ALGANT Master Thesis in Mathematics

UNIFORMIZATION RESULTS FOR SOME CLASSES OF ANALYTIC STACKS

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To my parents and my sister. . .

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Introduction

There are several reasons for studying stacks. One is for sure to construct a *moduli space* for certain mathematical objects, like elliptic curves and G-bundles.

Another motivation comes from the quotient of a manifold by the action of some Lie group. Indeed, suppose a Lie group G acts on a complex manifold X. Then, X/G is not necessarily a manifold. For example, if $X = \mathbb{D}^*$ is the punctured unit disc, acted on by a rotation group by some irrational angle τ such that $\frac{\tau}{\pi} \notin \mathbb{Q}$, then the quotient is not Hausdorff. We want to construct an object, which we are going to call the *quotient stack* $[X/G]$, which carries all the information of the naive quotient X/G, but can be manipulated in a similar way as complex manifolds.

According to the *Quotient Manifold Theorem*, if G is a Lie group acting properly and freely on X, then the quotient X/G is a manifold and $[X/G] \cong X/G$. In general, the quotient stack $[X/G]$ carries all the information of the naive quotient X/G (which is its *coarse moduli space*), together with an additional structure: points are allowed to have automorphisms. In the example of elliptic curves, $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ is a Riemann Surface, but it is not isomorphic to the quotient stack $[\mathfrak{H}/SL_2(\mathbb{Z})]$, for the latter has non-trivial inertia groups.

All the complex manifolds can be regarded as stacks, thanks to a fully faithful embedding **Comp** $\to$ **Stacks** which we construct in Section 1.1. The converse is not true (for example, as mentioned above, points of a stack carry the additional structure of the inertia groups). In order to manipulate stacks *as* complex manifolds, we need the notion of an atlas. An atlas for a stack $\mathcal{M}$ is a morphism of stacks from (a stack equivalent to) a complex manifold to $\mathcal{M}$, satisfying some *surjectivity* properties. The data of a stack $\mathcal{M}$ together with such an atlas is called an *analytic stack*. In Section 1.3 we make this definition more precise. Our definition will allow us to generalize some properties of morphisms of complex manifolds for the case of morphisms of stacks, in such a way that these definitions agree with the classical ones whenever the stacks are equivalent to a manifolds, and that they do not depend on the choice of an atlas.

Analytic stacks might be very hard to study, but quotient stacks are much easier to deal with, because their geometry is, in some sense, equivalent to the G-equivariant geometry of the manifold. In fact, points are G-orbits, inertia groups are stabilizers, and so forth. That is why one would like to always think of stacks as of quotient stacks.

This thesis provides two structure results which allow us to study stacks as quotient stacks: the first one is a classical theorem due to Deligne and Mumford claiming that any stack satisfying the Deligne-Mumford hypotheses is locally a quotient stack by the action of a finite Lie group on a simply connected complex manifold. The proof allows us to determine precisely how the action is made, in terms of the source and target maps of the groupoid associated with the stack. The second result is due to Behrend and Noohi, and is a generalization of the Uniformization Theorem for Riemann Surfaces for the case of Deligne-Mumford analytic stacky curves. This result allows us to classify the uniformizable Deligne-Mumford curves as *global* quotient stacks of a discrete group (which is precisely the fundamental group of the curve). Since we know the discrete groups acting on the simply connected Riemann Surfaces, this is enough to classify all the uniformizable Deligne-Mumford curves.

The first chapter of the thesis contains some basic theory about stacks. Section 1.4 deals with the Deligne-Mumford local quotient characterization. The second chapter develops some homotopy theory for analytic stacks, and links it to the theory of covering spaces for them. We also mention a generalization of Van Kampen theorem for analytic stacks. The third chapter is about the uniformization result proven by Behrend and Noohi in [1].

Chapter 4 deals with a slightly different case: we consider stacks with proper diagonal, and try to prove that, locally around any point, they can be regarded as quotient stacks by come action of the inertia group at the point (which we prove to be a compact Lie group). The result fails if we consider stacks of groupoids over the category **Comp** of complex manifolds (see Example 4.1). We thus change our setting by looking at stacks over the category **Diff** of differentiable real manifolds. All the theory developed in Chapter 1 still holds true, and the counterexample above fails. I was not able to complete the proof, which is left as a conjecture.

Chapter 1

Basics on analytic stacks

1.1 Stacks as pseudo-functors

In this chapter, we will mostly follow [2] and [7]. Let **Gpd** be the category of groupoids and **Comp** be the category of complex manifolds.

Definition 1.1. A *prestack* (of groupoids) over **Comp** is a pseudo-functor

$$\mathcal{M} : \mathbf{Comp} \to \mathbf{Gpd}$$

i.e. a contravariant functor $\mathbf{Comp}^{op} \to \mathbf{Gpd}$ such that:
- $id_X^* \cong id_{\mathcal{M}(X)}$ for any complex manifold X (where $f^* := \mathcal{M}(f)$ for any morphism f in **Comp**)
- for any pair of composable morphisms $(f : Y \to X, g : Z \to Y)$, there is a natural transformation $\phi_{f,g} : f^* \circ g^* \cong (g \circ f)^*$ which is associative on any triple of composable morphisms.

Definition 1.2. A *stack* (of groupoids) over **Comp** is a prestack $\mathcal{M}$ satisfying the following gluing conditions*:

1. on objects: Given an open covering $(U_i)_i$ of a manifold X, objects $P_i \in \mathcal{M}(U_i)$ and isomorphisms $\varphi_{ij} : P_i|_{U_i \cap U_j} \to P_j|_{U_i \cap U_j}$ which satisfy the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on the threefold intersection $U_i \cap U_j \cap U_k$, there is an object $P \in \mathcal{M}(X)$ together with isomorphisms $\varphi_i : P|_{U_i} \to P_i$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$.

2. on morphisms: Given a complex manifold X, objects $P, P' \in \mathcal{M}(X)$, an open covering $(U_i)_i$ of X and isomorphisms $\varphi_i : P|_{U_i} \to P'|_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there exists a unique $\varphi : P \to P'$ such that $\varphi_i = \varphi|_{U_i}$.

* Notation: as for sheaves, for simplicity, one writes $|_U$ instead of $j^*(\bullet)$, whenever $j : U \hookrightarrow X$ is an open embedding. For double and triple intersections, in order for the notation to make sense, one needs to consider the double and triple inclusions.

Example 1.1. The functor

$$\underline{BG} : \mathbf{Comp} \to \mathbf{Gpd}$$

assigning to any complex manifold X the groupoid $\underline{BG}(X)$ of G-bundles on it is a stack over **Comp**. More precisely, at the level of objects

$$\underline{BG}(X) = \langle P \to X \ G\text{-bundle} \rangle$$

where a morphism from $P \to X$ to $P' \to X$ is a G-equivariant isomorphism $P \to P'$. At the level of morphisms, given a map $f : Y \to X$ in **Comp**, we have

$$BG(f) = f^* : \underline{BG}(X) \to \underline{BG}(Y)$$

sending a G-bundle $P \to X$ to its pullback $P \times_X Y \to Y$ given by the following cartesian square

$$\begin{tikzcd} P \arrow[r] & X \\ P \times_X Y \arrow[u] \arrow[r] & Y \arrow[u, "f"'] \end{tikzcd}$$

Since objects and morphisms glue, this is indeed a stack.

Remark 1.1. Stacks over **Comp** form a 2-category, which we denote by **Stacks**. Morphisms of stacks $F : \mathcal{M} \to \mathcal{N}$ are given by collections of functors

$F_X : \mathcal{M}(X) \to \mathcal{N}(X)$, and, for any morphism $f : Y \to X$ in **Comp**, there is a natural transformation $F_f : F_Y \circ \mathcal{M}f \cong \mathcal{N}f \circ F_X$

$$\begin{tikzcd}
\mathcal{M}(X) \arrow[r, "\mathcal{M}f"] \arrow[d, "F_X"'] & \mathcal{M}(Y) \arrow[d, "F_Y"] \arrow[dl, phantom, "\Leftarrow F_f"] \\
\mathcal{N}(X) \arrow[r, "\mathcal{N}f"'] & \mathcal{N}(Y)
\end{tikzcd}$$

Remark 1.2. Given a complex manifold X, one can construct a stack $\underline{X}$ associating to any complex manifold Y the groupoid given by the set $\underline{X}(Y) := Hom_{\mathbf{Comp}}(Y, X)$, endowed with the trivial groupoid structure, i.e. fixing the identities to be the only possible morphisms. $\underline{X}$ is indeed a stack, since morphisms of complex manifolds can be glued. This construction gives a fully faithful embedding **Comp** $\hookrightarrow$ **Stacks**. One often drops the underline in the notation, and says that a stack *is* a complex manifold when it *corresponds to* a complex manifold by this embedding. It is common to use normal letters (e.g. X, Y...) to denote stacks corresponding to complex manifolds, and curly letters (e.g. $\mathcal{M}$, $\mathcal{N}$...) otherwise.

Example 1.2. Let G be a Lie group acting on a complex manifold X. One defines the *quotient stack*

$$[X/G](T) := \left\langle (P, p : P \to T, f : P \to X) \;\middle|\; \begin{array}{c} P \text{ is acted on by } G \\ p \text{ is a } G\text{-bundle} \\ f \text{ is } G\text{-equivariant} \end{array} \right\rangle$$

for any complex manifold T. The morphisms in this groupoid are defined to be the G-equivariant isomorphisms commuting with the maps to X.

Later on this chapter we will see that quotient stacks form a very nice class of stacks. Indeed, the geometry of a quotient stack $[X/G]$ is strictly related to the G-equivariant geometry of X.

Remark 1.3. For $X = pt$ and G acting trivially on it, the quotient stack

$[X/G]$ is the stack $\mathbf{BG}$ classifying G-bundles. Indeed:

$$[pt/G](T) = \left\langle (P, p : P \to T, f : P \to pt) \;\middle|\; \begin{array}{c} P \text{ is acted on by } G \\ p \text{ is a } G\text{-bundle} \\ f \text{ is } G\text{-equivariant} \end{array} \right\rangle \cong$$

$$\cong \left\langle (P, p : P \to T) \;\middle|\; \begin{array}{c} P \text{ is acted on by } G \\ p \text{ is a } G\text{-bundle} \end{array} \right\rangle = \mathbf{BG}(T)$$

Remark 1.4. If a Lie group G acts properly and freely on the complex manifold X, then X/G is also a complex manifold, and the natural projection $X \to X/G$ is a G-bundle. In this case, $[X/G] \cong X/G$, i.e. for any complex manifold T,

$$[X/G](T) \cong Hom(T, X/G).$$

Indeed, given

$$(p : P \to T, f : P \to X) \in [X/G](T)$$

f induces a map $\overline{f} : P/G = T \to X/G$. Note that the following diagram is cartesian

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ P/G \arrow[r] & X/G \end{tikzcd}$$

thus $P \cong P/G \times_{X/G} X$. So the inverse map sends $\overline{f} : T \to X/G$ to the two projections

$$(T \times_{X/G} X \to T, T \times_{X/G} X \to X).$$

Definition 1.3. We say that m is a *point* of a stack $\mathcal{M}$, and write $m \in \mathcal{M}$, if there exists a singleton $*$ (thought of as a complex manifold) such that $m \in \mathcal{M}(*)$.

As a 2-category, the category of stacks satisfies the 2-Yoneda lemma. In fact:

Lemma 1.1.1. *(Yoneda Lemma for Stacks)*
For any $\mathcal{M} \in$ **Stacks** *and* $X \in$ **Comp**, *there is a canonical equivalence of categories*

$$\mathcal{M}(X) \cong Hom_{\mathbf{Stacks}}(\underline{X}, \mathcal{M})$$

Proof. (Idea)
Given $P \in \mathcal{M}(X)$, we define a morphism of stacks $F_P : X \to \mathcal{M}$ sending any object $f \in \underline{X}(Y)$ (which is simply a morphism $Y \to X$) to $f^*(P) \in \mathcal{M}(Y)$. For any morphism $\phi : P \to P'$ in $\mathcal{M}(X)$, we define a natural transformation $F_P \to F_{P'}$ by $f^*\varphi : f^*P \to f^*P'$.
Conversely, given a morphism $F : X \to \mathcal{M}$, we just send it to the object $F(id_X) \in \mathcal{M}(X)$.
To prove the Lemma one needs to make sure that the compositions of these two maps are equivalent to the two identity functors. □

Example 1.3. By Yoneda, a point $m \in \mathcal{M}$ canonically corresponds to a $*$-point of $\mathcal{M}$, i.e. a morphism of stacks $* \to \mathcal{M}$.

Example 1.4. The points of a quotient stack $[X/G]$ are in correspondence with the points of X/G (the naive quotient, thought of as a topological space). Indeed, by definition

$$[X/G](*) = \langle (p : P \to * \ G\text{-bundle}, f : P \to X \ G\text{-equivariant}) \rangle$$

but a G-bundle on a singleton has to be trivial, hence $P \cong G \times * \cong G$, and we get

$$[X/G](*) = \langle f : G \to X \ G\text{-equivariant} \rangle$$

Given a G-equivariant map $f : G \to X$, we get a point $x \in X$ given by $x = f(1)$. Viceversa, given a point $x \in X$, we have a G-equivariant map $f : G \to X$ given by $f(g) = g.x$. If, instead of $x \in X$, we take $y = g'.x$ in the G-orbit Gx of x, we get a G-equivariant map f' such that

$$f'(g) = g.y = (gg').x$$

We claim that f and f' are isomorphic in the groupoid $[X/G](*)$. Indeed, the morphism $\varphi : G \to G$ such that $\varphi(g) = gg'$ is a G-equivariant isomorphism commuting with the maps to X, meaning that the following triangle commutes

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "f'"'] & & G' \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

1.2 Fibered products of Stacks

In the next sections, we will try to extend properties of complex manifolds to stacks. The usual way to do it is by a base change. First, one needs the notion of a pullback:

Definition 1.4. Given a diagram of morphisms of stacks

$$\begin{tikzcd} & \mathcal{M} \arrow[d, "F"] \\ \mathcal{M}' \arrow[r, "G"'] & \mathcal{N} \end{tikzcd}$$

the fibered product $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ is the stack given by, for every complex manifold X, the groupoid

$$(\mathcal{M} \times_{\mathcal{N}} \mathcal{M}')(X) = \langle (f, g, \varphi) | f : X \to \mathcal{M}, g : X \to \mathcal{M}', \varphi : F \circ f \Rightarrow G \circ g \rangle$$

where morphisms $(f, g, \varphi) \to (f', g', \varphi')$ are pairs of morphisms

$$(\psi_{f,f'} : f \to f', \psi_{g,g'} : g \to g')$$

such that

$$\varphi' \circ F(\psi_{f,f'}) = G(\psi_{g,g'}) \circ \varphi.$$

Remark 1.5. Since $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{N}$ are stacks, objects and morphisms glue, so the fibered product is also a stack. In fact, it is a pullback in the 2-category of stacks. The usual properties of pullbacks are satisfied, for example

- $\mathcal{M} \times_{\mathcal{M}} \mathcal{M}' \cong \mathcal{M}'$
- (commutativity) $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}' \cong \mathcal{M}' \times_{\mathcal{N}} \mathcal{M}$
- (associativity) $(\mathcal{L} \times_{\mathcal{X}} \mathcal{M}) \times_{\mathcal{Y}} \mathcal{N} \cong \mathcal{L} \times_{\mathcal{X}} (\mathcal{M} \times_{\mathcal{Y}} \mathcal{N})$

Example 1.5. Let $(t_1, t_2) : T \to \mathcal{M} \times \mathcal{M}$, that we can view, thanks to Yoneda, as $t_1, t_2 \in \mathcal{M}(T)$. Let $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the diagonal morphism. Then,

$$(T \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})(S) \cong \langle (f, s, \varphi) | f : S \to T, s \in \mathcal{M}(S), \varphi(s, s) \Rightarrow (f^*t_1, f^*t_2) \rangle \cong$$

$$\cong \langle (f, s, \varphi_1, \varphi_2) | f : S \to T, s \in \mathcal{M}(S), \varphi_1 : s \Rightarrow f^*t_1, \varphi_2 : s \Rightarrow f^*t_2 \rangle$$

By calling $\psi := \varphi_2 \circ \varphi_1^{-1}$, one gets

$$(T \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})(S) \cong \langle (f, \psi) | f : S \to T, \psi : f^*t_1 \Rightarrow f^*t_2 \rangle$$

In the particular case of $t_1 = t_2 =: t$, one gets

$$(T \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})(S) \cong \langle (f, \psi) | f : S \to T, \psi \in Aut(f^*t) \rangle$$

In the case $T = \mathcal{M}$ and $(t_1, t_2) = \Delta$, one gets the fibers over Δ:

$$(\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})(S) \cong \langle (x \in \mathcal{M}(S), \psi \in Aut(x)) \rangle$$

which we call the *inertia stack* of $\mathcal{M}$.

Example 1.6. Let $(\pi, \pi) : X \times X \to \mathcal{M} \times \mathcal{M}$ be a morphism of stacks, and let $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the diagonal morphism. Then

$$(X \times_{\mathcal{M}} X)(S) \cong \langle (f, g, \varphi) | f, g : S \to X, \varphi : \pi \circ f \Rightarrow \pi \circ g \rangle \cong$$

$$\cong \langle ((f, g) : S \to X \times X, s : S \to \mathcal{M}, \psi : (x, x) \cong (\pi \circ f, \pi \circ g)) \rangle \cong$$

$$\cong ((X \times X) \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})\,(S)$$

1.3 Analytic Stacks

Up to now, stacks over **Comp** are still too crude to do geometry with. One would like to manipulate them as if they were complex manifolds. In order to make our categorical definition more geometrical, we need to give sense to the notion of an *atlas*.

An atlas for a stack $\mathcal{M}$ will be a morphism of stacks of the form $Y \to \mathcal{M}$, where Y is a manifold, satifsfying some *surjectivity* properties. In order to make this more precise, one requires the definition

Definition 1.5. A morphism of stacks $\mathcal{M} \to \mathcal{N}$ is said to be *representable* if, for any morphism of stacks $Y \to \mathcal{N}$, the fibered product $\mathcal{M} \times_{\mathcal{N}} Y$ is a stack which is equivalent to some complex manifold.

The property of being representable is stable under composition and pullbacks:

Lemma 1.3.1. 1. *If $F : \mathcal{L} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{N}$ are representable, then $G \circ F$ is representable*

2. *If $F : \mathcal{M} \to \mathcal{N}$ is representable and $G : \mathcal{L} \to \mathcal{N}$ is arbitrary, then the projection $\mathcal{L} \times_{\mathcal{N}} \mathcal{M} \to \mathcal{L}$ is representable*

Proof. 1. Given a morphism $Y \to \mathcal{N}$, we want to check that the fibered product $Y \times_{\mathcal{N}} \mathcal{L}$ is a complex manifold. But we know that $Y \times_{\mathcal{N}} \mathcal{M}$ is a complex manifold, so also $(Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{L}$ is (because G and F are representable).

$$\begin{tikzcd}
(Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{L} \arrow[r, dotted] \arrow[d, dotted] & \mathcal{L} \arrow[d, "F"] \\
Y \times_{\mathcal{N}} \mathcal{M} \arrow[r, dotted] \arrow[d, dotted] & \mathcal{M} \arrow[d, "G"] \\
Y \arrow[r] & \mathcal{N}
\end{tikzcd}$$

The natural isomorphism $Y \times_{\mathcal{N}} \mathcal{L} \cong (Y \times_{\mathcal{N}} \mathcal{M}) \times_{\mathcal{M}} \mathcal{L}$ completes the proof.

2. Given a morphism $Y \to \mathcal{L}$, we want to check that the fibered product $Y \times_{\mathcal{L}} (\mathcal{L} \times_{\mathcal{N}} \mathcal{M})$ is a complex manifold.

$$
\begin{tikzcd}
Y \times_{\mathcal{N}} \mathcal{M} \arrow[dr, dotted, "\cong"] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& Y \times_{\mathcal{L}} (\mathcal{L} \times_{\mathcal{N}} \mathcal{M}) \arrow[r, dotted] \arrow[d, dotted] & \mathcal{L} \times_{\mathcal{N}} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "F"] \\
& Y \arrow[r] & \mathcal{L} \arrow[r, "G"'] & \mathcal{N}
\end{tikzcd}
$$

But $Y \times_{\mathcal{N}} \mathcal{M}$ is a complex manifold, for F is representable, hence, again, the natural isomorphism $Y \times_{\mathcal{L}} (\mathcal{L} \times_{\mathcal{N}} \mathcal{M}) \cong Y \times_{\mathcal{N}} \mathcal{M}$ completes the proof.

$\square$

One can now give a surjectivity notion on a representable morphism, in the following way:

Definition 1.6. A morphism of stacks $f : \mathcal{M} \to \mathcal{N}$ is a *submersion* if it is representable and if for any morphism $Y \to \mathcal{N}$ from a complex manifold Y the base extension $Y \times_{\mathcal{N}} \mathcal{M} \to Y$ is a submersion (in the sense of complex manifolds: it is a map whose differential is surjective at any point).

Analogously, f is *surjective* if it is so for complex manifolds under any such base change.

Remark 1.6. If the two stacks are manifolds, this definition agrees with the classical one, for being a submersion is a property of complex manifolds which is invariant under base change. In particular, the condition of being a submersion of complex manifolds allows to give to the fiber product a complex manifold structure.

Definition 1.7. A morphism $Y \to \mathcal{M}$ from a complex manifold Y is said to be an *atlas* for the stack $\mathcal{M}$ if it is a representable surjective submersion.

Definition 1.8. We say that a stack (over **Comp**) $\mathcal{M}$ is an *analytic stack* if there exists an atlas $p : Y \to \mathcal{M}$ for it.

Example 1.7. If X is a complex manifold acted on by a Lie group G, the quotient stack $[X/G]$ is an analytic stack. An atlas for it is the morphism $\underline{X} \to [X/G]$ corresponding, by Yoneda, to

$$(G \times X, G \times X \to X \text{ trivial G-bundle}, G \times X \to X \text{ action}) \in [X/G](X)$$

One directly checks that the diagram

$$\begin{tikzcd} G \times X \arrow[r, "proj"] \arrow[d, "act"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & {[X/G]} \end{tikzcd}$$

is cartesian. Moreover, as we are going to see in Lemma 1.3.2, checking the representability on one atlas is enough. Hence, π is a representable submersion. We will often call π the *canonical atlas* for the quotient stack $[X/G]$.

Remark 1.7. The reason why quotient stacks are so nice to study is that their geometry is strictly related to the G-equivariant geometry of the complex manifold. Indeed, giving a point $m \in [X/G]$ is the same as giving an orbit $Gx \subset X$, where $\pi(x) = m$. Moreover, the automorphism group of m is

$$Aut(m) = I_x = s^{-1}(\{x\}) \cap t^{-1}(\{x\})$$

where $s = proj : G \times X \to X$ and $t = act : G \times X \to X$ (as will become more clear in the section regarding analytic groupoids). Hence,

$$Aut(m) = proj^{-1}(\{x\}) \cap act^{-1}(\{x\}) =$$

$$= (G \times x) \cap \{(g, y) \in G \times X | g.y = x\} \cong$$

$$\cong \{g \in G | g.x = x\} = Stab_G(x)$$

For a more complete characterization of how the two geometries are related, the reader may look at the *dictionary* provided in section 3.1 of [8].

Remark 1.8. Analytic stacks form a full sub-2-category of the 2-category of stacks.

Definition 1.9. A morphism of stacks $\mathcal{M} \to \mathcal{N}$ is said to be *weakly representable* (or *spacelike representable*) if, for any atlas $Y \to \mathcal{N}$, the fiber product $\mathcal{M} \times_{\mathcal{N}} Y$ is equivalent to some complex manifold.

Our definition of atlas is good, because it allows to check global properties locally, i.e. on one atlas.

Lemma 1.3.2. *A morphism of analytic stacks $F : \mathcal{M} \to \mathcal{N}$ is weakly representable if and only if there exists one atlas $Y \to \mathcal{N}$ such that the projection $\mathcal{M} \times_{\mathcal{N}} Y \to \mathcal{M}$ is again an atlas.*

Proof. (Idea) One implication follows trivially by the natural isomorphism $T \times_{\mathcal{M}} (\mathcal{M} \times_{\mathcal{N}} Y) \cong T \times_{\mathcal{N}} Y$ for any $T \to \mathcal{M}$.
For the other, assume we are given another atlas $Z \to \mathcal{N}$ for $\mathcal{N}$, we need to check that $Z \times_{\mathcal{N}} \mathcal{M}$ is a manifold. Let us distinguish between two cases:
Special case: Suppose that the atlas $Z \to \mathcal{N}$ factors through $Y \to \mathcal{N}$. In this case, the thesis follows by the natural isomorphism $Z \times_{\mathcal{N}} \mathcal{M} \cong Z \times_Y (Y \times_{\mathcal{N}} \mathcal{M})$.

$$\begin{tikzcd}
Z \times_{\mathcal{N}} \mathcal{M} \arrow[dr, "\cong"] \arrow[ddr, bend right] \arrow[drrr, bend left=10] & & & \\
& Z \times_Y (Y \times_{\mathcal{N}} \mathcal{M}) \arrow[r] \arrow[d] & \mathcal{M} \times_N Y \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
& Z \arrow[r] & Y \arrow[r] & \mathcal{N}
\end{tikzcd}$$

General case: The idea here is to use the fact that $Y \to \mathcal{N}$ is an atlas to find (locally) a section of $Z \times_{\mathcal{N}} Y \to \mathcal{Z}$. This will allow (locally) to get back to the special case above. Then, one just glues. $\square$

Remark 1.9. The previous Lemma is false if one requires the morphism $F : \mathcal{M} \to \mathcal{N}$ to be just representable. For example, consider the morphism $F : \mathbb{C}^2 \to \mathbb{C}$ given by $(x, y) \mapsto xy$.

$$\begin{tikzcd}
\{0\} \times_{\mathbb{C}} \mathbb{C}^2 \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[d, "F"] \\
\{0\} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

The fiber product

$$\{0\} \times_{\mathbb{C}} \mathbb{C}^2 = F^{-1}(\{0\}) = \{(x,y) \in \mathbb{C}^2 | x = 0 \vee y = 0\}$$

is not a manifold, so F is not representable. But the Lemma tells us that F is weakly representable, because the identity $\mathbb{C} \to \mathbb{C}$ pullbacks to the identity $\mathbb{C}^2 \to \mathbb{C}^2$, and they are both atlases.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r] \arrow[d, "id"'] & \mathbb{C} \arrow[d, "id"] \\ \mathbb{C}^2 \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

Example 1.8. For any analytic stack $\mathcal{M}$, the diagonal morphism

$$\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$$

is weakly representable, but it may not be representable. Indeed:
Given an atlas $\pi : X \to \mathcal{M}$, we have the cartesian diagram:

$$\begin{tikzcd} \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \arrow[r] \arrow[d, "p_1"'] & X \times X \arrow[d, "(\pi,\pi)"] \\ \mathcal{M} \arrow[r, "\Delta"'] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

where (π, π) is an atlas for $\mathcal{M} \times \mathcal{M}$. By the previous Lemma, it is enough to check that

$$p_1 : \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \to \mathcal{M}$$

is an atlas. Clearly

$$\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \cong X \times_{\mathcal{M}} X$$

is a complex manifold (because π is an atlas), and given any $Y \to \mathcal{M}$ one has the diagram

$$\begin{tikzcd} Y \times_{\mathcal{M}} \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \arrow[r] \arrow[d] & \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \arrow[r] \arrow[d] & X \times X \arrow[d, "(\pi,\pi)"] \\ Y \arrow[r] & \mathcal{M} \arrow[r, "\Delta"'] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

and one concludes thanks to the isomorphism

$$Y \times_{\mathcal{M}} \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} (X \times X) \cong Y \times_{\mathcal{M}\times\mathcal{M}} (X \times X)$$

and the fact that (π, π) is an atlas.
To check that Δ may not be representable, think at the diagram

$$\begin{tikzcd}
\mathbb{C}^3 \times_{\mathbb{C}^2} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}^3 \arrow[d, "F"] \\
\mathbb{C} \arrow[r, "\Delta"'] & \mathbb{C}^2
\end{tikzcd}$$

where F is given by $(x, y, z) \mapsto (x, yz)$. The fiber product is not a complex manifold, hence Δ is not representable.

Some properties of morphisms between complex manifolds are invariant under base change by a submersion, meaning that, if a morphism of complex manifolds $Y \to X$ satisfies the property $\mathbf{P}$ (for example, $\mathbf{P}$ =local homeomorphism, open embedding, closed embedding, submersion, covering map, finite fibers, proper...), then for any submersion $Z \to X$ the base extension $Y \times_X Z \to Z$ satisfies $\mathbf{P}$.

In the stack case, we want to define properties as generalizations of the ones of complex manifolds. For properties which are invariant under base change by a submersion, the usual way to do it is by the following definition:

Definition 1.10. Let $\mathbf{P}$ be a property of morphisms of complex manifolds which is invariant under base change by a submersion. We say that a morphism of stacks $\mathcal{M} \to \mathcal{N}$ satisfies $\mathbf{P}$ if it is weakly representable and for any atlas $Y \to \mathcal{N}$ the base extension $Y \times_{\mathcal{N}} \mathcal{M} \to Y$ satisfies $\mathbf{P}$ (as a morphism of complex manifolds).

Definition 1.11. We say that a stack $\mathcal{M}$ is an *open substack* (resp. *closed substack*) of a stack $\mathcal{N}$ if there exists an open (resp. closed) embedding $\mathcal{M} \hookrightarrow \mathcal{N}$ (in the sense of the previous definition).

Remark 1.10. With an argument analogous to the one used in the proof of Lemma 1.3.2, one proves that checking the above properties after a base change along one single atlas is enough.

Definition 1.12. We define an *open substack* (resp. *closed substack*) of a stack $\mathcal{N}$ to be an open (resp. closed) embedding $\mathcal{M} \hookrightarrow \mathcal{N}$ (in the sense of the previous definition). By abuse of notation, we will sometimes just say that $\mathcal{M}$ is an open substack of $\mathcal{N}$.

Example 1.9. The quotient stack $[X/G]$ is an open substack of $[Y/G]$ if and only if X is an open submanifold of Y.

Remark 1.11. It follows from the previous considerations that atlases are stable under pullbacks, meaning that, given a morphism of analytic stacks $\mathcal{M} \to \mathcal{N}$ and an atlas $X \to \mathcal{N}$, the projection $X \times_{\mathcal{N}} \mathcal{M} \to \mathcal{M}$ is again an atlas.

Definition 1.13. Given a morphism of analytic stacks $\mathcal{M} \to \mathcal{N}$ and an atlas $\pi : X \to \mathcal{N}$, the projection $X \times_{\mathcal{N}} \mathcal{M} \to \mathcal{M}$ is called a *base change atlas* for $\mathcal{M}$ along π.

Definition 1.14. Given a point $m : * \to \mathcal{M}$ and an open immersion $i : \mathcal{U} \to \mathcal{M}$, we say that $m \in \mathcal{U}$ if m factors through the open immersion

$$\begin{tikzcd} & \mathcal{U} \arrow[d, "i"] \\ * \arrow[ur, dotted] \arrow[r, "m"'] & \mathcal{M} \end{tikzcd}$$

Remark 1.12. The previous definition is equivalent to saying that, given an atlas $\pi : X \to \mathcal{M}$, a base change atlas $\pi' : X \times_{\mathcal{M}} \mathcal{U} \to \mathcal{U}$ and a lift $x \in X$ of m, x also lies on $X \times_{\mathcal{M}} \mathcal{U}$ (viewed as a submanifold of X).

Definition 1.15. Given an analytic stack $\mathcal{M}$ and a point $m \in \mathcal{M}$, we say that an open substack $\mathcal{U}$ of $\mathcal{M}$ is an *open neighbourhood* of m in $\mathcal{M}$ if m also lies in $\mathcal{U}$ (as in the previous Remark and Definition).

Definition 1.16. We say that two open substacks $\mathcal{U}, \mathcal{V}$ of a stack $\mathcal{M}$ *intersect* (or *have non-empty intersection*) if there exists a point of $\mathcal{V}$ which also lies in $\mathcal{U}$ (of course, the definition is symmetric in $\mathcal{U}$ and $\mathcal{V}$).

Definition 1.17. We say that an open analytic substack $\mathcal{U}$ of an analytic stack $\mathcal{M}$ is *dense* in $\mathcal{M}$ if it is dense after a base change by an atlas.

Remark 1.13. Let $\mathcal{U}$ be an open dense substack of $\mathcal{M}$, $\pi : X \to \mathcal{M}$ an atlas for $\mathcal{M}$, $m \in \mathcal{M}$ and $x \in X$ such that $\pi(x) = m$. Then, given an open neighbourhood $\mathcal{V}$ of m in $\mathcal{M}$, it follows from the above remarks that $\mathcal{V}$ intersects $\mathcal{U}$.

1.4 Deligne-Mumford analytic stacks

We previously pointed out that quotient stacks are particularly easy to study. That is because the geometry of a quotient stack $[X/G]$ is the same as the G-equivariant geometry of X. The points of $[X/G]$ are just G-orbits, the inertia groups are just the stabilizers, the properties of the diagonal reflects on the morphisms $G \times X \to X \times X$ sending (g, x) to $(x, g.x)$. For this reason, one would like to think of stacks as of quotient stacks. There is a very well known class of stacks, namely the Deligne-Mumford stacks, which happen to be, locally around every point, isomorphic to quotient stacks. In this section, we will develop some theory about these stacks, and give them a local characterization.

Definition 1.18. An analytic stack $\mathcal{M}$ is said to be a *Deligne-Mumford analytic stack* (or *DM-analytic stack*) if the following two conditions are satisfied:
- there exists an atlas $\pi : X \to \mathcal{M}$ which is a local homeomorphism
- the diagonal morphism $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is closed with finite fibers

Example 1.10. Let G be a discrete Lie group acting on some manifold X, and let $\mathcal{M} := [X/G]$. Considering the canonical atlas $\pi : X \to \mathcal{M}$, we have a natural isomorphism $G \times X \cong X \times_{\mathcal{M}} X$, which leads to the following cartesian diagram

$$\begin{tikzcd}
G \times X \arrow[r, "(proj{,}act)"] \arrow[d] & X \times X \arrow[d, "(\pi{,}\pi)"] \\
\mathcal{M} \arrow[r, "\Delta_{\mathcal{M}}"'] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

Hence, $\Delta_{\mathcal{M}}$ is closed and with finite fibers if and only if $G \times X \to X \times X$ is. It the latter is closed, then the orbits of G are closed, and it has finite fibers if and only if G acts with finite stabilizers.
Coming to the atlas, it is clear from the definition and from the following cartesian square

$$\begin{tikzcd}
G \times X \arrow[r, "act"] \arrow[d, "proj"'] & X \arrow[d, "\pi"] \\
X \arrow[r, "\pi"'] & \mathcal{M}
\end{tikzcd}$$

that π is a local homeomorphism, for the projection and the action are (since G is discrete).

Example 1.11. As a particular case of the previous example, one may take the action of the group $G = SL_2(\mathbb{Z})$ acting on the Poincaré upper half-plane $X = \mathfrak{H}$. This action is closed and with finite stabilizers, hence the moduli stack of elliptic curves $\mathcal{M}_{1,1} := [\mathfrak{H}/SL_2(\mathbb{Z})]$ is a Deligne-Mumford stack.

Example 1.12. Let $\mathbb{D}$ be the complex unit disk, and consider a group action given by a rotation around the center. The center has infinite inertia group, so the fibers are not finite and $[\mathbb{D}/G]$ cannot be Deligne-Mumford.

Example 1.13. Let $\mathbb{D}^*$ be the punctured complex unit disk, and consider a group action given by the rotation around the center by an irrational number $\tau \notin \pi\mathbb{Q}$. Then, the map $\mathbb{Z}\tau \times \mathbb{D}^* \to \mathbb{D}^* \times \mathbb{D}^*$ given by $(n\tau, x) \mapsto (x, xe^{in\tau})$ is not closed, hence $[\mathbb{D}^*/\mathbb{Z}\tau]$ is not a DM-analytic stack.

Remark 1.14. Deligne-Mumford analytic stacks form a full sub-2-category of the 2-category of analytic stacks.

The following definitions will come useful for stating the main result:

Definition 1.19. An analytic stack $\mathcal{M}$ is said to be *connected* if it has no proper open-closed substacks.

Definition 1.20. Let $\mathcal{M}$ be an analytic stack acted on by a Lie group G which fixes a point $m \in \mathcal{M}$. The action is said to be *mild* (or that G acts *mildly*) at m if for any open neighbourhood $\mathcal{U}$ of m in $\mathcal{M}$ there exists an open sub-neighbourhood $\mathcal{U}'$ of m in $\mathcal{U}$ which is G-invariant.

We will now prove the main result of this section. The slogan is that

Any Deligne-Mumford analytic stack is locally the quotient stack of an action of some finite Lie group on a simply connected complex manifold.

The proof of the result is going to tell us precisely how this quotient is made. The group will just be the inertia group, and the action will be determined (locally) by the source and target maps of the groupoid associated with our stack.

Theorem 1.4.1. *Let $\mathcal{M}$ be a Deligne-Mumford analytic stack and $\pi : X \to \mathcal{M}$ be an atlas for it, given by a local homeomorphism. For any point $m \in \mathcal{M}$ and any $\mathcal{U}'$ open neighbourhood of m in $\mathcal{M}$, then there exists an open neighbourhood $\mathcal{U}$ of m in $\mathcal{U}'$ such that $\mathcal{U} \cong [V/H]$, with V a simply connected complex manifold and $H := I_x$ the inertia group at x, where x is a point lying above m (i.e. such that $\pi(x) = m$). Moreover, this action on V is mild around x.*

Proof. Consider the cartesian square

$$\begin{tikzcd} X \times_{\mathcal{M}} X \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

Define $R := X \times_{\mathcal{M}} X$. Since π is a local homeomorphism, s and t also are. Now, consider the cartesian square (recall that $R \cong (X \times X) \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M})$):

$$\begin{tikzcd} R \arrow[r, "(s,t)"] \arrow[d] & X \times X \arrow[d, "(\pi,\pi)"] \\ \mathcal{M} \arrow[r, "\Delta"'] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

Since the diagonal Δ is closed with finite fibers, $\overline{\Delta} := (s,t)$ also is. Hence,

$$H = s^{-1}(x) \cap t^{-1}(x) = \overline{\Delta}^{-1}(x,x) \subset R$$

is a finite group.

Thus, one can take, for any $h \in H$, some disjoint open neighbourhoods W'_h of h in R which are mapped homeomorphically by s into the neighbourhoods $s(W'_h)$ of x in X. By taking

$$W_h := W'_h \cap s^{-1}(V') \qquad \forall h \in H$$

where $x \in V' := \cap_{h\in H} s(W'_h)$, one gets an open covering

$$s : \coprod_{h\in H} W_h =: W \to V'.$$

Since $W \subseteq s^{-1}(V') = V' \times_{\mathcal{M}} X$ is an open subset, then

$$Z := (V' \times_{\mathcal{M}} X) \setminus W$$

is a closed subset of $V' \times_{\mathcal{M}} X$. Since Δ is closed onto its image, by definition, $\overline{\Delta}$ also is, so that $\overline{\Delta}(Z)$ is closed in $\overline{\Delta}(V' \times_{\mathcal{M}} X) = V' \times X$, and it does not contain (x,x), by construction. Hence, there exists an open neighbourhood U of x in X (choose it to be simply connected) such that $U \times U$ is an open neighbourhood of (x,x) in $(V' \times X) \setminus \overline{\Delta}(Z)$. Consider now $\overline{\Delta}^{-1}(U \times U) = U \times_{\mathcal{M}} U$, and call $U_h := (U \times_{\mathcal{M}} U) \cap W_h$. By construction, $U \times_{\mathcal{M}} U = \coprod_{h\in H} U_h$, and $[U \times_{\mathcal{M}} U \rightrightarrows U]$ is isomorphic to an action groupoid, as follows:

$$\begin{tikzcd} \coprod U_h \arrow[r, "\sim", "\varphi"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & H \times U \arrow[d, shift right, "proj"'] \arrow[d, shift left, "act"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

where φ is given by

$$U_h \ni (u \to u') \mapsto (h,u)$$

which is a homeomorphism, because it is a homeomorphism onto U for any connected component U_h (given by $s|U_h$). The action is given by

$$act(h,u) = h.u = t(s^{-1}(u) \cap U_h)$$

This is a well-defined action, and, by construction, it is mild around the fixed point x.

$\square$

Corollary 1.4.2. *Let $\mathcal{M}$ be a DM-analytic stack such that the inertia groups at any point are trivial. Then, $\mathcal{M}$ is (equivalent to) a complex manifold.*

Proof. Locally around any point, our stack is of the form $[V/H]$, with V a complex manifold and H is the automorphism group at the point, which is trivial by hypothesis. $\square$

Remark 1.15. The statement in the corollary does *not* hold for an arbitrary analytic stack. Indeed, as in Example 1.13, we can take the punctured unit disc $\mathbb{D}^*$ acted on by a rotation group G inducing a rotation around the center by an irrational number $\tau \notin \pi\mathbb{Q}$. The quotient stack has only trivial inertia groups, by construction, but is not Deligne-Mumford (the diagonal is not closed). In fact, any G-orbit is not closed. Taking an orbit of a point, say Gx, and a point in its boundary, say $y \in \overline{Gx} \setminus Gx$, one has that any open neighbourhood of Gy (which is just a neighbourhood for the class of y in the quotient stack) also contains Gx. Hence, $[\mathbb{D}^*/G]$ is not Hausdorff.

1.5 Dimension of an analytic stack

As for complex manifolds, a very useful notion is the one of *dimension*. Again, we are going to use the fact that the relative dimension of complex manifolds is invariant under base change and extend this notion to analytic stacks.

Definition 1.21. Given a representable submersion of analytic stacks $\mathcal{M} \to \mathcal{N}$, one defines the *dimension of the fibers* $rel.dim(\mathcal{M}/\mathcal{N})$ to be the dimension of the fibers of $\mathcal{M} \times_{\mathcal{N}} X \to X$ for one (equivalently, any) atlas $X \to \mathcal{N}$.

Definition 1.22. The *dimension* of a connected analytic stack $\mathcal{M}$ is defined to be

$$dim(\mathcal{M}) := dim(X) - rel.dim(X/\mathcal{M})$$

for one atlas $X \to \mathcal{M}$, with X connected.

Remark 1.16. This definition is invariant on the choice of the atlas. Indeed, locally any other atlas $X' \to \mathcal{M}$ factors through $X \to \mathcal{M}$, and

$$dim(\mathcal{M}) = dim(X) - rel.dim(X/\mathcal{M}) =$$

$$= (dim(X') - rel.dim(X'/X)) - rel.dim(X/\mathcal{M}) =$$

$$= dim(X') - rel.dim(X'/\mathcal{M}).$$

Example 1.14. Suppose $\mathcal{M}$ is a DM-analytic stack. Then, locally, it can be written as a quotient stack $[X/G]$, with X a complex manifold and G a finite Lie group acting on it. Recall that, given the canonical atlas $\pi : X \to [X/G]$, one has the cartesian diagram

$$\begin{tikzcd} G \times X \arrow[r, "act"] \arrow[d, "proj"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

Then, by definition, $dim([X/G]) = dim(X) - rel.dim(X/[X/G])$, where $rel.dim(X/[X/G])$ is the dimension of the fibers of the projection $G \times X \to X$, which is just the complex dimension of G. Since G is finite, $dim(G) = 0$, hence

$$dim([X/G]) = dim(X)$$

Definition 1.23. An *analytic stacky curve* (or just a *curve*) is a 1-dimentional analytic stack.

In chapter, we are going to use the fact that, if $\mathcal{M}$ is a DM-curve, then locally we can write it as $[X/G]$, where X is a 1-dimentional complex manifold (i.e. a Riemann Surface) and G is a finite Lie group acting on it. Indeed, the structure of local quotient $[X/G]$ follows from Theorem 1.4.1, while the fact that $dim(X) = 1$ follows from the previous Example.

1.6 Analytic stacks as groupoids

Given an analytic stack $\mathcal{M}$ and an atlas $\pi : X \to \mathcal{M}$ for it, we have the cartesian diagram

$$\begin{tikzcd} X \times_{\mathcal{M}} X \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

The two projections s and t can be chosen to be the source and target maps of a groupoid $[X \times_{\mathcal{M}} X \rightrightarrows X]$. The composition is given by the commutative triangle

$$\begin{tikzcd} (X \times_{\mathcal{M}} X) \times_{\mathcal{M}} (X \times_{\mathcal{M}} X) \arrow[r, dotted] \arrow[d, "\sim"'] & X \times_{\mathcal{M}} X \\ X \times_{\mathcal{M}} X \times_{\mathcal{M}} X \arrow[ur, "p_{1,3}"'] & \end{tikzcd}$$

where $p_{1,3}$ is the projection on the first and third factor. Since π is representable, this is a well-defined analytic groupoid. In fact, the inverse is given by

$$i : X \times_{\mathcal{M}} X \to X \times_{\mathcal{M}} X$$

sending (f, g, φ) to (g, f, φ^{-1}), while the identity is given by

$$e : X \to X \times_{\mathcal{M}} X$$

sending f to (f, f, id).

One can actually reverse this construction, and get an analytic stack out of any analytic groupoid. Both the constructions are functorial, and give rise to an equivalence between the 2-category of analytic stack (with a fixed atlas) and the 2-category of analytic groupoids (cfr. chapter 3 of [7]).

1.7 The coarse moduli space

One way to think of an analytic stack is to imagine a complex manifold with an additional structure: a point is not just a point, but a *cluster of* *equivalent points.* The equivalences are just the 2-isomorphisms in the 2-category of analytic stacks. Every point in a cluster comes naturally with an inertia group of self-identifications, and all the points in the cluster have isomorphic inertia groups. Imagining to cut by all these self-identifications, one is left with a *coarse moduli space.*

The intuition suggests to think of stacks as of their coarse moduli space, where every point has a group attached to it (the inertia group). Unfortunately, this is not a faithful picture in general (for example, there are stacks with only trivial inertia groups, but which are not equivalent to complex manifolds), but it represents a nice way to think of stacks in most cases (for example, for Deligne-Mumford orbifold curves, as we are going to see in Remark 3.6).

In this section we will try to make the intuition provided above more precise. First, note that, given an analytic stack $\mathcal{M}$ and an atlas $\pi : X \to \mathcal{M}$ for it, the analytic groupoid

$$[R := X \times_{\mathcal{M}} X \rightrightarrows X]$$

induces a canonical relation $\sim_R$ on X, given by

$$x_1 \sim_R x_2 \Leftrightarrow \exists r \in R \text{ such that } s(r) = x_1 \text{ and } t(r) = x_2 \qquad \forall x_1, x_2 \in X$$

where s and t are the source and target maps of the groupoid. Since on the groupoid we have identity, inverse and multiplication, this is a well-defined equivalence relation. Call $\mathcal{M}_{mod} := X/\sim_R$ and $p : X \to \mathcal{M}_{mod}$ the canonical projection. $\mathcal{M}_{mod}$ is a topological space with the quotient topology induced from X.

Thanks to the commutativity of the square

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

we have that

$$x_1 \sim_R x_2 \Leftrightarrow \pi(x_1) = \pi(x_2) \qquad \forall x_1, x_2 \in X$$

Hence, p factors uniquely through π, meaning that there exists a unique map $\pi_{mod} : \mathcal{M} \to \mathcal{M}_{mod}$ such that the following triangle commutes:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "p"'] & \mathcal{M} \arrow[dl, dotted, "\pi_{mod}"] \\ \mathcal{M}_{mod} & \end{tikzcd}$$

More precisely, $\pi_{mod}(m)$ is defined by

$$\pi_{mod}(m) = p(x)$$

for any $x \in X$ such that $\pi(x) = m$.

Definition 1.24. We call the topological space $\mathcal{M}_{mod}$ defined above the *coarse moduli space* associated with (or the *underlying space* of) $\mathcal{M}$.

Remark 1.17. One checks that the definition of π_{mod} does not depend on the choice of the atlas $\pi : X \to \mathcal{M}$.

Remark 1.18. π_{mod} is functorial, meaning that if $F : \mathcal{M} \to \mathcal{N}$ is a morphism of analytic stacks, $\pi_{\mathcal{N}} : Y \to \mathcal{N}$ is an atlas and $\pi_{\mathcal{M}} : X \to \mathcal{N}$ is the corresponding base change atlas for $\mathcal{M}$, then F induces $F_{mod} : \mathcal{M}_{mod} \to \mathcal{N}_{mod}$. Indeed, if $x_1, x_2 \in X$ are equal in $\mathcal{M}_{mod}$ (meaning that $\pi_{\mathcal{M}}(x_1) = \pi_{\mathcal{M}}(x_2)$), then also $(f \circ \pi_{\mathcal{M}})(x_1) = (f \circ \pi_{\mathcal{M}})(x_2)$. By the commutativity of the cartesian square

$$\begin{tikzcd} X \arrow[r, "\pi_{\mathcal{M}}"] \arrow[d, "f"'] & \mathcal{M} \arrow[d, "F"] \\ Y \arrow[r, "\pi_{\mathcal{N}}"'] & N \end{tikzcd}$$

one gets $F \circ \pi_{\mathcal{M}} = \pi_{\mathcal{N}} \circ f$. Hence, $(\pi_{\mathcal{N}} \circ f)(x_1) = (\pi_{\mathcal{N}} \circ f)(x_2)$, meaning that $f(x_1)$ and $f(x_2)$ are equivalent in N_{mod}.

A priori, $\mathcal{M}_{mod}$ is just a topological space, but there are some cases in which it is actually a manifold. For example, in the case of an orbifold curve (cfr Remark 3.8).

Example 1.15. The coarse moduli space of a quotient stack $\mathcal{M} = [X/G]$ is the naive quotient $\mathcal{M}_{mod} = X/G$. Indeed, in this setting, $R \cong G \times X$, the source map is the action and the target map is the projection. The equivalence relation is given by

$$x_1 \sim_R x_2 \Leftrightarrow \exists g \in G \text{ such that } g.x_1 = x_2.$$

Example 1.16. The two quotient stacks $[\mathfrak{H}/SL_2(\mathbb{Z})]$ and $[\mathfrak{H}/\mathbb{P}SL_2(\mathbb{Z})]$ have the same coarse moduli space, but they are different as stacks (the inertia groups are different).

Chapter 2

Homotopy theory and covering spaces for analytic stacks

In this chapter we are going to develop some homotopy theory for analytic stacks. One can define the homotopy groups in a similar way to the case of topological spaces, and prove that classical results, like the Van Kampen theorem, also hold in the stack setting. The fundamental group will also come handy in order to classify the covering spaces of an analytic stack.

2.1 Homotopy groups of pointed analytic stacks

In stacks, points are allowed to have automorphisms. That is why to develop some homotopy theory for analytic stacks we are going to need to consider the inertia groups at the points. First, let me introduce some notation from [4], for I like it a lot.

Notation 1. Given two points $m, m' : * \to \mathcal{M}$ of an analytic stack $\mathcal{M}$, we call a *hidden path* a 2-morphism $m \Rightarrow m'$, and we denote it by squiggly arrows: $m \rightsquigarrow m'$.

Remark 2.1. The term *hidden paths* refers to the fact that these paths are not visible on the coarse moduli space of $\mathcal{M}$.

Notation 2. Given a point $m \in \mathcal{M}$, we will call the set of automorphisms of m the *hidden fundamental group* (or *inertial fundamental group*) of $\mathcal{M}$ at m, and denote it by $\pi_1^h(\mathcal{M}, m)$.

Remark 2.2. We will see (Remark 2.9) how to build a loop out of the hidden fundamental group. In fact, these loops will correspond to *constant* loops at the point.

According to our previous notation, if $\pi : X \to \mathcal{M}$ is an atlas and $x \in X$ such that $\pi(x) = m$,

$$\pi_1^h(\mathcal{M}, m) = Aut_{\mathcal{M}}(m) = I_x = s^{-1}(\{x\}) \cap t^{-1}(\{x\})$$

where s and t are given by the following cartesian square

$$\begin{tikzcd} X \times_{\mathcal{M}} X \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

Definition 2.1. We define a *triple* to be a triple $(\mathcal{M}, \mathcal{A}, i)$, where $i : \mathcal{A} \to \mathcal{M}$ is a morphism of stacks (not necessarily an embedding). We may say that $\mathcal{A}$ is a *base stack* for $\mathcal{M}$. We usually drop i from the notation.

Notation 3. When $\mathcal{A} = *$ is (equivalent to) a point, we drop it from the notation and say that $(\mathcal{M}, m)$ is a *pointed* analytic stack, where $m : * \to \mathcal{M}$ is a $*$-point of $\mathcal{M}$. We may say that m is a *base point* for $\mathcal{M}$.

Definition 2.2. A *map of triples* is a triple $(f, g, \phi) : (\mathcal{M}, \mathcal{A}, i) \to (\mathcal{N}, \mathcal{B}, j)$, where $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{A} \to \mathcal{B}$ are morphisms of stacks and $\phi : j \circ g \Rightarrow f \circ i$ is a 2-morphism. We will usually drop g and ϕ in the notation.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "g"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "j"] \arrow[dl, phantom, "\overset{\phi}{\swarrow}"] \\ \mathcal{M} \arrow[r, "f"'] & \mathcal{N} \end{tikzcd}$$

Notation 4. For pointed stacks, we just need to give a pair $(f, \phi) : (\mathcal{M}, m) \to (\mathcal{N}, n)$ with $f : \mathcal{M} \to \mathcal{N}$ and $\phi : n \rightsquigarrow f(m)$ (just look at $m : * \to \mathcal{M}$ and $n : * \to \mathcal{N}$ as $*$-points over the same $*$). We will usually drop ϕ in the notation.

Definition 2.3. Given two morphisms of triples $(f, g, \phi), (f', g', \phi') : (\mathcal{M}, \mathcal{A}, i) \to (\mathcal{N}, \mathcal{B}, j)$, an *identification* from f to g will be a pair (ψ, ψ'), where $\psi : f \Rightarrow f'$, $\psi' : g \Rightarrow g'$ are such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\psi'"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "j"] \arrow[dl, Rightarrow, "\phi'"', "\phi"] \\
\mathcal{M} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{N}
\end{tikzcd}$$

Again, we may drop ψ' in the notation.

Remark 2.3. Pointed analytic stacks naturally form a 2-category.

Definition 2.4. Let $f, g : (\mathcal{M}, \mathcal{A}, i) \to (\mathcal{N}, \mathcal{B}, j)$ be maps of triples. A *homotopy* from f to g is a triple $(H, \epsilon_0, \epsilon_1)$ as follows:

- $H : (I \times \mathcal{M}, I \times \mathcal{A}) \to (\mathcal{N}, \mathcal{B})$ is a map of triples, where I stands for (the stack associated to) the real interval $[0, 1]$.

- Denoting by H_0 and H_1 the maps of triples obtained by restrincting H to $\{0\} \times \mathcal{M}$ and $\{1\} \times \mathcal{M}$, respectively, $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are identifications.

Remark 2.4. It is straightforward to check that homotopy gives a well-defined equivalence relation between maps of triples. We denote by

$$[(\mathcal{M}, \mathcal{A}), (\mathcal{N}, \mathcal{B})]$$

such an equivalence class.

Remark 2.5. An identification ψ between two maps of pairs

$$f, g : (\mathcal{M}, \mathcal{A}) \to (\mathcal{N}, \mathcal{B})$$

can be regarded as a homotopy, simply by defining $H = f \circ proj$

$$\begin{tikzcd} (I \times \mathcal{M}, I \times \mathcal{A}) \arrow[rr, dotted, "H"] \arrow[dr, "proj"'] & & (\mathcal{N}, \mathcal{B}) \\ & (\mathcal{M}, \mathcal{A}) \arrow[ur, "f"'] & \end{tikzcd}$$

Clearly $H_0 = H_1 = f$, so one can define $\epsilon_0 := id : f \Rightarrow f$ and $\epsilon_1 := \psi : f \Rightarrow g$

Remark 2.6. Given a $*$-point $m : * \to \mathcal{M}$, we have a natural morphism of triples (which we call again m, by slight abuse of notation) $m : (*, *, id) \to (\mathcal{M}, *, m)$.

$$\begin{tikzcd} * \arrow[r, "="] \arrow[d, "="'] & * \arrow[d, "m"] \\ * \arrow[r, "m"'] & \mathcal{M} \end{tikzcd}$$

Definition 2.5. In the notation of the previous remark, a *path* from m to n in $\mathcal{M}$ is defined to be a homotopy between m and n. A *loop* at m is simply a path from m to itself.

Definition 2.6. An analytic stack $\mathcal{M}$ is said to be *path connected* if for any pair of points (m, n) there is a path from m to n in $\mathcal{M}$.

Remark 2.7. One checks, just writing down the definitions, that giving a loop at m in $\mathcal{M}$ is essentially the same as giving a morphism of pointed stacks $(S^1, x) \to (\mathcal{M}, m)$. In fact, the former is the data of a map of triples $H : (I \times *, I \times *) \to (\mathcal{M}, *)$, together with 2-morphisms $\epsilon_0 : m \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow m$ (so that we get $H_0 \Rightarrow H_1$), while the latter is given by a morphism $f : S^1 \to \mathcal{M}$ (which is the same as a morphism $F : I \to \mathcal{M}$ such that $F(0) = F(1)$) and a 2 morphism $m \rightsquigarrow f(x)$ (which, by choosing $x = F(0) = F(1)$, is the same as $m \Rightarrow F(0) = F(1)$).
This justifies the following definition.

Definition 2.7. The *fundamental group* of a pointed analytic stack $(\mathcal{M}, m)$ is defined as $\pi_1(\mathcal{M}, m) := [(S^1, x), (\mathcal{M}, m)]$.

In view of the previous remark, a representative for a class is essentially a loop at m, i.e. a homotopy from m to itself.

Remark 2.8. One checks that $\pi_1(\mathcal{M}, m)$ is actually a group (see, for example, [2], §17). Moreover, π_1 gives a well-defined functor from the category of pointed stacks to the category of groups.

Remark 2.9. There is a natural morphism $\omega_m : \pi_1^h(\mathcal{M}, m) \to \pi_1(\mathcal{M}, m)$. Indeed, we already pointed out that an identification between maps of pairs can be regarded as a homotopy. More precisely, given a hidden path at m $\gamma : m \rightsquigarrow m$ in $\pi_1^h(\mathcal{M}, m)$, we define $\omega_m(\gamma) \in \pi_1(\mathcal{M}, m)$ to be the class of the constant loop at m. A representative for this class is a homotopy from m to itself, as follows:

- H is given by the following commutative triangle

$$\begin{tikzcd}
(I \times *, I \times *) \arrow[rr, dotted, "H"] \arrow[dr, "proj"'] & & (\mathcal{M}, *) \\
& (\mathcal{M}, *) \arrow[ur, "m"'] &
\end{tikzcd}$$

- $H_0 = H_1 = m$, hence one defines

$$\epsilon_0 := id : m \rightsquigarrow m$$

  and

$$\epsilon_1 := \gamma : m \rightsquigarrow m.$$

The maps ω_m are also functorial with respect to pointed maps, meaning that, if $f : (\mathcal{M}, m) \to (\mathcal{N}, n)$ is a map of pointed stacks and f_* is the induced map on the hidden fundamental groups, then the following square commutes:

$$\begin{tikzcd}
\pi_1^h(\mathcal{M}, m) \arrow[r, "w_m"] \arrow[d, "f_*"'] & \pi_1(\mathcal{M}, m) \arrow[d, "\pi_1 f"] \\
\pi_1^h(\mathcal{N}, n) \arrow[r, "w_n"] & \pi_1(\mathcal{N}, n)
\end{tikzcd}$$

Remark 2.10. Note that, when the stack is path connected, the fundamental group does not depend on the choice of the base point (to see it, as for topological spaces, just compose with a path from two different base points to get an isomorphism between the two corresponding fundamental groups). In this case, we will often omit the point in the notation, and just write $\pi_1(\mathcal{M})$.

Definition 2.8. A path connected analytic stack $\mathcal{M}$ is said to be *simply connected* if its fundamental group is trivial.

Later on this chapter, we will use hidden fundamental groups to study the covering spaces of pointed analytic stacks.

2.2 Covering spaces of analytic stacks

In this section we will study covering spaces for analytic stacks.

Definition 2.9. We say that a morphism of connected analytic stacks $\mathcal{M} \to \mathcal{N}$ is a *covering map* (or that $\mathcal{M}$ is a *covering space* for $\mathcal{N}$) if it is a local homeomorphism and if for any point $m \in \mathcal{M}$ there is a local homeomorphism $\psi : \mathcal{U} \to \mathcal{M}$ such that $m \in Im(\psi)$ and such that the fiber product $\mathcal{U} \times_{\mathcal{M}} \mathcal{N}$ is isomorphic to a disjoint union of copies of $\mathcal{U}$ itself

$$\begin{tikzcd} \coprod \mathcal{U} \cong \mathcal{U} \times_{\mathcal{M}} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N} \arrow[d, "f"] \\ \mathcal{U} \arrow[r] & \mathcal{M} \end{tikzcd}$$

A covering space is said to be a *universal covering space* if it is simply connected.

Definition 2.10. Let $\mathcal{M}$ be a connected analytic stack, $m : * \to M$ a point of $\mathcal{M}$. We define the category $\mathcal{C}_{\mathcal{M}}$ associated with $\mathcal{M}$ as follows:

$$Ob(\mathcal{C}_{\mathcal{M}}) = \{(\mathcal{N}, f) | \mathcal{N} \text{ analytic stack}, f : \mathcal{N} \to \mathcal{M} \text{ covering map}\}$$

$$Hom_{\mathcal{C}_{\mathcal{M}}}((\mathcal{N},f),(\mathcal{L},g)) = \{(a:\mathcal{N}\to\mathcal{L},\phi: f\Rightarrow g\circ a)\}/\sim$$

where $\sim$ is defined by

$$(a,\phi)\sim(b,\phi)\Leftrightarrow \exists\Gamma: a\Rightarrow b \text{ such that } g(\Gamma)\circ\phi=\psi$$

One can also define a functor

$$F_m:\mathcal{C}_{\mathcal{M}}\to\pi_1(\mathcal{M},m)-\mathbf{Sets}$$

as follows

$$F_m(\mathcal{N},f)=\{(n,\phi)|n:*\to\mathcal{N}\text{ point of }\mathcal{N},\phi:m\rightsquigarrow f(n)\text{ hidden path}\}/\sim_F$$

where $\sim_F$ is defined by

$$(n,\phi)\sim_F(l,\psi)\Leftrightarrow\exists\beta:n\rightsquigarrow l\text{ such that }f(\beta)\circ\phi=\psi$$

The action of $\pi_1(\mathcal{M},m)$ on $F_m(\mathcal{N},f)$ is defined as follows. Let $\gamma: I\to\mathcal{M}$, together with hidden paths $\epsilon_0: m\rightsquigarrow\gamma(0)$ and $\epsilon_1:\gamma(1)\rightsquigarrow m$, represent a loop at m. Consider the cartesian diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathcal{N} \arrow[d, "f"] \\ I \arrow[r, "\gamma"'] & \mathcal{M} \end{tikzcd}$$

with $E:=I\times_{\mathcal{M}}\mathcal{N}$. Since f is a covering map, the projection $E\to I$ also is, hence E is isomorphic to a disjoint union of copies of I. The pullback of ϵ_0 and ϵ_1 gives us natural bijections $\overline{\epsilon_0}: E_0\to F_m(\mathcal{N},f)$ and $\overline{\epsilon_1}: F_m(\mathcal{N},f)\to E_1$. Hence, $\overline{\epsilon_0}\circ\overline{\epsilon_1}: F_m(\mathcal{N},f)\to F_m(\mathcal{N},f)$ gives a well-defined action on $F_m(\mathcal{N},f)$. Indeed, the same argument, applied to $I\times I$, one shows that this action is independent of the choice of the representative in the homotopy class of γ, and it is straightforward to check that the action respects composition of loops.

Remark 2.11. The category $\mathcal{C}_{\mathcal{M}}$ above, together with the *fundamental functor* F_m, forms a *Galois category*. The interested reader may find [4] a good reference for the studying of this functor in the algebraic setting.

Remark 2.12. In the definition of the action of $\pi_1(\mathcal{M}, m)$ on $F_m(\mathcal{N})$, let γ be a constant loop, coming from an element $\alpha \in \pi_1^h(\mathcal{M}, m)$, as in Remark 2.9. Then, $\gamma : I \to \mathcal{M}$ is represented by the constant loop at m, together with identifications $\epsilon_0 = id : m \rightsquigarrow m$ and $\epsilon_1 = \alpha : m \rightsquigarrow m$. Hence, the action will send $\overline{(n, \phi)} \in F_m(\mathcal{N}, f)$ to $\overline{(n, \phi \circ \alpha)}$.

Lemma 2.2.1. *Let $f : (\mathcal{N}, n) \to (\mathcal{M}, m)$ be pointed covering map, $\gamma : I \to \mathcal{M}$ a loop at m in $\mathcal{M}$ (with identifications $\epsilon_0 : m \rightsquigarrow \gamma(0)$ and $\epsilon_1 : \gamma(1) \rightsquigarrow m$, as above). Then, the following are equivalent:*

1. *(n, id) is a fixed point for the action of γ on $F_m(\mathcal{N}, f)$*
2. *there exists a loop $\tilde{\gamma}$ at n in $\mathcal{N}$ lying over γ*
3. *the class of γ in $\pi_1(\mathcal{M}, m)$ is in $\pi_1 f(\pi_1(\mathcal{N}, n))$*

Proof. [$1 \Rightarrow 2$] As before, we have the cartesian square

$$\begin{tikzcd} E \arrow[r, "p_{\mathcal{N}}"] \arrow[d, "p_I"'] & \mathcal{N} \arrow[d, "f"] \\ I \arrow[r, "\gamma"'] & \mathcal{M} \end{tikzcd}$$

with $E := I \times_{\mathcal{M}} \mathcal{N}$. Let $n_0 \in E_0$ and $n_1 \in E_1$ be such that $\tilde{\epsilon_0}(n_0) = n$ and $\tilde{\epsilon_1}(n) = n_1$. Since the action of γ leaves n invariant (i.e. $(\tilde{\epsilon_0} \circ \tilde{\epsilon_1})\overline{(n, id)} = \overline{(n, id)}$), n_0 and n_1 must lie in the same layer of E, meaning that there must exist a section $s : I \to E$ of p_I such that $s(0) = n_0$ and $s_1 = n_1$. $\tilde{\gamma} := s \circ p_{\mathcal{N}}$ will be the sought loop at n.
[$2 \Rightarrow 3$] Just think at the commutative triangle

$$\begin{tikzcd} & \mathcal{N} \arrow[d, "f"] \\ I \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & \mathcal{M} \end{tikzcd}$$

[$3 \Rightarrow 1$] Since the action of γ on $f_m(\mathcal{N}, f)$ is independent of the choice of the homotopy class of γ, we may assume that there is a lift $\tilde{\gamma}$. The argument $1 \Rightarrow 2$ reversed gives the proof. $\square$

Lemma 2.2.2. *Let $f : (\mathcal{N}, n) \to (\mathcal{M}, m)$ be pointed covering map. Then, the induced map $\pi_1 f : \pi_1(\mathcal{N}, n) \to \pi_1(\mathcal{M}, m)$ is injective.*

Proof. Let α, β be loops at n in $\mathcal{N}$ whose images in $\mathcal{M}$ are homotopic, meaning that there exists a homotopy $H(I \times S^1, I \times *) \to (\mathcal{M}, m)$ between $f \circ \alpha$ and $f \circ \beta$. If $A := \cup(\{1\} \times S^1) \cup (I \times *)$, one can glue and get a natural map $g : A \to \mathcal{N}$ whose restriction to $(\{0\} \times S^1)$ (resp. $(\{1\} \times S^1)$, $(I \times *)$) is naturally identified with α (resp. β, the constant map), and such that $f \circ g : A \to \mathcal{M}$ is naturally identified with $H|_A$ [for example, see Theorem 16.2 of [2]]. Now, set $Z := (I \times S^1) \times_{\mathcal{M}} \mathcal{N}$ and consider the cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, shift left, "q"] & \mathcal{N} \arrow[d, "f"] \\ I \times S^1 \arrow[r, "H"'] \arrow[u, dotted, shift left, "s"] \arrow[ur, dotted, "F"] & \mathcal{M} \end{tikzcd}$$

This base change gives us a covering map $q : Z \to I \times S^1$, where Z is a manifold. This gives us (locally) a section $s : I \times S^1 \to Z$, and $F = H \circ s$ is the sought homotopy. $\square$

Corollary 2.2.3. *Let $f : (\mathcal{N}, n) \to (\mathcal{M}, m)$ be pointed covering map. Then, $\pi_1 f(\pi_1(\mathcal{N}, n)) \leq \pi_1(\mathcal{M}, m)$.*

There is actually a more general result:

Theorem 2.2.4. *Let $(\mathcal{M}, m)$ be a connected pointed analytic stack. Then, there is a correspondence between the subgroups of $\pi_1(\mathcal{M}, m)$ and the isomorphism classes of pointed covering maps for $(\mathcal{M}, m)$ (as objects of $\mathcal{C}_{\mathcal{M}}$).*

The correspondence is simply given by $\pi_1 f$, where $f : (\mathcal{N}, n) \to (\mathcal{M}, m)$ is a covering map. For a proof, see Theorem 18.19 and Corollary 18.20 of [2].

Corollary 2.2.5. *Any connected pointed analytic stack has a universal cover (corresponding to the trivial subgroup in Theorem 2.2.4).*

Proposition 2.2.6. *Let $f : (\mathcal{M}, m) \to (\mathcal{N}, n)$ be a map of pointed stacks. According to Remark 2.9, we have a commutative square*

$$\begin{tikzcd} \pi_1^h(\mathcal{M}, m) \arrow[r, "w_m"] \arrow[d, "f_*"'] & \pi_1(\mathcal{M}, m) \arrow[d, "\pi_1 f"] \\ \pi_1^h(\mathcal{N}, n) \arrow[r, "w_n"] & \pi_1(\mathcal{N}, n) \end{tikzcd}$$

If f is a covering map, the square is cartesian.

Proof. We claim that for any $\gamma \in \pi_1^h(\mathcal{N}, n)$, if $\omega_m(\gamma) \in Im(\pi_1 f)$, then there exists a unique $\alpha \in \pi_1^h(\mathcal{M}, m)$ such that $f_*(\alpha) = \gamma$. The uniqueness follows from the injectivity of f_*. As for the existence, the fact that $\omega_m(\gamma) \in Im(\pi_1 f)$ implies that the action of γ on $F_m(\mathcal{N})$ leaves (n, id) invariant (thanks to Lemma 2.2.1). This means, thanks to Remark 2.12, that $(n, id) \sim_F (n, \gamma)$, i.e. (Definition 2.10) there exists $\alpha : n \rightsquigarrow n$ such that $f(\alpha) \circ id = \gamma$. $\square$

Corollary 2.2.7. *Let $(\mathcal{N}, n)$ be a pointed analytic stack and $\tilde{\mathcal{N}}$ a universal cover for it. Then, for any $\tilde{n}$ lying over n there is an isomorphism $\pi_1^h(\tilde{\mathcal{N}}, \tilde{n}) \cong ker(\omega_n)$.*

Proof. Follows directly from Proposition 2.2.6 applied to the cartesian square

$$\begin{tikzcd} \pi_1^h(\mathcal{M}, m) \arrow[r] \arrow[d, "f_*"'] & 1 \arrow[d, "\pi_1 f"] \\ \pi_1^h(\mathcal{N}, n) \arrow[r] & \pi_1(\mathcal{N}, n) \end{tikzcd}$$

$\square$

2.3 The importance of the hidden fundamental groups

Hidden fundamental groups will be crucial in the local discussion of orbifolds in the next chapter. First, let us mention a result which will come useful later:

Theorem 2.3.1. *Suppose $f : \mathcal{N} \to \mathcal{M}$ is a covering map of analytic stacks, with $\mathcal{M}$ Deligne-Mumford. Then, $\mathcal{N}$ is also Deligne-Mumford.*

For a proof, see Proposition 18.25 of [2].

Theorem 2.3.2. *Let $\mathcal{M}$ be a connected Deligne-Mumford analytic stack, and let $\tilde{\mathcal{M}}$ be its universal cover. Then, the following are equivalent:*

1. *The morphisms $\omega_m : \pi_1^h(\mathcal{M}, m) \to \pi_1(\mathcal{M}, m)$ are injective for any $m \in \mathcal{M}$.*
2. *$\tilde{\mathcal{M}}$ is a manifold*
3. *$\mathcal{M} \cong [V/G]$ for some manifold V acted on by a discrete group G.*

Proof. [1 ⇒ 2] By Corollary 2.2.7, $\pi_1^h(\tilde{\mathcal{M}}, m) \cong ker(\omega_m) \cong 1$ for any $m \in \tilde{\mathcal{M}}$. Since $\mathcal{M}$ is Deligne-Mumford, by Theorem 2.3.1 $\tilde{\mathcal{M}}$ also is Deligne-Mumford, and so it has to be a manifold (Corollary 1.4.2).
[2 ⇒ 3] Consider $R := \tilde{\mathcal{M}} \times_{\mathcal{M}} \tilde{\mathcal{M}}$.

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & \tilde{\mathcal{M}} \arrow[d] \\ \tilde{\mathcal{M}} \arrow[r] & \mathcal{M} \end{tikzcd}$$

The groupoid $[R \rightrightarrows \tilde{\mathcal{M}}]$ has source and target maps which are covering maps, and $\tilde{\mathcal{M}}$ is simply connected. Hence, R can be written as a disjoint union of copies of $\tilde{\mathcal{M}}$, indexed by $\pi_1(\mathcal{M})$, each of which mapping homeomorphically to $\tilde{\mathcal{M}}$ via the source and target maps. Hence, with an argument analogous to the one used at the end of the proof of Theorem 1.4.1, one gets that $[R \rightrightarrows \tilde{\mathcal{M}}]$ is isomorphic to the action groupoid of an action of $\pi_1(\mathcal{M})$ on $\tilde{\mathcal{M}}$. Just define $V := \tilde{\mathcal{M}}$ and $G := \pi_1(\mathcal{M})$.
[3 ⇒ 1] $\mathcal{M} = [V/G]$ has a covering stack V with only trivial hidden fundamental groups (because it is a manifold). By Corollary 2.2.7, all the morphisms ω_m have trivial kernel, i.e. they are injective. □

Remark 2.13. Note that the proof of [2 ⇒ 3] gives us a precise characterization of a case when a connected Deligne-Mumford analytic stack is (globally) a quotient stack. More explicitly, if $\mathcal{M}$ has a universal cover $\tilde{\mathcal{M}}$ which is a manifold, then $\mathcal{M}$ is a global quotient stack stack of the form $[\tilde{\mathcal{M}}/\pi_1(\mathcal{M})]$.

Remark 2.14. If $\mathcal{M}$ is globally a quotient stack of the form $[V/G]$, with V simply connected and G discrete, then the argument in the proof of [2 ⇒ 3] (taking the universal cover $V \to [V/G]$) tells us that G is isomorphic to $\pi_1(\mathcal{M})$.

2.4 Van Kampen Theorem for analytic stacks

In this section we will discuss how the Van Kampen Theorem can be generalized for analytic stacks. As for topological spaces, one defines the fundamental groupoid:

Definition 2.11. Let $\mathcal{M}$ be an analytic stack. The *fundamental groupoid* $\Pi(\mathcal{M})$ of $\mathcal{M}$ is the groupoid given by

$$Ob(\Pi(\mathcal{M})) = \{x : * \to \mathcal{M}\}$$

$$Hom_{\Pi(\mathcal{M})}(x, y) = \{\textit{homotopy classes of paths from } x \text{ to } y\}$$

Since homotopy is an equivalence relation, $\Pi(\mathcal{M})$ is a well defined groupoid.

As for topological spaces, the following holds:

Theorem 2.4.1. *Let $\mathcal{M}$ be an analytic stack, let $\mathcal{X}, \mathcal{Y}$ be two open substacks such that $\mathcal{X} \cup \mathcal{Y} = \mathcal{M}$. Then, the diagram*

$$\begin{tikzcd} \Pi(\mathcal{X} \cap \mathcal{Y}) \arrow[r] \arrow[d] & \Pi(\mathcal{X}) \arrow[d] \\ \Pi(\mathcal{Y}) \arrow[r] & \Pi(\mathcal{M}) \end{tikzcd}$$

*is a pushout square in the category **Gpd** of groupoids.*

Proof. See Theorem 5.10 of [3] for a proof.
The idea is to take an atlas $\pi : M \to \mathcal{M}$ which is a classifying space for $\mathcal{M}$ (it always exists, thanks to Theorem 6.3 of [6]). By base change, one gets atlases $X, Y, X \cap Y$ for $\mathcal{X}, \mathcal{Y}, \mathcal{X} \cap \mathcal{Y}$ which are still classifying spaces. Since for $M, X, Y, X \cap Y$ the classical version of the result holds, then all the above-mentioned classifying spaces induce equivalences among the fundamental groupoids (because classifying spaces induce an equivalence between the fundamental groupoids). □

In a completely analogous way as for topological spaces (see, for example, [13]), the previous result allows to prove the Van Kampen Theorem:

Corollary 2.4.2 (Van Kampen Theorem). *Let $\mathcal{M}$ be an analytic stack, let $\mathcal{X}, \mathcal{Y}$ be two open substacks such that $\mathcal{X} \cup \mathcal{Y} = \mathcal{M}$. Assume $\mathcal{X} \cap \mathcal{Y}$ is path connected, and choose an arbitrary point $m : * \to \mathcal{X} \cap \mathcal{Y}$. Then, there is a natural isomorphism:*

$$\pi_1(\mathcal{M}, m) \cong \pi_1(\mathcal{X}, m) *_{\pi_1(\mathcal{X} \cap \mathcal{Y}, m)} \pi_1(\mathcal{Y}, m)$$

This version of Van Kampen Theorem will come useful in the next chapter, for calculating the fundamental group of orbifolds.

Chapter 3

Uniformization of Deligne-Mumford curves

For Riemann Surfaces, the Uniformization Theorem states:

Theorem 3.0.1. *Any Riemann Surface has a universal cover given by a simply connected Riemann Surface. The simply connected Riemann Surfaces are conformally equivalent to either $\mathbb{C}$ (the complex plane), $\mathfrak{H}$ (the Poincaré upper half-plane) or $\mathbb{P}^1_{\mathbb{C}}$ (the complex projective line).*

Proof. See, for example, Theorem 4.17.2 of [16]. □

A similar result holds for simply connected DM-stacky curves:

Theorem 3.0.2. *Any Deligne-Mumford curve has a universal cover given by a simply connected Deligne-Mumford curve. The simply connected Deligne-Mumford curves are equivalent to either $\mathbb{C}$, $\mathfrak{H}$ or $\mathcal{P}(m, n)$ (the weighted projective line of type (m, n)), for $m, n \in \mathbb{Z}_{\geq 1}$.*

In this chapter, we will study this result, which was proven by K. Behrend and B. Noohi in [1]. First, we will give a proof for the case of an orbifold. Then, we are going to extend the result to any DM-curve.

3.1 Orbifolds

A nice class of analytic stacks is given by orbifolds, which are stacks that resemble manifolds *almost everywhere*.

Definition 3.1. An analytic stack $\mathcal{M}$ is an orbifold if there exists an open dense substack $X \hookrightarrow \mathcal{M}$ which is a manifold (we may say that $\mathcal{M}$ is *generically* a manifold).

Definition 3.2. A point m of a stack $\mathcal{M}$ is said to be an *orbifold point* if for any open neighbourhood $\mathcal{U}$ of m in $\mathcal{M}$ there exists a point $m' \in \mathcal{U}$ such that the inertia groups at m and m' are different.

Remark 3.1. A point m of an orbifold $\mathcal{M}$ is an orbifold point if, and only if, it has non-trivial inertia group. Indeed, all the inertia groups of a manifold are trivial.

Example 3.1. The moduli stack of elliptic curves $\mathcal{M}_{1,1} := [\mathbb{H}/SL_2(\mathbb{Z})]$ is *not* an orbifold, because of the trivial action of $\{\pm 1\}$. But by quotienting $SL_2(\mathbb{Z})$ by the group $\{\pm 1\}$, one gets $\mathbb{P}SL_2(\mathbb{Z})$, and $[\mathbb{H}/\mathbb{P}SL_2(\mathbb{Z})]$ has an open dense substack with trivial inertia groups. In fact, we only have two points with non-trivial inertia group. Since the stack is Deligne-Mumford, this is enough to conclude that it is an orbifold.

Definition 3.3. Given $m, n \in \mathbb{Z}_{\geq 1}$, consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2 \setminus (0,0)$ given by $t.(x,y) := (t^m x, t^n y)$. We define the *weighted projective line* of type (m,n)

$$\mathcal{P}(m,n) := \left[\frac{\mathbb{C}^2 \setminus (0,0)}{\mathbb{C}^\times}\right]$$

.

Remark 3.2. The points of the coarse moduli space of a weighted projective line are just the points of the (classical) complex projective line $\mathbb{P}^1_{\mathbb{C}}$. Indeed, according to Section 1.7,

$$\mathcal{P}(m,n)_{mod} = \mathbb{C}^2 \setminus \{(0,0)\}/\sim$$

where $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists

$$(t, (x, y)) \in \mathbb{C}^\times \times (\mathbb{C}^2 \setminus \{(0, 0)\})$$

such that

$$proj(t, (x, y)) = (x, y) = (x_1, y_1)$$

and

$$act(t, (x, y)) = (t^m x, t^n y) = (x_2, y_2).$$

Equivalently, $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $t \in \mathbb{C}^\times$ such that $(t^m x_1, t^n y_1) = (x_2, y_2)$. A bijection between $\mathcal{P}(m, n)_{mod}$ and $\mathbb{P}^1_{\mathbb{C}}$ is given by the map

$$\begin{aligned} \mathcal{P}(m, n)_{mod} &\longrightarrow \mathbb{P}^1_{\mathbb{C}} \\ \overline{(x, y)} &\longmapsto [x^{\frac{n}{d}} : y^{\frac{m}{d}}] \end{aligned}$$

where $d = gcd(m, n)$.

But the points of a weighted projective line come with an additional structure, i.e. they are allowed to have automorphisms. More precisely, given $x, y \in \mathbb{C}^\times$, the stabilizer group of (x, y) is

$$Stab_{\mathbb{C}^\times}(x, y) = \{t \in \mathbb{C}^\times | (t^m x, t^n y) = (x, y)\} = \{t \in \mathbb{C}^\times | t^m = 1 = t = n\} \cong \mathbb{Z}/d\mathbb{Z}$$

with $d = gcd(m, n)$. Analogously,

$$Stab_{\mathbb{C}^\times}(x, 0) = \{t \in \mathbb{C}^\times | t^m = 1\} \cong \mathbb{Z}/m\mathbb{Z}$$

and

$$Stab_{\mathbb{C}^\times}(0, y) = \{t \in \mathbb{C}^\times | t^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}$$

Hence, for $m \neq n$, $\overline{(1, 0)}$ and $\overline{(0, 1)}$ are the only possible orbifold points.

$\mathcal{P}(m, n)$ is an orbifold if, and only if, m and n are coprime (so that all the inertia groups are trivial, except for at most the two orbifold points above).

By quotienting $\mathbb{C}^\times$ by the rotation group $\mathbb{Z}/d\mathbb{Z}$, with $d = gcd(m, n)$, we get

$$\left[\frac{\mathbb{C}^2 \setminus (0, 0)}{\mathbb{C}^\times / \mathbb{Z}_d}\right] \cong \mathcal{P}\left(\frac{m}{d}, \frac{n}{d}\right)$$

which is always an orbifold.

Moreover, $\mathcal{P}(m,n)$ is simply connected. Indeed, if $\tilde{\mathcal{P}}(m,n)$ is a universal cover and

$$\mathbb{C}^2 \setminus \{(0,0)\} \to \mathcal{P}(m,n)$$

is the canonical atlas, call

$$Z := (\mathbb{C}^2 \setminus \{(0,0)\}) \times_{\mathcal{P}(m,n)} \tilde{\mathcal{P}}(m,n)$$

the fiber product. If by absurd $\tilde{\mathcal{P}}(m,n) \neq \mathcal{P}(m,n)$, then Z would be homeomorphic to a non-trivial disjoint union of copies of $\mathbb{C}^2 \setminus \{(0,0)\}$.

$$\begin{tikzcd} \coprod \mathbb{C}^2 \smallsetminus \{(0,0)\} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \tilde{\mathcal{P}}(m,n) \arrow[d, "\varphi"] \\ \mathbb{C}^2 \smallsetminus \{(0,0)\} \arrow[r, "\pi"'] & \mathcal{P}(m,n) \end{tikzcd}$$

Note that π is a representable surjective submersion with fibers isomorphic to $\mathbb{C}^\times$ (hence, connected), so also p_1 has these properties, which implies that $\tilde{\mathcal{P}}(m,n)$ is not connected (for Z is not). This is a contradiction, since $\tilde{\mathcal{P}}(m,n)$ is simply connected. This statement can also be proven by a homotopy fiber sequence argument.

Definition 3.4. Given a complex manifold X and a Lie group G acting trivially on it, we define $B_X G := [X/G]$ to be the *trivial gerbe* (or the *classifying gerbe*) of G over X.

Definition 3.5. Given an analytic stack $\mathcal{N}$, we say that an analytic stack $\mathcal{M}$ is a *gerbe* over $\mathcal{N}$ if we are given a surjective submersion $f : \mathcal{M} \to \mathcal{N}$ such that:

- f has local sections, meaning that there exists an atlas $X \to \mathcal{N}$ and a section $s : X \to \mathcal{M}$ for $f|_X$.
- locally over $\mathcal{N}$ all the objects of $\mathcal{M}$ are isomorphic, meaning that for any complex manifold X, $\forall n \in \mathcal{N}(X)$ and lifts $m_1, m_2 \in \mathcal{M}(X)$ there exists an open covering $(U_i)_i$ of X such that $m_1|U_i \cong m_2|_{U_i}$ $\forall i$

Remark 3.3. Any trivial gerbe is a gerbe. Indeed, when a Lie group G acts trivially on a complex manifold X, the map $[X/G] \to X$ makes the classifying gerbe $B_X G$ into a gerbe in a natural way.

Definition 3.6. Given a a morphism of analytic stacks $\mathcal{M} \to \mathcal{N}$, we say that $\mathcal{M}$ is a G-gerbe over $\mathcal{N}$ if it is locally (on $\mathcal{N}$) equivalent to the classifying stack of G.

Example 3.2. The moduli stack of elliptic curves $\mathcal{M}_{1,1} = [\mathbb{H}/SL_2(\mathbb{Z})]$ is a $\mathbb{Z}_2$-gerbe over the orbifold $[\mathbb{H}/\mathbb{P}SL_2(\mathbb{Z})]$.

Example 3.3. The weighted projective line $\mathcal{P}(m,n)$ is a $\mathbb{Z}_d$-gerbe over the orbifold $\mathcal{P}(\frac{m}{d}, \frac{n}{d})$, with $d = gcd(m,n)$.
One can actually prove that all the simply connected H-gerbes over an orbifold $\mathcal{P}(m,n)$, with m,n coprime positive integers, are the weighted projective lines $\mathcal{P}(m',n')$, with m',n' arbitrary positive integers (see Corollary 6.3 of [1]).

The construction of the previous two examples can always be done for DM-analytic stacks, as we are about to see.

Proposition 3.1.1. *Any connected Deligne-Mumford analytic stack $\mathcal{M}$ is an H-gerbe over an orbifold $\mathcal{N}$, for some finite group H.*

Proof. Assume $\mathcal{M} = [X/G]$ for some complex manifold X and some finite group G acting on it. Let $H := \cap_{x\in X} Stab_G(x)$ be the subgroup of G given by elements acting trivially on X. H is finite, because G is, and it is a normal subgroup of G. We claim that there exists an open dense submanifold $U \subset X$ such that G/H acts freely on it. Take $U := X \setminus \cup_{g\in G} X^g$, where X^g are the points of X fixed by g. X^g is closed, so U is open in X. U is also dense in X: indeed, if by absurd it were not dense, there would exist a point $x \in X$ with an open neighbourhood $V \subset X^g$. But the action of g fixes $\emptyset \neq V \subset X$, so $g = id$. This proves that $[\frac{U}{G/H}]$ is an open dense substack of $[\frac{X}{G/H}]$ given by a manifold, hence $[\frac{X}{G/H}]$ is an orbifold, and $\mathcal{M}$ is an H-gerbe over it.

In the general case, one just proves the statement locally (the Deligne-Mumford hypothesis allows us to describe $\mathcal{M}$ locally as a quotient stack by a finite group action). One just has to check that the group H is the same everywhere, meaning that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-empty open substacks of $\mathcal{M}$ which are respectively an H_1-gerbe and an H_2-gerbe over some orbifold, then $H_1 = H_2$. But since the stack is connected, $\mathcal{U}_1$ and $\mathcal{U}_2$ intersect, so $H_1 = H_2$ in the intersection, hence everywhere. □

Remark 3.4. If $\mathcal{M}$ is an H-gerbe over a connected complex manifold T (i.e., the orbifold $\mathcal{N}$ in the previous proposition is a connected manifold), then $\mathcal{M} \cong B_X G$. Indeed, locally $\mathcal{M} \cong [X/G]$, and we constructed $\mathcal{N} = [\frac{X}{G/H}]$, where H was the normal subgroup of G given by the elements acting trivially on $\mathcal{M}$. If $\mathcal{N} = T$ is a manifold, then in the proof above $G = H$ (because all the elements of G act trivially on $\mathcal{M}$), so $\mathcal{N} \cong T \cong X$ and $\mathcal{M} \cong B_X G$. Since the manifold is connected, we can extend the construction everywhere on X (as in the proof, we can patch the H's) and get that $\mathcal{M} \cong B_X G$, i.e. $\mathcal{M}$ is globally a trivial gerbe.

Remark 3.5. The previous Proposition fails to be true if one does not require $\mathcal{M}$ to be connected. For example, if $\mathcal{M}$ is the disjoint union of the two trivial gerbes $B_X G_1$ and $B_X G_2$, for some manifold X, with G_1 and G_2 not isomorphic.

3.2 The orbifold case

Definition 3.7. Given the manifold $\mathbb{D}$ (resp. $\mathbb{C}$) and the action of the rotation group μ_n on it, with $n \in \mathbb{Z}_{\geq 1}$, we define $\mathcal{D}_n := [\mathbb{D}/\mu_n]$ (resp. $\mathcal{C}_n := [\mathbb{C}/\mu_n]$).

Theorem 3.2.1. *Let $\mathcal{M}$ be a DM-analytic orbifold whose coarse moduli space is $\mathbb{D}$ (resp. $\mathbb{C}$), with at most one orbifold point. Then, $\mathcal{M}$ is isomorphic to $\mathcal{D}_n$ (resp. $\mathcal{C}_n$) for some $n \in \mathbb{Z}_{\geq 1}$.*

Proof. Assume that the coarse moduli space of $\mathcal{M}$ is $\mathbb{D}$ (the other case is proven analogously).
If $\mathcal{M}$ has no orbifold points, then all its inertia groups are trivial, hence $\mathcal{M}$ is simply a manifold (Corollary 1.4.2). This means that $\mathcal{M} \cong \mathbb{D} \cong \mathcal{D}_1$.
Suppose $\mathcal{M}$ has precisely one orbifold point, say m. Since it is Deligne-Mumford, there exists an open neighbourhood $\mathcal{U}$ of m in $\mathcal{M}$ such that $\mathcal{U} \cong [V/G]$, for some simply connected manifold V and some action of the finite group $G = I_m$. By Theorem 2.3.2, the morphism $\omega_m : I_m \to \pi_1(\mathcal{U}, m)$ is injective (in fact, it is also an isomorphism, thanks to Remark 2.14). But by Van Kampen

$$\pi_1(\mathcal{M}) \cong \pi_1(\mathcal{M} \setminus \{m\}) *_{\pi_1(\mathcal{U} \setminus \{m\})} \pi_1(\mathcal{U}) \cong \pi_1 \mathcal{U}$$

where the second isomorphism follows from the fact that both $\mathcal{M} \setminus \{m\}$ and $\mathcal{U} \setminus \{m\}$ are isomorphic to a punctured disc (since they are orbifolds with only orbifold point m). Hence $I_m \to \pi_1(\mathcal{M}, m)$ is also injective. Since I_x is trivial for any $x \neq m$, $I_x \to \pi_1(\mathcal{M}, x)$ is injective $\forall x \in \mathcal{M}$, hence $\mathcal{M}$ is uniformizable (again, by Theorem 2.3.2). By the Uniformization Theorem of Riemann Surfaces, its universal cover has to be either $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}^1(\mathbb{C})$. But $\mathbb{C}$ and $\mathbb{P}^1(\mathbb{C})$ cannot occur, for otherwise we would have surjections $\mathbb{C} \to \mathcal{M} \to \mathcal{M}_{mod} \cong \mathbb{D}$ or $\mathbb{P}^1(\mathbb{C}) \to \mathcal{M} \to \mathcal{M}_{mod} \cong \mathbb{D}$, which cannot happen. Hence, $\mathcal{M} \cong [\mathbb{D}/H]$. Since $\mathcal{M}$ has a unique orbifold point, the action has a unique orbit $\mathcal{O}$ whose elements have non-trivial stabilizers. By removing this orbit from $\mathbb{D}$, we find a covering space for the punctured disk $\mathbb{D}^*$. But the only possible covering spaces for $\mathbb{D}^*$ are $\mathbb{D}$ and $\mathbb{D}^*$, hence $\mathcal{O}$ is either empty or a singleton. But $\mathcal{O}$ has to be non-empty, since we assumed that there is at least one orbifold point, so it is just a point. After some change of coordinates, we can assume that it is the center of the disc. Now, H is a finite group acting on a disk and fixing only the center, hence it has to be a rotation group μ_n for some $n \in \mathbb{Z}_{\geq 2}$, thanks to the Schwarz Lemma. It follows that $\mathcal{M} \cong [\mathbb{D}/\mu_n] \cong \mathcal{D}_n$. □

Remark 3.6. Thanks to the previuos proposition, one can now uniquely define an orbifold curve $\mathcal{M}$ simply by giving its coarse moduli space $\mathcal{M}$

(which has to be a 1-dimentional complex manifold, i.e. a Riemann Surface), and a finite collection of points on it (the orbifold points), together with some positive integers (the orders of each orbifold point).

Remark 3.7. If an orbifold curve $\mathcal{M}$ has at least one orbifold point m, it cannot be simply connected. Indeed, one can just apply Van Kampen and use an induction on the orbifold points.

Remark 3.8. In Section 1.7 we contructed the coarse moduli space of an analytic stack $\mathcal{M}$: if $[R \rightrightarrows X]$ is the groupoid associated with $\mathcal{M}$, then $\mathcal{M}_{mod} := X/\sim_R$, as a topological space with the quotient topology. In the case of orbifold curves, the coarse moduli space inherits a natural structure of complex manifold. Indeed, since it is an orbifold, there is an open dense substack equivalent a manifold, and this gives us a complex structure away from the orbifold points. Given an orbifold point $m \in \mathcal{M}$, by the Theorem there is an open neighbourhood of m given by $\mathcal{D}_n \cong [\mathbb{D}/\mu_n]$. Its coarse moduli space (around m) is just the naive quotient $\mathbb{D}/\mu_n$, which has a natural structure of Riemann Surface induced by the one on $\mathbb{D}$.

Remark 3.9. In Remark 3.3 we showed that the points of the coarse moduli space of a weighted projective line $\mathcal{P}(m,n)$ are in bijection with the points of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. If $(m,n) = 1$, $\mathcal{P}(m,n)$ is an orbifold, hence its coarse moduli space has a Riemann Surface structure, thanks to Remark 3.8. Since the coarse moduli space of a simply connected Riemann Surface is simply connected (we will give an argument for this later on this chapter), we get that $\mathcal{P}(m,n)_{mod}$ is a simply connected Riemann Surface whose points are in correspondence with the ones of $\mathbb{P}^1_{\mathbb{C}}$. Hence, it has to be conformally equivalent to $\mathbb{P}^1_{\mathbb{C}}$.

Theorem 3.2.2. *Let $\mathcal{M}$ be a simply connected orbifold curve. Then, $\mathcal{M}$ is equivalent to either $\mathbb{C}$, $\mathfrak{H}$ or $\mathcal{P}(m,n)$, for some coprime $m,n \in \mathbb{Z}_{\geq 1}$.*

Proof. First of all, note that $\mathcal{M}_{mod}$ must also be simply connected, for otherwise a non trivial covering space $\mathcal{M}_{mod} \to \mathcal{M}_{mod}$ would pull back to a non trivial covering space for $\mathcal{M}$ which contradicts the simply connectedness of $\mathcal{M}$. Indeed, a non trivial covering space for $\mathcal{M}_{mod}$ would give us a cartesian square

$$\begin{tikzcd} \tilde{\mathcal{M}} \arrow[r] \arrow[d] & \tilde{\mathcal{M}}_{mod} \arrow[d] \\ \mathcal{M} \arrow[r, "\pi_{mod}"'] & \mathcal{M}_{mod} \end{tikzcd}$$

and the fact that $\mathcal{M}$ is simply connected implies that $\tilde{\mathcal{M}} \to \mathcal{M}$ has a section. This gives us a map

$$\begin{tikzcd} \tilde{\mathcal{M}} \arrow[r] \arrow[d] & \tilde{\mathcal{M}}_{mod} \arrow[d] \\ \mathcal{M} \arrow[r, "\pi_{mod}"'] \arrow[ur, dotted] & \mathcal{M}_{mod} \end{tikzcd}$$

But $\tilde{\mathcal{M}}_{mod}$ is a manifold, hence the map $\mathcal{M} \to \tilde{\mathcal{M}}_{mod}$ must factor through the coarse moduli space. This means that the non-trivial covering space has a section

Now, by the Uniformization Theorem for Riemann Surfaces, $\mathcal{M}_{mod}$ has to be isomorphic to either $\mathbb{C}$, $\mathfrak{H}$ or $\mathbb{P}^1_{\mathbb{C}}$. If there are no orbifold points, the statement follows from the Uniformization Theorem for Riemann Surfaces (Theorem 3.0.1). Suppose there are some orbifold points $P_1, ..., P_n$. Then, the coarse moduli space of $\mathcal{M}$ cannot be neither $\mathbb{C}$ or $\mathfrak{H}$, otherwise, by Remark 3.7 the fundamental group would not be trivial. Hence, the coarse moduli space is $\mathbb{P}^1_{\mathbb{C}}$. Let $X := \mathcal{M} \setminus \{P_1, ..., P_n\}$. Then, X is just a manifold, isomorphich to $\mathbb{P}^1_{\mathbb{C}}$ minus n points. Its fundamental group has the form:

$$\pi_1(X) \cong < \rho_1, ..., \rho_n | \Pi \rho_i = 1 >$$

where the ρ_i's are loops around the P_i's. By sticking the orbifold points back in, we introduce the relations $\rho_i^{n_i} = 1 \ \forall i$, hence

$$1 = \pi_1(\mathcal{M}) \cong < \rho_1, ..., \rho_n | \Pi \rho_i = 1, \rho_i^{n_i} = 1 \ \forall i >$$

We have the following possibilities:

- If there is precisely one orbifold point, of order n, then $\mathcal{M}$ is naturally isomorphic to $\mathcal{P}(1,n) \cong \mathcal{P}(n,1)$. Indeed, an automorphism of $\pi_{mod}(\mathcal{P}(m,n)) \cong \mathbb{P}^1_{\mathbb{C}}$ pull backs to an automorphism of $\mathcal{P}(m,n)$, hence we can change coordinates on $\mathbb{P}^1_{\mathbb{C}}$ and send the orbifold point to the origin (resp. to ∞). We know that $\mathcal{P}(1,n)$ and $\mathcal{P}(n,1)$ have coarse moduli space $\mathbb{P}^1_{\mathbb{C}}$ and only one orbifold point at the origin (resp. at ∞), and by Remark 3.6 this is enough to conclude

- If there are two orbifold points, of order m and n, then

  $$\pi_1(\mathcal{M}) \cong < \rho, \tau | \rho\tau = 1, \rho^m = 1, \tau^n = 1 > \cong < \rho | \rho^d = 1 >$$

  where $d = gcd(m,n)$, hence the only possibility for $\pi_1(\mathcal{M})$ to be trivial is that $d = 1$, i.e. m and n are coprime. Again, Remark 3.6 allows us to conclude.

- If there are more than two orbifold points, $\pi_1(\mathcal{M})$ has no chance of being trivial (as before, one can exibit an element with order ≥ 1).

□

3.3 The general case

We can now discuss the general proof for Theorem 3.0.2.

Proof. (Theorem 3.0.2)
Assume $\mathcal{M}$ is a simply connected DM-analytic curve. Then, by Proposition 3.1.1, $\mathcal{M}$ is an H-gerbe over some orbifold $\mathcal{U}$, for some finite group H. The orbifold $\mathcal{U}$ has to be simply connected. Indeed, if $\tilde{\mathcal{U}}$ is a universal cover for $\mathcal{U}$ and $\tilde{\mathcal{M}} := \tilde{\mathcal{U}} \times_{\mathcal{U}} \mathcal{M}$, then $\tilde{\mathcal{M}}$ is a universal cover for $\mathcal{M}$. But $\mathcal{M}$ is simply connected, hence $\tilde{\mathcal{M}} \cong \mathcal{M}$, so also $\tilde{\mathcal{U}} \cong \mathcal{U}$. By Theorem 3.2.2, $\mathcal{U}$ is isomorphic to either $\mathbb{C}$, $\mathfrak{H}$ or $\mathcal{P}(m,n)$, for some coprime $m, n \in \mathbb{Z}_{\geq 1}$. If $\mathcal{U}$ is $\mathbb{C}$ or $\mathfrak{H}$, then $\mathcal{M} \cong \mathcal{U}$. Indeed, by Remark 3.4, $\mathcal{M} \cong U \times [*/H]$ and, since $\mathcal{M}$ is simply connected, $[*/H]$ also has to be, and this can only happen if

H is trivial (thanks to Theorem 2.3.2).If $\mathcal{U} \cong \mathcal{P}(m,n)$, with m, n coprime positive integers, then $\mathcal{M}$ has to be isomorphic to $\mathcal{P}(m',n')$, for some m', n' arbitrary positive integers (Corollary 6.3 of [1]). □

As for Riemann Surfaces, we have:

Definition 3.8. A Deligne-Mumford curve is said to be

- *Euclidean*, if its universal cover is $\mathbb{C}$
- *Hyperbolic*, if its universal cover is $\mathfrak{H}$
- *Spherical*, if its universal cover is $\mathcal{P}(m,n)$

Example 3.4. As pointed out in Example 3.1, the moduli stack of elliptic curves

$$\mathcal{M} \cong \mathcal{M}_{1,1} = [\mathfrak{H}/SL_2(\mathbb{Z})]$$

is a $\mathbb{Z}/2\mathbb{Z}$-gerbe over $\mathcal{U} := [\mathfrak{H}/\mathbb{P}SL_2(\mathbb{Z})]$. Its universal cover is $\mathfrak{H}$, and the covering map is given by the canonical atlas $\mathfrak{H} \to \mathcal{U}$. Hence, $\mathcal{M}_{1,1}$ is hyperbolic.

Definition 3.9. A Deligne-Mumford analytic curve is said to be *uniformizable* if its universal cover is a manifold (in fact, a Riemann Surface, thanks to Example 1.14).

Remark 3.10. It follows from Theorem 2.3.2 and Remark 2.13 that any uniformizable Deligne-Mumford curve $\mathcal{M}$ has the form $[\tilde{\mathcal{M}}/\pi_1(\mathcal{M})]$, where $\tilde{\mathcal{M}}$ is a Riemann Surface. More explicitly:

- Any Euclidean DM-curve $\mathcal{M}$ has the form $[\mathbb{C}/\pi_1(\mathcal{M})]$
- Any hyperbolic DM-curve $\mathcal{M}$ has the form $[\mathfrak{H}/\pi_1(\mathcal{M})]$
- Any spherical DM-curve $\mathcal{M}$ with universal cover $\mathbb{P}^1_{\mathbb{C}}$ has the form $[\mathbb{P}^1_{\mathbb{C}}/\pi_1(\mathcal{M})]$

Remark 3.11. This is a very nice result, because we know how such actions are made. Indeed:

- $Aut(\mathbb{C}) = \{z \mapsto az + b | a, b \in \mathbb{C}, a \neq 0\}$
- $Aut(\mathfrak{H}) = \mathbb{P}SL_2(\mathbb{R})$
- $Aut(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{P}SL_2(\mathbb{C})$

(see, for example, Theorem 4.17.3 of [16]). Hence, studying the discrete subgroups of these groups is enough to classify all the uniformizable DM-analytic curves.

Chapter 4

Differentiable stacks with proper diagonal

This chapter deals with an approach to a generalization of the Deligne-Mumford result discussed in Chapter 1 (Theorem 1.4.1). From now on, we are going to change our setting, and work with differentiable real manifolds, instead of with the complex ones. The reason why we do this will become clear later on in this chapter (see Example 4.1).

All the theory developed so far for complex manifolds can be rewritten in terms of real manifolds. For instance, a *prestack* of groupoids over the category **Diff** of differentiable real manifolds is going to be a pseudo-functor

$$\mathbf{Diff} \to \mathbf{Gpd}.$$

The gluing axioms are the same as in Definition 1.2. A *differentiable stack* is going to be the data of a stack (of groupoids over **Diff**) $\mathcal{M}$ together with an *atlas*, i.e. a surjective submersion $X \to \mathcal{M}$, where X is a real manifold. As for complex manifolds, the property of being a submersion has to be checked after a base change with any other $Y \to \mathcal{M}$, with $Y \in \mathbf{Diff}$.

The theory in this chapter will regard differentiable stacks with proper diagonal. Let us start with a result:

Lemma 4.0.1. *Let $\mathcal{M}$ be a differentiable stack with proper diagonal, and let $X \to \mathcal{M}$ be an atlas for it. Then, the inertia group I_m at any point $m \in \mathcal{M}$ is a compact Lie group.*

Proof. Lift a point $m \in \mathcal{M}$ to a point $x \in X$ such that $\pi(x) = m$. Let $K := I_x = s^{-1}(x) \cap t^{-1}(x) = x \times_{\mathcal{M}} x$ be the inertia group at m. We want to show that K can be equipped with the structure of a manifold.

Consider the cartesian square

$$\begin{tikzcd} X \times_{\mathcal{M}} X \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

Since π is a surjective submersion, s and t also are. Thanks to the *submersion theorem* for differentiable manifolds, $x \times_{\mathcal{M}} X = s^{-1}(x)$ can be naturally equipped with the structure of a manifold. We can thus choose a connected open neighbourhood $B_\epsilon(x) \subset \mathbb{R}^m$ of $x \times_{\mathcal{M}} x$ in $x \times_{\mathcal{M}} X$.

Let $\bar{t}$ be the restriction of t to $x \times_{\mathcal{M}} X$. Since t is a submersion, there exists an open subset $U \subset B_\epsilon(x)$ where the rank $rk(\bar{t})$ is constant. By the *rank theorem* (see, for example, Theorem 5.4 of [14]) $\bar{t}|_U^{-1}(y) \subset U$ is a manifold $\forall y \in X$. In particular, there exists $y \in X$ such that $x \times_{\mathcal{M}} y$ has a neighbourhood $V := \bar{t}|_U^{-1}(y)$ which is a manifold. If we are able to *translate* this neighbourhood and make it into a neighbourhood of the identity $id : x \to x \in K$, then we can conclude that the group K has a manifold structure.

Fix an element $\varphi : x \to y$ in V. For any other element $\psi : x \to y$ in V, by precomposing with φ^{-1} we get the sought translation. In fact, $\varphi^{-1}(V)$ it a neighbourhood of the identity $x \to x$, and it is given by a manifold.

Hence, K is a Lie group. The fact that it is compact follows from the fact that the diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is proper

$$\begin{tikzcd}
K \arrow[r, dotted] \arrow[d, hook, dotted] & (x,x) \arrow[d, hook, dotted] \\
R \arrow[r, "(s,t)"] \arrow[d] & X \times X \arrow[d, "(\pi,\pi)"] \\
\mathcal{M} \arrow[r, "\Delta"'] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

□

Remark 4.1. In Chapter 1 we saw that, if an analytic stack has diagonal which is closed and with finite fibers, the inertia groups are finite (complex) Lie groups. The previous Lemma shows that, if a differentiable stack has diagonal which is closed and with compact fibers (i.e. the diagonal is proper), the inertia groups are compact (real) Lie groups.

In the complex case, the Deligne-Mumford result (Theorem 1.4.1) assures that, whenever an analytic stack has an atlas given by a local homeomorphism, if the diagonal is closed and with finite fibers, then the stack is locally (around any point) a quotient stack, by some action of the inertia group at the point (which is a finite Lie group). We are trying to generalize this result in the case of differentiable stacks. The question is the following:

QUESTION: *Is any differentiable stack with proper diagonal locally the quotient stack of an action of some compact Lie group on a real manifold?*

Example 4.1. The claim is false if we work with (complex) analytic stacks. For instance, let $\mathfrak{H}$ be the Poincaré upper half-plane, and let $\mathbb{E}_{\mathfrak{H}}$ be the family of all the elliptic curves. Consider the map

$$\varphi : \mathbb{E}_{\mathfrak{H}} \to \mathfrak{H}$$

associating to any elliptic curve $E_\tau := \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ the point $\tau \in \mathfrak{H}$.
Take the groupoid

$$[\mathbb{E}_{\mathfrak{H}} \rightrightarrows \mathfrak{H}]$$

with source and target maps given by φ. Let $\mathcal{M}$ be the associated (complex) analytic stack. We thus have a cartesian square:

$$\begin{tikzcd} \mathbb{E}_{\mathfrak{H}} \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \mathfrak{H} \arrow[d, "\pi"] \\ \mathfrak{H} \arrow[r, "\pi"'] & \mathcal{M} \end{tikzcd}$$

In this setting, the QUESTION above cannot be true, since a point $\tau \in \mathfrak{H}$ is mapped by the canonical atlas $\pi : \mathfrak{H} \to \mathcal{M}$ to $[\tau/\mathbb{E}_\tau]$, and there is no way $\mathbb{E}_\tau$ could act in a neighbourhood of the point τ. Indeed, if $\tau \neq \tau'$, the automorphism groups E_τ and E'_τ are non-isomorphic elliptic curves. Hence, there is no neighbourhood of $\pi(\tau)$ which can be of the form $[X/E_\tau]$, since in such a quotient all the stabilizer groups are subgroups of E_τ.

Vice versa, in the real setting, $\mathbb{E}_{\mathfrak{H}}$ is diffeomorphic to $S^1 \times S^1$, and the problem above does not occur.

First steps towards answering the QUESTION

Let $\mathcal{M}$ be a differentiable stack with proper diagonal, $\pi : X \to \mathcal{M}$ being an atlas for it. Let s and t be the source and target maps of the associated groupoid. Fix a point $m \in \mathcal{M}$, and a lift $x \in X$ such that $\pi(x) = m$. By Lemma 4.0.1, the inertia group K at m is a compact Lie group. We would like to show that there exists a neighbourhood $\mathcal{U}$ of m in $\mathcal{M}$ such that $U \cong [\mathcal{U}_x/K]$ for some action of K on an open neighbourhood $\mathcal{U}_x$ of x in X.

Since π is a surjective submersion, s and t also are. Hence, by the submersion theorem, $\mathcal{O}_x := X \times_{\mathcal{M}} x = t^{-1}(x)$ is a manifold, containing K as a submanifold. Let $s_x := s|_{\mathcal{O}_x} : \mathcal{O}_x \to X$. Pick a disc $\mathbb{D}_x \subset X$ containing x which is transversal to s_x, i.e. such that

$$T_x\mathbb{D}_x \oplus im(ds_x) = T_xX.$$

This disc is going to have dimension $dim(\mathbb{D}_x) = dim(\mathcal{M}) + dim(K)$. Indeed,

$$dim(\mathbb{D}_x) = dim(T_x\mathbb{D}_x) = dim(T_xX) - dim(im(ds_x))$$

where $dim(im(ds_x)) = dim(\mathcal{O}_x) - dim(ker(s_x))$.
But

$$dim(ker(ds_x)) = dim(T_{(id:x\to x)}K) = dim(K),$$

while

$$dim(\mathcal{O}_x) = rel.dim(X/\mathcal{M}) = dim(X) - dim(\mathcal{M}).$$

Hence,

$$dim(\mathbb{D}_x) = dim(X) - (dim(X) - dim(\mathcal{M}) - dim(K)) =$$

$$= dim(\mathcal{M}) + dim(K),$$

as we wanted.

We now claim that the map $t_x := t|_{\mathbb{D}_x\times_{\mathcal{M}}X} : \mathbb{D}_x \times_{\mathcal{M}} X \to X$ has surjective differential at the point $(id : x \to x) \in K$. In order to check it, it is enough to show that

$$T_{(id:x\to x)}X \times_{\mathcal{M}} \{x\} + T_{(id:x\to x)}\mathbb{D}_x \times_{\mathcal{M}} X = T_{(id:x\to x)}X \times_{\mathcal{M}} X,$$

which follows directly from a dimension count. Indeed,

$$\begin{cases} dim(T_{(id:x\to x)}X \times_{\mathcal{M}} \{x\}) = dim(X) - dim(\mathcal{M}) \\ dim(T_{(id:x\to x)}\mathbb{D}_x \times_{\mathcal{M}} X) = dim(\mathbb{D}_x) + dim(X) - dim(\mathcal{M}) = dim(K) + dim(X) \\ dim(T_{(id:x\to x)}X \times_{\mathcal{M}} X) = 2dim(X) - dim(\mathcal{M}) \end{cases}$$

The equality now follows from the fact that

$$dim(T_{(id:x\to x)}X \times_{\mathcal{M}} \{x\} \cap T_{(id:x\to x)}\mathbb{D}_x \times_{\mathcal{M}} X) = dim(K),$$

which is true, since $\mathbb{D}_x$ was chosen to be transversal to s_x (see, for example, Theorem 5.5.3 of [15]).

Since we showed that t_x has surjective differential at the point $(id : x \to x) \in K$, we can conclude that there exists a neighbourhood $\mathcal{U}_x$ of x in $\mathbb{D}_x$ such that the composition $\varphi_U : \mathcal{U}_x \hookrightarrow X \to \mathcal{M}$ is a submersion (not necessarily surjective!). Now, we choose a neighbourhood $\mathcal{U}$ of m in $\mathcal{M}$ such that the image of $\varphi_{\mathcal{U}}$ is precisely $\mathcal{U}$. Thus, we have constructed a surjective submersion $\mathcal{U}_x \to \mathcal{U}$, which is an atlas for $\mathcal{M}$ around the point m.

To complete the proof, we would like to find an action of K on $\mathcal{U}_x$ such that $\mathcal{U} \cong [\mathcal{U}_x/K]$. I conjecture that it is possible to do so. More precisely:

Conjecture: *It is possible to shrink $\mathcal{U}_x$ and $\mathcal{U}$ further so that $\mathcal{U} \cong [\mathcal{U}_x/K]$ for some action of K on $\mathcal{U}_x$. If this were true, we would get a cartesian square*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & x \arrow[d] \\ \mathcal{U}_x \arrow[r] & U \end{tikzcd}$$

Let us discuss one example:

Example 4.2. Let $(S^1)^2$ act on $\mathbb{C}^2$ as

$$(t_1, t_2).(z_1, z_2) = (t_1 z_1, t_2 z_2),$$

and consider the quotient stack $\mathcal{M} := [\mathbb{C}^2/(S^1)^2]$. The stabilizers are as follows:

$$Stab_{(S^1)^2}(z_1, z_2) \cong \begin{cases} (S^1)^2 & \text{if } z_1 = z_2 = 0 \\ S^1 & \text{if } z_1 z_2 = 0 \text{ and } z_1 + z_2 \neq 0 \\ 0 & \text{if } z_1 \neq 0 \neq z_2 \end{cases}$$

Hence, every point has compact inertia group. We want to write $\mathcal{M}$, locally around every point, as a quotient stack by some action of the stabilizer group at the point.

- For the points $(z_1, z_2) \in \mathbb{C}^2$, with $z_1 \neq 0 \neq z_2$ (i.e. far from the axis), the action is free, hence the stack is locally a differentiable manifold.

- Around the origin, we can just take $\mathcal{M} = [\mathbb{C}^2/(S^1)^2]$ (or any open substack given by the quotient of the action of $(S^1)^2$ on some ball around the origin).

- Given a point of the form $(z, 0)$, with $z \neq 0$ [and analogously for $(0, z)$], we can take, for example, a slice $X = \mathbb{R}z \times \mathbb{C} \cong \mathbb{R}^3$, where the stabilizer $\{1\} \times S^1 \cong S^1$ acts as

$$t.(rz, w) = (rz, tw),$$

where $r \in \mathbb{R}$, $w \in \mathbb{C}$. The quotient stack $[X/S^1]$ is the open neighbourhood that we are looking for.

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