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Hopf algebras, Hopf monads and derived categories of sheaves

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To the memory of my father, Theodosios V. Aravanis.

Lefkada, 27 March 1961 - Athens, 4 May 2015

The windows

In these dark rooms where I live out
empty days, I circle back and forth
trying to find the windows.
It will be a great relief when a window opens.
But the windows are not there to be found—
or at least I cannot find them. And perhaps
it is better that I don't find them.
Perhaps the light will prove another tyranny.
Who knows what new things it will expose?

C.P. Cavafy [1]

[1] translation of Edmund Keeley and Philip Sherrard, appeared in *Collected poems (Edited George Savvides) of C.P. Cavafy*, published by London: Hogarth Press, 1984

Abstract

The aim of this thesis is to explain how the theory of Hopf monads on monoidal categories can be used to investigate the Hopf algebra object in a category of which objects are complexes of sheaves on a smooth complex projective variety.

In particular, we associate to a smooth complex projective variety X the category of orbits of the bounded derived category of coherent sheaves of the variety under the double shift functor and discuss its structure. We explain why the derived functors on the level of bounded derived categories of coherent sheaves induce functors on the level of categories of orbits. We prove that if the variety is of even dimension and has trivial canonical bundle, then the Serre functor on the orbit category is trivialised. As a direct application of this, we obtain functors on the level of orbit categories which have the same left and right adjoint functor.

Next, we work in a general categorical level considering rigid monoidal categories and a pair of adjoint functors defined between them. Moreover, we assume that the right adjoint is a strong symmetric monoidal functor and has a right quasi-inverse which is also a strong monoidal functor. We prove that every such pair of adjoint functors defines an augmented Hopf monad. From the theory of augmented Hopf monads of Brugières, Lack and Virelizier we obtain that these Hopf monads are equivalent to central Hopf algebras.

We explain why the orbit category of an even dimensional smooth complex projective variety X with trivial canonical bundle is a rigid symmetric monoidal category. In addition we explain why the diagonal embedding $\Delta : X \to X \times X$ of the variety gives rise to a pair of adjoint functors on the level orbit categories such that the right adjoint is strong symmetric monoidal and has a right quasi-inverse. As a result, we obtain a Hopf monad on the orbit category and pin down the Hopf algebra with explicit morphisms.

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Contents

Introduction

The aim of this thesis is to investigate the Hopf algebra object in a symmetric monoidal category of which objects are bounded complexes of sheaves and its morphisms are formally inverting quasi-isomorphisms. In particular, the category in which we are interested in is obtained as the category of orbits of the bounded derived category of coherent sheaves on a smooth complex projective variety under an autoequivalence, which for our purposes is the double shift functor. In order to pin down the Hopf algebra object we apply the theory of Hopf monads as developed in [8] and [7]. Hopf monads consist of generalisations of Hopf algebras to a non-braided setup.

In the following, we start our narrative with Hopf algebras with the scope to highlight those properties which led various mathematicians to introduce generalisations of Hopf algebras. Then, we discuss Hopf monads trying to make clear the analogy between Hopf algebras and Hopf monads. After that we discuss which is our contribution in this thesis, stating what was known before and which are our results. In the final part of this introduction, we explain where this story is heading or better, our expectations of how our results could be useful.

Hopf algebras

Hopf algebras are algebraic objects which emerged in the field of topology in the 1940s to investigate problems related to the cohomology of compact Lie groups, see [10]. In the last four decades, Hopf algebras have become extremely popular due to work of Drinfel'd and Jimbo on quantum groups and their applications in low dimensional topology, representation theory and topological quantum field theory.

A *Hopf algebra H* over a field k of characteristic zero is a bialgebra over k with an invertible map $S : H \to H$, called the antipode. Here, *bialgebra* means that H is given an algebra and a coalgebra structure in a compatible way. In other words, H has an associative product map, a unit map, a coassociative coproduct map and a counit map such that the product and the unit are morphisms of coalgebras or equivalently the coproduct and the counit are morphisms of algebras. Examples of Hopf algebras are the group algebra $k[G]$ of a finite group G and the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over k.

Bialgebras and Hopf algebras over a field k are characterised by the structure of the associated category of modules. In particular, if we consider a bialgebra H over k then the associated category of left H-modules is a monoidal category, see [20, Proposition XI.3.1]. If moreover on the bialgebra H is given an invertible antipode and so H is a Hopf algebra, then the associated category of the modules of this Hopf algebras is a monoidal category with left and right duals, see [20, Example 1, p. 347].

If we replace the category of k-vector spaces with a braided monoidal category $\mathscr{C}$ then one can define analogous notions of a bialgebra and of a Hopf algebra in $\mathscr{C}$ and give analogous characterisations for the associated categories of modules, see [8, Chapter 6]. In particular, if $\mathscr{C}$ is a braided monoidal category and H is a bialgebra in $\mathscr{C}$ then H is a Hopf algebra object in $\mathscr{C}$ if and only if the category of representations of H is a monoidal category with duals, see [38, Lemma 6.1] for details.

Street in [37] gave another characterisation of Hopf algebras in braided monoidal categories with the use of the notion of the *fusion operator*, generalising in an abstract categorical notions of the work of Baaj and Skandalis [5] on operator algebras. To be more precise, for a bialgebra $(H, m, u, \delta, \epsilon)$ in a braided monoidal category with product m, unit u, coproduct δ and counit ϵ, the fusion operator of H is the morphism $V : H \otimes H \to H \otimes H$ defined by $V = (\mathrm{id} \otimes m) \circ (\delta \otimes \mathrm{id})$ see [37, Proposition 1.1]. If moreover H is a Hopf algebra with an invertible antipode $S : H \to H$ then the inverse of the fusion operator is given by $V^{-1} = (\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id})$, see [37, Proposition1.2].

Now, a natural question which arises is if there is a generalisation of the notion of a Hopf algebra in monoidal categories which are not braided. The answer to this question is affirmative and is the notion of a Hopf monad which was introduced by Moerdijk [30] and studied in a deeper level by Brugières and Virelizier in [8] and in their joint work with Lack [7].

Hopf monads

Moerdijk in [30] based on the correspondence between bialgebras over a field k and monoidal structures on the associated category of modules, generalised the notion of a bialgebra to non-braided monoidal categories. Instead of an algebra in some braided category, he considered a *monad* on the monoidal category $\mathscr{C}$. A *monad* on a category is an endofunctor of the category with natural transformations called the product and the unit which satisfy similar axioms to that of an algebra. In an equivalent way, a monad can be defined as an algebra object of the category of endofunctors of $\mathscr{C}$. One can obtain monads on a monoidal category $\mathscr{C}$ from an adjunction between functors or an algebra object of the category. More specifically, if $\mathscr{C}$ is a monoidal category and A is an algebra in $\mathscr{C}$ then the endofunctor $T' = A \otimes -$ is a monad on $\mathscr{C}$. Also, if $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ is a pair of adjoint functors between monoidal categories with U being the right adjoint, then the endofunctor $T = UF$ on $\mathscr{C}$ is a monad on $\mathscr{C}$. For a monad on a category $\mathscr{C}$, one can define an action of the monad on the objects of the category.

Now, the analogous notion of a bialgebra on the level of monads is the notion of a *bimonad* or otherwise of an *opmonoidal monad*. In other words, a bimonad is the monad in the category of opmonoidal functors and opmonoidal natural transformations. Examples of bimonads on monoidal categories are obtained either from bialgebras in braided monoidal categories or from adjunctions between functors. In particular, if $\mathscr{C}$ is a monoidal category and A is a bialgebra in $\mathscr{C}$ then the monad $T' = A\otimes -$ has the structure of a bimonad on $\mathscr{C}$. Moreover, if we consider an adjunction $F \dashv U$ between functors defined on monoidal categories such that the right adjoint U is a strong monoidal functor, then the monad $T = UF$ is a bimonad. Moerdijk proved in [30, Theorem 7.1] that bimonad structures on a monad T corresponds to monoidal structures on the associated category of the modules of the monad T.

The analogue of an antipode for a bimonad was established by Brugières and Virelizier in [8]. In particular, they considered bimonads on monoidal categories with left duals, or right duals or both and proved in [8, Theorem 3.8] that a bimonad on a monoidal category $\mathscr{C}$ with left duals (respectively with right duals) has a left antipode (respectively a right antipode) if and only if the category of T-modules is monoidal and has left duals (respectively right duals). Then, they defined the notion of a Hopf monad as a bimonad on a monoidal category $\mathscr{C}$ with antipodes. A further generalisation of the definition of a Hopf monad on a monoidal category was given in [7], which was based on the notion of the fusion operator by Street.

Although a Hopf algebra in a braided monoidal category gives rise to a Hopf monad on the category, it is not true in general that all Hopf monads are coming from Hopf algebras. Under which circumstances a Hopf monad is coming from a Hopf algebra was studied in [7]. Of particular interest is [7, Corollary 5.9] where it is proved that the only Hopf monads which are obtained from a Hopf algebra are those which are defined on braided monoidal categories.

Contribution of this thesis

The aim of this thesis is to bring together the theory of Hopf monads of Brugières, Lack and Virelizier with derived categories of coherent sheaves. Consider a smooth complex projective variety X and let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X. In the case where X is a complex manifold, an example of a coherent sheaf is a holomorphic vector bundle. Let $C(X)$ be the category of cochain complexes in $\mathrm{Coh}(X)$. Then, bounded derived category of coherent sheaves $D^b\mathrm{Coh}(X)$ is obtained from $C(X)$ by formally inverting quasi-isomorphisms in $\mathrm{Coh}(X)$. Its objects are essentially cochain complexes of coherent sheaves on X. The ultimate goal is to investigate the Hopf algebra object in the bounded derived category of coherent sheaves on a smooth complex projective variety X.

Now, the category $D^b\operatorname{Coh}(X)$ is a symmetric monoidal category, with the monoidal product to be the derived tensor product and monoidal unit to be the structure sheaf of the variety X. Moreover, $D^b\operatorname{Coh}(X)$ is a category with duals. In particular, for every complex $\mathscr{E}^\bullet$ of $D^b\operatorname{Coh}(X)$ its right dual is defined to be the object $(\mathscr{E}^\bullet)^\vee := R\mathcal{H}\mathrm{om}(\mathscr{E}^\bullet, \mathscr{O}_X)$ where $R\mathcal{H}\mathrm{om}$ denotes the right derived functor of the internal hom functor $\mathcal{H}\mathrm{om}$. Then, a natural question to ask is if $D^b\operatorname{Coh}(X)$ is the module category of a Hopf algebra and which is explicitly this Hopf algebra.

A candidate for the Hopf algebra follows from the work of Kapranov [17] and the subsequent work of Markarian [29], Ramadoss [31] and Roberts and Willerton [33] where it is proved that for a complex manifold X, the shifted holomorphic tangent sheaf $\mathscr{T}_X[-1]$ of X is a Lie algebra object in $D^b\operatorname{Coh}(X)$ and the Hochschild cochain complex which is defined by $\mathscr{U} := R\pi_{1,*}R\mathcal{H}\mathrm{om}(\mathscr{O}_X, \mathscr{O}_X)$ is the universal enveloping algebra object for the Lie algebra $\mathscr{T}_X[-1]$. Moreover, Roberts and Willerton proved that the object $\mathscr{U}$ acts on all the objects of $D^b\operatorname{Coh}(X)$. Markarian [29] and Ramadoss [31] went through explicit calculations considering the Bar resolution of $R\mathcal{H}\mathrm{om}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$, unlike Roberts and Willerton [33] who worked at a more categorical level. Furthermore, in [33, Section 7.7] is given an equivalent definition of the object $\mathscr{U}$ as $\Delta^! R\Delta_* \mathscr{O}_X$ where $\Delta : X \to X \times X$ is the diagonal embedding, $R\Delta_*$ is the derived pushforward functor and $\Delta^!$ is the right adjoint of $R\Delta_*$ obtained by Grothendieck Verdier duality. In other words, one can say that the object $\mathscr{U}$ is obtained applying the composite functor $\Delta^! R\Delta_*$ on the structure sheaf of the complex manifold X.

It is then natural to ask if the object $\mathscr{U}$ of $D^b\operatorname{Coh}(X)$ is a Hopf algebra object. Willerton in [40] hinted that the theory of Hopf monads on monoidal categories with duals as developed by Brugières and Virelizier [8] can be applied on the case of $D^b\operatorname{Coh}(X)$. Moreover, Calaque, Caldararu and Tu proved in [9, Section 3] that the endofunctor $\mathscr{U} \otimes -$ of $D^b\operatorname{Coh}(X)$ is a bimonad.

Although, $D^b\operatorname{Coh}(X)$ has nice properties, i.e. is a symmetric monoidal category with duals, is the module category of the Lie algebra $\mathscr{U}$ and the functor $\Delta^! R\Delta_*$ is the monad obtained by the adjunction $R\Delta_* \dashv \Delta^!$, it is not straightforward that $\Delta^! R\Delta_*$ is a bimonad since $\Delta^!$ is not a strong monoidal functor. To overpass this difficulty, we considered the category $D^{\mathrm{or}}(X)$, called the orbit category, which is obtained as the fixed points category of $D^b\operatorname{Coh}(X)$ by the functor $[2] : D^b\operatorname{Coh}(X) \to D^b\operatorname{Coh}(X)$ following the work of Keller [21, 22]. The category $D^{\mathrm{or}}(X)$ has as objects the same as $D^b\operatorname{Coh}(X)$ and the hom spaces are of the form

$$\operatorname{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{D^b\operatorname{Coh}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[2n])$$

where $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ are objects of $D^{\mathrm{or}}(X)$. The category $D^{\mathrm{or}}(X)$ is a symmetric monoidal category with duals and for any object $\mathscr{E}^\bullet$ of $D^{\mathrm{or}}(X)$ it is true that $\mathscr{E}^\bullet[2] \cong \mathscr{E}^\bullet$. Moreover, if $f : X \to Y$ is a smooth map between smooth projective varieties, then the derived functors which are defined on the level of derived categories of coherent sheaves induce functors on the level of orbit categories and satisfy the same adjunctions.

In Section 7.1, we consider X to be an even dimensional smooth complex projective variety with trivial canonical bundle and let $D^{\mathrm{or}}(X)$ the associated orbit category of X. The Serre functor of $D^b \mathrm{Coh}(X)$ induces a Serre functor on $D^{\mathrm{or}}(X)$ and in the case at hand, the Serre functor is isomorphic to the identity functor, see Lemma 7.1.13. As a result, we obtain that the right adjoint $\Delta^!$ of $R\Delta_*$ is isomorphic to $L\Delta^*$ on the level of orbit categories, see Proposition 7.1.14. Symbolically we have $\Delta^! \simeq L\Delta^*$ and so, we obtain the adjunction $R\Delta_* \dashv L\Delta^*$. Now, $L\Delta^*$ is a strong monoidal functor and applying the theory of Hopf monads, the endofunctor $L\Delta^* R\Delta_*$ of $D^{\mathrm{or}}(X)$ is a Hopf monad, see Proposition 7.1.15. Since we are interested in investigating the Hopf algebra object in $D^{\mathrm{or}}(X)$, we prove also that $L\Delta^* R\Delta_*$ is an augmented Hopf monad on $D^{\mathrm{or}}(X)$ with left regular augmentation map. Then, by the correspondence between augmented Hopf monads with left regular augmentation maps and Hopf algebras in the centre of the category, as established in [7, Theorem 5.17 and Corollary 5.18] gives that the object $L\Delta^* R\Delta_* \mathscr{O}_X$ is a Hopf algebra object in $D^{\mathrm{or}}(X)$ and $D^{\mathrm{or}}(X)$ is a module category of the Hopf algebra of $\mathscr{U}$, see Theorem 7.3.1.

Our proof of the fact that $L\Delta^* R\Delta_*$ is an augmented Hopf monad relies on the structure of the functor $L\Delta^* R\Delta_*$. The functor $L\Delta^*$ is not only a strong monoidal functor but moreover has a right quasi inverse. In particular, if $\pi_1 : X \times X \to X$ is the projection map on the first coordinate, then its derived pullback $L\pi_1^* : D^b \mathrm{Coh}(X) \to D^b \mathrm{Coh}(X \times X)$ is defined on the level of bounded derived categories of sheaves which induce a map on the level of orbit categories $L\pi_1^* : D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(X \times X)$. Then, it is true that

$$L\Delta^* \circ L\pi_1^* \cong L(\pi \circ \Delta)^* = \mathrm{id}^* = \mathrm{id}.$$

In Section 7.2 we prove that for a pair of adjoint functors $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ defined between symmetric monoidal categories with duals such that the right adjoint is a strong monoidal functor and has a right quasi-inverse, then, the functor $T = UF$ on $\mathscr{C}$ is an augmented Hopf monad.

A natural question which follows then is what could be the use of $\mathscr{U}$ as a Hopf algebra in the $D^{\mathrm{or}}(X)$. Our expectation and our ultimate goal are to get more insight towards a rigorous formulation of the Rozansky-Witten topological quantum field theory. It is Rozansky-Witten theory which motivated Roberts and Willerton to study the object $\mathscr{U}$ and the results of [33]. In the following, we review some basics of this physical theory.

Rozansky-Witten theory

Rozansky-Witten theory [34] is a three-dimensional topological sigma model of which path integral integrates over all maps from a three dimensional topological manifold M to a hyperkähler manifold X. A hyperkähler manifold is a Riemannian manifold of real dimension $4n$, with three compatible complex structures and a holomorphic symplectic form. An example of a hyperkähler manifold is a $K3$ surface which is defined to be a compact complex surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$.

The invariant which assigns this sigma model to a three-manifold M is the partition function $Z_X(M)$ of the theory, also called *path integral*, which is not well defined from a mathematical point of view. An explicit perturbative expansion of the partition function over all Feynman diagrams, which in this case are just trivalent graphs, had been made by Rozansky and Witten and has the form

$$Z_X(M) = \sum_{\Gamma} b_\Gamma(X) I_\Gamma(M)$$

where Γ are trivalent graphs, $b_\Gamma(X)$ are complex-valued functions depending on X and $I_\Gamma(M)$ are integrals depending on M. It is the complex-valued functions $b_\Gamma(X)$ which were studied by Roberts and Willerton in [33] in a categorical level with the use of the derived category of sheaves on a holomorphic symplectic manifold.

Moreover, Rozansky and Witten conjectured that the Hilbert space $\mathcal{H}_{\Sigma_g}$ which is assigned to a genus g Riemann surface Σ_g is given by

$$\mathcal{H}_{\Sigma_g} = \sum_{q=0}^{\dim_{\mathbb{C}} X} H^q_{\bar{\partial}}\big(X, (\textstyle\bigwedge^* T_X)^{\otimes g}\big).$$

where T_X is the holomorphic tangent bundle on X. This conjecture was verified in [34] for Riemann surfaces of genus zero and genus one and for X being a $K3$ surface.

An interesting question which can be asked and still there is no clear answer is what a rigorous mathematical topological quantum field theory of Rozansky-Witten theory would look like. In the following, we review the axiomatic definition of a topological quantum field theory.

Mathematical definition of a topological quantum field theory

A simple definition of a *topological quantum field theory* in any dimension n, due to Atiyah [4], is as a rule which assigns a finite dimensional vector space $\tau(N)$ to a $(n-1)$-dimensional oriented closed manifold N and a linear map $\tau(N_1) \to \tau(N_2)$ to a n-dimensional compact oriented cobordism M from N_1 to N_2, i.e. a n-dimensional manifold M such that its boundary ∂M can be written as $\partial M = \overline{N_1} \sqcup N_2$ (where $\overline{N_1}$ is N_1 with reversed orientation). Considering any compact oriented closed n-dimensional manifold as a cobordism from the empty set to the empty set, this rule should assign a complex number. This number is called in physics terms the *partition function* of the topological quantum field theory. In an equivalent way, a topological quantum field theory in dimension n is defined as a symmetric monoidal functor $\mathscr{Z}$ from the cobordism category $\mathrm{Cob}_{n,n-1}$ to the category of vector spaces $\mathrm{Vect}(k)$ over a field k. The monoidal product of the cobordism category is the disjoint union and the monoidal product of the category of vector spaces is the tensor product of k-vector spaces. The unit objects are the empty set, considered as $(n-1)$-dimensional manifold and the field k respectively.

Atiyah's axioms [4] lay the foundations for a mathematical study of topological quantum field theories. However, this list of axioms doesn't provide information for manifolds of codimension bigger than one. For example, in dimension three in which we are mainly interested in a topological quantum field theory assigns a number to a closed 3-manifold M^3 and a vector space to a closed surface Σ but provides no information to 1-dimensional or zero-dimensional manifolds i.e. to circles, intervals and points. Much work has been done towards a definition of a topological quantum field theory which extends Atiyah's definition providing information for such submanifolds, see for example the work of Lurie [27], Schommer-Pries [35], Khovanov [24]. In the following, we discuss Khovanov's definition of a 3d-topological quantum field theory with corners [24].

Following Khovanov, a topological quantum field theory with corners in dimension three is a 2-functor F from the 2-category of three dimensional cobordisms with corners to the 2-category of all additive categories. In other words, this 2-functor assigns an additive category $F(K)$ to a closed oriented 1-dimensional manifold K, a functor $F(M)$ to an oriented 2-dimensional cobordism M and a natural transformation of functors $F(N)$ to an oriented 3-dimensional cobordism with corners N. Moreover, Khovanov proved in [24] that for a 3-dimensional topological quantum field theory with corners F, the functor $F(M)$ which is assigned to a 2-dimensional cobordism with boundary M, has simultaneously the same left and right adjoint functor; symbolically if G is a left adjoint to $F(M)$ then $G \dashv F(M) \dashv G$. Another important observation for 3d topological quantum field theories with corners is due to Crane and Yetter. In particular, in [11] it is proved that a 3d-topological quantum field theory with corners assigns to the torus with one disk removed a Hopf algebra object in the category which is assigned to the circle.

In view of the above observations, we are in the situation of having a category which is related with Rozansky-Witten theory through the work of Roberts and Willerton [33] and for which we have identified the Hopf algebra. Our hope and our expectation are that the orbit category would be useful to determine a rigorous three-dimensional topological quantum field theory for Rozansky-Witten theory.

Structure of the thesis

In Chapter 1 we give basic definitions about categories, functors and adjunctions for the reader who is not familiar with this language. Then we discuss additive and abelian categories, Serre functors on arbitrary k-linear Hom-finite categories and orbit categories of additive categories following the work of Keller.

In Chapter 2 we review the basics on closed symmetric monoidal categories and discuss monoidal and opmonoidal functors. Emphasis is given on rigid monoidal categories.

In Chapter 3 we introduce the notion of the derived category of an abelian category and discuss its strucure. Then, we discuss how derived functors on the level of derived categories are defined.

In Chapter 4 we introduce the bounded derived category of coherent sheaves on a smooth complex projective variety. Then, we discuss how to obtain derived functors on the level bounded derived categories of coherent sheaves. Emphasis is given in the various adjunctions which are formed between the derived functors. We conclude the chapter discussing the strongly dualisable complexes in the bounded derived category of coherent sheaves.

In Chapter 5 we review the basics of Hopf algebras in braided monoidal categories. Emphasis is given in the notion of the fusion operator due to Street.

In Chapter 6 we discuss Hopf monads on monoidal categories. We start explaining the notion of a monad and then explain why the notion of a bimonad consists of a generalisation of a bialgebra. Then, we discuss Hopf monads on monoidal categories.

In Chapter 7 which is the main chapter of this thesis we bring together all the above notions. In particular, we discuss the orbit category of the bounded derived category of coherent sheaves on a smooth complex projective variety and we prove that in the case where the variety X is even dimensional and has trivial canonical bundle then the Serre functor trivialises. Moreover, we prove that given a pair of adjoint functors defined between rigid symmetric monoidal categories such that the right adjoint is a strong monoidal functor and has a right quasi-inverse then the associated adjunction monad is an augmented Hopf monad with left regular augmentation map. Finally, apply this general theory to the case of the orbit category of an even dimensional smooth complex projective variety with trivial canonical and pin down the Hopf algebra object in this category.

In the Appendix 8 we give the definition of a triangulated category and the proofs of some technical lemmas.

Chapter 1

On categories and Serre functors

In this thesis, we will be mostly interested in categories and functors with specified structure and so this chapter serves as a first introduction on these topics. In particular in Section 1.1 we recall the basic definitions of a category, of a functor between categories and of a natural transformation between functors. In Section 1.2 we study the important notion of an adjunction between functors. In Section 1.3 we discuss for first time categories with extra structure. In particular, we cover the material of additive categories, of abelian categories and functors between them. Having defined the notion of an additive category we define the very important notion of a Serre functor in Section 1.4. Serre functors will appear again in Chapter 3. We close this chapter with Section 1.5 where we discuss orbit categories following the work of Keller. Our particular interest will be in orbit categories of derived categories, where the latter is discussed in Chapter 3.

1.1 Categories, functors and natural transformations

In this section, we cover the basic notions of a category, a functor and of a natural transformation. Our main reference for this section is [25].

We start with the definition of a category.

Definition 1.1.1. A *category* $\mathscr{C}$ consists of the following data.

1. A collection of objects which will be denoted by $\mathrm{Ob}(\mathscr{C})$. Its elements will be denoted by X, Y, Z.

2. For any two X and Y of $\mathrm{Ob}(\mathscr{C})$, a collection of sets $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ of which elements are called *maps* or *arrows* from X to Y.

3. For any X, Y and Z of $\mathrm{Ob}(\mathscr{C})$, a map

$$\mathrm{Hom}_{\mathscr{C}}(Y,Z)\times\mathrm{Hom}_{\mathscr{C}}(X,Y)\to\mathrm{Hom}_{\mathscr{C}}(X,Z),\qquad (g,f)\mapsto g\circ f \tag{1.1.1}$$

   called the *composition*.

4. For each object X of $\mathscr{C}$ an element id_X of $\mathrm{Hom}_{\mathscr{C}}(X,X)$ called the *identity* on X.

All the above satisfy the following:

- $(h\circ g)\circ f = h\circ(g\circ f)$ for each $f\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$, $g\in\mathrm{Hom}_{\mathscr{C}}(Y,Z)$ and $h\in\mathrm{Hom}_{\mathscr{C}}(Z,W)$;
- $f\circ\mathrm{id}_X = f = \mathrm{id}_Y\circ f$ for each $f\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

Examples of categories which we will be interested in are the following.

Example 1.1.2. For a field k of characteristic zero, there is a category $\mathrm{Vect}(k)$ of vector spaces over k and linear maps between them.

Example 1.1.3. Let R be a ring. Then, there is a category $\mathrm{Mod}(R)$ of which objects are left R-modules and its morphisms are morphisms between left R-modules.

Example 1.1.4. Ler R be a commutative ring. Then, there is a category $\mathrm{Ch}(R)$ of which objects are $\mathbb{Z}$-graded chain complexes of left R-modules with differetial of degree -1 and the morphisms are maps between chain complexes of left R-modules which commute with the differentials.

Maps between categories are called *functors*. To be explicit, we give the following definition.

Definition 1.1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F:\mathscr{C}\to\mathscr{D}$ assigns an object $F(X)$ of $\mathscr{D}$ to an object X of $\mathscr{C}$ and a morphism $F(f):F(X)\to F(Y)$ of $\mathscr{D}$ to a morphism $f:X\to Y$ in $\mathscr{C}$ such that the following axioms are satisfied:

- $F(\mathrm{id}_X)=\mathrm{id}_{F(X)}$ for each object X of $\mathscr{C}$;
- $F(g\circ f)=F(g)\circ F(f)$ whenever the composition $X\xrightarrow{f}Y\xrightarrow{g}Z$ is well defined.

Maps between functors are called *natural transformations*. To be explicit, we give the following definition.

Definition 1.1.6. A *natural transformation* between functors $F,G:\mathscr{C}\to\mathscr{D}$ is a family of morphisms

$$\phi_X : F(X)\to G(X) \tag{1.1.2}$$

for all X of $\mathscr{C}$ such that the following diagram commutes for all $f:X\to Y$.

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] & F(Y) \arrow[d, "\phi_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Remark 1.1.7. The above diagram is called the *naturality square* of the natural transformation ϕ and will be extremely useful to prove the commutativity of various diagrams.

Remark 1.1.8. Let $\mathscr{C}$ be a category. Then, the endofunctors of $\mathscr{C}$ form a category of which objects are functors $F : \mathscr{C} \to \mathscr{C}$ and its morphisms are natural transformations. This category is denoted by $\mathrm{End}(\mathscr{C})$.

1.2 Adjunctions

In this section we discuss the notion of an adjunction between functors. Adjunctions consist of one of the main notions of this thesis. In particular, it will be the adjunction between functors which will determine the structures that will want to study further. We start recalling the definition.

Definition 1.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ be functors. We say that the functor F is *left adjoint* to U and the functor U is *right adjoint* to F and write $F \dashv U$ if

$$\mathrm{Hom}_{\mathscr{D}}(F(A), B) \simeq \mathrm{Hom}_{\mathscr{C}}(A, U(B)) \tag{1.2.1}$$

naturally in $A \in \mathscr{C}$ and $B \in \mathscr{D}$. An *adjunction* between F and G is a choice of an natural ismorphism (1.2.1).

The following correspondence will be useful for applications. Actually, we will view adjunctions through the prism of the following equivalence.

Theorem 1.2.2 ([25] p.53)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ be functors between these categories. Then the following are equivalent:*

- *adjunctions between F and U with F being a left adjoint to U;*
- *pairs of natural transformations* $(\mathrm{id}_{\mathscr{C}} \xrightarrow{\eta} UF, \quad FU \xrightarrow{\epsilon} \mathrm{Id}_{\mathscr{D}})$ *such that the following diagrams commute for all X of $\mathscr{C}$ and Y of $\mathscr{D}$.*

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FUF(X) \arrow[d, "\epsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} U(Y) \arrow[r, "\eta_{U(Y)}"] \arrow[dr, "\mathrm{id}_{F(Y)}"'] & UFU(Y) \arrow[d, "U(\epsilon_Y)"] \\ & F(Y) \end{tikzcd}$$

Notation 1.2.3. The natural transformation η is called the *adjunction unit* or simply *unit*, the natural transformation ϵ is called the *adjunction counit* or simply *counit* .

We will be interested in sheaves of modules later in this thesis and the proposition below will be helpful. First we fix some notation.

Notation 1.2.4. Let $f : X \to Y$ be a smooth map between topological spaces and denote by $Sh(X)$ and $Sh(Y)$ the categories of sheaves on X and Y respectively. Associated to f is the direct image functor $f_* : Sh(X) \to Sh(Y)$ and the inverse image functor $f^{-1} : Sh(Y) \to Sh(X)$. If $\mathscr{R}$ is a sheaf of rings on X then $f_*\mathscr{R}$ is a sheaf of rings on Y. Moreover, if $\mathscr{S}$ is a sheaf of rings on Y then $f^{-1}\mathscr{S}$ is a sheaf of rings on X. The functors f_* and f^{-1} above induce the following functors, denoted with the same notation

$$f_* : Sh_{\mathscr{R}}(X) \to Sh_{f_*\mathscr{R}}(Y), \quad f^{-1} : Sh_{\mathscr{S}}(Y) \to Sh_{f^{-1}\mathscr{S}}(X).$$

where $Sh_{\mathscr{R}}(X)$ and $Sh_{\mathscr{S}}(Y)$ denote now the categories sheaves of $\mathscr{R}$-modules over X and the sheaves of $\mathscr{S}$-modules over Y. Similarly, we denote by $Sh_{f_*\mathscr{R}}(Y)$ and $Sh_{f^{-1}\mathscr{S}}(X)$ the category of sheaves of $f_*\mathscr{R}$-modules over Y and the category of sheaves of $f^{-1}\mathscr{S}$-modules over X.

Proposition 1.2.5 ([18] Proposition 2.3.3)**.** *Let $f : X \to Y$ be a smooth map between topological spaces and let $\mathscr{S}$ be a sheaf of rings on Y. Then, there exists a natural isomorphism*

$$\mathrm{Hom}_{f^{-1}\mathscr{S}}(f^{-1}A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{S}}(A, f_*B)$$

where $\mathscr{S}$ is a sheaf of rings over Y, A is an object of $Sh_{\mathscr{S}}(Y)$ and B is an object of $Sh_{f^{-1}\mathscr{S}}(X)$. In other words, the functor $f^{-1} : Sh_{\mathscr{S}}(Y) \to Sh_{f^{-1}\mathscr{S}}(X)$ is a left adjoint to $f_* : Sh_{f^{-1}\mathscr{S}}(X) \to Sh_{\mathscr{S}}(Y)$.

1.3 Additive categories and abelian categories

In this section, we will discuss categories with extra structure and in particular, additive categories, abelian categories and functors between such categories. Abelian categories which are additive categories with extra structure will be used in Chapter 3 to define derived categories. Our main reference for this section is [12].

Definition 1.3.1. An *additive category* is a category $\mathscr{C}$ such that the following are satisfied.

1. The hom set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of arrows from X to Y is an abelian group, with operation the addition, and the composition law is biadditive with respect to this structure.

2. There exists an object of $\mathscr{C}$ called the *zero object* and which is denoted by 0 such that the hom sets $\mathrm{Hom}_{\mathscr{C}}(X, 0)$ and $\mathrm{Hom}_{\mathscr{C}}(0, X)$ are equal to the one element trivial group 0, for all X of $\mathscr{C}$.

3. For any X_1, X_2 of $\mathscr{C}$, there exists an object Y and morphisms $\pi_1 : Y \to X_1$, $\pi_2 : Y \to X_2$, $i_1 : X_1 \to Y$ and $i_2 : X_2 \to Y$ such that Y is the direct sum and the direct product of X_1 and X_2.

Example 1.3.2. The category of abelian groups is an additive category.

Example 1.3.3. Let R be a ring. Then the category $\mathrm{Mod}(R)$ of left R-modules is an additive category.

Definition 1.3.4. Let k be a field. An additive category $\mathscr{C}$ is said to be k-linear if for any object X and Y of $\mathscr{C}$ the hom sets $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ are k-linear vector spaces and the composition law is bilinear.

Functors between additive categories preserve the additive structure.

Definition 1.3.5. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between additive categories. Then, F is said to be an *additive functor* if the maps

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}\big(F(X), F(Y)\big) \tag{1.3.1}$$

are homomorphisms of abelian groups for all X and Y of $\mathscr{C}$. Similarly, F is said to be *k-linear* if homorphisms 1.3.1 are k-linear.

Next we define abelian categories. To do this, we need the notion of a kernel and of a cokernel of a morphism. We give these definitions first.

Definition 1.3.6. Let $\mathscr{C}$ be an additive category and $f : X \to Y$ be a morphism in $\mathscr{C}$.

1. The *kernel of f* is a pair (K,k) where K is an object of $\mathscr{C}$ and $k : K \to X$ is a morphism in $\mathscr{C}$ such that $f \circ k = 0$ and also, if there is another map $k' : K' \to X$ such that $f \circ k' = 0$ then there exists a unique $k'' : K' \to K$ such that $k \circ k'' = k'$.

2. The *cokernel of f* is a pair (C,c) where C is an object of $\mathscr{C}$ and $c : Y \to C$ is a morphism in $\mathscr{C}$ such that $c \circ f = 0$ and also, if there is another map $c' : Y \to C'$ such that $c' \circ f = 0$ then there exists a unique $c'' : C \to C'$ such that $c'' \circ c = c'$.

Remark 1.3.7. Although kernels and cokernels of maps in additive categories don't exist always, if they do exist then they are unique up to a unique isomorphism.

Notation 1.3.8. The kernel of a map $f : X \to Y$ in an additive category $\mathscr{C}$ will be denoted by $\ker(f)$. Similarly, the cokernel of $f : X \to Y$ in $\mathscr{C}$ will be denoted by $\mathrm{coker}(f)$.

Definition 1.3.9. Let $\mathscr{C}$ be an additive category. Then, $\mathscr{C}$ is said to be *abelian* for every morphism $f : X \to Y$ in $\mathscr{C}$ there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C \tag{1.3.2}$$

such that the following are true.

- The pair (K, k) is the kernel of f and the pair (C, c) is the cokernel of f.
- The pair (I, i) is the cokernel of the map k and the pair (I, j) is the kernel of the map c.
- It is true that $j \circ i = f$.

In the following, we collect examples of abelian categories which we would like to keep in mind.

Example 1.3.10. Let R be a commutative ring. Then, the category $\mathrm{Mod}(R)$ of left R-modules is abelian. Moreover, the full subcategory of finitely generated modules is an abelian category.

Example 1.3.11. Let X be a topological space. The category $Sh(X)$ of sheaves of abelian groups is an abelian category. Moreover, if we fix a sheaf of commutative rings on X, then the subcategory of sheaves of modules over this sheaf is abelian.

The following example of an abelian category is very important and the reader is advised to keep it in mind.

Example 1.3.12. Let X is a smooth complex projective variety. We denote its structure sheaf by $\mathscr{O}_X$. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{E}$ over X with a natural map of sheaves $\mathscr{O}_X \otimes \mathscr{E} \to \mathscr{E}$. A *coherent sheaf* on X is an $\mathscr{O}_X$-module which is locally a quotient of a finite-rank locally-free sheaf. Here, *locally free* means that any point has a neighbourhood U over which the sections are isomorphic to the sheaf $\mathscr{O}_U^{\oplus n}$, for some n. Examples of locally free sheaves are holomorphic vector bundles on a finite dimensional complex manifold.

In the following we discuss left exact and right exact functors. Such functors are defined between abelian categories and will be essential for Chapter 3 and in particular the study of derived functors.

Definition 1.3.13. A functor $F : \mathscr{C} \to \mathscr{D}$ between abelian categories is said to be *left exact* if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$ the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \tag{1.3.3}$$

is exact in $\mathscr{D}$. Similarly, the functor F is said to be a *right exact* if the sequence

$$F(X) \to F(Y) \to F(Z) \to 0 \tag{1.3.4}$$

is exact in $\mathscr{D}$. If F is left and right exact, then F is simply called *exact.*

Example 1.3.14. Let X and Y be objects of an abelian category $\mathscr{C}$. Then, the functors

$$\mathrm{Hom}(A, -) : \mathscr{C} \to \mathrm{Ab}, \quad \mathrm{Hom}(-, A) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab} \tag{1.3.5}$$

are left exact. Here, $\mathscr{C}^{\mathrm{op}}$ denotes the opposite category of $\mathscr{C}$.

Definition 1.3.15. Let $\mathscr{C}$ be an abelian category.

1. An object I of $\mathscr{C}$ is said to be *injective*, if the functor $\mathrm{Hom}(-,I) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ is right exact.
2. An object P of $\mathscr{C}$ is said to be *projective*, if the functor $\mathrm{Hom}(P,-) : \mathscr{C} \to \mathrm{Ab}$ is right exact.

1.4 Serre functors

Serre functors are functors which were defined by Bondal and Kapranov [6] for triangulated categories. Later this notion was generalised by Reiten and Van den Bergh [32] to any k-linear and Hom-finite categories. In this section, we make a first discussion about Serre functors for k-linear categories and we will revisit this notion on the level of derived categories of coherent sheaves in Chapter 4. Our presentation of this section follows [16] and [32] closely.

Definition 1.4.1. Let $\mathscr{A}$ be a k-linear category which is Hom-finite. A *Serre functor* is an additive functor $S : \mathscr{A} \to \mathscr{A}$ together with a natural isomorphism

$$\eta_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(Y,S(X))^{\vee} \tag{1.4.1}$$

for all X,Y objects of $\mathscr{A}$ and $(-)^{\vee} := \mathrm{Hom}_k(-,k)$.

The following proposition establishes a criterion for an endofunctor on a k-linear, Hom-finite category $\mathscr{A}$ with finite hom spaces to be a Serre functor.

Proposition 1.4.2 ([32] Proposition I.1.4)**.** *In order to define a Serre functor $(S,\eta_{X,Y})$ on a k-linear and Hom-finite category $\mathscr{A}$ it is necessary and sufficient to give the action of S on objects, as well as k-linear maps $\eta_X :$ $\mathrm{Hom}_{\mathscr{A}}(X,SX) \to k$ such that the composition*

$$\mathrm{Hom}_{\mathscr{A}}(X,Y) \times \mathrm{Hom}_{\mathscr{A}}(Y,SX) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{A}}(X,SX) \xrightarrow{\eta_X} k \tag{1.4.2}$$

yields a non-degenerate pairing for all X and Y of $\mathscr{A}$. If we are given η_X, then $\eta_{X,Y}$ is obtained from the pairing.

The following lemma will be important for applications in the next section.

Lemma 1.4.3 ([16] p.10)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be k-linear, hom finite categories. If $\mathscr{A}$ and $\mathscr{B}$ are endowed with Serre functors $S_{\mathscr{A}}$ and $S_{\mathscr{B}}$, then any k-linear equivalence $F : \mathscr{A} \to \mathscr{B}$ commutes with the Serre functors; symbolically*

$$F \circ S_{\mathscr{A}} \simeq S_{\mathscr{B}} \circ F.$$

The following theorem establishes a relation between Serre functors and adjoints.

Theorem 1.4.4 ([16] Remark 1.31)**.** *Let $U : \mathscr{A} \to \mathscr{B}$ be a functor between k-linear categories with finite dimensional hom-spaces and let $F : \mathscr{B} \to \mathscr{A}$ be a left adjoint to U; symbolically $F \dashv U$. Then the functor $S_{\mathscr{A}} \circ F \circ S_{\mathscr{B}}^{-1}$ is right adjoint to U; symbolically $U \dashv S_{\mathscr{A}} \circ F \circ S_{\mathscr{B}}^{-1}$.*

Proof. For an object A of $\mathscr{A}$ and an object B of $\mathscr{B}$ the following chain of maps gives the required

$$\begin{aligned} \mathrm{Hom}(A, S_{\mathscr{A}} \circ F \circ S_{\mathscr{B}}^{-1}(B)) &\simeq \mathrm{Hom}((F \circ S_{\mathscr{B}}^{-1})(B), A)^{\vee} && \text{(Serre duality)} \\ &\simeq \mathrm{Hom}(S_{\mathscr{B}}^{-1}(B), U(A))^{\vee} && (F \dashv U) \\ &\simeq \mathrm{Hom}(U(A), S_{\mathscr{B}}(S_{\mathscr{B}}^{-1})(B)) && \text{(Serre duality)} \\ &\simeq \mathrm{Hom}(U(A), B). && (S_{\mathscr{B}} \circ S_{\mathscr{B}}^{-1} = \mathrm{id}) \end{aligned}$$

□

1.5 Orbit categories after Keller

In this section, we introduce the notion of the orbit category of a k-linear category and discuss how to obtain functors on the level of orbit categories. Our presentation follows closely the work of Keller in [22] and [21].

Definition 1.5.1 (Keller [22])**.** Let $\mathscr{C}$ be a k-linear category and $F : \mathscr{C} \to \mathscr{C}$ be an autoequivalence. Then, the *orbit category* of $\mathscr{C}$ is defined to be the category $\mathscr{C}/F$ with objects the same as $\mathscr{C}$ and hom spaces to be defined by

$$\mathrm{Hom}_{\mathscr{C}/F}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X, F^n Y) \tag{1.5.1}$$

for all X, Y objects of $\mathscr{C}$. The composition of two morphisms $f : X \to F^n Y$ and $g : Y \to F^m Z$ is defined by $(F^n g) \circ f : X \to F^{m+n} Z$.

Notation 1.5.2. We will denote by $\pi : \mathscr{C} \to \mathscr{C}/F$ the canonical projection functor endowed with a natural isomorphism $\phi : \pi \circ F \xrightarrow{\sim} \pi$ given by the identity on FX, for any object X.

Lemma 1.5.3 ([2] p.115)**.** *Let $\mathscr{C}$ be a k-linear category and $F : \mathscr{C} \to \mathscr{C}$ be an autoequivalence. Then, $\mathscr{C}/F$ is a k-linear category.*

In order to define functors on the level of orbit categories we need the following definitions.

Definition 1.5.4. Let $\mathscr{C}$ and $\mathscr{D}$ be k-linear categories and $F : \mathscr{C} \to \mathscr{C}$ be an autoequivalence. A *left F-invariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a k-linear functor $H : \mathscr{C} \to \mathscr{D}$ endowed with a natural isomorphism $\eta : HF \xrightarrow{\sim} H$.

A *morphism* between F-invariant functors $(H_1, \eta_1) \to (H_2, \eta_2)$ is defined to be a morphism of functors $\gamma : H_1 \to H_2$ such that the following square commutes.

$$\begin{tikzcd} H_1F \arrow[r, "\eta_1"] \arrow[d, "\gamma F"'] & H_1 \arrow[d, "\gamma"] \\ H_2F \arrow[r, "\eta_2"'] & H_2 \end{tikzcd}$$

Notation 1.5.5. We will denote by $\mathrm{Fun}_{F-\mathrm{inv}}(\mathscr{C}, \mathscr{D})$ the category of F-invariant functors from $\mathscr{C}$ to $\mathscr{D}$.

An important example of such functors is the following one.

Example 1.5.6. The projection functor $\pi : \mathscr{C} \to \mathscr{C}/F$ with the natural isomorphism $\phi : \pi \circ F \xrightarrow{\sim} \pi$ is a F-invariant functor.

We can obtain more F-invariant functors by composing any functor of the form $H : \mathscr{C}/F \to \mathscr{D}$ with the natural projection functor $\pi : \mathscr{C} \to \mathscr{C}/F$. More explicitly, we have the following.

Example 1.5.7. Let $H : \mathscr{C}/F \to \mathscr{D}$ be any functor and (π, ϕ) as above. Then the composite $H \circ \pi : \mathscr{C} \to \mathscr{D}$ is an F-invariant functor where the natural isomorphism $(H \circ \pi) \circ F \xrightarrow{\sim} H \circ \pi$ is given by $H(\phi)$.

Moreover, the category of functors from the orbit category $\mathscr{C}/F$ to a category $\mathscr{D}$ is equivalent to F-invariants functors from $\mathscr{C}$ to $\mathscr{D}$. In particular, we have the following.

Proposition 1.5.8 ([21])**.** *Let $(\pi, \phi) : \mathscr{C} \to \mathscr{C}/F$ be the projection functor. Then, there is an equivalence between* $\mathrm{Fun}_k(\mathscr{C}/F, \mathscr{D})$ *and* $\mathrm{Fun}_{F-inv}(\mathscr{C}, \mathscr{D})$.

Proof. (Sketch) If G is a functor from $\mathscr{C}/F$ to $\mathscr{D}$ then $G \circ \pi : \mathscr{C} \to \mathscr{D}$ is an F-invariant morphism with natural isomorphism $(G \circ \pi) \circ F \xrightarrow{\sim} G \circ \pi$ given by $G(\phi)$. Moreover, for functors G_1, G_2 from $\mathscr{C}/F$ to $\mathscr{D}$ and $\gamma : G_1 \to G_2$ a morphism between them, it induces an F-invariant morphism $\gamma \circ \pi : G_1\pi \to G_2\pi$.

In the other direction, if (H, η) is an F-invariant functor from $\mathscr{C}$ to $\mathscr{D}$, then it induces a functor $\overline{H} : \mathscr{C}/F \to \mathscr{D}$ such that a morphism $f : X \to FY$ in $\mathscr{C}/F$ is mapped to the composite

$$HX \xrightarrow{Hf} HFY \xrightarrow{\sim} HY. \qquad \square$$

Definition 1.5.9. An *F-equivariant functor* on a k-linear category $\mathscr{C}$ with autoequivalence $F : \mathscr{C} \to \mathscr{C}$ is a k-linear functor $H : \mathscr{C} \to \mathscr{C}$ endowed with a natural isomorphism $\eta : HF \xrightarrow{\sim} FH$.

The composition of two equivariant functors (H_1,η_1) and (H_2,η_2) is defined as the functor H_1H_2 endowed with the composed isomorphism $(\eta_2 H_1)(H_2\eta_1)$.

Notation 1.5.10. We will denote by $\mathrm{Fun}_{F\text{-eq}}(\mathscr{C},\mathscr{C})$ the category of F-equivariant functors of $\mathscr{C}$.

Lemma 1.5.11 ([21])**.** *Let (H,η) be an F-equivariant functor on $\mathscr{C}$. Then, $\pi \circ H : \mathscr{C} \to \mathscr{C}/F$ is naturally an F-invariant functor.*

Proof. The natural isomorphism $(\pi H)\circ F \xrightarrow{\sim} (\pi H)$ which makes πH a F-invariant functor is given by the following composition.

$$\begin{tikzcd} \pi HF \arrow[r, "\pi\eta"] \arrow[dr, "\sim"'] & \pi FH \arrow[d, "\phi H"] \\ & \pi H \end{tikzcd}$$

□

Corollary 1.5.12. *Starting with an F-equivariant functor, we obtain naturally an F-invariant functor and from the equivalence (1.5.8) a functor on the level of orbit categories; symbolically*

$$\mathrm{Fun}_{F-\mathrm{equiv}}(\mathscr{C},\mathscr{C}) \to \mathrm{Fun}_{F-\mathrm{inv}}(\mathscr{C},\mathscr{C}/F) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}/F,\mathscr{C}/F)$$

Example 1.5.13. The functor F, which is used to define the orbit category, can be made into an F-equivariant functor with natural transformation $\eta = \mathrm{id}_{F^2}$. To see this, recall that that $\pi(X) = \mathrm{id}_X$ for all objects X and that the following diagram is commutative.

$$\begin{tikzcd} \pi FF \arrow[r, "\pi\,\mathrm{id}_{F^2}"] \arrow[dr, "\sim"'] & \pi FF \arrow[d, "\mathrm{id}_F"] \\ & \pi F \end{tikzcd}$$

Next, we discuss the notion of a Serre functor on orbit categories. Let $\mathscr{C}$ be a k-linear Hom-finite category and let $F : \mathscr{C} \to \mathscr{C}$ be an autoequivalence. Assume that $S_{\mathscr{C}}$ is a Serre functor on $\mathscr{C}$, as discussed in Definition 1.4.1. By Proposition 1.4.2 any Serre functor on a k-linear Hom-finite category is determined by the trace maps $t_X : \mathrm{Hom}_{\mathscr{A}}(X,SX) \to k$. Suppose that there exists $\sigma_F : F\circ S_X \to S_X \circ F$ such that

$$t_{FX}\big((\sigma_F X)\circ F(f)\big) = t_X(f)$$

for all morphisms $f : X \to SX$. To unwrap the last assumption, if $f\colon X \to SX$ then $t_{FX}\big((\sigma_F X)\circ F(f)\big)$ is the assignment which maps a number of the field k to the composite $FX \xrightarrow{Ff} FSX \xrightarrow{\sigma_F X} SF(X)$.

Lemma 1.5.14 (Keller [22]). *Let $\mathscr{C}$ be a k-linear Hom-finite category and let $F : \mathscr{C} \to \mathscr{C}$ be an autoequivalence and let $S_{\mathscr{C}}$ be a Serre functor on $\mathscr{C}$. Then, the pair (S_X, σ^{-1}) is an F-equivariant functor and so induces a Serre functor on the orbit category $\mathscr{C}/F$ with induced trace map given by*

$$t_{\pi X}(f) = \begin{cases} t_X(f) & \text{if} \quad f\colon X \to SX \\ 0 & \text{if} \quad f\colon X \to F^p SX \end{cases}.$$

Chapter 2

On monoidal categories and functors

The goal of this chapter is to study closed symmetric monoidal categories and the associated functors between them. Furthermore, we discuss braided monoidal categories and the important notion of rigid monoidal categories.

The material of this chapter is presented as follows. In Section 2.1 we introduce the notion of a monoidal category and discuss the notion of a monoidal functor. In Section 2.2 we will discuss another class of functors between monoidal categories, the opmoidal functors which will be of much interest in Chapter 6 and in Chapter 7. In Section 2.3 we discuss braided monoidal categories and symmetric monoidal categories in order to define Hopf algebras in braided monoidal categories in Chapter 5 and vast generalisations of Hopf algebras in a non-braided setup in Chapter 6. In Section 2.4 we discuss the important notion of a category with duals and in Section 2.5 we discuss the notion of a closed category.

2.1 Monoidal categories and monoidal functors

In this section, we introduce the notion of a monoidal category following [28] closely.

Definition 2.1.1. A *monoidal category* is the data $(\mathscr{C}, \otimes, 1_{\mathscr{C}}, \alpha, l, r)$ where $\mathscr{C}$ is a category, $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a bifunctor called the *monoidal product*, $1_{\mathscr{C}}$ is a specified object in $\mathscr{C}$ called the *unit object* and natural isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \qquad \text{(associator)}$$
$$l_X : 1 \otimes X \xrightarrow{\sim} X \qquad \text{(left unit)}$$
$$r_X : X \otimes 1 \xrightarrow{\sim} X \qquad \text{(right unit)}$$

defined for all objects X, Y and Z of $\mathscr{C}$ such that α, l, r make the following diagrams commutative.

(Pentagon)

$$\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\
\big((X \otimes Y) \otimes Z\big) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z} \otimes \mathrm{id}_W"'] & & X \otimes \big(Y \otimes (Z \otimes W)\big) \\
\big(X \otimes (Y \otimes Z)\big) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"'] & & X \otimes \big((Y \otimes Z) \otimes W\big) \arrow[u, "\mathrm{id}_X \otimes \alpha_{Y,Z,W}"']
\end{tikzcd}$$

(Triangle)

$$\begin{tikzcd}
 & X \otimes Y & \\
(X \otimes 1) \otimes Y \arrow[ur, "r_X \otimes \mathrm{id}_Y"] \arrow[rr, "\alpha_{X,1,Y}"'] & & X \otimes (1 \otimes Y) \arrow[ul, "\mathrm{id}_X \otimes l_Y"']
\end{tikzcd}$$

Example 2.1.2. Let k be a field of characteristic zero. Then, the category $\mathrm{Vect}_f(k)$ of finite dimensional vector spaces over k is a monoidal category with monoidal unit the field k and monoidal product the tensor product of k-vector spaces.

Example 2.1.3. Let $\mathscr{C}$ be a category. Then, the category $\mathrm{End}(\mathscr{C})$ of the endofunctors of $\mathscr{C}$ is a monoidal category with monoidal product the composition of functors and monoidal unit the identity functor.

Example 2.1.4. Ler R be a commutative ring. Then, the category $\mathrm{Ch}(R)$ of $\mathbb{Z}$-graded chain complexes of R-modules with differetial of degree -1 is a monoidal category with monoidal product being the graded tensor product which is defined by

$$(X \otimes_R Y)_n = \sum_{p+q=n} X_p \otimes_R Y_q, \qquad d(x \otimes y) = d(x) \otimes y + (-1)^{pq} x \otimes d(y).$$

for all complexes X and Y of $\mathrm{Ch}(R)$.

In the following, we review the basics of monoidal functors between monoidal categories following [7] closely.

Definition 2.1.5. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories. A *monoidal functor* from the $\mathscr{C}$ to $\mathscr{D}$ is a triple (F, F_0, F_2), where $F : \mathscr{C} \to \mathscr{D}$ denotes a functor with $F_0 : 1_{\mathscr{D}} \to F(1_{\mathscr{C}})$ being a morphism in $\mathscr{D}$ and $F_2 : \otimes_{\mathscr{D}} \circ (F \times F) \to F \circ \otimes_{\mathscr{C}}$ being a natural transformation and such that the following diagrams commute for all objects X, Y and Z of $\mathscr{C}$.

$$\begin{tikzcd}
F(X)\otimes F(Y)\otimes F(Z) \arrow[r, "\mathrm{id}_{F(X)}\otimes F_2(Y,Z)"] \arrow[d, "F_2(X,Y)\otimes \mathrm{id}_{F(Z)}"'] & F(X)\otimes F(Y\otimes Z) \arrow[d, "F_2(X,Y\otimes Z)"] \\
F(X\otimes Y)\otimes F(Z) \arrow[r, "F_2(X\otimes Y,Z)"'] & F(X\otimes Y\otimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{id}_{F(X)}\otimes F_0"] \arrow[d, "F_2(1,X)"'] \arrow[dr, "\mathrm{id}_{F(X)}"] & F(X)\otimes F(1_{\mathscr{C}}) \arrow[d, "F_2(X,1)"] \\
F(1_{\mathscr{C}})\otimes F(X) \arrow[r, "F_0\otimes \mathrm{id}_{F(X)}"'] & F(X\otimes 1_{\mathscr{C}})
\end{tikzcd}$$

Notation 2.1.6. A monoidal functor (F, F_0, F_2) is said to be *strong*, if F_0 and F_2 are isomorphisms. The monoidal functor (F, F_0, F_2) is called *strict* if F_0 and F_2 are identities.

Next, we define the notion of a monoidal natural transformation between monoidal functors.

Definition 2.1.7. A natural transformation $\phi : F \to U$ between monoidal functors is *monoidal* if the following diagrams commute for all for all X, Y of $\mathscr{C}$.

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "F_2(X,Y)"] \arrow[d, "\phi_X\otimes\phi_Y"'] & F(X\otimes Y) \arrow[d, "\phi_{X\otimes Y}"] \\
U(X)\otimes U(Y) \arrow[r, "U_2(X,Y)"'] & U(X\otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathscr{D}} \arrow[r, "F_0"] \arrow[dr, "U_0"'] & F(1_{\mathscr{C}}) \arrow[d, "\phi_1"] \\
 & U(1_{\mathscr{C}})
\end{tikzcd}$$

Two monoidal functors can be composed and the composite is a monoidal functor. More precisely, we have the following lemma.

Lemma 2.1.8 ([38] p.16)**.** *Let $F : \mathscr{A}_1 \to \mathscr{A}_2$ and $G : \mathscr{A}_2 \to \mathscr{A}_3$ be two monoidal functors between monoidal categories. Then, the functor $G \circ F : \mathscr{A}_1 \to \mathscr{A}_3$ is a monoidal functor with constraints $(GF)_2 = G(F_2) \circ G_2$ and $(GF)_0 = G(F_0) \circ G_0$.*

Lemma 2.1.9 ([13] Section 3)**.** *Let $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors between monoidal categories such that F is a strong monoidal functor and left adjoint to U. Then, it is defined a monoidal structure on U by*

$$U(X)\otimes U(Y) \xrightarrow{\eta} UF\big(U(X)\otimes U(Y)\big) \xrightarrow{\simeq} U\big(FU(X)\otimes FU(Y)\big) \xrightarrow{U(\epsilon\otimes\epsilon)} U(X\otimes Y)$$

$$1_{\mathscr{C}} \xrightarrow{\eta} UF(1_{\mathscr{C}}) \xrightarrow{U(F_0)} U(1_{\mathscr{D}})$$

for all X and Y objects of $\mathscr{D}$.

Corollary 2.1.10 ([13] Section 3)**.** *Let $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ be as the above Lemma. Then, it is defined the morphism*

$$\pi : X \otimes U(Y) \xrightarrow{\eta \otimes \mathrm{id}} UF(X) \otimes U(Y) \to U(F(X) \otimes Y) \tag{2.1.1}$$

for all X of $\mathscr{C}$ and Y of $\mathscr{D}$.

Remark 2.1.11. We have to underline here that π is not an isomorphism in general. If it is, then π is called the *projection formula* in [13].

2.2 Opmonoidal functors

In this section, we introduce the notion of an opmonoidal functor between monoidal categories. Our presentation follows [7] closely and it is similar to the material on monoidal functors. The reason for introducing and discuss in this section independently is that in Chapter 6 where we will discuss the generalisations of Hopf algebras in a non-braided setup, opmonoidal functors will play fundamental role.

Definition 2.2.1. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, 1_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, 1_{\mathscr{D}})$ be monoidal categories. An *opmonoidal functor* from $\mathscr{C}$ to $\mathscr{D}$ is a triple (F, F_2, F_0), where $F : \mathscr{C} \to \mathscr{D}$ is a functor, $F_2 : F \circ \otimes_{\mathscr{C}} \to \otimes_{\mathscr{D}} \circ (F \times F)$ is a natural transformation and $F_0 : F(1_{\mathscr{C}}) \to 1_{\mathscr{D}}$ is a morphism in $\mathscr{D}$ such that the following diagrams commute for all objects X, Y and Z of $\mathscr{C}$.

$$\begin{tikzcd}
F(X \otimes Y \otimes Z) \arrow[r, "F_2(X, Y \otimes Z)"] \arrow[d, "F_2(X \otimes Y, Z)"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "\mathrm{id}_{F(X)} \otimes F_2(Y, Z)"] \\
F(X \otimes Y) \otimes F(Z) \arrow[r, "F_2(X, Y) \otimes \mathrm{id}_{F(Z)}"'] & F(X) \otimes F(Y) \otimes F(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \arrow[r, "F_2(X, 1)"] \arrow[d, "F_2(1, X)"'] \arrow[dr, "\mathrm{id}_{F(X)}"] & F(X) \otimes F(1_{\mathscr{C}}) \arrow[d, "\mathrm{id}_{F(X)} \otimes F_0"] \\
F(1_{\mathscr{C}}) \otimes F(X) \arrow[r, "F_0 \otimes \mathrm{id}_{F(X)}"'] & F(X)
\end{tikzcd}$$

Notation 2.2.2. An opmonoidal functor (F, F_2, F_0) is said to be *strong*, if F_2 and F_0 are isomorphisms. Then (F, F_2^{-1}, F_0^{-1}) is a strong monoidal functor. An opmonoidal functor (F, F_2, F_0) is called *strict* if F_2 and F_0 are identities.

Next, we define the notion of an opmonoidal natural transformation between opmonoidal functors.

Definition 2.2.3. A natural transformation $\phi : F \to U$ between opmonoidal functors is *opmonoidal* if the following diagrams commute for all X, Y of $\mathscr{C}$.

$$\begin{tikzcd} F(X \otimes Y) \arrow[r, "F_2(X,Y)"] \arrow[d, "\phi_{X\otimes Y}"'] & F(X) \otimes F(Y) \arrow[d, "\phi_X \otimes \phi_Y"] \\ U(X \otimes Y) \arrow[r, "U_2(X,Y)"'] & U(X) \otimes U(Y) \end{tikzcd} \qquad \begin{tikzcd} F(1_{\mathscr{C}}) \arrow[r, "\phi_1"] \arrow[dr, "F_0"'] & U(1_{\mathscr{C}}) \arrow[d, "U_0"] \\ & 1_{\mathscr{D}} \end{tikzcd}$$

Composition of opmonoidal functors is again an opmonoidal functor as the following lemma establishes.

Lemma 2.2.4 ([38], p.16)**.** *Let $F : \mathscr{A}_1 \to \mathscr{A}_2$ and $G : \mathscr{A}_2 \to \mathscr{A}_3$ be two opmonoidal functors between monoidal categories. Then, $G \circ F : \mathscr{A}_1 \to \mathscr{A}_2$ is an opmonoidal functor with opmonoidal constraints $(UF)_2 = U_2 \circ U(F_2)$ and $(UF)_0 = U_0 \circ U(F_0)$.*

The following definition introduces a specific kind of adjunction and it will be of much interest in Chapter 6 and in Chapter 7.

Definition 2.2.5. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories and let $F : \mathscr{C} \to \mathscr{D}$ be a functor with right adjoint the functor $G : \mathscr{D} \to \mathscr{C}$. Then, the adjunction $F \dashv U$ is called an *opmonoidal adjunction* if both F and U are given the structure of opmonoidal functors and the adjunction unit $\eta : 1_{\mathscr{C}} \to UF$ and adjunction counit $\epsilon : FU \to 1_{\mathscr{D}}$ are opmonoidal natural transformations.

We will be particularly interested in is the case where for the adjuction $F \dashv U$ the right adjoint U is a strong monoidal functor.

Theorem 2.2.6 ([8] Theorem 2.6)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories. Assume also that there are functors $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ such that U is the right adjoint; symbolically $F \dashv U$. Then monoidal structures on U correspond to opmonoidal structures on F.*

If moreover, U is a strong monoidal functor, then the composite UF is an opmonoidal functor.

To make the above statement precise, the fact that U is a monoidal functor means that there is morphisms $U(X) \otimes U(Y) \to U(X \otimes Y)$ and $1_{\mathscr{C}} \to U(1_{\mathscr{D}})$ in $\mathscr{C}$ such that they satisfy the diagrams of Definition 2.1.5. Then the corresponding opmonoidal structure on F is defined to be

$$F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F\big(UF(X) \otimes UF(Y)\big) \to FU\big(F(X) \otimes F(Y)\big) \xrightarrow{\epsilon} F(X) \otimes F(Y)$$

and

$$F(1_{\mathscr{C}}) \to FU(1_{\mathscr{D}}) \xrightarrow{\epsilon} 1_{\mathscr{D}}$$

for all X and Y of $\mathscr{C}$.

Remark 2.2.7. The above theorem will be fundamental for the theory of bimonads, which will be discussed in 6.2.

In the remaining of this section, we assume that it is given a pair of adjoint functors of which the right adjoint is a strong monoidal functor and has a right quasi-inverse. Then, we define a specific natural transformation ψ and we state a number of results of which the proof can be found in the appendix. All the following will be used for the proof of Lemma 7.2.14 and so the reader may skip the next part and return when it will be needed.

Definition 2.2.8. Let $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors defined between monoidal categories such that the right adjoint U is a strong monoidal functor. Assume also that there exists a strong monoidal functor $W : \mathscr{C} \to \mathscr{D}$ which is also a right quasi-inverse to U; symbolically we have $UW \cong \mathrm{id}$. We define the natural transformation $\psi : F \to F(1) \otimes W(-)$ to be the composite

$$\begin{aligned} F(A) \to F(1 \otimes A) = F(1 \otimes UWA) &\xrightarrow{F(\eta \otimes \mathrm{id})} F\big(UF(1) \otimes UWA\big) \\ &\xrightarrow{\sim} FU(F(1) \otimes WA) \xrightarrow{\epsilon} F(1) \otimes WA \end{aligned}$$

for all A of $\mathscr{C}$. Moreover, the natural transformation ψ will be useful to prove some technical lemmas later.

Proposition 2.2.9. *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint F. Moreover, assume also that there exists a functor $W : \mathscr{C} \to \mathscr{D}$ such that $UW \cong \mathrm{id}$. Then the morphism $\gamma_{A,1} : F(1 \otimes A \otimes 1 \otimes B) \to F(1) \otimes WA \otimes F(1) \otimes WB$ defined to be the composite*

$$\gamma_{A,1} = \epsilon \circ FU\Big(\epsilon \otimes \epsilon \circ \big(F(U^2) \otimes F(U^2)\big)\Big) \circ F\big(U^2 \circ (\eta \otimes \eta) \circ (\eta_1 \otimes \mathrm{id}_A \otimes \eta_1 \otimes \mathrm{id}_B)\big)$$

makes the following diagram commutative for all A and B of $\mathscr{C}$.

$$\begin{tikzcd}
F(A \otimes B) \arrow[rr, "F_2(A,B)"] \arrow[d, "F(l_A^{-1} \otimes l_B^{-1})"'] & & F(A) \otimes F(B) \arrow[d, "F(l_A^{-1} \otimes l_B^{-1})"] \\
F(1 \otimes A \otimes 1 \otimes B) \arrow[dr, "\gamma_{A,1}"'] & & F(1 \otimes A) \otimes F(1 \otimes B) \arrow[dl, "\psi_{1\otimes A} \otimes \psi_{1\otimes B}"] \\
& F(1) \otimes WA \otimes F(1) \otimes WB &
\end{tikzcd}$$

Proposition 2.2.10. *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint F. Moreover, assume also that there exists a functor $W : \mathscr{C} \to \mathscr{D}$ such that $UW \cong \mathrm{id}$. Then the morphism $\zeta_{1,A} : F(1 \otimes 1 \otimes A \otimes B) \to F(1) \otimes F(1) \otimes WA \otimes WB$ defined to be the composite*

$$\zeta_{1,A} = \epsilon \circ FU\Big(\epsilon \otimes \epsilon \circ \big(F(U^2) \otimes F(U^2)\big)\Big) \circ F\big(U^2 \circ (\eta \otimes \eta) \circ (\eta_1 \otimes \eta_1 \otimes \mathrm{id}_A \otimes \mathrm{id}_B)\big)$$

makes the following diagram commutative for all A and B of $\mathscr{C}$.

$$\begin{tikzcd}
F(1 \otimes A \otimes B) \arrow[r, "F(l^{-1}_{1\otimes A \otimes B})"] \arrow[d, "\psi_{A\otimes B}"'] & F(1 \otimes 1 \otimes A \otimes B) \arrow[d, "\zeta_{A,1}"] \\
F(1) \otimes WA \otimes WB \arrow[r, "(UF)_2(1,1) \otimes \mathrm{id}_{WA\otimes WB}"] & F(1) \otimes F(1) \otimes WA \otimes WB
\end{tikzcd}$$

The next two Lemmas we will be essential to prove Proposition 2.2.13.

Lemma 2.2.11. *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint F. Moreover, assume also that there exists a functor $W : \mathscr{C} \to \mathscr{D}$ such that $UW \cong \mathrm{id}$. Then, the following diagram is commutative.*

$$\begin{tikzcd}
\big(UF(1) \otimes UF(1)\big) \otimes \big(UWA \otimes UWB\big) \arrow[r, "U^2 \otimes U^2"] \arrow[d, "\eta \otimes \eta"'] & U\big(F(1) \otimes F(1)\big) \otimes U\big(WA \otimes WB)\big) \arrow[ddd, "U^2"] \\
UF\big(UF(1) \otimes UF(1)\big) \otimes UF\big(UF(1) \otimes UWB)\big) \arrow[d, "U^2"'] & \\
U\Big(F\big(UF(1) \otimes UF(1)\big) \otimes F\big(UWA \otimes UWB\big)\Big) \arrow[d, "U\big(F(U^2) \otimes F(U^2)\big)"'] & \\
U\Big(FU\big(F(1) \otimes F(1)\big) \otimes FU(WA \otimes WB)\Big) \arrow[r, "U(\epsilon \otimes \epsilon)"'] & U\big(F(1) \otimes F(1) \otimes WA \otimes WB\big)
\end{tikzcd}$$

In a similar sense with Lemma 2.2.11, we have also the following.

Lemma 2.2.12. *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint F. Moreover, assume also that there exists a functor $W : \mathscr{C} \to \mathscr{D}$ such that $UW \cong \mathrm{id}$. Then, the following diagram is commutative.*

$$
\begin{tikzcd}
U(F(1)\otimes WA)\otimes U(F(1)\otimes WB) \arrow[dddd, "U^2"'] & (UF(1)\otimes UWA)\otimes(UF(1)\otimes UWB \arrow[l, "U^2\otimes U^2"'] \arrow[d, "\eta\otimes\eta"] \\
& UF(UF(1)\otimes UWA)\otimes UF(UF(1)\otimes UWB) \arrow[d, "U^2"] \\
& U\Big(F(UF(1)\otimes UWA)\otimes F(UF(1)\otimes UWB)\Big) \arrow[d, "U(F(U^2)\otimes F(U^2))"] \\
& \\
U(F(1)\otimes WA\otimes F(1)\otimes WB) & U(FU(F(1)\otimes WA)\otimes U(F(1)\otimes WB)) \arrow[l, "U(\epsilon\otimes\epsilon)"]
\end{tikzcd}
$$

For the next proposition, recall that in Proposition 2.2.9 and in Proposition 2.2.10 we defined morphisms

$$\gamma_{1,A} : F(1\otimes 1\otimes A\otimes B) \to F(1)\otimes F(1)\otimes WA\otimes WB$$

$$\zeta_{A,1} : F(1\otimes A\otimes 1\otimes B) \to F(1)\otimes WA\otimes F(1)\otimes WB$$

Proposition 2.2.13. *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint F. Moreover, assume also that there exists a functor $W : \mathscr{C} \to \mathscr{D}$ such that $UW \cong \mathrm{id}$. Then, the morphisms $\gamma_{1,A}$ and $\zeta_{A,1}$ make the following diagram commutative for every A and B of $\mathscr{C}$.*

$$
\begin{tikzcd}
F(1\otimes 1\otimes A\otimes B) \arrow[r, "F(\mathrm{id}_1\otimes\tau_{1,A}\otimes\mathrm{id}_B)"] \arrow[d, "\gamma_{1,A}"'] & F(1\otimes A\otimes 1\otimes B) \arrow[d, "\zeta_{A,1}"] \\
F(1)\otimes F(1)\otimes WA\otimes WB \arrow[r, "\mathrm{id}\otimes\tau_{F(1),WA}\otimes\mathrm{id}"'] & F(1)\otimes WA\otimes F(1)\otimes WB
\end{tikzcd}
$$

2.3 Symmetries, braidings and centres of categories

In this section, we discuss braided monoidal categories and symmetric monoidal categories. Then, we discuss the construction of the centre of a category which turns out to be always a braided category even if the original category is not braided.

Definition 2.3.1. Let $\mathscr{C}$ be a monoidal category. A *braiding* for $\mathscr{C}$ is a natural isomorphism $\tau : X\otimes Y \xrightarrow{\sim} Y\otimes X$ such that the following diagrams commute for all X, Y and Z objects of $\mathscr{C}$.

$$\begin{tikzcd}
X \otimes 1 \arrow[rr, "\tau_{X,1}"] \arrow[dr, "r_X"'] & & 1 \otimes X \arrow[dl, "l_X"] \\
& X &
\end{tikzcd}$$

$$\begin{tikzcd}
& (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[dr, "\mathrm{id}_Y \otimes \tau_{X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\tau_{X,Y} \otimes \mathrm{id}_Z"] \arrow[dr, "\alpha_{Y,Z,X}"'] & & & Y \otimes (Z \otimes X) \\
& X \otimes (Y \otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"'] & (Y \otimes Z) \otimes X \arrow[ur, "\alpha_{Y,Z,X}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "\tau_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "\alpha^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes \tau_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "\tau_{X,Z} \otimes \mathrm{id}_B"'] &
\end{tikzcd}$$

Notation 2.3.2. The bottom two diagrams in the definition above are called the *hexagons*.

Next, we introduce the notion of a symmetric monoidal category.

Definition 2.3.3. A braided monoidal category $\mathscr{C}$ is said to be symmetric if the natural isomorphism $\tau : X \otimes Y \xrightarrow{\sim} Y \otimes X$ makes the following diagram commutative for all X and Y objects of $\mathscr{C}$.

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\tau_{X,Y}"] \arrow[dr, "\mathrm{id}_{X\otimes Y}"'] & Y \otimes X \arrow[d, "\tau_{Y,X}"] \\
& X \otimes Y
\end{tikzcd}$$

We will be mainly interested in symmetric monoidal categories and in the following we provide some useful examples for the following.

Example 2.3.4. Let k be a field of characteristic $\neq 2$. Then, the category sVect_k of $\mathbb{Z}_2$-graded vector spaces with symmetry given by $\tau_{X,Y}(x \otimes y) = (-1)^{|x||y|} y \otimes x$ where $|x|, |y| \in \{0,1\}$, $x \in X$, $y \in Y$ and $X, Y \in \mathrm{sVect}$ is a symmetric monoidal category.

An important example is the following.

Example 2.3.5. Recall from Example 2.1.4 that that the category $\mathrm{Ch}(R)$ of $\mathbb{Z}$-graded chain complexes of R-modules is a monoidal category. Furthermore, $\mathrm{Ch}(R)$ is a symmetric monoidal category with symmetric $\tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ to be defined by

$$\tau(x \otimes y) = (-1)^{pq} y \otimes x \tag{2.3.1}$$

for all $x \in X_p$ and $y \in Y_q$.

To each monoidal category $\mathscr{C}$ one can assign a braided category, called the *centre of the category* which is the analogue of the centre of an algebra A over a field k of characteristic zero. The notion of the centre of a category will play an important notion the Chapter 6.5 where we will discuss Hopf monads represented by Hopf algebras. First, we introduce the notion of a half braiding.

Definition 2.3.6. Let $\mathscr{C}$ be a monoidal category and let A be an object of $\mathscr{C}$.

1. A *half braiding* for the object A is a pair (A, σ) where σ is a family of isomoprhisms $\sigma_X : A \otimes X \xrightarrow{\sim} X \otimes A$ natural in X of $\mathscr{C}$ such that

$$\sigma_{X \otimes Y} = (\mathrm{id}_X \otimes \sigma_Y) \circ (\sigma_X \otimes \mathrm{id}_Y) \tag{2.3.2}$$

   for all X, Y objects of $\mathscr{C}$.

2. The *centre of a category* $\mathscr{C}$ is the category $\mathscr{Z}(\mathscr{C})$ of which objects are half-braidings. Let (A, σ) and (B, ρ) be two half braidings in $\mathscr{Z}(\mathscr{C})$. Then, an moprhism of half braidings from (A, σ) to (B, ρ) is a map $f : A \to B$ in $\mathscr{C}$ such that the following diagram commutes for all X of $\mathscr{C}$.

$$\begin{tikzcd}
A \otimes X \arrow[r, "\sigma_X"] \arrow[d, "f \otimes \mathrm{id}_X"'] & X \otimes A \arrow[d, "\mathrm{id}_X \otimes f"] \\
B \otimes X \arrow[r, "\rho_X"'] & X \otimes B
\end{tikzcd}$$

Theorem 2.3.7 ([20] p.330)**.** *Let $\mathscr{C}$ be a monoidal category. Then, $\mathscr{Z}(\mathscr{C})$ is a braided monoidal cateogry with monoidal product defined by*

$$(A, \sigma) \otimes_{\mathscr{Z}(\mathscr{C})} (B, \rho) = \big(A \otimes B, (\sigma \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho)\big), \tag{2.3.3}$$

unit object defined by $1_{\mathscr{Z}(\mathscr{C})} = (1_{\mathscr{C}}, \mathrm{id})$ and braiding τ defined by

$$\tau_{(A,\sigma),(B,\rho)} = \sigma_B : (A, \sigma) \otimes (B, \rho) \to (B, \rho) \otimes (A, \sigma). \tag{2.3.4}$$

2.4 Rigid categories

In this section, we discuss categories with duals. Our presentation for this section follows [20, Section XIV.2] and [38, Section 1.6] closely.

Definition 2.4.1. A monoidal category $(\mathscr{C}, \otimes, 1)$ is said to have a *left duality* if for each object X of $\mathscr{C}$ there exist an object ${}^\vee X$ and morphisms

$$\mathrm{ev}_X : {}^\vee X \otimes X \to 1, \qquad \mathrm{coev}_X : 1 \to X \otimes {}^\vee X \tag{2.4.1}$$

such that the following relations, called the *snake relations for the left dual*, are satisfied.

$$(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X, \quad (\mathrm{ev}_X \otimes \mathrm{id}_{{}^\vee X}) \circ (\mathrm{id}_{{}^\vee X} \otimes \mathrm{coev}_X) = \mathrm{id}_{{}^\vee X} \tag{2.4.2}$$

In that case, we say that the object $({}^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$ is a *left dual* of X.

Remark 2.4.2. If the left dual of an object X of a monoidal category $\mathscr{C}$ exists then it is unique up to a unique isomorphism which preserves the evaluation and the coevaluation, for details see [38, Section 1.6.1]

In the following, we discuss a fundamental example for a category with left duality, that of finite dimensional vector spaces. First we fix some notation. For any two finite dimensional k-vector spaces V and W there exists an isomorphism

$$\phi_{W,V} : V \otimes \mathrm{Hom}(W, k) \xrightarrow{\sim} \mathrm{Hom}(W, V), \quad v \otimes f \mapsto \big(w \mapsto \phi_{W,V}(v \otimes f)(w)\big)$$

where $w \in W$, $v \in V$ and $f \in \mathrm{Hom}(W, k)$.

Example 2.4.3. Let $\mathrm{Vect}_f(k)$ be the monoidal category of finite dimensional vector spaces over k. For any k-vector space V with basis $\{v_i\}$ its left dual is defined as follows. Let ${}^\vee V$ be the k-vector space $\mathrm{Hom}_k(V, k)$ with basis denoted by $\{v^i\}$ and consider morphisms

$$\mathrm{ev}_V : {}^\vee V \otimes V \to k, \quad v^i \otimes v_j \mapsto v^i(v_j) = \delta_{ij} \tag{2.4.3}$$

and

$$\mathrm{coev}_V : k \to V \otimes {}^\vee V, \quad 1 \mapsto \phi^{-1}_{V,V}(\mathrm{id}_V) = \sum_i v_i \otimes v^i \tag{2.4.4}$$

where $\phi_{V,V} : V \otimes {}^\vee V \xrightarrow{\sim} \mathrm{Hom}(V, V)$ as defined in 2.4. Then, $({}^\vee V, \mathrm{ev}_V, \mathrm{coev}_V)$ satisfies the snake relations of Definition 2.4.2 and hence consists of a left dual for V. The construction of the left dual is independent of basis, see for details [20, Section II.3].

For the following remark, we will denote by $\mathscr{C}^{\mathrm{op}}$ the opposite category of $\mathscr{C}$ which has as objects the same as the objects of $\mathscr{C}$ and its morphisms are defined as follows. Let X and Y be objects of $\mathscr{C}^{\mathrm{op}}$, then $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathscr{C}}(X, Y)$. The composition in $\mathscr{C}^{\mathrm{op}}$ is defined by $g \circ^{\mathrm{op}} f := f \circ g$.

Remark 2.4.4. A left duality on a category $\mathscr{C}$ determines a functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ such that to each object X of $\mathscr{C}^{\mathrm{op}}$ assigns its left dual ${}^{\vee}X$ and to each morphism $f : X \to Y$ in $\mathscr{C}^{\mathrm{op}}$ assigns the map ${}^{\vee}f : {}^{\vee}Y \to {}^{\vee}X$ which is defined to be the following composite

$${}^{\vee}f := (\mathrm{ev}_Y \otimes \mathrm{id}_{{}^{\vee}X}) \circ (\mathrm{id}_{{}^{\vee}Y} \otimes f \otimes \mathrm{id}_{{}^{\vee}X}) \circ (\mathrm{id}_{{}^{\vee}Y} \otimes \mathrm{coev}_X).$$

The map ${}^{\vee}f$ is called the *left dual of* f.

Proposition 2.4.5 ([20] p.344)**.** *Let $\mathscr{C}$ be a monoidal category with left duality. Then, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{C}}(X \otimes Y, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X, W \otimes {}^{\vee}Y)$$

$$\mathrm{Hom}_{\mathscr{C}}({}^{\vee}X \otimes Y, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(Y, X \otimes W)$$

for all X, Y and Z objects of $\mathscr{C}$. In other words, the functor $- \otimes Y$ is left adjoint to $- \otimes {}^{\vee}Y$ and similarly, the functor ${}^{\vee}X \otimes -$ is left adjoint to the functor $X \otimes -$.

In a similar fashion, we define the category with a right duality as follows.

Definition 2.4.6. A monoidal category $(\mathscr{C}, \otimes, 1)$ is said to have a *right duality* if for each object X of $\mathscr{C}$ there exist an object $X^{\vee}$ and morphisms

$$\mathrm{ev}'_X : X \otimes X^{\vee} \to 1, \qquad \mathrm{coev}'_X : 1 \to X^{\vee} \otimes X \tag{2.4.5}$$

in the category $\mathscr{C}$ such that the following relations, called the *snake relations for the right dual,* are satisfied.

$$(\mathrm{id}_{X^{\vee}} \otimes \mathrm{ev}'_X) \circ (\mathrm{coev}'_X \otimes \mathrm{id}_{X^{\vee}}) = \mathrm{id}_{X^{\vee}}, \quad (\mathrm{ev}'_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \mathrm{coev}'_X) = \mathrm{id}_X \tag{2.4.6}$$

In that case, we say that the object $(X^{\vee}, \mathrm{ev}'_X, \mathrm{coev}'_X)$ is a *right dual* of X.

Remark 2.4.7. If the right dual of an object X of a monoidal category $\mathscr{C}$ exists then it is unique up to a unique isomorphism which preserves the evaluation and the coevaluation, for details see [38, Section 1.6.1]

Remark 2.4.8. A right duality on a category $\mathscr{C}$ determines a functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ such that to each object X of $\mathscr{C}^{\mathrm{op}}$ is assigned its right dual $X^{\vee}$ and each morphism $f : X \to Y$ in $\mathscr{C}^{\mathrm{op}}$ is assigned the right dual map $f^{\vee} : Y^{\vee} \to X^{\vee}$ defined to be the following composite.

$$f^{\vee} := (\mathrm{id}_{X^{\vee}} \otimes \mathrm{ev}'_Y) \circ (\mathrm{id}_{X^{\vee}} \otimes f \otimes \mathrm{id}_{Y^{\vee}}) \circ (\mathrm{coev}'_X \otimes \mathrm{id}_{Y^{\vee}})$$

Notation 2.4.9. A monoidal category $(\mathscr{C}, \otimes, 1_{\mathscr{C}})$ which has a left duality and a right duality is called a *rigid category.*

The following lemma concerns symmetric monoidal cateogories with either left or right duality and it will be useful for the following, since we will deal symmetric monoidal categories.

Lemma 2.4.10. *Let $\mathscr{C}$ be a symmetric monoidal category with a left duality. Then $\mathscr{C}$ has also a right duality. Similarly, if $\mathscr{C}$ is a symmetric monoidal category and has a right duality then has also a left duality.*

Proof. (Sketch) Assume that the category $\mathscr{C}$ has a left duality. In other words, for every object X of $\mathscr{C}$ there exist a triple $({}^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$. Then, the right dual is defined to be the triple $(X^\vee, \mathrm{ev}'_X, \mathrm{coev}'_X)$ where $X^\vee = {}^\vee X$ as objects and $\mathrm{ev}'_X := \mathrm{ev}_X \circ \tau_{X,X^\vee}$ and $\mathrm{coev}'_X := \tau_{X,{}^\vee X} \circ \mathrm{coev}_X$. The snake relations of the right dual follow from the snake relations of the left dual. □

In the following, we will consider a pair of categories with left duality and a strong monoidal functor between them and we will discuss relations between the left duals of each category. The case of categories with right duality is similar. We start with the fact that strong monoidal functors preserve objects with left duals. In particular, we have the following.

Proposition 2.4.11 ([38] p. 25)**.** *Let $\mathscr{C}$ be a category with left duality and denote by $F : \mathscr{C} \to \mathscr{D}$ a strong monoidal functor. Then, for an object X of $\mathscr{C}$ with left dual $({}^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$, the object $F({}^\vee X)$ is a left dual to $F(X)$ in $\mathscr{D}$.*

Sketch of the proof. We give here only the evaluation map $(\mathrm{ev}_X)^F : F({}^\vee X) \otimes F(X) \to 1_{\mathscr{D}}$ and the coevaluation map $(\mathrm{coev}_X)^F : 1_{\mathscr{D}} \to F(X) \otimes F({}^\vee X)$. For more details the reader is advised to see [38, Section 1.6.4]. The following composite gives the evaluation

$$F({}^\vee X) \otimes F(X) \xrightarrow{\sim} F({}^\vee X \otimes X) \xrightarrow{F(\mathrm{ev}_X)} F(1_{\mathscr{C}}) \xrightarrow{\sim} 1_{\mathscr{D}}$$

and the following composite gives the coevaluation map

$$1_{\mathscr{D}} \xrightarrow{\sim} F(1_{\mathscr{C}}) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes {}^\vee X) \xrightarrow{\sim} F(X) \otimes F({}^\vee X).$$ □

Remark 2.4.12. Similarly, if $\mathscr{C}$ is a category with right duality and $F : \mathscr{C} \to \mathscr{D}$ is a strong monoidal functor, then for an object X of $\mathscr{C}$ with right dual $X^\vee$, the object $F(X^\vee)$ is a right dual to $F(X)$ in $\mathscr{D}$.

Next, we define what is a lift of a left duality along a strong monoidal functor.

Definition 2.4.13. Let $F : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor between categories with left duality. Then, a *lift* of a left dual $({}^\vee d, \mathrm{ev}_d, \mathrm{coev}_d)$ in $\mathscr{D}$ along the strong monoidal functor F is a left dual $({}^\vee c, \mathrm{ev}_c, \mathrm{coev}_c)$ in $\mathscr{C}$ for all $c \in \mathscr{C}$ such that

$$\big(F({}^\vee c), (\mathrm{ev}_c)^F, (\mathrm{coev}_c)^F\big) = \big({}^\vee F(c), \mathrm{ev}_{F(c)}, \mathrm{coev}_{F(c)}\big)$$

For the following theorem, we use the following notation. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories. Then, we denote by $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ the opposite functor which as functor is equal to F.

Theorem 2.4.14 ([38] p. 25). *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with left duality and let $F : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor. Then, there is a natural isomorphism between the functors ${}^{\vee}(-) \circ F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ and $F \circ {}^{\vee}(-) : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. On the level of objects, the above natural isomorphism means that $F({}^{\vee}X) \xrightarrow{\sim} {}^{\vee}F^{\mathrm{op}}(X)$ for all objects X of $\mathscr{C}$.*

Proof. (Sketch) We will give here only the natural transformations ${}^{\vee}F^{\mathrm{op}}(X) \to F({}^{\vee}X)$ and $F({}^{\vee}X) \to {}^{\vee}F^{\mathrm{op}}(X)$ and for more details the reader is advised to see [38, Section 1.6.4].

$$\begin{aligned}
{}^{\vee}F^{\mathrm{op}}(X) &\to {}^{\vee}F^{\mathrm{op}}(X) \otimes 1_{\mathscr{D}} \to {}^{\vee}F^{\mathrm{op}}(X) \otimes F(1_{\mathscr{C}}) \to {}^{\vee}F^{\mathrm{op}}(X) \otimes F(X \otimes {}^{\vee}X) \\
&\to {}^{\vee}F^{\mathrm{op}}(X) \otimes \big(F(X) \otimes F({}^{\vee}X)\big) \to \big({}^{\vee}F^{\mathrm{op}}(X) \otimes F(X)\big) \otimes F({}^{\vee}X) \\
&\to 1_{\mathscr{D}} \otimes F({}^{\vee}X) \to F({}^{\vee}X)
\end{aligned}$$

$$\begin{aligned}
F({}^{\vee}X) &\to F({}^{\vee}X) \otimes 1_{\mathscr{D}} \to F({}^{\vee}X) \otimes F(1_{\mathscr{C}}) \to F({}^{\vee}X) \otimes \big(F(X) \otimes {}^{\vee}F(X)\big) \\
&\to \big(F({}^{\vee}X) \otimes F(X)\big) \otimes {}^{\vee}F(X) \to F({}^{\vee}X \otimes X) \otimes {}^{\vee}F(X) \\
&\to F(1_{\mathscr{C}}) \otimes {}^{\vee}F(X) \to 1_{\mathscr{D}} \otimes {}^{\vee}F(X) \to {}^{\vee}F(X)
\end{aligned}$$

□

Remark 2.4.15. Similarly, for categories $\mathscr{C}$ and $\mathscr{D}$ with right duality and a strong monoidal functor $F : \mathscr{C} \to \mathscr{D}$ between them, there is a natural isomorphism between the functors $(-)^{\vee} \circ F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ and $F \circ (-)^{\vee} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

The following theorem will be important in Section 7.2.

Theorem 2.4.16 ([40] Theorem 5.6). *If F is a strong monoidal functor, then U commutes with evaluation and coevalution. In other words, the diagrams below commute.*

$$\begin{tikzcd}
{}^{\vee}F^{\mathrm{op}}(X) \otimes F(X) \arrow[r] \arrow[dd] & 1 \\
& F(1) \arrow[u] \\
F({}^{\vee}X) \otimes F(X) \arrow[r] & F({}^{\vee}X \otimes X) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r] \arrow[d] & F^{\mathrm{op}}(X) \otimes {}^{\vee}F(X) \\
F(1) \arrow[d] & \\
F(X \otimes {}^{\vee}X) \arrow[r] & F(X) \otimes F({}^{\vee}X) \arrow[uu]
\end{tikzcd}$$

2.5 Closed categories

In this section, we discuss left closed categories and right closed categories following [7] and [23] closely.

Definition 2.5.1. Let $\mathscr{C}$ be a monoidal category. Then, $\mathscr{C}$ is said to be a *left closed category* if for each object X of $\mathscr{C}$ the functor $- \otimes X : \mathscr{C} \to \mathscr{C}$ has a right adjoint $\mathcal{H}\mathrm{om}(X,-) : \mathscr{C} \to \mathscr{C}$ with adjunction unit and adjunction counit to be defined by

$$\mathrm{coev}_Y^X : Y \to \mathcal{H}\mathrm{om}(X, Y \otimes X), \quad \mathrm{ev}_Y^X : \mathcal{H}\mathrm{om}(X,Y) \otimes X \to Y \tag{2.5.1}$$

such that the diagrams below commute for all objects X and Y of $\mathscr{C}$.

$$\begin{tikzcd}
Y \otimes X \arrow[r, "\mathrm{coev}^X_{Y\otimes X}"] \arrow[dr, "\mathrm{id}"'] & \mathcal{H}\mathrm{om}(X, Y \otimes X) \otimes X \arrow[d, "\mathrm{ev}^X_{X\otimes Y}"] \\
 & Y \otimes X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}\mathrm{om}(X,Y) \arrow[r, "\mathrm{coev}^X_{\mathcal{H}\mathrm{om}(X,Y)}"] \arrow[dr, "\mathrm{id}"'] & \mathcal{H}\mathrm{om}\big(X, \mathcal{H}\mathrm{om}(X,Y) \otimes X\big) \arrow[d, "\mathcal{H}\mathrm{om}(\mathrm{id}_X, \mathrm{ev}^X_Y)"] \\
 & \mathcal{H}\mathrm{om}(X,Y)
\end{tikzcd}$$

Notation 2.5.2. In the following, the functor $\mathcal{H}\mathrm{om}(X,-) : \mathscr{C} \to \mathscr{C}$ will be also called *internal hom functor.*

Categories with objects complexes will be of special importance in this thesis and the following example will be fundamental.

Example 2.5.3. For a commutative ring R the category $\mathrm{Ch}(R)$ of $\mathbb{Z}$-graded chain complexes of R-modules with differetial of degree -1 is a symmetric monoidal category, see Example 2.1.4 and the Example 2.3.5. Furthermore, the category $\mathrm{Ch}(R)$ is a left closed category with internal hom defined by

$$\mathcal{H}\mathrm{om}_R(X,Y)_n = \prod_i \mathcal{H}\mathrm{om}_R(X_i, Y_{i+n}), \qquad d(f)_i = d \circ f_i - (-1)^n f_{i-1} \circ d.$$

Categories with left duality can be made in to left closed categories. To be precise, we give the following example.

Example 2.5.4. Let $\mathscr{C}$ be a monoidal category with left duality, see Definition 2.4.1. Then, the category $\mathscr{C}$ is a left closed category with right adjoint to $- \otimes X$ to be given by $\mathcal{H}\mathrm{om}(X,Y) := Y \otimes {}^\vee X$ such that the counit is given by $\mathrm{ev}_Y^X := \mathrm{id}_Y \otimes \mathrm{ev}_X$ and the unit is given by $\mathrm{coev}_Y^X := \mathrm{id}_Y \otimes \mathrm{coev}_X$.

Next we define right closed categories.

Definition 2.5.5. Let $\mathscr{C}$ be a monoidal category. Then, $\mathscr{C}$ is said to be *a right closed category* if for each object X of $\mathscr{C}$ the functor $X \otimes - : \mathscr{C} \to \mathscr{C}$ has a right adjoint $[[X,-]] : \mathscr{C} \to \mathscr{C}$ with adjunction unit and adjunction counit given by

$$\widetilde{\mathrm{coev}}^X_Y : Y \to [[X, X \otimes Y]], \qquad \widetilde{\mathrm{ev}}^X_Y : X \otimes [[X,Y]] \to Y \tag{2.5.2}$$

for all objects X and Y of $\mathscr{C}$ satisfying relations similar to that of Definition 2.5.1.

In this thesis, we will be mostly interested in symmetric monoidal categories and especially in one category, that of the derived category of coherent sheaves, see Chapter 4. The following result establishes that a symmetric monoidal category which is left closed it is also right closed.

Lemma 2.5.6 ([23])**.** *Let $\mathscr{C}$ be a left closed and symmetric monoidal category. Then $\mathscr{C}$ is also a right closed category.*

Proof. (Sketch) Set $\widetilde{\mathrm{ev}}^X_Y : X \otimes [[X,Y]] \to Y$ and $\widetilde{\mathrm{coev}}^X_Y : Y \to [[X, X \otimes Y]]$ to be the composites $\widetilde{\mathrm{ev}}^X_Y := \mathrm{ev}^X_Y \circ \tau_{X,\mathcal{H}\mathrm{om}(X,Y)}$ and $\widetilde{\mathrm{coev}}^X_Y := \mathcal{H}\mathrm{om}(\mathrm{id}, \tau_{Y,X}) \circ \mathrm{coev}^X_Y$. Due to the symmetry relations of τ and triangle equations of ev^X_Y and coev^X_Y all the required triangles for being a right closed category are satisfied. □

Notation 2.5.7. A category which is left closed and right closed, will be referred simply as a *closed category*.

Next, we define the notion of a left closed functor.

Definition 2.5.8. Let $F : \mathscr{C} \to \mathscr{D}$ be a monoidal functor between left closed monoidal categories. Then, F is said to be a *left closed monoidal functor* if the map

$$\theta^l : F\mathcal{H}\mathrm{om}(X,Y) \to \mathcal{H}\mathrm{om}(FX, FY)$$

which is defined by

$$\begin{aligned} F\mathcal{H}\mathrm{om}(X,Y) &\xrightarrow{\mathrm{coev}^{FX}_{F\mathcal{H}\mathrm{om}(X,Y)}} \mathcal{H}\mathrm{om}(FX, F\mathcal{H}\mathrm{om}(X,Y) \otimes FX) \\ &\xrightarrow{\mathcal{H}\mathrm{om}(\mathrm{id},\beta)} \mathcal{H}\mathrm{om}(FX, FY) \end{aligned}$$

is an isomorphism, where β is the composite

$$\beta : F\mathcal{H}\mathrm{om}(X,Y) \otimes FX \to F(\mathcal{H}\mathrm{om}(X,Y) \otimes X) \xrightarrow{F(\mathrm{ev}^X_Y)} FY$$

for all X and Y objects of $\mathscr{C}$.

Similarly, it is defined the notion of a right closed monoidal functor.

Definition 2.5.9. Let $F : \mathscr{C} \to \mathscr{D}$ be a monoidal functor between right closed monoidal categories. Then, F is said to be a *right closed monoidal functor* if the map

$$\theta^r : F[[X,Y]] \to [[FX,FY]]$$

defined by

$$F[[X,Y]] \xrightarrow{\widetilde{\mathrm{coev}^{FX}_{F[[X,Y]]}}} [[FX, FX \otimes F[[X,Y]]]] \xrightarrow{[[\mathrm{id},\tilde{\beta}]]} [[FX,FY]] \tag{2.5.3}$$

is an isomorphism, where $\tilde{\beta}$ is the following

$$\beta : FX \otimes F[[X,Y]] \to F(X \otimes [[X,Y]]) \xrightarrow{F(\widetilde{\mathrm{ev}^X_Y})} FY$$

for all X and Y objects of $\mathscr{C}$.

Notation 2.5.10. A functor which is left closed and right closed will be called a *closed functor.*

In Chapter 7 we will be interested in closed functors between closed symmetric monoidal categories and the notion of a strong dualisable object of a category which we introduce below will be essential to determine whether a functor is a closed functor. In the following, we discuss strongly dualisable objects of a closed symmetric monoidal category following [13] closely.

Notation 2.5.11. In the following, we will denote by DX the object $\mathcal{H}\mathrm{om}(X, 1_{\mathscr{C}})$ of $\mathscr{C}$.

Definition 2.5.12. Let $\mathscr{C}$ be a closed symmetric monoidal category $\mathscr{C}$ and X an object of $\mathscr{C}$. Then, X is said to be *dualisable object* or a *strongly dualisable object* if the map

$$\nu : DX \otimes X \to \mathcal{H}\mathrm{om}(X,X) \tag{2.5.4}$$

defined to be the composite

$$DX \otimes X \xrightarrow{\mathrm{coev}^X_{DX\otimes X}} \mathcal{H}\mathrm{om}(X, (DX \otimes X) \otimes X) \xrightarrow{\mathcal{H}\mathrm{om}(\mathrm{id}_X,\gamma)} \mathcal{H}\mathrm{om}(X,X)$$

is an isomorphism where $\gamma : (DX \otimes X) \otimes X \xrightarrow{\mathrm{ev}^X_X \otimes \mathrm{id}_X} 1_{\mathscr{C}} \otimes X \cong X$.

Remark 2.5.13. It is worth mentioning here, that the map ν is defined as the adjunct of the map γ.

The following lemma, characterises the dual objects in closed symmetric monoidal categories.

Lemma 2.5.14 ([18] Proposition 2.2.9)**.** *Let $\mathscr{C}$ be a closed symmetric monoidal category and X be a strongly dualisable object of $\mathscr{C}$. Then, its dual is defined to be the object $\mathcal{H}\mathrm{om}(X, 1_{\mathscr{C}})$.*

The following lemma will be essential for determining whether the map θ of Definition 2.5.8 is an isomorphism.

Lemma 2.5.15 ([13]). *Let $\mathscr{C}$ be a closed symmetric monoidal category $\mathscr{C}$. Then, the map $v : DX \otimes Y \to \mathcal{H}\mathrm{om}(X, Y)$ which is obtained as the adjunct of the map*

$$DX \otimes Y \otimes X \cong DX \otimes X \otimes Y \xrightarrow{\mathrm{coev}^X_{DX\otimes X}} 1_{\mathscr{C}} \otimes Y \cong Y$$

is an isomorphism if either X or Y is a strongly dualisable object of $\mathscr{C}$.

Proposition 2.5.16 ([13]). *Let $F : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor between closed symmetric monoidal categories and let X an object of $\mathscr{C}$. If X is dualisable, then the objects DX, $F(X)$ and $D\big(F(X)\big)$ are dualisable and in particular we have*

$$F\big(DX\big) \cong D\big(F(X)\big). \tag{2.5.5}$$

The following theorem establishes that the existence of strongly dualisable objects in a closed symmetric monoidal category determines closed monoidal fucntors.

Theorem 2.5.17 ([13]). *Let $F : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor between closed symmetric monoidal categories and let $U : \mathscr{D} \to \mathscr{C}$ be the right adjoint of F and X of $\mathscr{C}$ is strongly dualizable object of $\mathscr{C}$. Then the natural map*

$$\theta : F\mathcal{H}\mathrm{om}(X, Y) \to \mathcal{H}\mathrm{om}\big(F(X), F(Y)\big)$$

as defined in 2.5.8 is an isomoprhism for all Y of $\mathscr{C}$. Moreover, the map

$$\pi : Y \otimes U(X) \to U(F(Y) \otimes X)$$

as defined in 2.1.1 is an isomorphism for all Y of $\mathscr{D}$. If all the objects of the category are strongly dualisable then the maps θ and π are isomorphisms.

Proof. By Lemma 2.5.15, the strong monoidality of F, the Proposition 2.5.16 and Lemma 2.5.15, we obtain the following chain of maps which proves that the map θ is an isomorphism.

$$F\mathcal{H}\mathrm{om}(X, Y) \cong F(DX \otimes Y) \cong F(DX) \otimes FY \cong D(F(X)) \otimes FY \cong \mathcal{H}\mathrm{om}(F(X), FY).$$

Now for the second map, we have the following chain of maps.

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{D}}\big(Z, Y \otimes U(X)\big) &\cong \mathrm{Hom}_{\mathscr{D}}\big(DY \otimes Z, U(X)\big) && \text{(Proposition 2.4.5)}\\
&\cong \mathrm{Hom}_{\mathscr{C}}\big(F(DY \otimes Z), X\big) && (F \dashv U)\\
&\cong \mathrm{Hom}_{\mathscr{C}}\big(F(DY) \otimes F(Z), X\big) && (F \text{ strong monoidal})\\
&\cong \mathrm{Hom}_{\mathscr{C}}\big(D\big(F(Y)\big) \otimes F(Z), X\big) && \text{(Proposition 2.5.16)}\\
&\cong \mathrm{Hom}_{\mathscr{C}}\big(F(Z), F(Y) \otimes X\big) && \text{(Proposition 2.4.5)}\\
&\cong \mathrm{Hom}_{\mathscr{D}}\big(Z, U(F(Y) \otimes X)\big) && (F \dashv U)
\end{aligned}$$

By Yoneda lemma we have that π is an isomorphism. □

Chapter 3

Derived categories of abelian categories

In this chapter, we discuss derived categories of abelian categories and how the derived functors between such categories are constructed. Derived categories were introduced by Grothendieck and Verdier as the suitable notion of a category on which the derived functors of homological algebra can be defined naturally. Our presentation follows [14] and [16] closely.

Notation 3.0.1. In this section, $\mathscr{A}$ will denote an abelian category, see Definition 1.3.9.

3.1 The derived category of an abelian category

In this section, we give the definition of the derived category of an abelian category and some properties. The basic idea is to define the derived category of an abelian category as the localisation of a category of complexes on a specific class of morphisms.

Definition 3.1.1. Let $\mathscr{A}$ be an abelian category. We define the category $\mathrm{Ch}(\mathscr{A})$ to be the category of cochain complexes in $\mathscr{A}$. Objects of $\mathrm{Ch}(\mathscr{A})$ are unbounded cochain complexes of the form

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots, \qquad A^i \in \mathscr{A}$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$ and its morphisms are maps between cochain complexes which commute with the differentials.

For example, a morphism $f^\bullet : A^\bullet \to B^\bullet$ between cochain complexes is depicted in the following diagram and it is true that $f^i \circ d_A^{i-2} = d_B^{i-1} \circ f^{i-1}$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-2}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A^{i-1}"] \arrow[d, "f^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"'] & B^i \arrow[r, "d_B^i"'] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Notation 3.1.2. We will denote by $\mathrm{Ch}^+(\mathscr{A})$ the subcategory of $\mathrm{Ch}(\mathscr{A})$ with objects $A^\bullet$ such that $A^i = 0$ for $i < i_0$ for some integer i_0. Similarly, we will denote by $\mathrm{Ch}^-(\mathscr{A})$ the subcategory of $\mathrm{Ch}(\mathscr{A})$ with objects with objects $A^\bullet$ such that $A^i = 0$ for $i > i_0$ for some integer i_0. Also, let $\mathrm{Ch}^b(\mathscr{A}) = \mathrm{Ch}^+(\mathscr{A}) \cap \mathrm{Ch}^-(\mathscr{A})$ with objects $A^\bullet$ such that $A^i = 0$ for $|i| > i_0$ for some integer i_0.

Proposition 3.1.3 ([16] Proposition 2.3)**.** *The category* $\mathrm{Ch}(\mathscr{A})$ *of complexes in* $\mathscr{A}$ *is an abelian category.*

Remark 3.1.4. There is a fully faithful functor $J : \mathscr{A} \to \mathrm{Ch}(\mathscr{A})$. Indeed, every object A of $\mathscr{A}$ is assigned to the complex depicted below

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

and any map $f : A \to B$ of objects of $\mathscr{A}$ is assigned to a map of complexes as it is depicted below.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Now, for a cochain complex $A^\bullet$ of $\mathrm{Ch}(\mathscr{A})$, the i-th cohomology object is the quotient

$$H^i(A^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1})$$

and the operation which assigns the i-th cohomology object to an object $A^\bullet$ of $\mathrm{Ch}(\mathscr{A})$ defines a functor

$$H^i : \mathrm{Ch}(\mathscr{A}) \longrightarrow \mathscr{A}. \tag{3.1.1}$$

What we are really want to do in the following is to consider complexes in some abelian category $\mathscr{A}$ up to some specific equivalence relation. The following definition is essential for doing this.

Definition 3.1.5. Let $\mathscr{A}$ be an abelian category and let $A^\bullet$ and $B^\bullet$ be objects of $\mathrm{Ch}(\mathscr{A})$. Then a map $f^\bullet : A^\bullet \to B^\bullet$ of cochain complexes is said to be a *quasi-isomorphism* if f induces isomorphisms on all cohomology objects. In other words, if the map $H^i(f^\bullet) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

Notation 3.1.6. Two complexes $A^\bullet$ and $B^\bullet$ are said to be *quasi-isomorphic* if they are related by a chain of quasi-isomorphisms; symbolically

$$A^\bullet \longleftarrow \mathscr{E}^\bullet \longrightarrow \cdots \longleftarrow \mathscr{F}^\bullet \longrightarrow B^\bullet$$

where $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ are objects of $\mathrm{Ch}(\mathscr{A})$.

In the following, we give a formal construction of the derived category of an abelian category via localisation.

Theorem 3.1.7 ([14] III.2)**.** *Let $\mathscr{A}$ be an abelian category and $\mathrm{Ch}(\mathscr{A})$ the category of cochain complexes in $\mathscr{A}$. Then, there exists a category $D(\mathscr{A})$ and a functor*

$$l : \mathrm{Ch}(\mathscr{A}) \to D(\mathscr{A})$$

such that the following are satisfied:

1. *Any quasi-isomorphism $f^\bullet : A^\bullet \to B^\bullet$ between complexes in $\mathrm{Ch}(\mathscr{A})$ is mapped to an isomorphim $l(f)$ in $D(\mathscr{A})$.*

2. *Any functor $G : \mathrm{Ch}(\mathscr{A}) \to \mathscr{C}$ which maps quasi-isomorphisms to isomorphisms can be factorised in a unique way through $D(\mathscr{A})$ i.e. there is a unique functor $F : D(\mathscr{A}) \to \mathscr{C}$ such that $G = F \circ l$; symbolically we have the following commutative triangle.*

$$\begin{tikzcd} \mathrm{Ch}(\mathscr{A}) \arrow[r, "l"] \arrow[d, "G"'] & D(\mathscr{A}) \arrow[dl, "F"] \\ \mathscr{C} & \end{tikzcd}$$

The category $D(\mathscr{A})$ is called the derived category of the abelian category of $\mathscr{A}$.

Recall from Remark 3.1.4 that an object A of $\mathscr{A}$ can be considered as a complex concentrated in degree zero. Then we have the following.

Corollary 3.1.8 ([14] III.5.2)**.** *The abelian category $\mathscr{A}$ is equivalent to the the full subcategory of $D(\mathscr{A})$ which consists of all complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$ for $i \neq 0$.*

This enable us to describe the derived category of an abelian category $\mathscr{A}$ as the category with the same objects as $\mathrm{Ch}(\mathscr{A})$ and morphisms formally inverting quasi-isomorphisms.

Remark 3.1.9. Similarly, localisation of the categories $\mathrm{Ch}^+(\mathscr{A})$, $\mathrm{Ch}^-(\mathscr{A})$ and $\mathrm{Ch}^b(\mathscr{A})$ on the class of quasi-isomorphisms yields the derived categories $D^+(\mathscr{A})$, $D^-(\mathscr{A})$ and $D^b(\mathscr{A})$.

Working with complexes it is natural to consider maps between cochain complexes up to homotopy. In the following, we introduce the homotopy category of $\mathrm{Ch}(\mathscr{A})$ and establish a relation between the derived category of an abelian category with the associated homotopy category. This relation will be useful later to discuss derived functors on the level of derived categories.

Definition 3.1.10. Let $f, g : A^\bullet \to B^\bullet$ be two maps between cochain complexes. We say that f and g are *homotopically equivalent* and write $f \sim g$, if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1}$ for all $i \in \mathbb{Z}$ such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_b^{i-1} \circ h^i.$$

Definition 3.1.11. Let $\mathscr{A}$ be an abelian category. Then, the *homotopy category* $\mathrm{Ho}(\mathscr{A})$ of $\mathscr{A}$ is the category of which objects are cochain complexes of objects in $\mathscr{A}$ and its morphisms are maps between cochain complexes which are homotopically equivalent.

Definition 3.1.12. Let $f^\bullet : A^\bullet \to B^\bullet$ a map of cochain complexes. The *mapping cone* of $f^\bullet$ is the complex $C(f^\bullet)$ defined by

$$C^\bullet(f) := A^{i+1} \oplus B^i, \qquad d^i_{C(f)} := \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix} \tag{3.1.2}$$

Notation 3.1.13. We will denote by $\mathrm{Ho}^+(\mathscr{A})$ the subcategory of $\mathrm{Ho}(\mathscr{A})$ with objects $A^\bullet$ such that $A^i = 0$ for $i < i_0$ for some integer i_0. Similarly, we will denote by $\mathrm{Ho}^-(\mathscr{A})$ the subcategory of $\mathrm{Ch}(\mathscr{A})$ with objects $A^\bullet$ such that $A^i = 0$ for $i > i_0$ for some integer i_0. Also, let $\mathrm{Ho}^b(\mathscr{A}) = \mathrm{Ho}^+(\mathscr{A}) \cap \mathrm{Ho}^-(\mathscr{A})$ which has objects $A^\bullet$ such that $A^i = 0$ for $|i| > i_0$ for some integer i_0.

The following result establishes the relation between $D(\mathscr{A})$ and $\mathrm{Ho}(\mathscr{A})$ for some abelian category $\mathscr{A}$.

Theorem 3.1.14 ([14] III.4.2)**.** *Let $\mathscr{A}$ be an abelian category and $\mathrm{Ho}(\mathscr{A})$ its homotopy category. Then, the localisation of $\mathrm{Ho}(\mathscr{A})$ at the class of quasi-isomorphisms is canonically isomorphic to the derived category $D(\mathscr{A})$. The same is true for $\mathrm{Ho}^*(\mathscr{A})$ and $D^*(\mathscr{A})$ for $* = +, -, b$.*

In the following, we give a description of the objects, of morphisms and the composition of morphisms in the derived category $D(\mathscr{A})$ of an abelian category $\mathscr{A}$.

Proposition 3.1.15 ([14] III.2.8)**.** *Let $\mathscr{A}$ be an abelian category.*

1. *The objects of $\mathrm{Ch}(\mathscr{A})$, of $\mathrm{Ho}(\mathscr{A})$ and $D(\mathscr{A})$ are the same; symbolically*

$$\mathrm{Ob}\left(\mathrm{Ch}^*(\mathscr{A})\right) = \mathrm{Ob}\left(\mathrm{Ho}^*(\mathscr{A})\right) = \mathrm{Ob}\left(D^*(\mathscr{A})\right)$$

for $* = b, +, -, \emptyset$.

2. *Let X, Y be two objects of $D(\mathscr{A})$. A morphism from X to Y is an equivalence class of* roofs, *i.e. of diagrams of the form*

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$

where s is a quasi-isomorphism and t is a map in $\mathrm{Ch}(\mathscr{A})$, *such that any two roofs (s_1, t_1) and (s_2, t_2) are equivalent if there exists a third roof (s', t') forming commutative diagrams of the following form.*

$$\begin{tikzcd} & & Z \arrow[dl, "s_3"'] \arrow[dr, "t_3"] & & \\ & X' \arrow[dl, "s_1"'] \arrow[drrr, "t_1"' near end] & & Y' \arrow[dr, "t_2"] \arrow[dlll, "s_2" near end] & \\ X & & & & Y \end{tikzcd}$$

where s_1, s_2 and s_3 are quasi-isomorphisms and t_1, t_2 and t_3 are morphisms of $\mathrm{Ch}(\mathscr{A})$. *The identity morphism* $\mathrm{id} : X \to Y$ *is the class of the roof* $(\mathrm{id}_X, \mathrm{id}_X)$. *The same is true for morphisms in $D^*(\mathscr{A})$, for $* = +, -, b$.*

3. *Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be two morphisms in $D(\mathscr{A})$ which are represented by the roofs (s, t) and (s', t'). To define the composition of f and g, we need a third roof (κ, λ) such that $t \circ \kappa = s' \circ \lambda$ of which the existence is proved in [14, III.2.6(b)], and the composite $g \circ f$ is represented by the class $(\kappa s, \lambda t)$. Diagrammatically we have the following.*

$$\begin{tikzcd} & & C \arrow[dl, "\kappa"'] \arrow[dr, "\lambda"] & & \\ & A \arrow[dl, "s"'] \arrow[dr, "t"] & & B \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\ X \arrow[rr, "f"'] & & Y \arrow[rr, "g"'] & & Z \end{tikzcd}$$

Proposition 3.1.16 ([14] III.4.5)**.** *Let $\mathscr{A}$ be an abelian and let $D(\mathscr{A})$ be the associated derived category. Then, $D(\mathscr{A})$ is an additive category.*

The derived category of an abelian category is not an abelian category. Instead it has the structure of a *triangulated category*. That is an additive category with an additive autoequivalence, called the *translation* or *shift* and a collection of diagrams called the *distinguished triangles*. In the case at hand, the additive autoequivalence is given by the shift functor $[1] : D(\mathscr{A}) \to D(\mathscr{A})$ which is defined as follows. On the level of objects, the complex $A^\bullet[1]$ is defined by $\left(A^\bullet[1]\right)^i := A^{i+1}$ with differential $d^i_{A^\bullet[1]}$ to be $(-1)^n d^{i+n}_{A^\bullet}$ and on the level of morphisms, the shifted morphism $f^\bullet[1]$ is the morphism of complexes $A^\bullet[1] \to B^\bullet[1]$ defined by $\left(f^\bullet[1]\right)^i := f^{i+1}$.

Next, we discuss the distinguished triangles in $\mathrm{Ho}(\mathscr{A})$ and so in $D(\mathscr{A})$. Assume first that $f^\bullet : A^\bullet \to B^\bullet$ is a morphism of cochain complexes and let $C^\bullet(f^\bullet)$ be the mapping cone of $f^\bullet$. Then, a distinguished triangle in $\mathrm{Ho}(\mathscr{A})$ is a diagram of the form

$$X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$$

which is isomorphic to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet(f) \to A^\bullet[1]$$

Theorem 3.1.17 ([14] IV.2)**.** *Let $\mathscr{A}$ be an abelian category. Then, the derived category of $\mathscr{A}$ is a triangulated category with triangles as discussed above.*

In what we will be really interested in are exact functors between triangulated categories. Below we give the precise definition.

Definition 3.1.18. An additive functor $F : \mathscr{A} \to \mathscr{B}$ between triangulated categories is said to be an *exact functor* if the following are satisfied:

- There exists a natural isomorphism $\phi : F \circ [1] \xrightarrow{\sim} [1] \circ F$
- Any distinguished triangle

  $$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

  in $\mathscr{A}$ is mapped to

  $$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi \circ F(W)} \big(F(X)\big)[1]$$

  which is a distinguished triangle in $\mathscr{B}$.

Lemma 3.1.19. *Let $\mathscr{A}$ be a triangulated category and $[1] : \mathscr{A} \to \mathscr{A}$ be the shift functor. Then, the functor $[1]$ is an exact functor between triangulated categories.*

Proof. This follows by taking $\phi = -\mathrm{id}_{[2]}$ in the Definition (3.1.18) and applying Lemma 8.1.3. □

Notation 3.1.20. In the following, when we mention that the suspension functor is an exact functor we will write $([1], -\mathrm{id}_{[2]})$.

In Section 1.4 we discussed the notion of a Serre functor on a k-linear, hom-finite category.

Theorem 3.1.21 (Bondal-Kapranov [6]). *Any Serre functor on a triangulated category over a field k is an exact functor.*

3.2 Derived functors

In this section, we discuss how an additive functor between abelian categories can be extended to a functor on the level of derived categories. A functor between derived categories should map quasi-isomorphisms to quasi-isomorphisms and distinguished triangles to distinguished. Our presentation follows [14] closely.

We start with the case of an exact functor between abelian categories. Then, the following proposition establishes that exact functors extend to functors between derived categories.

Proposition 3.2.1 ([14] III.6.2). *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive exact functor between abelian categories, see Definition 1.3.13. Then, the induced functor on the level of homotopy categories*

$$\mathrm{Ho}^*(F) : \mathrm{Ho}^*(\mathscr{A}) \to \mathrm{Ho}^*(\mathscr{B}), \qquad * = \emptyset, +, -, b$$

defined by $\mathrm{Ho}^*(F)(\mathscr{A}^\bullet) := F(A^i)$ for all A^i in $\mathscr{A}$ and $i \in \mathbb{Z}$, maps quasi-isomorphisms to quasi-isomorphisms and induces a functor on the level of derived categories

$$D^*(F) : D^*(\mathscr{A}) \to D^*(\mathscr{B}).$$

Moreover, the functor $D^*(F)$ is an exact functor. In particular, it maps distinguished triangles to distinguished triangles.

If we have a left exact functor between abelian categories, then its right derived functor is defined as follows.

Definition 3.2.2 ([14] III.6.6). Let $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor between abelian categories. Then, the *right derived functor of F* is a pair (RF, ϵ_F) such that $RF : D^+(\mathscr{A}) \to D^+(B)$ is an exact functor and $\epsilon_F : l_{\mathscr{B}} \circ \mathrm{Ho}^+(F) \to RF \circ l_{\mathscr{A}}$ is a natural transformation such that the following universal property is satisfied. For any exact functor $G : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ and any natural transformation $\epsilon : l_{\mathscr{B}} \circ \mathrm{Ho}^+(F) \to G \circ l_{\mathscr{A}}$ there exists a unique natural transformation $\eta : RF \to G$ such that the following diagram commutes.

$$\begin{tikzcd} & l_{\mathscr{B}} \circ \mathrm{Ho}^+(F) \arrow[dl, "\epsilon_F"'] \arrow[dr, "\epsilon"] & \\ RF \circ l_{\mathscr{A}} \arrow[rr, "\eta \circ l_{\mathscr{A}}"'] & & G \circ l_{\mathscr{A}} \end{tikzcd}$$

Remark 3.2.3. In a similar way, if $F' : \mathscr{A} \to \mathscr{B}$ is an additive right exact functor between abelian categories, then the *left derived functor of* F' is a pair $(LF', \epsilon_{F'})$ where $LF' : D^-(\mathscr{B}) \to D^-(\mathscr{A})$ and $\epsilon_{F'} : LF \circ l_{\mathscr{B}} \to l_{\mathscr{A}} \circ \mathrm{Ho}^-(F')$ such that a similar universal property is satisfied.

In the above we just gave the definition of a right derived functor of a left exact functor. Next, we describe how we construct such right derived functors. To do this, we need the notion of an adapted class.

Definition 3.2.4. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive left exact functor between abelian categories. Then, a class $\mathscr{R}$ of objects of $\mathscr{A}$ is said to be *adapted to F* if it is stable under finite direct sums and the following two conditions are satisfied.

- The functor F maps acyclic complexes of $\mathrm{Ch}^+(\mathscr{R})$ into acyclic complex of $\mathrm{Ch}^+(\mathscr{B})$.
- Any object of $\mathscr{A}$ is a sub-object of an object of $\mathscr{R}$.

Remark 3.2.5. In a similar way it is defined the notion of an adapted class for a right exact functor.

Moreover, we need the following proposition.

Proposition 3.2.6 ([14] III.5.4 and III.5.8)**.** *Let $\mathscr{A}$ be an abelian category with $\mathscr{R}$ being a class of objects of $\mathscr{A}$ adapted to a left exact functor $F : \mathscr{A} \to \mathscr{B}$ and $\mathscr{S}_{\mathscr{R}}$ be a class of quasi-isomorphisms in* $\mathrm{Ho}^+(\mathscr{R})$*. Then, $\mathscr{S}_{\mathscr{R}}$ is a localising class of morphisms in* $\mathrm{Ho}^+(\mathscr{R})$ *and the canonical functor*

$$\mathrm{Ho}^+(\mathscr{R})[\mathscr{S}_{\mathscr{R}}^{-1}] \to D^+(\mathscr{A})$$

is an equivalence of categories. A similar statement is true for right exact functors.

Following [14, III.5.5] we define the right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ as follows. First we fix an equivalence $\Theta^{-1} : D^+(\mathscr{A}) \to \mathrm{Ho}^+(\mathscr{R})[\mathscr{S}_{\mathscr{R}}^{-1}]$. Then, define $\widetilde{F} : \mathrm{Ho}^+(\mathscr{R})[\mathscr{S}_{\mathscr{R}}^{-1}] \to \mathrm{Ho}^+(\mathscr{B})$ by sending a complex $A^\bullet$ to $\left(\widetilde{F}(A^\bullet)\right)^i = F(A^i)$. Then define RF to be the composite $RF(A^\bullet) = (l_{\mathscr{B}} \circ \widetilde{F} \circ \Theta^{-1})(A^\bullet)$. Pictorially, we have the following diagram.

$$\begin{tikzcd}
\mathrm{Ho}^+(\mathscr{R})[\mathscr{S}_{\mathscr{R}}^{-1}] \arrow[r, "\widetilde{F}"] & \mathrm{Ho}^+(\mathscr{B}) \arrow[d, "l_{\mathscr{B}}"] \\
D^+(\mathscr{A}) \arrow[u, "\Theta^{-1}"] \arrow[r, "RF"'] & D^+(\mathscr{B})
\end{tikzcd}$$

A natural question is what an adapted class of an abelian category would look like. The following definition is in this direction.

Definition 3.2.7. Let $\mathscr{A}$ be an abelian category.

- The category $\mathscr{A}$ is said to be a category with *enough injectives* if for every object A of $\mathscr{A}$ there exists an injective morphism $A \to I$ with I to be an injective object of $\mathscr{A}$.
- The category $\mathscr{A}$ is said to be a category with *enough projectives* if for every object A of $\mathscr{A}$ there exists an projective morphism $P \to A$ with P to be an projective object of $\mathscr{A}$.

Theorem 3.2.8 ([14] III.6.12)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. Then the following are true.*

- *If $\mathscr{A}$ is a category with enough injectives, then the class $\mathscr{I}$ of injective objects of $\mathscr{A}$ is adapted to an additive left exact functor $F : \mathscr{A} \to \mathscr{B}$.*
- *If $\mathscr{A}$ is a category with enough projectives, then the class $\mathscr{P}$ of projective objects of $\mathscr{A}$ is adapted to an additive right exact functor $F' : \mathscr{A} \to \mathscr{B}$.*

Remark 3.2.9. We have to underline here that it is possible to construct derived functors between derived categories of abelian categories which have neither enough injectives nor enough projectives. Such an example is the abelian of coherent sheaves which we discuss in a later section.

The following result describes how we restrict a functor between unbounded above derived categories to bounded derived categories and it will be useful later, when we will discuss the construction of derived functors between bounded derived categories of coherent sheaves.

Proposition 3.2.10 ([16] Corollary 2.68)**.** *Let $\mathscr{A}$ be an abelian category with enough injectives and let $F : \mathrm{Ho}^+(\mathscr{A}) \to \mathrm{Ho}^+(\mathscr{B})$ be an exact functor which admits a right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*

Then, if all objects A of $\mathscr{A}$ is mapped to $RF(A) \in D^b(\mathscr{B})$ then any complex $A^\bullet$ of $D^b(\mathscr{A})$ is mapped to $RF(A^\bullet) \in D^b(\mathscr{B})$. In other words, RF descends an exact functor $RF : D^b(\mathscr{A}) \to D^b(\mathscr{B})$.

Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories. Then, the higher derived functors R^iF associated to F are just the composites of the cohomology functors $H^i : \mathrm{Ch}(\mathscr{B}) \to \mathrm{Ab}$ with RF; symbolically we have $R^iF = H^i \circ RF$. A similar statement is true for the left derived functors.

Theorem 3.2.11 ([16] Proposition 2.56)**.** *Let A,B be objects of the abelian category $\mathscr{A}$. Then there exists an isomorphism $\mathrm{Ext}^i_{\mathscr{A}}(A,B) \cong \mathrm{Hom}_{D(\mathscr{A})}(A, B[i])$. Moreover, this can be generalised also in the case where $A^\bullet$ and $B^\bullet$ are complexes. In this case, we have a $\mathrm{Ext}^i_{\mathscr{A}}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathscr{A})}(A^\bullet, B^\bullet[i])$.*

Chapter 4

Derived categories of coherent sheaves

In this section, we consider a smooth complex projective variety X and discuss the bounded derived category of coherent sheaves on X. We are in particular interested in the various geometric functors which are defined between bounded derived categories of sheaves and the adjunctions which are formed between these functors.

The material of this chapter is presented in the following order. In Section 4.1 we discuss basic properties of the derived category of coherent sheaves on a smooth complex projective variety which will be essential in the following. In Section 4.2 we explain how to obtain derived functor defined between bounded derived categories of coherent sheaves. In Section 4.3 we discuss various adjunctions between the derived functors of coherent sheaves. Actually, we will be mainly interested in the adjunctions between the derived functors in the following. In Section 4.4 we discuss the dual objects of the bounded derived category of coherent sheaves which will play a fundamental role in Chapter 7 where we will investigate the Hopf algebra object in the derived category of coherent sheaves. Our presentation of the above material follows [15] and [16] closely.

Notation 4.0.1. In this chapter X will denote a smooth complex projective variety. Even if all the definitions and results can be defined in greater generality i.e. for schemes, we choose to restrict ourselves to varieties since this is the case we are interested in.

4.1 Basic definitions

Let X be a smooth complex projective variety and let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X, see Example 1.3.12 for the definition of a coherent sheaf. Coherent sheaves will be denoted by $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ and bounded complexes of coherent sheaves by $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$.

Definition 4.1.1. Let X be a smooth complex projective variety. We define the *bounded derived category of coherent sheaves on* X to be the category $D^b\big(\mathrm{Coh}(X)\big)$ which is obtained by localising $\mathrm{Ch}^b\big(\mathrm{Coh}(X)\big)$ on the class of quasi-isomorphisms.

Notation 4.1.2. In the following, we will use the notation $D^b(X)$ for the bounded derived category of coherent sheaves on a smooth complex projective variety X.

By Theorem 3.1.17, we have that $D^b(X)$ is a triangulated category. In particular, the translation functor is given by the shift functor $[1] : D^b(X) \to D^b(X)$ which is defined as follows. On the level of objects, the complex $\mathscr{E}^\bullet[1]$ is defined by $\big(\mathscr{E}^\bullet[1]\big)^i := \mathscr{E}^{i+1}$ with differential $d^i_{\mathscr{E}^\bullet[1]}$ to be $(-1)^n d^{i+n}_{\mathscr{E}^\bullet}$ and on the level of morphisms, the shifted morphism $f^\bullet[1]$ is the morphism of complexes $\mathscr{E}^\bullet[1] \to \mathscr{F}^\bullet[1]$ defined by $\big(f^\bullet[1]\big)^i := f^{i+1}$.

In a similar way, we can define the shift functor $[n] : D^b(X) \to D^b(X)$ where the complex $\mathscr{E}^\bullet[n]$ is defined by $\big(\mathscr{E}^\bullet[n]\big)^i := \mathscr{E}^{i+n}$ and differential $d^i_{\mathscr{E}^\bullet[n]}$ to be $(-n)^n d^{i+n}_{\mathscr{E}^\bullet}$.

Coherent sheaves $\mathscr{E}$ and $\mathscr{F}$ on X can be considered as complexes concentrated in degree zero and $\mathrm{Hom}_{D^b(X)}(\mathscr{E},\mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ where the right hand side of the isomorphism is the space of $\mathscr{O}_X$-module homomorphisms. Moreover, we have that

$$\mathrm{Ext}^n(\mathscr{E},\mathscr{F}) = \mathrm{Hom}_{D^b(X)}(\mathscr{E},\mathscr{F}[n]) \tag{4.1.1}$$

or for more general bounded complexes of of sheaves $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ we have

$$\mathrm{Ext}^n(\mathscr{E}^\bullet,\mathscr{F}^\bullet) = \mathrm{Hom}_{D^b(X)}(\mathscr{E}^\bullet,\mathscr{F}^\bullet[n]). \tag{4.1.2}$$

The following two results characterise the bounded complexes of coherent sheaves.

Proposition 4.1.3 ([16] Proposition 3.13)**.** *Let X be a smooth complex projective variety of dimension n and $\mathscr{E}$ and $\mathscr{F}$ be coherent sheaves on X. Then* $\mathrm{Ext}^i(\mathscr{E},\mathscr{F}) = 0$ *for $i > n$.*

Proposition 4.1.4 ([16] Proposition 3.26)**.** *If X is smooth projective variety over a field k, then any $\mathscr{F}^\bullet \in D^b(X)$ is isomorphic to a bounded complex $\mathscr{G}^\bullet \in D^b(X)$ of locally free sheaves $\mathscr{G}^i$.*

We will be mainly interested in functors defined on the level of bounded derived categories of coherent sheaves and of particular importance is the Serre functor which we discuss now.

For a smooth complex projective variety X of dimension d with canonical bundle $\omega_X := \bigwedge^{\dim X} T^*_X$ there is a functor denoted by $S_X : D^b(X) \to D^b(X)$ which is defined by the composite

$$D^b(X) \xrightarrow{(-)\otimes\omega_X} D^b(X) \xrightarrow{[\dim(X)]} D^b(X). \tag{4.1.3}$$

Theorem 4.1.5 ([6]Bondal-Kapranov)**.** *The functor $S_X : D^b(X) \to D^b(X)$ defined by*

$$S_X(-) = (-) \otimes \omega_X[\dim(X)] \tag{4.1.4}$$

is a Serre functor for $D^b(X)$ in the sense of Definition 1.4.1.

4.2 Derived functors between derived categories of sheaves

In this section, we will discuss the construction of the derived functors which associated to a smooth map $f : X \to Y$ between smooth complex projective varieties. Recall from Section 3.2 that we constructed right derived functors for a left exact functor assuming that the category has enough injectives. Similarly, to define a left derived functor to a right derived were based on the fact that the abelian category has enough projectives. However, the abelian category of coherent sheaves on X contains neither enough injectives nor enough projectives due to the finiteness conditions of coherent sheaves. The idea is to define derived functors on the level of unbounded derived categories of quasi-coherent sheaves and then restrict these functors between bounded derived categories of coherent sheaves. For full details the reader is advised to see [16, Section 3.3].

The following result establish that the abelian category of quasi-coherent sheaves has enough injectives.

Theorem 4.2.1 ([15] II.7.18)**.** *Let X be a smooth complex projective variety. Then any quasi-coherent sheaf $\mathscr{E}$ on X admits a resolution*

$$0 \to \mathscr{E} \to I^0 \to I^1 \to \cdots \tag{4.2.1}$$

of quasi-coherent sheaves I^i which are injectives as $\mathscr{O}_X$-modules.

The following results we be useful in the following.

Theorem 4.2.2 ([15]Proposition 3.3)**.** *For a smooth complex projective variety X, the natural functor*

$$D^*(\mathrm{Qcoh}(X)) \to D^*_{qcoh}\big(Sh_{\mathscr{O}_X}(X)\big)$$

for $* = b, +$, is an equivalence of categories.

Proposition 4.2.3 ([16] Proposition 3.5)**.** *Let X be a smooth complex projective variety. Then, there is an equivalence of categories between the bounded derived category of coherent sheaves on X and the full triangulated subacategory of $D^b\big(\mathrm{Qcoh}(X)\big)$ which consists of bounded complexes of quasi-coherent sheaves with coherent cohomology; symbolically*

$$D^b(X) \xrightarrow{\simeq} D^b_{coh}\big(\mathrm{Qcoh}(X)\big) \subset D^b\big(\mathrm{Qcoh}(X)\big).$$

In the following, we explain how to obtain the derived direct image functor, the derived internal hom functor and the derived tensor product functor on the level of bounded derived categories of coherent sheaves on X. Once we will define them, then we will not need it again. What will be of particular interest for us and for our purposes is the adjunctions which are formed between these derived functors.

The derived direct image functor

Let $f\colon X \to Y$ be a smooth map of smooth complex projective varieties. On the level of quasi-coherent sheaves is defined the functor $f_*: \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ which is left exact. Since the category of quasi-coherent sheaves has enough injectives it is defined the right derived functor

$$Rf_*: D^+\big(\mathrm{Qcoh}(X)\big) \to D^+\big(\mathrm{Qcoh}(Y)\big). \tag{4.2.2}$$

In order to restrict the derived functor Rf_* to the level of bounded complexes, the following result is essential.

Theorem 4.2.4 ([16] Theorem 3.22)**.** *Let $f\colon X \to Y$ be a map of smooth complex projective varieties and $\mathscr{E}$ be a quasi-coherent sheaf on X. Then, the higher direct image sheaves $R^i f_*\mathscr{E}$ are trivial for $i > \dim(X)$.*

As a result, there exists the functor $Rf_*: D^b\big(\mathrm{Qcoh}(X)\big) \to D^b\big(\mathrm{Qcoh}(Y)\big)$. Now, in order to obtain Rf_* on the level of coherent sheaves we use the result below.

Theorem 4.2.5 ([16] Theorem 3.23)**.** *Let $f\colon X \to Y$ be a smooth map of smooth complex projective varieties and $\mathscr{F}$ be a coherent sheaf on X. Then, the higher direct images $R^i f_*(\mathscr{F})$ are coherent.*

To sum up the above discussion, we define the functor

$$Rf_*: D^b(X) \to D^b(Y)$$

to be the composition of the inclusion $D^b(X) \to D^b(\mathrm{Qcoh}(X))$ —see Proposition 4.2.3— followed by the functor $Rf_*: D^b\big(\mathrm{Qcoh}(X)\big) \to D^b\big(\mathrm{Qcoh}(Y)\big)$ and finally applying Theorem 4.2.5.

Remark 4.2.6. For the smooth map $f\colon X \to \mathrm{Spec}(k)$ the associated direct image functor f_* on the level of coherent sheaves is the global sections functor

$$\Gamma: \mathrm{Coh}(X) \to \mathrm{Vect}_f(k).$$

Its right derived functor is $R\Gamma: D^b(X) \to D^b\big(\mathrm{Vect}_f(k)\big)$.

Derived internal hom functor

For a quasi-coherent sheaf $\mathscr{E}$ on X is defined the left exact functor

$$\mathcal{H}om(\mathscr{E},-) : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X) \tag{4.2.3}$$

which is called the *internal hom functor*. If in particular $\mathscr{E}$ is coherent, then we have a left exact functor

$$\mathcal{H}om(\mathscr{E},-) : \mathrm{Coh}(X) \to \mathrm{Coh}(X) \tag{4.2.4}$$

and we have that $\Gamma \circ \mathcal{H}om(\mathscr{E},-) = \mathrm{Hom}(\mathscr{E},-)$ where Γ is the global section functor and $\mathrm{Hom}(\mathscr{E},-) : \mathrm{Coh}(X) \to D^b\big(\mathrm{Vect}_f(k)\big)$.

Now, since the category $\mathrm{Qcoh}(X)$ has enough injectives, the right derived functor $R\mathcal{H}om(\mathscr{E},-) : D^+\big(\mathrm{Qcoh}(X)\big) \to D^+\big(\mathrm{Qcoh}(X)\big)$ is well defined. Now, for any two quasi-coherent sheaves $\mathscr{E}$ and $\mathscr{F}$ we set

$$\mathcal{E}xt^i(\mathscr{E},\mathscr{F}) := R^i\mathcal{H}om(\mathscr{E},\mathscr{F}).$$

If $\mathscr{E}$ is a coherent sheaf, then we have that $\mathcal{E}xt^i(\mathscr{E},\mathscr{F})_x \simeq \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x)$ and if both $\mathscr{E}$ and $\mathscr{F}$ are coherent then, $\mathcal{E}xt^i(\mathscr{E},\mathscr{F})$ is coherent. As a result, we obtain the right derived functor

$$R\mathcal{H}om(\mathscr{E},-) : D^+(X) \to D^+(X). \tag{4.2.5}$$

The above functor can be defined for complexes of coherent sheaves and so we have the functor

$$R\mathcal{H}om^\bullet(-,-) : D^b(X)^{\mathrm{op}} \times D^b(X) \to D^b(X)$$

Remark 4.2.7. Since we work with a smooth complex projective variety X, by Proposition 4.1.4 we can replace any complex $\mathscr{E}^\bullet$ of $D^b(X)$ with a bounded complex of locally free sheaves and compute $R\mathcal{H}om^\bullet(\mathscr{E}^\bullet,-)$ as the underived $\mathcal{H}om^\bullet(\mathscr{E}^\bullet,-)$.

Derived tensor product

Let $\mathscr{E}$ be a coherent sheaf on a smooth complex projective variety X. Then, the functor $\mathscr{E}\otimes - : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ is a right exact functor. Since we work with a smooth projective variety X by Proposition 4.1.4 any coherent sheaf admits a resolution of locally free sheaves of length n. Tensoring $\mathscr{E}$ with a bounded above acyclic complex $\mathscr{F}^\bullet$ of locally free sheaves, then $\mathscr{E}\otimes\mathscr{F}^\bullet$ is still acyclic. That yields that the class of locally free sheaves in $\mathrm{Coh}(X)$ is adapted to the right exact functor $\mathscr{E}\otimes -$ and hence the left derived functor $\mathscr{E}\otimes^L - : D^-(X) \to D^-(X)$ exists and it is well defined. Since X is smooth and projective and combined this with Proposition 4.1.4 the derived tensor product functor restricts to bounded complexes.

$$\mathscr{E}\otimes^L - : D^b(X) \to D^b(X). \tag{4.2.6}$$

Replacing the coherent sheaf $\mathscr{E}$ by a complex of coherent sheaves $\mathscr{E}^\bullet$ and applying similar arguments with the above, we obtain the left derived functor

$$\mathscr{E}^\bullet \otimes^L - : D^-(X) \to D^-(X) \tag{4.2.7}$$

such that

$$(\mathscr{E}^\bullet \otimes \mathscr{F}^\bullet)^k := \bigoplus_{i+j=k} \mathscr{E}^i \otimes \mathscr{F}^j, \qquad d = d_{\mathscr{E}} \otimes 1 + (-1)^k 1 \otimes d_{\mathscr{E}} \tag{4.2.8}$$

and since X is a smooth projective variety this yields a functor

$$- \otimes^L - : D^b(X) \times D^b(X) \to D^b(X). \tag{4.2.9}$$

The following proposition states that the derived tensor product is a associative and symmetric.

Proposition 4.2.8 ([15] II.5.13)**.** *Let X be a smooth complex projective variety. Then, for all $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ of $D^b(X)$ there exist the following natural isomorphisms.*

$$\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{F}^\bullet \otimes^L \mathscr{E}^\bullet \tag{4.2.10}$$

$$(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet) \otimes^L \mathscr{G}^\bullet \xrightarrow{\sim} \mathscr{E}^\bullet \otimes^L (\mathscr{F}^\bullet \otimes^L \mathscr{G}^\bullet) \tag{4.2.11}$$

Proof. (Sketch) Any bounded complex in $D^b(X)$ can be replaced by a bounded complex of locally free sheaves since X is a smooth projective variety and Proposition 4.1.4. Hence, the derived tensor product in the natural isomorphisms (4.2.10) and (4.2.11) need not be derived anymore and the category of cochain complexes is a symmetric monoidal category under the underived tensor product. □

Notation 4.2.9. From now on, we will treat the bounded derived category $D^b(X)$ of a smooth complex projective variety X as a symmetric monoidal category under the derived tensor product with monoidal unit the structure sheaf of $\mathscr{O}_X$. In other words, that it satisfies all the axioms of a symmetric monoidal category as it was defined in Chapter 2.

The derived pullback

Let $f\colon X \to Y$ be a smooth map of smooth complex projective varieties. On the level of sheaves of $\mathscr{O}_X$ modules there exists the right exact functor

$$f^* : \mathrm{Sh}_{\mathscr{O}_Y}(Y) \to \mathrm{Sh}_{\mathscr{O}_X}(X) \tag{4.2.12}$$

which is defined to be the composition of the exact functor $f^{-1} : \mathrm{Sh}_{\mathscr{O}_Y}(Y) \to \mathrm{Sh}_{f^{-1}\mathscr{O}_Y}(X)$ with the right exact functor $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} - : \mathrm{Sh}_{f^{-1}\mathscr{O}_Y}(X) \to \mathrm{Sh}_{\mathscr{O}_X}(X)$. Then, the right derived functor of f^* is defined to be the functor

$$Lf^* := (\mathscr{O}_X \otimes^L_{f^{-1}\mathscr{O}_Y} -) \circ f^{-1} : D^-(Y) \to D^-(X). \tag{4.2.13}$$

and can be restricted to a functor on the level of bounded derived categories of coherent sheaves.

4.3 Relations between derived functors

In this section, we give some basic relations between derived functors on the level of bounded derived categories of coherent sheaves. Our presentation follows [15, II.5]

Notation 4.3.1. Although all the following results can be stated in greater generality and it is actually done in [15], we have adapted them to the case in which X is a smooth complex projective variety, since this is the case in which we are particularly interested.

Theorem 4.3.2 ([15]II.5.12)**.** *Let X be a smooth complex projective variety. For any $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$, $\mathscr{G}^\bullet$ complexes of $D^b(X)$ there exists a natural isomorphism*

$$R\mathcal{H}\mathrm{om}^\bullet(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet, \mathscr{G}^\bullet) \xrightarrow{\sim} R\mathcal{H}\mathrm{om}^\bullet\big(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}^\bullet(\mathscr{F}^\bullet, \mathscr{G}^\bullet)\big) \tag{4.3.1}$$

determined by the natural morphism of sheaves

$$\mathcal{H}\mathrm{om}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathcal{H}\mathrm{om}\big(\mathscr{E}, \mathcal{H}\mathrm{om}(\mathscr{F}, \mathscr{G})\big).$$

Moreover, taking $H^0 R\Gamma$ of both sides of the above isomorphism yields the following adjunction.

$$\mathrm{Hom}_{D^b(X)}\big(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet, \mathscr{G}^\bullet\big) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}\big(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)\big). \tag{4.3.2}$$

In other words, for any $\mathscr{E}^\bullet$ of $D^b(X)$ the derived tensor product $-\otimes^L \mathscr{E}^\bullet$ is left adjoint to the derived internal hom functor $R\mathcal{H}\mathrm{om}(\mathscr{E}^\bullet, -)$.

From the above theorem and Proposition 4.2.8 we have the following.

Lemma 4.3.3. *Let X be a smooth complex projective variety. Then, the category $D^b(X)$ of bounded complexes of coherent sheaves on a smooth complex projective variety X is a closed symmetric monoidal category.*

Next, we discuss the relations of the derived functors Lf^*, Rf_*, $f^!$ and $R\mathcal{H}\mathrm{om}$. Of particular interest will be the left and the right adjoint of the derived functor $Rf_*: D^b(X) \to D^b(Y)$ associated to a smooth map $f: X \to Y$.

First we give a proposition, which is essential for the next theorem.

Proposition 4.3.4 ([15] II.5.5)**.** *Let $f\colon X \to Y$ be a smooth map between smooth complex projective varieties. Then, for all $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ of $D^b(X)$ there exists a natural transformation*

$$Rf_* R\mathcal{H}\mathrm{om}^\bullet_X(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \to R\mathcal{H}\mathrm{om}^\bullet_Y(Rf_* \mathscr{E}^\bullet, Rf_* \mathscr{F}^\bullet) \tag{4.3.3}$$

determined from the natural isomorphism of sheaves

$$f_* \mathcal{H}\mathrm{om}_X(\mathscr{E}, \mathscr{F}) \to \mathcal{H}\mathrm{om}_Y(f_* \mathscr{E}, f_* \mathscr{F}).$$

Theorem 4.3.5 ([15]II.5.10)**.** *Let $f: X \to Y$ be a smooth map between smooth complex projective varieties. Then, for all $\mathscr{E}^\bullet$ of $D^b(Y)$ and $\mathscr{F}^\bullet$ of $D^b(X)$ there exists a natural isomorphism*

$$Rf_* R\mathcal{H}\mathrm{om}^\bullet_X(Lf^* \mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\sim} R\mathcal{H}\mathrm{om}^\bullet_Y(\mathscr{E}^\bullet, Rf_* \mathscr{F}^\bullet) \tag{4.3.4}$$

defined by composing the natural isomorphism (4.3.3) with the natural map id $\to$ $Rf_* Lf^*$ *in the first variable. Moreover, taking $H^0 R\Gamma$ of both sides of (4.3.4) yields the following adjunction.*

$$\mathrm{Hom}_{D^b(X)}(Lf^* \mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Y)}(\mathscr{E}^\bullet, Rf_* \mathscr{F}^\bullet). \tag{4.3.5}$$

In other words, the functor $Lf^*: D^b(Y) \to D^b(X)$ is the left adjoint to the functor $Rf_*: D^b(X) \to D^b(Y)$; symbolically $Lf^* \dashv Rf_*$.

For a smooth complex projective variety let ω_X be its canonical, also called the *relative dualizing sheaf* and let $f^!$ be the functor from $D^b(Y)$ to $D^b(X)$ defined by $f^!(\mathscr{E}^\bullet) := Lf^*(\mathscr{E}^\bullet)\omega_X \otimes f^*\omega_Y^{-1}[\dim(X) - \dim(Y)]$. We have the following adjunction.

Theorem 4.3.6 ([15] III.5)**.** *Let $f: X \to Y$ be a smooth map between smooth complex projective varieties. Then, for all $\mathscr{E}^\bullet$ of $D^b(X)$ and $\mathscr{F}^\bullet$ of $D^b(Y)$ there exists a natural isomorphism*

$$Rf_* R\mathcal{H}\mathrm{om}_X(\mathscr{E}^\bullet, f^! \mathscr{F}^\bullet) \xrightarrow{\sim} R\mathcal{H}\mathrm{om}_Y(Rf_* \mathscr{E}^\bullet, \mathscr{F}^\bullet). \tag{4.3.6}$$

by composing the homomorphism (4.3.3) with the natural isomorphism $f^! Rf_* \to$ id*, called the* trace isomorphism*, as defined in [15, III.4]. Moreover, taking $H^0 R\Gamma$ of both sides of (4.3.6) yields the following adjunction.*

$$\mathrm{Hom}_{D^b(X)}(\mathscr{E}^\bullet, f^! \mathscr{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Y)}(Rf_* \mathscr{E}^\bullet, \mathscr{F}^\bullet) \tag{4.3.7}$$

In other words, the functor $f^!: D^b(Y) \to D^b(X)$ is the right adjoint to the functor $Rf_*: D^b(X) \to D^b(Y)$; symbolically $Rf_* \dashv f^!$.

Proposition 4.3.7 ([26]Proposition3.2.4(a))**.** *Let $f: X \to Y$ be a smooth map of smooth complex projective varieties. Then, there is a natural isomorphism*

$$Lf^* (\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet) \xrightarrow{\sim} Lf^* \mathscr{E}^\bullet \otimes^L Lf^* \mathscr{F}^\bullet, \qquad Lf^* \mathscr{O}_Y \xrightarrow{\sim} \mathscr{O}_X \tag{4.3.8}$$

and satisfy the relations of Definition 2.1.5.

Since, Rf_* is the right adjoint of Lf^*, by the general theory of Lemma 2.1.9 Rf_* is becoming a monoidal functor. To be precise, for all $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ of $D^b(X)$ there exist natural maps

$$Rf_* \mathscr{E}^\bullet \otimes^L Rf_* \mathscr{F}^\bullet \to Rf_*(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet), \qquad \mathscr{O}_Y \to Rf_* \mathscr{O}_X. \tag{4.3.9}$$

The monoidality of Rf_* and the adjunction unit of the adjunction $Lf^* \dashv Rf_*$ gives rise the following natural isomorphism, which is referred as projection formula in [15, II.5.6].

Proposition 4.3.8 ([15] II.5.6)**.** *(Projection formula) Let X be a smooth map of smooth complex projective varieties. Then, for all $\mathscr{E}^\bullet$ of $D^b(X)$ and $\mathscr{F}^\bullet$ of $D^b(Y)$ there exists a natural isomorphism*

$$Rf_*\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet \xrightarrow{\sim} Rf_*(\mathscr{E}^\bullet \otimes^L Lf^*\mathscr{F}^\bullet) \tag{4.3.10}$$

defined by the following chain of maps.

$$Rf_*\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet \xrightarrow{\mathrm{id}\otimes\eta} Rf_*\mathscr{E}^\bullet \otimes^L Rf_*Lf^*\mathscr{F}^\bullet \xrightarrow{\sim} Rf_*(\mathscr{E}^\bullet \otimes^L Lf^*\mathscr{F}^\bullet)$$

4.4 Strongly dualisable complexes

In this section, we discuss the notion of a dual complex and of the strongly dualisable complex in the $D^b(X)$. We start with a result which will be useful for the following.

Proposition 4.4.1 ([15]Proposition 5.14)**.** *For a smooth complex projective variety X and $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ objects of $D^b(X)$ there exists a natural isomorphism*

$$R\mathcal{H}\mathrm{om}^\bullet(\mathscr{E}^\bullet,\mathscr{F}^\bullet)\otimes^L \mathscr{G}^\bullet \xrightarrow{\sim} R\mathcal{H}\mathrm{om}^\bullet(\mathscr{E}^\bullet,\mathscr{F}^\bullet\otimes^L\mathscr{G}^\bullet). \tag{4.4.1}$$

Idea of the proof. Since X is a smooth complex projective variety, any bounded complex of coherent sheaves is isomorphic to a complex of locally free sheaves, see Proposition (4.1.4). So the derived tensor product is computed as the usual tensor product and the required isomorphism is determined by the natural isomorphism of sheaves

$$\mathcal{H}\mathrm{om}(\mathscr{E},\mathscr{F})\otimes\mathscr{G} \xrightarrow{\sim} \mathcal{H}\mathrm{om}(\mathscr{E},\mathscr{F}\otimes\mathscr{G}). \tag{4.4.2}$$

which is discussed in detail in [18, Proposition2.2.9]. □

Lemma 4.4.2. *Let X be a smooth complex projective variety. Then, every complex $\mathscr{E}^\bullet$ in $D^b(X)$ is a strongly dualisable object, as defined in 2.5.12.*

Proof. This follows from Proposition 4.4.1 setting $\mathscr{F}^\bullet = \mathscr{O}_X$ and $\mathscr{G}^\bullet = \mathscr{E}^\bullet$. □

Since $D^b(X)$ is a closed symmetric monoidal category and has strongly dualisable objects then by Lemma 2.5.14 we have the following Proposition which will play crucial role for our work.

Proposition 4.4.3. *Let X be a smooth complex projective variety. Then, the bounded derived category of coherent sheaves is a rigid monoidal category. In particular, for any bounded complex $\mathscr{E}^\bullet$ its dual is the object $\mathscr{E}^{\bullet\vee}$ of $D^b(X)$ which is defined by*

$$\mathscr{E}^{\bullet\vee} := R\mathcal{H}\mathrm{om}(\mathscr{E}^\bullet,\mathscr{O}_X) \in D^b(X). \tag{4.4.3}$$

Lemma 4.4.4. *Let X be a smooth complex projective variety and $\mathscr{E}^\bullet$ an object of $D^b(X)$. Then it is true that*

$$\mathscr{E}^{\bullet\vee} = \prod_{n\in\mathbb{Z}} \mathcal{H}\mathrm{om}(\mathscr{F}^n, \mathscr{O}_X) \tag{4.4.4}$$

for some complex $\mathscr{F}^\bullet$ of locally free sheaves $\mathscr{F}^n$.

Proof. Since X is smooth and projective variety, then by Proposition 4.1.4 any bounded complex $\mathscr{E}^\bullet$ of coherent sheaves is isomorphic to a bounded complex $\mathscr{F}^\bullet$ of locally locally free sheaves $\mathscr{F}^i$. Therefore, $R\mathcal{H}\mathrm{om}(\mathscr{F}^\bullet, \mathscr{O}_X)$ need not be derived and so can be computed as $\mathcal{H}\mathrm{om}(\mathscr{F}^\bullet, \mathscr{O}_X)$. Now, since the functor $\mathcal{H}\mathrm{om}(-, \mathscr{O}_X)$ is a contravariant functor, sends colimits to limits and so we have

$$\mathcal{H}\mathrm{om}(\bigoplus_{n\in\mathbb{Z}} \mathscr{F}^n, \mathscr{O}_X) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}\mathrm{om}(\mathscr{F}^n, \mathscr{O}_X).$$

This completes the proof. $\square$

Chapter 5

Hopf algebras in braided categories

In this chapter, we outline the basics of Hopf algebras in braided monoidal categories. Of particular interest will be the correspondence between Hopf algebras and rigid monoidal structures on the associated category of representations. For example, the coproduct of a Hopf algebra determines the monoidal product of the category of representations.

The material of this chapter is presented in the following order. In Section 5.1 we cover the basics of algebras in braided monoidal categories and their modules. By reversing the arrows in the definition of an algebra, we obtain the notion of a coalgebra which is discussed in Section 5.2. In Section 5.3 we discuss those objects of a category which are endowed simultaneously with an algebra structure and a coalgebra structure in a compatible way. Such objects are called bialgebras. In Section 5.4 we discuss Hopf algebras which are defined to be bialgebras with an invertible antipode. In Section 5.5 we discuss the notion of a fusion operator as studied by Street [37]. The notion of a fusion operator was generalised appropriately by Brugières, Lack and Virelizier in [7] to define generalisations of Hopf algebras in braided monoidal categories to a non-braided setup. Our presentation on Hopf algebras follows [20] and [38] closely.

Notation 5.0.1. In this chapter, $(\mathscr{C}, \otimes, 1_{\mathscr{C}}, \tau)$ will be a braided monoidal category, a notion which was discussed in Section 2.3.

5.1 Algebras

In this section, we discuss the notion of an algebra in a braided monoidal category and the associated category of representations.

Definition 5.1.1. An object A in a monoidal category $\mathscr{C}$ is said to be an *algebra* if it is equipped with morphisms $m : A \otimes A \to A$ and $u : 1 \to A$ called the *product* and the *unit* such that the following diagrams commute.

$$
\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes m"'] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A\otimes u"] \arrow[dr, "\mathrm{id}_A"'] & A\otimes A \arrow[d, "m"] & A \arrow[l, "u\otimes \mathrm{id}_A"'] \arrow[dl, "\mathrm{id}_A"] \\
& A &
\end{tikzcd}
$$

Notation 5.1.2. The left diagram above expresses the *associativity* of the product and the right diagram expresses the *unitality* of the unit.

A *morphism* of algebras $f\colon (A,m,u)\to(A',m',u')$ is a morphism on the level of objects $f: A\to A'$ such that the following diagrams commute.

$$
\begin{tikzcd}
A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & A'\otimes A' \arrow[d, "m"] \\
A \arrow[r, "f"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "u_A"] \arrow[dr, "u_{A'}"'] & A \arrow[d, "f"] \\
& A'
\end{tikzcd}
$$

We will be interested in the category of modules of an algebra and so we recall this definition here.

Definition 5.1.3. A *left A-module* for an algebra (A,m,u) is a pair (X,r) where X is an object of $\mathscr{C}$ and $r: A\otimes X\to X$ is a morphism in $\mathscr{C}$ called the *action* such that the following diagrams commute.

$$
\begin{tikzcd}
A\otimes A\otimes X \arrow[r, "m\otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X\otimes r"'] & A\otimes X \arrow[d, "r"] \\
A\otimes X \arrow[r, "r"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "u\otimes \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"'] & A\otimes X \arrow[d, "r"] \\
& X
\end{tikzcd}
$$

A *morphism* between two left A-modules (M,r) and (N,s) is a morphism on the level of objects $f: M\to N$ in $\mathscr{C}$ such that the following diagram commutes.

$$
\begin{tikzcd}
A\otimes M \arrow[r, "r"] \arrow[d, "\mathrm{id}_A\otimes f"'] & M \arrow[d, "f"] \\
A\otimes N \arrow[r, "s"'] & N
\end{tikzcd}
$$

5.2 Coalgebras

In this section, we discuss the notion of a coalgebra which is obtained by reversing the arrows in the definition of an algebra. In particular, we have the following definition.

Definition 5.2.1. An object C of $\mathscr{C}$ is said to be a *coalgebra* if C is equipped with morphisms $\delta : C \to C \otimes C$ and $\epsilon : C \to 1$ called the *coproduct* and the *counit* respectively, such that the following diagrams commute.

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes C \arrow[d, "\delta \otimes \mathrm{id}_C"] \\
C \otimes C \arrow[r, "\delta \otimes \mathrm{id}_C"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C & C \otimes C \arrow[l, "\mathrm{id}_C \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}_C"] & C \\
& C \arrow[ul, "\mathrm{id}_C"] \arrow[u, "\delta"] \arrow[ur, "\mathrm{id}_C"'] &
\end{tikzcd}$$

The left diagram above expresses the *coassociativity* of the coproduct and the right diagram expresses the *counitality* of the counit. If moreover for the coproduct δ the following diagram commutes.

$$\begin{tikzcd}
& C \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
C \otimes C \arrow[rr, "\tau_{C,C}"'] & & C \otimes C
\end{tikzcd}$$

then the coalgebra (C, δ, ϵ) is said to be *cocommutative.*

Example 5.2.2. Let X be a set and denote by $k[X]$ the k-vector space $\bigoplus_{x \in X} kx$ with basis the elements of X. Then, $k[X]$ has a coalgebra structure given by

$$\delta(x) = x \otimes x, \qquad \epsilon(x) = 1 \tag{5.2.1}$$

for any x of X.

A *morphism* of coalgebras $f : (C, \delta, \epsilon) \to (C', \delta', \epsilon')$ is a morphism on the level of objects $f : C \to C'$ such that the following diagrams commute.

$$\begin{tikzcd}
C \otimes C \arrow[r, "f \otimes f"] & C' \otimes C' \\
C \arrow[u, "\delta"] \arrow[r, "f"'] & C' \arrow[u, "\delta'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, "\epsilon"'] & C' \arrow[d, "\epsilon'"] \\
& 1
\end{tikzcd}$$

5.3 Bialgebras

In this section, we discuss objects in a braided category which are endowed with an algebra and a coalgebra structure and these structures are compatible in the way which is described below. Such objects are called bialgebras. The structure of the bialgebra determines the monoidal product of the category of representations of this bialgebra in a specific way and it is discussed in Theorem 5.3.5.

Definition 5.3.1. A tuple (A,m,u,δ,ϵ) of $\mathscr{C}$ is said to be a *bialgebra* if (A,m,u) is an algebra and (A,δ,ϵ) is a coalgebra and moreover the following diagrams commute.

$$\begin{tikzcd}
A\otimes A \arrow[r, "m"] \arrow[d, "\delta\otimes\delta"'] & A \arrow[r, "\delta"] & A\otimes A \\
A\otimes A\otimes A\otimes A \arrow[rr, "\mathrm{id}_A\otimes\tau_{A,A}\otimes\mathrm{id}_A"'] & & A\otimes A\otimes A\otimes A \arrow[u, "m\otimes m"']
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "\delta"] \\
1\otimes 1 \arrow[r, "u\otimes u"'] & A\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, "u"] \arrow[dr, "\mathrm{id}"'] & & A \arrow[dl, "\epsilon"] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & 1\otimes 1 \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "\epsilon"'] & 1
\end{tikzcd}$$

Remark 5.3.2. In other words, a bialgebra in a braided monoidal category $\mathscr{C}$ is an object with an algebra structure and a coalgebra structure on it such that the product and the unit are morphisms of coalgebras or equivalently the coproduct and the counit are morphisms of algebras.

Remark 5.3.3. Note here that only one compatibility condition makes use the braiding of the category and it is the top diagram in the above definition.

Example 5.3.4. Recall from Example 5.2.2 that for a set X the vector space $k[X]$ is a coalgebra. If moreover X has an associative product $m : X\times X \to X$ and left and right units e, i.e. X is a monoid, then m and e induce an algebra structure on $k[X]$ which is compatible with the coalgebra structure of $k[X]$. In particular, the coproduct $\delta : k[X] \to k[X]\otimes k[X]$ and the counit $\epsilon : k[X] \to 1$ are algebra morphisms. Indeed, we have that

$$\delta(xy) = xy\otimes xy = (x\otimes x)(y\otimes y) = \delta(x)\delta(y)$$

and $\epsilon(xy) = 1 = \epsilon(x)\epsilon(y)$ for all x and y elements of X.

Let A be an algebra in a braided monoidal category $\mathscr{C}$. Then, for any two left A-modules (M,r) and (N,s), there is a naturally defined left $A\otimes A$-module structure on $M\otimes N$. If A is also a bialgebra i.e. A has a coalgebra structure on it in a compatible way with the algebra structure, then an A-module structure is defined on $M\otimes N$ as follows.

Theorem 5.3.5 ([38] Section 6.1.3)**.** *Let (A,m,u,δ,ϵ) a bialgebra in $\mathscr{C}$. The category* $\mathrm{Rep}(A)$ *is a monoidal category with monoidal unit the pair* $(1_{\mathscr{C}},\epsilon)$ *and monoidal product defined on objects by*

$$(M,r)\otimes(N,s) = \big(M\otimes N, (r\otimes s)\circ(\mathrm{id}_A\otimes\tau_{A,M}\otimes\mathrm{id}_N)\circ(\delta\otimes\mathrm{id}_{M\otimes N})\big).$$

5.4 Hopf algebras

In this section, we study Hopf algebras in a braided monoidal category. The difference between a bialgebra and a Hopf algebra is that the latter has an invertible endomorphism.

Definition 5.4.1. A *Hopf algebra* in a braided monoidal category $\mathscr{C}$ is a bialgebra (H,m,u,δ,ϵ) in $\mathscr{C}$ with an invertible map $S : H \to H$ called the antipode map such that S satisfies the following relation

$$m \circ (\mathrm{id} \otimes S) \circ \delta = u \circ \epsilon = m \circ (S \otimes \mathrm{id}) \circ \delta \tag{5.4.1}$$

Example 5.4.2. Recall from Example 5.3.4 that for a set X with a monoid structure on it, the vector space $k[X]$ is a bialgebra. Now, $k[X]$ has an antipode if and only if X is a group i.e. every element of X has an inverse. In this case, the antipode is defined to be $S(x) = x^{-1}$ for all $x \in X$.

In the following we describe a different perspective for $S : H \to H$, which is actually the way from which relation 5.4.1 is obtained. First we introduce the notion of the convolution algebra. Let (A,m,u) be an algebra in the braided category $\mathscr{C}$ and let (C,δ,ϵ) be a coalgebra in $\mathscr{C}$. Then, it is formed the k-vector space $\mathrm{Hom}_{\mathscr{C}}(C,A)$ of all maps from C to A. The *convolution* product for any two maps f and g of $\mathrm{Hom}_{\mathscr{C}}(C,A)$ is defined to the composite

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A. \tag{5.4.2}$$

Lemma 5.4.3 ([38] Section 6.2.1)**.** *Let (A,m,u) be an algebra in $\mathscr{C}$ and (C,δ,ϵ) be a coalgebra in $\mathscr{C}$.*

1. *Then, the triple* $(\mathrm{Hom}(C,A)), *, \eta \circ u)$ *is an algebra with product $*$ and unit $\eta \circ u$.*
2. *If $A = C = H$ then the antipode $S : H \to H$ is a left and right quasi-inverse to the identity map under the convolution product.*
3. *Let $(H,m,u,\delta,\epsilon,S)$ be a Hopf algebra in $\mathscr{C}$. Then, it is true that*

$$\delta \circ S = (S \otimes S) \circ \tau \circ \delta, \qquad u \circ S = u. \tag{5.4.3}$$

In Theorem 5.3.5 it is discussed that the category of left A-modules of a bialgebra A is monoidal with the monoidal product to be determined by the coproduct of A. If moreover A is a Hopf algebra, i.e. has an invertible antipode then the antipode and its inverse determines a left duality and a right duality to the monoidal category of left A-modules. The following theorem establishes this relation.

Theorem 5.4.4 ([38] Lemma 6.1)**.** *Let A be a bialgebra in a braided rigid category $\mathscr{C}$. Then, A is a Hopf algebra if and only if the monoidal category* $\mathrm{Rep}(A)$ *is rigid, in the sense of Section 2.4.*

5.5 Fusion operators

In [37] a characterisation of a Hopf algebra H in a braided monoidal category was given in view of an endomorphism V of $H\otimes H$. This endofunctor is called the *fusion operator*. It is the notion of the fusion operator which was generalised appropriately by Brugières, Lack and Virelizier in [7] in order to define generalisations of Hopf algebras in non-braided monoidal categories. In this section, we present the notion of a fusion operator, emphasising the example of the group algebra and the relation of the inverse of the fusion operator with the antipode of the Hopf algebra.

Proposition 5.5.1 (Street [37])**.** *Let $(\mathscr{C},\otimes,\tau)$ be a braided monoidal category with braiding $\tau_{X,Y}: X\otimes Y \xrightarrow{\sim} Y\otimes X$ for all X and Y of $\mathscr{C}$.*

1. *Let H is a bialgebra in $\mathscr{C}$ with the inverse braiding. The endomorphism $V: H\otimes H\to H\otimes H$ which is defined by $V=(1\otimes m)\circ(\delta\otimes \mathrm{id}_H)$ satisfies the following equation*
$$V_{23}\circ V_{12} = V_{12}\circ V_{13}\circ V_{23}$$
where $V_{13}=(\mathrm{id}_H\otimes\tau_{H,H})^{-1}\circ(\mathrm{id}_H\otimes V)\circ(\tau_{H,H}\otimes\mathrm{id}_H)^{-1}: H\otimes H\otimes H\to H\otimes H\otimes H$, $V_{12}=V\otimes \mathrm{id}_H$ and $V_{23}=\mathrm{id}_H\otimes V$. Equation (1) is called the fusion equation *and an endomorphism which satisfies it a* fusion operator.

2. *Assume also that H is a Hopf algebra in $\mathscr{C}$ with antipode $S: H\to H$. Then, the fusion operator V is invertible with inverse $V^{-1}: H\otimes H\to H\otimes H$ given by the composite*
$$V^{-1}=(\mathrm{id}_H\otimes m)\circ(\mathrm{id}_H\otimes S\otimes \mathrm{id}_H)\otimes(\delta\otimes\mathrm{id}_H).$$

Example 5.5.2. For a finite group G, the group algebra $k[G]$ is a bialgebra with coproduct defined by $\delta(g):=g\otimes g$ and counit $\epsilon(g)=1$ for $g\in G$. Consider the endomorphism $V_{kG}: k[G]\otimes k[G]\to k[G]\otimes k[G]$ defined to be the composite

$$g_1\otimes g_2 \xrightarrow{\delta\otimes\mathrm{id}} (g_1\otimes g_1)\otimes g_2 \xrightarrow{\mathrm{id}\otimes m} g_1\otimes g_1g_2$$

where g_1 and g_2 elements of G. Since, $k[G]$ is a Hopf algebra with antipode defined by $S(g)=g^{-1}$, the inverse $V^{-1}_{k[G]}$ of the endomorphism $V_{k[G]}$ is defined to be the composition

$$g_1\otimes g_1g_2 \xrightarrow{\delta\otimes\mathrm{id}} (g_1\otimes g_1)\otimes g_1g_2 \xrightarrow{\mathrm{id}\otimes S\otimes\mathrm{id}} g_1\otimes g_1^{-1}\otimes g_1g_2 \xrightarrow{\mathrm{id}\otimes m} g_1\otimes g_1^{-1}g_1g_2 = g_1\otimes g_2$$

for all g_1 and g_2 elements of G.

It is stated without proof in [37] that the antipode $S: H\to H$ of the Hopf algebra H in $\mathscr{C}$ is given by $S=(\epsilon\otimes\mathrm{id}_H)\circ V^{-1}\circ(\mathrm{id}_H\otimes\eta_1)$ where V^{-1} is the inverse of the fusion operator. In the remainder of this section we check that S is indeed the antipode. First we establish the commutativity of some diagrams which will be essential for the proof.

Lemma 5.5.3. *Let (H, m, η, δ, u) be a bialgebra in a braided monoidal category and let $V : H \otimes H \to H \otimes H$ its fusion operator. Then, the following diagrams are commutative.*

$$
\begin{tikzcd}
H \otimes H \arrow[r, "V"] \arrow[dr, "m"'] & H \otimes H \arrow[d, "\epsilon \otimes \mathrm{id}"] \\
& H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[r, "V"] & H \otimes H \\
H \arrow[u, "\mathrm{id} \otimes \eta_1"] \arrow[ur, "\delta"'] &
\end{tikzcd}
$$

Proof. For the commutativity of the left diagram we work as follows. First we unpack V and obtain the following commutative diagram due to interchange law and the counitarity axiom of the coalgebra.

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\delta \otimes \mathrm{id}"] \arrow[dr] \arrow[ddr, "m"'] & H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "\epsilon \otimes \mathrm{id} \otimes \mathrm{id}"] & H \otimes H \arrow[ddl, "\epsilon \otimes \mathrm{id}"] \\
& 1 \otimes H \otimes H \arrow[d, "m"] & \\
& H &
\end{tikzcd}
$$

For the commutativity of the right diagram we work as follows. First we uncpack V and obtain the following commutative diagram due to interchange law and the unitarity axiom of the algebra.

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\delta \otimes \mathrm{id}"] & H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes m"] & H \otimes H \arrow[ddl, "\delta"] \\
& H \otimes H \arrow[u, "\mathrm{id} \otimes \mathrm{id} \otimes \eta"] \arrow[ur] & \\
& H \arrow[u, "\delta"] \arrow[uul, "\mathrm{id} \otimes \eta"] &
\end{tikzcd}
$$

□

Lemma 5.5.4. *Let (H, m, η, δ, u) be a bialgebra in a braided monoidal category and let $V : H \otimes H \to H \otimes H$ its fusion operator. Then, the following diagrams are commutative.*

$$
\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "V \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & H \otimes H \otimes H \arrow[d, "\mathrm{id} \otimes m"'] \\
H \otimes H \arrow[r, "V"'] & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes V"] & H \otimes H \otimes H \\
H \otimes H \arrow[r, "V"'] \arrow[u, "\delta \otimes \mathrm{id}"'] & H \otimes H \arrow[u, "\delta \otimes \mathrm{id}"']
\end{tikzcd}
$$

Proof. For the commutativity of the left diagram we work as follows. First, we unpack V and obtain the following diagram.

$$
\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\delta \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[dr, "\delta \otimes \mathrm{id} \otimes \mathrm{id}"'] \arrow[dd, "\mathrm{id} \otimes m"'] & H \otimes H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes m \otimes \mathrm{id}"] & H \otimes H \otimes H \arrow[dd, "\mathrm{id} \otimes m"] \\
& H \otimes H \otimes H \otimes H \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes m"] & \\
H \otimes H \arrow[r, "\delta \otimes \mathrm{id}"'] & H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes m"'] & H \otimes H
\end{tikzcd}
$$

The commutative of the above boundary diagram follows from the interchange law and the associativity of the product m.

The commutativity of the right diagram follows similarly from the interchange law and the coassociativity of the coproduct as it shown below.

$$
\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes \delta \otimes \mathrm{id}"] & H \otimes H \otimes H \otimes H \arrow[r, "\mathrm{id} \otimes \mathrm{id} \otimes m"] & H \otimes H \otimes H \\
& H \otimes H \otimes H \otimes H \arrow[ur, "\mathrm{id} \otimes \mathrm{id} \otimes m"'] & \\
H \otimes H \arrow[uu, "\delta \otimes \mathrm{id}"] \arrow[r, "\delta \otimes \mathrm{id}"'] & H \otimes H \otimes H \arrow[u, "\delta \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes m"'] & H \otimes H \arrow[uu, "\delta \otimes \mathrm{id}"']
\end{tikzcd}
$$

□

Now, with the use of the above commutative diagrams and the fact that V is invertible, we check that S satisfies the relations 5.4.1. The commutative diagram below express the equality $m \circ (S \otimes \mathrm{id}) \circ \delta = \eta \circ \epsilon$.

$$\begin{tikzcd}
 & H \otimes H \otimes H \arrow[rr, "V^{-1} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"] \arrow[dr, phantom, "2"] & & H \otimes H \otimes H \arrow[d, "\mathrm{id} \otimes m"'] \arrow[dr, "(\epsilon \otimes \mathrm{id}) \otimes \mathrm{id}"] & \\
H \otimes H \arrow[ur, "(\mathrm{id} \otimes \eta_1) \otimes \mathrm{id}"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "1", near end] & H \otimes H \arrow[rr, "V^{-1}"] & & H \otimes H \arrow[dr, "\epsilon \otimes \mathrm{id}"'] \arrow[r, phantom, "3"] \arrow[dl, phantom, "5"] & H \otimes H \arrow[d, "m"] \\
H \arrow[u, "\delta"] \arrow[urrr, "\mathrm{id} \otimes \eta_1"] \arrow[r, phantom, "4"] \arrow[rr, "\epsilon"'] & & 1 \arrow[rr, "\eta"'] & & H
\end{tikzcd}$$

The commutativity of the triangle with number one is the unitary axiom of the algebra, see Definition 5.1.1. The commutativity of the square with number two is checked in Lemma 5.5.4. The commutativity of the square with number three is the interchange law between the product m and the counit ϵ. The commutativity of the triangle with number four is checked in Lemma 5.5.3.

Similarly, the diagram below express the relation $m \circ (\mathrm{id} \otimes S) \circ \delta = \eta \circ \epsilon$ of which commutativity follows for similar reasons with the above.

$$\begin{tikzcd}
 & H \otimes H \otimes H \arrow[rr, "\mathrm{id} \otimes V^{-1}"] \arrow[dr, phantom, "2"] & & H \otimes H \otimes H \arrow[dr, "\mathrm{id} \otimes (\epsilon \otimes \mathrm{id})"] \arrow[d, phantom, "3"] & \\
H \otimes H \arrow[ur, "\mathrm{id} \otimes (\mathrm{id} \otimes \eta_1)"] \arrow[r, phantom, "1"] & H \otimes H \arrow[u, "\delta \otimes \mathrm{id}"'] \arrow[rr, "V^{-1}"] \arrow[drrr, "\epsilon \otimes \mathrm{id}"'] \arrow[d, phantom, "4"] & & H \otimes H \arrow[u, "\delta \otimes \mathrm{id}"] \arrow[r, "\mathrm{id}"'] & H \otimes H \arrow[d, "m"] \arrow[d, phantom, "5", shift right=4ex] \\
H \arrow[u, "\delta"] \arrow[ur, "\mathrm{id} \otimes \eta_1"] \arrow[rr, "\epsilon"'] & & 1 \arrow[rr, "\eta"'] & & H
\end{tikzcd}$$

Chapter 6

Hopf monads on monoidal categories

The aim of this chapter is to discuss a generalisation of Hopf algebras due to Brugières, Lack and Virelizier [7]. This generalisation is called a Hopf monad and was established firstly for categories with duals in [8], based on previous work of Moerdijk [30].

The material which covered in this chapter is presented in the following order. In Section 6.1 we cover the basics of monads on a category and the associated category of modules of a monad. The analogue of a bialgebra on the level of monads, is called a bimonad, and introduced first by Moerdijk [30]. We discuss the theory of bimonads in Section 6.2. In Section 6.3 we discuss the notion of a Hopf monad as established in [7]. In particular, we explain how the authors of [7] generalised the notion of the fusion operator of Hopf algebra due to Street to defined the notion of a Hopf monad on a monoidal category. In Section 6.4 we discuss Hopf monads on monoidal categories which are obtained from pairs of adjoints functors. In particular, our goal in this section is two fold. On the one hand, we want to explain the theory of Hopf operators of an opmonoidal and how this is related with the fusion operators of the previous section. On the other hand, we want discuss the particular case of a Hopf monad on a rigid monoidal category which introduced by Brugières and Virelizier in [8]. Finally, in Section 6.5 we discuss under which circumstances a Hopf monad on a monoidal category is obtained from a Hopf algebra.

Our presentation of the above material follows [7], [8] and [38] closely.

6.1 Monads

In this section, we introduce the notion of a monad on a monoidal category. Our exposition follows [7] and [28] closely.

Definition 6.1.1. Let $\mathscr{C}$ be a category. A *monad* on $\mathscr{C}$ is a triple (T, μ, η), where $T : \mathscr{C} \to \mathscr{C}$ is a functor, $\mu : TT \to T$ is a natural transformation called the *product of* *the monad* and $\eta : 1_{\mathscr{C}} \to T$ is a natural transformation called the *unit of the monad* such that the following diagrams commute for all X of $\mathscr{C}$.

$$\begin{tikzcd} TTT(X) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{TX}"'] & TT(X) \arrow[d, "\mu_X"] \\ TT(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[d, "\eta_{TX}"'] \arrow[dr, "\mathrm{id}_{TX}"] & TT(X) \arrow[d, "\mu_X"] \\ TT(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

Remark 6.1.2. In an equivalent way, a monad T on a category $\mathscr{C}$ is the algebra object in the category $\mathrm{End}(\mathscr{C})$ of endofunctors of $\mathscr{C}$. It is worth mentioning here that $\mathrm{End}(\mathscr{C})$ is a monoidal category with monoidal product the composition of endofunctors and unit object the identity functor but it is neither symmetric nor braided.

Notation 6.1.3. In the following, we will refer to μ and η as product and unit and it will be clear from the context that we are referring to the product and to the unit of a monad.

Important examples of monads are obtained from adjoint functors and algebra objects in some category.

Example 6.1.4. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ be a functor with right adjoint $U : \mathscr{D} \to \mathscr{C}$; symbolically $F \dashv U$. Denote by $\eta : \mathrm{id}_{\mathscr{C}} \to UF$ the adjunction unit and by $\epsilon : FU \to \mathrm{id}_{\mathscr{D}}$ the adjunction counit. Then, the endofunctor $T := UF$ is a monad on $\mathscr{C}$, called the *adjunction monad*, with product $\mu := U(\epsilon_F)$ and unit the adjunction unit.

Example 6.1.5. Let (A, m, u) be an algebra object in a monoidal category $\mathscr{C}$ with product $m : A \otimes A \to A$ and unit $u : 1 \to A$. Then, the endofunctor $T' = A \otimes -$ with product $\mu = m \otimes \mathrm{id}$ and unit $\eta = u \otimes \mathrm{id}$, is a monad on $\mathscr{C}$.

Example 6.1.6. The identity functor $\mathrm{id} : \mathscr{C} \to \mathscr{C}$ is a monad with product and unit the identity.

Monads on a category $\mathscr{C}$ form a category. Since a monad is an algebra object in the category $\mathrm{End}(\mathscr{C})$, morphisms between monads are essentially morphisms between algebra objects. To be more explicit we have the following definition.

Definition 6.1.7. Let (T, μ, η) and (T', μ', η') be monads on a category $\mathscr{C}$. Then, a *morphism of monads* from (T, μ, η) to (T', μ', η') is a natural transformation $f : T \to T'$ such that the following diagrams commute for all X of $\mathscr{C}$.

$$\begin{tikzcd} TT(X) \arrow[r, "T(f_X)"] \arrow[d, "\mu_X"'] & T\big(T'(X)\big) \arrow[r, "f_{T'(X)}"] & T'T'(X) \arrow[d, "\mu'_X"] \\ T(X) \arrow[rr, "f_X"'] & & T'(X) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\eta'_X"'] & T(X) \arrow[d, "f_X"] \\ & T'(X) \end{tikzcd}$$

An action of a monad T on an object of a category $\mathscr{C}$ is defined as follows.

Definition 6.1.8. Let (T,μ,η) be a monad on a category $\mathscr{C}$. An *action* of T on an object X of $\mathscr{C}$ is a morphism $r : T(X) \to X$ in $\mathscr{C}$ such that the following diagrams commute.

$$\begin{tikzcd} TT(X) \arrow[r, "\mu_X"] \arrow[d, "T(r)"'] & T(X) \arrow[d, "r"] \\ T(X) \arrow[r, "r"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & T(X) \arrow[d, "r"] \\ & X \end{tikzcd}$$

The pair (X,r) is called a T*-module* in $\mathscr{C}$. A morphism of T-modules from (X,r) to (Y,s) is defined to be a map $f : X \to Y$ such that the following diagram commute.

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "r"'] & T(Y) \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Example 6.1.9. Let (A,m,u) be an algebra in a monoidal category $\mathscr{C}$ and denote by $T' = A\otimes -$ the associated monad. Then, the category of modules of the monad T' is the category of left A-modules of the algebra A.

Proposition 6.1.10 ([8] Lemma 1.7)**.** *Let $\mathscr{C}$ be a category, T and T' be monads on $\mathscr{C}$ and let $g : T \to T'$ be a natural transformation. Then, the following are equivalent.*

- *The natural transformation $g : T \to T'$ is a morphism of monads.*
- *For any T'-module (N,s) the module $(N, s\circ g_N)$ is a T-module.*

6.2 Bimonads

In this section, we discuss the analogue of a bialgebra in a braided monoidal category on the level of monads on any monoidal category. In particular, we start with a monad on a category and endow it with extra structure. We have the following definition.

Definition 6.2.1. A monad (T,μ,η) on a monoidal category $\mathscr{C}$ is said to be a *bimonad* if the functor $T : \mathscr{C} \to \mathscr{C}$ is opmonoidal and the product μ and the unit η of the monad are opmonoidal natural transformations.

Notation 6.2.2. Bimonads are also called opmonoidal monads meaning monads in the category of opmonoidal functors and opmonoidal natural transformations.

Bimonads will play a fundamental role in the following Chapter, where we will define a bimonad on the orbit category of the derived category of coherent sheaves. For this reason we consider it useful to unpack in the form of diagrams what does it mean that the functor T is opmonoidal and the natural transformation μ and η opmonoidal.

By Definitions 2.2.1 and 2.2.3, the fact that the functor $T : \mathscr{C} \to \mathscr{C}$ is opmonoidal is expressed as the commutativity of the following diagrams for all X, Y and Z objects of $\mathscr{C}$.

$$
\begin{tikzcd}
T(X \otimes Y \otimes Z) \arrow[r, "T_2(X, Y \otimes Z)"] \arrow[d, "T_2(X \otimes Y, Z)"'] & T(X) \otimes T(Y \otimes Z) \arrow[d, "\mathrm{id}_{T(X)} \otimes T_2(Y, Z)"] \\
T(X \otimes Y) \otimes T(Z) \arrow[r, "T_2(X, Y) \otimes \mathrm{id}_{T(Z)}"'] & T(X) \otimes T(Y) \otimes T(Z)
\end{tikzcd}
$$

$$
\begin{tikzcd}
T(X) \arrow[r, "T_2(X, 1)"] \arrow[d, "T_2(1, X)"'] \arrow[dr, "\mathrm{id}_{T(X)}"] & (X) \otimes T(1_{\mathscr{C}}) \arrow[d, "\mathrm{id}_{T(X)} \otimes T_0"] \\
T(1_{\mathscr{C}}) \otimes T(X) \arrow[r, "T_0 \otimes \mathrm{id}_{T(X)}"'] & T(X)
\end{tikzcd}
$$

In a similar fashion, the fact that the product $\mu : TT \to T$ is an opmonoidal natural transformation is expressed as the commutativity of the following diagrams for all X and Y of $\mathscr{C}$.

$$
\begin{tikzcd}
TT(X \otimes Y) \arrow[rr, "T(T_2(X, Y))"] \arrow[d, "\mu_{X \otimes Y}"'] & & T(T(X) \otimes T(Y)) \arrow[d, "T_2(TX, TY)"] \\
T(X \otimes Y) \arrow[dr, "T_2(X, Y)"'] & & TT(X) \otimes TT(Y) \arrow[dl, "\mu_X \otimes \mu_Y"] \\
& T(X) \otimes T(Y) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
T(T(1)) \arrow[r, "T(T_0)"] \arrow[d, "\mu_1"'] & T(1) \arrow[d, "T_0"] \\
T(1) \arrow[r, "T_0"'] & 1
\end{tikzcd}
$$

Similarly, the fact that the unit $\eta : \mathrm{id}_{\mathscr{C}} \to T$is an opmonoidal natural transformation expressed as the commutativity of the following diagrams for all X and Y of $\mathscr{C}$.

$$
\begin{tikzcd}
X \otimes Y \arrow[r, "\eta_{X\otimes Y}"] \arrow[dr, "\eta_X \otimes \eta_Y"'] & T(X\otimes Y) \arrow[d, "T_2(X,Y)"] \\
& T(X)\otimes T(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\eta_1"] \arrow[dr, "\mathrm{id}_1"'] & T(1) \arrow[d, "T_0"] \\
& 1
\end{tikzcd}
$$

Bimonads form a category in which the morphisms are defined as follows.

Definition 6.2.3. A *morphism* between two bimonads (T,μ,η) and (T',μ',η') on a category $\mathscr{C}$ is a moprhism of monads $f : T \to T'$ which is opmonoidal as natural transformation.

Important examples of bimonads are obtained from strong monoidal functors with left adjoints and bialgebras in braided monoidal categories.

Theorem 6.2.4 (Brugières-Virelizier [8] p. 690)**.** *Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor between monoidal categories with left adjoint $F : \mathscr{C} \to \mathscr{D}$. Then, the adjunction monad $T = UF$ is a bimonad with constraints*

$$T_2(X,Y) = U_2(FX,FY)\circ U\big(F_2(X,Y)\big), \qquad T_0 = U_0 \circ U(F_0) \tag{6.2.1}$$

for all X and Y of $\mathscr{C}$.

Opmonoidal adjunctions, as defined in 2.2.5, will be important in Section 6.4 where we will discuss Hopf monads obtained from adjoint functors. For the moment, we want to establish the following result which follows from Theorem 6.2.4.

Corollary 6.2.5. *Let $F \dashv U$ be an opmonoidal adjunction. Then, the adjunction monad $T = UF$ has a bimonad structure.*

Example 6.2.6 (Brugières-Virelizier [8] p.689)**.** Let (A,m,η) be an algebra in a braided monoidal category $\mathscr{C}$ of which the braiding is denoted $\tau_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$ for all X, Y of $\mathscr{C}$. Since A is an algebra, the functor $T' = A\otimes -$ is a monad on $\mathscr{C}$. If also are given maps $\delta : A \to A\otimes A$ and $\epsilon : A \to 1_{\mathscr{C}}$ then the following is true: the tuple $(A,m,\eta,\delta,\epsilon)$ is a bialgebra in $\mathscr{C}$ if and only if $T' = A\otimes -$ is a bimonad with opmonoidal constraint T_2' defined by

$$T_2'(X,Y) = (\mathrm{id}_A \otimes \tau_{A,X}\otimes \mathrm{id}_Y)\circ(\delta\otimes \mathrm{id}_{X\otimes Y})$$

for all X and Y of $\mathscr{C}$ and opmonoidal constraint T_0' to be defined by $T_0' = \epsilon\otimes\mathrm{id}$ where $\epsilon : A \to 1$ is the counit of the bialgebra A.

Example 6.2.7. The identity functor $\mathrm{id} : \mathscr{C} \to \mathscr{C}$ is a bimonad with opmonoidal structure given by identities.

In the same way that a bialgebra determines the monoidal structure of the associated category of representations see Theorem 5.3.5, a bimonad determines a monoidal structure on the associated category of modules. In particular, we have the following.

Theorem 6.2.8 (Moerdijk [30] Theorem 7.1)**.** *Let T be a monad on a monoidal category $\mathscr{C}$ and let $\mathscr{C}^T$ be the category of T-modules. Then, the following are equivalent.*

- *Bimonad structures on the monad T of $\mathscr{C}$.*
- *Monoidal structures on the category of modules $\mathscr{C}^T$ such that the forgetful functor $\mathscr{C}^T \to T$ is a strict monoidal functor. In particular, the monoidal product and the monoidal unit of $\mathscr{C}^T$ is defined by*

$$(M,r) \otimes_{\mathscr{C}^T} (N,s) = \big(M \otimes N, (r \otimes s) \circ T_2(M,N)\big), \quad 1_{\mathscr{C}^T} = (1, T_0)$$

for any T-modules (M,r) and (N,s).

Remark 6.2.9. It is worth mentioning here that Moerdijk in [30] used the nomenclature *Hopf monads* for the notion of a bimonad.

6.3 Fusion operators and Hopf monads

Brugières, Lack and Virelizier in [7] associated to any bimonad on a monoidal category two natural transformations, called *fusion operators* generalising the notion of a fusion operator of a bialgebra in a braided monoidal category as discussed in Section 5.5. With the use of the generalised fusion operators the notion of a bimonad can be made to a Hopf monad, a notion which was defined in [7] and consists of a generalisation of Hopf algebras in braided monoidal categories to a non braided setup. In the following, we outline the basics of fusion operators and Hopf monads. Our presentation follows [7] closely.

First, we introduce the notion of a left fusion operator and a right fusion operator.

Definition 6.3.1. Let T be a bimonad on $\mathscr{C}$. The *left fusion operator* of T is the natural transformation $H^l : T \circ \otimes \circ (\mathrm{id}_{\mathscr{C}} \times T) \to \otimes \circ (T \times T)$ defined by

$$H^l_{X,Y} = (\mathrm{id}_{TX} \otimes \mu_Y) \circ T_2(X, TY) : T(X \otimes TY) \to TX \otimes TY \tag{6.3.1}$$

for all X, Y of $\mathscr{C}$. Similarly, the *right fusion operator* of T is the natural transformation $H^r : T \circ \otimes \circ (T \times \mathrm{id}_{\mathscr{C}}) \to \otimes \circ (T \times T)$ defined by

$$H^r_{X,Y} = (\mu_X \otimes \mathrm{id}_{TY}) \circ T_2(TX, Y) : T(TX \otimes Y) \to TX \otimes TY \tag{6.3.2}$$

for all X, Y of $\mathscr{C}$.

Bimonads for which the associated fusion operators are invertible are called Hopf monads. More precisely, we have the following definition.

Definition 6.3.2. Let $\mathscr{C}$ be a monoidal category, T be a bimonad on $\mathscr{C}$ with left fusion operator H^l and right fusion operator H^r as defined above.

1. The bimonad T is said to be a *left Hopf monad* if its left fusion operator is an isomorphism.

2. The bimonad T is said to be a a *right Hopf* monad if its right fusion operator is an isomorphism.

3. The bimonad T is said to be a *Hopf monad* on $\mathscr{C}$ if both left and right fusion operators are isomorphisms.

Remark 6.3.3. It is worth mentioning here that the category on which the Hopf monad is defined is just a monoidal category.

Example 6.3.4. Let $(H, m, u, \delta, \eta, S)$ be a Hopf algebra in a braided monoidal category $\mathscr{C}$ with an invertible antipode S. Then, the endofunctor $T' = H \otimes -$ is a bimonad on $\mathscr{C}$ with constraints $T_2'(X,Y)$ and T_0' as discussed in Example 6.2.6. Define the left fusion operator $H^l_{X,Y} : H \otimes \big(X \otimes (H \otimes Y)\big) \to (H \otimes X) \otimes (H \otimes Y)$ to be the composite

$$\begin{aligned} H \otimes \big(X \otimes (H \otimes Y)\big) &\xrightarrow{\delta \otimes \mathrm{id}_{X \otimes (H \otimes Y)}} H \otimes H \otimes \big(X \otimes (H \otimes Y)\big) \\ &\xrightarrow{\mathrm{id}_A \otimes \tau_{H,X} \otimes \mathrm{id}_{H \otimes Y}} H \otimes X \otimes \big(H \otimes (H \otimes Y)\big) \\ &\xrightarrow{\mathrm{id}_{H \otimes X} \otimes \mu'_Y} (H \otimes X) \otimes (H \otimes Y) \end{aligned}$$

and the right fusion $H^r_{X,Y} : H \otimes \big((H \otimes X) \ \otimes Y\big) \to (H \otimes X) \otimes (H \otimes Y)$ defined to be the composite

$$\begin{aligned} H \otimes \big((H \otimes X) \otimes Y\big) &\xrightarrow{\delta \otimes \mathrm{id}_{(H \otimes X) \otimes Y}} H \otimes H \otimes \big((H \otimes X) \otimes Y\big) \\ &\xrightarrow{\mathrm{id}_H \tau_{H, H \otimes X} \otimes \mathrm{id}_Y} \big(H \otimes (H \otimes X)\big) \otimes H \otimes Y \\ &\xrightarrow{\mu'_X \otimes \mathrm{id}_{H \otimes Y}} (H \otimes X) \otimes (H \otimes Y). \end{aligned}$$

Since H is a Hopf algebra, using the antipode S we can find an inverse $(H^l)^{-1}$ for the left fusion operator. To be precise, the inverse $(H^l)^{-1}$ is defined to be the following composite.

$$
\begin{aligned}
H\otimes X\otimes H\otimes Y &\xrightarrow{\delta\otimes \mathrm{id}_X\otimes \mathrm{id}_{H\otimes Y}} H\otimes H\otimes X\otimes H\otimes Y\\
&\xrightarrow{\mathrm{id}_H\otimes \tau_{H,X}\otimes \mathrm{id}_{H\otimes Y}} H\otimes X\otimes H\otimes H\otimes Y\\
&\xrightarrow{\mathrm{id}_{H\otimes X}\otimes S\otimes \mathrm{id}_{H\otimes Y}} H\otimes X\otimes H\otimes H\otimes Y\\
&\xrightarrow{\mathrm{id}_{H\otimes X}\otimes \mu'_Y} H\otimes \big(X\otimes (H\otimes Y)\big)
\end{aligned}
$$

Similarly, the inverse $(H^r)^{-1}$ of the right fusion operator uses the inverse of the antipode $S^{-1}: H\to H$ and it is given by the following composite.

$$
\begin{aligned}
H\otimes X\otimes H\otimes Y &\xrightarrow{\tau_{H\otimes X,H}\otimes \mathrm{id}_Y} H\otimes (H\otimes X)\otimes Y\\
&\xrightarrow{\delta\otimes \mathrm{id}_{H\otimes X}\otimes \mathrm{id}_Y} \big((H\otimes H)\otimes H\otimes X\big)\otimes Y\\
&\xrightarrow{S^{-1}\otimes \mathrm{id}_H\otimes \mathrm{id}_{H\otimes X}\otimes \mathrm{id}_Y} H\otimes H\otimes H\otimes X\otimes Y\\
&\xrightarrow{\tau_{H,H}\otimes \mathrm{id}_{H\otimes X}\otimes \mathrm{id}_Y} H\otimes \big((H\otimes H\otimes X)\otimes Y\big)\\
&\xrightarrow{\mathrm{id}_H\otimes \mu'_X\otimes \mathrm{id}_Y} H\otimes \big((H\otimes X)\otimes Y\big)
\end{aligned}
$$

Remark 6.3.5. If we set $X=Y=1_{\mathscr{C}}$ in the definition of the left fusion operator, then we recover the formula for fusion operator as defined by Street, see Definition 5.5.1.

In the next chapter we will be interested in a rigid monoidal category and a pair of adjoint functors. The following theorem establishes that a pair of adjoint functors between rigid monoidal categories gives rise to a Hopf monad.

The following theorem will be fundamental for our work.

Theorem 6.3.6 ([8] p.701)**.** *Let $(F:\mathscr{C}\to\mathscr{D}, U:\mathscr{D}\to\mathscr{C})$ be a pair of adjoint between rigid monoidal categories, such that the right adjoint U is a strong monoidal functor. Then, the adjunction monad $T=UF$ is a Hopf monad.*

Remark 6.3.7. Hopf monads on rigid monoidal categories were studied by Brugières and Virelizier in [8]. In [7, Section 3.4] is explained why the definition of a Hopf monad with the use of the fusion as above consists of a generalisation of the Hopf monad on a rigid monoidal category.

The following is the analogue of the Theorem 5.4.4 on the level of Hopf monads with left duality.

Theorem 6.3.8 ([38] p. 166)**.** *Let (T,μ,η) be a bimonad on a category with left duality. Then the following are equivalent.*

- *The bimonad T is a left Hopf monad.*
- *The Hopf monad T has a left antipode. In particular, for each X of $\mathscr{C}$ there is a natural transformation $s^l_X : T({}^\vee X) \to {}^\vee X$ defined by*

$$s^l_X = \Big(\big(T_0 \circ T(\mathrm{ev}_{T(X)}) \circ (H^l_{{}^\vee T(X),X})^{-1}\big) \otimes {}^\vee\eta_X\Big) \circ \big(\mathrm{id}_{T({}^\vee T(X))} \otimes \mathrm{coev}_{T(X)}\big)$$

such that the following relations are satisfied.

$$T_0 \circ T(\mathrm{ev}_X) \circ T({}^\vee\eta_X \otimes \mathrm{id}_X) = \mathrm{ev}_{T(X)} \circ \big(s^l_{T(X)} T({}^\vee\mu_X) \otimes \mathrm{id}_{T(X)}\big) \circ T_2\big({}^\vee T(X), X\big)$$

$$(\eta_X \otimes \mathrm{id}_{{}^\vee X}) \circ \mathrm{coev}_X \circ T_0 = (\mu_X \otimes s^l_X) \circ T_2\big(T(X), {}^\vee T(X)\big) T(\mathrm{coev}_{T(X)})$$

- *The monoidal category of T-modules has a left duality which is computed from the left antipode as follows.*

$${}^\vee(X,r) = \big({}^\vee X, s^l_X \circ T({}^\vee r)\big), \qquad \mathrm{ev}_{(X,r)} = \mathrm{ev}_X, \qquad \mathrm{coev}_{(X,r)} = \mathrm{coev}_X\,.$$

In a similar way, we can consider a bimonad on a category with right duality. Then, there exists analogous statements for right Hopf bimonads, right antipodes and right duality on the category of the T-modules.

Example 6.3.9 ([8] p.699)**.** Let $\mathscr{C}$ be a braided category with left duality and let $(H, m, \eta, \delta, u, S)$ be a Hopf algebra in $\mathscr{C}$. Then, the bimonad $H \otimes (-)$ is a left Hopf monad with left antipode $s^l_X : H \otimes {}^\vee X \otimes {}^\vee H \to {}^\vee X$ given by

$$s^l_X = \big((\mathrm{ev}_H \circ \tau_{{}^\vee H,H}) \otimes \mathrm{id}_{{}^\vee X}\big) \circ \big(S \otimes \tau^{-1}_{{}^\vee H,{}^\vee X}\big).$$

The next lemma relates morphisms of Hopf monads, which are morphisms of the underlying bimonads and the left antipodes.

Lemma 6.3.10 ([8] p.702)**.** *Let (T, s^l_X) and $(T', \overline{s}^l_X)$ be Hopf monads with s^l and $\overline{s}^l$ their associated left antipodes on a monoidal category $\mathscr{C}$ with left duality. Then, a morphism $f : T \to T'$ of Hopf monads preserves the antipodes. That means that the following diagram commutes for any object X of $\mathscr{C}$.*

$$\begin{tikzcd}
T\big({}^\vee T'(X)\big) \arrow[r, "T({}^\vee f_X)"] \arrow[d, "f_{{}^\vee T'(X)}"'] & T'\big({}^\vee T'(X)\big) \arrow[d, "s^l_X"] \\
T\big({}^\vee T(X)\big) \arrow[r, "\overline{s}^l_X"'] & {}^\vee X
\end{tikzcd}$$

6.4 Hopf monads from Hopf operators

In this section, we discuss a way for obtaining Hopf monads starting from an opmonoidal adjunction. Our presentation follows [7] closely.

First, recall from Definition 2.2.5 that an adjunction $F \dashv U$ between functors $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ is said to be an opmonoidal adjunction if both functors F and G are opmonoidal and the adjunction unit and adjunction counit are opmonoidal natural transformations. To any such opmonoidal adjunction a pair of natural transformations, called Hopf operators is assigned. To be precise, we give the following definition.

Definition 6.4.1. A *left Hopf operator* of the opmonoidal adjunction $F \dashv U$ is the natural transformation $\mathbb{H}^l : F(1_{\mathscr{C}} \otimes U) \to F \otimes 1_{\mathscr{D}}$ defined by

$$\mathbb{H}^l_{X,Y} := (FX \otimes \epsilon_Y) \circ F_2(X, UY)$$

for all $X \in \mathscr{C}$ and $Y \in \mathscr{D}$. Similarly, the *right Hopf operator* of the opmonoidal adjunction $F \dashv U$ is the natural transformation $\mathbb{H}^r : F(U \otimes 1_{\mathscr{C}}) \to 1_{\mathscr{D}} \otimes F$ defined by

$$\mathbb{H}^r_{X,Y} := (\epsilon_Y \otimes FX) \circ F_2(UY, X)$$

for all $X \in \mathscr{C}$ and $Y \in \mathscr{D}$.

For any opmonoidal adjunction of the form $F \dashv U$, the endofunctor $T = UF$ of $\mathscr{C}$ has a bimonad stucture on it, see Corollary 6.2.5. Moreover, to any bimonad T is assigned fusion transformations, see Section 6.3. The following result establishes a relation between the Hopf operators of the opmonoidal adjunction $F \dashv U$ and the fusion operators of the bimonad $T = UF$.

Proposition 6.4.2 ([7] p.755)**.** *Let $T = UF$ be the bimonad of an opmonoidal adjunction $F \dashv U$. Then the fusion operators of T as defined in 6.3.1 are related with the Hopf operators of the opmonoidal adjunction by the following formuli.*

$$H^l_{X,Y} = U_2(FX, FY) \circ U(\mathbb{H}^l_{X,FY}), \qquad H^r_{X,Y} = U_2(FX, FY) \circ U(\mathbb{H}^r_{FX,Y}). \tag{6.4.1}$$

Proof. We will show the first formula for $H^l_{X,Y}$ and $\mathbb{H}^l_{X,FY}$. The second equation is proved in a similar fashion.

$$\begin{aligned} H^l_{X,Y} &\overset{(1)}{=} (\mathrm{id}_{TX} \otimes \mu_Y) T_2(X, TY) \\ &\overset{(2)}{=} (\mathrm{id}_{UFX} \otimes U(\epsilon_{FY})) U_2(FX, FUFY) U(F_2(X, UFY) \\ &\overset{(3)}{=} U_2(FX, FY) U((\mathrm{id}_{FX} \otimes \epsilon_{FY}) F_2(X, UFY) \\ &\overset{(4)}{=} U_2(FX, FY) U(\mathbb{H}_{X,FY}). \end{aligned}$$

Equalities (1) and (4) are the definitions of the fusion operator of the bimonad $T = UF$ and the Hopf operator of the opmonoidal adjunction $F \dashv U$. Equality (2) is the definition of the product map μ and of the natural transformation T_2. Finally equality (3) is due to the naturality of U_2. □

Opmonoidal adjunctions for which the Hopf operators are invertable are called Hopf adjunctions. To be explicit we have the following definition.

Definition 6.4.3. Let $F \dashv U$ be an opmonoidal adjunction.

1. The opmonoidal adjunction $F \dashv U$ is said to be *left Hopf adjunction* if $\mathbb{H}^l$ is invertible.

2. The opmonoidal adjunction $F \dashv U$ is said to be *right Hopf adjunction* if $\mathbb{H}^r$ is invertible.

3. The opmonoidal adjunction $F \dashv U$ is said to be a *Hopf adjunction* if both $\mathbb{H}^l$ and $\mathbb{H}^r$ are invertible.

Proposition 6.4.4 ([7] p.755)**.** *The monad of a left Hopf adjunction is a left Hopf monad. Similarly, the monad of a right Hopf adjunction is a right Hopf monad. Moreover, the monad of a Hopf adjunction is a Hopf monad.*

The following proposition consists of a criterion for an opmonoidal adjunction $F \dashv U$ to be a Hopf adjunction.

Proposition 6.4.5 ([7] p.760)**.** *Let $F \dashv U$ be an opmonoidal adjunction between left closed monoidal categories. Then, the adjunction $F \dashv U$ is a left Hopf adjunction if and only if U is a left closed. In particular, the inverse of H^l is given by*

$$\mathrm{ev}^{Y}_{F(X\otimes UY)} \circ \Big(\epsilon_{R\mathcal{H}\mathrm{om}(Y,F(X\otimes UY))} \circ F\big(\alpha^{-1} \circ R\mathcal{H}\mathrm{om}(\mathrm{id},\eta) \circ \mathrm{coev}^{UY}_{X}\big) \otimes \mathrm{id}\Big)$$

where $\alpha^{-1} : \mathcal{H}\mathrm{om}(UX, UY) \xrightarrow{\sim} U R\mathcal{H}\mathrm{om}(X,Y)$.

Similarly, if $F \dashv U$ is an opmonoidal adjunction between right closed monoidal categories then, the adjunction $F \dashv U$ is a right Hopf adjunction if and only if U is right closed.

The above proposition will be used as follows. Assume that we have an opmonoidal adjunction $F \dashv U$ where the functors are defined between closed symmetric monoidal categories. By proving that F is a left closed functor yields that the adjunction $F \dashv U$ is a Hopf adjunction and hence the associated bimonad is actually a Hopf monad.

6.5 Hopf algebras from Hopf monads

In Section 6.3 we were in the situation where a Hopf algerba H in a braided monoidal category $\mathscr{C}$ gives rise to a Hopf monad $T' := H \otimes -$ on $\mathscr{C}$. However it is not generally true that all Hopf monads are coming from Hopf algebras in some category. A detailed study of those Hopf monads which are represented by Hopf algebras has been done by Brugières, Lack and Virelizier in [7]. We present this theory here.

We start with a definition which is essential for the following.

Definition 6.5.1. Let T be an opmonoidal endofunctor of a monoidal category $\mathscr{C}$ and $e: T \to \mathrm{id}_{\mathscr{C}}$ be an opmonoidal natural transfromation. Define the natural transformations $u^e: T \to T(1) \otimes -$ and $v^e: T \to - \otimes T(1)$ by

$$u^e_X = (\mathrm{id}_{T(1)} \otimes e_X) \circ T_2(1, X), \qquad v^e_X = (e_X \otimes \mathrm{id}_{T(1)}) \circ T_2(X, 1). \tag{6.5.1}$$

The augmentation map e is said to be *left regular* if u^e is invertible.

If the above opmonoidal endofunctor is a bimonad and $e: T \to \mathrm{id}_{\mathscr{C}}$ is a morphism of bimonads we obtain the notion of an augmented bimonad.

Definition 6.5.2. Let $\mathscr{C}$ be a monoidal category. A bimonad T on $\mathscr{C}$ is *augmented* if it is endowed with a bimonad morphism $e: T \to 1_{\mathscr{C}}$. In this case, e is called an *augmentation* for the bimonad T.

Augmented bimonads for which the augmentation map is left regular define bialgebras in the center of the category $\mathscr{C}$.

Proposition 6.5.3 ([7] p.782)**.** *Let T be an augmented opmonoidal endofunctor of $\mathscr{C}$ such that the augmentation $e: T \to \mathrm{id}_{\mathscr{C}}$ is left regular and define the following natural transformation*

$$\sigma = v^e \circ (u^e)^{-1}: T(1) \otimes (-) \to (-) \otimes T(1). \tag{6.5.2}$$

Then, σ is a half braiding in $\mathscr{C}$ and $(T(1), \sigma)$ is a central coalgebra of $\mathscr{C}$ with coproduct $T_2(1,1)$ and counit T_0. Moreover, the natural isomorphism $u^e: T \to T(1) \otimes_\sigma -$ is an opmonoidal isomorphism.

The fact that u^e is an opmonoidal isomorphism means that $T' = T(1) \otimes_\sigma -$ is given an opmonoidal structure by

$$T'_2(X, Y) = (\mathrm{id}_{T(1)} \otimes \sigma_X \otimes \mathrm{id}_Y) \circ (T_2(1,1) \otimes X \otimes Y), \qquad T'_0 = T_0 \otimes \mathrm{id}.$$

If moreover, the above augmented opmonoidal endofunctor T is given the structure of a bimonad, see Definition 6.2.1 and $e: T \to \mathrm{id}$ is a morphism of bimonads, then $(T(1), \sigma, T_2(1,1), T_0)$ is a bialgebra in the centre of the category $\mathscr{C}$. More precisely, we have the following result.

Proposition 6.5.4 ([7] p. 783)**.** *Let (T, μ, η, e) be an augmented bimonad on $\mathscr{C}$ such that the augmentation $e: T \to \mathrm{id}_{\mathscr{C}}$ is left regular. Then, $\sigma := v^e \circ (u^e)^{-1}$ is a half braiding for $T(1)$ and the pair $(T(1), \sigma)$ is a bialgebra in the Drinfel'd centre $\mathcal{Z}(\mathscr{C})$ of $\mathscr{C}$ with product $m = \mu_1 \circ (u^e_{T(1)})^{-1}$, unit $u = \eta_1$, coproduct $T_2(1,1)$ and counit T_0. Moreover, $u^e: T \to T(1) \otimes_\sigma -$ is an isomorphism of bimonads.*

The following lemma gives an explicit formula for the inverse $(u^e_X)^{-1}$ in the case which T is an augmented Hopf monad.

Lemma 6.5.5 ([7] p.783)**.** *Let T be an augmented left Hopf monad on $\mathscr{C}$. Then, its augmentation $e : T \to \mathrm{id}_{\mathscr{C}}$ is left regular and*

$$(u^e)^{-1} = T(e) \circ H^{l^{-1}}_{1,-} \circ (\mathrm{id}_{T(1)} \otimes \eta)$$

All the above material lead to the following fundamental theorem which is the cornerstone of our work in the next chapter.

Theorem 6.5.6 ([7] p.784)**.** *Let $\mathscr{C}$ be a monoidal category. Then, there is a one to one correspodence between augmented Hopf monads on $\mathscr{C}$ such that augmentation map is left regular and Hopf algebras in the Drinfel'd centre $\mathscr{Z}(\mathscr{C})$ of the monoidal category $\mathscr{C}$.*

We have to make the point out here that the above theorem is true for Hopf monads on arbitrary monoidal categories for which no requirement about braiding or symmetry has been made. In the following proposition a further characterisation of Hopf monads on braided categories which are represented by Hopf algebras is given.

Proposition 6.5.7 ([7] p.780)**.** *Let T be a Hopf monad on a braided category $(\mathscr{C}, \otimes, 1, \tau)$. Then, the monad T is isomorphic to the Hopf monad $H \otimes -$ for some Hopf algebra H in $\mathscr{C}$ if and only if the monad T is endowed with an augmentation map $e : T \to \mathrm{id}$ which is compatible with the braiding τ of $\mathscr{C}$ in the following way*

$$(e_X \otimes \mathrm{id}_{T(1)}) \circ T_2(X, 1) = (e_X \otimes \mathrm{id}_{T(1)}) \circ \tau_{T(1), T(X)} \circ T_2(1, X) \tag{6.5.3}$$

for all X of $\mathscr{C}$.

The above proposition, in view of Theorem 6.5.6 tells us that if we have a braided monoidal category $\mathscr{C}$ and an augmented Hopf monad on $\mathscr{C}$ then T is isomorphic to the $T' = T(1) \otimes -$. Among other that means that the augmentation map e of the Hopf monad T and augmentation of T' which is defined by $e' = T_0 \otimes -$ coincide. Since, we couldn't find it written in [7] we make this check here.

Lemma 6.5.8. *For the augmented Hopf monads (T, e) and $(T', e' = T_0)$ it is true that $e = u^e \circ e'$.*

Proof. Unpacking the relation $e = u^e \circ e'$ comes down to check the commutativity of the diagram below.

$$
\begin{tikzcd}
UF(1\otimes X) \arrow[r] \arrow[d, "T_2(1,X)"'] \arrow[dr, "\mathrm{id}_{T(X)}"] & UFUW(1\otimes X) \arrow[r, "U(\epsilon_W)"] & UW(1\otimes X) \arrow[dd] \\
UF(1)\otimes UF(X) \arrow[r, "T_0\otimes \mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes e"'] & 1\otimes UF(X) \arrow[d, "\mathrm{id}\otimes e"] & \\
UF(1)\otimes X \arrow[r, "T_0\otimes \mathrm{id}"'] & 1\otimes X \arrow[r, "\mathrm{id}"'] & 1\otimes X
\end{tikzcd}
$$

Tthe commutativity of the top left triangle follows from the fact that T is a bimonad, see Definition 6.2 and the bottom left square from the interchange law. □

If moreover $\zeta : T(X) \to X$ is an action of the Hopf monad T on the objects of $\mathscr{C}$ then, from the fact that u^e is a natural isomorphism of monads, we can define an action of the Hopf monad T' on the objects of $\mathscr{C}$ by

$$\rho : T(1)\otimes X \xrightarrow{(u^e)^{-1}} T(X) \xrightarrow{\rho} X.$$

for any X of $\mathscr{C}$. Now, being both of them Hopf monads on rigid monoidal categories we can define antipodes and by Theorem 6.3.8 these Hopf monads act on the dual objects of the category $\mathscr{C}$. In the following, we denote the left antipode of T by s^l_X and the left antipode of T' by $\overline{s}^l_X$ and check the following, since we couldn't find it in [7].

Proposition 6.5.9. *Let $\mathscr{C}$ be a category with left duality and let T and $T' = T(1)\otimes -$ be isomorphic Hopf monads on $\mathscr{C}$. Then, the natural isomorphism u^e preserves the actions of T and T' on the left dual ${}^\vee X$ of $\mathscr{C}$.*

Proof. What we want to prove comes down to the proof of the commutativity of the following diagram.

$$
\begin{tikzcd}
T(1)\otimes {}^\vee X \arrow[dd, "\mathrm{id}\otimes {}^\vee\rho"'] & & T({}^\vee X) \arrow[ll, "u^e_{{}^\vee X}"'] \arrow[dl, "T({}^\vee\rho)"] \arrow[dd, "T({}^\vee\zeta)"] \\
& T({}^\vee X\otimes {}^\vee T(1)) \arrow[dl, "u^e_{{}^\vee X\otimes {}^\vee T(1)}"] \arrow[dr, "T({}^\vee u^e)"] & \\
T(1)\otimes {}^\vee X\otimes {}^\vee T(1) \arrow[dr, "\overline{s}^l_X"'] & & T({}^\vee T(X)) \arrow[dl, "s^l_X"] \\
& {}^\vee X &
\end{tikzcd}
$$

The commutativity of the top left square follows from the naturality of the natural transformation ϕ. The top right triangle commutes since we defined the action of the Hopf monad T' via the action of UF. Finally, the commutativity of the bottom square follows from Lemma 6.3.10. The proof is complete. $\square$

Chapter 7

Hopf algebras, Hopf monads and orbit categories

In this chapter we bring together the theory of Hopf monads on monoidal categories and the theory of derived categories of coherent sheaves on smooth complex projective varieties.

In Section 7.1 we associate to a smooth complex projective variety the orbit category $D^{\mathrm{or}}(X)$ which is obtained from the bounded derived category of X and the functor $[2] : D^b(X) \to D^b(X)$. Then, we prove that if the variety is even dimensional and has trivial canonical bundle, then the functor $T = L\Delta^* R\Delta_*$ is a Hopf monad on $D^{\mathrm{or}}(X)$, where $\Delta : X \to X \times X$ is the diagonal embedding and $L\Delta^*$, $R\Delta_*$ denote the associated functors of Δ on the level of orbit categories.

In Section 7.2 we are motivated from the structure of the orbit category $D^{\mathrm{or}}(X)$ and the properties of the functors $R\Delta_*$ and $L\Delta^*$ and we prove in a general categorical level the following: given a pair of adjoint functors (F, U) defined between rigid monoidal categories such that the right adjoint U is strong monoidal and for which there exists a right quasi-inverse W which is also strong monoidal and moreover the natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is a monoidal natural transformation, then the adjunction monad is an augmented Hopf monad.

In Section 7.3 we explain why the Hopf monad $T = \Delta^* \Delta_*$ is an example of the above general categorical setup and so T is an augmented Hopf monad. As an application of this general theory, in combination with the theory of Hopf monads, we obtain that the object $\mathscr{U} := \Delta^* \Delta_* \mathscr{O}_X$ is a cocommutative Hopf algebra object in $D^{\mathrm{or}}(X)$ of which the antipode is an involution. This is the main result of this thesis.

7.1 Hopf monad on the orbit category

In this section we consider a smooth complex projective variety X and its associated bounded derived category of coherent sheaves $D^b(X)$ and define the orbit category $D^{\mathrm{or}}(X)$ as in Section 1.5 where the essential autoequivalence of $D^b(X)$ is the shift functor $[2] : D^b(X) \to D^b(X)$. We prove that if X is an even dimensional smooth complex projective variety with trivial canonical bundle and $\Delta : X \to X \times X$ is the diagonal embedding, then the endofunctor $T = L\Delta^* R\Delta_*$ of $D^{\mathrm{or}}(X)$ is a Hopf monad on $D^{\mathrm{or}}(X)$.

Notation 7.1.1. In the following, we will denote by X a smooth complex projective variety, by $\Delta : X \to X \times X$ the diagonal embedding and by $D^b(X)$ the bounded derived category of coherent sheaves on X.

Definition 7.1.2. Let X be a smooth complex projective variety. We define the category $D^{\mathrm{or}}(X)$ to be the category with objects bounded cochain complexes of coherent sheaves on X and morphisms to be defined by

$$\mathrm{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}\left(\mathscr{E}^\bullet, \mathscr{F}^\bullet[2n]\right) \tag{7.1.1}$$

The composition of two morphisms $f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet[2n]$ and $g : \mathscr{F}^\bullet \to \mathscr{G}^\bullet[2m]$ is defined by $\left(g[2n]\right) \circ f : \mathscr{E}^\bullet \to \mathscr{G}^\bullet[2(m+n)]$.

An important property of the objects of $D^{\mathrm{or}}(X)$ which will be essential for our purposes is the following.

Lemma 7.1.3. *For each $\mathscr{E}^\bullet$ of $D^{\mathrm{or}}(X)$, there exists an isomorphism $\mathscr{E}^\bullet[2] \simeq \mathscr{E}^\bullet$.*

Proof. We will find morphisms $f : \mathscr{E}^\bullet[2] \to \mathscr{E}^\bullet$ and $g : \mathscr{E}^\bullet \to \mathscr{E}^\bullet[2]$ in $D^{\mathrm{or}}(X)$ such that $f \circ g = \mathrm{Id}_{\mathscr{E}^\bullet}$ and $g \circ f = \mathrm{Id}_{\mathscr{E}^\bullet[2]}$. Take $f = \mathrm{Id}_{\mathscr{E}^\bullet[2]}$ and $g = \mathrm{Id}_{\mathscr{E}^\bullet}$. Then, we compute that

$$f \circ g = (\mathrm{Id}_{\mathscr{E}^\bullet[2]})[-2] \circ \mathrm{Id}_{\mathscr{E}^\bullet} = \mathrm{Id}_{\mathscr{E}^\bullet}, \qquad g \circ f = \mathrm{Id}_{\mathscr{E}^\bullet}[2] \circ \mathrm{Id}_{\mathscr{E}^\bullet[2]} = \mathrm{Id}_{\mathscr{E}^\bullet[2]}$$

and the proof is complete. □

Corollary 7.1.4. *For each $\mathscr{E}^\bullet$ of $D^{\mathrm{or}}(X)$ and $n \in \mathbb{Z}$ is true that $\mathscr{E}^\bullet[2n] \simeq \mathscr{E}^\bullet$ for $n \in \mathbb{Z}$.*

The hom sets of the orbit category have the following property.

Lemma 7.1.5. *For all $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ of $D^{\mathrm{or}}(X)$ it is true that*

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet)\right) < \infty \tag{7.1.2}$$

Proof. This follows by the definition of the hom sets in the orbit category, see 7.1.1, and from the fact that $\dim_{\mathbb{C}}\left(\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}\left(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i]\right)\right) < \infty$ since X is a smooth complex projective variety and Proposition 4.1.3. □

In Section 4.2 we discussed derived functors on the level of bounded derived categories of coherent sheaves. A fundamental property of derived functors is that they are exact functors between triangulated categories. For example, if $f: X \to Y$ is a smooth map between smooth projective varieties, then the derived functor $Rf_*: D^b(X) \to D^b(Y)$ is equipped with a natural isomorphism $Rf_* \circ [2] \xrightarrow{\sim} [2] \circ Rf_*$, in other words Rf_* is a $[2]$-equivariant functor. Similarly, the derived functors $Lf^*: D^b(Y) \to D^b(X)$ and $f^!: D^b(Y) \to D^b(X)$ enjoy this property. From the general theory about functors on the level of orbit categories, see Section 1.5 we have the following lemma.

Lemma 7.1.6. *Let $f: X \to Y$ be a smooth map between smooth complex projective varieties. Then, the derived functors Rf_*, Lf^*, $f^!$ which are defined on the level of bounded derived categories induce functors on the level of orbit categories.*

In Section 4.3 we discussed various adjunctions between derived functors on the level of derived categories of coherent sheaves. In the following, we are verifying that these adjunctions hold also on the level of orbit categories.

Lemma 7.1.7. *Let $f: X \to Y$ be a smooth map of smooth complex projective varieties and let $Lf^*: D^{\mathrm{or}}(Y) \to D^{\mathrm{or}}(X)$ and $Rf_*: D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(Y)$ be the functors which are induced from the derived functor Lf^* and Rf_*. Then, there exists a natural isomorphism*

$$\mathrm{Hom}_{D^{\mathrm{or}}(X)}(Lf^*\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{or}}(Y)}(\mathscr{E}^\bullet, Rf_*\mathscr{F}^\bullet). \tag{7.1.3}$$

for all $\mathscr{E}^\bullet$ of $D^{\mathrm{or}}(Y)$ and $\mathscr{F}^\bullet$ of $D^{\mathrm{or}}(X)$.

Proof. By the definition of the hom-sets of the orbit category, the fact that Lf^* is left adjoint to Rf_* on the level of bounded derived categories, the exactness of the derived functors Rf_* and again the definition of the hom-sets of the orbit category we obtain the following chain of maps.

$$\begin{aligned}
\mathrm{Hom}_{D^{\mathrm{or}}(X)}(Lf^*\mathscr{E}^\bullet, \mathscr{F}^\bullet) &= \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(Lf^*\mathscr{E}^\bullet, \mathscr{F}^\bullet[2n]) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}\big(\mathscr{E}^\bullet, Rf_*(\mathscr{F}^\bullet[2n])\big) \qquad (Lf^* \dashv Rf_*) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathscr{E}^\bullet, (Rf_*\mathscr{F}^\bullet)[2n]) \\
&= \mathrm{Hom}_{D^{\mathrm{or}}(Y)}(\mathscr{E}^\bullet, Rf_*\mathscr{F}^\bullet).
\end{aligned}$$ □

In a similar way, we obtain for $Rf_*: D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(Y)$ its right adjoint.

Lemma 7.1.8. *Let $f: X \to Y$ be a smooth map of smooth complex projective varieties and let $f^!: D^{\mathrm{or}}(Y) \to D^{\mathrm{or}}(X)$ and $Rf_*: D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(Y)$. Then, there exist a natural isomorphism*

$$\mathrm{Hom}_{D^{\mathrm{or}}(Y)}(Rf_*\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{E}^\bullet, f^!\mathscr{F}^\bullet) \tag{7.1.4}$$

for all $\mathscr{E}^\bullet$ of $D^{\mathrm{or}}(X)$ and $\mathscr{F}^\bullet$ of $D^{\mathrm{or}}(Y)$.

The objects of orbit category are bounded complexes of coherent sheaves and so the derived tensor product as defined in 4.2 which turns $D^b(X)$ to a symmetric monoidal category, see Proposition 4.2.8 is well defined also on the level of orbit categories. The following lemma establishes that the orbit category $D^{\text{or}}(X)$ is a closed category, a notion which was discussed in Section 2.5.

Proposition 7.1.9. *Let X be a smooth complex projective variety. Then, there exists a natural isomorphism*

$$\operatorname{Hom}_{D^{\text{or}}(X)}\left(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet, \mathscr{G}^\bullet\right) \xrightarrow{\sim} \operatorname{Hom}_{D^{\text{or}}(X)}\left(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)\right). \tag{7.1.5}$$

for all $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ of $D^{\text{or}}(X)$. In other words, $D^{\text{or}}(X)$ is a closed symmetric monoidal category.

Proof. By the Definition 7.1.1 of the hom sets of $D^{\text{or}}(X)$ and by Proposition 4.3.2 we have the following chain of maps.

$$\begin{aligned}
\operatorname{Hom}_{D^{\text{or}}(X)}(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet, \mathscr{G}^\bullet) &= \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{D^b(X)}(\mathscr{E}^\bullet \otimes^L \mathscr{F}^\bullet, \mathscr{G}^\bullet[2n]) \\
&\simeq \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{D^b(X)}\left(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}^\bullet(\mathscr{F}^\bullet, \mathscr{G}^\bullet[2n]\right) \\
&\simeq \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{D^b(X)}\left(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}^\bullet(\mathscr{F}^\bullet, \mathscr{G}^\bullet)[2n]\right) \\
&= \operatorname{Hom}_{D^{\text{or}}(X)}\left(\mathscr{E}^\bullet, R\mathcal{H}\mathrm{om}(\mathscr{F}^\bullet, \mathscr{G}^\bullet)\right).
\end{aligned}$$

□

In Section 4.4 we discussed the construction of a dual object for a bounded complex of coherent sheaves $\mathscr{E}^\bullet$. Since the objects of the orbit category are the same as the bounded derived category and the internal hom functor is well defined we have the following.

Proposition 7.1.10. *Let X be a smooth complex projective variety. Then, the category $D^{\text{or}}(X)$ is a rigid monoidal category.*

In Theorem 4.1.5 we discussed the Serre functor on the bounded derived category of coherent sheaves and gave emphasis on the fact that S_X is an exact functor, see Theorem 3.1.21. In the following, we check that the Serre functor $S_X : D^{\text{or}}(X) \to D^{\text{or}}(X)$ defined by $S_X(-) = (-) \otimes \omega_X[\dim(X)]$ is indeed a Serre functor for the orbit category $D^{\text{or}}(X)$.

Lemma 7.1.11. *The functor $S_X(-) := (-) \otimes \omega_X[\dim X]$ is a Serre functor for $D^{\text{or}}(X)$.*

Proof. For any $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ in $D^{\text{or}}(X)$ we will prove that there is natural isomorphisms

$$\operatorname{Hom}_{D^{\text{or}}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\sim} \left(\operatorname{Hom}_{D^{\text{or}}(X)}(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes \omega_X[\dim X])\right)^\vee. \tag{7.1.6}$$

Starting with the left hand side of (7.1.6) then by the definition of Hom-spaces in $D^{\mathrm{or}}(X)$ and Serre duality on $D^b(X)$ we have the following.

$$\begin{aligned}
\mathrm{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{E}^\bullet,\mathscr{F}^\bullet) &:= \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathscr{E}^\bullet,\mathscr{F}^\bullet[2n]) \\
&\simeq \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathscr{F}^\bullet[2n],\mathscr{E}^\bullet[\dim X])^\vee \\
&\simeq \Big(\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathscr{F}^\bullet[2n],\mathscr{E}^\bullet\otimes^L\omega_X[\dim X])\Big)^\vee \quad \text{(Lemma 7.1.5)} \\
&\simeq \Big(\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathscr{F}^\bullet,\mathscr{E}^\bullet\otimes^L\omega_X[\dim X][-2n])\Big)^\vee \\
&\simeq \Big(\mathrm{Hom}_{D^{\mathrm{or}}(X)}(\mathscr{F}^\bullet,\mathscr{E}^\bullet\otimes^L\omega_X[\dim X])\Big)^\vee. \qquad \square
\end{aligned}$$

In the following, we will be interested in n-dimensional smooth complex projective varieties X with trivial canonical bundle $\omega_X := \bigwedge^n \Omega_X$. An example of such variety is a $K3$ surfaces which is defined below.

Example 7.1.12 ([16] Definition 10.1)**.** A $K3$ surface is a compact complex surface X with trivial canonical bundle, i.e. $\omega_X \simeq \mathscr{O}_X$ and $H^1(X,\mathscr{O}_X) = 0$.

By Theorem 6.3.6, we have that given a pair of adjoint functors $(F:\mathscr{C}\to\mathscr{D}, U:\mathscr{D}\to\mathscr{C})$ between rigid monoidal categories such that the right adjoint U is strong monoidal, we obtain a Hopf monad on $\mathscr{C}$. In the case at hand, for a smooth map $f: X\to Y$ between smooth projective varieties we obtain adjunctions $Lf^*\dashv Rf_*$ and $Rf_*\dashv f^!$ on the level of adjoint functors where only Lf^* is a strong monoidal functor, see Proposition 4.3.7. The following lemma is essential for obtaining pairs of adjoint functors with the right adjoint to be strong monoidal.

Lemma 7.1.13. *Let X be an even dimensional smooth complex projective variety with trivial canonical bundle. Then, the Serre functor S_X of $D^{\mathrm{or}}(X)$ is trivial; symbolically $S_X \cong \mathrm{id}$.*

Proof. The following chain of maps gives the required natural isomorphism.

$$\begin{aligned}
S_X(\mathscr{E}^\bullet) &:= \mathscr{E}^\bullet\otimes^L\omega_X[\dim(X)] \\
&\simeq \mathscr{E}^\bullet\otimes^L\mathscr{O}_X[\dim(X)] && (\omega_X\simeq\mathscr{O}_X) \\
&\simeq \mathscr{E}^\bullet[\dim(X)] \\
&\simeq \mathscr{E}^\bullet && (\mathscr{E}^\bullet\cong\mathscr{E}^\bullet[2]\text{ in } D^{\mathrm{or}}(X))
\end{aligned}$$

In particular, the isomorphism $\mathscr{E}^\bullet\cong\mathscr{E}^\bullet[2]$ is the identity. $\square$

As a direct application of the above lemma we obtain functors on the level of orbit categories with the same left and right adjoint functor. More explicitly, we have the following.

Proposition 7.1.14. *Let $f : X \to Y$ be a smooth map between even dimensional smooth complex projective varieties with trivial canonical bundles. Then, the functor $f^! : D^{\mathrm{or}}(Y) \to D^{\mathrm{or}}(X)$ is isomorphic to $Lf^* : D^{\mathrm{or}}(Y) \to D^{\mathrm{or}}(X)$. Moreover, the functor $Rf_* : D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(Y)$ has simultaneously the functor Lf^* as a left and right adjoint. Symbolically, $Lf^* \dashv Rf_* \dashv Lf^*$.*

Proof. By Theorem 4.3.6 we have that $f^!$ is a right adjoint to Rf_*. Since, Rf_* has as left adjoint the functor Lf^* and $D^{\mathrm{or}}(X)$ and $D^{\mathrm{or}}(Y)$ admit Serre functors, then by Theorem 1.4.4 we have that the right adjoint Rf_* is defined by $S_X \circ Lf^* \circ S_Y^{-1}$, where S_X and S_Y are the Serre functors on the associated orbit categories. From the uniqueness of the right adjoints we have that $f^! \simeq S_X \circ Lf^* \circ S_Y^{-1}$. Then, by Lemma 7.1.13 we have that $S_X = \mathrm{id} = S_Y$ for even dimensional Calabi-Yau varieties X and Y and the proof is complete. □

With the use of the above proposition, we obtain a Hopf monad on the orbit category $D^{\mathrm{or}}(X)$.

Proposition 7.1.15. *Let X be an even dimensional smooth complex projective variety with trivial canonical bundle. Then, the functor $T = L\Delta^* R\Delta_*$ is a Hopf monad on $D^{\mathrm{or}}(X)$, where $R\Delta_* : D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(X \times X)$ and $L\Delta^* : D^{\mathrm{or}}(X \times X) \to D^{\mathrm{or}}(X)$.*

Proof. Setting in Proposition 7.1.14 $X = Y$ and taking f to be the diagonal embedding $\Delta : X \to X \times X$, we obtain the adjunction $R\Delta_* \dashv L\Delta^*$ of which the counit is $\epsilon : R\Delta_* L\Delta^* \to \mathrm{id}$ and its unit is $\eta : \mathrm{id} \to L\Delta^* R\Delta_*$. Then, the functor $T = L\Delta^* R\Delta_*$ is the adjunction monad with product map $\mu := L\Delta^*(\epsilon_{R\Delta_*})$ and unit the adjunction unit η. Since, $L\Delta^*$ is a strong monoidal functor and $D^{\mathrm{or}}(X)$ is a rigid monoidal category, by Theorem 6.3.6 we obtain that $T = \Delta^* \Delta_*$ is a Hopf monad. □

Remark 7.1.16. The functor $L\Delta^* : D^{\mathrm{or}}(X \times X) \to D^{\mathrm{or}}(X)$ has more interesting properties. First of all, $L\Delta^*$ is a strong monoidal functor and so an opmonoidal functor. That means, that there are natural isomorphisms

$$(L\Delta^*)_2(\mathscr{G}^\bullet, \mathscr{I}^\bullet) : L\Delta^*(\mathscr{G}^\bullet \otimes \mathscr{I}^\bullet) \xrightarrow{\sim} L\Delta^* \mathscr{G}^\bullet \otimes L\Delta^* \mathscr{I}^\bullet, \qquad (L\Delta^*)_0 : L\Delta^* \mathscr{O}_{X \times X} \xrightarrow{\sim} \mathscr{O}_X$$

which satisfy relations of Definition 2.1.5 for any objects $\mathscr{G}^\bullet$ and $\mathscr{I}^\bullet$ of $D^{\mathrm{or}}(X \times X)$.

Moreover, $L\Delta^*$ has a right quasi-inverse. To be precise, let $\pi : X \times X \to X$ be the projection map on the first coordinate and let $L\pi^* : D^b(X) \to D^b(X \times X)$ be its left derived functor which induces a functor on the level of orbit categories which will also denote by $L\pi^* : D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(X \times X)$. Notice that $L\pi^*$ is also a strong monoidal functor as being the derived pullback, and so an opmonoidal functor. That means that there exist natural isomorphisms

$$(L\pi^*)_2(\mathscr{E}^\bullet, \mathscr{F}^\bullet) : L\pi^*(\mathscr{E}^\bullet \otimes \mathscr{F}^\bullet) \xrightarrow{\sim} L\pi^* \mathscr{E}^\bullet \otimes L\pi^* \mathscr{F}^\bullet, \qquad (L\pi^*)_0 : L\pi^* \mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_{X \times X}$$

which satisfy relations of Definition 2.1.5 for any $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ objects of $D^{\mathrm{or}}(X)$. Then, for the functors $L\Delta^*$ and $L\pi^*$ there exists the natural isomorphism

$$\alpha : L\Delta^* \circ L\pi^* \xrightarrow{\sim} L(\pi\circ\Delta)^* = L\mathrm{id}^* = \mathrm{id}. \tag{7.1.7}$$

where the first isomorphism holds by [15, Proposition 5.4] and so $L\pi^*$ is a right quasi-inverse for $L\Delta^*$.

Following [26, Section 3.6], the natural isomorhism $\alpha : L\Delta^*\, L\pi^* \xrightarrow{\sim} \mathrm{id}$ is monoidal, see Definition 2.1.7 and since $L\Delta^*$ and $L\pi^*$ are strong monoidal functors the natural isomorphism $\alpha : L\Delta^*\, L\pi^* \xrightarrow{\sim} \mathrm{id}$ is also an opmonoidal natural isomorphism. Indeed, the strong monoidality of $L\Delta^*$ and $L\pi^*$ yields that they are also opmonoidal functors and their composite $L\Delta^*\, L\pi^*$ is an opmonoidal functor with opmonoidal constraints to be given by $L\Delta^*\, L\pi^*$ are given by

$$(L\Delta^*\, L\pi^*)_2 :\; = (L\Delta^*)_2 \circ L\Delta^*(L\pi^*)_2, \qquad (L\Delta^*\, L\pi^*)_0 :\; = (L\Delta^*)_0 \circ L\Delta^*(L\pi^*)_0 \tag{7.1.8}$$

such that the relations of Definition 2.1.5 are satisfied for any two object $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ of $D^{\mathrm{or}}(X)$. Now, the fact that α is an opmonoidal natural isomoprhism means that the diagrams below are commutative. For simplicity we drop L from $L\Delta^*$ and $L\pi^*$.

$$\begin{tikzcd}
\Delta^*\pi^*(\mathscr{E}^\bullet\otimes\mathscr{F}^\bullet) \arrow[r, "\simeq"] \arrow[d, "\Delta^*(\pi^*)_2"'] & (\pi\circ\Delta)^*(\mathscr{E}^\bullet\otimes\mathscr{F}^\bullet) \arrow[r, "="] \arrow[dd, "((\pi\circ\Delta)^*)_2"'] & \mathrm{id}^*(\mathscr{E}^\bullet\otimes\mathscr{F}^\bullet) \arrow[r, "="] \arrow[dd, "(\mathrm{id}^*)_2"'] & \mathrm{id}(\mathscr{E}^\bullet\otimes\mathscr{F}^\bullet) \arrow[dd, "\mathrm{id}_2"] \\
\Delta^*(\pi^*\mathscr{E}^\bullet\otimes\pi^*\mathscr{F}^\bullet) \arrow[d, "(\Delta^*)_2"'] & & & \\
\Delta^*\pi^*\mathscr{E}^\bullet\otimes\Delta^*\pi^*\mathscr{F}^\bullet \arrow[r, "\simeq"] & (\pi\circ\Delta)^*\mathscr{E}^\bullet\otimes(\pi\circ\Delta)^*\mathscr{F}^\bullet \arrow[r, "="] & \mathrm{id}^*\mathscr{E}^\bullet\otimes\mathrm{id}^*\mathscr{F}^\bullet \arrow[r, "="] & \mathrm{id}\mathscr{E}^\bullet\otimes\mathrm{id}\mathscr{F}^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^*\pi^*(\mathscr{O}_X) \arrow[rr, "\simeq"] \arrow[dr, "\Delta^*(\pi^*)_0"'] & & (\pi\circ\Delta)^*(\mathscr{O}_X) \arrow[r, "="] \arrow[dd, "((\pi\circ\Delta)^*)_0"'] & \mathrm{id}^*(\mathscr{O}_X) \arrow[r, "="] \arrow[ddl] & \mathrm{id}(\mathscr{O}_X) \arrow[ddll] \\
& \Delta^*\mathscr{O}_{X\times X} \arrow[dr, "(\Delta^*)_0"'] & & & \\
& & \mathscr{O}_X & &
\end{tikzcd}$$

The only thing we need to check are the the far left pentagon on the top diagram and the far left square (triangle shape diagram) on the diagram above. Their commutativity follows from the fact that the morphisms 7.1.8 are compatible with pseudofunctoriality, see [1, Section 2.3, diagram (2.26)] and [26, Section 3.6].

This observation will be fundamental in the following. It is this structure of the right quasi-inverse of $L\Delta^*$, the fact that both $L\Delta^*$ and $L\pi^*$ are strong monoidal and the fact that $\alpha : L\Delta^* L\pi^* \xrightarrow{\sim} \mathrm{id}$ is a monoidal natural isomorphism which enable us to construct for T an augmentation map and turn T to an augmented Hopf monad.

7.2 Augmented Hopf monads on orbit categories

In the previous section, we proved that the functor $T = L\Delta^* R\Delta_*$ defined by the functors $R\Delta_* : D^{\mathrm{or}}(X) \to D^{\mathrm{or}}(X \times X)$ and $L\Delta^* : D^{\mathrm{or}}(X \times X) \to D^{\mathrm{or}}(X)$ is a Hopf monad on $D^{\mathrm{or}}(X)$ where X is an even dimensional smooth complex projective variety with trivial canonical bundle. Key role in the proof of this was the fact that on the level of orbit categories we have the adjunction $R\Delta_* \dashv L\Delta^*$ and that $D^{\mathrm{or}}(X)$ is a rigid monoidal category. The goal of this section is to prove that $T = L\Delta^* R\Delta_*$ is an augmented Hopf monad.

Actually, we prove something more general. Motivated from Remark 7.1.16 and in particular from the fact that the right adjoint $L\Delta^*$ has a right quasi-inverse $L\pi^*$ which is also a strong monoidal functor and moreover $\alpha : L\Delta^* L\pi^* \to \mathrm{id}$ is a monoidal natural isomorphism, we work in the following categorical setup. We assume that $\mathscr{C}$ and $\mathscr{D}$ are rigid symmetric monoidal categories and that $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ is a pair of adjoint functors such that the right adjoint U is a strong symmetric monoidal functor and has a right quasi-inverse W which is also a strong monoidal and moreover the natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is monoidal. We prove that the Hopf monad $T = UF$ is an augmented one of which the augmentation map is left regular.

Notation 7.2.1. In the following, we will denote by $\mathscr{C}$ and $\mathscr{D}$ monoidal categories. Moreover, we will assume also that there is a pair $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ of adjoint functors with U being the right adjoint and let the adjunction unit be $\eta : \mathrm{id} \to UF$ and the adjunction counit be $\epsilon : FU \to \mathrm{id}$. We assume also that U is a strong symmetric monoidal functor and has as a right quasi-inverse the functor $W : \mathscr{C} \to \mathscr{D}$; symbolically we have $\alpha : UW \xrightarrow{\sim} \mathrm{id}$.

Definition 7.2.2. Let $F \dashv U$ be the adjunction as discussed in Notation 7.2.1. We define $T = UF$ to be the associated monad with product $\mu := U(\epsilon_F)$ and unit the adjunction unit $\eta : \mathrm{id} \to UF$.

With the use of the right quasi-inverse W of U we define an action of the adjunction monad T on the objects of the category $\mathscr{C}$.

Lemma 7.2.3. *Let $T = UF$ be the monad on $\mathscr{C}$ as defined above. Then, for any object X of $\mathscr{C}$, the morphism $\rho : T(X) \to X$ of $\mathscr{C}$ defined by*

$$T(X) := UF(X) = UFUW(X) \xrightarrow{U(\epsilon_{WX})} UW(X) = X. \tag{7.2.1}$$

is an action of T on the objects of $\mathscr{C}$.

Proof. We will check that the relations of Definition 6.1.8 are satisfied. The commutative diagram below express the left diagram of Definition 6.1.8 where all the morphisms have been unpacked for the case at hand. We denote by $\alpha^{-1} : \mathrm{id} \to UW$ the inverse of α.

$$\begin{tikzcd}
UFUF(X) \arrow[r, "U(\epsilon_F)"] \arrow[d, "UFUF(\alpha_X^{-1})"'] & UF(X) \arrow[r, "UF(\alpha_X^{-1})"] & UFUW(X) \arrow[r, "U(\epsilon_W)"] & UW(X) \arrow[dddl, "\mathrm{id}"'] \arrow[ddd, "\alpha_X"] \\
UFUFUW(X) \arrow[d, "UFU(\epsilon_W)"'] & & & \\
UFUW(X) \arrow[d, "UF(\alpha_X)"'] \arrow[dr, "\mathrm{id}"] & & & \\
UF(X) \arrow[r, "UF(\alpha_X^{-1})"'] & UFUW(X) \arrow[r, "U(\epsilon_W)"'] & UW(X) \arrow[r, "\alpha_X"'] & X
\end{tikzcd}$$

The commutativity of the hexagon in the middle follows from the interchange law. Now, the commutativity of the triangles on the bottom right corner and on the bottom left corner follows from the fact that U has W a right quasi-inverse i.e. $UW \cong \mathrm{id}$. From this follows that the boundary diagram commutes. Now, the commutative diagram below express the right diagram of Definition 6.1.8 where all the morphisms have been unpacked for the case at hand.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[drr] \arrow[ddrrr, bend right, "\mathrm{id}"'] & UF(X) \arrow[r] & UFUW(X) \arrow[r, "U(\epsilon_W)"] & UW(X) \arrow[dd, "\mathrm{id}"] \\
& & UW(X) \arrow[u, "\eta_{UW}"] \arrow[ur, "\mathrm{id}"'] & \\
& & & X
\end{tikzcd}$$

The above commutative diagrams prove that $\rho : T(X) \to X$ is indeed an action of the monad T on the objects of the category. $\square$

Since $T = UF$ is the monad obtained from the adjunction $F \dashv U$ and U is strong monoidal we have that T is a bimonad. Assuming that there exists a strong monoidal functor W which is also a right quasi-inverse to U and assuming moreover that the natural isomorphism $\alpha : UW \to \mathrm{id}$ is monoidal, we will define now an augmentation map for the bimonad T.

Proposition 7.2.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories and consider the pair of adjoint functors $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ such that the right adjoint U is a strong monoidal functor. Denote by UF the adjunction bimonad. Let $W : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor which is right quasi-inverse to U i.e. that there exists a natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$. Assume also that $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is a monoidal natural transformation. Then, the natural transformation $e : UF \to \mathrm{id}$ defined by*

$$UF \xrightarrow{UF(\alpha^{-1})} UFUW \xrightarrow{U(\epsilon_W)} UW \xrightarrow{\alpha} \mathrm{id} \tag{7.2.2}$$

is an augmentation for the bimonad $T = UF$ of $\mathscr{C}$.

Notation 7.2.5. In the following, we will denote by $\alpha^{-1} : \mathrm{id} \to UW$ the inverse of the natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$.

For the strong monoidal functor U we will denote by $U_2 : U(X \otimes Y) \xrightarrow{\sim} UX \otimes UY$ and $U_0 : U(1) \xrightarrow{\sim} 1$ its opmonoidal constraints and by $U_2^{-1} : UX \otimes UY \to U(X \otimes Y)$ and $U_0^{-1} : 1 \to U(1)$ their inverses respectively, for all X and Y of $\mathscr{D}$. Similarly, for the strong monoidal functor W we have $W_2 : W(X' \otimes Y') \xrightarrow{\sim} WX' \otimes WY'$ with inverse $W_2^{-1} : WX' \otimes WY' \to W(X' \otimes Y')$ and $W_0 : W(1) \xrightarrow{\sim} 1$ with inverse $W_0^{-1} : 1 \to W(1)$, for all X' and Y' objects of $\mathscr{C}$.

The fact that $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is a monoidal natural transformation means that the following diagrams are commutative.

$$\begin{tikzcd}
UW(X \otimes Y) \arrow[r, "\alpha"] & \mathrm{id}(X \otimes Y) \arrow[dd, "\mathrm{id}_2^{-1}"] \\
U(WX \otimes WY) \arrow[u, "U(W_2^{-1})"] & \\
UW(X) \otimes UW(Y) \arrow[u, "U_2^{-1}"] \arrow[r, "\alpha \otimes \alpha"'] & \mathrm{id}(X) \otimes \mathrm{id}(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
UW(1) \arrow[rr, "\alpha"] & & \mathrm{id}(1) \\
& U(1) \arrow[ul, "U(W_0^{-1})"] & \\
& & 1 \arrow[ul, "U_0^{-1}"] \arrow[uu, "\mathrm{id}_0"']
\end{tikzcd}$$

Since U and W are strong monoidal functors, their composition is also a strong monoidal functor and so we obtain also that $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is an opmonoidal natural transformation. That means that the diagrams below commute.

$$\begin{tikzcd}
UW(X \otimes Y) \arrow[r, "\alpha"] \arrow[d, "U(W_2)"'] & \mathrm{id}(X \otimes Y) \arrow[dd, "\mathrm{id}_2"] \\
U(WX \otimes WY) \arrow[d, "U_2"'] & \\
UW(X) \otimes UW(Y) \arrow[r, "\alpha \otimes \alpha"'] & \mathrm{id}(X) \otimes \mathrm{id}(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
UW(1) \arrow[rr, "\alpha"] \arrow[dr, "U(W_0)"'] & & \mathrm{Id}(1) \arrow[dd, "\mathrm{id}_0"] \\
& U(1) \arrow[dr, "U_0"'] & \\
& & 1
\end{tikzcd}$$

Proof of Proposition 7.2.4. We will prove that e is a morphism of bimonads, in other words a moprhism of monads which is also opmonoidal as a natural transformation.

The following two commutative diagrams are the relations 6.1.7 and show that e is a moprhism of monads.

$$\begin{tikzcd}
UFUF \arrow[r, "U(\epsilon_F)"] \arrow[d, "UFUF\alpha"'] & UF \arrow[r, "UF(\alpha^{-1})"] & UFUW \arrow[r, "U(\epsilon_W)"] & UW \arrow[dd, "\alpha"] \\
UFUFUW \arrow[rr, "U(\epsilon_{FUW})"] \arrow[d, "UFU(\epsilon_W)"'] & & U(FUW) \arrow[dd, "U(\epsilon_W)"] & \\
UFUW \arrow[d, "\alpha"'] \arrow[dr, "UF(\mathrm{id})"] & & & \mathrm{id} \\
UF \arrow[r, "UF(\alpha^{-1})"'] & UFUW \arrow[r, "U(\epsilon_W)"'] & UW \arrow[r, "\alpha"'] & \mathrm{id} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id} \arrow[rr, "\eta"] \arrow[dr, "\alpha^{-1}"] \arrow[dd, "\mathrm{id}"'] & & UF \arrow[d, "UF(\alpha^{-1})"] \\
& UW \arrow[r, "\eta_{UW}"] \arrow[dr, "\mathrm{id}"'] & UFUW \arrow[d, "U(\epsilon_W)"] \\
\mathrm{id} & & UW \arrow[ll, "\alpha"]
\end{tikzcd}$$

Next, we prove that the natural transformation $e : T \to \mathrm{id}$ is an opmonoidal natural transformation, in other words we check that the relations 2.2.3 hold. The diagram below is the right relation of 2.2.3 adapted and expanded for our case.

$$\begin{tikzcd}
UF(1) \arrow[r, "UF(U_0^{-1})"] \arrow[d, "UF\alpha^{-1}"'] & UFU(1) \arrow[dd, "U(\epsilon)"] \\
UFUW(1) \arrow[d, "U(\epsilon_W)"'] & \\
UW(1) \arrow[d, "\alpha"'] & U(1) \arrow[d, "U_0"] \\
1 \arrow[r, "\mathrm{id}"'] & 1
\end{tikzcd}$$

To prove its commutativity we work as follows. Using the fact that W is a strong monoidal functor, in particular that the composite $W(1) \xrightarrow{W_0} 1 \xrightarrow{W_0^{-1}} W(1)$ is the identity on $W(1)$, and then applied the naturality of the counit, we expand the left vertical column of the above diagram as it is shown below.

$$\begin{tikzcd}
UF(1) \arrow[r, "UF\alpha^{-1}"] & UFUW(1) \arrow[r, "U(\epsilon_W)"] \arrow[dr, "UFU(W_0)"'] & UW(1) \arrow[r, "U(W_0)"] & U(1) \arrow[r, "U(W_0^{-1})"] & UW(1) \arrow[r, "\alpha"] & 1 \\
 & & UFU(1) \arrow[ur, "U(\epsilon)"'] & & &
\end{tikzcd}$$

Next, using the fact that U is a strong monoidal functor and in particular that the composite $U(1) \xrightarrow{U_0} 1 \xrightarrow{U_0^{-1}} U(1)$ is the identity on $U(1)$, we obtain the diagram below.

$$\begin{tikzcd}
UF(1) \arrow[rrrrr, "UF(U_0^{-1})"] \arrow[d, "UF(\alpha^{-1})"'] \arrow[ddrr, "\mathrm{id}"] & & & & & UFU(1) \arrow[dd, "U(\epsilon)"] \\
UFUW(1) \arrow[d, "UFU(W_0)"'] \arrow[drr, "UF(\alpha)"'] & & & & & \\
UFU(1) \arrow[d, "UF(U_0)"'] & & UF(1) \arrow[dll, "UF(\mathrm{id}_0)"] & & & U(1) \arrow[ddd, "U_0"] \\
UF(1) \arrow[d, "UF(U_0^{-1})"'] & & & & & \\
UFU(1) \arrow[d, "U(\epsilon)"'] & & & & & \\
U(1) \arrow[r, "U_0"'] & 1 \arrow[r, "U_0^{-1}"'] \arrow[rrr, bend left, "\mathrm{id}_0"] & U(1) \arrow[r, "U(W_0^{-1})"'] & UW(1) \arrow[r, "\alpha"'] & 1 \arrow[r, "\mathrm{id}"'] & 1
\end{tikzcd}$$

The commutativity of the above diagram follows from the fact that the natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is monoidal and opmonoidal.

To complete the proof that $e : T \to \mathrm{id}$ is a morphism of bimonads, we need to prove the commutativity of the next diagram below which corresponds to the left relation of 2.2.3, adapted and expanded for our case.

$$
\begin{tikzcd}
UF \arrow[r, "UF\alpha^{-1}"] \arrow[dd, "UF(\eta\otimes\eta)"'] \arrow[dr, "UF(\alpha^{-1}\otimes\alpha^{-1})"] & UFUW \arrow[r, "U(\epsilon_W)"] & UW \arrow[r, "\alpha"] & \mathrm{id} \arrow[dddddd, "\mathrm{id}_2"] \\
 & UF(UW\otimes UW) \arrow[r, "\mathrm{id}"] \arrow[d, "UF(\eta_U\otimes\eta_U)"'] & UF(UW\otimes UW) \arrow[dd, "UF(U_2^{-1})"] & \\
UF(UF\otimes UF) \arrow[r] \arrow[d, "UF(U_2^{-1})"'] & UF(UFUW\otimes UFUW) \arrow[ur, "UF(U(\epsilon)\otimes U(\epsilon))"] \arrow[d, "UF(U_2^{-1})"'] & & \\
UFU(F\otimes F) \arrow[r] \arrow[d, "U(\epsilon)"'] & UFU(FUW\otimes FUW) \arrow[r, "UFU(\epsilon\otimes\epsilon)"] \arrow[dd, "U(\epsilon)"] & UFU(W\otimes W) \arrow[dd, "U(\epsilon_{W\otimes W})"] & \\
U(F\otimes F) \arrow[d, "U_2"'] \arrow[dr] & & & \\
UF\otimes UF \arrow[d] & U(FUW\otimes FUW) \arrow[r, "U(\epsilon\otimes\epsilon)"] \arrow[dl, "U_2"] & U(W\otimes W) \arrow[dl, "U_2"] & \\
UFUW\otimes UFUW \arrow[r, "U(\epsilon)\otimes U(\epsilon)"'] & UW\otimes UW \arrow[rr, "\alpha\otimes\alpha"'] & & \mathrm{id}\otimes\mathrm{id}
\end{tikzcd}
$$

The commutativity of the boundary diagram follows from the triangle equations of the adjunction $F \dashv U$, the natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ and the naturality of the adjunction unit η and the adjunction counit ϵ. The tricky part is the commutativity of the far right decagonon for which we proceed as follows. First we expand the decagonon it as it is shown below. We do that using the fact that U and W are strong monoidal functors, in particular that the composite $U(1) \xrightarrow{U_0} 1 \xrightarrow{U_0^{-1}} U(1)$ is the identity on $U(1)$ and $W(1) \xrightarrow{W_0} 1 \xrightarrow{W_0^{-1}} W(1)$ is the identity on $W(1)$ as well as that $\alpha : UW \xrightarrow{\sim} \mathrm{id}$ is a natural isomorphism with inverse $\alpha^{-1} : \mathrm{id} \to UW$.

$$
\begin{tikzcd}
UF \arrow[r, "UF\alpha^{-1}"] \arrow[d, "UF(\alpha^{-1}\otimes\alpha^{-1})"'] \arrow[dr] & UFUW \arrow[r, "U(\epsilon_W)"] & UW \arrow[r, "\alpha"] & \mathrm{id}\otimes\mathrm{id} \\
UF(UW\otimes UW) \arrow[r, "UF(\alpha\otimes\alpha)"'] \arrow[d, "UF(U_2^{-1})"'] & UF(\mathrm{id}\otimes\mathrm{id}) \arrow[dddl, "UF(\mathrm{id}_2^{-1})"] & & UW\otimes UW \arrow[u, "\alpha\otimes\alpha"'] \\
UFU(W\otimes W) \arrow[d, "UFU(W_2^{-1})"'] & & \mathrm{id} \arrow[uur, "\mathrm{id}_2"] & U(W\otimes W) \arrow[u, "U_2"'] \\
UFUW \arrow[d, "UF\alpha"'] & & & UW \arrow[u, "U(W_2)"'] \arrow[ul, "\alpha"] \\
UF \arrow[d, "UF\alpha^{-1}"'] & & & \mathrm{id} \arrow[u, "\alpha^{-1}"'] \arrow[uul, "\mathrm{id}"] \\
UFUW \arrow[r, "U(\epsilon_W)"] \arrow[d, "UFU(W_2)"'] & UW \arrow[rr, "\mathrm{id}"] \arrow[drr, "U(W_2)"'] & & UW \arrow[u, "\alpha"'] \\
UFU(W\otimes W) \arrow[rrr, "U(\epsilon_{W\otimes W})"'] & & & U(W\otimes W) \arrow[u, "U(W_2^{-1})"']
\end{tikzcd}
$$

The final essential bit for the commutativity of the internal diagram is the fact that α is a monoidal and an opmonoidal natural isomorphism. □

Lemma 7.2.6. *Let $\mathscr{C}$ be a symmetric monoidal category and consider a pair of adjoint functors $(F:\mathscr{C}\to\mathscr{D}, U:\mathscr{D}\to\mathscr{C})$ such that the right adjoint U is a strong monoidal functor. Let W be a strong monoidal functor which is right quasi-inverse to U i.e. there exists a natural isomorphism $\alpha: UW \xrightarrow{\sim} \mathrm{id}$. Assume also that α is a monoidal natural transformation. Then, the associated bimonad $T = UF$ of $\mathscr{C}$ with augmentation map $e: T\to\mathrm{id}$ as defined above is compatible with the symmetry of $\mathscr{C}$ as follows*

$$(e_X\otimes\mathrm{id}_{T(1)})\circ T_2(X,1) = (e_X\otimes\mathrm{id}_{T(1)})\circ\tau_{T(1),T(X)}\circ T_2(1,X) \tag{7.2.3}$$

for all objects X of $\mathscr{C}$.

Proof. First observe that we have the following commutative diagram.

$$\begin{tikzcd}
UF(X\otimes 1) \arrow[r, "UF(\tau)"] \arrow[d, "UF(\eta\otimes\eta)"'] & UF(1\otimes X) \arrow[d, "UF(\eta\otimes\eta)"] \\
UF\big(UF(X)\otimes UF(1)\big) \arrow[r, "UF(\tau)"] \arrow[d] & UF\big(UF(1)\otimes UF(X)\big) \arrow[d] \\
UFU\big(F(X)\otimes F(1)\big) \arrow[r, "UFU(\tau)"] \arrow[d, "U(\epsilon)"'] & UFU\big(F(1)\otimes F(X)\big) \arrow[d, "U(\epsilon)"] \\
U\big(F(X)\otimes F(1)\big) \arrow[r, "U(\tau)"] \arrow[d] & U\big(F(1)\otimes F(X)\big) \arrow[d] \\
UF(X)\otimes UF(1) \arrow[r, "\tau"] & UF(1)\otimes UF(X)
\end{tikzcd}$$

We read the diagram from the top to the bottom. The commutativity of the top square follows from the naturality of τ. The commutativity of the third from the top square follows from the naturality of the counit ϵ. The commutativity of the second and of the fourth square follows from the fact that U is a symmetric monoidal functor. The proof of the commutativity of the above hexagon is now complete.

Next, we unpack the morphisms of equation 7.2.3 and obtain the diagram below.

$$\begin{tikzcd}
T(X) \arrow[r, "T(l^{-1})"] \arrow[d, "T(r^{-1})"'] & T(1\otimes X) \arrow[r, "T_2(1,X)"] \arrow[dl, "T(\tau_{1,X})"] & T(1)\otimes T(X) \arrow[d, "\tau_{T(1),T(X)}"] \\
T(X\otimes 1) \arrow[d, "T_2(X,1)"'] & & T(X)\otimes T(1) \arrow[d, "e_X\otimes \mathrm{id}_{T(1)}"] \\
T(X)\otimes T(1) \arrow[rr, "e_X\otimes \mathrm{id}_{T(1)}"'] \arrow[urr, "\mathrm{id}\otimes\mathrm{id}"] & & X\otimes T(1)
\end{tikzcd}$$

The commutativity of the triangle on the top left corner follows from the symmetry of the tensor product of $D^{\mathrm{or}}(X)$. The commutativity of the hexagon in the middle was discussed above. The proof is now complete. □

Remark 7.2.7. As part of the proof we have that $T_2(X,1) = \tau_{T(1),T(X)} \circ T_2(1,X)$. This will be important for the following.

Remark 7.2.8. There is a good reason for assuming that $\mathscr{C}$ is symmetric monoidal category and U is symmetric monoidal functor. The example which motivates all the above for the above is the orbit category $D^{\mathrm{or}}(X)$ of an even dimensional smooth complex projective variety with trivial canonical bundle which is a symmetric monoidal category. Moreover, the main example for U which we have in our mind is the functor $L\Delta^* : D^{\mathrm{or}}(X \times X) \to D^{\mathrm{or}}(X)$.

Recall from Definition 6.5.1 that the augmentation map $e : T \to \mathrm{id}$ of an opmonoidal endofunctor T is said to be left regular, if the natural transformation $u^e : T \to T(1) \otimes -$ which is given by $u^e = (\mathrm{id}_{T(1)} \otimes e_X) \circ T_2(1, X)$ is invertible. Now, if the categories $\mathscr{C}$ and $\mathscr{D}$ are rigid categories, then by Theorem 6.3.6 we have that the bimonad $T = UF$ which is obtained from the adjunction $F \dashv U$ as discussed in Notation 7.2.1 is a Hopf monad on $\mathscr{C}$. Then, applying Lemma 6.5.5 we obtain that the augmentation map is a left regular. We have the following.

Proposition 7.2.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be rigid and symmetric monoidal categories and consider a pair of adjoint functors $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ such that the right adjoint U is a strong symmetric monoidal functor. Let $W : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor which is a right quasi-inverse to U i.e. there exists a natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$. Assume also that α is a monoidal natural transformation. Then, the adjunction monad $T = UF$ is an augmented Hopf monad on $\mathscr{C}$ with left regular augmentation map where the latter defined in Proposition 7.2.4.*

Now, looking in Lemma 6.5.5 we observe the following. The inverse of the natural transformation u^e is defined with the use of the inverse of the left fusion operator H^l which is assigned to any bimonad T on a monoidal category $\mathscr{C}$, see Section 6.3. In the following, we explain why assuming $\mathscr{C}$ to be a rigid monoidal category $\mathscr{C}$ the left fusion operator H^l is invertible.

First, recall from Section 6.4 that to any opmonoidal adjunction $F \dashv U$, see Definition 2.2.5, there are associated natural transformations called Hopf operators, see Definition 6.4.1. Opmonoidal adjunctions with invertible Hopf operators are called Hopf adjunctions. Now, the Hopf operators associated to an opmonoidal adjunction and the fusion operators associated to the bimonad obtained from the opmonoidal adjunction are related as in Proposition 6.4.2. So, to prove that the left fusion operator H^l associated to a bimonad is invertible is enough to prove that the left Hopf operator associated to the adjunction from which the bimonad is obtained is invertible.

By Proposition 6.4.5 we have that an opmonoidal adjunction $F \dashv U$ between left closed monoidal categories is a Hopf adjunction if and only if the right adjoint U is a left closed functor, for the latter see Definition 2.5.8. Conisdering that $\mathscr{C}$ is symmetric monoidal category with left duality then it has also right duality, see Lemma 2.4.10. Now, a category with left duality is a left closed category, see Example 2.5.4 and so if $\mathscr{C}$ is symmetric monoidal category with left duality then it is a closed category. Now, a strong monoidal functor on a closed symmetric monoidal category is a left closed, see Theorem 2.5.17 and so the opmonoidal adjunction $F \dashv U$ is a

Hopf adjunction. As a result the left Hopf operator is invertible and the left fusion operator is invertible too.

In the following, we discuss further the natural transformation $u^e : T \to T(1)\otimes -$ defined by $u^e = (\mathrm{id}_{T(1)} \otimes e_X) \circ T_2(1, X)$. In particular, we construct another natural transformation $\phi : UF \to UF(1) \otimes (-)$ with the used of 2.1.1 and then we prove that ϕ has the form $u^e = (\mathrm{id}_{T(1)} \otimes e_X) \circ T_2(1, X)$. The motivation for doing this is that we want to understand in more depth the adjunction $R\Delta_* \dashv L\Delta^*$ on the level of orbit categories which is obtained by Proposition 7.1.14 with $X = Y$ and f to be the diagonal map and compare it with natural isomorphisms on the level of bounded derived categories of coherent sheaves.

Definition 7.2.10. Let $U : \mathscr{D} \to \mathscr{C}$ be a strong monoidal functor with left adjoint $F : \mathscr{C} \to \mathscr{D}$ and let $W : \mathscr{C} \to \mathscr{D}$ be a strong monoidal functor such that W is a right quasi-inverse of U i.e. there exists a natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$. Then, there exists a natural transformation $\phi : UF \to UF(1) \otimes (-)$ defined to be the composite

$$\begin{aligned} UF(A) &\to UF(1 \otimes A) \cong UF(1 \otimes UWA) \to UF\big(UF(1) \otimes UWA\big) \\ &\to UFU\big(F(1(\otimes WA)\big) \xrightarrow{U(\epsilon)} U\big(F(1) \otimes WA\big) \\ &\to UF(1) \otimes UWA \cong UF(1) \otimes A \end{aligned}$$

for all A of $\mathscr{C}$.

Remark 7.2.11. Alternatively, ϕ is defined by applying the strong monoidal functor U on the natural transformation $\psi : F \to F(1) \otimes W(-)$ as defined in 2.2.8. We want to think the natural transformation ψ as the analogue of the natural isomorphism

$$\begin{aligned} L\pi_1^*(-) \otimes^L \mathscr{O}_\Delta &\cong L\pi_1^*(-) \otimes^L R\Delta_* \mathscr{O}_X \\ &\cong R\Delta_*(L\Delta^* L\pi_1^*(-) \otimes^L \mathscr{O}_X) && \text{(Proposition 4.3.8)} \\ &\cong R\Delta_*(\mathrm{id}(-) \otimes^L \mathscr{O}_X) && (L\Delta^* \circ L\pi^* = \mathrm{id}) \\ &\cong R\Delta_*\big((-) \otimes^L \mathscr{O}_X\big) \\ &\cong R\Delta_*(-) \end{aligned}$$

Observe that both the above natural isomorphism and ψ are obtained using the general categorical formula 2.1.1. The difference is that they are applied from different adjunctions. Our case at hand which motivates this general categorical approach is the adjunction $R\Delta_* \dashv L\Delta^*$ where the above natural isomorphism

$$L\pi_1^*(-) \otimes^L \mathscr{O}_\Delta \cong R\Delta_*(-)$$

is obtained from the adjunction $L\Delta^* \dashv R\Delta_*$.

Proposition 7.2.12. *Let $T = UF$ be the bimonad obtained from $F \dashv U$ with the properties as stated in Notation 7.2.1. Then, the natural isomorphism $\phi \colon UF \to UF(1) \otimes -$ is of the form $u^e_X = (\mathrm{id}_{T(1)} \otimes e) \circ T_2(1, X)$ where T_2 is the opmonoidal constraint of T and e the augmentation map of T.*

Proof. Unpacking the morphisms of the relation which we want to prove we obtain the diagram below. Its commutativity follows from the triangle relations of the adjunction $F \dashv U$, the strong monoidality of U and the interchange law.

$$\begin{tikzcd}
UF \arrow[r] \arrow[d] & UF\big(UF \otimes UF\big) \arrow[dd] \arrow[ddr] & \\
UF\big(UF \otimes UW\big) \arrow[d] \arrow[dr] & & \\
UF(UF \otimes UW) \arrow[d] & UF(UF \otimes UFUW(X)) \arrow[l] \arrow[d] & UFU(F \otimes F) \arrow[dl] \arrow[dd] \\
UFU(F \otimes W) \arrow[d] & UFU(F \otimes FUW) \arrow[l] \arrow[d] & \\
U(F \otimes W) \arrow[d] & U(F \otimes FUW) \arrow[l] \arrow[d] & U(F \otimes F(X)) \arrow[l] \arrow[d] \\
UF \otimes UW \arrow[d] & UF \otimes UFUW \arrow[l] & UF \otimes UF \arrow[l] \\
UF \otimes \mathrm{id} & &
\end{tikzcd}$$

□

The category of augmented Hopf monads on monoidal category for which the augmentation map is left regular is by Theorem 6.5.6 equivalent to the category of Hopf algebras in the Drinfel'd centre of the category $\mathscr{C}$. Now from this theorem, by Proposition 6.5.7 and Lemma 7.2.6 we obtain the following.

Proposition 7.2.13. *Let $\mathscr{C}$ and $\mathscr{D}$ be rigid symmetric monoidal categories and let $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors such that the right adjoint U is a strong symmetric monoidal functor. Assume also that there is a strong monoidal functor $W : \mathscr{C} \to \mathscr{D}$ which is right quasi-inverse to U i.e. there exists a natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$. Assume also that α is a monoidal natural transformation. Then, the following are true.*

1. *The object $T(1) := UF(1)$ is a Hopf algebra in $\mathscr{C}$ with product $m = \mu_1 \circ \phi^{-1}_{T(1)}$, coproduct $\delta = T_2(1,1)$, unit $u = \eta_1$, counit $\epsilon = T_0$ and antipode*

$$S = e_{T(1)} \circ (H^l_{1,1})^{-1} \circ (\mathrm{id}_{T(1)} \otimes \eta_1).$$

2. *The Hopf monad $T = UF$ is isomorphic to the Hopf monad $T' = UF(1) \otimes -$ on $\mathscr{C}$.*

The opmonoidal constraint T'_2 of T' is given by $T'_2(X,Y) = (\mathrm{id}_{T(1)} \otimes \tau_{T(1),X} \otimes Y) \circ (T_2(1,1) \otimes \mathrm{id})$. In the following, we make the following calculation with the use of the natural isomorphism ϕ as defined in Definition 7.2.10.

Lemma 7.2.14. *The morphism of bimonads $\phi : UF \to UF(1) \otimes (-)$ preserves the opmonoidal constraints, in other words the following diagram commutes.*

$$\begin{tikzcd}
UF(X \otimes Y) \arrow[rr, "(UF)_2"] \arrow[d] & & UF(X) \otimes UF(Y) \arrow[dd, "\phi_X \otimes \phi_Y"] \\
UF(1 \otimes X \otimes Y) \arrow[d, "\phi_{X \otimes Y}"'] & & \\
UF(1) \otimes X \otimes Y \arrow[dr, "(UF)_2 \otimes \mathrm{id}"'] & & UF(1) \otimes X \otimes UF(1) \otimes Y \\
& UF(1) \otimes UF(1) \otimes X \otimes Y \arrow[ur, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] &
\end{tikzcd}$$

Proof. We prove the commutativity of the boundary diagram due to commutativity of all the internal diagrams below.

$$\begin{tikzcd}
UF(X \otimes Y) \arrow[rr, "(UF)_2"] \arrow[d] \arrow[dr] & & UF(X) \otimes UF(Y) \arrow[dd, "\phi_X \otimes \phi_Y"] \\
UF(1 \otimes X \otimes Y) \arrow[d, "\phi_{X \otimes Y}"'] \arrow[dr] & UF(1 \otimes X \otimes 1 \otimes Y) \arrow[dr] & \\
UF(1) \otimes X \otimes Y \arrow[dr, "(UF)_2 \otimes \mathrm{id}"'] & UF(1 \otimes 1 \otimes X \otimes Y) \arrow[u] \arrow[d] & UF(1) \otimes X \otimes UF(1) \otimes Y \\
& UF(1) \otimes UF(1) \otimes X \otimes Y \arrow[ur, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] &
\end{tikzcd}$$

The commutativity of the square on the top left corner we have the following commutative diagram

$$
\begin{tikzcd}
1 \otimes 1 \otimes X \otimes Y \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[ddd, "l_1 \otimes \mathrm{id} \otimes \mathrm{id}"'] & 1 \otimes (X \otimes 1) \otimes Y \arrow[d, "\mathrm{id} \otimes \alpha_{X,1,Y}"] \arrow[dddl, "\mathrm{id} \otimes r_X \otimes \mathrm{id}"'] \\
 & 1 \otimes X \otimes (1 \otimes Y) \arrow[d, "l_X \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[ddl, "\mathrm{id} \otimes \mathrm{id} \otimes l_Y"] \\
 & X \otimes (1 \otimes Y) \arrow[d, "\mathrm{id} \otimes l_Y"] \\
1 \otimes X \otimes Y \arrow[r, "l_X \otimes \mathrm{id}"'] & X \otimes Y
\end{tikzcd}
$$

where $\alpha_{X,1,Y}$ is the associator. Indeed, by the naturality of the natural isomorphism $l : 1 \otimes X \xrightarrow{\sim} X$ follows that

$$\mathrm{id}_1 \otimes l_X = l_{1 \otimes X} = l_1 \otimes \mathrm{id}_X .$$

From this, the far left vertical arrow is becoming $\mathrm{id}_1 \otimes l_X \otimes \mathrm{id}_Y$ and in combination with the triangle relation of the symmetry τ, see 2.3.3, follows that the top left triangle is commutative. The commutativity of the triangle in the middle follows from the triangle relation of the associator α, see 2.1.1 and the commutativity of the bottom triangle follows from the interchange law. Then, applying the functor UF on the above diagram we obtain the commutativity of the pentagon.

The commutativity of the remaining diagrams follow from technical lemmas who have stated in Section 2.2. In particular for the commutativity of the square on the top right corner we apply in the diagram of Lemma 2.2.9 the strong monoidal functor U and taking in to account that $UW \cong \mathrm{id}$. In a similar way the commutativity of the triangle on the bottom and on the left of the diagram follows from the application of the strong monoidal functor on the commutative diagram of Lemma 2.2.10 and the strong monoidal functor U. To complete the proof, the commutativity of the bottom right square follows from the application of the strong monoidal functor U on the commutative diagram of Lemma 2.2.13. □

7.3 Hopf algebra object in the orbit category

In this section, we put together all the pieces and discuss the Hopf algebra object in the orbit category $D^{\mathrm{or}}(X)$ of an even dimensional smooth complex projective variety with trivial canonical bundle.

Recall from Proposition 7.1.15 that the functor $T = L\Delta^* R\Delta_*$ is a Hopf monad on $D^{\mathrm{or}}(X)$, of which product we denote by $\mu : L\Delta^* R\Delta_* L\Delta^* R\Delta_* \to L\Delta^* R\Delta_*$. Fundamental role for proving this was the fact that $D^{\mathrm{or}}(X)$ is a rigid symmetric monoidal category and T is the adjunction monad of the adjunction $R\Delta_* \dashv L\Delta^*$ where $L\Delta^*$ is a strong monoidal functor. Denote by $\epsilon : R\Delta_* L\Delta^* \to \mathrm{id}$ the adjunction counit.

Motivated from the structure of the functors $R\Delta_*$ and $L\Delta^*$ on the level of orbit categories and in particular the fact that $L\Delta^*$ has a right quasi-inverse, see Remark 7.1.16, we considered in Section 7.2 a more general categorical level.

In particular, given a pair of adjoint functors $(F : \mathscr{C} \to \mathscr{D}, U : \mathscr{D} \to \mathscr{C})$ between monoidal categories such that the left adjoint U is strong monoidal and assuming U has a right quasi-inverse functor $W : \mathscr{C} \to \mathscr{D}$ i.e. there exists a natural isomorphism $\alpha : UW \xrightarrow{\sim} \mathrm{id}$, such that W is a strong monoidal and that α is a monoidal natural transformation, then it is proved in Proposition 7.2.4, that the associated adjunction bimonad $T = UF$ is augmented.

In Lemma 7.2.6 we considered the same setup with Proposition 7.2.4 and assuming that the categories are morever symmetric, we proved that the above augmentation map is compatible with the symmetry of the category $\mathscr{C}$.

Last but not least, a natural isomorphism $\phi : UF \to UF(1) \otimes -$ was constructed in Definition 7.2.10 where 1 is the unit object of $\mathscr{C}$ which we want to think about as the analogue of the natural isomorphism obtained from the projection formula for the adjunction $R\Delta_* \dashv L\Delta^*$ on the level of orbit categories.

The main result of this thesis is the following.

Theorem 7.3.1. *Let X be an even dimensional smooth complex projective variety with trivial canonical bundle and $D^{\mathrm{or}}(X)$ the associated orbit category. Then the following are true.*

1. *The functor $T = L\Delta^* R\Delta_*$ is an augmented Hopf monad on $D^{\mathrm{or}}(X)$ with augmentation map $e : L\Delta^* R\Delta_* \to \mathrm{id}$ to be given by the composite*

$$L\Delta^* R\Delta_* \xrightarrow{\sim} L\Delta^* R\Delta_* L\Delta^* L\pi^* \xrightarrow{L\Delta^*(\epsilon_{L\pi^*})} L\Delta^* L\pi^* \xrightarrow{\sim} \mathrm{id}.$$

   Moreover, the above augmentation map is compatible with the symmetry of the category $D^{\mathrm{or}}(X)$.

2. *The object $\mathscr{U} := L\Delta^* R\Delta_* \mathscr{O}_X$ is a Hopf algebra in $D^{\mathrm{or}}(X)$. The product map is given by $m = \mu_{\mathscr{O}_X} \circ \phi^{-1}_{T(\mathscr{O}_X)}$, the unit map by $u = \eta_{\mathscr{O}_X}$, the coproduct $\delta = T_2(\mathscr{O}_X, \mathscr{O}_X)$, the counit by $\epsilon = T_0$, the antipode*

$$S = e_{T_{\mathscr{O}_X}} \circ (H^l_{\mathscr{O}_X,\mathscr{O}_X})^{-1} \circ (\mathrm{id}_{T(\mathscr{O}_X)} \otimes \eta_{\mathscr{O}_X})$$

   and the inverse of the antipode is given by

$$S^{-1} = e_{T(\mathscr{O}_X)} \circ (H^r_{\mathscr{O}_X,\mathscr{O}_X})^{-1} \circ (\eta_{\mathscr{O}_X} \otimes \mathrm{id}_{T(\mathscr{O}_X)}).$$

   Moreover, the Hopf algebra $\mathscr{U}$ is cocommutative (that is, $\delta = \tau_{U,U} \circ \delta$) and involutory (that is, $S^2 = \mathrm{id}_U$).

3. *There is a natural isomorphism of Hopf monads between $T = L\Delta^* R\Delta_*$ and $T' = L\Delta^* R\Delta_* \mathscr{O}_X \otimes -$ on $D^{\mathrm{or}}(X)$.*

To sum up, we pinned down the Hopf algebra object in the orbit category of an even dimensional smooth complex projective variety with trivial canonical bundle, based on the rigid and symmetric monoidal structure of the orbit category, analysing the properties of the functors $R\Delta_*$ and $L\Delta^*$ and proving firstly that on the orbit category we have a generalised version of a Hopf algebra.

Chapter 8

Appendix

8.1 Triangulated categories

In this section, we give the definition of a Triangulated category. See [15] for more details.

Definition 8.1.1. An additive category $\mathscr{C}$ is said to be *triangulated* if it is given an additive equivalence

$$\Sigma : \mathscr{C} \to \mathscr{C},$$

called the *suspension functor* or the *shift functor* or the *translation functor*, and a set of *distinguished triangles* of the form

$$X \to Y \to Z \to \Sigma(X) \tag{8.1.1}$$

such that the following axioms (TR0)–(TR5) are satisfied for all objects X, Y, Z of $\mathscr{C}$.

(TR0) Any triangle of the form

$$X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma(X) \tag{8.1.2}$$

is distinguished.

(TR1) Any triangle isomorphic to a distinguished triangle is distinguished.

(TR2) Any morphism $f : X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma(X). \tag{8.1.3}$$

(TR3) The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \tag{8.1.4}$$

is a distinguished triangle if and only if the triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y) \tag{8.1.5}$$

is a distinguished triangle.

(TR4) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \tag{8.1.6}$$

and

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X') \tag{8.1.7}$$

and morphisms $f : X \to X'$, $g : Y \to Y'$ which commute with u and u' then there exists a morphism $h : Z \to Z'$, not necessarily unique, such that (f, g, h) is a morphism of the first triangle into the second.

(TR5) *(Octahedral axiom)*

Given distinguished triangles $X \xrightarrow{u} Y \to Z' \to \Sigma X, Y \xrightarrow{v} Z \to X' \to \Sigma Y$ and $X \xrightarrow{vu} Z \to Y' \to \Sigma X$, there exists a distinguished triangle $Z' \to Y' \to X' \to \Sigma Z'$ making the following diagram commutative.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\mathrm{id}_{\Sigma X}"] \\
X \arrow[r, "vu"] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"] & \Sigma Y \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z
\end{tikzcd}$$

Remark 8.1.2. There are many variations of the axiom (TR5) in the bibliography. Here, we gave that one which appears in Kashiwara and Schapira [18].

The following lemma will be used in the following. Although seems to be known, we were unable to find a proof in the literature and so we give one below.

Lemma 8.1.3. *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is a distinguished triangle in $\mathscr{C}$ then*

$$\Sigma X \xrightarrow{\Sigma u} \Sigma Y \xrightarrow{\Sigma v} \Sigma Z \xrightarrow{-\Sigma w} (\Sigma X)[1] \tag{8.1.8}$$

is a distinguished triangle too.

Proof. By the third axiom of triangulated categories (TR3) we obtain that

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$$

is a distinguished triangle. Applying twice the axiom (TR3) yields the following distinguished triangles.

$$Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma u} \Sigma(Y) \xrightarrow{-\Sigma(v)} \Sigma(Z)$$
$$\Sigma X \xrightarrow{-\Sigma u} \Sigma Y \xrightarrow{-\Sigma v} \Sigma Z \xrightarrow{-\Sigma w} \Sigma^2 A \tag{8.1.9}$$

By the commutativity of the diagram below

$$\begin{tikzcd}
\Sigma X \arrow[r, "-\Sigma u"] \arrow[d, "\mathrm{id}"'] & \Sigma Y \arrow[r, "-\Sigma v"] \arrow[d, "-\mathrm{id}"] & \Sigma Z \arrow[r, "-\Sigma w"] \arrow[d, "\mathrm{id}"] & \Sigma^2 A \arrow[d, "\mathrm{id}"] \\
\Sigma X \arrow[r, "\Sigma u"'] & \Sigma Y \arrow[r, "\Sigma v"'] & \Sigma Z \arrow[r, "-\Sigma w"'] & \Sigma^2 A
\end{tikzcd}$$

we obtain that the distinguished triangle 8.1.9 is isomorphic to the

$$\Sigma X \xrightarrow{\Sigma u} \Sigma Y \xrightarrow{\Sigma v} \Sigma Z \xrightarrow{-\Sigma w} \Sigma^2 A.$$

Applyin axiom (TR2) we obtain the required distinguished triangle. This completes the proof. $\square$

8.2 Some technical proofs

In this section, we provide detailed proofs for the statements stated at the end of Section 2.2. Each subsection is entitled with the proof of the proposition or lemma.

8.2.1 Proof of proposition 2.2.9

We will verify the commutativity of the following diagram, which is the expansion of the arrows the

$$\begin{tikzcd}
F(A\otimes B) \arrow[r, "F_2"] \arrow[d] & F(A)\otimes F(B) \arrow[d] \\
F\big((1\otimes A)\otimes(1\otimes B)\big) \arrow[r, "F_2"] \arrow[d, "F(\eta\otimes \mathrm{id}\otimes\eta\otimes\mathrm{id})"'] & F(1\otimes A)\otimes F(1\otimes B) \arrow[ddd, "F(\eta\otimes\mathrm{id})\otimes F(\eta\otimes\mathrm{id})"] \\
F\Big(\big(UF(1)\otimes UWA\big)\otimes\big(UF(1)\otimes UWB\big)\Big) \arrow[d, "F(\eta\otimes\eta)"'] & \\
F\Big(UF\big(UF(1)\otimes UWA\big)\otimes UF\big(UF(1)\otimes UWB\big)\Big) \arrow[d, "F(U^2)"'] & \\
FU\Big(F\big(UF(1)\otimes UWA\big)\otimes F\big(UF(1)\otimes UWB\big)\Big) \arrow[r, "\epsilon"] \arrow[d] & F\big(UF(1)\otimes UWA\big)\otimes F\big(UF(1)\otimes UWB\big) \arrow[d] \\
FU\Big(FU\big(F(1)\otimes WA\big)\otimes FU\big(F(1)\otimes WB\big)\Big) \arrow[r, "\epsilon"] \arrow[d] & FU\big(F(1)\otimes WA\big)\otimes FU\big(F(1)\otimes WB\big) \arrow[d] \\
FU\Big(F(1)\otimes WA\otimes F(1)\otimes WB\Big) \arrow[r, "\epsilon"] & F(1)\otimes WA\otimes F(1)\otimes WB
\end{tikzcd}$$

The commutativity of the top square follows from the naturality of F_2. Now for the commutativity of the hexagon in the middle we underline that the left hand arrow is $F(\eta\otimes\mathrm{id}\otimes\eta\otimes\mathrm{id})$ followed by $F_2(UF(1)\otimes WA, UF(1)\otimes WB)$, see Definition 2.2.1 and so the commutativity of hexagon follows from the naturality of F_2. The commutativity of the bottom two squares follows again from the naturality of ϵ.

8.2.2 Proof of Proposition 2.2.10

The commutativity of the above diagram comes down to the verification of the commutativity of the diagram in the next page. first observe that we have the following commutative diagram.

$$
\begin{tikzcd}
UF(1\otimes 1)\otimes U(WA\otimes WB) \arrow[dr, "(UF)_2(1,1)\otimes \mathrm{id}"] \arrow[ddr, "U(F_2)\otimes \mathrm{id}"'] \arrow[ddd, "U^2"'] & \\
 & UF(1)\otimes UF(1)\otimes U(WA\otimes UWB) \arrow[ddd, bend left, "U^2\otimes \mathrm{id}"] \\
 & U(F(1)\otimes F(1))\otimes U(WA\otimes WB) \arrow[u, "U_2\otimes \mathrm{id}"'] \arrow[dd, "\mathrm{id}"] \arrow[dddl, "U^2"'] \\
U(F(1\otimes 1)\otimes WA\otimes UWB) \arrow[dd, "U(F_2\otimes \mathrm{id}\otimes \mathrm{id})"'] & \\
 & U(F(1)\otimes F(1))\otimes U(WA\otimes UWB) \arrow[dl, "U^2"] \\
U(F(1)\otimes F(1)\otimes WA\otimes WB) &
\end{tikzcd}
$$

Applying the functor F on the above diagram we obtain the commutative diagram C in the next page. The square on the top left corner commutes from the naturality of $\eta\otimes\mathrm{id}\otimes\mathrm{id}$. The second square from the top and on the left commutes due from the naturality of $\mathrm{id}\otimes U^2$. The third square from the top and on the left commutes by the naturality of U^2. Now, the top right triangle is commutative due to the bimonad relation (6.2). The second square from the top and on the right commutes due to interchange law of morphisms.The bottom square is the definition of the $F_2(1,1)$, see 2.2.1. The commutativity of the hexagon follows from the naturality of ϵ.

$$
\begin{tikzcd}
F(1\otimes A\otimes B) \arrow[r, "F(l_1^{-1}\otimes \mathrm{id}_A\otimes \mathrm{id}_B)"] \arrow[d, "F(\eta\otimes \mathrm{id}\otimes \mathrm{id})"'] & F(1\otimes 1\otimes A\otimes B) \arrow[d, "F(\eta\otimes \mathrm{id}\otimes \mathrm{id})"] \arrow[dr, bend left, "F(\eta\otimes\eta\otimes \mathrm{id}\otimes \mathrm{id})"] & \\
F(UF(1)\otimes UWA\otimes UWB) \arrow[r, "F(UF(l_1^{-1})\otimes \mathrm{id}\otimes \mathrm{id})"] \arrow[d, "F(\mathrm{id}\otimes U^2)"'] & F(UF(1\otimes 1)\otimes UWA\otimes UWB) \arrow[r, "F((UF)_2(1,1)\otimes \mathrm{id}\otimes \mathrm{id})"] \arrow[d, "F(\mathrm{id}\otimes U^2)"] & F(UF(1)\otimes UF(1)\otimes UWA\otimes UWB) \arrow[d, "F(\mathrm{id}\otimes \mathrm{id}\otimes U^2)"] \\
F(UF(1)\otimes U(WA\otimes WB)) \arrow[r, "F(UF(l_1^{-1})\otimes \mathrm{id})"] \arrow[d, "F(U^2)"'] & F(UF(1\otimes 1)\otimes U(WA\otimes WB)) \arrow[r, "F((UF)_2(1,1)\otimes \mathrm{id})"] \arrow[d, "F(U^2)"] \arrow[dr, phantom, "C"] & F(UF(1)\otimes UF(1)\otimes U(WA\otimes WB)) \arrow[d, "F(U^2\otimes \mathrm{id})"] \\
FU(F(1)\otimes WA\otimes WB) \arrow[r, "FU(F(l_1^{-1})\otimes \mathrm{id}\otimes \mathrm{id})"] \arrow[d, "\epsilon"'] & FU(F(1\otimes 1)\otimes WA\otimes WB) \arrow[dr, "F(U(F_2(1,1)\otimes \mathrm{id}\otimes \mathrm{id}))"'] & F(U(F(1)\otimes F(1))\otimes U(WA\otimes WB)) \arrow[d, "F(U^2)"] \\
F(1)\otimes WA\otimes WB \arrow[d, "F(l_1^{-1})\otimes \mathrm{id}\otimes \mathrm{id}"'] & & FU(F(1)\otimes F(1))\otimes WA\otimes WB) \arrow[d, "\epsilon"] \\
F(1\otimes 1)\otimes WA\otimes WB \arrow[rr, "F_2(1,1)\otimes \mathrm{id}\otimes \mathrm{id}"'] \arrow[d, "F(\eta\otimes\eta)\otimes \mathrm{id}\otimes \mathrm{id}"'] & & F(1)\otimes F(1)\otimes WA\otimes WB \\
F(UF(1)\otimes UF(1))\otimes WA\otimes WB \arrow[rr, "F(U^2))\otimes \mathrm{id}\otimes \mathrm{id}"'] & & FU(F(1)\otimes F(1))\otimes WA\otimes WB \arrow[u, "\epsilon\otimes \mathrm{id}\otimes \mathrm{id}"']
\end{tikzcd}
$$

8.2.3 Proof of Lemma 2.2.11

The commutativity of the required diagrams comes down to the verification of the diagram in the next page where the commutativity of the top square, the middle triangle and the bottom square on the left follows from the naturality of the monoidal U^2. The far right diagram is the triangle equality for the unit and the counit of the adjunction $F \dashv U$.

$$
\begin{tikzcd}
\big(UF(1)\otimes UF(1)\big)\otimes\big(UWA\otimes UWB)\big) \arrow[r, "U^2\otimes U^2"] \arrow[d, "\eta\otimes\eta"'] & U(F(1)\otimes F(1))\otimes U(WA\otimes WB) \arrow[d, "\eta_U\otimes\eta_U"] \arrow[dddd, bend left, "\mathrm{Id}"] \\
UF\big(UF(1)\otimes UF(1)\big)\otimes UF\big(UF(1)\otimes UWB)\big) \arrow[d, "U^2"'] \arrow[dr, bend left, "UF(U^2)\otimes UF(U^2))"'] & \\
U\Big(F\big(UF(1)\otimes UF(1)\big)\otimes F\big(UWA\otimes UWB\big)\Big) \arrow[d, "U(F(U^2)\otimes F(U^2))"'] & U\Big(FU(F(1)\otimes F(1)\Big)\otimes U\Big(FU(WA\otimes WB)\Big) \arrow[dl, bend left, "U^2"'] \arrow[dd, "U(\epsilon)\otimes U(\epsilon)"] \\
U\Big(FU\big(F(1)\otimes F(1)\big)\otimes FU\big(WA\otimes WB\big)\Big) \arrow[d, "U(\epsilon\otimes\epsilon)"'] & \\
U\big(F(1)\otimes F(1)\otimes WA\otimes WB\big) & U\big(F(1)\otimes F(1)\big)\otimes U\big(WA\otimes WB)\big) \arrow[l, "U^2"']
\end{tikzcd}
$$

8.2.4 Proof of Proposition 2.2.13

The commutativity of the above diagram comes down to the verification of the diagram below, which is obtained by expanding the arrows $\gamma_{1,A}$ and $\zeta_{A,1}$ as shown below.

$$\begin{tikzcd}
F(1\otimes 1\otimes A\otimes B) \arrow[r, "F(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] \arrow[d, "F(\eta\otimes\eta\otimes\mathrm{id}\otimes\mathrm{id})"'] & F(1\otimes A\otimes 1\otimes B) \arrow[d, "F(\eta\otimes\mathrm{id}\otimes\eta\otimes\mathrm{id})"] \\
F\big((UF(1)\otimes UF(1))\otimes(UWA\otimes UWB)\big) \arrow[r] \arrow[d, "F(\eta\otimes\eta)"'] & F\big((UF(1)\otimes UWA)\otimes(UF(1)\otimes UWB)\big) \arrow[d, "F(\eta\otimes\eta)"] \\
F\big(UF(UF(1)\otimes UF(1))\otimes UF(UWA\otimes UWB)\big) \arrow[d, "F(U^2)"'] & F\big(UF(UF(1)\otimes UWA)\otimes UF(UF(1)\otimes UWB)\big) \arrow[d, "F(U^2)"] \\
FU(F(UF(1)\otimes UF(1))\otimes F(UWA\otimes UWB) \arrow[d, "FU(F(U^2)\otimes F(U^2))"'] & FU(F(UF(1)\otimes UWA)\otimes F(UF(1)\otimes UWB) \arrow[d, "FU(F(U^2)\otimes F(U^2))"] \\
FU(FU(F(1)\otimes F(1))\otimes FU(WA\otimes WB)) \arrow[d, "FU(\epsilon\otimes\epsilon)"'] & FU(FU(F(1)\otimes WA)\otimes FU(F(1)\otimes WB)) \arrow[d, "FU(\epsilon\otimes\epsilon)"] \\
FU(F(1)\otimes F(1)\otimes WA\otimes WB) \arrow[r, "FU(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] \arrow[d, "\epsilon"'] & FU(F(1)\otimes WA\otimes F(1)\otimes WB) \arrow[d, "\epsilon"] \\
F(1)\otimes F(1)\otimes WA\otimes WB \arrow[r, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] & F(1)\otimes WA\otimes F(1)\otimes WB
\end{tikzcd}$$

The commutativity of the top square and of the bottom square are commutative follows from naturality of the symmetry. For the middle diagram we work as follows. First apply the functor F to the commutative diagrams of Lemma 2.2.11 and Lemma 2.2.12 and so the middle diagram is simplified to the following.

$$
\begin{tikzcd}
F\big((UF(1)\otimes UF(1))\otimes(UWA\otimes UWB)\big) \arrow[dd, "F(U^2\otimes U^2)"'] \arrow[dr, "F(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] & \\
& F\big((UF(1)\otimes UWA)\otimes(UF(1)\otimes UWB)\big) \arrow[dd, "F(U^2\otimes U^2)"] \\
F(U(F(1)\otimes F(1))\otimes U(WA\otimes WB)) \arrow[dd, "F(U^2)"'] & \\
& F(U(F(1)\otimes WA)\otimes U(F(1)\otimes WB)) \arrow[dd, "F(U^2)"] \\
FU(F(1)\otimes F(1)\otimes WA\otimes WB) \arrow[dr, "FU(\mathrm{id}\otimes\tau\otimes\mathrm{id})"'] & \\
& FU(F(1)\otimes WA\otimes F(1)\otimes WB)
\end{tikzcd}
$$

The left and the right columns of the above diagram simplifies further as it is shown in the next two pages in landscape position.

$$\begin{tikzcd}
((UF(1)\otimes UF(1))\otimes UWA)\otimes UWB \arrow[d, "(U^2\otimes \mathrm{id})\otimes \mathrm{id}"'] \arrow[dr, "\alpha\otimes \mathrm{id}"] & & \\
(U(F(1)\otimes F(1))\otimes UWA)\otimes UWB \arrow[d, "\alpha_{U(F(1)\otimes F(1)),UWA,UWB}"'] \arrow[dr, "U^2"] & (UF(1)\otimes (UF(1)\otimes UWA))\otimes UWB \arrow[dr, "U^2"] & \\
U(F(1)\otimes F(1))\otimes (UWA\otimes UWB) \arrow[d, "\mathrm{id}\otimes U^2"'] & U((F(1)\otimes F(1))\otimes WA)\otimes UWB \arrow[dd, "U^2"'] \arrow[dr, "U(\alpha)\otimes \mathrm{id}"] & (UF(1)\otimes U(F(1)\otimes WA))\otimes UWB \arrow[d, "U^2"'] \\
U(F(1)\otimes F(1))\otimes U(WA\otimes WB) \arrow[d, "U^2"'] & & (U(F(1)\otimes (F(1)\otimes WA))\otimes UWB \arrow[d, "U^2"'] \\
U((F(1)\otimes F(1))\otimes (WA\otimes WB)) \arrow[r, "U(\alpha)\otimes \mathrm{id}"] & U(((F(1)\otimes F(1))\otimes WA)\otimes WB) \arrow[r, "U(\alpha\otimes \mathrm{id})"] & U((F(1)\otimes (F(1)\otimes WA))\otimes WB)
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & ((UF(1)) \otimes UWA) \otimes UF(1)) \otimes UWB \arrow[dl, "\alpha \otimes \mathrm{id}"'] \arrow[d, "(U^2 \otimes \mathrm{id}) \otimes \mathrm{id}"] \\
 & (UF(1) \otimes (UWA \otimes UF(1))) \otimes UWB \arrow[dl, "U^2"'] & (U(F(1) \otimes WA) \otimes UF(1)) \otimes UWB \arrow[dl, "U^2"] \arrow[d, "\alpha_{U(F(1) \otimes F(1)), UWA, UWB}"] \\
(UF(1) \otimes U(WA \otimes F(1))) \otimes UWB \arrow[d, "U^2"'] & U((F(1) \otimes WA) \otimes F(1)) \otimes UWB \arrow[dl, "U(\alpha) \otimes \mathrm{id}"] \arrow[dd, "U^2"'] & U(F(1) \otimes WA) \otimes (UF(1) \otimes UWB) \arrow[d, "\mathrm{id} \otimes U^2"] \\
(U(F(1) \otimes (WA \otimes F(1)))) \otimes UWB \arrow[d, "U^2"'] & & U(F(1) \otimes WA) \otimes U(F(1) \otimes WB) \arrow[d, "U^2"] \\
U((F(1) \otimes (WA \otimes F(1))) \otimes WB) & U(((F(1) \otimes WA) \otimes F(1)) \otimes WB) \arrow[l, "U(\alpha \otimes \mathrm{id})"'] & U((F(1) \otimes WA) \otimes (F(1) \otimes WB)) \arrow[l, "U(\alpha) \otimes \mathrm{id}"']
\end{tikzcd}
$$

After the simplification, we have to prove the commutativity of the diagram below.

$$\begin{tikzcd}
UF(1)\otimes UF(1)\otimes UWA\otimes UWB \arrow[r, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes U^2\otimes\mathrm{id}"'] & UF(1)\otimes UWA\otimes UF(1)\otimes UWB \arrow[d, "\mathrm{id}\otimes U^2\otimes\mathrm{id}"'] \\
UF(1)\otimes U(F(1)\otimes WA)\otimes UWB \arrow[r, "\mathrm{id}\otimes U(\tau)\otimes\mathrm{id}"] \arrow[d, "U^2\otimes\mathrm{id}"'] & UF(1)\otimes U(WA\otimes F(1))\otimes UWB \arrow[d, "U^2\otimes\mathrm{id}"'] \\
U(F(1)\otimes(F(1)\otimes WA))\otimes UWB \arrow[r, "U(\mathrm{id}\otimes\tau)\otimes\mathrm{id}"] \arrow[d, "U^2"'] & U(F(1)\otimes(WA\otimes F(1)))\otimes UWB \arrow[d, "U^2"'] \\
U((F(1)\otimes(F(1)\otimes WA))\otimes WB) \arrow[r, "U(\mathrm{id}\otimes\tau\otimes\mathrm{id})"] & U((F(1)\otimes(WA\otimes F(1)))\otimes UWB)
\end{tikzcd}$$

The commutativity of the top square follows from the fact that U is a symmetric monoidal functor. The commutativity of thehe second square and of the third squares follows from the naturality of $U^2\otimes\mathrm{id}$ and U^2 respectively and the proof is complete.

In Definition 7.2.10 we defined a natural transformation ϕ from $T = UF$ to $T' = UF(1)\otimes -$ and in Proposition 7.2.12 we proved that ϕ is of the form u^e. In the following, we check that ϕ is an opmonoidal natural transformation.

Lemma 8.2.1. *The morphism of bimonads $\phi : UF \to UF(1)\otimes(-)$ preserves the opmonoidal constraints, in other words the following diagram commutes.*

$$\begin{tikzcd}
UF(X\otimes Y) \arrow[rr, "(UF)_2"] \arrow[d] & & UF(X)\otimes UF(Y) \arrow[dd, "\phi_X\otimes\phi_Y"] \\
UF(1\otimes X\otimes Y) \arrow[d, "\phi_{X\otimes Y}"'] & & \\
UF(1)\otimes X\otimes Y \arrow[dr, "(UF)_2\otimes \mathrm{id}"'] & & UF(1)\otimes X\otimes UF(1)\otimes Y \\
& UF(1)\otimes UF(1)\otimes X\otimes Y \arrow[ur, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] &
\end{tikzcd}$$

Proof. We prove the commutativity of the boundary diagram due to commutativity of all the internal diagrams below.

$$\begin{tikzcd}
UF(X\otimes Y) \arrow[rr, "(UF)_2"] \arrow[d] \arrow[dr] & & UF(X)\otimes UF(Y) \arrow[dd, "\phi_X\otimes\phi_Y"] \\
UF(1\otimes X\otimes Y) \arrow[d, "\phi_{X\otimes Y}"'] \arrow[dr] & UF(1\otimes X\otimes 1\otimes Y) \arrow[dr] & \\
UF(1)\otimes X\otimes Y \arrow[dr, "(UF)_2\otimes \mathrm{id}"'] & UF(1\otimes 1\otimes X\otimes Y) \arrow[u] \arrow[d] & UF(1)\otimes X\otimes UF(1)\otimes Y \\
& UF(1)\otimes UF(1)\otimes X\otimes Y \arrow[ur, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] &
\end{tikzcd}$$

The commutativity of the square on the top left corner we have the following commutative diagram

$$\begin{tikzcd}
1\otimes 1\otimes X\otimes Y \arrow[r, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[ddd, "l_1\otimes\mathrm{id}\otimes\mathrm{id}"'] & 1\otimes (X\otimes 1)\otimes Y \arrow[d, "\mathrm{id}\otimes\alpha_{X,1,Y}"] \arrow[dddl, "\mathrm{id}\otimes r_X\otimes\mathrm{id}"'] \\
& 1\otimes X\otimes (1\otimes Y) \arrow[d, "l_X\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[ddl, "\mathrm{id}\otimes\mathrm{id}\otimes l_Y"] \\
& X\otimes (1\otimes Y) \arrow[d, "\mathrm{id}\otimes l_Y"] \\
1\otimes X\otimes Y \arrow[r, "l_X\otimes\mathrm{id}"'] & X\otimes Y
\end{tikzcd}$$

where $\alpha_{X,1,Y}$ is the associator. Indeed, by the naturality of the natural isomorphism $l : 1 \otimes X \xrightarrow{\sim} X$ follows that

$$\mathrm{id}_1 \otimes l_X = l_{1\otimes X} = l_1 \otimes \mathrm{id}_X .$$

From this, the far left vertical arrow is becoming $\mathrm{id}_1 \otimes l_X \otimes \mathrm{id}_Y$ and in combination with the triangle relation of the symmetry τ, see 2.3.3, follows that the top left triangle is commutative. The commutativity of the triangle in the middle follows from the triangle relation of the associator α, see 2.1.1 and the commutativity of the bottom triangle follows from the interchange law. Then, applying the functor UF on the above diagram we obtain the commutativity of the pentagon.

The commutativity of the remaining diagrams follow from technical lemmas who have stated in Section 2.2. In particular for the commutativity of the square on the top right corner we apply in the diagram of Lemma 2.2.9 the strong monoidal functor U and taking in to account that $UW \cong \mathrm{id}$. In a similar way the commutativity of the triangle on the bottom and on the left of the diagram follows from the application of the strong monoidal functor on the commutative diagram of Lemma 2.2.10 and the strong monoidal functor U. To complete the proof, the commutativity of the bottom right square follows from the application of the strong monoidal functor U on the commutative diagram of Lemma 2.2.13. □

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