\mu_r)^2 = 0$).
In the R-sector, we have to be more careful because here there exist the zero modes b^μ_0 which, according to (6.6), satisfy a $D-$dimensional Dirac-algebra. We assume for D to be even[81] and so we can arrange the zero modes into the following pairs of operators:

$$\begin{aligned} b_{00} &\equiv \frac{1}{\sqrt{2}}(b^1_0 - b^0_0) \qquad b^\dagger_{00} \equiv \frac{1}{\sqrt{2}}(b^1_0 + b^0_0) \\ b_{0j} &\equiv \frac{1}{\sqrt{2}}(b^{2j+1}_0 - ib^{2j}_0) \qquad b^\dagger_{0j} \equiv \frac{1}{\sqrt{2}}(b^{2j+1}_0 + ib^{2j}_0) \quad . \end{aligned} \tag{6.9}$$

By construction, these operators obey $\{b_{0i}, b^\dagger_{0j}\} = \delta_{ij}$ and, therefore, the space of ground states in the R-sector can be generated with b_{0i} from a state $|R\rangle$ satisfying

$$\begin{aligned} b^\dagger_{0i}|R\rangle &= 0 \quad i \in \{0, ..., D/2-1\} \\ b^\mu_n|R\rangle &= 0 \quad n > 0 \quad , \end{aligned} \tag{6.10}$$

[81] Later, we will see that the Weyl anomaly cancels in the superstring if and only if $D = 10$.

from which we learn that the space of ground states has real dimension $2^{D/2}$. From this space it is now possible to get the whole Hilbert space $\mathcal{H}_R$ of the R-sector, which is spanned by the states of the form

$$b^{\mu_1}_{n_1}...b^{\mu_i}_{n_i}\prod_{l=0}^{\frac{D-2}{2}} b^{\epsilon_l}_{0l}\,|R\rangle \quad \text{where } n_1<...<n_i<0 \quad \text{and} \quad \epsilon_l\in\{0,1\} \quad . \tag{6.11}$$

So far we have built the Hilbert spaces $\mathcal{H}_{R/NS}$ corresponding to the R- and NS-sectors of the right movers ψ^μ_-; clearly, the same construction holds for the left moving bit of the theory (ψ^μ_+) and we could analogously define $\bar{\mathcal{H}}_{R/NS}$. Thus, the Hilbert space of the full theory is given by

$$\begin{aligned}&\Big(\mathcal{H}_R\oplus\mathcal{H}_{NS}\Big)\otimes\Big(\bar{\mathcal{H}}_R\oplus\bar{\mathcal{H}}_{NS}\Big)\cong\\ &\quad\cong\Big(\mathcal{H}_R\otimes\bar{\mathcal{H}}_R\Big)\oplus\Big(\mathcal{H}_R\otimes\bar{\mathcal{H}}_{NS}\Big)\oplus\Big(\mathcal{H}_{NS}\otimes\bar{\mathcal{H}}_R\Big)\oplus\Big(\mathcal{H}_{NS}\otimes\bar{\mathcal{H}}_{NS}\Big)\\ &\quad=\mathcal{H}_{RR}\oplus\mathcal{H}_{RNS}\oplus\mathcal{H}_{NSR}\oplus\mathcal{H}_{NSNS}\quad ,\end{aligned} \tag{6.12}$$

where in the last line we defined - with an obvious notation - the Hilbert spaces $\mathcal{H}_{RR},\mathcal{H}_{RNS},\mathcal{H}_{NSR},\mathcal{H}_{NSNS}$. All these four spaces obviously carry a representation of the Virasoro algebra. Indeed, following a standard route, we could determine the energy-momentum tensor T^ψ and decompose it into the modes

$$\begin{aligned}L^R_m&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\Big(n+\frac{m}{2}\Big):b_{-n}\cdot b_{n+m}:+\frac{D}{16}\delta^R_{m,0}\\ L^{NS}_m&=\frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\Big(r+\frac{m}{2}\Big):b_{-r}\cdot b_{m+r}:\quad ,\end{aligned} \tag{6.13}$$

where, as always, the index m for the modes is integer in the R-sector and half-integer in the NS-sector; then, it is not difficult to compute that these modes satisfy the Virasoro algebra with central charge $c=D/2$ (as opposed to $c=D$ in the case of D free bosons).

6.3 The classical supersymmetric (field) theory

We can combine the fields ψ^μ with the fields X^μ and obtain the action

$$S[X^\mu,\psi^\mu]=-\frac{1}{4\pi}\int d\sigma^0 d\sigma^1\Big(\frac{1}{\alpha'}\partial_\alpha X^\mu\partial^\alpha X_\mu+\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\Big)\quad . \tag{6.14}$$

Being just the sum of the two models we studied before, one can guess that the analysis of this system trivially reduces to the combination of the results that we already know for the free bosons and free fermions. This is true (and this is the reason why we spent some time by studying also the fermionic action alone) but it is not the whole story, because the action (6.14) enjoys a new global symmetry - the supersymmetry - whose transformations on fields are

$$\begin{aligned}\delta X^\mu&=\sqrt{\frac{\alpha'}{2}}\bar{\epsilon}\psi^\mu\\ \delta\psi^\mu&=\frac{1}{\sqrt{2\alpha'}}\rho^\alpha\partial_\alpha X^\mu\epsilon\quad ,\end{aligned} \tag{6.15}$$

where ϵ is a two-component Grassmann odd valued constant.
Supersymmetry implies a rigid mathematical structure and, in few words, it associates a supersymmetric partner to every field in the theory. For example, the superpartner of the total energy-momentum tensor $T=T^X+T^\psi$ is the *super energy-momentum tensor* G_α

$$G_\alpha\equiv\frac{i}{\sqrt{2\alpha'}}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu \tag{6.16}$$

which can be handily written with the lightcone coordinates as

$$G_\pm\equiv\sqrt{\frac{2}{\alpha'}}(\psi_\mu)_\pm\partial X^\mu_\pm\quad . \tag{6.17}$$

The super energy momentum tensor G is conserved and traceless[82] - as the energy-momentum tensor is. The tracelessness condition shows that G has the same number (i.e. 2) of independent components of T[83], whereas the conservation equation $\partial^\alpha G_\alpha = 0$ states that the two components correspond to the left- and right-movers of G (i.e. $G(z)$ and $\bar{G}(\bar{z})$).

6.4 The quantum supersymmetric (field) theory

The Hilbert space of the quantum theory has clearly four different sectors, because it is the tensor product of the Hilbert space of the bosonic theory (which is unique) and of that one of the fermionic theory (which can be NSNS, RR, NSR, RNS).
On all these spaces the Virasoro modes $L_n = L_n^X + L_n^\psi$ act; L_n^X commutes with L_n^ψ (the α_n^μ's don't talk with the b_n^μ's) and, thus, the central charge c of the system is the sum of the central charges: $c = c^X + c^\psi = D + D/2$.
Of course, on these spaces also the superpartners of the Virasoro modes act. These are the modes of the super energy-momentum tensor G_α which has conformal weight 3/2 and which can be expanded as

$$G(z) = \begin{cases} \sum_{n\in\mathbb{Z}} G_n z^{-n-\frac{3}{2}} & \text{R-sector} \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}} & \text{NS-sector} \end{cases} \quad , \tag{6.18}$$

with the modes $G_{n/r}$ given in terms of the α_n^μ, b_n^μ and b_r^μ according to

$$\begin{aligned} G_n &= \sum_{m\in\mathbb{Z}} \alpha_m^\mu b_{n-m}^\nu \eta_{\mu\nu} \qquad \text{R-sector} \\ G_r &= \sum_{m\in\mathbb{Z}} \alpha_m^\mu b_{r-m}^\nu \eta_{\mu\nu} \qquad \text{NS-sector} \quad . \end{aligned} \tag{6.19}$$

With the last two formulas one can determine the commutation relations of G_r, G_n with L_n and with each other. The set of commutation relations $[L_n, L_m]$, $[L_m, G_r]$ and $\{G_r, G_s\}$ is the *N=1 super-Virasoro algebra* which, in the NS-sector, reads as

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8} m(m^2-1)\delta_{m+n,0} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^s - \frac{1}{4}\right)\delta_{r+s,0} \quad ; \end{aligned} \tag{6.20}$$

in the R-sector, it is the same, with $r, s \to n, m$.

6.5 The action in superconformal gauge

The action for the superstring is a long expression that we don't write, because it is not particularly enlightnening (look at [17]). The important thing is that it enjoys enough symmetries (local supersymmetry, local Weyl, diffeomorphisms invariance, local super Weyl, 2−dimensional local Lorentz) that allow us to work - locally - with the so-called *superconformal gauge*, in which the action takes the simple form of the supersymmetric field theory presented in the last section:

$$S[X^\mu, \psi^\mu] = -\frac{1}{4\pi} \int d\sigma^0 d\sigma^1 \left(\frac{1}{\alpha'} \partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) \quad . \tag{6.21}$$

As we saw for the bosonic string, in order to fix the gauge in the proper manner, we should resort to the Faddeev-Popov procedure (so as to introuce the ghost system and the measure on the moduli space) and

[82] To be more precise, it is $\rho-$traceless: $\rho^\alpha G_\alpha = 0$.
[83] As we know, out of the four components as $T_{\alpha\beta}$, only two of them are independent. Both G_0 and G_1 have two components so (also) G has four components; but $\rho^\alpha G_\alpha = 0$ kills two of them.

to the BRST quantization (so as to impose the right physical condition on states). We are not going to redo all the Fadeev-Popov and BRST analysis also for the superstring and we prefer to simply sketch the analogy and the differences with the bosonic case.
In few words, we should remember that, after fixing the gauge:

- we still have to impose the right constraints on physical states. In the case of the bosonic string we had to impose the Virasoro constraints, now we have to impose the super-Virasoro ones.

- we have to add a new ghost system in the game, the so-called $\beta-\gamma$ *ghost system* which arises from fixing the local supersymmetry. This is a first order lagrangian system (as that one described in full generality in section 4.11) characterized by $\epsilon=-1$ and $\lambda=3/2$; to be more explicit, the β and γ fields are commuting fields of half-integer spin, whose conformal dimensions are respectively $(3/2,0)$ and $(-1/2,0)$. The $\beta-\gamma$ ghost system has central charge $c_{\beta\gamma}=11$ and a backgroung charge $Q_{\beta\gamma}=2$. As for the $b-c$ ghost system, this charge requires for a sensible scattering amplitude to have the right number of zero mode insertions of the β and γ fields. Note also that the total central charge of the superstring is

$$c^{tot}=c^{X}+c^{\psi}+c^{bc}+c^{\beta\gamma}=D+D/2-26+11 \quad ,$$

  so, if we don't want a Weyl anomaly, we have to require D to be $D=10$. As in the bosonic string, it is possible to show that the requirement of the critical condition on D is needed to eliminate negative norm states from the Hilbert space of the theory[84].

- in addition to the moduli, now there are also the *super-moduli* in the game, which are parameters that describe the new geometrical features of the worldsheet. Indeed, the worldsheet is not a Riemann surface anymore, but a *super-Riemann surface*, namely a Riemann surface equipped with a *spin structure* and whose transition functions are *superconformal transformations*. For more details about the geometry of such a surface, refer to [2]. For us, it is enough to know that the super-moduli can be integrated out at the cost of introducing new operators in the scattering amplitudes, the so-called *picture changing operators*; this is explained very well in [5]. Then, we can still use the formula (5.51) for the S-matrix, provided that we leave as understood the sum over the spin structure in the path integral $\langle ...\rangle$ and provided that we insert the right number of PCO in a proper way (see later).

Now we are going to briefly explain what the super-Virasoro constraints are and how to obtain the spectrum of the theory. In particular, this will lead us in a natural way to the Type IIA/B superstrings; after describing a bit of their spectrum, we will be able to construct the massless spectrum of the heterotic SO(32) superstring, which is the theory with which we will deal in the last chapter.
Then, we will introduce the picture changing operator which, being the essential new feature of the superstring, deserves its own section.

6.6 The super-Virasoro constraints

Not surprisingly, if we followed the BRST quantization, we would find that we need to impose as constraints not only the vanishing of the modes of T, but also of those of its superpartner G. As in the bosonic string, we impose the constraints in the weak sense: physical states have to be annihilated by the non-negative modes of T and G.

R-sector The Hilbert space corresponding to the unconstrained R-sector of the right-moving half of the theory is $\mathcal{H}_R^{X\psi}\equiv\mathcal{H}^X\otimes\mathcal{H}_R^{\psi}$, where, with obvious notation, we denoted with $\mathcal{H}^X$ and $\mathcal{H}_R^{\psi}$ the Hilbert spaces of the right-moving bosons and Ramond-fermions. The space $\mathcal{H}_R^{X\psi}$ carries the action of an N=1 superconformal algebra with generators L_n and G_n. We expect L_0 to be affected by a normal order ambiguity (in the bosonic case we have to impose, as a constraint, L_0-1 instead of L_0 alone). In the case

[84] To be more precise, in the R-sector the condition $D=10$ is all what we need to recover unitarity; indeed, in the R-sector, the normal order ambiguity constant of L_0 is fixed -as we will see - by the super-Virasoro algebra. Instead, in order to remove all the negative norm states in the NS-sector, we have to require - in addition to $D=10$ - for the normal order ambiguity constant in the NS-sector to assume the value 1/2.

of $G_0 = \sum \alpha^\mu_{-n} b^\nu_n \eta_{\mu\nu}$, instead, there is no such problem because it is expressed in terms of commuting operators (α^μ_n don't see b^μ_n) and we can impose $G_0 \left|\text{phys}\right\rangle = 0$. Then, from the super-Virasoro algebra we see that

$$2G_0^2 = \{G_0, G_0\} = 2L_0 - \frac{D}{8}$$

and we learn that the normal ordering ambiguity for L_0 is $-D/16$, because consistency requires $L_0 \left|\text{phys}\right\rangle = D/16 \left|\text{phys}\right\rangle$. Thus, the constrained theory is defined by the states $\left|\text{phys}\right\rangle \in \mathcal{H}^{X\psi}_R$ such that

$$\begin{aligned} L_n \left|\text{phys}\right\rangle &= 0 \quad n > 0 \\ L_0 \left|\text{phys}\right\rangle &= \frac{D}{16} \left|\text{phys}\right\rangle = \frac{5}{8} \left|\text{phys}\right\rangle \\ G_n \left|\text{phys}\right\rangle &= 0 \quad n \geq 0 \quad . \end{aligned} \tag{6.22}$$

By looking at (6.13) we see that the on-shell condition $L_0 - D/16 = 0$ can be simply rewritten as

$$0 = L_0 - \frac{D}{16} = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{+\infty} n N_n^\alpha + \sum_{n=1}^{+\infty} n N_n^b \quad , \tag{6.23}$$

where we denoted with N_n^α (N_n^b) the number of α's (b's) operators with index $-n$. So we conclude that the mass formula in this sector is given by

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} n N_n \quad , \tag{6.24}$$

with N_n that counts the number of creation operators α^μ_{-n} and b^μ_{-n} together. We immediately see that in the R-sector there is no tachyon, given that $M^2 \geq 0$. The massless states are created by the zero modes $\alpha^\mu_0 \sim p^\mu$ and b^μ_{0l}:

$$\left|p\right\rangle \otimes \prod_{l=0}^{4} b_{0l}^{\epsilon_l} \left|R\right\rangle \quad , \tag{6.25}$$

where for the fermionic excitation we used the same notation introduced around formula (6.11). Being the momentum of the centre of mass of a massless excitation, we can always suppose that $p^\mu \sim (1, -1, 0, ..., 0)$ and it is then immediate to see that the $0 = G_0 \sim \alpha^\mu_0 b^\nu_0 \eta_{\mu\nu} \sim p^\mu_0 b^\nu_0 \eta_{\mu\nu}$ constraint on the state (6.25) implies

$$(b_0^0 + b_0^1) \left(\left|p\right\rangle \otimes \prod_{l=0}^{4} b_{0l}^{\epsilon_l} \left|R\right\rangle \right) = 0 \quad ,$$

which means that the operator $b^\dagger_{00} \sim b^0_0 + b^1_0$ must annihilate the massless state (6.25). In other words, the massless state (6.25) must not contain the creation operator b_{00} and, at the massless level, we have the following 16 states

$$\left|p\right\rangle \otimes \prod_{l=1}^{4} b_{0l}^{\epsilon_l} \left|R\right\rangle \quad . \tag{6.26}$$

Under the action of the little group $SO(D-2) = SO(8)$, these 16 states decompose into two inequivalent irreducible representations, that can be distinguished by the number of fermionic modes: the states with $\sum_{l=1}^4 \epsilon_l$ even form the spinor representation $\mathbf{8}_S$ and those with $\sum_{l=1}^4 \epsilon_l$ odd form the spinor representation $\mathbf{8}_C$.

NS-sector The Hilbert space corresponding to the non-constrained NS-sector of the right-moving half of the theory is $\mathcal{H}^{X\psi}_{NS} \equiv \mathcal{H}^X \otimes \mathcal{H}^\psi_{NS}$. It carries the action of the N=1 superconformal algebra generated

by the modes L_n and G_r (remember that $r \in \mathbb{Z} + \frac{1}{2}$).
We constrained the Hilbert space by defining the physical states as those states $|\text{phys}\rangle \in \mathcal{H}_{NS}^{X\psi}$ that satisfy

$$\begin{aligned} L_n \,|\text{phys}\rangle &= 0 \quad n > 0 \\ L_0 \,|\text{phys}\rangle &= a_{NS} \,|\text{phys}\rangle \\ G_r \,|\text{phys}\rangle &= 0 \quad r \geq \frac{1}{2} \quad . \end{aligned} \tag{6.27}$$

This time, the order ambiguity constant a_{NS} is not determined by the super-Virasoro algebra and, as in the case of the bosonic string, it is fixed by the no-ghost theorem; indeed, it is possible to show that, in order to remove the negative norm states from the theory, we have to impose $D = 10$ *and*

$$a_{NS} = 1/2 \quad . \tag{6.28}$$

The on-shell condition $(L_0 - 1/2)\,|\text{phys}\rangle = 0$ gives

$$M^2 = -\frac{2}{\alpha'} + \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \frac{n}{2} N_{\frac{n}{2}} \quad , \tag{6.29}$$

where we denoted with $N_{\frac{n}{2}}$ the number of α^μ's and b^μ's modes with index[85] $-\frac{n}{2}$ (with $n > 0$). From this formula, we learn that there is the tachyonic mode $|p\rangle \otimes |0\rangle_{NS}$, where $|0\rangle_{NS}$ is the ground state in the NS-sector defined by (6.7). At the first excited level, we find the massless states

$$\xi_\mu b^\mu_{-\frac{1}{2}} \,|p^\mu\rangle \otimes |0\rangle_{NS} \tag{6.30}$$

with polarization vector ξ^μ satisfying the transversal constraint $\xi_\mu p^\mu = 0$ because of

$$0 = G_{\frac{1}{2}} \left(\xi_\mu b^\mu_{-\frac{1}{2}} \,|p^\mu\rangle \otimes |0\rangle \right) \sim b^\nu_{\frac{1}{2}} p_\nu \left(\xi_\mu b^\mu_{-\frac{1}{2}} \,|p^\mu\rangle \otimes |0\rangle \right) = \xi_\mu p^\mu \left(|p^\mu\rangle \otimes |0\rangle \right) \quad . \tag{6.31}$$

We mention that in the construction of the constrained Hilbert spaces of the R- and NS- sector, we have always left as understood that states are defined up to null states, as in the case of the bosonic string. For example, the state (6.30) has zero norm iff $\xi^2 = 0$ and, because of $\xi_\mu p^\mu = 0$, this means that the state (6.30) with transversal polarization tensor defines a physical state up to the sum of longitudinal null excitations. Once we remove this longitudinal excitation by defining the proper equivalence class, we are left with the massless vector representation $\mathbf{8}_V$ of $SO(8)$.

6.7 The GSO projection and the spectrum of Type IIA/B superstring

In the last paragraph we have seen that we have a tachyon in the NS-sector and that there is no space-time supersymmetry (between the massless space-time bosons in the representation $\mathbf{8}_V$ and the fermions in the representation $\mathbf{8}_S \oplus \mathbf{8}_C$), because we have too many fermions. The *Gliozzi-Scherk-Olive (GSO)* projection consists of neglecting the states containing an even number of fermionic generators b^μ's: the tachyon is removed from the NS-sector, along with one of the fermionic octoplets.
To construct this projection, we need two operators $\Gamma_\pm$ which commute with all bosonic operators α^μ_n and that anti-commute with the fermionic ones: $\Gamma_\pm b^\mu_l = -b^\mu_l \Gamma_\pm$ ($l \in \mathbb{Z}/2$). Then, they are uniquely defined in terms of the fermionic operators b^μ_l by their action on the vacuum states, namely by

$$\Gamma_\pm \,|p\rangle \otimes |0\rangle_{NS} = -\,|p\rangle \otimes |0\rangle_{NS} \qquad \Gamma_\pm \,|p\rangle \otimes |R\rangle = \pm\,|p\rangle \otimes |R\rangle \quad . \tag{6.32}$$

With the GSO projection, we keep only those states that have eigenvalue $+1$ under the action of $\Gamma_\pm$: the tachyon of the NS-sector and half of the ground states of the R-sector are eliminated.

[85]Here we are sloppy only to make our formulae shorter. When we say that $N_{\frac{n}{2}}(\alpha^\mu)$ counts the number of α^μ's modes with index $-n/2$ we leave as understood that $N_{\frac{n}{2}}(\alpha^\mu) = 0$ when n is odd.

To be more precise, let's consider the Hilbert space $\mathcal{H}'$ of the theory before the projection. It consists of the linear combination of the Hilbert spaces that we built in the last paragraph[86], i.e.

$$\mathcal{H}' \equiv \mathcal{H}'_{NSNS} \oplus \mathcal{H}'_{RR} \oplus \mathcal{H}'_{NSR} \oplus \mathcal{H}'_{RNS} \quad , \tag{6.33}$$

where we introduced a $'$-label to stress that $\mathcal{H}'_\bullet$ is the Hilbert space obtained by $\mathcal{H}_\bullet$ after imposing the super-Virasoro constraints. The Hilbert space $\mathcal{H}'$ admits two types of projection operators:

$$\begin{aligned} \Pi_A &\equiv \Pi_+\bar{\Pi}_- \qquad (\text{or, } \Pi_A \equiv \Pi_-\bar{\Pi}_+) \\ \Pi_B &\equiv \Pi_+\bar{\Pi}_+ \qquad (\text{or, } \Pi_B \equiv \Pi_-\bar{\Pi}_-) \end{aligned} \tag{6.34}$$

with $\Pi_\pm \equiv (1+\Gamma_\pm)/2$.

Type IIB If we implement the projection by means of the operator Π_B (we keep only those states of $\mathcal{H}'$ that have Π_B-eigenvalue $+1$), we obtain the spectrum of the so-called *Type IIB superstring theory.* In the

- NSNS-sector: the tachyon is removed and, at the massless level, we have

$$\mathbf{8}_V \otimes \mathbf{8}_V \cong \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_V \quad , \tag{6.35}$$

  where we displayed, on the right-hand side, the usual decomposition of the product of two vectors into the trace, antisymmetric, symmetric traceless tensors; as we already know, these correspond to a space-time dilaton, Kalb-Ramond field and graviton.

- RR-sector: the massless excitations are

$$\mathbf{8}_S \otimes \mathbf{8}_S \cong \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_V \quad , \tag{6.36}$$

  which correspond to a $0-$, $2-$ and $4-$form.

- RNS-sector: the massless excitations are

$$\mathbf{8}_V \otimes \mathbf{8}_S \cong \mathbf{8}_S \oplus \mathbf{56}_S \quad , \tag{6.37}$$

  which are the left-handed dilatino and the left-handed gravitino.

- NSR-sector: the massless excitations are

$$\mathbf{8}_S \otimes \mathbf{8}_V \cong \mathbf{8}_S \oplus \mathbf{56}_S \quad , \tag{6.38}$$

  which give another left-handed dilatino and another left-handed gravitino.

Note that NSNS- and RR-sectors represent space-time bosons, whereas NSR- and RNS-sectors give space-time fermions, and that the number of the massless fermions matches with the number of the massless bosons. The spectrum is chiral[87]: the massless spectrum of the Type IIB superstring ($N=2$ supersymmetry) gives Type IIB supergravity in 10 dimensions.

Type IIA If we implement the projection by means of the operator Π_A (we keep only those states of $\mathcal{H}'$ that have Π_A-eigenvalue $+1$), we obtain the spectrum of the so-called *Type IIA superstring theory.* In the

- NSNS-sector: the tachyon is removed and, at the massless level, we obtain the dilaton, the Kalb-Ramond field and the graviton, exactly as before.

[86] To streamline the notation, we drop the $X\psi$ label from the Hilbert spaces; for example, we write $\mathcal{H}_{NSR}$ instead of $\mathcal{H}^{X\psi}_{NSR}$, because at this point of the thesis it is obvious that we are dealing with both fermions and bosons at the same time.

[87] In the sense that both gravitinos (dilatinos) are left-handed. If one had used $\Pi_B = \Pi_-\bar{\Pi}_-$ instead of $\Pi_B = \Pi_+\bar{\Pi}_+$, the two gravitinos (dilatinos) would have been right-handed; the spectrum would have been the same, because right/left is only a matter of conventions.

- RR-sector: the massless excitations are

$$\mathbf{8}_C \otimes \mathbf{8}_S \cong \mathbf{8}_V \oplus \mathbf{56}_V \quad , \tag{6.39}$$

which correspond to a 1− and 3−form.

- RNS-sector: the massless excitations are

$$\mathbf{8}_V \otimes \mathbf{8}_S \cong \mathbf{8}_S \oplus \mathbf{56}_S \quad , \tag{6.40}$$

which are the left-handed dilatino and the left-handed gravitino.

- NSR-sector: the massless excitations are

$$\mathbf{8}_C \otimes \mathbf{8}_V \cong \mathbf{8}_C \oplus \mathbf{56}_C \quad , \tag{6.41}$$

which give a right-handed dilatino and a right-handed gravitino.

As before, the number of the massless fermions (NSR- and RNS-sectors) matches with the number of the massless bosons (NN- and RR- sectors). The spectrum is not chiral: the massless spectrum of Type IIA superstring ($N = 2$ supersymmetry) gives Type IIA supergravity in 10 dimensions.

We like to conclude this chapter with two comments.
First, we have seen that with both types of superstrings we have recovered a supersymmetric massless spectrum, but supersymmetry should be a property of the whole of the spectrum; it is indeed possible to show that supersymmetry holds also at every massive level of the spectrum.
Second, to keep things simple, we presented the GSO-projection as a dirty trick introduced by hand in the theory; but it is not a dirty trick, because, in the covariant NSR-formulation discussed here, the GSO-projection is implemented in the path-integral by summing over all the spin structures (refer to [17] for more details). Given that the sum over spin structures has to be performed for every scattering amplitude, we understand that, regardless of the number and nature of the external states, the states killed by the GSO projection will never appear as modes exchanged in string scattering experiments and this means that removing them from the spectrum is a consistent procedure[88].

6.8 The heterotic superstring theory

So far we have discussed superstring theories with N=2 supersymmetry in 10 dimensions, namely with 32 supercharges. In this chapter, by following the approach of [20] and [21], we are going to mention other two superstring teories that, instead, have the minimal number of supercharges, i.e. 16. They have N=1 supersymmetry and they are tachyon free. They are the so-called *heterotic superstring theories* and can be effectively[89] described by N=1 supergravity in 10 dimensions. The latter is a chiral theory which admits two kinds of massless supermultiplets:

- the N=1 vector multiplet which, under the action of the little group $SO(8)$ decomposes into

$$\mathbf{8}_V \oplus \mathbf{8}_{S/C} \quad , \tag{6.42}$$

namely into a vector boson of eight physical polarizations and into eight fermionic degrees of freedom.

[88] This is in apparent contrast with what happens for the bosonic string. For example, one could compute the 4-point scattering amplitude among massless bosonic strings and the result would have a pole (in the Mandelstam variable t) for every state of the spectrum. In particular, the first pole would appear at $t = M^2 = -4/\alpha'$, namely for the tachyon, which then cannot be removed from the spectrum. In the superstring, instead, the states killed by the GSO projection like the tachyon simply are not exchanged among strings, do not appear in loops.

[89] In the limit in which we get rid of the massive spectrum by taking $\alpha' \to 0$. We have also to assume that higher loops contributions (which are suppressed by additional power of g_s) can be neglected and we have to work at the weak coupling $g_s \ll 1$.

- the N=1 graviton multiplet which, under the action of $SO(8)$, decomposes into

$$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{56}_{S/C} \oplus \mathbf{8}_{S/C} \quad , \tag{6.43}$$

namely into a dilaton, the Kalb-Ramond field B, a graviton, a gravitino and a dilatino.

We can use one graviton supermultiplet and a certain copies of the vector supermultiplet to build the N=1 supergravity action as (we write down only its bosonic part)

$$S_{10} \sim \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left[\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{2}(\tilde{H}_3)^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr(|F_2^2|^2) \right] \quad , \tag{6.44}$$

where: κ_{10} plays the role of the Newton constant in 10 dimensions, $\mathcal{R}$ is the Ricci scalar associated to the spacetime metric $G_{\mu\nu}$, ϕ is the dilaton and F_2 denotes the field strength of a Yang-Mills gauge field A_μ with gauge group G_A and with coupling constant g_{10}. Note that the kinetic term H_3^2 for the Kalb-Ramond field ($H_3 = dB$) is contained - along with interactions between the gauge and gravity content of the theory - in the term $\tilde{H}_3^2$, because we defined

$$\tilde{H}_3 \equiv dB - \frac{\kappa_{10}^2}{g_{10}^2} \Omega_{CS}(A) \quad , \tag{6.45}$$

with $\Omega_{CS}(A)$ given by the so-called Chern-Simon 3-form of the gauge field, i.e.

$$\Omega_{CS}(A) = \mathrm{tr}\Big(A \wedge dA - \frac{1}{3} A \wedge A \wedge A \Big) \quad .$$

At the classical level, this action enjoys diffeomorphism invariance, gauge invariance and N=1 local supersymmetry invariance. It is possible to show that, in order to avoid anomalies and, thus, negative norm states in the quantum theory, we have to require the Lie algebra of G_A to be a 496-dimensional Lie algebra isomorphic to that one of $SO(32)$ or to that one of $E_8 \times E_8$.
It is then intuitive that - correspondingly - there are only two consistent heterotic superstring theories, from which the $SO(32)$ and the $E_8 \times E_8$ supergravity limits can be recovered. We will present only the $SO(32)$ heterotic superstring, given that it is the one that will be used in the last chapters. But let's first present the fundamental idea underlying the heterotic string.
The simplest way to obtain an N=1 supersymmetry (in spacetime) is to combine an N=1 (spacetime) supersymmetric spectrum for the right movers with a (spacetime) non-supersymmetric spectrum for the left movers. We stress the fact that we have already met an N=1 supersymmetric spectrum for one single mover of the theory. Indeed, if we look back into our discussion of Type II superstring theories, we see that, after the GSO projection, in the right moving sector we got massless modes in the representation $\mathbf{8}_V \oplus \mathbf{8}_{S/C}$ that we then tensored with the representation $\mathbf{8}_V \oplus \mathbf{8}_{S/C}$ of the massless excitations coming from the left-moving sector in order to obtain the N=2 supersymmetric spectrum of Type II theories[90]. But from (6.42) we immediately recognize $\mathbf{8}_V \oplus \mathbf{8}_{S/C}$ as the representation of the vector supermultiplet for N=1 supersymmetry! In light of this, we can take the supersymmetric N=1 right moving sector of the Type II superstring and combine it with the non-supersymmetric left moving sector of the bosonic string. By recasting this goal in the language of the CFT, we need to build a theory with central charges $(c, \bar{c}) = (15, 26)$. Since we want to describe a theory propagating in a ten dimensional Minkowski space, for sure we need 10 bosonic fields X^μ and these fields contribute with $(c^X, \bar{c}^X) = (10, 10)$. On the right moving sector then we add 10 fermions ψ^μ so as to reach a supersymmetric rigth moving sector with the desired $c = 15$. In the left sector, we are tempted to add other 16 bosonic fields X^μ so as to reach $\bar{c} = 26$, but this would also bring $c = 15$ to $c = 31$; we remember that the 2-dimensional Majorana fermions have two components, one of which is purely right/left-moving, so we add into the system 32 left moving Majorana fields. We end up with the following action:

$$\begin{aligned} S_{het}[X^\mu, \psi^\mu, \lambda^\alpha] \equiv & \frac{1}{\pi\alpha'} \int d\sigma^0 d\sigma^1 \partial_+ X^\mu \partial_- X_\mu + \\ & + \frac{1}{2\pi} \int d\sigma^0 d\sigma^1 \big(\lambda^\alpha \partial_- \lambda_\alpha + \psi_-^\mu \partial_+ \psi_{-,\mu} \big) \quad , \end{aligned} \tag{6.46}$$

[90] To be more precise, the massless spectrum of Type IIB is $(\mathbf{8}_V \oplus \mathbf{8}_S) \otimes (\mathbf{8}_V \oplus \mathbf{8}_S)$, whereas the massless spectrum of Type IIA is $(\mathbf{8}_V \oplus \mathbf{8}_C) \otimes (\mathbf{8}_V \oplus \mathbf{8}_S)$.

where $\alpha \in \{1, ..., 32\}$.
With this action, we can describe different theories that can be distinguished by their GSO-projection and by the choice of boundary conditions for the left-moving fermions λ^α. It turns out[91] that there are only two consistent boundary conditions for the 32 fermion. The first choice consists of treating all of them on the same footing: we can impose the same boundary conditions on every λ^α and this clearly brings us to a model with an $so(32)$ symmetry. Alternatively, we can split the 32 fermions into two groups of 16 fermions each and independently choose for these two groups the boundary conditions; this is the $E_8 \times E_8$ heterotic superstring.
In each case, the Hilbert space of the right-sector will be generated by the α^μ_{-n} (modes of X^μ) and $b^\mu_{-n/r}$ (modes of ψ^μ) and will be subject to the super-Virasoro constraints of the N=1 superconformal algebra, whereas the Hilbert space of the left-sector will be generated by the α^μ_{-n} and by the $c^\alpha_{-n/r}$ (modes of λ^α) and will be subject to the Virasoro constraints. As we know, the formula for the mass spectrum comes, in both cases, from the L_0 constraint (or - equivalently - from the $\bar{L}_0$ constraint[92]):

$$M^2 = \frac{4}{\alpha'}(-a^* + L_0') = \frac{4}{\alpha'}(-\bar{a}^* + \bar{L}_0') \quad , \tag{6.47}$$

where we added the $'-$label in order to denote the L_0, $\bar{L}_0$ Virasoro generators without the contributions from the zero modes. The values of the order ambiguity constant a^* (and $\bar{a}^*$) depend on the kind of constraints. In the case of Virasoro constraints, we have $a^* = 1$, as we saw for the bosonic string; for the super-Virasoro constraint we have $a^* = 1/2$ in the NS-sector and $a^* = 5/8$ in the R-sector, as we discussed for the superstring. The expressions for L_0' and $\bar{L}_0'$ are

$$L_0' = \sum_{n=1}^{+\infty} \frac{n}{2} N_{\frac{n}{2}} + \frac{n_P^f}{16} \qquad \bar{L}_0' = \sum_{n=1}^{+\infty} \frac{n}{2} \bar{N}_{\frac{n}{2}} + \frac{\bar{n}_P^f}{16} \quad , \tag{6.48}$$

where $N_{\frac{n}{2}}$ ($\bar{N}_{\frac{n}{2}}$) counts all the $-n/2$ operator modes (fermionic and bosonic) of the right (left) sector, whereas the real number n_P^f ($\bar{n}_P^f$) counts the right- (left-) moving fermions with periodic boundary conditions (see (6.13)). With these formulae for L_0' and $\bar{L}_0'$, we can rewrite (6.47) in a more direct way as

$$M^2 = \frac{4}{\alpha'}\Big(K + \sum_{n=1}^{+\infty} \frac{n}{2} N_{\frac{n}{2}}\Big) = \frac{4}{\alpha'}\Big(\bar{K} + \sum_{n=1}^{+\infty} \frac{n}{2} \bar{N}_{\frac{n}{2}}\Big) \quad , \tag{6.49}$$

where

$$K \equiv -a^* + \frac{n_P^f}{16} \qquad \bar{K} \equiv -\bar{a}^* + \frac{\bar{n}_P^f}{16} \tag{6.50}$$

are the lowest eigenvalue of M^2 in the right- and left- sector.
With these formulae, now we are ready to discuss the spectrum of the heterotic string.

The $SO(32)$ Heterotic superstring If we impose the same boundary conditions on all the left-moving 32 fermions λ^α, then we obtain two sectors for the left-moving bit of the theory: one when all of them satisfy periodic (P) boundary conditions, the other one when they are subject to anti-periodic (A) boundary conditions. Thus, according to formula (6.49), to find the spectrum of this theory we have to find the values of the constants K and $\bar{K}$ for all the four sectors (P,R), (P,NS), (A,R) and (A,NS).
In the right-sector we have N=1 supersymmetry (Super-Virasoro constraints) and

$$K = -a^* + \frac{n_P^f}{16} = \begin{cases} -\frac{1}{2} + \frac{0}{16} = -\frac{1}{2} & \text{NS-sector} \\ -\frac{5}{8} + \frac{10}{16} = 0 & \text{R-sector} \end{cases} \tag{6.51}$$

[91] It is enough to compute the one-loop partition function of the theory defined by (6.46) and to require it to be modular invariant.
[92] Remember that the level matching condition still holds, even if we're describing the left/right-sectors of the theory with different fields; as we have already explained, the states of the theory must be invariant under the action of $L_0 - \bar{L}_0$, otherwise not all the points of the closed string would be indistinguishable.

so we recover formulae (6.29) and (6.24).
In the left-sector, we have the Virasoro constraint ($\bar{a}^* = 1$) and $\bar{K}$ is

$$\bar{K} = -\bar{a}^* + \frac{\bar{n}_P^f}{16} = \begin{cases} -1 + \frac{32}{16} = 1 & \text{P-sector} \\ -1 + \frac{0}{16} = -1 & \text{A-sector} \end{cases} \quad ; \tag{6.52}$$

please note that in the periodic-sector we must have $M^2 \geq \frac{4}{\alpha'}(+1) > 0$ and therefore here we cannot have contributions to the massless spectrum.
Anyway, with formulae (6.52) and (6.51), the mass formula (6.49) is determined; to determine the spectrum, we need only to find the proper GSO projection.
For the right-sector, we can recycle the projection operators $\Pi_\pm = (\frac{1+\Gamma_\pm}{2})$, with $\Gamma_\pm$ defined around (6.32); we know that this projection removes the tachyon from the NS-sector and thus imposes $M^2 \geq 0$: because of the level matching condition, this means that, regardless of the GSO projection used in the left-half of the theory, the heterotic superstring is tachyon free. In particular, we can select a GSO projection[93] on the left-sector that keeps the tachyon $|A\rangle$ of the A-sector ($\bar{K} = -1 < 0$ for anti-periodic boundary conditions) and this allows us to obtain a richer massless spectrum on the left-half of the theory, as now we are going to explain.
To study the massless spectrum of the theory it is enough to analyze the case in which all the 32 left-moving fermions have anti-periodic boundary conditions, because in the P-sector we have $M^2 \geq \frac{4}{\alpha'} > 0$; the massless spectrum of the $SO(32)$ heterotic string arises from the (A,NS) and (A,R) sectors. We already know that the NS-sector gives massless bosons in the $\mathbf{8}_V$ representation and that the R-sector yields fermions in the $\mathbf{8}_{S/C}$ representation; thus, the contribution to the massless spectrum coming from the right-movers is

$$(\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{8}_{S/C}, \mathbf{1}) \quad ,$$

where we denoted each representation with two labels $(\mathbf{a}, \mathbf{b})$, because there are two symmetries in the game: $\mathbf{a}$ specifies the representation of the little group $so(8)$, whereas $\mathbf{b}$ refers to the representation of the $so(32)$ gauge symmetry (which affects only the left-sector, so it is trivially represented on the right-sector). To find the contribution to the massless spectrum coming from the left-movers, we have to look at the mass-formula, which, in the A-sector, reads as

$$M^2 = \frac{4}{\alpha'}\Big(-1 + \sum_{n=1}^{+\infty} \frac{n}{2} \bar{N}_{\frac{n}{2}}\Big) \quad ; \tag{6.54}$$

we immediately see that we can obtain $M^2 = 0$ in two ways:

- we can act on the ground state $|k\rangle \otimes |A\rangle$ with the modes α^μ_{-1}; the GSO projection keeps this states simply because it doesn't kill the anti-periodic ground state $|A\rangle$. As usual, we have to impose the physical state condition on these 10 states and after removing the null norm states we end up with the $\mathbf{8}_V$ representation of $so(8)$; note that we have built this representation without the fermionic modes c^α's and this means that, to be more precise, we have obtained the representation $(\mathbf{8}_V, \mathbf{1})$.

- we can also act with a couple of fermionic modes $c^\alpha_{-1/2}$; indeed, the states $c^\alpha_{-1/2} c^\beta_{-1/2} |p\rangle \otimes |A\rangle$ are not removed by the GSO projection, because they contain an even number of fermionic modes. These are $32 * 32/2 = 496$ states that form the adjoint representation of $so(32)$, that in our conventions we denote as $(\mathbf{1}, \mathbf{496})$ (the states have no spacetime indices, so $so(8)$ is trivially represented on them).

Putting together the massless contributions from the right- and left- movers, we finally get that the massless spectrum of the $SO(32)$ heterotic superstring is given by

$$\Big[(\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{8}_{S/C}, \mathbf{1})\Big] \otimes \Big[(\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{496})\Big] \quad , \tag{6.55}$$

[93] We only mention that for the left-sector, in case of anti-periodic conditions, we use the projection operator given by

$$\bar{\Pi}^A_\pm \equiv \frac{1}{2}(1 - \bar{\Gamma}^A_\pm) \quad , \tag{6.53}$$

where $\bar{\Gamma}^A_\pm$ are the analogue of the $\Gamma_\pm$ of formula (6.32): they are defined such that they anti-commutes with the modes c^α_r and they are -1 on the ground state; note that the ground state $|A\rangle$ of the anti-periodic sector and all the states that are built from it with an even number of fermionic modes c^α_r are not projected out.
In case of periodic conditions, we use similar operators $\bar{\Pi}^P_\pm$.

which can be decomposed into the following irreducible representations

$$\begin{aligned}&\Big[(\mathbf{8}_V,\mathbf{1})\oplus(\mathbf{8}_{S/C},\mathbf{1})\Big]\otimes\Big[(\mathbf{8}_V,\mathbf{1})\oplus(\mathbf{1},\mathbf{496})\Big]=\\&=(\mathbf{1},\mathbf{1})\oplus(\mathbf{28},\mathbf{1})\oplus(\mathbf{35},\mathbf{1})\oplus(\mathbf{8}_V,\mathbf{496})\oplus\text{fermionic partners}\end{aligned}\tag{6.56}$$

which are precisely the dilaton, the Kalb-Ramond field, the graviton, the $so(32)-$ vector (and all their fermionic partners) of the N=1 supergravity with gauge group $G_A = SO(32)$.

6.9 The PCO

In bosonic string theory, we saw that the $b-c$ ghost system was essential to build BRST-invariant vertex operators. Clearly, the same will still hold for the superstring, and we expect that some bits of the $\beta-\gamma$ ghost system should be included in the construction of the covariant vertex operators of the superstring. We will be interested in the heterotic superstring so, from now on, we will focus on the right-half of the theory (in the left-sector there is no supersymmetry, no $\beta-\gamma$ system).
It is immediate to guess, by analogy with (4.119), that a BRST-invariant state should have the following form

$$\underbrace{|p\rangle}_{\text{centre of mass momentum}}\otimes\underbrace{|c\rangle}_{bc\text{ sector}}\otimes\underbrace{|V^{X\psi}\rangle}_{\text{NS/R sector}}\otimes\underbrace{|q\rangle}_{\beta\gamma\text{ sector}}\quad,\tag{6.57}$$

where:

- we introduced $|p\rangle = e^{ip\cdot X}\,|0\rangle_X$ so as to give motion to the string in the 10-dimensional spacetime (remember (4.138), where we showed that the insertion of $e^{ip\cdot X}$ gives momentum to the centre of mass of the string). Note that this is the only way to get a non-vanishing momentum for the centre of mass of the string, because the equations of motions for the fermionic fields ψ^μ are $\partial_\pm\psi^\mu_\mp = 0$ and this means that ψ^μ cannot have a linear term in the time variable σ^0 (σ^2) *on the cylinder*[94]; this is in agreement with the interpretation of the fields ψ^μ, which are not embeddings in spacetime so they should not give information about the centre of mass of the string.

- we inserted $|c\rangle = c(0)\,|0\rangle_{bc}$, as in (4.119). Remember that - locally, in the superconformal gauge - the fields X^μ, ψ^μ, b, c, β, γ are not interacting; thery are independent[95], so we can recycle our knowledge about the bosonic string.

- the state $|V^{X\psi}\rangle$ is created from the vacuum of the matter sector ($|0\rangle_{NS}$ or $|R\rangle$) by an operator which can have both X^μ and ψ^μ dependence. Clearly, out of the four states appearing in (6.57), this is the only one that is subjected to the restrictions given by the GSO projection.

- the state $|q\rangle = U_q(0)\,|0\rangle_{\beta\gamma}$ is created from the $PSL(2,\mathbb{C})$ vacuum of the $\beta\gamma$ theory by an operator U_q. This is in close analogy with what we studied for the bc system; because of the non-unitarity of the ghost system ($h(c) = -1 < 0$ for the conformal ghost system, $h(\gamma) = -1/2 < 0$ for the superconformal one) the $PSL(2,\mathbb{C})$ vacuum is not the ground state and, to obtain the state of lowest energy, we have to act on it with some ghosts, as we are going to explain in more detail now.

Fermi and Bose sea level Let's focus on a first order Lagrangian system, as those ones discussed in full generality in section[96] 4.11. According to (4.73), the energy of the system is given by $L_0 = \sum_{m\in\mathbb{Z}} m : b_{-m}c_m :$ and it is not difficult to find that $[L_0, c_n] = -nc_n$, namely that

$$L_0\big(c_n\,|0\rangle\big) = -n\big(c_n\,|0\rangle\big)\quad.\tag{6.58}$$

[94] Instead, the equations of motion for the bosonic fields X^μ are of the second order ($\partial_-\partial_+X^\mu = 0$) and this allows X^μ to have a non-vanishing linear term $\alpha_0^\mu\sigma^0 \sim p^\mu\sigma^0$ on the cylinder.

[95] From a global point of view, all these fields are not completely independent and, because of supersymmetry, their boundary conditions must be chosen according (see [17]). For us, it is enough to know that, for consistency, the boundary conditions of the β and γ fields are precisely the same of those imposed on the fields ψ^μ, and now formula (4.68) should be clear, where we labeled the modes of β and γ in the R-sector (NS-sector) with integer (half-integer) numbers.

[96] As we did in 4.11, again we mainly refer to [7].

We also know (see (4.70)) that

$$c_n |0\rangle = 0 \quad \text{for } n \geq \lambda, \tag{6.59}$$

and this implies that

$$L_0\big(c_n |0\rangle\big) = -n\big(c_n |0\rangle\big) \neq 0 \quad \text{for } n < \lambda \quad . \tag{6.60}$$

We have just obtained the key result: for $0 < n < \lambda$, c_n lowers the energy of the $PSL(2,\mathbb{C})$ vacuum $|0\rangle$.
For example, in the case of the conformal ghost system, $\lambda = 2$ and, according to (6.60), the ground state of the system must be built from $|0\rangle$ by acting with c_1; we thus recover the well-known ground state of the bc ghost system, i.e. $c_1 |0\rangle = c(0) |0\rangle = |c\rangle$; we stress that there are no other possible ground states, because c_1 is Grassmann odd: we can act on $|0\rangle$ with c_1 only one time and this means that the system of the anticommuting ghosts b, c has a spectrum which is bounded from below[97]. The situation is reminescent of systems with Fermi statistics, in which a Fermi-sea level must be specified; we can build states with negative energy, because we can define different vacua by specifying to which level the Fermi-sea is filled. These vacua are stable thanks to Pauli's exclusion principle and this is in agreement with the fact that the spectrum is bounded from below.
Instead, the superconformal ghost system ($\lambda = 3/2$) consists of Grassmann even fields, and, obviously, this is a remarkable difference. Acording to (6.60), we can build the ground state of the $\beta\gamma$ system by acting on the vacuum by means of γ_n with $0 < n < 3/2$. To be more concrete, let's consider the NS-sector, where the modes are labeled by half-integer numbers . This means that we can reach the ground state by acting with $\gamma_{1/2}$ on $|0\rangle$[98]; $\gamma_{1/2}$ is a bosonic operator, we can apply it an arbitrary number of times and, thus, the spectrum of the $\beta\gamma$ theory is unbounded from below, because the possible ground states $(\gamma_{1/2})^n |0\rangle$ have energy $-n/2$ (with n arbitrary). This is in apparent contrast with the case of the Fermi-statistics, where the bounded-from-below-spectrum gives us a clear notion of ground state; in the case of Bose-statistics there is no natural and unique choice for the ground state. Anyway, as in the Fermi case, we define a ground state by specifying the energy level below which all the levels of the "Bose-sea" are filled. This time there is no exclusion principle that guarantees the stability of the ground state, so one would guess that, regardless of the ground state that we can specify, the system would collapse to the infinite bottom of the spectrum; but, our $\beta\gamma$ system is a free theory and, without interaction, transitions between levels are not allowed. So, in this case, it makes sense to define a ground state even for the Bose-sea level. We define the possible ground states $|q\rangle$ by

$$\gamma_n |q\rangle = 0 \quad \text{ for } \; n \geq q + \lambda = q + \frac{3}{2} \quad , \tag{6.61}$$

which, for consistency (see (4.70)) requires also

$$\beta_n |q\rangle = 0 \quad \text{ for } \; n > -q - \lambda = -q - \frac{3}{2} \quad , \tag{6.62}$$

where we left as understood that $q \in \mathbb{Z}$ for the NS-sector and $q \in \mathbb{Z} + \frac{1}{2}$ for the R-sector.
Now we must solve the constraints (6.61), (6.62) to find the state $|q\rangle$ and it is precisely at this point that the bosonization of the first order Lagrangian systems reveals itself to be very handy.
Let's remind us that the bosonization of the $\beta\gamma$ is encoded in the following expressions

$$\begin{gathered}
\beta(z) = e^{-\phi(z)}\partial\xi(z) \qquad\qquad \gamma(z) = e^{\phi(z)}\eta(z) \\
\phi(z)\phi(w) \sim -\log(z-w) \qquad\qquad \xi(z)\eta(w) \sim \frac{1}{z-w} \\
j^{\beta\gamma}(z) = -:\beta(z)\gamma(z): = -\partial\phi(z) \quad ,
\end{gathered} \tag{6.63}$$

[97] Actually, from (6.58) we see that c_0 commutes with L_0 so also the state $c_0 c_1 |0\rangle = |\partial cc\rangle$ is a possible ground state. But it clearly has the same energy as $|c\rangle$, so the spectrum of the bc system is still bounded from below; anyway, as we know, this state can be neglected, when the bc system is seen as part of string theory, so we consider only $|c\rangle$.

[98] One can equivalently reformulate the argument in the R-sector, by considering γ_1 instead of $\gamma_{1/2}$.

where η and ξ are fermionic fields with conformal dimensions 1 and 0, $c^{\eta\xi} = -2$ and $Q^{\eta\xi} = -1$. One can find that (6.61) and (6.62) are satisfied[99] by

$$|q\rangle \equiv e^{q\phi(0)} |0\rangle_{\beta\gamma} \quad , \tag{6.65}$$

where we denoted, as usual, with $|0\rangle_{\beta\gamma}$ the $PSL(2,\mathbb{C})$ vacuum of the $\beta\gamma$ system. We stress the obvious fact that the possible ground states $|q\rangle$ can be distinguished by their conformal properties, which are encoded in (4.99) and that here are summarised:

$$\begin{aligned} L_0^{\beta\gamma} |q\rangle &= -\frac{q}{2}\left(q + Q^{\beta\gamma}\right) |q\rangle = -\frac{q}{2}(q+2)\,|q\rangle \\ j_0^{\beta\gamma} |q\rangle &= q\,|q\rangle \quad . \end{aligned} \tag{6.66}$$

We have just understood the meaning of each component of formula (6.57). We want to keep track of the $q-$charge of the ground state selected for the $\beta\gamma$ system, so we will add a label q to the fixed version F_q of the vertex operator:

$$|F_q\rangle \cong |p\rangle \otimes |c\rangle \otimes |V^{X\psi}\rangle \otimes |q\rangle \quad . \tag{6.67}$$

To be more precise, with the $q-$label we will denote the so-called *picture number* of the vertex operator. The picture number N_p is a new quantum number which naturally appears in the superconformal ghost system and which is defined as

$$N_p \equiv \oint \frac{dz}{2\pi i}\big(\xi(z)\eta(z) - \partial\phi(z)\big) \quad , \tag{6.68}$$

such that the original β, γ ghosts have picture number zero. In our approach[100], the picture number arises through the bosonization, when we have to choose a ground state for the $\beta\gamma$ system.
Clearly, in the context of the superconformal ghost system, there isn't a particular ground state $|q\rangle$ which is preferred out of all the possible ones. On the other hand, in the context of the superstring, it does exist a favourite ground state $|q\rangle$ because, in order to build a BRST-invariant state, the vertex operator F_q must be characterized by a precise picture number[101], as now we are going to show for vertex operators corresponding to massless states in the NS-sector (which precisely are the kind of vertex operators with which we will have do deal in the next chapter).
The masslessness condition on the state $|p\rangle$ fixes the conformal dimension of $e^{ip\cdot X}$ to be 0, because, as we know from the bosonic string (see (4.137)), we have $h(e^{ip\cdot X}) \sim p^2$. If the state comes from the NS-sector, we have to impose the corresponding super-Virasoro constraint[102] on the state $|V^{X\psi}\rangle = V^{X\psi}(0)\,|0\rangle_{NS}$ and (6.28) fixes the conformal dimension of $V^{X\psi}$ to be $1/2$. A physical state must be invariant under gauge

[99]For example,

$$\begin{aligned} \beta_n |q\rangle &= \oint \frac{dz}{2\pi i} z^{n+\frac{1}{2}} \beta(z) : e^{q\phi(0)} : |0\rangle = \\ &= \oint \frac{dz}{2\pi i} z^{n+\frac{1}{2}} e^{-\phi(z)}\partial\xi(z) : e^{q\phi(0)} : |0\rangle = \\ &= -\oint \frac{dz}{2\pi i} z^{n+\frac{1}{2}} \partial\xi(z) e^{-q\langle\phi(z)\phi(0)\rangle} : e^{-\phi(z)+q\phi(0)} : |0\rangle = \\ &= -\oint \frac{dz}{2\pi i} z^{n+\frac{1}{2}+q} : \partial\xi(z) e^{-\phi(z)+q\phi(0)} : |0\rangle = \\ &= -\oint \frac{dz}{2\pi i} z^{n+\frac{1}{2}+q} \Big(O(1)\Big)\,|0\rangle \quad , \end{aligned} \tag{6.64}$$

where, in the last step, we rewrote the normal ordered product as $O(1)$ because, by definition, the normal order product is regular ar $z = 0$ and it can be thus expanded as a positive power series in the variable z; so, in the case also $n + \frac{1}{2} + q$ was non-negative, then $\beta_n |q\rangle = 0$ vanishes. We have recovered equation (6.62).
[100]Which is that one proposed by D. Friedan, E. Martinec and S. Shenker in [7].
[101]The situation is similar to that one that we met during the construction of the physical states of the bosonic string: according to the bc ghost system, both the states $|c\rangle$ and $|\partial cc\rangle$ can be equivalently used as ground states, something that is not anymore true in string theory where, because of the BRST-invariance, we have to reject all the vertex operators built on $|\partial cc\rangle$.
[102]We are reasoning by analogy with the bosonic string, where we saw that the BRST-invariance of a physical state was equivalent to the Virasoro constraints.

transformations, so the corresponding operator (here F_q) must have, in particular, vanishing conformal dimension, i.e.

$$\begin{aligned}0 &\overset{!}{=} h(F_q) = h(e^{ip\cdot X}) + h(c) + h(V^{X\psi}) + h(e^{q\phi}) = 0 - 1 + \frac{1}{2} + h(e^{q\phi}) \\ &\overset{(6.66)}{=} -\frac{1}{2} - \frac{q}{2}(q+2) \quad \Longleftrightarrow \quad q = -1 \quad .\end{aligned} \tag{6.69}$$

If we want the operator F_q corresponding to the state (6.67) to describe a physical (BRST-invariant) massless state in the NS-sector, then we necessarily have to fix its picture number to be -1; NS (fixed) vertex operators with $q = -1$ are said to be in the *canonical picture*. Analogously, one can find[103] that the canonical picture in the massless R-sector is given by $q = -1/2$.
Since the OPE between operators like $e^{q_1\phi(z)}$ and $e^{q_2\phi(w)}$ reads

$$\begin{aligned}e^{q_1\phi(z)}e^{q_2\phi(w)} =& e^{q_1 q_2\langle\phi(z)\phi(w)\rangle} : e^{q_1\phi(z)+q_2\phi(w)} := \\ =& (z-w)^{-q_1 q_2} : e^{(q_1+q_2)\phi(w)} + O(1) : \quad ,\end{aligned} \tag{6.70}$$

it is apparent that vertex operators in non-canonical pictures appear in the theory. In order to understand the meaning of states with arbitrary picture number (whose existence is required for the closure of the OPE algebra) we need to resort to the BRST-formalism, because, as we are going to explain, we must learn how to change picture number of a vertex operator in a "BRST-invariant"-way.
Let's suppose that we want to construct a correlation function that is manifestly BRST-invariant, so it must consist of a bunch of BRST-invariant operators. Let's suppose that we want to compute the S-matrix element for a process involving $n+m$ vertex operators, n (m) of which correspond to massless states coming from the NS (R) sector. In order to obtain a BRST-invariant expression, we could insert all the vertex operators in their canonical pictures, and this would bring the picture number of the correlation function to $-n-m/2$. But, as we know, the $\beta\gamma$ system is anomalous; to be more precise, the anomaly affects the ghost-number current and (4.88) reads as

$$\nabla^z j_z^{\beta\gamma}(z) = \frac{1}{4}Q^{\beta\gamma}R = \frac{1}{2}R \quad . \tag{6.71}$$

This is a local expression that can be integrated to give the superconformal analogue of equation (5.37), namely

$$N_\gamma - N_\beta = -\frac{Q^{\beta\gamma}}{2}\Xi = -2(1-g) = 2g-2 \quad . \tag{6.72}$$

According to the bosonization of the $\beta\gamma$ system, we have $j^{\beta\gamma} = - : \beta\gamma := -\partial\phi$, so the $\beta\gamma$ anomaly doesn't affect the $\xi\eta$ system and it is translated into an anomaly of the ϕ-system. Each $\gamma = e^{\phi}\eta \sim e^{\phi}$ carries one unit of $\phi-$charge, whereas each $\beta = e^{-\phi}\partial\xi \sim e^{-\phi}$ decreases by -1 the $\phi-$charge; (6.72) thus states that the total $\phi-$charge of the insertions must be $2g-2$.
This means that we have to introduce a BRST-invariant expression in the correlation function which must be able to rise the ϕ-charge of the correlation function by $2g-2+n+m/2$ units.
The right way to do it is by defining the so-called *picture changing operator (PCO)* χ, which is BRST-invariant by construction, as we can see from

$$\chi(z) \cong \{Q_B, \xi(z)\} \tag{6.73}$$

[103] The super-Virasoro constraints in the R-sector give $h(V^{X\psi}) = 5/8$, see the second line of (6.22). Thus, by requiring the conformal dimension of F_q to vanish (as we have done in (6.69)), we would obtain two possible choices: $q = -1/2$ and $q = -3/2$. As discussed in [22], we have to reject the choice $q = -3/2$, because only $q = -1/2$ reproduces the right super-Virasoro constraints (i.e. all conditions appearing in (6.22)) from BRST-invariance.

and by introducing $2g-2+n+m/2$ copies of this operator in the correlation function.
The BRST-charge of the superstring theory is given by

$$\begin{aligned}
Q_B &= Q_0+Q_1+Q_2\\
Q_0 &= \oint \frac{dw}{2\pi i} c(w)\Big(T^{tot}(w) - :\partial(c(w)b(w)):\Big)\\
Q_1 &= -\oint \frac{dw}{2\pi i} :\gamma(w)G^{X,\psi}(w) := -\oint \frac{dz}{2\pi i} :\eta(w)e^{\phi(w)}G^{X,\psi}(w):\\
Q_2 &= -\frac{1}{4}\oint \frac{dw}{2\pi i} :b(w)\gamma^2(w) := \frac{1}{4}\oint \frac{dz}{2\pi i} :b(w)e^{2\phi(w)}\eta^2(w): \quad ,
\end{aligned} \tag{6.74}$$

where:

- Q_0 is the same[104] BRST operator as in the bosonic theory which clearly acts - on the matter fields and on the $\beta\gamma$ system - as conformal transformation with parameter $\sim c$;
- Q_1 is the "super-analogue" of Q_0, giving that it generates superconformal transformation with parameter $\sim \gamma$ on the matter sector of the theory;
- Q_2 is needed for the nilpotency of the BRST algebra.

Given the BRST-charge, one can compute the PCO and its explicit expression reads as

$$\chi(w) =: \{Q_B, \xi(w)\} :=: \left[e^{\phi}G^{X\psi} + \underbrace{c\partial\xi - \frac{1}{2}\partial\eta e^{2\phi}b - \frac{1}{4}\eta\partial_w(e^{2\phi}b)}_{\text{pure ghost contribution}}\right]\Bigg|_w : \tag{6.75}$$

The PCO has picture number[105] $+1$ so it increases by one unit the picture number of the (fixed) vertex operator F_q on which it acts:

$$F_{q+1}(z) = \lim_{w\to z}\chi(w)F_q(z) = \chi(z)F_q(z) \quad , \tag{6.76}$$

and F_{q+1} is usually written as

$$\begin{aligned}
F_{q+1}(z) =&\chi(z)F_q(z) = \{Q_B,\xi(z)\}F_q(z) \stackrel{*}{=} \{Q_B,\xi(z)\}F_q(z) + \xi(z)[Q_B, F_q(z)] =\\
=&\{Q_B, \xi(z)F_q(z)\} =\\
=&\oint_{C(z)} \frac{dw}{2\pi i} j_B(w)\xi(z)F_q(z) \quad ,
\end{aligned} \tag{6.77}$$

where in the (*) step we used $[Q_B, F_q]=0$, namely the fact that the fixed version of the vertex operator (i.e. the version with the c factor) is BRST-invariant.
If we think about the $\phi-$charge anomaly, we understand that the most important piece of the PCO (6.75) is $e^{\phi}G^{X\psi}$. It is the only piece of the PCO that carries one unit of $\phi-$charge, so, if we insert $2g-2+n+m/2$ PCO's in the correlation function, then $e^{\phi}G^{X\psi}$ is the piece that - for sure - will saturate the $\phi-$anomaly and it will let the correlation function be non-vanishing. This is the reason why some authors (see [17], for instance) sloppily write

$$\chi(w) = e^{\phi}G^{X\psi} + \ldots \quad ; \tag{6.78}$$

this notation has the advantage of giving an immediate interpretation of the PCO: it essentially acts on the matter fields of the theory as a superconformal transformation with parameter $\sim e^{\phi}$; from this point of view, it is even more clear that all the Bose-sea levels must be physically equivalent, given that we can pass from one to the other one by means of a remnant of the gauge redundancy of the superstring.

[104] This is true if we treat β, γ as extra matter fields; indeed, compare Q_0 with the BRST charge of (4.109): we have only substituted $T^X + T^{bc}$ with $T^X + T^{\psi} + T^{bc} + T^{\beta\gamma} = T^{tot}$.

[105] Indeed, $N_p(\xi) = +1$; moreover, $N_p(Q_B) = 0$, because Q_B doesn't need the bosonization of the $\beta\gamma$ system to be defined and the picture number appears only when we separate the ϕ-sector from the $\xi\eta-$sector.

Anyway, we like to stress that all the pure ghost contributions that we neglected in (6.78) (and which are instead present in formula (6.75)) are indispensable to change the picture number of the vertex operator in a manifestly BRST-invariant way and it can happen that some of them do contribute to the final result of the correlation function. Actually, the BRST-invariance of the PCO (6.75) is so manifest that one could naively guess that χ is also BRST-exact in which case its insertion would make vanish any correlation function of BRST-invariant operators. This is however not the case, since the $\beta\gamma$ algebra (and consequently also Q_B) only contains $\partial\xi$, but not the constant zero mode ξ_0 of ξ. The latter is Grassmann odd, so the Hilbert space $\mathcal{H}_{\phi\eta\xi}$ of the $\phi\xi\eta-$theory is twice as large as the Hilbert space $\mathcal{H}_{\beta\gamma}$ of the $\beta\gamma-$system:

$$\mathcal{H}_{\phi\eta\xi} = \mathcal{H}_{\beta\gamma} \oplus \xi_0 \mathcal{H}_{\beta\gamma} \quad ; \tag{6.79}$$

therefore, we can specify $\mathcal{H}_{\beta\gamma}$ as

$$\mathcal{H}_{\beta\gamma} = \{ |\psi\rangle \in \mathcal{H}_{\phi\eta\xi} \mid \eta_0 |\psi\rangle = 0 \} \quad . \tag{6.80}$$

The Hilbert space $\mathcal{H}_{\phi\xi\phi}$ carries a reducible representation of the $\beta\gamma-$algebra (equivalently, of the *small algebra* $\phi\partial\xi\eta$), whereas it hosts an irreducible representation of the $\beta\gamma\xi_0$ algebra (equivalently, of the *large algebra* $\phi\xi\eta$). The reader should be aware that the distinction between the small and large algebra is not a detail - not at all! - and it can be a source of troubles, for practical computations. On the other hand, this distincion turns out to be essential to prove that we can arbitrarily attach the $2g-2+n+m/2$ PCO insertions to the vertex operators of the correlation function, as we are now going to argue for the case of the sphere.

Let's suppose that we have distributed our PCOs among the $n+m$ vertex operators such that in the correlation function we end up with having a couple of the latters (let them be called $F_{q_1}(z_1)$ and $F_{q_2}(z_2)$) in the q_1 and q_2 picture at the positions z_1 and z_2. Since none of the $n+m$ vertex operators depends on ξ_0, we can switch from the small to the large algebra by inserting it in the path integral and by integrating over it, because ξ_0 is Grassmann odd: $\int d\xi_0 \xi_0 = 1$. Actually, we can replace ξ_0 with $\xi(z)$ for an arbitrary z because

$$\int D[\xi'] d\xi_0 \xi_0 g(\xi') = \int D[\xi'] d\xi_0 \xi(z) g(\xi') \quad ,$$

where g is an arbitrary function and ξ' denotes the non-zero mode part of ξ ($\xi(z) = \xi_0 + \xi'(z)$). In other words, we can attach $\xi(z)$ to any of the vertex operators of the correlation function, say to $F_{q_1}(z_1)$ (so $z = z_1$). Now let's rewrite $F_{q_2}(z_2)$ in terms of a PCO and of $F_{q_2-1}(z_2)$, namely as (see (6.77))

$$F_{q_2}(z_2) = \oint_{C_{z_2}} \frac{dw}{2\pi i} j_B(w) \xi(z_2) F_{q_2-1}(z_2) \quad .$$

We deform the integration contour by pulling it off the back of the sphere; due to the BRST invariance it passes through all vertex operators except for $\xi(z_1)F_{q_1}(z_1)$, which becomes $F_{q_1+1}(z_1)$. Then, the integral $\int d\xi_0$ (that in the correlation function is sitting on the left of F_{q_1+1}) can be moved[106] to the left of $\xi(z_2)$ and, as at the beginning, we can use $\int d\xi_0 \xi(z_2) = 1$ to soak up $\xi(z_2)$ and to turn back to the small algebra again. Schematically, we have just shown that our correlation function can be equivalently written as

$$\langle ... F_{q_1}(z_1) ... F_{q_2}(z_2) ... \rangle = \langle ... F_{q_1+1}(z_1) ... F_{q_2-1}(z_2) ... \rangle \quad ,$$

where the dots denote the presence of the other insertions of the correlation function. Obviously, this expression can be reformulated in a more explicit way as

$$\langle ... F_{q_1}(z_1) ... \chi(z_2) F_{q_2-1}(z_2) ... \rangle = \langle ... \chi(z_1) F_{q_1}(z_1) ... F_{q_2-1}(z_2) ... \rangle \quad ,$$

and we learn that our $2g-2+n+m/2$ PCO's can be distributed in the correlation functions as we prefer. This freedom is a great advantage, because - as long as our correlation function has a $2g-2$ total picture number - we can distribute the PCOs so as to obtain an expression as simple as possible

[106] To be honest, in doing this we get a minus sign every time that we permute the position in the correlation function of ξ_0 with another Grassmann odd field. But we get the same sign when we deform the integration contour of j_B, because also the position of the latter - which is Grassmann odd too- in the correlation function must be changed.

from a computational point of view: sometimes, it is easier and shorter to change the picture number of a particular vertex operator, out of those present in the correlation functions. Other times, we want to exploit this freedom to change the picture number of a vertex operator whose position is not fixed, but so far, we have always worked with the vertex operators in their Fixed version F_q. We know that in the S-matrix some of them will appear in their Integrated version I_q (defined by $F_q(z) = c(z)I_q(z)$) and now it's time to learn how to change the picture numbers also of these operators. We would like to mimick (6.77) and define I_{q+1} as $I_{q+1} = [Q_B, \xi I_q]$, but it turns out that we have to slightly modify this expression, as we are now going to show with an integrated vertex operator corresponding to a massless state of the NS-sector. To be even more explicit, let's take it in the canonical picture: $I_{-1} \sim e^{ip\cdot X} V^{X\psi} e^{-\phi}$; this vertex operator has no c-ghost insertions at all, so[107]

$$\begin{aligned} I_0(z) =& [Q_B, \xi(z) I_{-1}(z)] \sim [Q_0, \xi(z) I_{-1}(z)] = \\ =& \oint_{C(z)} \frac{dw}{2\pi i} c(w) \Big(T^{tot}(w) - :\partial(c(w)b(w)): \Big) \xi(z) I_{-1}(z) = \\ =& \oint_{C(z)} \frac{dw}{2\pi i} c(w) \Big[\frac{\xi(z)I_{-1}(z)}{(z-w)^2} + \frac{\partial(\xi(z)I_{-1}(z))}{z-w} + O(1) \Big] + \\ & - \oint_{C(z)} \frac{dw}{2\pi i} \overbrace{c(w) : \partial(c(w) b}(w)) : \xi(z) I_{-1}(z) = \\ =& c(z)\partial(\xi(z)I_{-1}(z)) + \partial c(z) \xi(z) I_{-1}(z) = \\ =& \partial(c\xi I_{-1})(z) \quad ; \end{aligned} \tag{6.81}$$

this result is unacceptable, because it sits in the large algebra. In order for I_0 to be in the small algebra, we have to subtract this problematic contribution and, by following the approach of [23], we define

$$I_0(z) = [Q_B - Q_0, \xi(z) I_q(z)] = [Q_B, \xi(z) I_{-1}(z)] - \partial(c\xi I_{-1})(z) \quad . \tag{6.82}$$

We could perform a similar construction for all the I_q regardless of the picture number q and so we define

$$I_{q+1}(z) \cong [Q_B - Q_0, \xi(z) I_q(z)] = [Q_B, \xi(z) I_q(z)] - \partial(c\xi I_q)(z) \quad ; \tag{6.83}$$

this guess turns out to be the right one given that, for each q, we recover the expected relation among the fixed and integrated operators, namely $\{Q_B, I_{q+1}\} = \partial F_{q+1}$, as it is easy to see

$$\begin{aligned} \{Q_B, I_{q+1}\} =& \Big\{ Q_B, [Q_B, \xi I_q] - \partial(c\xi I_q) \Big\} = \\ =& \{Q_B, -\partial(c\xi I_q)\} = \{Q_B, \partial(\xi c I_q)\} = \\ =& \{Q_B, \partial(\xi F_q)\} = \partial\{Q_B, \xi F_q\} = \\ =& \partial F_{q+1} \quad , \end{aligned} \tag{6.84}$$

where in the last step we used $F_{q+1} = \{Q_B, \xi F_q\}$.
The most important lesson that we learn from (6.83) is that

$$\begin{aligned} I_{q+1} =& [Q_B, \xi I_q] + \partial(\xi c I_q) = \{Q_B, \xi\} I_q - \xi\{Q_B, I_q\} + \partial\xi c I_q + \xi\partial F_q = \\ =& \chi I_q - \xi\partial F_q + \partial\xi c I_q + \xi \partial F_q = \\ =& \Big(\chi + \partial\xi c\Big) I_q = \\ =& \chi^M I_q \quad ; \end{aligned} \tag{6.85}$$

in other words, we can raise the picture number of an integrated vertex operator by acting on it with the operator χ^M defined by

$$\chi^M(z) \cong \chi(z) + \partial\xi(z)c(z) \quad , \tag{6.86}$$

and that we like to call the *moving PCO*.

[107] Let's focus on Q_0 because it is the only problematic piece. Clearly, the contribution from Q_1 doesn't trigger any problem, because Q_1 is not sensitive to the presence of the c-ghost, see (6.74). It is also clear that the contribution from Q_2 doesn't give any bad surprise in $[Q_2, \xi(z)I_{-1}(z)]$: we know that $[Q_2, \xi(z)F_{-1}(z)]$ is acceptable and that $Q_2 \sim b$, so, if we remove the c-insertion from F_{-1} we even lose terms from $[Q_2, \xi(z)F_{-1}(z)]$.

6.10 Degeneration limit

We have stressed a lot that we have the freedom to distribute the PCOs as we prefer. This is not completely true, because we have a constraint on their positions: they must be chosen in such a way that they behave well under the so-called *degeneration limit*, as explained - for example - by A. Sen et al. in [24]. The mathematical issues underlying the degeneration limit of a Riemann surface with punctures are very involved (see [22]); here we give only the practical rule of thumb that lets one obtain the right correlation function.

Let's consider a worldsheet $\Sigma_{g,m}$ of genus g with m NS vertex operators[108], and let's suppose that we have enough CKG transformations to fix the positions of m_F of them. We need to insert $2g-2+m$ PCOs; we introduce a χ everytime we want to change the picture of a fixed vertex operator, otherwise we use the moving version χ^M.

We say that the Riemann surface $\Sigma_{g,m}$ is falling into a degeneration limit when one of its moduli approaches the boundary of the moduli space.

For the moduli coming from the metric, the boundary is simply the boundary of the corresponding fundamental region; for example, for the torus whose metric moduli space is represented with the fundamental region F_0, the boundary is the region $\tau_2 = +\infty$.

For the moduli coming from the positions of the integrated vertex operators, it is possible to show[109] that the boundary of moduli space corresponds to those versions of $\Sigma_{g,m}$ in which one of the integrated vertex operators comes together with another vertex operator. To be more concrete, let's take into con-

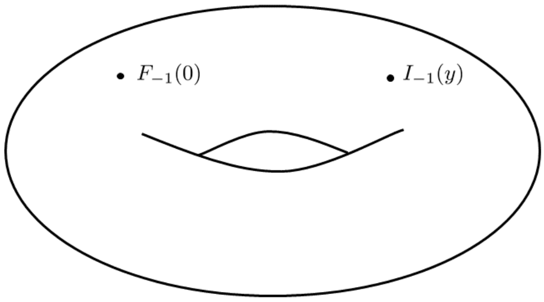

Figure 17

Torus with a fixed vertex operator at $z=0$ and an integrated vertex operator at $z=y$. Both of them come from the NS-sector and they are in the canonical picture.

sideration the case of a torus (parametrized with z) with two vertex operators; we know that for $g=1$ we can fix the position of only one vertex operator (let's call it F_{-1} and let's put it at the point labeled by $z=0$) so the other vertex operator must be in the integrated form (let's call it $I_{-1}(y)$, with $z=y$ describing any point of the worldsheet); see Figure 17. We obtain a degeneration limit when $y \to 0$. It is possible to show that the torus falling into this limit is conformally equivalent to a Riemann sphere which hosts the two vertex operators and which is connected to a torus by means of a "NS-propagator", a long cylinder (a propagating closed string) whose ends are characterized by -1 picture number, as we represented in Figure 18 . A similar description of the degeneration limit holds for every Riemann surface and the boundary of the moduli space associated to the positions of the vertex operators in $\Sigma_{g,m}$ can be represented by a set[110] of degenerating Riemann surfaces, each of which consists of two Riemann surfaces Σ_{g_1,m_1}, Σ_{g_2,m_2} connected with the NS-propagator ($g = g_1 + g_2$ and $m = m_1 + m_2$).

The rule of thumb: we locate the PCOs in such a way that we restore the right picture number $2g_i - 2$ in both Riemann surfaces Σ_{g_1,m_1}, Σ_{g_2,m_2} into which the worldsheet splits in the case of a degeneration

[108] See [4] for the Ramond case, which is more troublesome and which will be out of our interest.

[109] Here the point where we should resort to powerful tools/results of algebraic geometry is, because we should deal with the so-called "Deligne-Mumford compactification" of the moduli space of a Riemann surface with punctures; see [22].

[110] Clearly, there is a degenerating version of $\Sigma_{g,m}$ for each possible degeneration limit. In the case of the torus with two insertions, only the degeneration limit $y \to 0$ is possible, so in this case the boundary of the moduli space is described only by the torus with $y = 0$.

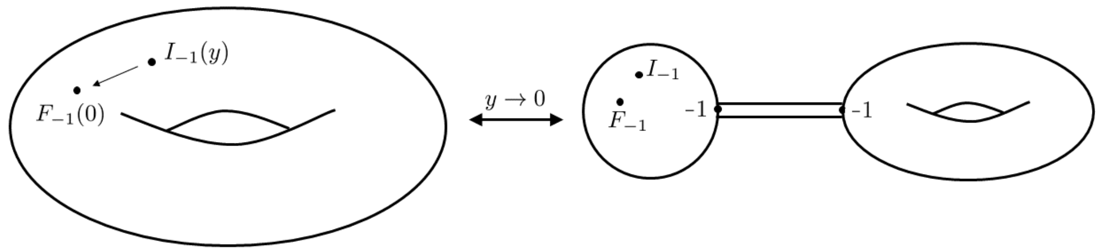

Figure 18
Representation of the limit version of the Riemann surface $\Sigma_{1,2}$ which sits at the boundary of the moduli space associated with the positions of the vertex operators. On the right, the cylinder connecting the sphere and the torus has to be thought of as very long. The ends of this cylinder carry picture number -1 (which is the canonical picture of the NS vertex operators), because this degeneration limit is obtained by taking a particular value for the position of an NS-vertex operator.

limit.
For example, in the case of $y \to 0$ for the torus with two vertex operators, we obtain a sphere with $N_p(F_{-1}) + N_p(I_{-1}) - 1 = -3$ and a torus with $N_p = -1$ (remember that each end of the NS-propagator contributes as an NS-insertion in the canonical picture). Both the sphere and the torus developed in the degeneration limit need one PCO. We can then satisfy this requirement, by:

- fixing one χ PCO at a point of the original torus, say $z = u_1$ (with u_1 fixed);
- locating a moving PCO $\chi^M(u_2)$ "nearby" the position y of the integrated vertex operator $I_{-1}(y)$ so, when $y \to 0$ and $I_{-1}(y)$ flees to the sphere with $F_{-1}(0)$, $\chi^M(u_2)$ must follow them. For instance, we can set $u_2 = \alpha y$, with $\alpha \in \mathbb{C}\backslash\{0\}$ constant ($\alpha = 1$ corresponds to working with $I_0(y)$ instead of $I_{-1}(0)$).

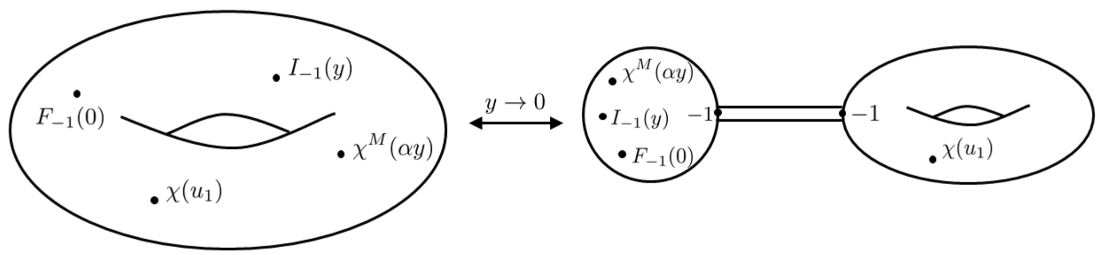

Figure 19
On the left, a possible way to distribute the two PCOs on $\Sigma_{1,2}$. As we can see on the right, this choice is compatible with the degeneration limit $y \to 0$.

This choice gives the right correlation function, because in the $y \to 0$ limit we have the picture represented in Figure 19, and we see that both the sphere and the torus have the right picture number (respectively -2 and 0). We could also have decided to fix $\chi(u_1)$ to u_1 as before and to fix another $\chi(u_2)$ in $u_2 = 0$ so as to work with $F_0(0)$ instead of $F_{-1}(0)$; actually, this correspond to setting $\alpha = 0$ in the previous case, because $\chi^M(0)F_{-1}(0) = \chi(0)F_{-1}(0)$.

So far, we have gathered all the basic background about perturbation theory in string theory. In the next chapter, we are going to use what we have learnt to understand the spontaneous supersymmetry breaking that happens at one loop for particular compactifications of the heterotic superstring; after presenting the physical problem, we will evaluate the entity of the spontaneous breaking, by computing the mass splitting that appears among fields of the same supermultiplet.

7 Spontaneous supersymmetry breaking

In this chapter, we will briefly introduce the problem and then we will delve into calculations. We will refer to [17] for all the details about compactifications and to [2, 25, 26, 3, 4] for physical and computational issues.

We know that the heterotic superstring is characterized by N=1 supersymmetry in D=10 dimensions. We can reach the phenomenologically desirable N=1 supersymmetry in $d = 4$ dimensions (that is, one gravitino, four supercharges) by compactifying the theory on a Calabi-Yau three-fold (CY_3), namely on a compact complex manifold of real dimension 6 that is characterized by $SU(3)$ holonomy group.
The complete form of $\tilde{H}_3$ is

$$\tilde{H}_3 = dB - \frac{\kappa_{10}^2}{g_{10}^2}\Big(\Omega_{CS}(A) - \Omega_{CS}(\omega)\Big) \quad ,$$

were we denoted with $\Omega_{CS}(\omega)$, $\Omega_{CS}(A)$ the Cern-Simon 3-forms for the spin connection ω and for the gauge connection A:

$$\begin{aligned} \Omega_{CS}(A) &= \mathrm{tr}\Big(A \wedge F_2 - \frac{1}{3}A \wedge A \wedge A\Big) \qquad & d\Omega_{CS}(A) &= \mathrm{tr}\Big(F_2 \wedge F_2\Big) \\ \Omega_{CS}(\omega) &= \mathrm{tr}\Big(\omega \wedge R - \frac{1}{3}\omega \wedge \omega \wedge \omega\Big) \qquad & d\Omega_{CS}(\omega) &= \mathrm{tr}\Big(R \wedge R\Big) \quad . \end{aligned} \tag{7.1}$$

In a topologically non-trivial situation, the spin connection, gauge field and Chern-Simons forms are not globally defined. At best, we can cover our manifold with open sets on each of which these are defined with suitable relations imposed in the overlapping regions (as it happens for the gauge field configuration of Wu-Yang's monopole). However, the gauge invariant field strength $\tilde{H}_3$ must be globally defined, since it is a physical object (for instance, the energy contains a term $\sim (\tilde{H}_3)^2$). To see what this implies, note that the Bianchi-identity reads as

$$d\tilde{H}_3 \propto -\mathrm{tr}\Big(F_2 \wedge F_2\Big) + \mathrm{tr}\Big(R \wedge R\Big) \quad ; \tag{7.2}$$

now, let $\mathcal{S}_4$ be a closed (i.e. with no boundaries) four dimensional submanifold in space-time. In general, $\int_{\mathcal{S}_4} \mathrm{tr}\Big(R \wedge R\Big)$ and $\int_{\mathcal{S}_4} \mathrm{tr}\Big(F_2 \wedge F_2\Big)$ may be non-trivial topological invariants, but (7.2) implies

$$\int_{\mathcal{S}_4} \Big[\mathrm{tr}\Big(R \wedge R\Big) - \mathrm{tr}\Big(F_2 \wedge F_2\Big)\Big] = \int_{\mathcal{S}_4} d\tilde{H}_3 = 0 \quad , \tag{7.3}$$

where in the last step we used Stokes theorem and the fact that $\tilde{H}_3$ is globally defined. We have just obtained a restriction on possible compactifications of the string theory: the cohomology class represented by $\mathrm{tr}\Big(R \wedge R\Big) - \mathrm{tr}\Big(F_2 \wedge F_2\Big)$ must be zero. The simplest solution to this constraint is achieved by the so-called *standard embedding of the spin connection in the gauge connection*, which consists of switching on a background field A such that $A = \omega$; note that in this way we can then set $\tilde{H}_3 = 0$ everywhere and this makes life easier (compactifications with fluxes are more involved). For a Calabi-Yau manifold, the spin connection takes value in the Lie Algebra of $SU(3)$ and the standard embedding choice thus breaks the $SO(32)$ and $E_8 \times E_8$ gauge group of the heterotic superstring to $SU(3) \times [U(1) \times SO(26)]$ and $SU(3) \times [E_6 \times E_8]$, where we denoted the unbroken subgroup between the square brackets; we will consider only the $SO(32)$ heterotic superstring, because it is precisely the unbroken $U(1)$ factor that may trigger the spontaneous supersymmetry breaking, by leading to a Fayet-Iliopoulos D-term.

It turns out that - at one loop level - the $U(1)$ factor is affected by a variety of anomalies[111]. All these anomalies are removed - at one loop level - by the so-called *Green-Schwarz mechanism*, which is

[111] Anomalies involving three $U(1)$ gauge bosons, one $U(1)$ and two $SO(26)$ gauge bosons, one $U(1)$ gauge boson and two gravitons. Note that the $U(1)$ and the $SO(26)$ gauge bosons that we have just mentioned are those obtained by decomposing, under $SU(3) \times SO(26) \times U(1)$ the adjoint representation of $SO(32)$ as

$$\mathbf{496} = (\mathbf{8},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{325})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3},\mathbf{26})_{\mathbf{1}} \oplus (\mathbf{3},\mathbf{1})_{\mathbf{-2}} \oplus (\bar{\mathbf{3}},\mathbf{26})_{\mathbf{-1}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}} \quad .$$

In this language, the $U(1)$ gauge boson is $(\mathbf{1},\mathbf{1})_{\mathbf{0}}$, whereas the $SO(26)$ gauge boson is $(\mathbf{1},\mathbf{325})_{\mathbf{0}}$ (clearly, they are singlets under $SU(3)$).

essentially a way to give a Stückelberg mass to the U(1) gauge boson, which acquires the longitudinal degrees of freedom by incorporating the Hodge-dual of the anti-symmetric d=4 tensor $B_{\mu\nu}$.

At one-loop order, Green-Schwarz interactions I^{10}_{G-S} in the 10-dimensional spacetime

$$I^{10}_{G-S} \sim \int_{\mathbb{R}^4\times CY_3} B\wedge tr\big(F_2\wedge F_2\wedge F_2\wedge F_2\big)$$

are generated; B is the usual two-form field coming from the NS-sector and F_2 is the field strength of the $SO(32)$ gauge vector A_μ. Because of the standard embedding choice ($A_\mu = \omega_\mu$), some of the background gauge fields (those with index μ corresponding to an internal direction, because only in this case $\omega_\mu \neq 0$) possess expectation values; we can then isolate from I^{10}_{G-S} the contribution coming from the internal space and we find that the Green-Schwarz interaction I^4_{G-S} in the 4-dimensional bulk is given by

$$I^4_{G-S} \sim p\int_{\mathbb{R}^4} B\wedge \hat{F}_2$$

where we denoted with $\hat{F}_2$ the field strength of the anomalous photon $\hat{A}_\mu$ and where the real number p takes into account the integration over the CY_3, that is

$$p \sim \int_{CY_3} tr_{SU(3)}F_2\wedge F_2\wedge F_2 \quad .$$

To understand the effect of the I^4_{G-S} interaction, it is convenient to dualize the purely four-dimensional part of B to a scalar field a; the $B\wedge\hat{F}_2$ then dualizes to $\hat{A}^\mu\partial_\mu a$ and $\tilde{H}^2_3$ will give a term like

$$D_\mu a D^\mu a = (\partial_\mu a + p\hat{A}_\mu)(\partial^\mu a + p\hat{A}^\mu) \quad .$$

In other words, the anomalous photon $\hat{A}$ acquires a Stückelberg mass $\sim p^2$ and the B field is not invariant under the gauge transformation, because the $U(1)$-gauge transformation $\hat{A}_\mu \mapsto \hat{A}_\mu - \partial_\mu s$ must be accompanied by $a\mapsto a+ps$.
From the perspective of the N=1 supersymmetry in $d=4$ dimensions:

- the field a is the imaginary part of the scalar component of the chiral multiplet S_c

$$S_c = e^{-2\phi} - ia + \theta^\alpha\kappa_\alpha + \ldots \quad , \tag{7.4}$$

  where we denoted with ϕ the dilaton ($g_s = e^\phi$) and with κ_α the dilatino;
- the $U(1)$ gauge field $\hat{A}_\mu$ is part of a vector multiplet, whose auxiliary field we call $\hat{D}$;
- there are some massless charged chiral multiplets

$$S_{\rho_i} = \rho_i + \theta^\alpha(\psi_i)_\alpha + \ldots \tag{7.5}$$

  that arise in the four dimensional bulk after the expansion of the ten-dimensional $SO(32)$ vector multiplet. We denote their $U(1)$ charges as e_i.

The potential energy of the effective low energy field theory is

$$V = \frac{\hat{D}^2}{2g_s^2} \quad . \tag{7.6}$$

The important thing to know is that the expectation value of the D-term $\hat{D}$ - beyond getting a natural contribution from the massless charged chiral multiplets ρ_i ($\sim\sum_i e_i|\rho_i|^2$) - now is also influenced by the chiral multiplet S_c, according to

$$\hat{D} = \frac{p}{\operatorname{Re} S_c} + \sum_i e_i|\rho_i|^2 = pg_s^2 + \sum_i e_i|\rho_i|^2 \quad ; \tag{7.7}$$

intuitively, this happens because all the $U(1)$-charged fields contribute to the auxiliary $\hat{D}$ term and, as we have explained above, at one loop level the a field (which is a component of the S_c chiral multiplet) acquires a $U(1)$-charge proportional to p. Thus, because of the mass acquired by the $U(1)$-photon $\hat{A}_\mu$, the D-term is corrected - at one loop level (g_s^2) - by the parameter p (which is a topological invariant of CY_3: it's half of its Euler number).
This is the only known superstring model in which supersymmetry can be spontaneously broken in perturbation theory, despite being unbroken at tree-level. Indeed, if we start, at the tree-level, with the supersymmetry preserving ($V = 0$) vacuum $< \phi_i >= 0$ (let's suppose $e_i \neq 0$ for a partiular i), then, at the loop level, we get $V = p^2 g_s^2/2 > 0$ and supersymmetry is broken. As expected, $V = p^2 g_s^2/2 > 0$ only if $p \neq 0$, because it is the presence of p that induces $\hat{A}$ to become massive and the mass of $\hat{A}$ is what precisely relates S_c ($a \in S_c$) to $\hat{D}$ (which belongs to the same supermultiplet as $\hat{A}^\mu$).
The fact that p is proportional to the Euler characteristic of CY_3 suggests to treat p essentially as an index: thus, it shouldn't receive contributions from the massive modes and, in order to analyze the supersymmetry breaking, it should be enough to examine the massless spectrum of the compactified theory. Indeed, the supersymmetry breaking can be detected by looking at the mass splitting among the fields of the supermultiplet S_{ρ_i} that is developed at one loop level. Actually this is already apparent in formulae (7.6) and (7.7), from which we can see that the field ρ_i corresponding to $e_i \neq 0$ acquires a mass $m^2 = e_i p$. Instead, its superpartner $(\psi_i)_\alpha$ cannot acquire a mass because terms like $(\bar{\psi}_i)_\alpha (\psi_i)_\alpha$ are not Lorentz invariant ($(\bar{\psi}_i)_\alpha$ and $(\psi_i)_\alpha$ have opposite chirality), while $(\psi_i)_\alpha (\psi_i)_\alpha$ and $(\bar{\psi}_i)_\alpha (\bar{\psi}_i)_\alpha$ do not conserve the $U(1)$ charge.
We'll focus on the scalar ρ_i which is a singlet of $SO(26)$ (and carries $U(1)$ charge 2). Our goal is to compute the mass term for ρ_i, namely the two point correlation function $\rho_i \rho_i^*$. Clearly, this field comes from the NS-sector of the theory so we know how the vertex operators corresponding to ρ_i and ρ_i^* should look like; in their fixed versions, they are

$$F_{-1} = \bar{c}c \underbrace{e^{-\phi} V e^{ik_1 \cdot X}}_{I_{-1}} \qquad F_{-1}^* = \bar{c}c \underbrace{e^{-\phi} V^* e^{ik_2 \cdot X}}_{I_{-1}^*} \quad , \tag{7.8}$$

where we added also a $\bar{c}$ in addition to the formula (6.57), because in (6.57) we considered only the right sector of the theory. As we discussed in (6.57), the operators V and V^* are very important, because they contain all the physical information about the nature of the particle. Here they are made of the degrees of freedom associated with the compact directions, consistently with the fact that ρ_i is a singlet of $SO(26)$. We like to streamline our presentation by saying that all what we need to know about V and V^* is that they are Grassmann odd primary operators of conformal dimensions $(1, \frac{1}{2})$ (so F_{-1} and F_{-1}^* have vanishing conformal dimensions) which satisfy the following OPEs:

$$\begin{aligned} G^{X\psi}(z)V(w,\bar{w}) &= G^{int}(z)V(w,\bar{w}) = -\frac{1}{z-w}\tilde{V}(w,\bar{w}) + O(z-w) \\ G^{X\psi}(z)V^*(w,\bar{w}) &= G^{int}(z)V^*(w,\bar{w}) = -\frac{1}{z-w}\tilde{V}^*(w,\bar{w}) + O(z-w) \\ V(z,\bar{z})V^*(w,\bar{w}) &= -\frac{q}{\bar{z}-\bar{w}}V_{\hat{D}}(z,\bar{z}) + O(\bar{z}-\bar{w}) \quad . \end{aligned} \tag{7.9}$$

In these formulae:

- $\tilde{V}$ and $\tilde{V}^*$ are Grassmann even (they are the superpartners of V and V^*) vertex operators of dimensions $(1, 1)$ of the CFT associated with CY_3. According to (6.78), $\chi \sim e^\phi G^{X\psi}$, so we expect the operators $\tilde{V}$ and $\tilde{V}^*$ to appear when we'll change the pictures of V and V^*. Anyway, we are not interested in giving a precise description of $\tilde{V}$ and $\tilde{V}^*$, because we will see that they will not influence the final result of our computation;

- $V_{\hat{D}}$ is the vertex operator of conformal dimensions $(1, 1)$ that represents the $\hat{D}$ term discussed above, namely the auxiliary field contained in the supermultiplet of the $U(1)$ anomalous photon $\hat{A}_\mu$.
Unfortunately, there is not a systematic theory of correlation functions with insertions of vertex operators for auxiliary fields (as opposed to vertex operators associated to physical states), and there is no standard recipe to build such a vertex operator. Nevertheless, $V_{\hat{D}}$ appears in the literature in a number of interesting calculations that suggest to interpret it as the vertex operator of $\hat{D}$.

Clearly, the most important hint in this direction comes from the fact that, under the space-time supersymmetry, it transforms as it should, namely into the gaugino vertex operator.

- the coefficient q with which $V_{\hat{D}}$ appears in the product (7.9) depends only on the $U(1)$ charges of the operators V and V^*; in other words: $q \propto e_i$, where e_i is the charge of ρ_i.

We stress one more time that we will treat V and V^* as "black boxes" satisfyng the relations (7.9). We prefer this approach, because in this way we can present the strategy that will let us perform the computation in a neater fashion; of course, one can always find the explicit[112] form for the vertex operators V and V^* and check that formulae (7.9) indeed hold.
We have to compute the 2-point function of the vertex operators F_{-1} and F^*_{-1} at one loop and extract the mass term from it (i.e. the term that survives for $k_1^2 = k_2^2 = 0$, which are the on-shell conditions for F_{-1} and F^*_{-1}, see later). So we have to insert them on a torus with two PCOs and the positions of the latter must be chosen according to respect the degeneration limit, as explained at the end of the last chapter.
We describe the torus as we explained in section 5.5. In particular, we can use the fundamental region F_b defined by (5.61) and the "global" chart given by the coordinate z:

$$\begin{aligned} z \cong& z + 2\pi \\ z \cong& z + 2\pi\tau \quad ; \end{aligned} \tag{7.10}$$

as discussed in 5.5, this variable doesn't allow us to properly describe the points of the torus corresponding to C_{-b}, but this is not a problem, since it is a zero-measure set and the only Dirac delta with which we have to deal will be developed at the position of F_{-1}, that we fix at $z = 0$ (and $\bar{z} = 0$), far from C_{-b} (and $\bar{C}_{-b}$).
We have to take into account the b-insertions associated to the moduli of the torus with two punctures:

- for the metric moduli τ and $\bar{\tau}$, we already computed that these are

$$b_\tau = \frac{1}{2\pi i} \oint_{C_a} dz 2\pi b(z) \qquad \bar{b}_{\bar{\tau}} = \frac{1}{2\pi i} \oint_{\bar{C}_a} d\bar{z} 2\pi \bar{b}(\bar{z}) \quad , \tag{7.11}$$

  where C_a is the circle in the torus corresponding to $\mathrm{Im}\, z = a$.

- for the moduli corresponding to the position $z = y$ and $\bar{z} = \bar{y}$ of F^*_{-1}, the proper b-insertions will transform $F^*_{-1}(y, \bar{y})$ into $I^*_{-1}(y, \bar{y})$, as explained in (5.53).

We have fixed $F_{-1}(z = 0, \bar{z} = 0)$ at $z = \bar{z} = 0$ and we have $I^*_{-1}(z = y, \bar{z} = \bar{y})$ that is free to move on the torus. Then, according to the proper treatment of the degeneration limit, we can fix a PCO $\chi(z)$ at $z = u_1$ and insert a moving PCO $\chi^M(z)$ at $z = u_2 = \alpha y$, precisely as we explained at the end of the last chapter.
To sum up, the vertex operators are distributed as in Figure 20 and our 2-point function will be proportional to

$$\begin{aligned} &\int_{F_0} d^2\tau \int_{F_b} d^2 y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \chi^M(\alpha y) I^*_{-1}(y, \bar{y}) \right\rangle \quad = \\ =& \int_{F_0} d^2\tau \int_{F_b} d^2 y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[\chi(\alpha y) + \partial\xi(\alpha y) c(\alpha y) \Big] I^*_{-1}(y, \bar{y}) \right\rangle \quad , \end{aligned} \tag{7.12}$$

where, in the last step, we used the definition (6.86) of χ^M.
This correlation function is the right one only for $\alpha = 1$. In the last chapter, while we were searching for the expression of the moving PCO (see (6.85)), we assumed that it was always acting precisely at the

[112] But this requires a sound background in string compactifications.
In the special case that CY_3 is a Calabi-Yau orbifold (we mean the quotient of the six-dimensional torus $\mathbb{R}/\Lambda$ by a finite group of symmetries of Λ that preserves N=1 supersymmetry in four dimensions) the operators V and V^* are simply given by

$$V \sim \epsilon_{mpq} \psi^m \lambda^p \lambda^q \qquad V^* \sim \epsilon_{\bar{m}\bar{p}\bar{q}} \psi^{\bar{m}} \lambda^{\bar{p}} \lambda^{\bar{q}} \quad ,$$

where m, p, q are $SU(3)$ indices. Note that, as we mentioned, V has $U(1)$ charge 2 (and V^* has charge -2).

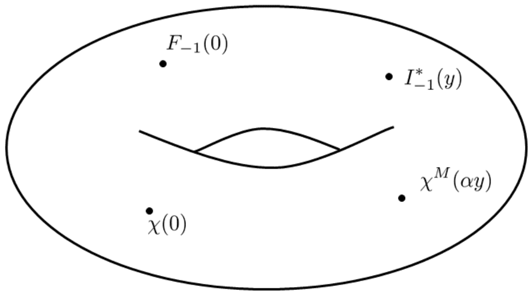

Figure 20
The distribution of the vertex operators and of the PCOs chosen for the computation.

same spot where the integrated vertex operator I_q was located; clearly, in our correlation function, this corresponds to setting $\alpha = 1$, because I_{-1} is inserted at $z = y$, whereas χ^M is inserted at $z = \alpha y$.
One can ask whether the form of the moving PCO that is not acting precisely at the spot of an integrated vertex operator is different from the χ^M that we found in (6.86). The answer is yes, they enter the correlation functions in a slightly different way. Unfortunately, to understand the "more general version of the moving PCO", one should resort to a heavy formalism, that is discussed, for example, in [4]. We haven't found an intuitive way to present it, so we only say that the starting point for our computation is not (7.12), but

$$\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha \partial\xi(\alpha y) c(y)\Big] I^*_{-1}(y) \right\rangle \quad . \tag{7.13}$$

The reader should note the tiny difference between (7.13) and (7.12), which lies in the substitution

$$c(\alpha y) \mapsto \alpha c(y)$$

and which vanishes for $\alpha = 1$, as expected.
Only in the second case (with $\alpha c(y)$) we get a result (the right one) which is independent of the value of α. We are going to perform the calculation with (7.13), because we like the reassuring idea of a PCO that can be located on the worldsheet wherever we want (as long as we respect the degeneration limit). The reader who doesn't like to take "God-given" expressions can set $\alpha = 1$, start with (7.12) and find the same final result, by following precisely the same steps that we are going to present below.

Fou our convenience, here we write again all the formulae that we will need for the computation:

$$\begin{aligned}
T^{X\psi}(w) &= -\partial X^\mu(w)\partial X_\mu(w) + \psi^\mu(w)\partial\psi_\mu(w) \\
T(w) &= T^{X\psi}(w) + T^{\rm int}(w) \\
T^{\beta\gamma}(w) &= -\frac{3}{2}\beta(w)\partial\gamma(w) - \frac{1}{2}\partial\beta(w)\gamma(w) \\
T^{bc}(w) &= 2\partial c(w) b(w) + c(w)\partial b(w) \\
T^{tot}(w) &= T^{X\psi}(w) + T^{int}(w) + T^{\beta\gamma}(w) + T^{bc}(w) \\
G^{X\psi}(w) &= -\psi^\mu(w)\partial X_\mu(w) \\
G(w) &= G^{X\psi}(w) + G^{int}(w) \\
\gamma(w) &= e^{\phi(w)}\eta(w) \\
\beta(w) &= e^{-\phi(w)}\partial\xi(w) \\
Q_B = Q_B^R + Q_B^L &= \oint_{C_0} \frac{dw}{2\pi i} j_B(w) + \oint_{\bar{C}_0} \frac{d\bar{w}}{2\pi i} \bar{j}_B(\bar{w})
\end{aligned} \tag{7.14}$$

$$
\begin{aligned}
j_B(w) = &: c(w)\Big(T^{X\psi}(w) + T^{int}(w) + T^{\beta\gamma}(w) + \partial c(w) b(w)\Big) : + \\
&- : \gamma(w) G(w) : -\frac{1}{4} : \beta(w)\gamma^2(w) := \\
= &: c(w)\Big(T^{tot}(w) - \partial(c(w)b(w))\Big) : -: \gamma(w)G(w) :+ \\
&- \frac{1}{4} : b(w)\gamma^2(w) : \\
\bar{j}_B(\bar{w}) = &\bar{c}(\bar{w})\Big(\bar{T}^{X\psi}(\bar{w}) + \bar{T}^{int}(w) + \bar{T}^{bc}(\bar{w}) - \partial\big(\bar{c}(\bar{w})\bar{b}(\bar{w})\big)\Big) \\
\chi(w) =&: \{Q_B^R, \xi(w)\} :=: \Big[e^{\phi}G + c\partial\xi - \frac{1}{2}\partial\eta e^{2\phi} b - \frac{1}{4}\eta\partial_w(e^{2\phi}b)\Big]\Big|_w : \\
c(z)b(w) = &\frac{1}{z-w} + \ldots \\
\xi(z)\eta(w) = &\frac{1}{z-w} + \ldots \\
e^{q_1\phi(z)} e^{q_2\phi(z)} = &(z-w)^{-q_1 q_2} : e^{(q_1+q_2)\phi(w)} + \ldots : \\
\partial X(z) : e^{ikX(w)} : = &-i\frac{\alpha' k}{2} : e^{ikX(w)} : \frac{1}{z-w} + \ldots \\
T^{X\psi}(z) : e^{ikX(w)} : = &\frac{\alpha' k^2/4}{(z-w)^2} : e^{ikX(w)} : + \frac{\partial_w}{(z-w)} : e^{ikX(w)} : + \ldots \\
: e^{ik_1 X(z)} :: e^{ik_2 X(w)} : = &(z-w)^{\alpha'(k_1 k_2)} : e^{i(k_1+k_2)X(w)}(1 + O(z-w)) : \quad .
\end{aligned}
\tag{7.15}
$$

Note that we have slightly changed the conventions. In particular, we will refer with X^μ and ψ^μ only to the non-compact directions ($\mu = 0, 1, 2, 3$); then, the energy-momentum tensor T for the matter sector splits as $T = T^{X\psi} + T^{\text{int}}$, and analogous relations hold for its superpartner ($G = G^{X\psi} + G^{\text{int}}$), where we labeled with "*int*" the contributions coming from the internal CFT.
We stress that now we are working on the torus, so the Green functions among the fields are in general different (much more complicated!) from those that we have in the complex plane and that we used to study each CFT; this is nothing strange, since Green's functions are the solutions to particular PDEs and the latter are sensitive to the boundary conditions of the problem. This implies that the OPEs among fields on the torus will be different from the OPEs among the same fields on the sphere/complex plane. Nevertheless, we expect the divergent parts of the OPEs to be independent of the worldsheet, because they arise only when two vertex operators come together. This is the reason why the OPEs that we have just written look like the same of those ones that we have already met when we studied conformal field theories on the complex plane/sphere. Locally one should not feel the boundary conditions and this is a great advantage for our calculations, because all the result will come from a Dirac delta; this is a lucky case: for this scattering amplitude, we don't have to be worried by the (non-singular) corrections to the OPEs due to the periodic boundary conditions of the torus and we can use the (divergent parts of the) OPES as if we were on the sphere.

7.1 The bone of the calculation

The most important step to get the right result is the following simple and seemingly innocent one: in the correlation function, we substitute $\chi(u_1)$ with

$$
\chi(u_1) = \chi(0) + [\chi(u_1) - \chi(0)] \quad . \tag{7.16}
$$

We'll separately analyze the $\chi(0)$ and the $\chi(u_1) - \chi(0)$ contributions.

Let's start with the first one. When $\chi(0)$ is inserted in the correlation function, it changes the picture number of $F_{-1}(0,0)$ and we obtain $F_0(0)$, namely (see Appendix A for the computation)

$$\begin{aligned} F_0(0) =& \chi(0)F_{-1}(0,0) = \chi(0)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} = \\ =& :\bar{c}(0)c(0)\Big[\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}: -\frac{1}{4}:\bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}: \end{aligned} \tag{7.17}$$

In other words, $\chi(0)$ transforms the correlation function into

$$\begin{aligned} \int_{F_0} d^2\tau \int_{F_b} d^2y \Big\langle b_\tau \bar{b}_{\bar{\tau}} \Big\{ :\bar{c}(0)c(0)\Big[\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}: -\frac{1}{4}:\bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}: \Big\}\times \\ \times\Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big]I^*_{-1}(y,\bar{y})\Big\rangle \quad . \end{aligned} \tag{7.18}$$

As we will show later, the correlation function (7.13) is independent of the value of α, so we can take the limit $\alpha \to 1$ in (7.18). As we know, this correspond to changing the picture number of $I^*_{-1}(y,\bar{y})$ and we get (see Appendix B)

$$\begin{aligned} I^*_0(y) =& \lim_{\alpha\to 1}[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)]I^*_{-1}(0,0) = \chi^M(0)I^*_{-1}(0,0) = \\ =& \Big[\tilde{V}^*(y,\bar{y}) - \frac{i}{2}k_2\cdot\psi(y)V^*(y,\bar{y})\Big]e^{ik_2\cdot X(y,\bar{y})} \quad . \end{aligned} \tag{7.19}$$

This means that (7.18) becomes

$$\begin{aligned} \int_{F_0} d^2\tau \int_{F_b} d^2y \Big\langle b_\tau \bar{b}_{\bar{\tau}} \Big\{ :\bar{c}(0)c(0)\Big[\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}: -\frac{1}{4}:\bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}: \Big\}\times \\ \times\Big[\tilde{V}^*(y,\bar{y}) - \frac{i}{2}k_2\cdot\psi(y)V^*(y,\bar{y})\Big]e^{ik_2\cdot X(y,\bar{y})}\Big\rangle \quad ; \end{aligned} \tag{7.20}$$

from this expression, we can drop the terms that don't respect the constraint given by the anomaly of the $\beta\gamma$ system, which requires - as we know- that the ϕ-charge of the insertions must be $2g-2=0$, so (7.20) is

$$\begin{aligned} \int_{F_0} d^2\tau \int_{F_b} d^2y \Big\langle b_\tau \bar{b}_{\bar{\tau}} :\bar{c}(0)c(0)\Big[\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}:\times \\ \times:\Big[\tilde{V}^*(y,\bar{y}) - \frac{i}{2}k_2\cdot\psi(y)V^*(y,\bar{y})\Big]e^{ik_2\cdot X(y,\bar{y})}:\Big\rangle. \end{aligned} \tag{7.21}$$

In this expression, we can distinguish three components: the contribution without any ψ^μ, a piece which is linear in ψ^μ and a bit with quadratic dependence on ψ^μ. The first term vanishes because of the sum over the spin structure, that, in our formalism, is hidden in $\langle ... \rangle$; this claim is true because of the "usual" GSO cancellation between spin structures that also leads to the vanishing of the 1-loop cosmological constant for superstrings in[113] $\mathbb{R}^{10}$. Also the contribution with linear dependence on ψ^μ vanishes, this time because of the four-dimensional Lorentz invariance of the two dimenional field theory. In other words, (7.21) reduces to the contribution quadratic in ψ^μ, which is

$$\int_{F_0} d^2\tau \int_{F_b} d^2y \Big\langle b_\tau \bar{b}_{\bar{\tau}} :\bar{c}(0)c(0)\Big[-\frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}::\Big[-\frac{i}{2}k_2\cdot\psi(y)V^*(y,\bar{y})\Big]e^{ik_2\cdot X(y,\bar{y})}:\Big\rangle. \tag{7.22}$$

By Lorentz invariance, the $\psi^\mu\psi^\nu$ correlator is proportional to $\eta^{\mu\nu}$, so (7.22) contains an

$$k_1 \cdot k_2$$

[113] In our case, we have compactified six dimensions, but it can be shown that this difference doesn't matter: the fermionic partition function still vanishes after the sum over spin structures.

overall factor. We have to remember that ρ_i is massless at the tree level, so the on-shell condition (which comes from the BRST invariance of the vertex operator) gives $k_1^2 = 0$ and - obviously - the same holds for k_2, $k_2^2 = 0$. As in QFT, also in string theory we have a Dirac delta in front of the S-matrix element that forces the momentum conservation, so we are actually working with $k_1^\mu + k_2^\mu = 0$. This means that

$$0 = (k_1^\mu + k_2^\mu)^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = 2k_1 \cdot k_2 \quad , \tag{7.23}$$

and (7.22) must vanish.
We have thus shown that the contribution to our original correlation function of the $\chi(0)$ coming from formula (7.16) is zero. All the result must come from the term $\chi(u_1) - \chi(0)$ of (7.16), as we are now going to show.

In other words, we are left with computing

$$\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\chi(u_1) - \chi(0)\Big] F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle \quad , \tag{7.24}$$

provided that we take the limit $\alpha \to 1$, because only in this case we are sure that this term is *all* the contribution that the result gets. Clearly, we are free to take this limit when we want so we postpone it till the end of the computation, so as to avoid - in the meantime - dealing with a variety of terms resulting from changing the picture of $I^*_{-1}(y,\bar{y})$.
We rewrite $\chi(u_1) - \chi(0)$ as

$$\chi(u_1) - \chi(0) = \{Q_B^R, \xi(u_1) - \xi(0)\} \tag{7.25}$$

and we deform the BRST-contour so as to get BRST commutators/anticommutators for the other insertions. We obtain (see Appendix C)

$$\begin{aligned}
&\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big\{Q_B^R, \xi(u_1) - \xi(0)\Big\} F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle = \\
&\quad = -\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle \Big\{Q_B^R, b_\tau\Big\} \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle + \\
&\quad + \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \Big\{Q_B^R, \bar{b}_{\bar{\tau}}\Big\} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle + \\
&\quad + \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] \Big[Q_B^R, F_{-1}(0,0)\Big] \Big[\chi(\alpha y) + \alpha\partial\xi(\alpha y)c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle + \\
&\quad + \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \Big[Q_B^R, \chi(\alpha y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle + \\
&\quad + \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \chi(\alpha y) \Big[Q_B^R, I^*_{-1}(y,\bar{y})\Big] \right\rangle + \\
&\quad + \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \alpha \Big\{Q_B^R, \partial\xi(\alpha y)\Big\} c(y) I^*_{-1}(y,\bar{y}) \right\rangle + \\
&\quad - \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \alpha\partial\xi(\alpha y) \Big\{Q_B^R, c(y) I^*_{-1}(y,\bar{y})\Big\} \right\rangle \quad .
\end{aligned} \tag{7.26}$$

Clearly, from the definition of the PCO, we get

$$\begin{aligned}
&[Q_B^R, \chi(\alpha y)] = 0 \\
&\{Q_B^R, \partial\xi(\alpha y)\} = \partial\chi(\alpha y) \quad ;
\end{aligned} \tag{7.27}$$

moreover, from the BRST characteristics of the fixed and integrated vertex operators, we know that

$$\begin{aligned}
[Q_B^R, F_{-1}(0,0)] &= 0 \\
[Q_B^R, I^*_{-1}(y,\bar{y})] &= [Q_B^R, e^{-\phi(y)} V^*(y,\bar{y}) e^{ik_2 \cdot X(y)}] = \partial_y \Big(c(y) e^{-\phi(y)} V^*(y,\bar{y}) e^{ik_2 \cdot X(y)}\Big) = \partial_y \Big(c(y) I^*_{-1}(y,\bar{y})\Big) \\
\{Q_B^R, c(y) I^*_{-1}(y,\bar{y})\} &= \{Q_B^R, c(y) e^{-\phi(y)} V^*(y,\bar{y}) e^{ik_2 \cdot X(y)}\} = 0 \quad ,
\end{aligned} \tag{7.28}$$

which are formulae that we also explicitly show in Appendix D.
Finally, there are the BRST anticommutators with the b-ghost insertions. Obviously,

$$\{Q_B^R, \bar{b}_\tau\} = 0 \quad , \tag{7.29}$$

becuase the left and right sector of the theory are independent. It's more involved, instead, to show that

$$\{Q_B^R, b_\tau\} \tag{7.30}$$

generates a total derivative with respect to τ which vanishes after the integration over the fundamental domain of the metric moduli space[114].
Thus, the net contribution that we get from (7.26) is

$$\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0) \Big[\chi(\alpha y)\partial_y\big(c(y)I^*_{-1}(y,\bar{y})\big) + \alpha\partial\chi(\alpha y)c(y)I^*_{-1}(y,\bar{y})\Big]\right\rangle \quad , \tag{7.32}$$

where, as always, we have been careful to distinguish the partial derivative with respect to the argument (∂) with the partial derivative ∂_y with respect to y; it is then immediate to see that(7.32) can be written as a total derivative, namely as

$$\int_{F_0} d^2\tau \int_{F_b} d^2y\, \partial_y \left[\left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] F_{-1}(0,0)\chi(\alpha y)c(y)I^*_{-1}(y,\bar{y})\right\rangle\right] \quad , \tag{7.33}$$

which is a compact expression corresponding to (let's insert the formulae for $F_{-1}(0,0)$ and $I^*_{-1}(y,\bar{y})$)

$$\int_{F_0} d^2\tau \int_{F_b} d^2y\, \partial_y \left[\left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1) - \xi(0)\Big] \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\chi(\alpha y)c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}\right\rangle\right] \quad . \tag{7.34}$$

Because of the constraint on the ϕ-charge, only the components of $\chi(\alpha y)$ with ϕ-charge equal to 2 can contribute to (7.33); according to (6.75), these are

$$\chi(\alpha y)\Big|_{\phi\text{- charge}=2} = -\frac{1}{2}\partial\eta(\alpha y)e^{2\phi(\alpha y)}b(\alpha y) - \frac{1}{4}\eta(\alpha y)\partial\big(e^{2\phi(\alpha y)}b(\alpha y)\big) \quad . \tag{7.35}$$

[114]We don't give the proof, we only roughly outline it. By using the fundamental relation $\{Q_B^R, b(z)\} = T^{\text{tot}}(z)$ we get that (rewrite b_τ as in (5.39))

$$\{Q_B^R, b_\tau\} = \Big\{Q_B^R, \frac{1}{4\pi}(b, \partial_\tau \hat{h})\Big\} = \frac{1}{4\pi}(T^{\text{tot}}, \partial_\tau \hat{h}) \sim \partial_\tau S^{\text{tot}} \quad ,$$

where, in the last step, we used the fact that $T^{\text{tot}} \sim \frac{\delta S^{\text{tot}}}{\delta h}$ (here S^{tot} is the total action, namely the sum of the action of the superstring and the actions of the ghost systems). So the first line of the right-hand side of (7.26) can be written as (we hide $\int_{F_b} d^2y$ and the presence of the other operators in the dots)

$$\int_{F_0} d^2\tau < \{Q_B^R, b_\tau\}... > \sim \int_{F_0} d^2\tau < \partial_\tau S^{\text{tot}}... > \sim \int_{F_0} d^2\tau \int D[\Phi]e^{-S^{\text{tot}}[\Phi]}\partial_\tau S^{\text{tot}}[\Phi]... \quad = \\ \sim \int_{F_0} d^2\tau \int D[\Phi]\partial_\tau\Big(e^{-S^{\text{tot}}[\Phi]}\Big)... \quad ,$$

where we denoted with $D[\Phi]$ the path-integral measure of all the fields Φ of the theory. Then, with some work (read around formula (2.194) of [6] to get an idea), it is possible to show that we can bring the derivative ∂_τ out of $< ... >$ and we end up with

$$\int_{F_0} d^2\tau < \{Q_B^R, b_\tau\}... > \sim \int_{F_0} d^2\tau \int D[\Phi]\partial_\tau\Big(e^{-S^{\text{tot}}[\Phi]}...\Big) \sim \int_{F_0} d^2\tau\partial_\tau\Bigg(< ... >\Bigg) \quad , \tag{7.31}$$

where $\Bigg(< ... >\Bigg)$ is shown to be zero for $\tau_2 \to +\infty$. Thus, (7.31) vanishes, because the only boundary that F_0 has is at $\tau_2 \to +\infty$, where the integrand is zero.

By taking the limit $\alpha \to 1$, we get (see Appendix (E))

$$\chi(\alpha y)\Big|_{\phi\text{- charge}=2} c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = -\frac{1}{4}\eta(y)e^{\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} \quad ; \tag{7.36}$$

which can be inserted into (7.34) to obtain

$$-\frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y\, \partial_y \left[\left\langle b_\tau \bar{b}_{\bar{\tau}} \Big[\xi(u_1)-\xi(0)\Big]\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\eta(y)e^{\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}\right\rangle\right] \quad . \tag{7.37}$$

This is a total derivative in y. The torus has no boundary, so we get a non-vanishing result only if the integrand contributes with a singularity like

$$\frac{1}{\bar{y}} \quad ,$$

because in this case we can use the well-known formula

$$\partial_y \frac{1}{\bar{y}} = 2\pi\delta(y,\bar{y}) \quad . \tag{7.38}$$

In particular, this means that all the result to our correlation function comes from the limit $y \to 0$ and we can use the OPEs to understand what the integrand of (7.37) will become in this limit. Because of the on-shell condition on k_1, k_2 and because of (7.23), we have

$$: e^{ik_1\cdot X(0,0)} :: e^{ik_2\cdot X(y,\bar{y})} := |-y|^{\alpha' k_1\cdot k_2} : e^{i(k_1+k_2)X(y,\bar{y})} + O(y) := 1 + O(y) \quad . \tag{7.39}$$

The other possible OPEs in the game are

$$\begin{aligned} \Big[\xi(u_1)-\xi(0)\Big]\eta(y) &= \frac{1}{y} + O(y) \\ e^{-\phi(0)}e^{\phi(y)} &= -y + O(y^2) \end{aligned} \tag{7.40}$$

and, most importantly, the OPE (7.9), namely

$$V(0,0)V^*(y,\bar{y}) = \frac{q}{\bar{y}}V_{\hat{D}}(0,0) + O(y) \quad . \tag{7.41}$$

The only way to get, in the integrand, a Dirac delta that is not multiplied by a positive power of y is to pick up the first terms from all these OPEs:

$$\begin{aligned} &+\frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y\, \partial_y \left[\left\langle b_\tau \bar{b}_{\bar{\tau}} \bar{c}(0)c(0)\Big[\xi(u_1)-\xi(0)\Big]\eta(y)e^{-\phi(0)}e^{\phi(y)}V(0,0)V^*(y,\bar{y})e^{ik_1\cdot X(0,0)}e^{ik_2\cdot X(y,\bar{y})}\right\rangle\right] = \\ &= \frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y\, \partial_y \left[\left\langle b_\tau \bar{b}_{\bar{\tau}} \bar{c}(0)c(0)\underbrace{\Big[\xi(u_1)-\xi(0)\Big]\eta(y)e^{-\phi(0)}e^{\phi(y)}V(0,0)V^*(y,\bar{y})e^{ik_1\cdot X(0,0)}e^{ik_2\cdot X(y,\bar{y})}}_{\frac{1}{y}\times(-y)\times\frac{q}{\bar{y}}V_{\hat{D}}(0,0)\times 1 = -\frac{q}{\bar{y}}V_{\hat{D}}(0,0)}\right\rangle\right] = \\ &= -\frac{q}{4}\int_{F_0} d^2\tau \left\langle b_\tau \bar{b}_{\bar{\tau}} \bar{c}(0)c(0)V_{\hat{D}}(0,0)\right\rangle \times \int_{F_b} d^2y\partial_y \frac{1}{\bar{y}} = \\ &= -\frac{\pi q}{2}\int_{F_0} d^2\tau \left\langle b_\tau \bar{b}_{\bar{\tau}} \bar{c}(0)c(0)V_{\hat{D}}(0,0)\right\rangle \quad . \end{aligned} \tag{7.42}$$

We have thus found that at one-loop level a mass term $\rho_i\bar{\rho}_i$ is generated and this is proportional to the expectation value of the D-term.
The computation explicitly shows that the perturbation theory around the vacuum $<\rho_i>=0$ (that is supersymmetric at the tree-level) leads - at one-loop level - to a mass splitting affecting the chiral multiplet S_{ρ_i}. Note that our calculations have been performed around the vacuum $<\rho_i>=0$, because we supposed ρ_i to be massless at the tree-level (according to formulae (7.6) and (7.7), if $<\rho_i>\neq 0$ then the field ρ_i would be massive already at tree level).

7.2 The independence of α

More times we have mentioned that we are free to distribute the PCOs as we prefer, as long as the degeneration limit is respected. With our calculation, we can explicitly check that this is the case. Indeed:

- the final result doesn't depend on the fixed position u_1 of the PCO $\chi(u_1)$. We note that all the dependence on u_1 is erased at the very end of the calculation, thanks to the first line of (7.40). Actually, at this comment, the careful reader should be very upset. All the computation was based on the innocent (innocent not at all) equation (7.16), where we wrote $\chi(u_1) = \chi(0) + (\chi(u_1) - \chi(0))$ and then we showed that the contribution coming from $\chi(0)$ alone is zero (see the comments under (7.22)). But we have just noted that all the contribution coming from $\chi(u_1) - \chi(0)$ is due to $-\chi(0)$, because of the first OPE in (7.40). It seems that we have reached an absurd: how is it possible for $\chi(0)$ to give zero contribution in the first case and, at the same time, to give all the non-vanishing contribution in the second case? We have to be careful. It is wrong to say that in the second case (when there is $\chi(u_1) - \chi(0)$ in the correlation function) all the result comes only from $-\chi(0)$. Even though the OPE (7.40) seems to suggest that only $\xi(0)$ is important for the calculations, the latter are right only when $\xi(0)$ is accompanied by $\xi(u_1)$ in $\xi(u_1) - \xi(0)$. This is because of the tricky nature of the bosonization of the $\beta\gamma$ system, which requires a lot of attention, in particular at loop levels, where the distinction between the small and large algebra becomes essential. A detailed analysis of this kind of issues is beyond the purposes of this thesis, so we refer to [4]. It turns out that on surfaces with genus $g \geq 1$ the bosonization of the superconformal system brings into the game unphysical singularities (the so-called *spurious singularities*) which make life harder. To avoid this kind of problems, we should always work with the small algebra. This means that we have to plan our manipulations so as to avoid working with the zero mode ξ_0: the field ξ can explicitly appear only through its derivatives $\partial^n \xi(z)$ and through differences of ξs (like $\xi(u_1) - \xi(0)$). It is precisely from this point of view that also $\chi(u_1)$ contributes to the final result, because $\xi(u_1)$ is necessary to remove the zero mode ξ_0 from $\xi(0)$;

- the correlation function (7.13) is independent of the value of α. By taking the total derivative with respect to α of formula (7.13), we get

$$
\begin{aligned}
&\frac{d}{d\alpha}\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[\chi(\alpha y) + \alpha \partial\xi(\alpha y) c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle = \\
&= \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[y\{Q_B^R, \partial\xi(\alpha y)\} + \partial\xi(\alpha y) c(y) + \alpha y \partial^2\xi(\alpha y) c(y)\Big] I^*_{-1}(y,\bar{y}) \right\rangle \\
&= \int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[y\{Q_B^R, \partial\xi(\alpha y)\} I^*_{-1}(y,\bar{y}) + \partial_y \Big(y \partial\xi(\alpha y)\Big) c(y) I^*_{-1}(y,\bar{y})\Big] \right\rangle \quad ,
\end{aligned}
\tag{7.43}
$$

where we substituted, in the first line, $\chi(\alpha y)$ with its definition $\chi(\alpha y) = \{Q_B^R, \xi(\alpha y)\}$, because now we want to deform the BRST-contour so as to obtain BRST anticommutators/commutators on the other fields, as we did in (7.26). As before, the anticommutor $\{Q_B^R, b_\tau\}$ gives a total derivative which integrates to zero on the metric moduli space, and $\{Q_B^R, \bar{b}_{\bar{\tau}}\}$ vanishes as also $[Q_B^R, F_{-1}(0,0)]$ does too; on the other hand, $[Q_B^R, I^*_{-1}(y,\bar{y})] = \partial_y\big(c(y) I^*_{-1}(y,\bar{y})\big)$, as we saw in (7.28). Thus, we can rewrite (7.43) as

$$
\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[y \partial\xi(\alpha y) \partial_y \Big(c(y) I^*_{-1}(y,\bar{y})\Big) + \partial_y \Big(y \partial\xi(\alpha y)\Big) c(y) I^*_{-1}(y,\bar{y})\Big] \right\rangle \quad ,
\tag{7.44}
$$

which is simply

$$
\int_{F_0} d^2\tau \int_{F_b} d^2y \;\partial_y\; \left\langle b_\tau \bar{b}_{\bar{\tau}} \chi(u_1) F_{-1}(0,0) \Big[y \partial\xi(\alpha y) c(y) I^*_{-1}(y,\bar{y})\Big] \right\rangle \quad .
\tag{7.45}
$$

Again, the torus has no boundary so (7.45) is non-zero only if a Dirac-delta is developed. We can obtain a term $\sim \frac{1}{\bar{y}}$ in the integrand only when the two vertex operators $F_{-1}(0,0) \sim V(0,0)$ and

$I_{-1}^*(y,\bar{y}) \sim V^*(y,\bar{y})$ come together, so we need to analyze what happens in the $y \to 0$ limit:

$$\begin{aligned}
F_{-1}(0,0)\Big[y\partial\xi(\alpha y)c(y)I_{-1}^*(y,\bar{y})\Big] &= \\
&= y\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\partial\xi(\alpha y)c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&= y\bar{c}(0)\underbrace{c(0)c(y)}_{\sim y+O(y^3)}\underbrace{e^{-\phi(0)}e^{-\phi(y)}}_{-\frac{1}{y}+O(1)}\underbrace{V(0,0)V^*(y,\bar{y})}_{\frac{q}{\bar{y}}V_{\hat{D}}+O(y,\bar{y})}\underbrace{e^{ik_1\cdot X(0,0)}e^{ik_2\cdot X(y,\bar{y})}}_{1+O(y)}\partial\xi(\alpha y) = \\
&\sim \frac{y}{\bar{y}} \quad ,
\end{aligned} \tag{7.46}$$

where, in the last step, we used the OPEs (7.39), (7.41), (7.40) and the fact that the ghost c is a Grassmann odd variable (i.e. $c(0)c(0) = 0$ which means that $c(0)c(y) = O(y)$).
Thus, the integrand in (7.45) doesn't develop a Dirac delta and it integrates to zero: the correlation function (7.13) is independent of the value of α.

7.3 The old computation

We want to conclude this thesis by presenting how our correlation function used to be computed in the 80's (see [26] and [3], for example).
At that time, it was not clear that the PCOs should be placed according to the degeneration limit and, to compute the mass-term $\rho_i\rho_i^*$, they legitimately thought to locate one PCO on the top of each vertex operator.
This choice corresponds, in our language, to set $\alpha = 1$ and $u_1 = 0$, so the starting point of the old computation is precisely our formula (7.18) (which - remember it! - was the "$\chi(0)$" contribution of fomula (7.16)). Then, we can follow the same steps as before and arrive again at (7.22), namely at

$$-\frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau\bar{b}_{\bar{\tau}}\bar{c}(0)c(0)\Big[k_1\cdot\psi(0)V(0,0)\Big]e^{ik_1\cdot X(0,0)}\Big[k_2\cdot\psi(y)V^*(y,\bar{y})\Big]e^{ik_2\cdot X(y,\bar{y})}\right\rangle \quad . \tag{7.47}$$

Obviously, they knew that this term vanishes if we mantain both momentum conservation and the on-shell (BRST-invariance) condition for our vertex operators, because (7.47) is proportional to $k_1 \cdot k_2$. The Dirac delta that forces the conservation of the momentum of the system comes, in the path integral approach, from the integration of the zero modes of the field X^μs, so they proposed to perform this integration as the very last step of the computation. From a practical point of view, the right result should come, in their opinion, from imposing the conditions

$$\begin{aligned} k_1^2 = k_2^2 &= 0 \\ k_1 \cdot k_2 &= \epsilon \end{aligned} \tag{7.48}$$

on formula (7.47) and from performing the limit $\epsilon \to 0$ at the very end of the calculations. This strategy will lead to a non-zero result because the $k_1 \cdot k_2$ factor coming from $< k_1 \cdot \psi(0) k_2 \cdot \psi(y) >$ multiplies an integral that diverges as $1/k_1 \cdot k_2$. This singularity arises near $y = \bar{y} = 0$, where the following OPEs hold

$$\begin{aligned}
e^{ik_1\cdot X(0,0)}e^{ik_2\cdot X(y,\bar{y})} &= \frac{1}{|y|^{k_1\cdot k_2}}e^{i(k_1+k_2)\cdot X(y,\bar(y))}(1+O(|y|^2)) \xrightarrow{\epsilon\to 0} \frac{1}{|y|^{k_1\cdot k_2}}(1+O(|y|^2)) \\
\psi^\mu(0)\psi^\nu(y) &= -\frac{\eta^{\mu\nu}}{y} + O(y) \\
V(0,0)V^*(y,\bar{y}) &= \frac{q}{\bar{y}}V_{\hat{D}}(0,0) + O(y,\bar{y}) \quad .
\end{aligned} \tag{7.49}$$

So, we have

$$
\begin{aligned}
-\frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar\tau}\bar{c}(0)c(0)\left[k_1\cdot\psi(0)V(0,0)\right]e^{ik_1\cdot X(0,0)}\left[k_2\cdot\psi(y)V^*(y,\bar{y})\right]e^{ik_2\cdot X(y,\bar{y})}\right\rangle = \\
=\frac{1}{4}\int_{F_0} d^2\tau \int_{F_b} d^2y \left\langle b_\tau \bar{b}_{\bar\tau}\bar{c}(0)c(0)\underbrace{k_1\cdot\psi(0)k_2\cdot\psi(y)}_{-\frac{k_1\cdot k_2}{y}+O(y)}\underbrace{V(0,0)V^*(y,\bar{y})}_{\frac{q}{\bar{y}}V_{\hat{D}}(0,0)+O(y,\bar{y})}\underbrace{e^{ik_1\cdot X(0,0)}e^{ik_2\cdot X(y,\bar{y})}}_{\frac{1}{|y|^{k_1\cdot k_2}}(1+O(|y|^2))}\right\rangle = \\
=-q\frac{k_1\cdot k_2}{4}\int_{F_0} d^2\tau \langle b_\tau \bar{b}_{\bar\tau}\bar{c}(0)c(0)V_{\hat{D}}(0,0)\rangle \int_{F_b} d^2y\,\frac{1}{|y|^{2+k_1\cdot k_2}} + \big(\text{vanishing terms for } k_1\cdot k_2\to 0\big) = \\
=-q\frac{k_1\cdot k_2}{4}\int_{F_0} d^2\tau \langle b_\tau \bar{b}_{\bar\tau}\bar{c}(0)c(0)V_{\hat{D}}(0,0)\rangle \frac{2\pi}{-k_1\cdot k_2};
\end{aligned}
\tag{7.50}
$$

note that the integral[115] $\int_{F_b} d^2y\,\frac{1}{|y|^{2+k_1\cdot k_2}}$ is clearly positive and this is consistent with the result $2\pi/(-k_1\cdot k_2)$ because, in order to make it convergent, we have to take the $k_1\cdot k_2\to 0$ limit from below, namely $k_1\cdot k_2\to 0^-$.
The result of the amplitude is then

$$
q\frac{\pi}{2}\int_{F_0} d^2\tau \langle b_\tau \bar{b}_{\bar\tau}\bar{c}(0)c(0)V_{\hat{D}}(0,0)\rangle \quad , \tag{7.51}
$$

which is precisely the result that we computed with our approach, see (7.42), except for a sign. Intuitively, the results (7.42) and (7.51) differ for a sign because, according to our strategy, the final result comes from the $\chi(u_1)-\chi(0)\sim-\chi(0)$ contribution ("only "$-\chi(0)$ gave a contribution!), whereas, according to the old approach, it instead arises from the insertion of $\chi(0)$.
Note that the sign is relevant because if $m^2_{\rho_i\rho_i^*}<0$, then a tachyonic direction is developed and the classical vacuum becomes, at one loop level, unstable[116]. If we want to pick up the right sign, we need to distribute the PCOs according to the degeneration limit.
The reason why the old approach gives the right result (modulo a sign) despite the fact that momentum

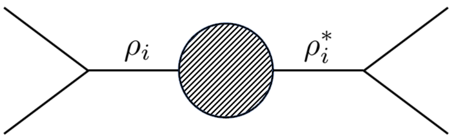

Figure 21
The mass shift of the massless particle ρ_i can be computed slightly off-shell by treating ρ_i as a resonance in a scattering amplitude with four external particles.

conservation was only imposed at the end can be understood by computing another correlation function. One can consider a scattering amplitude in which the particle ρ_i appears as a resonance or intermediate state. Such an amplitude (see Figure 21) is affected by the mass shift of the ρ_i particle, but now the latter can be slighlty off-shell. So k_1^2 and k_2^2 are not properly zero and, correspondingly, we can consider $k_1\cdot k_2\neq 0$, as we have done throughout this section. The mass shift δm^2 of the ρ_i particle appears in the perturbative computation of the scattering amplitude of Figure 21 as the coefficient of a double pole, because of the usual expansion

$$
\frac{1}{k^2+\delta m^2}=\frac{1}{k^2}-\frac{1}{k^2}\delta m^2\frac{1}{k^2}+\ldots \tag{7.52}
$$

[115] We don't compute the integral here because it is straightforward; it is enough to note that all the contribution in which we are interested comes from the region around $y=\bar{y}=0$, so we can equivalently calculate the integral $\int_{B(0,\delta)} d^2y\,\frac{1}{|y|^{2+k_1\cdot k_2}}$, where $B(0,\delta)$ is a little ball of radius δ centered in $y=0$.

[116] In this case, according to (7.6) and (7.7), we expect a supersymmetric stable vacuum to be developed at $|\rho_i|^2=-pg_s^2/e_i$ provided that there are no D-terms for the other fields.

Appendix A Calculation for (7.17)

In this appendix we will be very precise with the computations, because we want explicitly show that if we are interested in a local issue (as changing the picture of a vertex operator is) then we don't have to worry about all the non-singular corrections that the OPEs get from the boundary conditions of the surface and we can pretend to be on the sphere.

First of all, we clarify our conventions.
In QFT, Wick's theorem states that a time ordered product of normal ordered bunches of fields can be simplified to a sum over all possible contractions performed over fields evaluated at different time. In radial quantization, we analogously have that radial ordered product of normal ordered bunches of fields can be simplified to a sum over all possible contractions performed over fields positioned at different radii. With the label "NER" we will precisely mean that we have to do all the contractions among Not Equal Radii fields. Also,

- $\overbracket{\phi(w)\phi(0)}$ is the 2-fields contraction. It means that we have substitute these two fields with their correlation function;
- $\overbrace{\phi(w)^n\phi(0)^m}^{c}$ will mean that we are taking into consideration all the possible terms that arise by taking c times the 2-fields-contractions out of the n copies of the field $\phi(w)$ and the m copies of the field $\phi(0)$. For example,
$$\phi\overbracket{(w)^n\phi(0)}^m \equiv \overbrace{\phi(w)^n\phi(0)^m}^{1} \quad .$$

Finally, we say that in all the appendices we will write $\int_{C(z)} dw$ instead of $\int_{C(z)} \frac{dw}{2\pi i}$, where, as usual, $C(z)$ denotes a circle around z.

We need to compute

$$\lim_{w\to 0} \chi(w) : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \tag{A.1}$$
$$= : \bar{c}(0)c(0)\{\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\}e^{ik_1\cdot X(0,0)} : -\frac{1}{4} : \bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \tag{A.2}$$

We have to use the definition[117] of

$$\chi(w) =: \{Q_B^R, \xi(w)\} :=: \left[\underbrace{c\partial\xi}_{1.} + \underbrace{e^{\phi}G}_{2.} - \underbrace{\frac{1}{2}\partial\eta e^{2\phi}b}_{3.} - \underbrace{\frac{1}{4}\eta\partial_w(e^{2\phi}b)}_{4.}\right]\Big|_w : \tag{A.3}$$

So, when $\chi(w)$ approaches $:\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:$ we get 4 contributions:

1. $\lim_{w\to 0} : c(w)\partial\xi(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :$ here there are no possible contractions so Wick theorem gives

$$\lim_{w\to 0} : c(w)\partial\xi(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \tag{A.4}$$
$$= \lim_{w\to 0} : c(w)\partial\xi(w)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \tag{A.5}$$
$$= \lim_{w\to 0} \left[: c(0)\partial\xi(0)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : + O(w)\right] = \tag{A.6}$$
$$= 0 \tag{A.7}$$

where the we have exploited the standard fact $c(0)c(0) = 0$.

[117]Please note that, whenever not precised, ∂ is a partial derivation with respect to the argument of the function on which it is acting.

2. $\lim_{w\to 0}\,:e^{\phi(w)}G(w)::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:$ which we are going to compute by using Wick theorem

$$
\begin{aligned}
&:e^{\phi(w)}G(w)::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:=\\
&=\Big[:e^{\phi(w)}G(w)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:\Big]\Big|_{NER}=\\
&=-\Big[:e^{\phi(w)}e^{-\phi(0)}G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\Big]\Big|_{NER}=
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
&=-\Big[\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n!\,m!}:\phi(w)^n\phi(0)^mG(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\Big]\Big|_{NER}=\\
&=-\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n!\,m!}:\phi(w)^n\phi(0)^mG(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=1}^{\infty}\sum_{m,n=c}^{\infty}\frac{(-1)^m}{n!\,m!}:\overbrace{\phi(w)^n\phi(0)^m}^{c}G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=1}^{\infty}\sum_{m,n=c-1}^{\infty}\frac{(-1)^m}{n!\,m!}:\overbrace{\phi(w)^n\phi(0)^m}^{c-1}\overline{G(w)\bar{c}(0)c(0)V}(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=1}^{\infty}\sum_{m,n=c-1}^{\infty}\frac{(-1)^m}{n!\,m!}:\overbrace{\phi(w)^n\phi(0)^m}^{c-1}\overline{G(w)\bar{c}(0)c(0)V(0,0)e}^{ik_1\cdot X(0,0)}:\;=\\
&=-\sum_{c=0}^{\infty}\sum_{m,n=c}^{\infty}\frac{(-1)^m}{n!\,m!}\binom{n}{c}\binom{m}{c}c!\,\langle\phi(w)\phi(0)\rangle^c\times\\
&\qquad\times:\phi(w)^{n-c}\phi(0)^{m-c}G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=1}^{\infty}\sum_{m,n=c-1}^{\infty}\frac{(-1)^m}{n!\,m!}\binom{n}{c-1}\binom{m}{c-1}(c-1)!\,\langle\phi(w)\phi(0)\rangle^{c-1}\times\\
&\qquad\times:\phi(w)^{n-c+1}\phi(0)^{m-c+1}\overline{G(w)\bar{c}(0)c(0)V}(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=1}^{\infty}\sum_{m,n=c-1}^{\infty}\frac{(-1)^m}{n!\,m!}\binom{n}{c-1}\binom{m}{c-1}(c-1)!\,\langle\phi(w)\phi(0)\rangle^{c-1}\times\\
&\qquad\times:\phi(w)^{n-c+1}\phi(0)^{m-c+1}\overline{G(w)\bar{c}(0)c(0)V(0,0)e}^{ik_1\cdot X(0,0)}:\;=\\
&=-\sum_{c=0}^{\infty}\frac{\big(-\langle\phi(w)\phi(0)\rangle\big)^c}{c!}\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n!\,m!}\times\\
&\qquad\times:\phi(w)^n\phi(0)^mG(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=0}^{\infty}\frac{\big(-\langle\phi(w)\phi(0)\rangle\big)^c}{c!}\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n!\,m!}\times\\
&\qquad\times:\phi(w)^n\phi(0)^m\overline{G(w)\bar{c}(0)c(0)V}(0,0)e^{ik_1\cdot X(0,0)}:+\\
&\quad-\sum_{c=0}^{\infty}\frac{\big(-\langle\phi(w)\phi(0)\rangle\big)^c}{c!}\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n!\,m!}\times\\
&\qquad\times:\phi(w)^n\phi(0)^m\overline{G(w)\bar{c}(0)c(0)V(0,0)e}^{ik_1\cdot X(0,0)}:=
\end{aligned}
\tag{A.9}
$$

$$
=-e^{-\langle\phi(w)\phi(0)\rangle}\,:e^{\phi(w)}e^{-\phi(0)}:\,\Big[:G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\Big]\Big|_{NER}
\tag{A.10}
$$

Now we can use the following relations

$$
\begin{aligned}
\langle \phi(w)\phi(0)\rangle &= -\log(w) + O(w) \\
: e^{\phi(w)} e^{-\phi(0)} : &= 1 + O(w) \\
G(w)V(0,0) &= G^{int}(w)V(0,0) = \\
&= -\frac{1}{w}\tilde{V}(0,0) + O(1) \\
G(w) : e^{ik_1\cdot X(0,0)} : &= G^{X\psi}(w) : e^{ik_1\cdot X(0,0)} := \\
&=: -\psi(w)\cdot\partial X(w) :: e^{ik_1\cdot X(0,0)} := \\
&= \frac{i}{2w} : k_1\cdot\psi(0) e^{ik_1\cdot X(0,0)} : + O(1)
\end{aligned}
\tag{A.11}
$$

so as to obtain

$$
\begin{aligned}
&= -\, e^{ln(w)+O(w)} \big(1+O(w)\big) \Big[: G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} : \Big]\Big|_{NER} = \\
&=(-w+O(w)) \big(1+O(w)\big) \Big[: G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} : \Big]\Big|_{NER} = \\
&=(-w+O(w)) \big(1+O(w)\big) \; : G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} : + \\
&\quad + (-w+O(w)) \big(1+O(w)\big) \; : G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} : + \\
&=(-w+O(w)) \big(1+O(w)\big) \; : \bar{c}(0)c(0)\Big(-\frac{1}{w}\tilde{V}(0,0) + O(1)\Big) e^{ik_1\cdot X(0,0)} : + \\
&\quad + (-w+O(w)) \big(1+O(w)\big) \; : \bar{c}(0)c(0)V(0,0) \times \\
&\qquad\qquad \times \Big(-\frac{i}{2w} k_1\cdot\psi(0)e^{ik_1\cdot X(0,0)} : + O(1)\Big) := \\
&= : \bar{c}(0)c(0)\tilde{V}(0,0)e^{ik_1\cdot X(0,0)} : - : \bar{c}(0)c(0)\frac{i}{2}k_1\cdot\psi(0)V(0,0)e^{ik_1\cdot X(0,0)} : + O(w)
\end{aligned}
\tag{A.12}
$$

Finally, taking the $\lim_{w\to 0}$ we end up with

$$
\begin{aligned}
\lim_{w\to 0} : e^{\phi(w)} G(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
=: \bar{c}(0)c(0)\Big[\tilde{V}(0,0) - \frac{i}{2}k_1\cdot\psi(0)V(0,0)\Big] e^{ik_1\cdot X(0,0)} :
\end{aligned}
\tag{A.13}
$$

3. $\lim_{w\to 0} : \frac{1}{2}\partial\eta(w)e^{2\phi(w)}b(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}$: here we can use Wick theorem to go through similar calculations as we have just done for the previous contribution so as to obtain the result analogous to (0.10):

$$
\begin{aligned}
&: \frac{1}{2}\partial\eta(w)e^{2\phi(w)}b(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&=\Big[: \frac{1}{2}\partial\eta(w)e^{2\phi(w)}b(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \Big]\Big|_{NER} = \\
&= -\, e^{-2\langle\phi(w)\phi(0)\rangle} \; : e^{2\phi(w)}e^{-\phi(0)} : \Big[: \frac{1}{2}\partial\eta(w)b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} : \Big]\Big|_{NER} = \\
&= -\, e^{-2\langle\phi(w)\phi(0)\rangle} \; : e^{2\phi(w)}e^{-\phi(0)} :: \frac{1}{2}\partial\eta(w)b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} :
\end{aligned}
\tag{A.14}
$$

By using the standard relations

$$
\begin{aligned}
\langle \phi(w)\phi(0)\rangle &= -\log(w) + O(w) \\
\langle b(w)c(0)\rangle &= \frac{1}{w} + O(1)
\end{aligned}
\tag{A.15}
$$

this contribution reads

$$
\begin{aligned}
&= -\left(w^2 + O(w^3)\right)\ :\left(e^{\phi(0)} + O(w)\right):\ \frac{1}{2}\left(-\frac{1}{w} + O(w)\right)\ :\partial\eta(w)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:= \\
&= \frac{1}{2}\left(w + O(w^2)\right)\ :\left(e^{\phi(0)} + O(w)\right)::\left(\partial\eta(0) + O(w)\right)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:
\end{aligned}
\tag{A.16}
$$

and we see that it vanishes by taking the $w \to 0$ limit.

4. $\lim_{w\to 0}:\frac{1}{4}\eta(w)\partial_w(e^{2\phi(w)}b(w))::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:$ as first step, we are going to use Leibniz rule to split it into 2 terms, that is

$$
\begin{aligned}
&:\frac{1}{4}\eta(w)\partial_w\left(e^{2\phi(w)}b(w)\right)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:= \\
=&\frac{1}{4}:\partial_w e^{2\phi(w)}e^{-\phi(0)}\eta(w)b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:+ \\
&+\frac{1}{4}:e^{2\phi(w)}e^{-\phi(0)}\eta(w)\partial b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:
\end{aligned}
\tag{A.17}
$$

With both of these terms we could go through all the possible contractions as we have done when computing the second contribution and end up with results analogous to (0.10), that is:

$$
\begin{aligned}
&:\frac{1}{4}\eta(w)\partial_w\left(e^{2\phi(w)}b(w)\right)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:= \\
=&\frac{1}{4}\left[:\partial_w e^{2\phi(w)}e^{-\phi(0)}\eta(w)b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\right]\Bigg|_{NER}+ \\
&+\frac{1}{4}\left[:e^{2\phi(w)}e^{-\phi(0)}\eta(w)\partial b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\right]\Bigg|_{NER} \\
=&\frac{1}{4}\partial_w\left(e^{-2\langle\phi(w)\phi(0)\rangle}:e^{2\phi(w)}e^{-\phi(0)}:\right)\left[:\eta(w)b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\right]\Bigg|_{NER}+ \\
&+\frac{1}{4}e^{-2\langle\phi(w)\phi(0)\rangle}:e^{2\phi(w)}e^{-\phi(0)}:\left[:\eta(w)\partial b(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}:\right]\Bigg|_{NER}
\end{aligned}
\tag{A.18}
$$

And using the usual formulae for the contractions (see 0.15) we obtain

$$
\begin{aligned}
=&\frac{1}{4}\partial_w\left(e^{-2\langle\phi(w)\phi(0)\rangle}:e^{2\phi(w)}e^{-\phi(0)}:\right):\eta(w)\overbrace{b(w)\bar{c}(0)c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:+ \\
&+\frac{1}{4}e^{-2\langle\phi(w)\phi(0)\rangle}:e^{2\phi(w)}e^{-\phi(0)}:\eta(w)\partial\overbrace{b(w)\bar{c}(0)c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:= \\
=&\frac{1}{4}\partial_w\left((w^2+O(w^3)):e^{\phi(0)}+O(w):\right)\left(-\frac{1}{w}+O(w)\right):\eta(w)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:+ \\
&+\frac{1}{4}((w^2+O(w^3)):e^{\phi(0)}+O(w):\partial\left(-\frac{1}{w}+O(w)\right):\eta(w)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:= \\
=&-\frac{1}{2}:e^{\phi(0)}\eta(0)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:+\frac{1}{4}:e^{\phi(0)}\eta(0)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:+O(w)= \\
=&\frac{1}{4}:e^{\phi(0)}\eta(0)\bar{c}(0)V(0,0)e^{ik_1\cdot X(0,0)}:+O(w)
\end{aligned}
\tag{A.19}
$$

This means that after taking the $w \to 0$ limit we end up with

$$
\begin{aligned}
&\lim_{w\to 0}:\frac{1}{4}\eta(w)\partial_w(e^{2\phi(w)}b(w))::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:= \\
=&\frac{1}{4}:\bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:
\end{aligned}
\tag{A.20}
$$

Coming to a conclusion, we can sum all these contributions together and we finally obtain

$$\begin{aligned}
&\lim_{w\to 0}\chi(w):\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:=\\
&=\lim_{w\to 0}:\left[c\partial\xi+e^{\phi}G-\frac{1}{2}\partial\eta e^{2\phi}b-\frac{1}{4}\eta\partial_w(e^{2\phi}b)\right]\Big|_w::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:=\\
&=\lim_{w\to 0}:\left[e^{\phi(w)}G(w)-\frac{1}{4}\eta\partial_w(e^{2\phi(w)}b(w))\right]::\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:=\\
&=:\bar{c}(0)c(0)\left[\tilde{V}(0,0)-\frac{i}{2}k_1\cdot\psi(0)V(0,0)\right]e^{ik_1\cdot X(0,0)}:-\frac{1}{4}:\bar{c}(0)\eta(0)e^{\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:
\end{aligned} \tag{A.21}$$

Note that, as expected, when we compute the OPE among the PCO and a vertex operator, we get a non-zero contribution only from the divergent part of the OPEs, because any vertex operator (regardless of its picture) is something local and should not contain any information on the world sheet on which it is inserted. Locally, one should not feel the boundary conditions.

Appendix B Calculation for (7.19)

We have to compute

$$\lim_{\alpha\to 1}\big(\chi(\alpha y)+\alpha\partial\xi(\alpha y)c(y)\big):e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:= \tag{B.1}$$

In this limit, the $c(\alpha y)\partial\xi(\alpha y)$ term from $\chi(\alpha y)$ cancels the $\alpha\partial\xi(\alpha y)c(\alpha y)$ term and this means that if we want to change the picture number of an integrated vertex operator, we have to act on it with the moving operator χ^M:

$$\begin{aligned}
&\chi^M(y)\equiv\chi(y)+\partial\xi(y)c(y)=\\
&\quad=:\underbrace{e^{\phi(y)}G(y)}_{1.}-\frac{1}{2}\underbrace{\partial\eta(y)e^{2\phi(y)}b(y)}_{2.}-\frac{1}{4}\underbrace{\eta(y)\partial_y(e^{2\phi(y)}b(y))}_{3.}:
\end{aligned} \tag{B.2}$$

Given that the operator $:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:$ has no c insertions, the contributions 2. and 3. vanishes in the $\lim_{w\to y}$ and we are left with

$$\begin{aligned}
&\lim_{w\to y}:e^{\phi(w)}G(w)::e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:=\lim_{w\to y}\Big[:e^{\phi(w)}G(w)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big]_{NER}=\\
&=\lim_{w\to y}(-1)e^{-\langle\phi(w)\phi(y)\rangle}:e^{\phi(w)-\phi(y)}\overline{G(w)V^*}(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:+\\
&\quad+\lim_{w\to y}(-1)e^{-\langle\phi(w)\phi(y)\rangle}:e^{\phi(w)-\phi(y)}\overline{G(w)V^*(y,\bar{y})e}^{ik_2\cdot X(y,\bar{y})}:=\\
&=\left[\tilde{V}(y,\bar{y})-\frac{i}{2}k_2\cdot\psi(y)V^*(y,\bar{y})\right]e^{ik_2\cdot X(y,\bar{y})}\quad.
\end{aligned} \tag{B.3}$$

Appendix C BRST-contour deformation

Here we show how the deformation of the BRST-contour integral should be done. We illustrate this with a general example, then the application to (7.26) will be straightforward.

Let's consider a correlation function like

$$\Big\langle A(z_A)B(z_B)\Big\{Q_B^R,C(z_C)\Big\}D(z_D)\Big\rangle=\oint_{C(z_C)}dw\,\Big\langle A(z_A)B(z_B)j_B^R(w)C(z_C)D(z_D)\Big\rangle\quad, \tag{C.1}$$

where Q_B^R is the holomorphic bit of the BRST charge and the capital letters $I(z_I)$ denote local operators inserted at the positions z_I on (the fundamental region of) the torus. In the second expression, we used

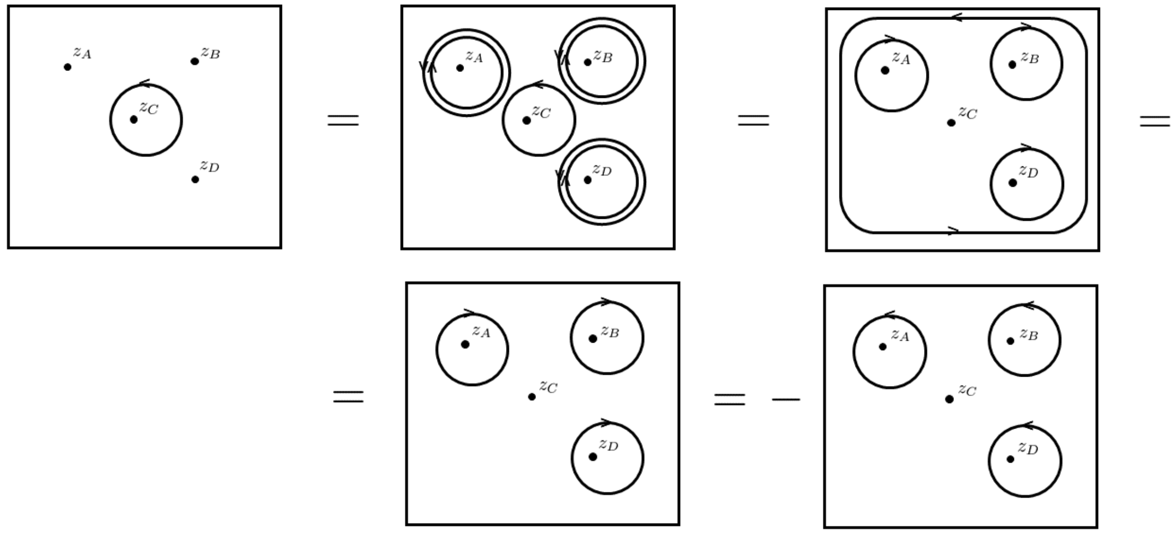

Figure 22
Deformation of the contour integral on the torus.

(4.35) to write the anticommutator in terms of the BRST-integral along a circle C_{z_C} that is centered on z_C and which is counterclockwise oriented.

We can deform C_{z_C} as depicted in Figure 22 to obtain

$$\begin{aligned}\oint_{C(z_C)} dw \left\langle A(z_A)B(z_B)j_B^R(w)C(z_C)D(z_D)\right\rangle = &-\oint_{C(z_A)} dw \left\langle A(z_A)B(z_B)j_B^R(w)C(z_C)D(z_D)\right\rangle + \\ &-\oint_{C(z_B)} dw \left\langle A(z_A)B(z_B)j_B^R(w)C(z_C)D(z_D)\right\rangle + \\ &-\oint_{C(z_D)} dw \left\langle A(z_A)B(z_B)j_B^R(w)C(z_C)D(z_D)\right\rangle \quad ,\end{aligned} \tag{C.2}$$

where all the contours C_{z_I} are counterclockwise oriented around the insertion at z_I. If we want to rewrite this expression in terms of BRST- commutators/anticommutators of the fields A, B, D, we need to bring j_B^R to the close left of each operator, so then we can appply (4.35) again. In doing so, we have to be careful, because swapping j_B^R with a fermionic operator brings a sign into the game. To be more concrete, let's consider the case in which A, B, C are Grassmann odd and D is Grassmann even. Then, (C.4) reads as

$$\begin{aligned}\left\langle A(z_A)B(z_B)\left\{Q_B^R, C(z_C)\right\}D(z_D)\right\rangle = &-\oint_{C(z_A)} dw \left\langle j_B^R(w)A(z_A)B(z_B)C(z_C)D(z_D)\right\rangle + \\ &+\oint_{C(z_B)} dw \left\langle A(z_A)j_B^R(w)B(z_B)C(z_C)D(z_D)\right\rangle + \\ &+\oint_{C(z_D)} dw \left\langle A(z_A)B(z_B)C(z_C)j_B^R(w)D(z_D)\right\rangle \quad ;\end{aligned} \tag{C.3}$$

now, by using (4.35) we reach the final expression

$$\Big\langle A(z_A)B(z_B)\Big\{Q_B^R, C(z_C)\Big\}D(z_D)\Big\rangle = -\oint_{C(z_A)} dw \Big\langle\Big\{Q_B^R, A(z_A)\Big\}B(z_B C(z_C)D(z_D)\Big\rangle + \\ + \oint_{C(z_B)} dw \Big\langle A(z_A)\Big\{Q_B^R, B(z_B)\Big\}C(z_C)D(z_D)\Big\rangle + \\ + \oint_{C(z_D)} dw \Big\langle A(z_A)B(z_B)C(z_C)\Big[Q_B^R, D(z_D)\Big]\Big\rangle \quad . \tag{C.4}$$

We mention that this kind of mannipulations can be done also for (compact, oriented) higher genus surfaces, because these can be represented - as the torus - as spaces obtained by introducing proper identifications on the complex plane.

Appendix D Calculation for (7.28)

We have to compute the commutator of the first vertex operator with the holomorphic bit of the BRST-charge, that is

$$[Q_B^R\,, :\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:] = ? \tag{D.1}$$

where

$$\begin{aligned} Q_B^R &= \oint dw j_B^R(w) \\ j_B^R(w) &=: c(w)\Big(T^{X\psi}(w) + T^{int}(w) + T^{\beta\gamma}(w) + \partial c(w) b(w)\Big) : + \\ &\qquad - :\gamma(w)G(w): -\frac{1}{4}:\beta(w)\gamma^2(w): = \\ &= \underbrace{:c(w)\Big(T^{tot}(w) - \partial(c(w)b(w))\Big):}_{1.} - \underbrace{:\gamma(w)G(w):}_{2.} - \underbrace{\frac{1}{4}:b(w)\gamma^2(w):}_{3.} \end{aligned} \tag{D.2}$$

Here, T^{tot} is the holomorphic part of the sum of all the energy momentum tensors (matter, bc ghost system, $\beta\gamma$ system) and G is the holomorphic matter part of the worldsheet supersymmetry current. To be more precise:

$$\begin{aligned} T^{tot}(w) &= T^{X\psi}(w) + T^{int}(w) + T^{\beta\gamma}(w) + T^{bc}(w) \\ T^{X\psi}(w) &= -\partial X(w)\cdot\partial X(w) + \psi(w)\cdot\partial\psi(w) \\ T^{\beta\gamma}(w) &= -\frac{3}{2}\beta(w)\partial\gamma(w) - \frac{1}{2}\partial\beta(w)\gamma(w) \\ T^{bc}(w) &= 2\partial c(w)b(w) + c(w)\partial b(w) \\ G(w) &= G^{X\psi}(w) + G^{int}(w) = -\psi(w)\cdot\partial X(w) + G^{int}(w) \end{aligned} \tag{D.3}$$

Turning back to the computation of (D.1), we have three contributions, which correspond to the three terms which $j_B^R(w)$ consists of. These are:

1. $[\oint dw\ :\ c(w)\Big(T^{tot}(w) - \partial(c(w)b(w))\Big)\ :,:\ \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\ :]$ of course, here it is essential to remember that we fixed the position of the first vertex operator (as the presence of $\bar{c}(0)c(0)$ in it testifies) so : $\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}$: is a primary operator of conformal

dimension 0; said another way, the most singular term of its OPE with T^{tot} goes like a simple pole.

$$
\begin{aligned}
&[\oint dw : c(w)\Big(T^{tot}(w) - \partial(c(w)b(w))\Big) :, : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :] = \\
&\quad = \oint_{C(0)} dw : c(w)T^{tot}(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : + \\
&\qquad - \oint_{C(0)} dw : c(w)\partial(c(w)b(w)) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = \oint_{C(0)} dw\Big[: c(w)T^{tot}(w)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \Big]_{NER} + \\
&\qquad - \oint_{C(0)} dw\Big[: c(w)\partial(c(w)b(w))\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \Big]_{NER} = \\
&\quad = \oint_{C(0)} dw : c(w)\Big[T^{tot}(w)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\Big]_{NER} : + \\
&\qquad - \oint_{C(0)} dw : c(w)\partial(c(w)b(w))\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : = \\
&\quad = \oint_{C(0)} dw : c(w)\Big[\frac{1}{w}\partial\Big(\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\Big) + O(1)\Big] : + \\
&\qquad - \oint_{C(0)} dw : c(w)\partial\Big(- \frac{1}{w}c(w) + O(w)\Big)\bar{c}(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = : c(0)\partial\Big(\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}\Big) : + \\
&\qquad - \oint_{C(0)} dw : c(w)\Big(- \frac{1}{w}\partial c(w) + O(1)\Big)\bar{c}(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = : c(0)\bar{c}(0)\partial c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : + \\
&\qquad + : c(0)\partial c(0)\bar{c}(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = 0
\end{aligned}
\tag{D.4}
$$

2. $[\oint dw : \gamma(w)G(w) :, : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :]$ which is

$$
\begin{aligned}
&[\oint dw : \gamma(w)G(w) :, : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :] = \\
&\quad = \oint_{C(0)} dw : \gamma(w)G(w) :: \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = \oint_{C(0)} dw \Big[: \eta(w)e^{\phi(w)}G(w)\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \Big]_{NER} = \\
&\quad = \oint_{C(0)} dw\, e^{-\langle\phi(w)\phi(0)\rangle} : e^{\phi(w)-\phi(0)} \Big[\eta(w)G(w)\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)}\Big]_{NER} := \\
&\quad = \oint_{C(0)} dw\, e^{-\langle\phi(w)\phi(0)\rangle} : e^{\phi(w)-\phi(0)}\eta(w)\overline{G(w)\bar{c}(0)c(0)V}(0,0)e^{ik_1\cdot X(0,0)} : + \\
&\qquad + \oint_{C(0)} dw\, e^{-\langle\phi(w)\phi(0)\rangle} : e^{\phi(w)-\phi(0)}\eta(w)\overline{G(w)\bar{c}(0)c(0)V(0,0)e}^{ik_1\cdot X(0,0)} : = \\
&\quad = \oint_{C(0)} dw(w+O(w^2)) : (1+O(w))(\eta(0)+O(w))\bar{c}(0)c(0)\times \\
&\qquad\qquad \times \left(\frac{1}{w}\tilde{V}(0,0)+O(1)\right)e^{ik_1\cdot X(0,0)} : + \\
&\qquad + \oint_{C(0)} dw(w+O(w^2)) : (1+O(w))(\eta(0)+O(w))\bar{c}(0)c(0)V(0,0)\times \\
&\qquad\qquad \times \left(\frac{i}{2w} : k\cdot\psi(0)e^{ik_1\cdot X(0,0)} : +O(1)\right) := \\
&\quad = \oint_{C(0)} dw O(1) = \\
&\quad =0
\end{aligned}
\tag{D.5}
$$

3. $[\oint dw\frac{1}{4} : b(w)\gamma^2(w) :, : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :]$ which is

$$
\begin{aligned}
&[\oint dw\frac{1}{4} : b(w)\gamma^2(w) :, : \bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} :] = \\
&\quad =\frac{1}{4}\oint dw \Big[: b(w)\eta^2(w)e^{2\phi(w)}\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)} : \Big]_{NER} = \\
&\quad = -\frac{1}{4}\oint dw\, e^{-2\langle\phi(w)\phi(0)\rangle} : e^{2\phi(w)-\phi(0)}\overline{b(w)\eta^2(w)\bar{c}(0)c}(0)V(0,0)e^{ik_1\cdot X(0,0)} : = \\
&\quad = -\frac{1}{4}\oint dw\,(w^2+O(w^3)) : (e^{\phi(0)}+O(w))\times \\
&\qquad\qquad \times\left(\frac{1}{w}+O(1)\right)(\eta^2(0)+O(w))\bar{c}(0)c(0)V(0,0)e^{ik_1\cdot X(0,0)} := \\
&\quad = \oint dw\, O(w) = \\
&\quad =0
\end{aligned}
\tag{D.6}
$$

In this computations, the $\bar{c}$ ghost of the vertex operator has been a spectator. We have never contracted it with something else, becuase we are computing the commutator of the vertex operator with the *holomorphic* bit of the BRST-charge. We can drop it and the final result of the (anti)commutator will not change, so we have actually proved[118] both

[118] More formally, we have

$$
\begin{aligned}
[Q_B^R, : \bar{c}(y)c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} :] =& \{Q_B^R, \bar{c}(y)\} : c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} : + \\
& - \bar{c}(y)\{Q_B^R, : c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} : \} = \\
=& - \bar{c}(y)\{Q_B^R, : c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} : \}
\end{aligned}
\tag{D.7}
$$

$$\left[Q_B^R\,,\,:\bar{c}(0)c(0)e^{-\phi(0)}V(0,0)e^{ik_1\cdot X(0,0)}:\right]=0 \tag{D.8}$$

and

$$\left\{Q_B^R\,,\,:c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\right\}=0 \tag{D.9}$$

It is useful to understand how this anticommutator changes if we remove also the c ghost from it, which is equivalent to understand how an integrated vertex operator behaves under the action of the holomorphic part of the BRST charge:

$$\left[Q_B^R\,,\,:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\right]\ =? \tag{D.10}$$

As we have just done, we are going to perform this computation by splitting it into three contributions, that is

$$\begin{aligned}&\left[Q_B^R\,,\,:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\right]\ =\\
&=\underbrace{\oint_{C(y)}dw\ :c(w)\Big(T^{tot}(w)-\partial(c(w)b(w))\Big)::e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:}_{1.}+\\
&\quad-\underbrace{\oint_{C(y)}dw\ :\gamma(w)G(w)::e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:}_{2.}+\\
&\quad-\frac{1}{4}\underbrace{\oint_{C(y)}dw\ :b(w)\gamma^2(w)::e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:}_{3.}\end{aligned} \tag{D.11}$$

where

1. given that $:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:$ is a primary operator of conformal dimension 1 without any b or c dependece, the first contribution simplifies to

$$\begin{aligned}&\oint_{C(y)}dw\ :c(w)T^{tot}(w)::e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:=\\
&\quad=\oint_{C(y)}dw\ :c(w)\Big[T^{tot}(w)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}\Big]_{NER}:=\\
&\quad=\oint_{C(y)}dw\ :c(w)\frac{1}{(w-y)^2}e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:+\\
&\qquad+\oint_{C(y)}dw\ :c(w)\frac{1}{w-y}\partial_y\Big(e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big)+O(1)\\
&\quad=\oint_{C(y)}dw\ :c(w)\frac{1}{(w-y)^2}e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:+\\
&\qquad+\oint_{C(y)}dw\ :c(w)\frac{1}{w-y}\partial_y\Big(e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big)+\\
&\qquad+\oint_{C(y)}dw\ :c(w)O(1):=\\
&\quad=\oint_{C(y)}dw\ :\Big(c(y)+(w-y)\partial c(y)+O((w-y)^2)\Big)\frac{1}{(w-y)^2}\times\\
&\qquad\qquad\times e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:+\\
&\qquad+\oint_{C(y)}dw\ :\Big(c(y)+(w-y)\partial c(y)+O((w-y)^2)\Big)\frac{1}{w-y}\times\\
&\qquad\qquad\times\partial_y\Big(:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big)=\\
&\quad=\partial_y\Big(:c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big)\end{aligned} \tag{D.12}$$

2. to compute the second term we have to go through the same computations that we did to arrive at the result D.5. Nothing changes, because there both c and $\bar{c}$ were spectators. So this contribution vanishes again.

3. following the same procedure that lead us to the the result D.6, it is immediate to see that also this contribution vanishes.

So we have proved that

$$\left[Q_B^R\,,\,:e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\right] = \partial_y\Big(:c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})}:\Big) \tag{D.13}$$

Appendix E Calculation for (7.36)

Now we will be very sloppy, given that we showed in Appendix A how computations should be done in a diligent way.

We have to compute

$$\chi(w)\Big|_{\phi\text{- charge}=2} c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = -\frac{1}{4}\eta(y)e^{\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} \quad , \tag{E.1}$$

in the $w \to y$ limit.
We have two contributions, because

$$\chi(w)\Big|_{\phi\text{- charge}=2} = -\frac{1}{2}\partial\eta(w)e^{2\phi(w)}b(w) - \frac{1}{4}\eta(w)\partial\big(e^{2\phi(w)}b(w)\big) \quad . \tag{E.2}$$

- The first one is zero, because

$$\begin{aligned}
&-\frac{1}{2}\partial\eta(w)e^{2\phi(w)}b(w)c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&= -\frac{1}{2}\partial\eta(w):\Big[(w-y)^2e^{2\phi(w)-\phi(y)} + O((w-y)^3)\Big]:\Big[\frac{1}{w-y} + O(w-y)\Big]V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&= O(w-y) \stackrel{w\to y}{\longrightarrow} 0
\end{aligned} \tag{E.3}$$

- instead, the second one reads as

$$\begin{aligned}
&-\frac{1}{4}\eta(w)\partial_w\Big(e^{2\phi(w)}b(w)\Big)c(y)e^{-\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&= -\frac{1}{4}\eta(w)\partial_w\Bigg\{:\Big[(w-y)^2e^{2\phi(w)-\phi(y)} + O((w-y)^3)\Big]:\Big[\frac{1}{w-y} + O(w-y)\Big]\Bigg\}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&= -\frac{1}{4}\eta(w)\partial_w:\Bigg\{(w-y)e^{2\phi(w)-\phi(y)} + O((w-y)^2)\Bigg\}:V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} = \\
&\stackrel{w\to y}{\longrightarrow} -\frac{1}{4}\eta(y)e^{\phi(y)}V^*(y,\bar{y})e^{ik_2\cdot X(y,\bar{y})} \quad .
\end{aligned} \tag{E.4}$$

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