Characterizations of Integral Input-to-State Stability for systems with multiple invariant sets

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***Abstract*—We extend the classical integral Input-to-State Stability (iISS) theory to systems evolving on complete Riemannian manifolds and admitting multiple disjoint invariant sets, so as to allow a much broader variety of dynamical behaviors of interest. Building upon a recent extension of the Input-to-State (ISS) theory for this same class of systems, we provide characterizations of the iISS concept in terms of dissipation inequalities and integral estimates as well as connections with the Strong iISS notion. Finally, we discuss some examples within the domain of mechanical systems.**

***Index Terms*—Stability of nonlinear systems, Lyapunov methods, Manifolds**

I. Introduction

In the last 30 years, the study of stability and sensitivity to disturbances for general nonlinear systems has largely gained new valid tools by the introduction of the Input-to-State Stability (ISS) paradigm [23], [24]. The definition of ISS implies the qualitative property of small overshoot with small disturbances and initial conditions, and thus represents a measure of performance in the qualitative analog of "finite $\mathcal{L}^2$ gain" (nonlinear $\mathcal{H}^\infty$). As a weaker but still very meaningful notion of stability, the Integral Input-to-State Stability (iISS) has been introduced in [22], [5]. The definition of iISS implies the qualitative property of small overshoot when initial conditions are small and disturbances have finite energy, and thus represents a measure of performance in the qualitative nonlinear analog of "finite $\mathcal{H}^2$ norm" for linear systems . Applications of the iISS property address the stabilization and disturbance attenuation of systems with bounded controls [17], nonlinear cascades [8], large-scale systems via decentralized output-feedback control [13], systems in block strict-feedback form via output regulation [12], [14], and hybrid switched systems [19].

In their classical formulation, both ISS and iISS operate globally on systems defined in Euclidean space and having a single equilibrium at the origin. The definition potentially allows to characterize stability with respect to arbitrary compact invariant sets, provided that these sets are simultaneously Lyapunov stable and globally attractive. These requirements hamper a global analysis of many dynamical behaviors of interests, such as multistability, periodic oscillations, chaos, just to name a few. As an attempt to overcome such limitations, the almost global stability property was introduced in [21], and short afterwards, almost Input-to-State Stability [1] for systems admitting exogenous disturbances. The key idea of this approach is to replace Lyapunov function by suitable density function whose explicit construction still generates some difficulties. More recently, the need for conditions involving density functions was removed in the case of systems with exponentially unstable equilibria thanks to a careful application of integral manifold theory [4]. The authors in [2] have instead shown that the most natural way of still conducting a global analysis in the ISS sense is to relax the Lyapunov stability requirement [11] (rather than the global attractiveness), under relatively mild additional assumptions.

Within the framework introduced in [2], we extend the iISS notion to systems evolving on complete Riemannian manifolds and possibly exhibiting multiple disjoint compact invariant sets. Specifically, we introduce a notion of iISS which is weaker than the classical one given in [22], that is the conjunction of two properties: global attractiveness with zero disturbances and the bounded-energy bounded-state property. Indeed, only such weaker notion is consistent with the lack of Lyapunov stability which is typical of multistable systems, while still providing equivalent characterizations in terms of Lyapunov-like and LaSalle-like dissipation inequalities.

Outline of Paper: Basic definitions and main equivalences for iISS are presented in Section II. Section III provides the main proofs of this paper. When characterizing ISS and iISS in terms of Lyapunov-like inequalities, an intermediate stability concept arises, namely the Strong iISS notion, which is discussed in the multistable setting in Section IV. Section V collects examples of systems, within the mechanical domain, which fulfill the iISS or Strong iISS properties. Conclusive remarks are discussed in Section VI.

Notation: The symbol $\mathfrak{d}(x_1, x_2)$ denotes the Riemannian distance between two points x_1 and x_2 of a Riemannian manifold M, according with the Riemannian metric g. For $v \in T_M x$, denote $\|v\|_g$ denote the norm $g_x(v,v)^{\frac{1}{2}}$ induced by g at x.

For a set $S \subset M$ define $|\cdot|_S$ as

$$|x|_S = \inf_{a \in S} \mathfrak{d}(x, a).$$

For a compact set $A \subset M$, $\eth(A)$ and $\operatorname{int}\{A\}$ respectively denote the boundary and the interior of A. If w is a vector in the Euclidean space, i.e. an input, notation $|w|$ will indicate the standard Euclidean norm; otherwise, if $w \in M$, we define $|w| := \mathfrak{d}(w, x_0)$ with x_0 being the designated origin element of M. For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define its

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Manuscript received April 19, 2005; revised September 17, 2014.

infinity norm over the time interval $[t_1, t_2]$ with respect to the origin and with respect to a compact set S as follows:

$$\left\|d_{[t_1,t_2]}\right\| = \text{ess}\sup_{t_1 \le t \le t_2} |d(t)|,$$
$$\left\|d_{[t_1,t_2]}\right\|_S = \text{ess}\sup_{t_1 \le t \le t_2} |d(t)|_S,$$

and denote $\|d\| := \left\|d_{[0,+\infty)}\right\|$ and $\|d\|_S := \left\|d_{[0,+\infty)}\right\|_S$. We define the saturation function as $\text{sat}\{x\} := \text{sign}\{x\}\min\{1,|x|\}$.

II. Definitions and main result

Let M be a $\mathcal{C}^2$, complete, connected, n-dimensional Riemannian manifold without boundary and with g its associated Riemannian metric. Let D be a closed subset of $\mathbb{R}^m$ containing the origin. Consider the system:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where $f(x,d) : M \times D \to T_x M$ is a locally Lipschitz continuous mapping (in the sense of Definition 13), with state x taking value in M and $d(\cdot)$ any locally essentially bounded and measurable input signal taking values in D. We denote with $\mathcal{M}_D$ such class of input signals. We denote by $X(t,x;d)$ the uniquely defined solution of (1) at time t fulfilling $x(0) = x$ under the input $d(\cdot)$.

It is useful to recall here the classical definition of the integral ISS property [22] and, in particular, to generalize it for systems evolving on manifolds.

Definition 1: System (1) is said to be *integral ISS (*iISS*) in the classical sense* or, in short, *classical iISS* with respect to a compact set $\mathcal{A}$ and input d if there exist functions $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for all $x \in M$ and all $d(\cdot) \in \mathcal{M}_D$, the solution $X(t,x;d)$ is defined for all $t \ge 0$ and

$$\alpha\left(|X(t,x;d)|_{\mathcal{A}}\right) \le \beta\left(|x|_{\mathcal{A}}, t\right) + \int_0^t \gamma(|d(s)|)\, ds \tag{2}$$

for all $t \ge 0$.
Note that, if in inequality (2) we set $d(t) \equiv 0$ for all $t \ge 0$, we recover the inequality $\alpha\left(|X(t,x;d)|_{\mathcal{A}}\right) \le \beta\left(|x|_{\mathcal{A}}, t\right)$, which has been shown to be equivalent to global asymptotic stability with zero inputs (0-GAS) [18]. The 0-GAS property is in turn equivalent to the conjunction of two properties: global Lyapunov stability and global attractiveness. For this reason, the aforementioned lack of global Lyapunov stability - typical of multistable systems - enforces the adoption of a weaker notion of the integral ISS property for this kind of systems, as in the following.

The class of multistable systems of interest is defined as in [2] by considering the unperturbed system:

$$\dot{x}(t) = f(x(t), 0). \tag{3}$$

We assume that all solutions of the unperturbed system (3) are bounded forward in time and that there exists a non-empty ω-limit set. Consider a compact invariant set $\mathcal{W}$ containing all α- and ω-limit sets of (3). Crucial in the stability analysis by means of Lyapunov-like analytical tools would then be the notion of decomposition for a compact invariant set and the notion of absence of cycles in the decomposition itself, as detailed in the Definitions 2 and 3 as follows.

Definition 2: Let $\Lambda \subset M$ be a compact and invariant set for (3). A *decomposition* of Λ is a finite, disjoint family of compact invariant sets $\Lambda_1, \dots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

Each Λ_i will then be called an *atom* of the decomposition.
For an invariant set Λ, its attraction and repulsion subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \left\{x \in M : |X(t,x;0)|_\Lambda \to 0 \text{ as } t \to +\infty\right\},$$
$$\mathfrak{R}(\Lambda) = \left\{x \in M : |X(t,x;0)|_\Lambda \to 0 \text{ as } t \to -\infty\right\}.$$

Define a relation between $\Lambda_i \subset M$ and $\Lambda_j \subset M$ by $\Lambda_i \prec \Lambda_j$ if $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_j) \neq \emptyset$ (this relation implies that there is no solution connecting set Λ_j with set Λ_i.

Definition 3: Let $\Lambda_1, \dots, \Lambda_k$ be a decomposition of Λ, then

1) An *r-cycle* ($r \ge 2$) is an ordered r-tuple of distinct indices $i_1, \dots, i_r$ such that $\Lambda_{i_1} \prec \cdots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2) A *1-cycle* is an index i such that $[\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)] - \Lambda_i \neq \emptyset$.
3) A *filtration ordering* is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \le j$.

Throughout the paper, we will use the following assumption on $\mathcal{W}$ which is crucial for the construction of a Lyapunov function for the unperturbed system (as carried out in Appendix C in [2]):

Assumption 1 (No cycle condition): The set $\mathcal{W}$ admits a finite decomposition, i.e. $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which shows no cycle between the $\mathcal{W}_i$s and which satisfies a filtration ordering as detailed in Definitions 2 and 3.
Note that Assumption 1 basically rules out the presence of heteroclinic cycles and homoclinic orbits or, more precisely, any such cycles would typically need to be included as atoms of a coarser decomposition. (e.g., the theory does not apply to non-dissipative Hamiltonian systems).

We are going to list several interesting properties for system (1) with $\mathcal{W}$ satisfying Assumption 1. The conjunction of the properties as in Definitions 4 and 5 will then provide our weaker notion of integral ISS for multistable systems. We remark that most of the following properties are direct extensions of those introduced in [5], [6], and [3].

Definition 4 (UBEBS): System (1) is said to have the *uniform bounded-energy bounded-state* (UBEBS) property if, for some $\alpha, \gamma, \sigma \in \mathcal{K}_\infty$ and some positive constant c_u, the following estimate holds for all $t \ge 0$, all $x \in M$ and all $d(\cdot) \in \mathcal{M}_D$:

$$\alpha(|X(t,x;d)|_{\mathcal{W}}) \le \gamma(|x|_{\mathcal{W}}) + \int_0^t \sigma(|d(s)|)ds + c_u. \tag{4}$$

Definition 5 (zero-GATT): System (1) is said to have the *zero-global attraction (zero-GATT)* property with respect to a compact invariant set $\mathcal{W}$, if every trajectory $X(t,x;0)$ of the unperturbed system (3) satisfies

$$\lim_{t \to +\infty} |X(t,x;0)|_{\mathcal{W}} = 0. \tag{5}$$

Definition 6: System (1) is said to be ***iISS in the multistable sense*** with respect to the set $\mathcal{W}$ and the input $d(\cdot)$ if and only if it satisfies Assumption 1 and the UBEBS and zero-GATT properties.

Definition 7 (BEWCS): System (1) is said to satisfy the *Bounded Energy Weakly Converging State* (BEWCS) property if there exists a function $\tilde{\sigma} \in \mathcal{K}_\infty$ such that the following holds for all $x \in M$, and all $d(\cdot) \in \mathcal{M}_D$:

$$\int_0^{+\infty} \tilde{\sigma}\left(|d(s)|\right) \, ds < +\infty \;\Rightarrow\; \liminf_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} = 0. \tag{6}$$

Definition 8 (BESCS): System (1) is said to satisfy the *Bounded Energy Strongly Converging State* (BESCS) property if there exists a function $\tilde{\sigma} \in \mathcal{K}_\infty$ such that the following holds for all $x \in M$, and all $d(\cdot) \in \mathcal{M}_D$:

$$\int_0^{+\infty} \tilde{\sigma}\left(|d(s)|\right) \, ds < +\infty \;\Rightarrow\; \lim_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} = 0. \tag{7}$$

Definition 9 (iISS-Lyapunov function): A $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is called an ***iISS-Lyapunov function*** for system (1) if there exist functions $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$, and $c \geq 0$, and a continuous positive definite function α_3 such that, for all $x \in M$,

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c), \tag{8}$$

and the following dissipation inequality hold for all $(x, d) \in M \times D$:

$$DV(x) f(x, d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|). \tag{9}$$

Remark 1: Note that, in contrast to the classical definition of iISS-Lyapunov function as given in [5], function $V(x)$ in (8) is bounded from above by an increasing function of $|x|_{\mathcal{W}}$ which is not necessarily class $\mathcal{K}$ (as c may be positive). Setting $c = 0$ would in fact imply 0-GAS of the set $\mathcal{W}$ (by standard arguments as in [18]), which cannot be the case for decompositions with multiple connected components (lack of Lyapunov stability).

Definition 10 (Smooth dissipativity): System (1) is said to be *smoothly dissipative* if there exists a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$, functions $\alpha_1, \alpha_2, \sigma \in \mathcal{K}_\infty$, a continuous positive-definite function α_4, and a continuous output map $h : M \to \mathbb{R}^q$ with $h(x) = 0$ whenever $|x|_{\mathcal{W}} = 0$, such that (8) holds and the following dissipation inequality holds for all $(x, d) \in M \times D$:

$$DV(x) f(x, d) \leq -\alpha_4(|h(x)|) + \sigma(|d|). \tag{10}$$

Definition 11 (Weak zero-detectability): System (1) with output $h(x)$ is said to be *weakly zero-detectable* if $h(X(t, x; 0)) \equiv 0$ for all t implies $|X(t, x; 0)|_{\mathcal{W}} \to 0$ as $t \to +\infty$.

In our main result we will also compare the previous definitions with the following apparently weaker stability estimate which mixes integral and sup norms and is satisfied if, for some $\alpha, \beta, \gamma, \sigma \in \mathcal{K}_\infty$ and some positive constant c_m, we have for all $t \geq 0$, all $x \in M$ and all $d(\cdot) \in \mathcal{M}_D$:

$$\begin{aligned} \alpha(|X(t, x; d)|_{\mathcal{W}}) &\leq \beta(|x|_{\mathcal{W}}) \\ &+ \int_0^t \sigma(|d(s)|) ds + \gamma\left(\left\|d_{[0,t]}\right\|\right) + c_m. \end{aligned} \tag{11}$$

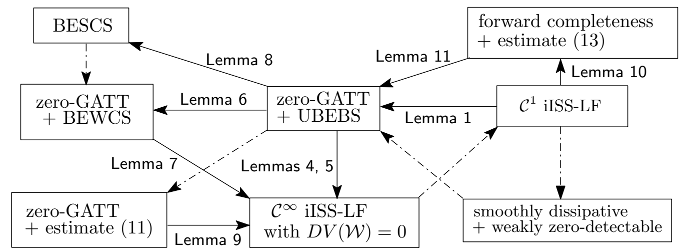

Fig. 1. Road-map of the proof of Theorem 1. Dash-dotted lines refers to those proofs directly sketched below the statement of the Theorem.

Moreover, in the context of operators with finite $\mathcal{H}_\infty$ norm, it is useful to introduce the following notion of mixed integral norm, inspired by [6]. We say that system (1) satisfies an $\mathcal{L}_p$-to-$\mathcal{L}_q$ norm with $p \neq q$ if, for some $\sigma \in \mathcal{K}_\infty$, we have for all $t \geq 0$, all $x \in M$ and all $d(\cdot) \in \mathcal{M}_D$:

$$\begin{aligned} &\left(\int_0^t \left(|X(s, x; d)|_{\mathcal{W}}\right)^q ds\right)^{1/q} \leq \\ &\qquad \leq \left(|x|_{\mathcal{W}}^p + \int_0^t \sigma(|u(s)|)^p \, ds + c\right)^{1/p}. \end{aligned} \tag{12}$$

Estimate (12) for $p \neq q$ is actually equivalent to the following:

$$\int_0^t \alpha(|X(t, x; d)|_{\mathcal{W}}) \leq \chi\left(|x|_{\mathcal{W}} + \int_0^t \sigma(|d(s)|) ds + \chi_0\right), \tag{13}$$

for some $\alpha, \chi, \sigma \in \mathcal{K}_\infty$ and a constant $\chi_0 > 0$.

Theorem 1: Consider a nonlinear system as in (1) and let $\mathcal{W}$ be a compact invariant set containing all α and ω limit sets of (3) as in Assumption 1. Then the following properties are equivalent:

1) zero-GATT and UBEBS;
2) existence of a smooth iISS-Lyapunov function V such that $DV(x) = 0$ for all $x \in \mathcal{W}$;
3) existence of a $\mathcal{C}^1$ iISS-Lyapunov function V;
4) existence of an output function that makes the system smoothly dissipative and weakly zero-detectable;
5) zero-GATT and BEWCS;
6) BESCS;
7) zero-GATT and mixed estimate (11);
8) forward completeness and integral estimate (13).

Proof: Implication 2 $\Rightarrow$ 3 is trivial. Implication 3 $\Rightarrow$ 4 holds true as it can be seen by setting $h(x) := \alpha_3(|x|_{\mathcal{W}})$ and α_4 equal to the identity function. Moreover, weak zero-detectability is obtained by noticing that $h(x) = 0$ implies $x \in \mathcal{W}$ by positive-definiteness of α_3. We are now going to show that 4 $\Rightarrow$ 1. The UBEBS property directly follows by applying the same arguments given in Lemma 1 for inequality (9) to inequality (10). The zero-detectability condition allows us to invoke the LaSalle's invariance principle to inequality (10) with $d \equiv 0$ for all t therefore yielding the zero-GATT property. In order to show that 6 $\Rightarrow$ 5, note that zero-GATT follows from BESCS by setting $d(t) \equiv 0$ for all $t \geq 0$, whereas BESCS $\Rightarrow$ BEWCS trivially follows by the definitions of $\limsup$, $\liminf$, and $|\cdot|_{\mathcal{W}} \geq 0$. Regarding implication 1 $\Rightarrow$ 7, note that mixed estimate (11) simply follows from the UBEBS property by adding the sup norm $\gamma(\left\|d_{[0,t]}\right\|)$ for an arbitrary $\gamma \in \mathcal{K}_\infty$ function. A diagram for the proof of all other implications is given in Figure 1. □

Remark 2: Integral ISS in the classical sense (Definition 1) clearly implies integral ISS in the multistable sense (Definition 6) while the reverse is not true in general, i.e. whenever $\mathcal{W}$ admits a decomposition with multiple atoms. However, if $\mathcal{W}$ consists of a single connected component, integral ISS in the multistable sense is actually equivalent to integral ISS in the classical sense. The latter statement is proven by observing that, if there is only one atom $\mathcal{W}$, then the set $\mathcal{D}$ in (17) and $\mathcal{W}$ are equivalent, thus Lemma 5 proves classical iISS of $\mathcal{W}$.

Remark 3: One may wonder whether the definition of integral Input-to-State *Stability* is not appropriate or even misleading in the multistability context due to the fundamental lack of Lyapunov *stability* in autonomous systems with globally attractive multiple invariant sets. However, as explained in Remark 2, our definition still implies stability in case of $\mathcal{W}$ being not decomposed in more than one connected component. The latter property then justifies the adoption of term integral Input-to-State *Stability* which naturally generalizes - and retains continuity with - the standard ISS framework.

III. Main Proofs

In this Section, we provide all remaining proofs for Theorem 1, as sketched in Figure 1. In all subsequent Lemmas, we let Assumption 1 be satisfied for system (1).

Lemma 1 (iISS-Lyapunov ⇒ UBEBS): If system (1) admits an iISS-Lyapunov function then it satisfies the zero-GATT and UBEBS properties.

Proof: The zero-GATT property follows by standard Lyapunov arguments by considering $d(t) = 0$ for all $t \geq 0$, integrating (9) along solutions of the unperturbed system. The UBEBS property follows considering that:

$$DV(x)f(x,d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) \leq \gamma(|d|), \tag{14}$$

for all $(x,d) \in M \times D$. Now pick any trajectory $X(t,x;d)$ corresponding to an input $d(\cdot)$ and initial state x. By integration of (14) along solutions of system (1) it follows that:

$$V(X(t,x;d)) - V(x) \leq \int_0^t \gamma(|d(s)|)ds. \tag{15}$$

It then follows from (8) that:

$$\begin{aligned}\alpha_1(|X(t,x;d)|_{\mathcal{W}}) &\leq \alpha_2(|x|_{\mathcal{W}} + c) + \int_0^t \gamma(|d(s)|)ds \\ &\leq \alpha_2(2|x|_{\mathcal{W}}) + \alpha_2(2c) + \int_0^t \gamma(|d(s)|)ds,\end{aligned}$$

which concludes our proof. □

Instrumental, in the proofs of the following Lemmas, will be the definition of a certain set $\mathcal{D}$ which, under zero-GATT of $\mathcal{W}$, represents a global (Lyapunov stable) attractor for system (3). Consider a monotonically increasing sequence of compact subsets[1] $M_1 \subset M_2 \subset \cdots \subset M$ with the property that:

$$M = \bigcup_{n=1}^{+\infty} \text{int}\{M_n\} \ . \tag{16}$$

Denote by $X(t,S)$ the attainable set of (3) at time t from initial conditions in S. Consider the set:

$$\mathcal{D} = \bigcup_{n=1}^{+\infty} \bigcap_{t\geq 0} X(t, M_n) \ . \tag{17}$$

It is shown in Lemma 3 in [2] that the set $\mathcal{D}$ is positively invariant, compact, globally asymptotically stable for (3), and satisfies $\mathcal{W} \subseteq \mathcal{D}$, therefore it is also true that:

$$|x|_{\mathcal{D}} \leq |x|_{\mathcal{W}} \leq |x|_{\mathcal{D}} + c \ , \tag{18}$$

for some non-negative constant c.

The following two Lemmas are obtained by adapting the arguments in Lemma IV.10 and Proposition II.5 of [5] to the case of systems evolving on manifolds and having multiple invariant sets.

Lemma 2: Let system (1) be zero-GATT. Then, there exist a smooth function $U : M \to \mathbb{R}$, $\mathcal{K}_\infty$ functions $\upsilon_1, \upsilon_2, \upsilon, \delta$, and a positive constant c such that:

$$\upsilon_1(|x|_{\mathcal{W}}) \leq U(x) \leq \upsilon_2(|x|_{\mathcal{W}} + c) \tag{19}$$

$$DU(x)f(x,d) \leq -\upsilon(|x|_{\mathcal{W}}) + \delta(|x|_{\mathcal{W}})\delta(|d|), \tag{20}$$

for all $x \in M$ and $d \in D$. Moreover, $DU(x) = 0$ for all $x \in \mathcal{W}$.

Proof: Due to the zero-GATT property, the arguments presented in [2, Section III C] can be adapted for the case of $d \equiv 0$, so as to infer the existence of a smooth function $U : M \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$ functions $\upsilon_1, \upsilon_2, \upsilon$, and a positive constant c such that:

$$\begin{aligned}&\upsilon_1(|x|_{\mathcal{W}}) \leq U(x) \leq \upsilon_2(|x|_{\mathcal{W}} + c) \\ &DU(x)f(x,0) \leq -\upsilon(|x|_{\mathcal{W}}).\end{aligned}$$

In [2, Section III C], function $U(x)$ is obtained as the sum of two functions: $U_1(x)$ with the property $DU_1(x) = 0$ if $x \in \mathcal{D}$ (the same as in Theorem 4); $U_2(x)$ with the property $DU_2(x) = 0$ if $x \in \mathcal{W}$. Since (18) holds, we obtain $DU(x) = 0$ if $x \in \mathcal{W}$.

Consider now the following function:

$$\tilde{\gamma}(r,s) := \max_{|x|_{\mathcal{W}} \leq r, |d| \leq s} |DU(x)|\,|f(x,d) - f(x,0)| \ . \tag{21}$$

Note that the correspondence $C(r,s) = \{x \in M, d \in D \mid |x|_{\mathcal{W}} \leq r, |d| \leq s\}$ is compact-valued and upper and lower hemicontinuous. Moreover function $(x,d) \to |DU(x)|\,|f(x,d) - f(x,0)|$ is continuous by smoothness of U and continuity of f. Therefore, $\tilde{\gamma}$ is continuous by the maximum theorem and nondecreasing with respect to each argument. Moreover, we can easily see that $\tilde{\gamma}$ vanishes for $s = 0$ (trivial) and $r = 0$ (since $DU(x) = 0$

[1] Existence of a monotonically increasing sequence of compact subsets of M as in (16) follows from completeness of the manifold [10] and from the fact that a locally compact and connected metric space M is second countable [15].

at $x \in \mathcal{W}$). Hence, it can be majorized by a function $\gamma(r,s) := \tilde{\gamma}(r,s) + r + s$ separately of class $\mathcal{K}_\infty$ in each argument. Corollary IV.5 in [5] shows that, for a function $\gamma(t,s)$ separately of class $\mathcal{K}_\infty$ in each argument, there exist some functions $\delta \in \mathcal{K}_\infty$ such that $\gamma(t,s) \le \delta(t)\delta(s)$. Then, it follows from (20) that:

$$\begin{aligned} DU(x)f(x,d) &= DU(x)\left[(f(x,d) - f(x,0)) + f(x,0)\right] \\ &\le |DU(x)|\,|f(x,d) - f(x,0)| - \upsilon(|x|_{\mathcal{W}}) \\ &\le -\upsilon(|x|_{\mathcal{W}}) + \delta(|x|_{\mathcal{W}})\delta(|d|) \ . \end{aligned} \tag{22}$$

□

Lemma 3: Let system (1) be zero-GATT. Then, there exist a smooth function $U : M \to \mathbb{R}$, $\mathcal{K}$ functions υ_1, υ_2, a $\mathcal{K}$ function δ, a continuous positive-definite function ϖ, and a positive constant c such that:

$$\upsilon_1(|x|_{\mathcal{W}}) \le U(x) \le \upsilon_2(|x|_{\mathcal{W}} + c) \tag{23}$$

$$DU(x)f(x,d) \le -\varpi(|x|_{\mathcal{W}}) + \delta(|d|), \tag{24}$$

for all $x \in M$ and $d \in D$. Moreover, $DU(x) = 0$ for all $x \in \mathcal{W}$.

Proof: Due to Assumption 1 and the zero-GATT property, Lemma 2 holds for some smooth function $\tilde{U}(x)$, and $\mathcal{K}_\infty$ functions $\tilde{\upsilon}_1, \tilde{\upsilon}_2, \tilde{\upsilon}, \delta$ and positive constant $\tilde{c}$ such that:

$$\tilde{\upsilon}_1(|x|_{\mathcal{W}}) \le \tilde{U}(x) \le \tilde{\upsilon}_2(|x|_{\mathcal{W}} + \tilde{c})$$

for all $x \in M$, and

$$D\tilde{U}(x)f(x,d) \le -\tilde{\upsilon}(|x|_{\mathcal{W}}) + \delta(|x|_{\mathcal{W}})\delta(|d|) \tag{25}$$

for all $x \in M$ and $d \in D$. Define $\pi(\cdot)$ of class $\mathcal{K}$ as follows:

$$\pi(r) = \int_0^r \frac{ds}{1 + \chi(s)} \tag{26}$$

with χ a suitable continuous, positive, and increasing function to be defined later. Composing π with $\tilde{U}$ and taking derivatives yields, by virtue of (25):

$$\begin{aligned} D[(\pi \circ \tilde{U})(x)]f(x,d) &= \frac{D\tilde{U}(x)f(x,d)}{1 + \chi(\tilde{U}(x))} \\ &\le -\frac{\tilde{\upsilon}(|x|_{\mathcal{W}})}{1 + \chi(\tilde{U}(x))} + \frac{\delta(|x|_{\mathcal{W}})\delta(|d|)}{1 + \chi(\tilde{U}(x))} \end{aligned} \tag{27}$$

which can be rewritten as (24) by selecting:

$$\begin{aligned} U &:= \pi \circ \tilde{U}, \\ \chi(\cdot) &:= \delta(\tilde{\upsilon}_1^{-1}(\cdot)), \\ \varpi(\cdot) &:= \frac{\tilde{\upsilon}(|x|_{\mathcal{W}})}{1 + \delta(\tilde{\upsilon}_1^{-1}(\tilde{\upsilon}_2(\cdot + c)))}. \end{aligned}$$

Note that, by the chain rule,

$$\frac{\partial U(x)}{\partial x} = \left.\frac{\partial \pi(s)}{\partial s}\right|_{s=\tilde{U}(x)} \cdot \frac{\partial \tilde{U}(x)}{\partial x} = 0 \tag{28}$$

for all $x \in \mathcal{W}$. □

We bring to the attention of the reader that if function U in Lemma 3 were proper, i.e. $\nu_1 \in \mathcal{K}_\infty$, then it would qualify as an iISS-Lyapunov function. Unfortunately, this is not necessarily the case, therefore U needs to be used in addition to a proper function W as described in the proof of Lemma 4.

From this point onwards, let the system (1) have the zero-GATT and UBEBS properties. Therefore, let constant $c > 0$ and functions $\upsilon_1, \upsilon_2, \delta \in \mathcal{K}$, ϖ continuous positive-definite, and U non-negative be given as in Lemma 3 and let constant $c_u > 0$ and functions $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ be given as in Definition 4.

Lemma 4: Assume that system (1) has the zero-GATT property. Assume furthermore that system (1) is iISS in the classical sense with respect to the set $\mathcal{D}$ and input $d(\cdot)$. Then, there exists a smooth iISS-Lyapunov function as in Definition 9 and such that $DV(x) = 0$ for all $x \in \mathcal{W}$.

Proof: By virtue of Theorem 6, classical iISS with respect to $\mathcal{D}$ and input $d(\cdot)$ implies the existence of a function W, and functions $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\sigma} \in \mathcal{K}_\infty$ and a continuous positive-definite function $\bar{\alpha}_3$ such that:

$$\bar{\alpha}_1(|x|_{\mathcal{D}}) \le W(x) \le \bar{\alpha}_2(|x|_{\mathcal{D}}) \tag{29}$$

$$DW(x)f(x,d) \le -\bar{\alpha}_3(|x|_{\mathcal{D}}) + \bar{\sigma}(|d|) \ . \tag{30}$$

By combining (29)-(30) with inequalities (23)-(24) in Lemma 3 and by taking into account the bounds (18), we have that:

$$\begin{aligned} \upsilon_1(|x|_{\mathcal{W}}) + \bar{\alpha}_1(|x|_{\mathcal{D}}) &\le U(x) + W(x) \\ &\le \upsilon_2(|x|_{\mathcal{W}} + c) + \bar{\alpha}_2(|x|_{\mathcal{W}}) \end{aligned} \tag{31}$$

$$\begin{aligned} D\left[U(x) + W(x)\right]f(x,d) &\le -\varpi(|x|_{\mathcal{W}}) - \bar{\alpha}_3(|x|_{\mathcal{D}}) \\ &\quad +\bar{\sigma}(|d|) + \delta(|d|) \ . \end{aligned} \tag{32}$$

By introducing the $\mathcal{K}_\infty$ function

$$\alpha_1(s) = \begin{cases} \upsilon_1(s) & \text{if } s \le c \\ \upsilon_1(c) + \bar{\alpha}_1(s - c) & \text{if } s > c \end{cases} ,$$

inequalities (31)-(32) read as inequalities (8)-(9) with $\alpha_2 = \upsilon_2 + \bar{\alpha}_2$, $\alpha_3 = \varpi$, $V = U + W$, and $\gamma = \bar{\sigma} + \delta$. □

Lemma 5: Assume that system (1) has the zero-GATT and UBEBS properties. Then system (1) is iISS in the classical sense with respect to the set $\mathcal{D}$.

Proof: Lemma 3 in [2] proves that zero-GATT with respect to $\mathcal{W}$ implies zero-GAS with respect to $\mathcal{D}$. Furthermore, system (1) is UBEBS with respect to the set $\mathcal{D}$. In fact, the UBEBS property with respect to the set $\mathcal{W}$ can be combined with bounds (18), so as to obtain the following:

$$\begin{aligned} &\alpha(|X(t,x;d)|_{\mathcal{D}}) \le \alpha(|X(t,x;d)|_{\mathcal{W}}) \\ &\le \gamma(|x|_{\mathcal{W}}) + \int_0^t \sigma(|d(s)|)ds + c_u \\ &\le \gamma(|x|_{\mathcal{D}} + c) + \int_0^t \sigma(|d(s)|)ds + c_u \\ &\le \gamma(2|x|_{\mathcal{D}}) + \int_0^t \sigma(|d(s)|)ds + (\gamma(2c) + c_u) \ , \end{aligned}$$

which implies the UBEBS property with respect to the set $\mathcal{D}$. By virtue of Theorem 5, it is concluded that system (1) is iISS in the classical sense with respect to the set $\mathcal{D}$ and input $d(\cdot)$. □

Lemma 6: If system (1) satisfies the zero-GATT and UBEBS property then it also satisfies the BEWCS property.

Proof: (*UBEBS ⇒ BEWCS*) It has been shown in Lemma 2 that, under Assumption 1, the zero-GATT property yields the existence of a smooth function $V : M \to \mathbb{R}$, functions $\nu_1, \nu_2, \nu, \delta \in \mathcal{K}_\infty$, and a positive constant $c > 0$ such that inequalities (19)-(20) follow. Consider the functions α, γ, σ and the constant $c_u > 0$ as in the UBEBS estimate (4). Let $\tilde{\sigma} \in \mathcal{K}_\infty$ be defined as $\tilde{\sigma}(s) := \max\{\sigma(s), \delta(s)\}$. Consider a signal $d(\cdot)$ which satisfies the left-hand side of the implication (6). It then immediately follows that:

$$\int_0^{+\infty} \sigma(|d(s)|)\, ds = C_{\sigma,d} < +\infty\ ,$$
$$\int_0^{+\infty} \delta(|d(s)|)\, ds = C_{\delta,d} < +\infty, \tag{33}$$

for some positive constants $C_{\sigma,d}$ and $C_{\delta,d}$. Moreover, boundedness of trajectories for all $t \geq 0$ follows from the UBEBS property:

$$\begin{aligned} |X(t)|_{\mathcal{W}} &\leq \alpha^{-1}\left(\gamma(|x|_{\mathcal{W}}) + \int_0^t \sigma(|d(s)|)ds + c_u\right) \\ &\leq C_{x,d} < +\infty \end{aligned} \tag{34}$$

with $C_{x,d} := \alpha^{-1}(\gamma(|x|_{\mathcal{W}}) + C_{\sigma,d} + c_u)$ and where we have used the short-hand notation $X(t) := X(t, x; d)$. Due to boundedness of trajectories (34), estimate (20) can be rewritten as:

$$DV(x)f(x,d) \leq -\nu(|x|_{\mathcal{W}}) + \delta(C_{x,d})\delta(|d|), \tag{35}$$

thus the function $V(x)$ qualifies as an ISS Lyapunov function. We now assume by contradiction that:

$$\liminf_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} = \epsilon > 0, \tag{36}$$

for some positive constant ϵ, which implies the existence of time $T > 0$ such that

$$|X(t)|_{\mathcal{W}} \geq \frac{\epsilon}{2}, \tag{37}$$

for all $t \geq T$. Estimate (35) can be integrated over the time interval $[T, t]$ to obtain:

$$\begin{aligned} &V(X(t)) - V(X(T)) \leq \\ &\qquad -\int_T^t \nu(|X(s)|_{\mathcal{W}})\, ds + \delta(C_{x,d}) \int_T^t \delta(|d(s)|)\, ds \end{aligned} \tag{38}$$

Due to inequality (37) and to ν being a $\mathcal{K}_\infty$ function, it is easy to see that:

$$\int_T^{+\infty} \nu(|X(s)|_{\mathcal{W}})\, ds \geq \int_T^{+\infty} \nu\left(\frac{\epsilon}{2}\right) ds = +\infty. \tag{39}$$

The terms in estimate (38) can be rearranged in such a way to obtain:

$$\begin{aligned} \int_T^t \nu(|X(s)|_{\mathcal{W}})\, ds &\leq V(X(T)) + \delta(C_{x,d}) \int_T^t \delta(|d|)\, ds \\ &\leq \nu_2((|X(T)|_{\mathcal{W}}) + c + \delta(C_{x,d}) \int_T^t \delta(|d|)\, ds. \end{aligned} \tag{40}$$

The joint contribution of (33), (34), and (39) to estimate (40) yields the contradiction:

$$+\infty \leq \nu_2(C_{x,d}) + c + \delta(C_{x,d})\, C_{\delta,d} < +\infty.$$

□

Lemma 7: Let Assumption 1 hold. If system (1) satisfies the BEWCS property then it admits an iISS Lyapunov function.

Proof: The zero-GATT property of the set $\mathcal{W}$ ensures that the set $\mathcal{D}$ as defined in (17) is positively invariant, compact, and globally asymptotically stable for (3). Moreover, by taking the $\liminf$ on both sides of inequality (18), it follows:

$$\liminf_{t\to+\infty} |X(t)|_{\mathcal{D}} \leq \liminf_{t\to+\infty} |X(t)|_{\mathcal{W}} = 0, \tag{41}$$

By virtue of Theorem 7, the BEWCS property wrt the set $\mathcal{D}$ as in (41) together with zero-GAS of $\mathcal{D}$ ensures classical iISS of (1) wrt to $\mathcal{D}$. In virtue of Lemma 4, we can then conclude the existence of an iISS-Lyapunov function as in Definition 9.

□

Lemma 8: If system (1) satisfies the zero-GATT and UBEBS property then it also satisfies the BESCS property.

Proof: Due to zero-GATT, we consider a smooth function $V : M \to \mathbb{R}$, functions $\nu_1, \nu_2, \nu, \delta \in \mathcal{K}_\infty$, and a positive constant $c > 0$ such that inequalities (19)-(20) hold. Due to UBEBS, we consider functions α, γ, σ and the constant $c_u > 0$ as in (4). Consider the following function:

$$\tilde{\rho}(r,s) := \max_{|x|_{\mathcal{W}} \leq r, |d| \leq s} |f(x,d) - f(x,0)|.$$

Notice that $\tilde{\rho}(r,s)$ is continuous, nondecreasing with respect to each argument and vanishes for $s = 0$. Hence, it can be majorized by a function $\rho(r+1, s)$ separately of class $\mathcal{K}_\infty$. By [5, Corollary IV.5] (and the same proof applies to norm induced by the metric g), there exists $\rho_x, \rho_d \in \mathcal{K}_\infty$ such that:

$$\begin{aligned} \|f(x,d)\|_g &\leq \|f(x,d) - f(x,0)\|_g + \|f(x,0)\|_g \\ &\leq \rho_x(|x|_{\mathcal{W}} + 1)\rho_d(|d|) + \|f(x,0)\|_g. \end{aligned}$$

Let $\tilde{\sigma} \in \mathcal{K}_\infty$ be defined as $\tilde{\sigma}(s) := \max\{\sigma(s), \delta(s), \rho_d(s)\}$, Pick any initial condition $x \in M$ and any input signal $d(\cdot) \in \mathcal{M}_D$ such that:

$$\int_0^{+\infty} \tilde{\sigma}(|d(s)|)\, ds < +\infty. \tag{42}$$

We have proven in Lemma 6 that boundedness of trajectories holds for all $t \geq 0$, namely

$$|X(t,x;d)|_{\mathcal{W}} \in \mathcal{X}_{x,d} := \{x \in M \mid |x|_{\mathcal{W}} \leq C_{x,d}\}. \tag{43}$$

Thus estimate (20) can be rewritten as (39) and integrated over the time $[T, t]$ as in estimate (40). As a result, we obtain that:

- V is bounded from above along trajectories, and is bounded from below by zero as in (19);
- V cannot "increase too much", namely for all $\epsilon > 0$ there exists T_ϵ such that, for all $t > T_\epsilon$:

$$V(X(t)) \leq V(T_\epsilon) + \epsilon \tag{44}$$

- the following integral is finite:

$$\int_0^{+\infty} \nu(|X(s,x;d)|_{\mathcal{W}})\, ds < +\infty. \tag{45}$$

Moreover, we have proven in Lemma 6 that the BEWCS property holds, and therefore, for all $\varepsilon > 0$, there exists a diverging sequence of times $\{\underline{t}_{n,\varepsilon}\}_{n\in\mathbb{N}}$ such that $|X(\underline{t}_{n,\varepsilon}, x; d)|_{\mathcal{W}} \leq \varepsilon$.

We recall that the following holds true over the time interval $[t_0, t]$:

$$\mathfrak{d}\left(X(t), X(t_0)\right) \leq \int_{t_0}^{t} \left\| f\left(x(s), d(s)\right)\right\|_g ds. \quad (46)$$

Let $\bar{F}_0$ and $\bar{\rho}_x$ be defined as follows:

$$\bar{F}_0 := \max_{x \in \mathcal{X}_{x,d}} \left\| f\left(x, 0\right)\right\|_g \quad (47)$$

$$\bar{\rho}_x := \max_{x \in \mathcal{X}_{x,d}} \rho_x(|x|_{\mathcal{W}} + 1). \quad (48)$$

Estimate (46) can be rewritten using definitions (47) and (48) as:

$$\mathfrak{d}\left(X(t), X(t_0)\right) \leq \bar{F}_0(t - t_0) + \bar{\rho}_x \int_{t_0}^{t} \rho_d(|d(s)|)\, ds; \quad (49)$$

for all $[t_0, t]$.

We assume by contradiction that $\limsup_{t\to+\infty} |X(t, x; d)|_{\mathcal{W}} \neq 0$, namely that there exists $\bar{\varepsilon} > 0$ and a diverging sequence of times $\{\bar{t}_{n,\bar{\varepsilon}}\}_{n\in\mathbb{N}}$ such that:

$$|X(\bar{t}_{n,\bar{\varepsilon}}, x; d)|_{\mathcal{W}} > \bar{\varepsilon}. \quad (50)$$

By the BEWCS property, we can select $\varepsilon := \bar{\varepsilon}/2$ so as to obtain the sequence $\left\{\underline{t}_{n,\bar{\varepsilon}/2}\right\}_{n\in\mathbb{N}}$ such that:

$$|X(\underline{t}_{n,\bar{\varepsilon}/2}, x; d)|_{\mathcal{W}} \leq \bar{\varepsilon}/2. \quad (51)$$

We can therefore select a subsequence of $\left\{\underline{t}_{n,\bar{\varepsilon}/2}\right\}_{n\in\mathbb{N}}$, say $\{t_m\}_{m\in\mathbb{N}}$, such that, for all $m \in \mathbb{N}$, at least one element of $\{\bar{t}_{n,\bar{\varepsilon}}\}_{n\in\mathbb{N}}$ belongs to the interval $[t_m, t_{m+1}]$. In other words, we select the t_ms in such a way to obtain at least one "spike" of $|X(t, x; d)|_{\mathcal{W}}$ in the interval $[t_m, t_{m+1}]$. We then make the following definitions for all $m \in \mathbb{N}$. Let $t_{m,B}$ denote the first occurrence of $\bar{t}_{n,\bar{\varepsilon}}$ in the interval $[t_m, t_{m+1}]$, namely the time at which the first "spike" occurs in the interval $[t_m, t_{m+1}]$. Let $t_{m,A} \in [t_m, t_{m,B}]$ denote the last time that the state $X(t, x; d)$ leaves the set $\mathcal{P} := \{x \in M \mid |x|_{\mathcal{W}} \leq \bar{\varepsilon}/2\}$, and thus we have

$$|X(t, x; d)|_{\mathcal{W}} \geq \bar{\varepsilon}/2. \quad (52)$$

for all $t \in [t_{m,A}, t_{m,B}]$. The sequences in consideration are depicted in Figure 2.

Claim 1: $\lim_{m\to+\infty} (t_{m,B} - t_{m,A}) = 0$.

Proof: Since $t_{m,B} - t_{m,A} \geq 0$ by definition, it is enough to prove that $\limsup_{m\to+\infty} (t_{m,B} - t_{m,A}) = 0$. By contradiction, there exists a constant $\pi > 0$ and a diverging monotone subsequence $\{\tilde{m}_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $t_{\tilde{m}_n,B} - t_{\tilde{m}_n,A} > \pi$. By embedding (52), we can then write:

$$\begin{aligned} &\int_0^{+\infty} \nu\left(|X(s, x; d)|_{\mathcal{W}}\right)\, ds \\ &\quad \geq \sum_{n=1}^{+\infty} \int_{t_{\tilde{m}_n,A}}^{t_{\tilde{m}_n,B}} \nu\left(|X(s, x; d)|_{\mathcal{W}}\right)\, ds \\ &\quad \geq \sum_{n=1}^{+\infty} \int_{t_{\tilde{m}_n,A}}^{t_{\tilde{m}_n,B}} \nu(\frac{\bar{\varepsilon}}{2})\, ds \\ &\quad \geq \sum_{n=1}^{+\infty} \int_{t_{\tilde{m}_n,A}}^{t_{\tilde{m}_n,A}+\pi} \nu(\frac{\bar{\varepsilon}}{2})\, ds = +\infty \end{aligned} \quad (53)$$

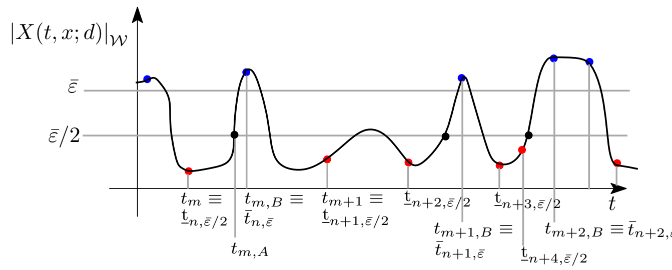

Fig. 2. "Spikes" in $|X(t, x; d)|_{\mathcal{W}}$. Points at $\{\bar{t}_{n,\bar{\varepsilon}}\}_{n\in\mathbb{N}}$ and $\left\{\underline{t}_{n,\bar{\varepsilon}/2}\right\}_{n\in\mathbb{N}}$ are depicted in blue and red, respectively. The t_m subsequence is highlighted on the t-axis.

which represents a contradiction with (45). □

Define $x_{m,A} = X(t_{m,A}, x; d)$ and $x_{m,B} = X(t_{m,B}, x; d)$. By definition, we have, for all $m \in \mathbb{N}$:

$$|x_{m,A}|_{\mathcal{W}} = \bar{\varepsilon}/2 \text{ and } |x_{m,B}|_{\mathcal{W}} \geq \bar{\varepsilon}. \quad (54)$$

Inequality (49) can then be rewritten as:

$$\mathfrak{d}\left(x_{m,B}, x_{m,A}\right) \leq \bar{F}_0(t_{m,B} - t_{m,A}) + \bar{\rho}_x \int_{t_{m,A}}^{t_{m,B}} \rho_d(|d(s)|)\, ds. \quad (55)$$

By virtue of Claim 1 and finiteness of the integral (42), it follows that

$$\lim_{m\to+\infty} \mathfrak{d}\left(x_{m,B}, x_{m,A}\right) = 0, \quad (56)$$

which implies

$$\lim_{m\to+\infty} |x_{m,A}|_{\mathcal{W}} = \lim_{m\to+\infty} |x_{m,B}|_{\mathcal{W}},$$

thus representing a contradiction with (54).

□

Lemma 9: The zero-GATT property and mixed estimate (11) imply the existence of an iISS-Lyapunov function.

Proof: We have already mentioned that the set $\mathcal{D}$ as defined in (17) is globally asymptotically stable for sytsem (3). Due to bounds (18), mixed estimate (11) can be rewritten in terms of the distance to the set $\mathcal{D}$, as follows:

$$\begin{aligned} \alpha(|X(t, x; d)|_{\mathcal{D}}) &\leq \beta(2|x|_{\mathcal{D}}) \\ &\quad + \int_0^t \sigma(|d(s)|)ds + \gamma\left(\left\|d_{[0,t]}\right\|\right) + \bar{c}_m, \end{aligned} \quad (57)$$

with $\bar{c}_m := c_m + \beta(2c)$.

Claim 2: Estimate (57) actually holds with $\bar{c}_m = 0$.

Proof: As shown in [5, Lemma IV.10], global asymptotic stability of $\mathcal{D}$ implies the existence of a smooth function $U : M \to \mathbb{R}$ and of $\mathcal{K}_\infty$ functions $\nu_1, \nu_2, \nu, \delta$ such that:

$$\nu_1(|x|_{\mathcal{D}}) \leq U(x) \leq \nu_2(|x|_{\mathcal{D}}) \quad (58)$$

$$DU(x)f(x, d) \leq -\nu(|x|_{\mathcal{D}}) + \delta(|x|_{\mathcal{D}})\delta(|d|). \quad (59)$$

By integrating inequality (59) over the time interval $[0,t]$ and combining it with (57) we obtain the following kind of estimate:

$$U(X(t,x;d)) - U(x) \leq \int_0^t \left[\delta(|d(s)|) \cdot (\delta\circ\alpha^{-1})\Big(\beta(2|x|_{\mathcal{D}}) + \gamma(\|d\|) + \int_0^s \sigma(|d(s')|)\,ds' + \bar{c}_m\Big)\,ds\right] \leq$$
$$\leq \left[\int_0^t \delta(|d(s)|)\,ds\right] \cdot \cdot \left[\tilde{\beta}(|x|_{\mathcal{D}}) + \tilde{\gamma}_A(\|d\|) + \tilde{\gamma}_B\left(\int_0^t \sigma(|d(s)|)\,ds\right) + \tilde{c}_m\right], \quad (60)$$

with $\tilde{\beta}(r) := (\delta\circ\alpha^{-1})(4\beta(2r))$, $\tilde{\gamma}_A(r) := (\delta\circ\alpha^{-1})(4\gamma(r))$, $\tilde{\gamma}_B(r) := (\delta\circ\alpha^{-1})(4r)$, and $\tilde{c}_m := (\delta\circ\alpha^{-1})(4\bar{c}_m)$, for all $r\geq 0$. Define $\kappa(r) := \max\{\delta(r),\kappa(r)\}$ and $v(r) := r^2 + \tilde{\gamma}_B(r) + \tilde{c}_m r$ for all $r\geq 0$. By recalling that $AB \leq A^2+B^2$ for all non-negative constants A, B, estimate (60) reads as:

$$U(X(t,x;d)) - U(x) \leq \tilde{\beta}(|x|_{\mathcal{D}})^2 + v\left(\int_0^t \kappa(|d(s)|)\,ds\right) + \tilde{\gamma}_A(\|d\|)^2 + \left(\int_0^t \kappa(|d(s)|)\,ds\right)^2. \quad (61)$$

Set $\alpha(r) := \nu_1(r)$, $\hat{\beta}(r) := \nu_2(r) + \tilde{\beta}(r)^2$, $\hat{\gamma}(r) := \tilde{\gamma}_A(r)^2$, and $\hat{\chi}(r) := \chi(r) + r^2$ for all $r\geq 0$. Then, estimate (61) is rewritten as:

$$\hat{\alpha}(|X(t,x;d)|_{\mathcal{D}}) \leq \hat{\beta}(|x|_{\mathcal{D}}) + \hat{\chi}\left(\int_0^t \sigma(|d(s)|)ds\right) + \hat{\gamma}\left(\|d\|\right). \quad (62)$$

By evaluating both sides of inequality (62) with function $\hat{\chi}(\frac{1}{3}\,\cdot)$, we conclude that indeed an estimate of type (57) holds with $\bar{c}_m = 0$. □

Since such an estimate exists with respect to the zero-GAS set $\mathcal{D}$, we can invoke the results in [6, Theorem 1] to conclude that system (1) is iISS in the classical sense with respect to $\mathcal{D}$. By hypothesis, (1) satisfies the zero-GATT property with respect to $\mathcal{W}$. By virtue of Lemma 4, we conclude the existence of an iISS Lyapunov function as in Definition 9. □

Lemma 10: If there exists an iISS-Lyapunov function V, then system (1) satisfies estimate (13).

Proof: The existence of an iISS function V implies that inequalities (8) and (9) hold. The UBEBS property (4) is also implied by (9) as in Lemma 1 and can be equivalently formulated as:

$$|X(t,x;d)|_{\mathcal{W}} \leq \kappa_1(|x|_{\mathcal{W}}) + \kappa_2\left(\int_0^t \gamma\left(|d(s)|\,ds\right)\right) + \kappa_3, \quad (63)$$

for $\kappa_1(\cdot) := (\alpha_1^{-1}\circ 3\alpha_2)(2\cdot)$, $\kappa_2(\cdot) := \alpha_1^{-1}(3\cdot)$, and $\kappa_3 := (\alpha_1^{-1}\circ 3\alpha_2)(2c)$. We can then integrate the dissipation inequality (9) over the time interval $[0,t]$ and combine it with (63) so as to obtain:

$$V\left(X(t,x;d)\right) - V(x) \leq \int_0^t \gamma(|d(s)|)\,ds - \left(\int_0^t \alpha_{31}(|X(s,x;d)|_{\mathcal{W}})\,ds\right) \cdot \cdot\, \alpha_{32}\left(\kappa_1(|x|_{\mathcal{W}}) + \kappa_2\left(\int_0^t \gamma(|d(s)|)\,ds\right) + \kappa_3\right), \quad (64)$$

where we have made use of [5, Lemma IV.1] for the definition of $\alpha_{31}\in\mathcal{K}_\infty$ and $\alpha_{32}\in\mathcal{L}$ such that $\alpha_3 \geq \alpha_{31}(r)\,\alpha_{32}(r)$ for all $r\geq 0$. We introduce the following definitions:

$$\begin{aligned} \eta(r) &:= \frac{1}{\alpha_{32}(r)} - \frac{1}{\alpha_{32}(0)} \\ \chi_1(r) &:= \max\{(\eta\circ 3\kappa_1)(r)\,,\ \alpha_2(r)\} \\ \chi_2(r) &:= \max\{(\eta\circ 3\kappa_2)(r)\,,\ r\} \\ \chi_3 &:= \eta(3\kappa_3) \\ q &:= \frac{1}{\alpha_{32}(0)} + \chi_3\ ,\bar{c} := \alpha_2(2c). \end{aligned} \quad (65)$$

It follows from (64) and definitions (65) that:

$$\int_0^t \alpha_{31}(|X(t,x;d)|_{\mathcal{W}}) \leq \leq \left[\frac{1}{\alpha_{32}(0)} + \eta\left(\kappa_1(|x|_{\mathcal{W}}) + \kappa_2\left(\int_0^t \gamma(|d(s)|)\,ds\right) + \kappa_3\right)\right] \cdot \cdot \left[V(x) + \int_0^t \gamma(|d(s)|\,ds)\right]. \quad (66)$$

The right-hand side of (66) is majorized by

$$\left[q + \chi_1(|x|_{\mathcal{W}}) + \chi_2\left(\int_0^t \gamma(|d(s)|)\,ds\right)\right] \cdot \cdot \left[\chi_1(|x|_{\mathcal{W}}) + \bar{c} + \chi_2\left(\int_0^t \gamma(|d(s)|\,ds)\right)\right]$$
$$\leq q\bar{c} + (q+\bar{c})\,\chi_1(|x|_{\mathcal{W}}) + (q+\bar{c})\,\chi_2\left(\int_0^t \gamma(|d(s)|\,ds)\right) + \left(\chi_1(|x|_{\mathcal{W}}) + \chi_2\left(\int_0^t \gamma(|d(s)|\,ds)\right)\right)^2,$$

Set $\bar{\chi}_1(r) := (q+\bar{c})\,\chi_1(r) + \chi_1(r)^2$ and $\bar{\chi}_2(r) := (q+\bar{c})\,\chi_2(r) + \chi_2(r)^2$ for all $r\geq 0$. We obtain:

$$\int_0^t \alpha_{31}(|X(t,x;d)|_{\mathcal{W}}) \leq q\bar{c} + \bar{\chi}_1(|x|_{\mathcal{W}}) + \bar{\chi}_2\left(\int_0^t \gamma(|d(s)|\,ds)\right),$$

which indeed represents an estimate of type (13) by setting $\alpha := \alpha_{31}$, $\sigma := \gamma$, $\chi_0 = 1$, and $\chi(r) := 3\max\{q\bar{c}r, \tilde{\chi}_1(r), \tilde{\chi}_2(r)\}$. □

Lemma 11: If system (1) is forward complete and satisfies an estimate of type (13), then it satisfies the zero-GATT and UBEBS properties.

Proof: Step 1: UBEBS. Assume estimate (13) holds with $\alpha,\chi,\sigma\in\mathcal{K}_\infty$ and $\chi_0 > 0$. Forward completeness of (1) (see [7, Corollary 2.3]) translates into estimate:

$$|X(t,x;d)| \leq \kappa_1(t) + \kappa_2(|x|) + \kappa_3\left(\int_0^t \gamma(|d(s)|)\,ds\right) + \kappa_4, \quad (67)$$

with $\kappa_1, \kappa_2, \kappa_3, \gamma \in \mathcal{K}_\infty$ and $\kappa_4 > 0$. By embedding bounds

$$|x| \le \nu_3(|x|_\mathcal{W}) + g_3$$
$$|x|_\mathcal{W} \le \nu_4(|x|) + g_4,$$

estimate (67) reads as:

$$|X(t,x;d)|_\mathcal{W} \le \tilde\kappa_1(t) + \tilde\kappa_2(|x|) + \tilde\kappa_3\left(\int_0^t \gamma(|d(s)|)\,ds\right) + \tilde\kappa_4, \quad (68)$$

with $\tilde\kappa_1 := \nu_4 \circ 4\kappa_1$, $\tilde\kappa_2 := \nu_4 \circ 4\kappa_2 \circ 2\nu_3$, $\tilde\kappa_3 := \nu_4 \circ 4\kappa_3$, and $\tilde\kappa_4 := \nu_4(4\kappa_4) + \nu_4(4\kappa_2(2g_3)) + g_4$. Without loss of generality, we can take $\tilde\kappa_2$ such that $\tilde\kappa_2(r) \ge r$ for all $r \ge 0$. Set $\tilde\sigma := \max\{\gamma, \sigma\}$. We also define, for each $r \ge 0$:

$$m(r) := \sup\{|X(t,x;d)|_\mathcal{W} : t \ge 0, |x|_\mathcal{W} \le r, d \in \mathcal{E}_r\}. \quad (69)$$

with $\mathcal{E}_r := \left\{d(\cdot) \in \mathcal{M}_D : \int_0^{+\infty} \tilde\sigma(|d(s)|)\,ds \le r\right\}$. Note that m is a nondecreasing function. The main technical step is in showing the following

Claim 3: For each $r \ge 0$,

$$m(r) \le \tilde\kappa_1\left(\frac{\chi(2r+\chi_0)}{\alpha(r)}\right) + \tilde\kappa_2(r) + \tilde\kappa_3(r) + \tilde\kappa_4 =: M(r).$$

Proof: Note that $M(r) \ge r$ for all $r \ge 0$ due to our choice of $\tilde\kappa_2$. Pick any $r > 0$, then pick any state x and input $d(\cdot)$ such that $|x|_\mathcal{W} \le r$ and $d \in \mathcal{E}_r$. We need to show that, for all $t \ge 0$, $|X(t,x;d)|_\mathcal{W} \le M(r)$. By contradiction, select $T > 0$ so that $|X(T,x;d)|_\mathcal{W} > M(r)$. Let

$$\tau := \sup\{t \le T : |X(t,x;d)|_\mathcal{W} \le r\}.$$

Clearly, $|X(t,x;d)|_\mathcal{W} \ge r$ for all $t \in [\tau, T]$. It follows from (13) that:

$$\alpha(r)(T-\tau) \le \int_0^T \alpha(|X(t,x;d)|_\mathcal{W})\,ds \le \chi(2r+\chi_0). \quad (70)$$

Consider now $\tilde x := X(\tau,x;d)$, $\tilde d(\cdot) = d(\cdot - \tau)$, and $\tilde T := T - \tau$. We then have:

$$|X(T,x;d)|_\mathcal{W} = |X(\tilde T, \tilde x; \tilde d|_\mathcal{W}$$
$$\le \tilde\kappa_1(\tilde T) + \tilde\kappa_2(r) + \tilde\kappa_3(r) + \tilde\kappa_4 = M(r),$$

which represents a contradiction. □

We can now pick any state x and input $d(\cdot)$, and let

$$r := \max\left\{|x|_\mathcal{W}, \int_0^{+\infty} \tilde\sigma(|d(s)|)\,ds\right\}.$$

By definition of m,

$$|X(t,x;d)|_\mathcal{W} \le m(r) \le \beta(r + c_u), \quad (71)$$

for all $t \ge 0$, where β is any $\mathcal{K}_\infty$ function and c_{ubebs} is any constant such that $m(r) \le \beta(r + c_u)$ for all $r \ge 0$. With $\alpha_u = \beta^{-1}$ we conclude the UBEBS estimate

$$\alpha_u(|X(t,x;d)|_\mathcal{W}) \le |x|_\mathcal{W} + \int_0^t \tilde\sigma(|d(s)|)\,ds + c_u \quad (72)$$

as wanted.

Step 2: zero-GATT. Set $d(t) \equiv 0$ for all $t \ge 0$. Then estimate (13) reads as

$$\int_0^t \alpha(|X(s,x;0)|_\mathcal{W})\,ds \le \chi(|x|_\mathcal{W} + \chi_0). \quad (73)$$

We claim that estimate (73) implies weak convergence of the trajectories to $\mathcal{W}$, namely $\liminf_{t\to+\infty} |X(t,x;0)|_\mathcal{W} = 0$. Indeed, by contradiction there exists some positive constant $\epsilon > 0$ and a time $T > 0$ such that $|X(t,x;0)|_\mathcal{W} \ge \epsilon$ for all $t \ge T$. It then follows that:

$$\int_0^\infty \alpha(|X(s,x;0)|_\mathcal{W})\,ds \ge \int_T^\infty \alpha(\epsilon)\,ds = +\infty,$$

which represents a contradiction with (73). Furthermore, we claim that estimate (73) implies strong convergence of the trajectories to $\mathcal{W}$, namely $\limsup_{t\to+\infty} |X(t,x;0)|_\mathcal{W} = 0$. Indeed, the proof of the claim follows by contradiction along the lines of the proof of Lemma 8 by simply setting $d(\cdot) \equiv 0$ and by noting that the UBEBS property (72) implies boundedness of trajectories, namely $|X(t,x;0)|_\mathcal{W} \in \mathcal{X}_x$ with $\mathcal{X}_x := \{y \in M : \alpha_u(|y|_\mathcal{W}) \le |x|_\mathcal{W} + c_u\}$, and hence we can set $\bar F_0 := \max_{y \in \mathcal{X}_x} |f(y,0)|$. □

IV. Link with Strong iISS

In order to characterize robustness properties which can be considered halfway between ISS and iISS, an intermediate property was introduced in [9] and denoted as Strong iISS. The definition in [9] is directly extended to systems with multiple invariant sets as in the following

Definition 12 (Strong iISS): System (1) is said to be *Strongly iISS* if it has the properties:

- zero-GATT and UBEBS;
- asymptotic gain (AG) with respect to small inputs, namely there is a function $\eta \in \mathcal{K}$ and a positive constant R such that:

$$\|d\| \le R \;\Rightarrow\; \limsup_{t\to+\infty} |X(t,x;d)|_\mathcal{W} \le \kappa(\|d\|), \quad (74)$$

  for all $x \in M$ and all $d(\cdot) \in \mathcal{M}_D$.

The following Theorems 2 and 3 represent sufficient conditions for verifying the Strong iISS property and are adapted from [9].

Theorem 2: Assume that there exists a proper $\mathcal{C}^1$ function $V : M \to \mathbb{R}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}$, a continuous positive-definite function γ, and a positive constant c such that (8) holds together with the following dissipation in equality:

$$DV(x)f(x,d) \le -\alpha(|x|_\mathcal{W}) + \gamma(|d|), \quad (75)$$

for all $(x,d) \in M \times D$. Then the system (1) is Strongly iISS.

Proof: Zero-GATT and UBEBS properties are estabilished by dissipation inequality (75) as shown in Theorem 1. We are now going to prove the AG property with respect to small inputs. Pick R such that $\gamma(R) \le \frac{1}{4}\alpha(+\infty)$. By definition of R, α is invertible in $[0, 2\gamma(R)]$. Then, the following implication holds true:

$$|x|_\mathcal{W} \ge \alpha^{-1}(2\gamma(|d|)) \Rightarrow \dot V \le -\frac{\alpha(|x|_\mathcal{W})}{2}$$

for all $d \leq R$. Assume that $R \geq \|d\|$. Define $r := \|d\|$, $\mu := \alpha^{-1}(2\gamma(r))$, $\eta := \alpha_2(\mu+c)$, and the set $\Omega_\eta := \{x|V(x) \leq \eta\}$. For a point x not in of Ω_η or at the boundary $\eth\Omega_\eta$, it holds that:

$$\begin{aligned} V(x) \geq \alpha_2(\mu+c) \quad &\Rightarrow \quad |x|_{\mathcal{W}} \geq \mu \\ &\Rightarrow \quad \dot{V} \leq -\frac{\alpha(|x|_{\mathcal{W}})}{2} \leq -\frac{\alpha(\mu)}{2} \end{aligned} \quad (76)$$

Integration of (76) along a trajectory $X(t,x;d)$ yields:

$$V(X(t,x;d)) < V(x) - \frac{\alpha(\mu)}{2}\, t$$

therefore the solution enters the set Ω_η in a finite time T and never leaves it again. For a point $z \in \Omega_\eta$ we can write $\alpha_1(|z|_{\mathcal{W}}) \leq V(z) \leq \eta$, thus leading to the estimate

$$|X(t,z;d)|_{\mathcal{W}} \leq \kappa(\|d\|) \ .$$

for all $t \geq T$, with $\kappa(s) := \alpha^{-1}\left(\alpha_2\left(\alpha^{-1}(2\gamma(s))\right)\right)$. □

Theorem 3: Assume that there exists a proper $\mathcal{C}^1$ function $V : M \to \mathbb{R}$, functions $\alpha_1, \alpha_2, \eta \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$, a continuous positive-definite function ρ, and a positive constant c such that (8) holds together with the following dissipation in equality:

$$DV(x)f(x,d) \leq -\rho(|x|_{\mathcal{W}}) + \eta\left(\max\{0; |d| - \kappa(|x|_{\mathcal{W}})\}\right) \ , \quad (77)$$

for all $(x,d) \in M \times D$. Then the system (1) is Strongly iISS.

Proof: Notice that (77) can be equivalently written in the following decomposed form:

$$\dot{V} \leq -\rho(|x|_{\mathcal{W}}) + \eta(|d|) \quad (78)$$

$$|d| \leq \kappa(|x|) \Rightarrow \dot{V} \leq \rho(|x|_{\mathcal{W}}) \ . \quad (79)$$

Inequality (78) yields the UBEBS and zero-GATT properties. For the AG property wrt small inputs, pick R such that $R < \kappa(\infty)$. Then, for all $|d| \leq R$, it holds that:

$$|x|_{\mathcal{W}} \geq \kappa^{-1}(|d|) \Rightarrow \dot{V} \leq \rho(|x|_{\mathcal{W}}).$$

Following the trace provided in the proof of Theorem 2, we conclude that (1) enjoys the AG property wrt small inputs. □

V. Examples

A. Pendulum with vanishing friction

Consider the following set of differential equations, describing the motion of a 1-link actuated pendulum:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -a\sin(\theta) - \frac{b\omega}{1+\omega^2} + d \ , \end{aligned} \quad (80)$$

where the state $x := (\theta,\omega)^\top$ lies on the cylinder $M := \mathbb{S} \times \mathbb{R}$ and the disturbance is denoted by $d(t)$. The unperturbed system admits two equlibria $[0,0]$ and $[\pi,0]$, the latter being a saddle-point. The compact invariant set $\mathcal{W} = \{[0,0],[\pi,0]\}$ satisfies Assumption 1 because of the following. Consider the mechanical energy of the pendulum, that is $E(x) = \omega^2/2 - a\cos(\theta) + a$. For the case $d \equiv 0$ the dissipation inequality $\dot{E} = -b\omega^2/\left(\omega^2+1\right) \leq 0$ holds, which, together with the fact that there is no trajectory on the line $\omega = 0$ connecting the equilibria, implies Assumption 1. Notice that $|x|_{\mathcal{W}} = \sqrt{\omega^2 + \min\{|\theta|, |\theta-\pi|\}^2}$ and $\varepsilon|x|^2_{\mathcal{W}} \leq E(x)$ for some sufficiently small $\varepsilon > 0$.

Consider the following Lyapunov function:

$$V(x) = \ln\left(E(x)+1\right) \ , \quad (81)$$

which is well-defined since $E(x) \geq 0$. It then follows that:

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \ , \quad (82)$$

with $\alpha_1(s) := \ln\left(\varepsilon s^2 + 1\right)$ being a $\mathcal{K}_\infty$ function. Taking derivatives of V along the solutions of (80) yields:

$$\begin{aligned} \dot{V}(x) &= \frac{\dot{E}(x)}{E(x)+1} \\ &= \frac{\omega\left(-\frac{b\omega}{1+\omega^2} - a\sin\theta + d\right) + a\sin\theta\,\omega}{E(x)+1} \\ &= -\underbrace{\frac{b\,\omega^2}{(\omega^2+1)\,(E(x)+1)}}_{h(x)} + \underbrace{\frac{\omega}{E(x)+1}}_{\leq 1}\, d \\ &\leq -\alpha_4(|h(x)|) + |d| \ , \end{aligned} \quad (83)$$

with α_4 being the identity function and $h(x)$ being the output function as defined in (83). The largest invariant set with $h(x) \equiv 0$ is $\mathcal{W}$, thus weak zero-detectability holds. Inequality (83) shows smooth dissipativity. Therefore, by means of Theorem 1, system (80) has the zero-GATT and UBEBS properties and possesses a smooth iISS-Lyapunov function.

We now prove that system (80) is not Strong iISS. Specifically we will prove that there is no input threshold R (as in Definition 12) below which the system fulfills the asymptotic gain (AG) property. To this end, it is enough to show that for an arbitrarily bounded input $d(\cdot)$ there exists an initial condition $x \in M$ such that the norm of the state grows indefinitely, namely $\limsup_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} = +\infty$. Consider the input:

$$d := \frac{b\omega}{1+\omega^2} + h\ \mathsf{sat}\{\omega\} \leq b + h, \quad (84)$$

with $h > 0$. We are now going to find the initial condition $\omega(0)$ and the constant h such that $|d(t)|$ has an arbitrary upper bound for all $t \geq 0$ and $\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = +\infty$. Observe that, with input (84), the energy $E(t)$ can never decrease because

$$\dot{E}(t) = h\,\omega\ \mathsf{sat}\{\omega\} \geq 0. \quad (85)$$

Assume that the energy at time $t = 0$ is large enough, i.e. $E(0) \gg 2a$.

Claim 4: $\lim_{t\to+\infty} E(t) = +\infty$.

Proof: Since $\dot{E}(t) \geq 0$ for all $t \geq 0$, we can assume by contradiction that $\lim_{t\to+\infty} E(t) = \bar{E} < +\infty$. By virtue of Barbalat's lemma, it holds that $\lim_{t\to+\infty} \dot{E}(t) = 0$. From (85), it follows that $\lim_{t\to+\infty} \omega(t) = 0$. Then, for all $\varepsilon > 0$ arbitrarily small, there exists $T_\varepsilon > 0$ such that $|\omega(t)| \leq \varepsilon$, for all $t \geq T_\varepsilon$. It thus follows that $E(t) = 0.5\omega(t)^2 + a - a\cos(\theta(t)) \leq 0.5\varepsilon^2 + 2a$ for all $t \geq T_\varepsilon$, which represents a contradiction with assumption $E(0) \gg 2a$ and $\dot{E} \geq 0$. □

Since $E(\theta,\omega) := 0.5\omega^2 + a - a\cos(\theta) \leq 0.5\omega^2 + 2a$, Claim 4 implies $\lim_{t\to+\infty} \omega(t) = +\infty$, and thus $\lim_{t\to+\infty} |X(t)|_{\mathcal{W}} = +\infty$. We have just proved that input (84) yields no AG property. We are now going to prove that input (84) can be made arbitrarily small and still $\lim_{t\to+\infty}|X(t)|_{\mathcal{W}} = +\infty$. Indeed, pick $\bar{d} > 0$ arbitrarily small. Select $\bar{\omega} > 0$ and $\bar{h} > 0$ so that $\frac{b\omega}{1+\omega^2} + \bar{h}\ \mathrm{sat}\{\omega\} \le \bar{d}$. By choosing the initial conditions so that $E(0) \gg 2a$, we have shown that $\lim_{t\to+\infty}\omega(t) = +\infty$, and therefore there exists $\bar{T} > 0$ such that $\omega(t) \ge \bar{\omega}$ for all $t \ge \bar{T}$. If we now consider $(\theta(\bar{T}), \omega(\bar{T}))$ as the new initial conditions and $h := \bar{h}$ in (84), then we would still have $\lim_{t\to+\infty}|X(\bar{T}+t)|_{\mathcal{W}} = +\infty$ together with $\|d(\bar{T}+\cdot)\| \le \bar{d}$.

B. Duffing system with vanishing friction

For the Duffing system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{bx_2}{1+x_2^2} + x_1 - x_1^3 + d \ , \end{aligned} \tag{86}$$

the set $\mathcal{W}$ is composed by three equilibria, namely $\mathcal{W} = \{(0,0), (-1,0), (0,1)\}$, among which the origin is a saddle point whereas the remaining two equilibria are locally asymptotically stable. Following the same arguments as given for the pendulum case, Assumption 1 holds true. The set-point distance is then defined as $|x|_{\mathcal{W}} = \sqrt{\min\{|x_1|, |x_1-1|, |x_1+1|\}^2 + x_2^2}$. The mechanical energy of the Duffing system

$$E(x) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{4}$$

satisfies the relation $\frac{1}{2}|x|^2_{\mathcal{W}} \le E(x)$. Consider the Lyapunov function $V(x) = \ln(E(x)+1)$, which satisfies the relation $\alpha_1(|x|_{\mathcal{W}}) \le V(x)$ with $\alpha_1(s) := \ln\left(\frac{1}{2}s^2+1\right)$. Taking derivatives of V along the solutions of (86) yields:

$$\begin{aligned} \dot{V}(x) &= -\underbrace{\frac{b\,x_2^2}{(x_2^2+1)\,(E(x)+1)}}_{h(x)} + \underbrace{\frac{x_2}{E(x)+1}}_{\le 1}\,d \\ &\le -\alpha_4(|h(x)|) + |d| \ , \end{aligned} \tag{87}$$

with α_4 being the identity function and $h(x)$ as defined in (87). The largest invariant set with $h(x) \equiv 0$ is $\mathcal{W}$, thus weak zero-detectability holds. Inequality (87) shows smooth dissipativity. Therefore, by means of Theorem 1, system (86) fulfills the zero-GATT and UBEBS properties and possesses a smooth iISS-Lyapunov function.

We now prove that system (86) is not ISS (in the sense of [2]), as previously shown for the pendulum example. Consider $d = \frac{bx_2}{1+x_2^2} + h\mathrm{sat}\{x_2\} \le b + h$, with $h > 0$. The closed-loop system reads as:

$$\dot{x}_1 = x_2 \tag{88}$$
$$\dot{x}_2 = x_1 - x_1^3 + h\,\mathrm{sat}\{x_2\} \ . \tag{89}$$

The rate of change of the energy $E(x)$ is given as:

$$\dot{E}(x) = x_2\,\mathrm{sat}\{x_2\} \ge 0 \tag{90}$$

for all $t \ge 0$. Consider an initial condition $x \in \mathbb{R}^2$. For the sake of readability, we will make use of the notation $X(t) = X(t,x;d)$ and $X_i(t) = X_i(t,x;d)$ with $i = 1,2$. To the end of showing that $\limsup_{t\to+\infty}|X(t)|_{\mathcal{W}} = +\infty$, we assume by contradiction that this limit superior is bounded by a finite value. Hence, we assume that $\limsup_{t\to+\infty}E(X(t)) = \bar{E} < +\infty$. Due to monotonicity in (90), it actually holds that $\lim_{t\to+\infty}E(X(t)) = \bar{E} < +\infty$. By applying Barbalat's lemma twice, it follows that:

$$\begin{aligned} \lim_{t\to+\infty}\dot{E}(X(t)) &= 0 \\ \Longrightarrow \lim_{t\to+\infty}X_2(t) &= 0 \\ \lim_{t\to+\infty}\dot{X}_2(t) &= 0 \ . \end{aligned} \tag{91}$$

Pick any $\varepsilon > 0$. The limits in (91) imply the existence of a M_ε such that for all $t > M_\varepsilon$ it holds:

$$|X_2(t)| < \varepsilon, \ \ |\dot{X}_2(t)| < \varepsilon, \tag{92}$$

and thus $\frac{1}{2}|X_2(t)|^2 < \frac{1}{2}\varepsilon^2$ for all $t > M_\varepsilon$. Using the previous relation and the fact that the energy $E(\cdot)$ can never decrease along the trajectories, it follows:

$$E(X(t)) - \frac{1}{2}\varepsilon^2 < \frac{1}{4}X_1(t)^4 + \frac{1}{4} - \frac{1}{2}X_1(t)^2, \tag{93}$$

for all $t > M_\varepsilon$. As a further consequence of equations (89) and (92), it holds for all $t > M_\varepsilon$ that

$$|-X_1(t)^3 + X_1(t)| < 2\varepsilon. \tag{94}$$

It is possible to show that, by selecting proper initial conditions yielding large values of $E(X(0))$, there always exists $\varepsilon > 0$ sufficiently small such that inequality (93) contradicts inequality (94). It can be concluded that system (86) does not have the AG property and thus it is not ISS (in the sense of [2]). Unfortunately, we could not show that input $d = \frac{bx_2}{1+x_2^2} + h\,\mathrm{sat}\{x_2\} \le b+h$ can be made arbitrarily small, and thus we could not prove nor disprove Strong iISS of the Duffing system (88)-(89).

C. Tracking velocity fields in robotic manipulators

Consider the general equations of motion for a robotic manipulator:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau + d \tag{95}$$

with $q, \dot{q}, \ddot{q}$ the joint angles, velocities and accelerations, $M(q)$ the inertia matrix, $C(q,\dot{q})$ the Coriolis matrix, $G(q)$ the potential vector field, τ the vector of all available control torques, and d disturbances occurring in the joint dynamics. Suppose that it is desired for the robotic manipulator to track a reference velocity field with desired reference dynamics:

$$\dot{q} = f_d(q) \ . \tag{96}$$

Let $\mathcal{W}_q$ denote the set of all invariant solutions of (96). Let $\mathcal{W}$ denote the following set:

$$\mathcal{W} = \left\{ \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \in \mathbb{S}^2 \times \mathbb{R}^2 \mid q \in \mathcal{W}_q\,,\ \dot{q} = f_d(q) \right\} \ . \tag{97}$$

Suppose that for the desired reference dynamics (96) an iISS-Lyapunov function $\mu(q_d)$ exists, namely there exists $\mathcal{K}_\infty$

functions μ_1, μ_2, a continuous positive-definite function μ_3 and a positive constant c_q such that the following holds:

$$\begin{aligned}\mu_1(|q_d|_{\mathcal{W}_q}) \leq \mu(q_d) \leq \mu_2(|q_d|_{\mathcal{W}_q} + c_q)\\ D\mu(q_d) f_d(q_d) \leq -\mu_3(|q_d|_{\mathcal{W}_q}) \ .\end{aligned}$$

Let $F(q, \dot{q}) := \dot{q} - f_d(q)$. Consider the control law:

$$\tau = -KF(q,\dot{q}) - D\mu(q) + M(q)\left(Df_d(q)\dot{q}\right) + C(q,\dot{q})f_d(q) \ , \tag{98}$$

with a positive-definite gain matrix K. We want to prove that the closed-loop system obtained by applying the control law (98) to the plant (95), namely

$$M(q)\dot{F}(q,\dot{q},\ddot{q}) + C(q,\dot{q})F(q,\dot{q}) + KF(q,\dot{q}) + D\mu(q) = d \ , \tag{99}$$

has the UBEBS and zero-GATT properties with respect to the invariant set $\mathcal{W}$ and the disturbance d. To this end, let $V(q, \dot{q})$ be the following Lyapunov function:

$$V(q,\dot{q}) = \mu(q) + \frac{1}{2}F(q,\dot{q})^\top M(q) F(q,\dot{q}). \tag{100}$$

Let the state x be defined as $x := (q^\top, \dot{q}^\top)^\top$. The distance from the invariant set $\mathcal{W}$ is then given by:

$$|x|_{\mathcal{W}} := \sqrt{|q|^2_{\mathcal{W}_q} + |F(q,\dot{q})|^2} \ . \tag{101}$$

Due to $M(q)$ being the inertia matrix of a mechanical system, we can conclude that there exists functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive constant c such that:

$$\begin{aligned}&\alpha_1(|x|_{\mathcal{W}}) \leq \mu_1(|q_d|_{\mathcal{W}_q}) + \underline{\varepsilon}|F|^2 \leq V(q,\dot{q})\\ &\leq \mu_2(|q_d|_{\mathcal{W}_q} + c_q) + \overline{\varepsilon}|F|^2 \leq \alpha_2(|x|_{\mathcal{W}} + c) \ .\end{aligned}$$

Taking derivatives of V along the solutions of (99) yields:

$$\begin{aligned}\dot{V}(q,\dot{q}) &= D\mu(q)\dot{q} - F(q,\dot{q})^\top KF(q,\dot{q}) + F(q,\dot{q})^\top d\\ &\leq -\mu_3(|q_d|_{\mathcal{W}_q}) - c_1|F|^2 + c_2|d|^2 \ ,\end{aligned} \tag{102}$$

for sufficiently small positive constant c_1 and sufficiently large positive constant c_2. By setting $\alpha_3(|x|_{\mathcal{W}}) = \mu_3(|q|_{\mathcal{W}_q}) + c_1|F|^2$, we can conclude that $V(q, \dot{q})$ is an iISS-Lyapunov function for system (99), therefore system (99) has the UBEBS and zero-GATT properties. Note that, in case of $\mu_3 \in \mathcal{K}_\infty$ (respectively $\mu_3 \in \mathcal{K} - \mathcal{K}_\infty$), system (99) would be ISS (respectively Strongly iISS, as in Theorem 2).

As an example of a robotic manipulator tracking a velocity vector field, we consider the following desired reference dynamics for a 2-link manipulator (see Figure 3 for its phase plot):

$$f_d(q_d) = \begin{pmatrix} q_{d1}\\ q_{d2}\end{pmatrix} S(q_d) + \begin{pmatrix} q_{d2}\\ -q_{d1}\end{pmatrix} \ , \tag{103}$$

with $S(q_d) := 1 - q_{d1}^2 - q_{d2}^2$. In the example, the set of all invariant solutions for (96) is $\mathcal{W}_q = \{(0,0), \Gamma_q\}$ with the origin $(0,0)$ being an unstable equilibrium and $\Gamma_q = \left\{q_d \in \mathbb{S}^2 \mid |q_d| = 1\right\}$ being an asymptotically stable limit cycle. Let $\mu(q_d)$ be defined as $\mu(q_d) := \frac{1}{4}S(q_d)^2$. By defining

$$|q_d|_{\mathcal{W}_q} := \min\{|q_d|, |1 - |q_d||\} \ , \tag{104}$$

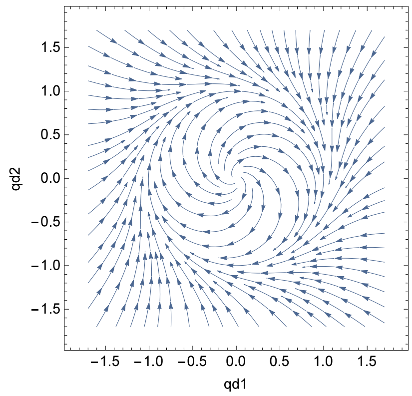

Fig. 3. Phase plot of the reference dynamics (103).

it holds that $\mu_1(|q_d|_{\mathcal{W}_q}) \leq \mu(q_d)$ with $\mu_1(r) := \frac{1}{2}r^2$. Taking derivatives of $\mu(q_d)$ along the solutions of (103) yields:

$$D\mu(q_d) f_d(q_d) = -S(q_d)^2|q_d|^2 =: -\mu_3(|q_d|_{\mathcal{W}_q}) \ . \tag{105}$$

By applying all previous considerations, we conclude that the 2-link manipulator with control law (98) has the UBEBS and zero-GATT properties with respect to disturbance d and the invariant set $\mathcal{W}$ as defined in (97). Specifically in this example, since μ_3 is not only positive-definite but also a $\mathcal{K}_\infty$ function of the distance $|\cdot|_{\mathcal{W}_q}$, not only iISS but also ISS in the sense of [2] actually holds.

VI. Conclusion

The central idea of this paper is to extend the iISS notion to the case of systems evolving on Riemannian manifolds and having multiple disjoint compact invariant sets, within the framework introduced in [2]. By means of Lyapunov/LaSalle-like dissipation inequalities and estimates of the integral type, the iISS property can be inferred for a broad class of nonlinear systems exhibiting many dynamical behaviors of interest, such as multistability, periodic orbits, chaos, just to name a few. We have also shown extension of the Strong iISS notion to the case of systems having multiple disjoing compact invariant sets. Applicability of the proposed framework is demonstrated on several examples of mechanical systems, which have also shown how the type of stability in the presence of inputs is altered by the friction models and, therefore, by the dissipation rates. Further investigations along the lines of this paper might address the stability of nonlinear cascades and feedback interconnections of iISS systems with multiple invariant sets.

Appendix

Classical iISS for systems on manifolds

In this appendix, we show how some results on the classical integral ISS property can be proven for systems evolving on manifolds. It turns out that most proofs cannot be adapted to the manifold case in a straightforward fashion, unless few assumptions and facts are established beforehand, as in the following.

Let M denote a n-dimensional connected C^2 Riemannian manifold without boundary. Let $\mathcal{A}$ be a compact subset of M. Consider system (1) with vector field f.

Fact 1: Manifold M is assumed to be complete. By virtue of the Hopf-Rinow theorem, geodesical completeness and completeness as a metric space are equivalent. Furthermore, completeness implies compactness of all closed and bounded subsets of M (see [10]). Then, any set of the form $\{x \in M \;:\; |x|_{\mathcal{A}} \leq r, r \geq 0\}$ is compact [2].

Fact 2: Vector field f is assumed to be a locally Lipschitz mapping in the sense of Definition 13.

Fact 3: Gronwall lemma applies to systems evolving on manifolds as estabilished in Corollary 1.

All proofs of subsequent Theorems 4, 5, and 6 necessarily make implicit use of Facts 1, 2, and 3.

Theorem 4: Assume that system (3) is zero-GAS with respect to $\mathcal{A}$, i.e. there exists a class-$\mathcal{KL}$ function β such that

$$|X(t,x;0)|_{\mathcal{A}} \leq \beta\left(|x|_{\mathcal{D}}, t\right).$$

Then there exists a smooth function $V : M \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha$ such that the following inequalities hold for all $x \in M$:

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{106}$$

$$DV(x)f(x,0) \leq -\alpha(|x|_{\mathcal{A}}). \tag{107}$$

Proof: Proof follows along the lines of Section 6 in [18]. □

Theorem 5: Assume that system (1) satisfies the UBEBS property in the classical sense with respect to set $\mathcal{A}$ and input $d(\cdot)$, i.e. there exist class-$\mathcal{K}_\infty$ functions α, γ, σ such that the following inequality:

$$\alpha(|X(t,x;d)|_{\mathcal{A}}) \leq \gamma(|x|_{\mathcal{W}}) + \int_0^t \sigma(|d(s)|)ds + c_u,$$

holds for all $x \in M$ and $d(\cdot)$. Assume furthermore that system (3) is zero-GAS with respect to $\mathcal{A}$. Then it is integral ISS in the classical sense with respect to set $\mathcal{A}$ and input $d(\cdot)$.

Proof: Proof follows along the lines of Section 2.1 in [6]. □

Theorem 6: Assume that system (1) is integral ISS in the classical sense with respect to the compact set $\mathcal{A}$ and input $d(\cdot)$, i.e. inequality (2) holds for some class-$\mathcal{K}_\infty$ function α, some class-$\mathcal{KL}$ function β, and some class-$\mathcal{K}$ function γ. Then there exists a smooth function $V : M \to \mathbb{R}_{\geq 0}$, class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \delta$, and a positive definite function ϖ such that inequality (106) holds together with inequality

$$DV(x)f(x,d) \leq -\varpi(|x|_{\mathcal{A}}) + \delta(|d|), \tag{108}$$

for all $x \in M$ and all $d \in D$.

Proof: Proof follows along the lines of Section IV in [5]. □

Theorem 7: Assume that system (1) satisfies the BEWCS property in the classical sense with respect to set $\mathcal{A}$ and input $d(\cdot)$, i.e. there exists a class-$\mathcal{K}_\infty$ function such that the following holds for all $x \in M$, and all $d(\cdot) \in \mathcal{M}_D$:

$$\int_0^{+\infty} \sigma\left(|d(s)|\right)\, ds < +\infty \;\Rightarrow\; \liminf_{t\to+\infty} |X(t,x;d)|_{\mathcal{A}} = 0.$$

Assume furthermore that system (3) is globally Lyapunov stable with respect to $\mathcal{A}$. Then it is integral ISS in the classical sense with respect to set $\mathcal{A}$ and input $d(\cdot)$.

Proof: Proof follows along the lines of Theorem 4.8 in [3]. □

Gronwall estimate for systems on manifolds

Notation: Let M denote a n-dimensional connected $\mathcal{C}^2$ Riemannian manifold without boundary. Let $\nabla : T_xM \times T_xM \to T_xM$ denote the Levi-Civita connection on M and $\exp_x : T_xM \to M$ the exponential map at $x \in M$. Denote by $\mathcal{T}_{\gamma,t_1,t_2}$ the parallel transport along a smooth curve $\gamma : \mathbb{R} \to M$ from the frame at $\gamma(t_1)$ to the frame at $\gamma(t_2)$. Let $L(T_xM)$ denote the vector space of linear endomorphisms on a tangent space T_xM. Let $\|v\|_g$ denote the norm $g(v,v)^{\frac{1}{2}}$ under the action of the Riemannian metric g. As before, $\mathfrak{d}(p,q)$ denotes the Riemannian distance between two points $p, q \in M$.

Definition 13: A vector field f on M is called locally Lipschitz continuous if, for all compact sets $\mathcal{K} \subset M$, there exists a constant $C_{\mathcal{K}}$ such that, for all $x \in \mathcal{K}$, there exists a neighborhood $U \subset T_xM$ of 0 such that $\exp_x : T_xM \to M$ is bijective on U and, for all $v \in U$, it holds:

$$\left\|\mathcal{T}_{\exp_x(tv),1,0} f(\exp_x(v)) \;-\; f(x)\right\|_{g_x} \leq C_{\mathcal{K}} \|v\|_{g_x}. \tag{109}$$

The constant $C_{\mathcal{K}}$ is said to be the Lipschitz constant of f in $C_{\mathcal{K}}$.

This definition of Lipschitz continuity is equivalent to Lipschitz continuity in local coordinates, therefore we can conclude that the set of points $\Omega(f)$ where f is not differentiable, has zero measure. For points $x \in M \setminus \Omega(f)$ and $v \in T_xM$ the covariant derivative $\nabla_v f(x)$ can be defined by

$$\lim_{h\to 0} \frac{\mathcal{T}_{\gamma,h,0}(f(\gamma(t))) - f(x)}{h}$$

for a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$, $\gamma'(0) = v$. We would like to extend this definition to arbitrary points $x \in M$, therefore we introduce a set-valued generalization of the covariant derivative for nonsmooth vector fields:

Definition 14: Let f be a Lipschitz continuous vector field on M. The generalized covariant derivative of f at $x \in M$ is

$$\nabla f(x) = \text{Conv}\,\Big\{A \in L(T_xM) \;:\; \exists (x_k) \subset M \setminus \Omega(f), \\ x_k \to x, A = \lim_{k\to\infty} \nabla X(x_k)\Big\}$$

[2] Lack of geodesical completeness for a manifold M may generate subsets which are bounded and closed (in the manifold topology, i.e. the topology induced by its maximal atlas) but not complete (in the metric space topology, i.e. the topology induced by the Riemannian metric. Consider the manifold $M = \mathbb{R}^2$ with Riemannian metric $\mathfrak{g} = 2dx \otimes dy/(x^2+y^2)$. It can verified that the curve $\gamma(t) := (1/(1-t), 0)$ is a geodesic of M which is not complete. Therefore any bounded and closed subset containing the origin cannot be contained in any geodesic ball, and thus is not compact.

However, in the context of dynamical systems, completeness is obtained without loss of generality. In fact, the state-space is normally defined as a manifold and the Riemannian metric structure is a posteriori assigned in order to induce a notion of distance. In this respect then, if the manifold M is embedded in $\mathbb{R}^N$ for some $N \geq n$, then it can be made into a geodesically complete one just by taking the Riemannian metric induced by the Euclidean norm of the ambient space.

where Conv denotes the convex hull of a subset of $L(T_xM)$. For a Lipschitz continuous vector field f, formula (109) can be replaced by $Av \leq C_{\mathcal{K}} v$ whenever $A \in \nabla f(x)$.

The following Lemmas provide a local Gronwall estimate for dynamical systems evolving on Riemannian manifolds.

Lemma 12 (Gronwall estimate for system (3)): Let $c_0 : [a,b] \to M, \tau \mapsto c_0(\tau)$ be a C^2 curve in a Riemannian manifold (M,g). Let $f \in \mathrm{Vec}(M)$ be a locally Lipschitz continuous vector field on M and set $c(t,\tau) := \varphi_t^f(c_0(\tau))$ where φ_t^f is the flow of f. Choose T such that φ^f is defined on $[0,T] \times c_0([a,b])$. Then, denoting by $l(t)$ the length of $\tau \mapsto c(t,\tau)$, we have

$$l(t) \leq l(0) e^{C_U t} \;\; \forall t \in [0,T]$$

where C_U is the Lipschitz constant in $U := \{c(t,\tau) \in M \;:\; t \in [0,T], \tau \in [a,b]\}$ according to Definition 13.

Proof: The proof follows along the lines of Proposition 1.1 in [16] with only minor modifications. Let $\tau \mapsto c(0,\tau)$ be parameterized by arc length, $\tau \in [0, l(0)]$. Since φ_t^f is a local diffeomorphism $g(\partial_\tau c, \partial_\tau c) > 0$ on $[0,T] \times [a,b]$.

From the theory of ODEs it follows that $t \to c(t,\tau)$ is differentiable and has a Lipschitz continuous first derivative. By hypothesis and φ_t being a diffeomorphism, it holds that $\tau \mapsto c(t,\tau)$ is Lipschitz continuous as well. Therefore, we can apply Theorem 9 in [20] in order to show that $\partial_t \partial_\tau c = \partial_\tau \partial_t c$ almost everywhere. Since the Levi-Civita connection ∇ is torsion free, we then have $\nabla_{\partial_t} \partial_\tau c = \nabla_{\partial_\tau} \partial_t c$. Then:

$$\begin{aligned}
l(s) - l(0) &= \int_0^s \partial_t l(t)\, dt = \int_0^s \partial_t \int_0^{l(0)} \|\partial_\tau c(t,\tau)\|_g \, d\tau \, dt \\
&= \int_0^s \int_0^{l(0)} \frac{\partial_t \, g\left(\partial_\tau c(t,\tau), \partial_\tau c(t,\tau)\right)}{2\|\partial_\tau c(t,\tau)\|_g} \\
&= \int_0^s \int_0^{l(0)} \frac{g\left(\nabla_{\partial_t} \partial_\tau c(t,\tau), \partial_\tau c(t,\tau)\right)}{\|\partial_\tau c(t,\tau)\|_g} \\
&= \int_0^s \int_0^{l(0)} \frac{g\left(\nabla_{\partial_\tau} \partial_t c(t,\tau), \partial_\tau c(t,\tau)\right)}{\|\partial_\tau c(t,\tau)\|_g} \\
&\leq \int_0^s \int_0^{l(0)} \|\nabla_{\partial_\tau} \partial_t c(t,\tau)\|_g \, d\tau \, dt \\
&= \int_0^s \int_0^{l(0)} \left\|\nabla_{\partial_\tau c(t,\tau)} f\right\|_g d\tau \, dt \\
&\leq C_U \int_0^s \int_0^{l(0)} \|\partial_\tau c(t,\tau)\|_g \, d\tau \, dt \\
&= C_U \int_0^s l(t)\, dt.
\end{aligned}$$

The Claim follows by applying Gronwall's inequality. □

Corollary 1: Let (M,g) be a connected C^2 Riemannian manifold, $f \in \mathrm{Vec}(M)$ be a locally Lipschitz continuous vector field on M, and let $p_0, q_0 \in M$. Let $p(t) = \varphi_t^f(p_0)$, $q(t) = \varphi_t^f(q_0)$. Consider $c(t,\cdot) : [a,b] \to M, \tau \mapsto c(t,\tau)$ a family of geodesic curves parameterized by $t \in [0,T]$ whose arc length $l(t) := \int_a^b \|\partial_\tau c(t,\tau)\|_g \, d\tau$ equals $\mathfrak{d}(p(t), q(t))$. Suppose that C_U is the Lipschitz constant in $U := \{c(t,\tau) \in M \;:\; t \in [0,T], \tau \in [a,b]\}$ according to Definition 13. Then:

$$\mathfrak{d}(p(t), q(t)) \leq \mathfrak{d}(p_0, q_0)\, e^{C_U\, t} \;\; \forall t \in [0,T] \tag{110}$$

Furthermore, if $p(t), q(t)$ evolve in a bounded set $U \subset M$ for all $t \geq 0$, there exists a constant C_U such that (110) holds with $T = +\infty$.

Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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