# Perturbation theory and singular perturbations for Input-to-State Multistable systems on manifolds

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***Abstract*—We consider the notion of Input-to-State Multistability, which generalizes ISS to nonlinear systems evolving on Riemannian manifolds and possessing a finite number of compact, globally attractive, invariant sets, which in addition satisfy a specific condition of acyclicity. We prove that a parameterized family of dynamical systems whose solutions converge to those of a limiting system inherits such Input-to-State Multistability property from the limiting system in a semi-global practical fashion. A similar result is also established for singular perturbation models whose boundary-layer subsystem is uniformly asymptotically stable and whose reduced subsystem is Input-to-State Multistable. Known results in the theory of perturbations, singular perturbations, averaging, and highly oscillatory control systems, are here generalized to the multistable setting by replacing the classical asymptotic stability requirement of a single invariant set with attractivity and acyclicity of a decomposable invariant one.**

***Index Terms*—Input-to-state stability (ISS), Manifolds, Averaging, singular perturbations**

## I. Introduction

### A. Background

The analysis of intrinsic robustness of stability properties with respect to perturbations and input disturbances constitutes a widely recurrent theme in nonlinear system theory. The preservation of stability in regular perturbations, i.e. small changes of the vector field continuously depending on a given parameter, has been investigated since the 1940s and summarized in [34, Chapter VI] and [12, Section 56]. Singular perturbation theory is typically referenced whenever time scale separation plays a major role in control engineering problems [16], [15]. Robustness of asymptotic stability in differential inclusions and complete hybrid systems under admissible perturbation radiuses is addressed in [5] and [3] respectively. Averaging techniques are largely employed to yield practical stability results under rapidly varying input disturbances [24]. In particular, practical stabilization techniques by means of high oscillatory controls have been developed in [29], [17], [18], [19], [21]. More recently, the authors in [31], [19] used a Lyapunov and a trajectory-based approach to prove robustness of time-varying nonlinear systems whose solutions continuously depend on a small parameter, thus enclosing most of the hitherto-presented robustness problems in a semi-global practical framework.

Studying intrinsic robustness of stability properties in presence of exogenous disturbances has been made possible by replacing the uniform asymptotic stability requirement with the Input-to-State Stability (ISS) property [26], [25]. Building upon [19], the authors in [20] have shown that ISS is preserved - in a semi-global practical fashion - in families of systems whose solutions continuously depend on a small parameter. Similarly, in singular perturbation models, ISS of the whole system can be inferred from ISS of the reduced one [4], [30].

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Manuscript received April 19, 2005; revised September 17, 2014.

However, all aforementioned contributions focus on stability of a single connected attractor, e.g. an equilibrium point, whereas dynamics of interest in system biology, mechanics, and electronics, sometimes require a global analysis of the so-called "multistable" systems. The term encompasses a variety of non-trivial dynamical behaviors - almost global stability, multiple equilibria, periodicity, almost periodicity, chaos - and commonly refers to the existence of a compact invariant set which is simultaneously globally attractive and decomposable as a disjoint union of a finite number of compact invariant subsets. Typically, such set is not Lyapunov stable and, for this reason, the standard aforementioned approaches fail to work in a multistable setting. However, the authors in [22], [7] have shown that the most natural way of conducting a global multistability analysis is to relax the Lyapunov stability requirement - rather than the global attractiveness - of the invariant set, under the relatively-mild additional assumption of acyclicity and decomposability of the invariant set. The notion of multistability arising from this assumption and detailed in Section II has led - within the context of systems subject to exogeneous disturbances - to the concept of Input-to-State Multistability [1], which generalizes ISS to multistable systems evolving on Riemannian manifolds.

### B. Contributions

The object of our study is twofold. First, in Section III, we are concerned with the preservation of the aforementioned notion of Input-to-State Multistability in families of nonlinear time-varying systems $\Sigma_\varepsilon$ whose solutions satisfy continuity with respect to the parameter $\varepsilon > 0$ while their vector fields might not. It is assumed that such solutions converge uniformly on compact time intervals, as $\varepsilon \downarrow 0$, to those of a system $\bar{\Sigma}$ which qualifies as Input-to-State Multistable with respect to an acyclic invariant set $\mathcal{W}$ and some input $u$. Our first contribution establishes semi-global practical Input-to-State Multistability of $\Sigma_\varepsilon$ with respect to $\mathcal{W}$, i.e. solutions departing from arbitrarily large initial conditions and subject to arbitrarily large inputs asymptotically approach a geodesic ball centred at $\mathcal{W}$ and whose radius is proportional to the norm of the input, whenever $\varepsilon$ is small enough. The starting point of this study is [19], where the authors only focused on stability of the origin for systems evolving in Euclidean space. This result builds upon and generalizes [19] to multistable systems with exogenous inputs and evolving on Riemannian manifolds. It also encompasses [22, Theorem 2] where conditions are proposed for ruling the so-called "$\Omega$-explosions" in families $\Sigma_\varepsilon$ whose vector fields are specifically time-invariant, smooth over the manifold, and continuous in $\varepsilon$. Furthermore, assuming continuity of solutions - and not vector fields - with respect to $\varepsilon$ finds immediate application in systems where averaging techniques are employed or where practical stabilization is achieved by means of highly oscillatory controls [19].

Our second contribution, addressed in Section IV, focuses on singular perturbation models and establishes their semi-global practical Input-to-State Multistability under the hypothesis of having a uniformly asymptotically stable boundary-layer subsystem and an Input-to-State Multistable reduced subsystem. By making use of our first aforementioned contribution on perturbations, this results generalizes [4] to multistable systems with external inputs and evolving on Riemannian manifolds. Similarly as in [15, Theorem 11.4], we also show the role played by hyperbolicity of fixed points into establishing semi-global Input-to-State Multistability in a *non-practical* fashion. Among the possible applications, multistable singular perturbation models are ubiquitous in genetic regulatory networks where the transcription dynamics (RNA production) is generally much faster than the translation one (production of proteins). In fact, the system biologist typically ignores the first one and resorts to the analysis of the second, and our contribution provides a plausible theoretical foundation for this accepted methodology.

### C. Notation

$\mathfrak{d}[x,y]$ is the Riemannian distance between $x$ and $y$. If $D$ is a set and $x$ a point, $|x|_D$ denotes the set-point distance $\min_{y\in D}\mathfrak{d}[x,y]$ whenever it exists. $\mathfrak{B}(c,r)$ is the closed geodesic ball of radius $r>0$ centered at $c$, i.e. $\mathfrak{B}(c,r) := \{y :\ \mathfrak{d}[y,c] \le r\}$. $\mathfrak{B}(D,r)$ is the closed geodesic ball $\mathfrak{B}(D,r) := \{y :\ |y|_D \le r\}$. If $\mathcal{M}$ is a manifold, $x^{\mathcal{M}}_{ORIG} \in \mathcal{M}$ denotes the "origin" of $\mathcal{M}$. If $\mathcal{M} = \mathbb{R}^n$, then $x^{\mathcal{M}}_{ORIG} = x^{\mathbb{R}^n}_{ORIG} = 0$. If $\mathcal{M}$ is a manifold and $x \in \mathcal{M}$, then $|x|$ denotes $|x|_{x^{\mathcal{M}}_{ORIG}}$. Observe that, if $x \in \mathbb{R}^n$, then $|x|$ denotes the standard Euclidean norm of $x$. The boundary of a set $D$ is denoted as $\eth D$. The interior of a set $D$ is denoted as $\operatorname{int} D$. Either $\mathfrak{X}(\mathcal{S}_1,\mathcal{S}_2;\mathcal{S}_1)$ or $\mathfrak{X}(\mathcal{S}_2,\mathcal{S}_1;\mathcal{S}_1)$ will denote the set of continuous vector fields on manifold $\mathcal{S}_1$ with arguments in $\mathcal{S}_2$. $\mathfrak{X}^2(\mathcal{S};\mathcal{S})$ denotes the set of $\mathcal{C}^2$-differentiable vector fields on manifold $\mathcal{S}$. If $\mathcal{U} \subseteq \mathbb{R}^m$, then $\mathcal{L}(\mathcal{U})$ will denote the space of all measurable and essentially-bounded signals taking values in $\mathcal{U}$ over infinite time intervals. If $v \in \mathcal{L}(\mathcal{U})$, then $\|v\|$ denotes the infinity norm of $v$, i.e. $\|v\| := \sup_{t\in\mathbb{R}}\{|v|\}$. When $r \ge 0$ is a constant, $\mathcal{L}(r)$ will denote the space of all signals $v \in \mathcal{L}(\mathcal{U})$ such that $\|v\| \le r$. Let $\Delta$ and $\Delta^k$ respectively denote a time-shifting operator on $\mathcal{L}(r)$ and its iteration, where $\Delta u(t) := u(t+1)$ for any $u \in \mathcal{L}(r)$ and any $t \in \mathbb{R}$ in the domain of $u$. If $\mathcal{M}$ is a manifold, $\mathfrak{g}$ its Riemannian metric, and $x \in \mathcal{M}$, then $|v|_{\mathfrak{g}}$ denotes the Riemmanian norm of some $v \in T_x\mathcal{M}$.

## II. Preliminaries

### A. Convergence of trajectories

Let $\mathcal{M}$ be an $n$-dimensional, connected, and complete Riemannian manifold without boundary and let $\mathfrak{g}$ be the Riemannian metric associated with $\mathcal{M}$[1]. Let $\mathcal{U} \subseteq \mathbb{R}^m$ denote the $m$-dimensional space where the input signals take values. Consider two vector fields $f_\varepsilon(\cdot,\cdot,\cdot) \in \mathfrak{X}(\mathbb{R},\mathcal{M},\mathcal{U};\mathcal{M})$ and $g(\cdot,\cdot) \in \mathfrak{X}(\mathcal{M},\mathcal{U};\mathcal{M})$ with $f_\varepsilon$ being parametrized by a small $\varepsilon > 0$. Consider the family of systems evolving on $\mathcal{M}$ and parameterized by $\varepsilon$:

$$\Sigma_\varepsilon \ : \ \dot{x} = f_\varepsilon(t,x,u), \tag{1}$$

the non-autonomous autonomous system on $\mathcal{M}$:

$$\bar{\Sigma} \ : \ \dot{x} = g(x,u), \tag{2}$$

and their autonomous counterparts:

$$\Sigma_{\varepsilon,0} \ : \ \dot{x} = f_\varepsilon(t,x,0), \tag{3}$$

$$\bar{\Sigma}_0 \ : \ \dot{x} = g(x,0), \tag{4}$$

[1]A manifold $\mathcal{M}$ is said to be *connected* if it is not the union of two disjoint open sets, is said to have *no boundary* if every point belonging to $\mathcal{M}$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$, and is said to be *complete* if every Cauchy sequence of points in $\mathcal{M}$ converges in $\mathcal{M}$. The property of *geodesical completeness* is the property of every maximal geodesic $\gamma(t)$ on $\mathcal{M}$ being extendible for all $t \in \mathbb{R}$. Geodesical completeness relates to the notion of completeness of $\mathcal{M}$ as a metric space via the Hopf-Rinow theorem ([6]) and implies compactness of all closed and bounded sets of $\mathcal{M}$.

where $f_\varepsilon, g$ satisfy the following mild regularity conditions and assumption on convergence of solutions of $\Sigma_\varepsilon$ - uniformly on compact time intervals - to those of $\bar{\Sigma}$ as $\varepsilon \downarrow 0$.

*Assumption 1 (regularity of vector fields):* For each $\varepsilon$, $f_\varepsilon(t,x,u)$ is continuous in $t \in \mathbb{R}_{\ge 0}$, $x \in \mathcal{M}$, $u \in \mathcal{U}$, and is locally Lipschitz continuous on $\mathcal{M}$ uniformly with respect to $t$. Vector field $g(x,u)$ is locally Lipschitz continuous for $x \in \mathcal{M}$, uniformly in $u \in \mathcal{U}$.

Let $\phi_\varepsilon(t,x;u), \psi(t,x;u)$ respectively denote the unique solutions of systems (1) and (2) with initial condition at $x \in \mathcal{M}$ and input $u \in \mathcal{L}(\mathcal{U})$.

*Assumption 2 (Convergence of solutions under forcing):* For any triple $(T,r,d)$ of strictly positive real numbers and compact set $K \subset \mathcal{M}$, there exists a strictly positive real number $\varepsilon^\star$ such that, for all $\varepsilon \in (0,\varepsilon^\star)$, all $x \in K$, all $u \in \mathcal{L}(r)$, there exists a $y \in \mathcal{M}$ such that

$$\begin{aligned} &\phi_\varepsilon(t,x;u) \text{ exists } \ \forall t \in [0,T], \text{ and} \\ &\mathfrak{d}\left[\phi_\varepsilon(t,x;u), \psi(t,y;u)\right] \le d, \ \ \forall t \in [0,T]. \end{aligned} \tag{5}$$

*Remark 1:* The aforementioned convergence on compact time intervals goes beyond continuity of solutions for vector fields continuously depending on a parameter $\varepsilon$. In fact, since continuity of $f_\varepsilon$ wrt $\epsilon$ is not required, Assumption 2 is also applicable to fast time-varying systems as in averaging theory [19], [31] and to highly oscillatory systems as in [29]. Furthermore, in most cases it holds that $y = x$ in (5).

### B. Multistability

Instrumental in the proof of our main results will be the notions of decomposition and filtration ordering for a number of compact and invariant sets of the autonomous system (4), as in the following.

*Definition 1:* A *decomposition* for a compact and invariant set $\mathcal{W}$ is a finite family of disjoint, compact, and invariant sets $\mathcal{W}_1,\ldots,\mathcal{W}_N$ - referred to as the *atoms* of the decomposition - such that $\mathcal{W} = \bigcup_{i=1}^N \mathcal{W}_i$.

*Definition 2:* The *basins of attraction and repulsion* for a set $\mathcal{W}$ are defined as:

$$\mathfrak{A}(\mathcal{W}) = \{x \in \mathcal{M} :\ \lim_{t\to+\infty} |\psi(t,x)|_{\mathcal{W}} = 0\},$$

$$\mathfrak{R}(\mathcal{W}) = \{x \in \mathcal{M} :\ \lim_{t\to-\infty} |\psi(t,x)|_{\mathcal{W}} = 0\}.$$

*Definition 3:* A *connecting orbit* between two disjoint sets $\mathcal{W}_1, \mathcal{W}_2$ exists if $\mathfrak{A}(\mathcal{W}_1) \cap \mathfrak{R}(\mathcal{W}_2) \ne \emptyset$ and is denoted as $\mathcal{W}_1 \prec \mathcal{W}_2$.

*Definition 4:* Let $\mathcal{W}_1,\ldots,\mathcal{W}_N$ be a decomposition of a compact and invariant set $\mathcal{W}$.

1) A 1-*cycle* is an index $i \in \{1,\ldots,N\}$ such that $\mathfrak{A}(\mathcal{W}_i) \cap \mathfrak{R}(\mathcal{W}_i) \setminus \mathcal{W}_i$.
2) An *r-cycle* ($r \ge 2$) is an ordered $r$-tuple of distinct indices $i_1,\ldots,i_r \in \{1,\ldots,N\}$ such that $\mathcal{W}_{i_1} \prec \mathcal{W}_{i_2} \prec \cdots \prec \mathcal{W}_{i_r} \prec \mathcal{W}_{i_1}$.
3) A *filtration ordering* is a numbering of the $\mathcal{W}_i$s so that $\mathcal{W}_i \prec \mathcal{W}_j \Rightarrow i < j$ with $i,j \in \{1,\ldots,N\}$.

*Remark 2:* The existence of a filtration ordering automatically rules out the existence of 1- and $r$-cycles, namely it rules out the existence of homoclinic trajectories and heteroclinic cycles among the atoms of $\mathcal{W}$. This property is typically instrumental in the construction of smooth Lyapunov functions for (4) (see [1], [22], [9]), and it will turn out to be instrumental in the proof of our main result as well (see Section V-C).

In this paper, we focus our attention to a specific set compact and invariant set $\mathcal{W}$ which admits a given decomposition $\mathcal{W}_1,\ldots,\mathcal{W}_N$ as in Definition 1. Let $\alpha(x)$ and $\omega(x)$ respectively denote the $\alpha$- and $\omega$-limit sets of $x \in \mathcal{M}$.

*Definition 5:* A $\mathcal{W}$*-limit set* for (4) is any compact and invariant set $\mathcal{W}$ satisfying the inclusion:

$$\mathcal{W} \supseteq \bigcup_{x\in\mathcal{M}} \alpha(x) \cup \omega(x). \tag{6}$$

If, in addition, the given decomposition $\mathcal{W}_1, \ldots, \mathcal{W}_N$ admits a filtration ordering, then $\mathcal{W}$ is called an *acyclic* $\mathcal{W}$*-limit set*.

*Remark 3:* The term "acyclic" refers only to the requirement of having neither heteroclinic nor homoclinic cycles among the atoms of the decomposition, and must not be confused with the presence of any limit cycles within each atom, which is actually allowed. In fact, if an homoclinic/heteroclinic cycle to some atom $\mathcal{W}_i$ existed, one would typically resort to a coarser decomposition where $\mathcal{W}_i$ and the entire homoclinic/heteroclinic cycle to $\mathcal{W}_i$ are embedded together in a larger single atom.

*Remark 4:* The existence of a $\mathcal{W}$-limit set automatically entails compactness of all $\alpha$- and $\omega$-limit sets of $\bar{\Sigma}_0$.

*Remark 5:* The identification of the $\mathcal{W}$-limit set typically is a non-trivial operation. While in planar systems it might be straightforward to apply standard techniques such as Poincaré-Bendixson theorem to identify $\mathcal{W}$ and check its acyclicity, much more involved would be the analysis for systems of higher dimensions. In this case, one must resort - whenever possible - to other system properties such as monotonicity [2] (e.g. in biological systems), passivity and dissipativity (e.g. in mechanical systems).

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Assume that $\mathcal{W}$ is a globally attractive $\mathcal{W}$-limit set for (4). We can then define, for $i \in \{1, \ldots, N\}$, the sets:

$$A_i := \bigcup_{j\le i} \mathfrak{R}(\mathcal{W}_j), \quad B_i := \bigcup_{j>i} \mathfrak{A}(\mathcal{W}_j). \tag{7}$$

*Lemma 1:* The following properties are true for all $i \in \{1, \ldots, N\}$:

(i) $A_i$ and $B_i$ are closed, invariant, and $A_i \cap B_i = \emptyset$;

(ii) if $x \in \mathcal{M} \setminus B_i$, then $\lim_{t\to+\infty} |\psi(t,x)|_{A_i} = 0$; if $x \in \mathcal{M} \setminus A_i$, then $\lim_{t\to-\infty} |\psi(t,x)|_{B_i} = 0$ or $\lim_{t\to-\infty} |\psi(t,x)| = +\infty$;

(iii) there exist a closed neighborhood $\mathcal{A}_i$ of $A_i$ and a closed neighborhood $\mathcal{B}_i$ of $B_i$ such that $\mathcal{A}_i \cap B_i = \mathcal{B}_i \cap A_i = \emptyset$ and $\mathcal{A}_i \cup \mathcal{B}_i = \mathcal{M}$; in particular, one may select $\mathcal{A}_N = \mathcal{M}$ and $\mathcal{B}_N = \emptyset$;

(iv) $A_i$ is locally asymptotically stable on $\mathcal{A}_i$; $B_i$ is locally asymptotically anti-stable on $\mathcal{B}_i$.

*Proof:* See [1, Appendix C] and [9, Lemma 7]. ■

## III. Perturbation Theory

### A. Semi-global practical Input-to-State Multistability

In this Section, we focus on a notion of multistability for systems with external inputs and provide sufficient conditions for a parameterized family of nonlinear systems to preserve such property in a semi-global practical fashion. Such notion of multistability will be a generalization [1] of the classical definition of ISS [26], [25], as recalled in the following.

*Definition 6:* System (2) is said to be *Input-to-State Stable* with respect to set $\mathcal{S}$ and input $u$ if there exist a class-$\mathcal{KL}$ function $\beta$ and a class-$\mathcal{K}_\infty$ function $\eta_u$ satisfying:

$$\begin{aligned} |\psi(t,x;u)|_{\mathcal{S}} &\le \beta(|x|_{\mathcal{S}}, t) + \eta_u(\|u\|), \\ &\forall t \ge 0,\ \forall x \in \mathcal{M},\ \forall u \in \mathcal{L}(\mathcal{U}). \end{aligned} \tag{8}$$

In [1], the authors have introduced the following notion of multistability for systems with external inputs. Let $\mathcal{W} \subseteq \mathcal{M}$ be a compact and invariant set for (4).

*Definition 7:* System (2) is said to be *Input-to-State Multistable* wrt set $\mathcal{W}$ and input $u$ if $\mathcal{W}$ is an acyclic $\mathcal{W}$-limit set for (4) and there exists a class-$\mathcal{K}_\infty$ function $\eta_u$ satisfying the Asymptotic Gain (AG) property:

$$\limsup_{t\to+\infty} |\psi(t,x;u)|_{\mathcal{W}} \le \eta_u(\|u\|), \quad \forall x \in \mathcal{M},\ \forall u \in \mathcal{L}(\mathcal{U}). \tag{9}$$

*Definition 8:* System (4) is said to be *globally multistable* wrt set $\mathcal{W}$ if $\mathcal{W}$ is an acyclic $\mathcal{W}$-limit set and $\limsup_{t\to+\infty} |\psi(t,x;0)|_{\mathcal{W}} = 0$ for any $x \in \mathcal{M}$.

*Remark 6:* If $\mathcal{W}$ consists of a single connected compact component (e.g. a continuum of fixed-points), then $\mathcal{W}$ cannot be decomposed in multiple atoms (according to Definition 1), thus $N = 1$, and it can be proved that $\mathcal{W} = A_1 = A_N$, with $A_1, A_N$ defined as in (7). In particular, in this case Input-to-State Multistability of $\mathcal{W}$ implies Input-to-State Stability of $\mathcal{W}$ (see Lemma 2). Conversely, if the decomposition of $\mathcal{W}$ consists of multiple components, i.e. $N > 1$, then, typically, $\mathcal{W}$ is not Lyapunov stable (see [1], [10]), and no class-$\mathcal{KL}$ function $\beta$ exists so as to satisfy (8), thus one must relax the Lyapunov stability requirement and use asymptotic estimates like (9). However, due to the filtration property, no solution of (4) is attracted in backward time to the atom $\mathcal{W}_1$ whereas, in forward time, $\mathcal{W}_1$ attracts all solutions in a small enough neighborhood, and for this reason it is always asymptotically stable, at least locally.

The semi-global practical counterparts of Definitions 6 and 7 are respectively given in the following Definitions 9 and 10.

*Definition 9:* Flow $\phi_\varepsilon$ is said to be *semi-globally practically Input-to-State Stable* wrt set $\mathcal{S}$ and input $u \in \mathcal{L}(\mathcal{U})$ if there exist a class-$\mathcal{KL}$ function $\beta$ and a class-$\mathcal{K}_\infty$ function $\eta_u^\star$ such that:

$$\begin{aligned} &\forall d_1, d_2, d_u > 0,\ \exists \varepsilon^\star > 0 :\ \forall \varepsilon \in (0, \varepsilon^\star), \\ &\forall x \text{ with } |x|_{\mathcal{S}} \le d_1,\ \forall u \in \mathcal{L}_\infty(d_u), \\ &|\phi_\varepsilon(t,x;u)|_{\mathcal{S}} \le \beta(|x|_{\mathcal{S}}, t) + \eta_u^\star(\|u\|) + d_2. \end{aligned} \tag{10}$$

*Definition 10:* Flow $\phi_\varepsilon$ is said to be *semi-globally practically Input-to-State Multistable* wrt set $\mathcal{W}$ and input $u \in \mathcal{L}(\mathcal{U})$ if there exists a class-$\mathcal{K}_\infty$ function $\eta_u^\star$ such that:

$$\begin{aligned} &\forall d_1, d_2, d_u > 0,\ \exists \varepsilon^\star > 0 :\ \forall \varepsilon \in (0, \varepsilon^\star), \\ &\forall x \text{ with } |x| \le d_1,\ \forall u \in \mathcal{L}_\infty(d_u), \\ &\limsup_{t\to+\infty} |\phi_\varepsilon(t,x;u)|_{\mathcal{W}} \le \eta_u^\star(\|u\|) + d_2. \end{aligned} \tag{11}$$

*Definition 11:* Flow $\phi_\varepsilon$ is said to be *semi-globally practically multistable* wrt set $\mathcal{W}$ if:

$$\begin{aligned} &\forall d_1, d_2 > 0,\ \exists \varepsilon^\star > 0 : \\ &\forall \varepsilon \in (0, \varepsilon^\star),\ \forall x \text{ with } |x| \le d_1, \\ &\limsup_{t\to+\infty} |\phi_\varepsilon(t,x;0)|_{\mathcal{W}} \le d_2. \end{aligned} \tag{12}$$

We prove in Theorem 1 that Input-to-State Multistability of $\psi$ implies semi-global practical Input-to-State Multistability of $\phi_\varepsilon$. This is a three-step procedure where acyclicity of $\mathcal{W}$ plays a major role:

- we first prove semi-global practical ISS wrt the larger set $A_N$, so as to bound solutions of $\phi_\varepsilon$ whenever the norm of the input and the initial condition are bounded, for $\varepsilon$ small enough;
- second, we prove that acyclicity of $\mathcal{W}$ and the AG property imply a property of acyclicity under forcing, i.e. absence of homoclinic and heteroclinic cycles among the elements $\mathfrak{B}(\mathcal{W}_i, \breve{\eta}_u(r))$ for some $\mathcal{K}_\infty$ gain $\breve{\eta}_u$, whenever $\|u\| \le r$;
- finally, we assume by contradiction that $\phi_\varepsilon$ is not semi-global practical Input-to-State Multistable, and therefore there exist solutions that, no matter how small we select $\varepsilon$, will always escape the $\mathfrak{B}(\mathcal{W}_i, \breve{\eta}_u(d_u))$ balls, for some fixed input norm $d_u$. However, since semi-global practical ISS of $A_N$ confines such solutions to a bounded set, a careful application of Arzelà-Ascoli's theorem and [26, Lemma III.2] allows us to construct

an infinite series of solutions of $\psi$ (and inputs $u \in \mathcal{L}_r$) which travel in and out the $\mathfrak{B}(\mathcal{W}_i, \breve{\eta}_u(d_u))$ balls, therefore violating the acyclicity under forcing in $\mathcal{L}(r)$.

Semi-global practical ISS wrt to $A_N$ is formalized in the following Lemmas 2 and 3.

*Lemma 2:* If (2) is Input-to-State Multistable wrt set $\mathcal{W}$ and input $u$, then it is Input-to-State Stable wrt to $A_N$ and input $u$.

*Proof:* By virtue of Lemma 1, $A_N$ is compact and globally asymptotically stable (GAS) on $\mathcal{M}$ along the solutions of the autonomous system (4). Furthermore, since $\mathcal{W} \subseteq A_N$, it holds $|q|_{A_N} \leq |q|_{\mathcal{W}}$ for all $q \in \mathcal{M}$, and thus the AG property (9) wrt to $\mathcal{W}$ implies the AG property wrt $A_N$, i.e.

$$\limsup_{t\to+\infty} |\psi(t,x;u)|_{\mathcal{W}} \leq \eta_u(\|u\|), \quad \forall x \in \mathcal{M},\ \forall u \in \mathcal{L}(\mathcal{U}). \tag{13}$$

GAS of $A_N$ and the AG property (13) imply, via [26, Theorem 1] adapted to the manifold case, that (2) is ISS wrt to $A_N$ and input $u$. ■

*Lemma 3:* Let Assumption 1 and 2 hold. Then, if (2) is Input-to-State Multistable wrt an acyclic $\mathcal{W}$-limit set $\mathcal{W}$ and input $u$, the flow $\phi_\varepsilon$ is semiglobally practically Input-to-State Stable wrt set $A_N$ and input $u$.

*Proof:* By virtue of Lemma 2, system (2) is ISS wrt set $A_N$ and input $u$. Then, this Lemma follows via [20, Corollary 1], which can be easily adapted to the manifold case. ■

Acyclicity under forcing is formalized in the following Definitions 12, 13 and Lemma 4.

*Definition 12:* Let $r \geq 0$. Let $\mathcal{S}_1, \dots, \mathcal{S}_N$ be compact subsets of $\mathcal{M}$.

1) A 1-*cycle under forcing in* $\mathcal{L}(r)$ *among sets* $\mathcal{S}_1, \dots, \mathcal{S}_N$ is an index $i \in \{1, \dots, N\}$ such that there exist $x \notin \mathcal{S}_1 \cup \dots \cup \mathcal{S}_N$ and $u \in \mathcal{L}(r)$ satisfying $\lim_{t\to+\infty} |\psi(t,x;u)|_{\mathcal{S}_i} = \lim_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{S}_i} = 0$.
2) Given $h \in \{2, \dots, N\}$, a $h$-*cycle under forcing in* $\mathcal{L}(r)$ *among sets* $\mathcal{S}_1, \dots, \mathcal{S}_N$ is an ordered $h$-tuple of distinct indices $i_1, \dots, i_h \in \{1, \dots N\}$ such that there exist $x_{i_1}, \dots, x_{i_h} \notin \mathcal{S}_1 \cup \dots \cup \mathcal{S}_N$ and $u_{i_1}, \dots, u_{i_h} \in \mathcal{L}(r)$ satisfying

$$\begin{aligned}
&\lim_{t\to-\infty} |\psi(t,x_{i_1};u_{i_1})|_{\mathcal{S}_{i_1}} = \lim_{t\to+\infty} |\psi(t,x_{i_1};u_{i_1}))|_{\mathcal{S}_{i_2}} = 0,\\
&\dots,\\
&\lim_{t\to-\infty} |\psi(t,x_{i_{h-1}};u_{i_{h-1}})|_{\mathcal{S}_{i_{h-1}}} =\\
&\qquad \lim_{t\to+\infty} |\psi(t,x_{i_{h-1}};u_{i_{h-1}}))|_{\mathcal{S}_{i_h}} = 0,\\
&\lim_{t\to-\infty} |\psi(t,x_{i_h};u_{i_h})|_{\mathcal{S}_{i_h}} = \lim_{t\to+\infty} |\psi(t,x_{i_h};u_{i_h})|_{\mathcal{S}_{i_1}} = 0.
\end{aligned}$$

*Definition 13:* An acyclic $\mathcal{W}$-limit set $\mathcal{W}$, where $D := \min_{i\neq j} \mathfrak{d}[\mathcal{W}_i, \mathcal{W}_j]$ is the minimum distance among the atoms of its decomposition, is said to be *acyclic under forcing* if and only if there exists a class-$\mathcal{K}_\infty$ gain $\breve{\eta}_u$ such that, for any $d_u \geq 0$ with $\breve{\eta}_u(d_u) < D/2$, no cycle under forcing in $\mathcal{L}(d_u)$ exists among the balls $\mathfrak{B}(\mathcal{W}_i, \breve{\eta}_u(d_u))$.

*Lemma 4:* If system (2) is Input-to-State Multistable wrt set $\mathcal{W}$ and input $u$, then $\mathcal{W}$ is acyclic under forcing.

*Proof:* The Lemma is a direct consequence of Lemmas 12, 13. ■

We are now ready to state our main result.

*Theorem 1:* Let Assumption 1 and 2 hold. If (2) is Input-to-State Multistable wrt set $\mathcal{W}$-limit set $\mathcal{W}$ and input $u$, then (1) is semi-globally practically Input-to-State Multistable wrt set $\mathcal{W}$ and input $u$.

*Proof:* See Appendix A. ■

*Corollary 1:* Under the same assumptions of Theorem 1, if (4) is globally multistable wrt set $\mathcal{W}$, then (3) is semi-globally practically multistable wrt $\mathcal{W}$.

*Corollary 2:* Under the same assumptions of Theorem 1,

$$\begin{aligned}
&\forall d_1, d_2, d_u > 0,\ \ \exists \varepsilon^\star > 0 :\ \ \forall \varepsilon \in (0, \varepsilon^\star),\\
&\forall x \text{ with } |x| \leq d_1,\ \ \forall u \in \mathcal{L}_\infty(d_u),\\
&\limsup_{t\to+\infty} |\phi_\varepsilon(t,x;u)|_{\mathcal{W}} \leq \eta_u^\star(\limsup_{t\to+\infty} |u(t)|) + d_2.
\end{aligned} \tag{14}$$

*Proof:* By compactness of $\mathcal{W}$, there exists $Q \geq 0$ such that $|q| \leq |q|_{\mathcal{W}} + Q$ for all $q \in \mathcal{M}$. Pick $d_1, d_2, d_u > 0$. Select $\bar{d}_1 > \max\{Q + \eta_u^\star(d_u) + d_2,\ d_1\}$. Select $\varepsilon^\star(\bar{d}_1, d_2, d_u)$ as given by Theorem 1. It then follows from (11) that

$$\limsup_{t\to+\infty} |\phi_\varepsilon(t,x;u)| \leq Q + \eta_u^\star(d_u) + d_2. \tag{15}$$

for all $\varepsilon \in (0, \varepsilon^\star]$, $|x| \leq d_1$, and $u \in \mathcal{L}_\infty(d_u)$. Now, fix $\varepsilon \in (0, \varepsilon^\star]$, $|x| \leq d_1$, and $u \in \mathcal{L}_\infty(d_u)$. Then, (15) implies the existence of some $T_{\varepsilon,x,u} > 0$ such that

$$|\phi_\varepsilon(t,x;u)| \leq \bar{d}_1,\ \ \forall t \geq T_{\varepsilon,x,u}. \tag{16}$$

Now, let $r := \limsup_{t\to+\infty} |u(t)|$ and $\iota > 0$. Select $h > 0$ so that $\eta_u^\star(r+h) - \eta_u^\star(r) < \iota$. By definition of $r$, there exists $\tau_h > 0$ such that $|u(t)| \leq r+h$ for all $t \geq \tau_h$. Define $\bar{T} := \max\{T_{\varepsilon,x,u}, \tau_h\}$. Let $\tilde{\phi}_\varepsilon(t,\tilde{x},\tilde{u})$ be the response from the initial state $\tilde{x} := \phi_\varepsilon(\bar{T},x;u)$ and input $\tilde{u}$ defined as $\tilde{u}(s) := u(s+\bar{T})$ for all $s \geq 0$, i.e. $\tilde{\phi}_\varepsilon(s,\tilde{x},\tilde{u}) := \phi_\varepsilon(s+\bar{T},x;u)$ for all $s \geq 0$. First, we observe that $\|\tilde{u}\| = r+h$. Second, since (16) holds, we can apply (11) to $\tilde{\phi}_\varepsilon$ in order to have:

$$\limsup_{t\to+\infty} |\tilde{\phi}_\varepsilon(t,\tilde{x};\tilde{u})|_{\mathcal{W}} \leq \eta_u^\star(\limsup_{t\to+\infty} |u(t)|) + \iota + d_2. \tag{17}$$

Letting $\iota \to 0$ yields (14). ■

*Remark 7:* Theorem 1 establishes a result for flows $\phi_\varepsilon$ - and not for vector fields $f_\varepsilon$ - that are continuously depending on $\varepsilon$. For this reason, Theorem 1 applies to systems where averaging techniques [24] are typically employed or where practical stabilization is achieved by means of highly oscillatory controls [19].

*Remark 8:* Acyclicity of $\mathcal{W}$ in Theorem 1 is a necessary condition for semi-global practical Input-to-State Multistability of $\phi_\varepsilon$, as illustrated by the counterexample to Corollary 3 in Section V-B.

### B. Averaging application

A straightforward application of Theorem 1 and Corollary 1 lies in averaging theory, whose basic notions we recall in the following.

*Definition 14:* Consider the continuous vector field $f \in \mathfrak{X}(\mathbb{R}\times\mathcal{M})$. The *average* of $f$ is the vector field $\bar{f} \in \mathfrak{X}(\mathcal{M})$, defined as

$$\bar{f}(x) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(x,s)\,ds.$$

*Definition 15:* A vector field $f \in \mathfrak{X}(\mathbb{R}\times\mathcal{M})$ is called a *KBM-vector field* iff:

(i) $f(t,x)$ is continuous in $x \in \mathcal{M}$ and $t \geq 0$;
(ii) $f(t,x)$ is locally Lipschitz continuous in $x$ uniformly in $t$;
(iii) the limit in the definition of the average of $f$ is uniform on compact sets $K \subseteq \mathcal{M}$, namely for all compact sets $K$ and all $\varepsilon > 0$ there exists $\bar{T} > 0$ such that

$$\left| \bar{f}(x) - \frac{1}{T}\int_0^T f(s,x)\,ds \right| < \varepsilon \quad \forall x \in K,\ \forall T > \bar{T}.$$

Consider the time-varying system

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right) \tag{18}$$

and the corresponding averaged system:

$$\dot{x} = \bar{f}\,(x) \tag{19}$$

We recall here the following classical result in averaging theory [32], [24], [19],

*Proposition 1:* If $f$ is a KBM-vector field and $\bar{f}$ is its average, then the solutions of (18) converge uniformly on compact time intervals to the solutions of (19), i.e. Assumption 2 is satisfied with $\phi_\varepsilon$ and $\psi$ respectively denoting the unique solutions of systems (18) and (19).

By virtue of Corollary 1, we have the following result.

*Corollary 3:* If $f$ is a KBM-vector field, $\bar{f}$ is its average, and (19) is globally multistable wrt an acyclic $\mathcal{W}$-limit set $\mathcal{W}$, then (18) is semi-globally practically multistable wrt $\mathcal{W}$.

## IV. Singular perturbations

### A. Singular perturbations: general non-autonomous case

Let $\mathcal{X}$ be a $n_x$-dimensional, connected, and complete Riemannian manifold without boundary. Let $\mathcal{Z}$ be a $n_z$-dimensional Euclidean space. Let $\Theta \subseteq \mathbb{R}^m$ denote the $m$-dimensional space where the input signals take values. We are now going to focus on the so-called singular perturbation models evolving on $\mathcal{X} \times \mathcal{Z}$, and having the following state-space description:

$$\dot{x} = f(x, z, \theta(t), \varepsilon) \tag{20a}$$
$$\varepsilon \dot{z} = g(x, z, \theta(t), \varepsilon), \tag{20b}$$

where $x \in \mathcal{X}$, $z \in \mathcal{Z}$, $f \in \mathfrak{X}(\mathcal{X}, \mathcal{Z}, \Theta, \mathbb{R}_{\geq 0}; \mathcal{X})$, and $g \in \mathfrak{X}(\mathcal{X}, \mathcal{Z}, \Theta, \mathbb{R}_{\geq 0}; \mathcal{Z})$. Let $X_\varepsilon(t, w; \theta), Y_\varepsilon(t, w; \theta)$ denote the unique solution of system (20) with initial condition at $w := (x, y \in \mathcal{X} \times \mathcal{Y}$ and input $\theta \in \mathcal{L}(\Theta)$. Setting $\varepsilon = 0$ yields the so-called quasi-steady-state model:

$$\dot{x} = f(x, z_s, \theta(t), 0) \tag{21a}$$
$$0 = g(x, z_s, \theta(t), 0). \tag{21b}$$

The following assumption is made on the quasi-state model.

*Assumption 3:* The algebraic equation $0 = g(x, z_s, \theta, 0)$ possesses a unique real root

$$z_s = h(x, \theta), \tag{22}$$

for each value of $x \in \mathcal{X}$ and $\theta \in \Theta$.

Under such assumption, and due to time scale separation, it makes sense to consider the so-called reduced system:

$$\dot{x} = f(x, h(x, \theta), \theta, 0), \tag{23}$$

and its autonomous counter-part:

$$\dot{x} = f(x, h(x, 0), 0, 0). \tag{24}$$

Furthermore, by defining

$$y := z - h(x, \theta) \text{ and } \tau := \frac{t}{\varepsilon},$$

it becomes possible to consider the so-called boundary-layer system:

$$\frac{dy}{d\tau} = g(x, h(x, \theta) + y, \theta, 0). \tag{25}$$

In particular, the singular perturbation model (20) in the new coordinates $x, y$ reads as:

$$\frac{dx}{dt} = f(x, h(x, \theta) + y, \theta, \varepsilon) \tag{26a}$$

$$\varepsilon \frac{dy}{dt} = g(x, h(x, \theta) + y, \theta, \varepsilon) - \varepsilon \left[ \frac{\partial h}{\partial x} f(x, h(x, \theta) + y, \theta, \varepsilon) + \frac{\partial h}{\partial \theta} \dot{\theta} \right]. \tag{26b}$$

We prove in Theorem 2 that, if the boundary layer system (25) is globally asymptotically stable uniformly in $x, \theta$ and the reduced system (23) is Input-to-State Multistable wrt set $\mathcal{W}$ and input $\theta$, then the singular perturbation model (26) satisfies a semi-global practical Input-to-State Multistability property. In analogy with the proof of Theorem 1, this is a three-step procedure where acyclicity of $\mathcal{W}$ plays a major role:

- we first prove in Lemma 5 that a semi-global practical ISS property holds wrt $A_N$ for subsystem (26a) and wrt to the origin for subsystem (26b), therefore solutions of (26b) are ultimately bounded;
- second, by making use of Theorem 1 and Lemma 7, we prove Input-to-State Multistability of (26a) wrt to input $\theta$ and $\tilde{y} := B^{-1}(x)y$, for an appropriate state-dependent smooth gain $B(x)$;
- finally, boundedness of solutions allows us to conclude semi-global practical Input-to-State Multistability of (26a) wrt to $\theta, y$.

*Lemma 5:* Assume that:

- the reduced system (23) is Input-to-State Multistable with respect to an acyclic $\mathcal{W}$-limit set $\mathcal{W}$ and input $\theta$;
- the equilibrium $y = 0$ of the boundary layer system (25) is globally asymptotically stable, uniformly wrt $x(\cdot)$ and $\theta(\cdot)$.

Let $A_N := \bigcup_{i \in \{1,\ldots,N\}} \mathfrak{R}(\mathcal{W}_i)$ be defined along the solutions of the autonomous system (24) according to Definition 2. Then, there exist two class-$\mathcal{KL}$ functions $\beta_x, \beta_y$, a class-$\mathcal{K}_\infty$ function $\gamma^\star$, and, for any pair of positive real numbers $(d_1, d_2)$, there exists an $\varepsilon^\star > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\star]$, any initial condition $w := (x, y) \in \mathcal{X} \times \mathcal{Y}$, and any absolutely continuous function $\theta(t)$ satisfying $\max\left\{|x|_\mathcal{W}, |y|, \|\theta\|, \left\|\dot{\theta}\right\|\right\} \leq d_1$, it holds:

$$|X_\varepsilon(t, w; \theta)|_{A_N} \leq \beta_x \left(|x|_{A_N}, t\right) + \gamma^\star(\|\theta\|) + d_2, \quad \forall t \geq 0, \tag{27}$$
$$|Y_\varepsilon(t, w; \theta)| \leq \beta_y \left(|y|, \frac{t}{\varepsilon}\right) + d_2, \quad \forall t \geq 0. \tag{28}$$

Moreover, if $g$ does not depend on $\theta$, then $\theta(t)$ can simply be measurable and there is no requirement on $\left\|\dot{\theta}\right\|$ whenever it exists.

*Proof:* It follows along the lines of Lemma 2 that system (23) is ISS wrt to $A_N$ and input $\theta$. Then, this Lemma follows by virtue of [4, Theorem 1] which can be easily adapted to the manifold case. ∎

*Theorem 2:* Under the same assumptions of Lemma 5, system (26) is semi-globally practically Input-to-State Stable wrt set $\mathcal{W} \times \{0\}$ and input $\theta \in \mathcal{L}(\Theta)$. In particular, there exist a class-$\mathcal{KL}$ function $\beta_y$ and a class-$\mathcal{K}_\infty$ function $\eta^\star_\theta$ and, for any $d_1, d_2 > 0$, there exists an $\varepsilon^\star > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\star]$, any initial condition $w := (x, y) \in \mathcal{X} \times \mathcal{Y}$, and any absolutely continuous function $\theta(t)$ satisfying $\max\left\{|x|_\mathcal{W}, |y|, \|\theta\|, \left\|\dot{\theta}\right\|\right\} \leq d_1$, it holds:

$$\limsup_{t \to +\infty} |X_\varepsilon(t, w; \theta)|_\mathcal{W} \leq \eta^\star_\theta(\|\theta\|) + d_2, \tag{29}$$

$$|Y_\varepsilon(t, w; \theta)| \leq \beta_y \left(|y|, \frac{t}{\varepsilon}\right) + d_2, \quad \forall t \geq 0. \tag{30}$$

Moreover, if $g$ does not depend on $\theta$, then $\theta(t)$ can simply be measurable and there is no requirement on $\left\|\dot{\theta}\right\|$ whenever it exists.

*Proof:* See Appendix B. ∎

*Remark 9:* As shown by our counterexample in Section V-C, acyclicity of $\mathcal{W}$ is a necessary condition for establishing the results of Theorem 2, i.e. semi-global practical Input-to-State Multistability of singular perturbation model (26)

### B. Singular perturbation: hyperbolic autonomous case

In this Section, we provide sufficient conditions for multistability of singular perturbation models where inputs are not present and the invariant set $\mathcal{W}$ consists of hyperbolic fixed points only. Let

$f \in \mathfrak{X}(\mathcal{X}, \mathcal{Z}, \mathbb{R}_{\geq 0}; \mathcal{X})$, and $g \in \mathfrak{X}(\mathcal{X}, \mathcal{Z}, \mathbb{R}_{\geq 0}; \mathcal{Z})$. Consider the following state-space description:

$$\dot{x} = f(x, z, \varepsilon) \tag{31a}$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon). \tag{31b}$$

with state $(x, z) \in \mathcal{X} \times \mathcal{Z}$ and parameter $\varepsilon \geq 0$. Setting $\varepsilon = 0$ yields the so-called quasi-steady-state model:

$$\dot{x} = f(x, z_s, 0) \tag{32a}$$
$$0 = g(x, z_s, 0). \tag{32b}$$

The following assumption is made on the quasi-state model.

*Assumption 4:* The algebraic equation $0 = g(x, z_s, 0)$ possesses a unique real root:

$$z_s = h(x), \tag{33}$$

for each value of $x \in \mathcal{X}$.

As in Section IV-A, we consider the so-called reduced system:

$$\dot{x} = f(x, h(x), 0), \tag{34}$$

and, by defining $y := z - h(x)$ and $\tau := t/\varepsilon$, the boundary-layer system:

$$\frac{dy}{d\tau} = g(x, h(x) + y, 0). \tag{35}$$

In particular, the singular perturbation model (31) in the new coordinates $x, y$ reads as:

$$\frac{dx}{dt} = f(x, h(x) + y, \varepsilon) \tag{36a}$$
$$\varepsilon \frac{dy}{dt} = g(x, h(x) + y, \varepsilon) - \varepsilon \frac{\partial h}{\partial x} f(x, h(x) + y, \varepsilon). \tag{36b}$$

*Theorem 3:* Let Assumption 4 hold. Assume that:

- $f(\mathcal{W}_i, h(\mathcal{W}_i), \varepsilon) = 0$ and $g(\mathcal{W}_i, h(\mathcal{W}_i), \varepsilon) = 0$ for all $\varepsilon$ and all $i \in \{1, \ldots, N\}$;
- $f, g, h$ satisfy $\mathcal{C}^2$ continuity in their parameters; are bounded for $y$ in a bounded domain;
- reduced system (34) is Multistable wrt set $\mathcal{W}$;
- $f(x, h(x), 0) \in \mathfrak{X}^2(\mathcal{X}; \mathcal{X})$ and each $\mathcal{W}_i$, $i \in \{1, \ldots, N\}$, is a hyperbolic fixed point for the autonomous flow (34);
- the origin $y = 0$ of the boundary layer system (35) is globally exponentially stable, uniformly wrt $x(\cdot)$.

Then, for any $d_1 > 0$, there exists an $\varepsilon^\star > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\star]$ and any initial condition $w := (x, y) \in \mathcal{X} \times \mathcal{Y}$ satisfying $\max\{|x|_\mathcal{W}, |y|\} \leq d_1$, it holds:

$$\lim_{t \to +\infty} |X_\varepsilon(t, w)|_\mathcal{W} = 0, \tag{37}$$
$$\lim_{t \to +\infty} |Y_\varepsilon(t, w)| = 0. \tag{38}$$

*Proof:* By virtue of Lemma 6, for any $i \in \{1, \ldots, N\}$, there exist an open neighborhood $\bar{\mathcal{U}}_i$ a $\mathcal{C}^1$ function $V_i : \bar{\mathcal{U}}_i \to \mathbb{R}$ such that (43) holds. Without loss of generality, assume that the $\bar{\mathcal{U}}$s are pairwise disjoint. Now select $d_2 > 0$ such that:

$$\mathfrak{B}[\mathcal{W}, d_2] \subset \bigcup_{i=1}^{N} \bar{\mathcal{U}}_i.$$

Fix $d_1 > 0$ and let $\varepsilon_0^\star$ be given by Theorem 2 as $\varepsilon^\star(d_1, d_2/2)$. By virtue of Theorem 2, for any $\varepsilon \in (0, \varepsilon_0^\star)$ and any initial condition $w := (x, y) \in \mathcal{X} \times \mathcal{Y}$ satisfying $\max\{|x|_\mathcal{W}, |y|\} \leq d_1$, it holds:

$$\limsup_{t \to +\infty} |X_\varepsilon(t, w)|_\mathcal{W} \leq \frac{d_2}{2}, \tag{39}$$
$$\limsup_{t \to +\infty} |Y_\varepsilon(t, w)| \leq \frac{d_2}{2}, \quad \forall t \geq 0. \tag{40}$$

Since the $\bar{\mathcal{U}}_i$s are pairwise disjoint, it follows from (29) that solution $X_\varepsilon$ eventually enters some $i$th open neighborhood, i.e.

$$\forall w \;\; \exists T > 0, \;\; \exists i \in \{1, \ldots, N\} : \;\; \forall t \geq T \quad X_\varepsilon(t, w) \in \bar{\mathcal{U}}_i. \tag{41}$$

Observe that:

- due to $V_i$ being $\mathcal{C}^1$ and $dV_i(\mathcal{W}_i) = 0$, we have that $|dV_i(x)|_\mathfrak{g} \leq c_4 |x|_\mathcal{W}$ for some $c_4 > 0$;
- by virtue of [15, Lemma 9.8], since $g, h$ are $\mathcal{C}^2$ continuous and the origin of (35) is globally exponentially stable uniformly in $x$, there exist a $\mathcal{C}^1$ Lyapunov function $W(x, y)$ and positive constants $b_3, b_4, b_6$ that satisfy $d_y W(x, y) \cdot g(x, y + h(x), 0) \leq -b_3 |y|^2$, $|d_y W(x, y)| \leq b_4 |y|$, and $|d_x W(x, y)| \leq b_6 |y|^2$;
- due to Lipschitz continuity of $f$ and $f$, and due to $f(\mathcal{W}_i, h(\mathcal{W}_i), \varepsilon) = 0$ and $g(\mathcal{W}_i, h(\mathcal{W}_i), \varepsilon) = 0$ for all $i \in \{1, \ldots, N\}$, there exist positive constants $L_1, L_2 > 0$ such that

$$|f(x, y + h(x), \varepsilon) - f(x, y + h(x), 0)|_\mathfrak{g} \leq \varepsilon L_1 (|x|_\mathcal{W} + |y|)$$
$$|g(x, y + h(x), \varepsilon) - g(x, y + h(x), 0)| \leq \varepsilon L_2 (|x|_\mathcal{W} + |y|)$$

for all $x \in \bar{\mathcal{U}}_i$ and all $y \in \mathcal{Y}$.

Following [15, Theorem 11.4], it can then be proved that the Lyapunov function

$$\upsilon_i(x, y) := V_i(x) + W(x, y)$$

decreases along the solutions of (36) for any $i \in \{1, \ldots, N\}$, i.e. there exist $\varepsilon^\star \leq \varepsilon_0^\star$ and constants $a_1, a_2, a_3, a_4, a_5 > 0$ such that, for all $\varepsilon \in (0, \varepsilon^\star)$ and all $i \in \{1, \ldots, N\}$, there exist $a_x, a_y > 0$ satisfying

$$\begin{aligned} \frac{d\upsilon_i}{dt} &\leq -\begin{pmatrix} |x|_{\mathcal{W}_i} \\ |y| \end{pmatrix}^\top \begin{pmatrix} a_1 - \varepsilon a_2 & -a_5 \\ -a_5 & (a_3/\varepsilon) - a_4 \end{pmatrix} \begin{pmatrix} |x|_{\mathcal{W}_i} \\ |y| \end{pmatrix} \\ &\leq -a_x |x|^2_{\mathcal{W}_i} - a_y |y|^2, \end{aligned} \tag{42}$$

for all $x \in \bar{\mathcal{U}}_i$ and all $y \in \mathcal{Y}$. Fix $w = (x, y) \in \mathcal{X} \times \mathcal{Y}$ and $\varepsilon \in (0, \varepsilon^\star)$. finally, due to (40) and (41), convergence (37) and (38) simply follows from (42) and LaSalle's invariance principle. ■

*Lemma 6:* Assume that the reduced system (23) is Multistable with respect to an acyclic $\mathcal{W}$-limit set $\mathcal{W}$. In particular, assume that $f(x, h(x), 0) \in \mathfrak{X}^2(\mathcal{X}, \mathcal{X})$ and that each atom in the decomposition of $\mathcal{W}$, i.e. each $\mathcal{W}_i$, $i \in \{1, \ldots, N\}$, is a singleton consisting of a hyperbolic fixed point. Let $i \in \{1, \ldots, N\}$. There exist an open neighborhood $\bar{\mathcal{U}}_i$ of $\mathcal{W}_i := \{x_i\}$, a constant $c_i > 0$, and a $\mathcal{C}^1$ function $V_i : \bar{\mathcal{U}}_i \to \mathbb{R}$, such that $V_i(\mathcal{W}_i) = 0$, $dV_i(\mathcal{W}_i) = 0$ and:

$$dV_i(x) \cdot f(x, h(x), 0, 0)|_{t=0} \leq -c_i |x|^2_{\mathcal{W}_i} \;\; \text{for all } x \in \bar{\mathcal{U}}_i. \tag{43}$$

*Proof:* Hyperbolicity of fixed points and the $\mathcal{C}^2$-differentiability condition on vector field $f(x, h(x), 0)$ imply, via [13][2], the existence of an open neighborhood $\bar{\mathcal{U}}_i$ of $\mathcal{W}_i$ and a $\mathcal{C}^1$-diffeomorphism $\mathfrak{m}$ :

[2] It is mentioned in [23, Section 2.8] that, unlike Sternberg's result [28], Hartman's result [13] is established without the requirement of non-resonance of the hyperbolic fixed point, i.e. the abscence of any linear relationships of the kind $\exists k : \;\; \lambda_k = \sum_{j=1}^{n} m_j \lambda_j$ where the $\lambda_j$s are the eigenvalues of the linearization at the fixed point and the $m_j$s are any $n$ non-negative integers satisfying $\sum_{j=1}^{n} m_j \geq 2$.

$\bar{\mathcal{U}}_i \to \mathbb{R}^n$ with $\mathfrak{m}(\mathcal{W}_i) = 0$ such that the linearized flow[3] $\tilde{X}(t,\cdot)$ generated in local coordinates by

$$\dot{\tilde{x}} = Df(\mathcal{W}_i)\tilde{x} \tag{44}$$

and the nonlinear flow $X(t,x)$ generated by (24) are $\mathcal{C}^1$-conjugate on $\bar{\mathcal{U}}_i$, namely:

$$\mathfrak{m}(X(t,x)) = \tilde{X}(t,\mathfrak{m}(x)), \tag{45}$$

for all $x \in \bar{\mathcal{U}}_i$ and all $t \geq 0$ such that $X(t,x) \in \bar{\mathcal{U}}_i$. In particular, the diffeomorphism $\mathfrak{m}$ can be chosen in such a way to obtain the following decomposition of linearization (44):

$$\dot{\tilde{x}}_S = \Sigma_S \tilde{x}_S, \quad \dot{\tilde{x}}_U = \Sigma_U \tilde{x}_U, \tag{46}$$

where $\Sigma_S$ (respectively $\Sigma_U$) has eigenvalues in the open left (respectively right) half of the complex plane. Then, there exist matrices $P_S, P_U, Q_S, Q_U \succ 0$ such that:

$$\Sigma_S^\top P_S + P_S \Sigma_S \preceq -Q_S \prec 0$$
$$-\left(\Sigma_U^\top P_U + P_U \Sigma_U\right) \preceq -Q_U \prec 0.$$

Let $\tilde{V}_i : \mathbb{R}^n \to \mathbb{R}$ be defined as:

$$\tilde{V}_i(\tilde{x}) := \tilde{x}_S^\top P_S \tilde{x}_S - \tilde{x}_U^\top P_U \tilde{x}_U. \tag{47}$$

By taking the time derivative of (47) along the trajectories of (46), we obtain for any $x \in$ Range $h$

$$\begin{aligned}\dot{\tilde{V}}_i(\tilde{x}) &= -\tilde{x}_S^\top Q_S \tilde{x}_S - \tilde{x}_U^\top Q_U \tilde{x}_U \\ &\leq -\underline{\lambda}_{Q_S}|\tilde{x}_S|^2 - \underline{\lambda}_{Q_U}|\tilde{x}_U|^2 \leq -c_0|\tilde{x}|^2,\end{aligned} \tag{48}$$

where $c_0$ is a positive constant and $\underline{\lambda}_{Q_S}, \underline{\lambda}_{Q_U}$ denote the minimum eigenvalues of $Q_S$ and $Q_U$ respectively. Let $V_i(x) := \tilde{V}_i(\mathfrak{m}(x))$. By virtue of $\mathcal{C}^1$ conjugacy (45) and inequality (48), the time derivative of $V_i$ along the trajectories of the autonomous nonlinear system (24) reads as:

$$\begin{aligned}\frac{d}{dt}V_i(X(t,x))\bigg|_{t=0} &= \frac{d}{dt}\tilde{V}_i\left(\mathfrak{m}(X(t,x))\right)\bigg|_{t=0} \\ &= \frac{d}{dt}\tilde{V}_i\left(\tilde{X}(t,\mathfrak{m}(x))\right)\bigg|_{t=0} \leq -c_0|\mathfrak{m}(x)|^2,\end{aligned}$$

for all $x \in \bar{\mathcal{U}}_i$. Since $\mathfrak{m}$ is a diffeomorphism, it holds that $c_1\mathfrak{d}[x,y] \leq |\mathfrak{m}(x) - \mathfrak{m}(y)| \leq c_2\mathfrak{d}[x,y]$ for some $c_1, c_2 > 0$ and all $x, y \in \bar{\mathcal{U}}_i$, and thus $|\mathfrak{m}(x)| \geq c_1|x|_{\mathcal{W}_i}$ for all $x \in \bar{\mathcal{U}}_i$. Due to $V_i$ being a $\mathcal{C}^1$ function, it then follows that:

$$dV_i(x)\cdot f(x,h(x),0) = \frac{d}{dt}V_i(X(t,x))\bigg|_{t=0} \leq -c_i|x|^2_{\mathcal{W}_i}, \tag{49}$$

for all $x \in \bar{\mathcal{U}}_i$, with $c_i := c_0 c_1$. ■

## V. Examples

### A. Toggle switch with mRNA dynamics

Within the system biology community, the genetic toggle switch has become a classical example of gene-regulatory circuit exhibiting bistable behavior [11]. It consists of two promoters, i.e. the region of DNA where RNA polymerases start transcription, and two repressors, i.e. often proteins which bind to promoters to turn off the transcription of the gene encoded. As stated in [11], in the toggle switch each promoter is inhibited by the repressor that is transcribed by the opposing promoter. Let $x_1, x_2$ be the concentration of repressor 1 and 2 respectively, an let $z_1, z_2$ be the concentration of mRNA for repressor 1 and 2 respectively. In literature (see [33, Chapter 2] for a general presentation), dynamics of trascription (production of mRNA) and translation (production of proteins by ribosomes) are typically simplified into the following model:

$$\dot{z}_i = f_i(x_{3-i}) - \delta_i z_i \tag{50}$$
$$\dot{x}_i = \kappa_i z_i - \gamma_i x_i \qquad i \in \{1,2\}, \tag{51}$$

where $\delta_i, \kappa_i, \gamma_i > 0$ are constant rates and $f_i$ defined in the following. In the toggle switch, due to cross-repression, the presence of $x_1, x_2$ inhibits the transcription of gene $2, 1$ respectively. Repression dynamics is captured by production rates $f_1(\cdot), f_2(\cdot)$ in (50) and represented in the standard Hill function form as:

$$f_i(x_{3-i}) := \frac{\alpha_i}{1 + \left(\frac{x_{3-i}}{k_{3-i}}\right)^n}, \quad i \in \{1,2\}, \tag{52}$$

with some $n \in \mathbb{N}$. Typically, transcription dynamics are much faster than translation, i.e. $\alpha_i = \tilde{\alpha}_i/\varepsilon$ and $\delta_i = \tilde{\delta}_i/\varepsilon$ for $i \in \{1,2\}$, thus yielding a time-scale separation between (50) and (51). The steady state value of (50) is:

$$z_i^s = h_i(x) := \frac{\alpha_i}{\delta_i\left(1 + \left(\frac{x_{3-i}}{k_{3-1}}\right)^n\right)}, \quad i = \{1,2\},$$

which satisfies Assumption 4. Let $y_i := z_i - h_i(x)$ for $i \in \{1,2\}$. Boundary-layer system (35) then reads as

$$\frac{dy_i}{d\tau_i} = -\tilde{\delta}_i y_i,$$

and is clearly exponentially stable, uniformly in $x$. Reduced system (34) reads as:

$$\dot{x}_i = \frac{\beta_i}{\left(1 + \left(\frac{x_{3-i}}{k_{3-i}}\right)^n\right)} - \gamma_i x_i \qquad i \in \{1,2\}. \tag{53}$$

where $\beta_i := \kappa_i\alpha_i/\delta_i$ for $i \in \{1,2\}$. For the choice of parameters $\beta_i = 1, k_i = 1/2, \gamma_i = 1, n = 4$ with $i \in \{1,2\}$, system (53) has a globally attractive $\mathcal{W}$-limit set containing two hyperbolic stable equilibria and a hyperbolic saddle point, i.e. $\mathcal{W} := \{(\bar{x},\bar{y}),(\bar{y},\bar{x}),(1/2,1/2)\}$, where $\bar{x} \simeq 0.058$, $\bar{y} \simeq 0.999$ respectively are the smallest and greatest real solutions of equation $(x-1)(1+16x^4)^4 + 16x = 0$. It is easy to show that (53) has another invariant set: the separatrix $\{x = (x_1,x_2) \in \mathcal{X} : x_1 = x_2\}$. Furthermore, by making use of Bendixson's criterion, since the divergence of (53) is $-\gamma_1 - \gamma_2 = -2$ everywhere on $\mathcal{X}$, there is no closed orbit in $\mathcal{X}$, and thus $\mathcal{W}$ is an acyclic $\mathcal{W}$-limit set. By virtue of Theorem 3, we can conclude that for arbitrarily large initial conditions $w = (x,z) \in \mathcal{X} \times \mathcal{Z}$, there exist $\alpha_i^\star, \delta_i^\star$ with $i \in \{1,2\}$ such that:

$$\lim_{t\to+\infty}|X_{\alpha,\delta}(t,w)|_{\mathcal{W}} = \lim_{t\to+\infty}|Z_{\alpha,\delta}(t,w)|_{h(\mathcal{W})} = 0,$$

for any $\alpha_i > \alpha_i^\star$ and any $\delta_i > \delta_i^\star$, and where $X_{\alpha,\delta}, Z_{\alpha,\delta}$ denotes the flow of (50)-(52).

### B. Counterexample for Theorem 1

In this Section, we show that acyclicity of the $\mathcal{W}$-limit set of (2) is a necessary condition for semi-global practical multistability of (1). Consider the following system in polar coordinates:

$$\dot{z} = g(z,d) := \begin{cases} r(1-r^2) \\ \sin^2\left(\frac{\theta}{2}\right) + d, \end{cases} \tag{54}$$

with state $z = (r,\theta) \in \mathbb{R} \times \mathbb{S}$ and disturbance $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ being measurable and locally essentially bounded. The autonomous system $\dot{z} = g(z,0)$ has the following $\alpha$- and $\omega$-limit sets: the origin $w_1 = (0,\cdot)$ and the point $w_2 = (r,\theta) = (1,0)$. Let $\mathcal{W} = w_1 \cup w_2$ be the $\mathcal{W}$-limit set and observe that such decomposition is not acyclic

[3] With a slight abuse of notation, $Df(\mathcal{W}_i)$ in (44) denotes the state matrix of the linearization in local coordinates $\tilde{x}$.

due to the presence of an homoclinic cycle to $w_2$ whose image on the plane is $\Gamma := \{(r,\theta) : r = 1, \theta \in \mathbb{S}\}$, as depicted by the phase portrait in Figure 1. In order to prove that acyclicity of $\mathcal{W}$ is a necessary condition, we show that there exists a disturbance $\bar{d}(t)$ such that $\dot{z} = g(z, \bar{d}(t/\varepsilon))$ is not semi-globally practically multistable wrt $\mathcal{W}$, yet $g(z, \bar{d}(t))$ is a KBM-vector field with average $g(z, 0)$. To this aim, let:

$$\bar{d}(t) = \begin{cases} 1 & \text{for } t \in [k^2, k^2+1], \quad \forall\, k \in \mathbb{N} \\ 0 & \text{elsewhere.} \end{cases}$$

Observe that $\bar{d}(t)$ comprises a sequence of square pulses that have unit length, unit amplitude, and are increasingly spaced in time, thus

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \bar{d}(t) = 0.$$

Since such limit does not depend on $z$, we conclude that $g(z, \bar{d}(t))$ qualifies as a KBM-vector field with average $g(z, 0)$. In order to prove that $\dot{z} = g(z, \bar{d}(t/\varepsilon))$ is not semi-globally practically multistable wrt $\mathcal{W}$, we show that:

$$\begin{aligned} &\exists d_1 > 0, \;\; \exists d_2 > 0, \;\; \forall \varepsilon > 0 \text{ arbitrarily small }: \\ &\exists z_0 \text{ with } |z| \le d_1, \;\; \limsup_{t\to+\infty} |\phi_\varepsilon(t, z_0)|_{\mathcal{W}} > d_2. \end{aligned} \tag{55}$$

Let $d_1 = 1.5$ and $d_2 = 0.1$. Pick an arbitrarily small $\varepsilon > 0$. Observe that $\bar{d}(t/\varepsilon)$ comprises a sequence of square pulses that have length $\varepsilon$, unit amplitude, and are increasingly spaced in time. Pick any $z_0 = (r_0, \theta_0) \in \Gamma \setminus \{(1,0)\}$. Then, since $\dot{r} = 0$ and $\dot{\theta} = \sin^2(\theta/2) + \bar{d}(t/\varepsilon) > 0$ whenever $\theta \neq 0$, solution $\phi_\varepsilon(t, z)$ gets arbitrarily close to $(1,0)$. In particular, without loss of generality, there exists $k_1 \in \mathbb{N}$ such that $\theta_\varepsilon(\varepsilon k_1) \in [-\varepsilon/2, 0)$. At $t = \varepsilon k_1$, a square pulse of length $\varepsilon$ and amplitude 1 is applied to $\dot{\theta}$, thus yielding $\theta_\varepsilon(\varepsilon k_1 + \varepsilon) \ge \varepsilon/2$. Since $\dot{\theta} > 0$, a solution starting at $\theta_\varepsilon(\varepsilon k_1 + \varepsilon)$ follows the homoclinic orbit, thus there exists $t_1 \ge \varepsilon k_1 + \varepsilon$ such that $\theta(t_1) > d_2$, and there exists $k_2 \in \mathbb{N}$ such that $k_2 > k_1$ and $\theta_\varepsilon(\varepsilon k_2) \in [-\varepsilon/2, 0)$. By iterating the arguments above, we can construct an increasing sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that $\theta(t_n) > d_2$ for all $n \in \mathbb{N}$. Such sequence is unbounded due to $|\dot{\theta}|$ being bounded at all times. We can thus conclude that $\limsup_{t\to+\infty} |\phi_\varepsilon(t, z_0)|_{\mathcal{W}} > d_2$.

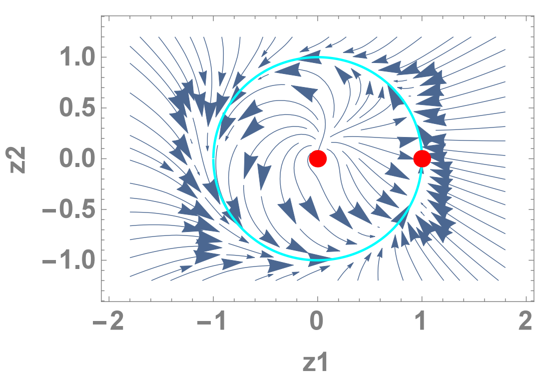


Fig. 1. *Example of Section V-B.* Phase portrait of system (54) with $d_\varepsilon(t) \equiv 0$ at all $t \ge 0$. The homoclinic cycle $\Gamma$ and the set $\mathcal{W} = w_1 \cup w_2$ are depicted in cyan and red respectively.

### C. Counterexample for Theorem 2

In this Section, we show by means of a counterexample that acyclicity of the $\mathcal{W}$-limit set of (2) is necessary for establishing the results of Theorem 2. Consider the following system in cylindrical coordinates:

$$\dot{x} = f(x,z) := z \tag{56}$$

$$\varepsilon\dot{z} = g(x,z) := \left(\sin(x/2)^2 - z\right)^3, \tag{57}$$

with state $x \in \mathbb{S}$ and $z \in \mathbb{R}$. Assumption 3 is satisfied with $z_s = h(x) := \sin(x/2)^2$. Boundary-layer system (25) reads as:

$$\frac{dy}{d\tau} = g(x, y + h(x)) = -y^3, \tag{58}$$

and its origin $y = 0$ is globally asymptotically stable, uniformly in $x \in \mathbb{S}$. It is then straightforward to show that reduced system

$$\dot{x} = f(x, h(x)) = \sin\left(\frac{x}{2}\right)^2 \tag{59}$$

has $w = \{0\}$ as the only $\alpha$- and $\omega$-limit set. Then, we may select select $\mathcal{W} = w$ as $\mathcal{W}$-limit set. Observe that our decomposition of $\mathcal{W}$ is not acyclic due to the presence of an homoclinic cycle to $w$ whose image on the circle is the circle itself.

In order to prove that acyclicity of $\mathcal{W}$ is a necessary condition, we show that (29) is not verified, i.e.:

$$\begin{aligned} &\exists d_1 > 0, \;\; \exists d_2 > 0, \;\; \exists \{\varepsilon_n\}_{n\in\mathbb{N}}, \;\; \exists w = (x,z) \in \mathbb{S}\times\mathbb{R} : \\ &\varepsilon_n \in (0, 1/n) \;\; \forall n \in \mathbb{N}, \;\; \max\{|x|_{\mathcal{W}}, |z|_{h(\mathcal{W})}\} \le d_1, \text{ and} \\ &\limsup_{t\to+\infty} |X_{\varepsilon_n}(t, w)|_{\mathcal{W}} > d_2. \end{aligned} \tag{60}$$

Let $d_1 = 1.5$ and $d_2 = 0.1$. Select any $n \in \mathbb{N}$ and any $\varepsilon_n \in (0, 1/n)$. In particular, we are going to show that for any $\varepsilon$ small enough, system (56)-(57) admits the existence of a limit cycle whose image on the $x$ coordinates is $\mathbb{S}$, and thus there exists a bounded $X_\varepsilon$ solution which is not eventually captured into the neighborhood $\mathbb{B}[\mathcal{W}, d_2]$. To this end, we will show that, for any $\varepsilon$ small enough, there exists a bounded region $R_\varepsilon$ of the $(\theta, z)$ cylinder where (56)-(57) has no fixed point and which captures any entering solution of (56)-(57) for all forward times. Finally, by an application of Poincaré-Bendixson theorem, we will conclude that all such solutions must converge to a limit cycle. Let $R_\varepsilon$ be the region of the cylinder $\mathbb{S}\times\mathbb{R}$ bounded above by the $\mathcal{C}^1$ curve $\bar{c}_\varepsilon : \mathbb{S} \to \mathbb{S}\times\mathbb{R}$ and below by the piecewise-$\mathcal{C}^1$ curve $\underline{c}_\varepsilon : \mathbb{S} \to \mathbb{S}\times\mathbb{R}$. Curve $\bar{c}_\varepsilon$ is defined as the mapping $\theta \mapsto (\theta, \bar{\gamma}_\varepsilon(\theta))$ where $\theta \in \mathbb{S}$ and $\bar{\gamma}_\varepsilon(\theta) := \sin^2(\theta/2) + \varepsilon^{(1/4)}$. Then, in order for all solutions of (56)-(57) with initial condition at $\bar{c}_\varepsilon(\theta)$ to enter $R_\varepsilon$ from above, the following transversality condition must be verified: $f[\bar{\gamma}_\varepsilon](\theta) < 0, \;\; \forall \theta \in \mathbb{S}$, where

$$f[\eta](\theta) := -\eta(\theta)\frac{\partial \eta(\theta)}{\partial \theta} + \frac{1}{\varepsilon}\left[\sin^2\left(\frac{\theta}{2}\right) - \eta(\theta)\right]^3.$$

It is easy to show that such condition is satisfied for all $\theta \in \mathbb{S}$ and all $\varepsilon \in (0, \varepsilon_0^\star)$, where $\varepsilon_0^\star := 3.16$.

Now, define $\underline{\gamma}_\varepsilon(\theta) := \varepsilon^3 \sin^2((\theta - \sqrt[3]{\varepsilon})/2)$ and $\underline{\nu}_\varepsilon(\theta) := -\underline{\gamma}_\varepsilon(\pi)\cos(\theta)$. Then curve $\underline{c}_\varepsilon$ is defined as the continuous concatenation of four $\mathcal{C}^1$ curves:

$$\begin{cases} \underline{c}_1 : & (\theta, \underline{\gamma}_\varepsilon(\theta)) & \text{for } \theta \in [-\pi, \; \underline{\theta}_\varepsilon], \\ \underline{c}_2 : & (\underline{\theta}_\varepsilon, z) & \text{for } z \in [0, \underline{\gamma}_\varepsilon(\underline{\theta}_\varepsilon)], \\ \underline{c}_3 : & (\theta, 0) & \text{for } \theta \in [\underline{\theta}_\varepsilon, \; \frac{\pi}{2}], \\ \underline{c}_4 : & (\theta, \underline{\nu}_\varepsilon(\theta)) & \text{for } \theta \in [\frac{\pi}{2}, \pi], \end{cases}$$

where $\underline{\theta}_\varepsilon > 0$ will be defined later. In order for all solutions of (56)-(57) with initial condition at $\underline{c}(\theta)$ to enter $R_\varepsilon$ from below, the following transversality conditions must be satisfied:

$$f[\underline{\gamma}_\varepsilon](\theta) > 0 \quad \forall\, \theta \in [-\pi,\ \underline{\theta}_\varepsilon], \tag{61}$$

$$\dot{\theta}(\theta, z) \geq 0 \quad \forall\, (\theta, z) \in \underline{c}_2, \tag{62}$$

$$\dot{z}(\theta, z) > 0 \quad \forall\, (\theta, z) \in \underline{c}_3, \tag{63}$$

$$f[\underline{\nu}_\varepsilon](\theta) > 0 \quad \forall\, \theta \in \left[\frac{\pi}{2}, \pi\right]. \tag{64}$$

It is straightforward to prove that there exists a small enough $\varepsilon_1^\star \in (0, \varepsilon_0^\star]$ such that conditions (62)-(64) are satisfied for all $\varepsilon \in (0, \varepsilon_1^\star]$. Regarding (61), it can be shown that there exists a small enough $\varepsilon_2^\star \in (0, \varepsilon_1^\star]$ which allows the use of linear approximation $\sin(\sqrt[3]{\varepsilon}) \simeq \sqrt[3]{\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_2^\star]$, which in addition implies:

$$f[\underline{\gamma}_\varepsilon](0) \simeq \frac{1}{8}\,\varepsilon^7 - \frac{1}{4^3}\varepsilon^{10} > 0,$$

for all $\varepsilon \in (0, \varepsilon_2^\star]$. It follows by continuity that, for all such $\varepsilon$s, there exists $\underline{\theta}_\varepsilon > 0$ such that:

$$f[\underline{\gamma}_\varepsilon](\theta) > 0, \quad \forall \theta \in [-\underline{\theta}_\varepsilon, \underline{\theta}_\varepsilon]. \tag{65}$$

We are now going to prove that $f[\underline{\gamma}_\varepsilon](\theta) > 0$ for all $\theta \in [-\pi, 0)$. Indeed, if $\theta < 0$, then:

$$f[\underline{\gamma}_\varepsilon](\theta) \geq \frac{1}{\varepsilon}\left(\sin^2\left(\frac{\theta}{2}\right) - \varepsilon^3\right)^3,$$

and thus $f[\underline{\gamma}_\varepsilon](\theta) > 0$ holds

$$\forall\, \theta \in \left[-\pi, -2\arcsin(\varepsilon^{3/2})\right). \tag{66}$$

We are now going to consider $\theta \in [-2\arcsin(\varepsilon^{3/2}),\, 0)$. Let $\varepsilon_3^\star \in (0, \varepsilon_2^\star]$ be such that linear approximation $\sin(s) \simeq s$ can be used for all $s \in [-\arcsin((\varepsilon_3^\star)^{3/2}) - (\varepsilon_3^\star)^{1/3}, 0]$. Under such approximation and given that $\theta < 0$, we have:

$$\begin{aligned}
f[\underline{\gamma}_\varepsilon](\theta) &= -\frac{1}{2}\varepsilon^6 \sin(\theta - \varepsilon^{1/3})\, \sin\left(\frac{\theta - \varepsilon^{1/3}}{2}\right)^2 + \\
&\quad \frac{1}{\varepsilon}\left[\sin^2\left(\frac{\theta}{2}\right) - \varepsilon^3\, \sin^2\left(\frac{\theta - \varepsilon^{1/3}}{2}\right)\right]^3 \\
&\geq \frac{1}{2}\varepsilon^6 \sin \varepsilon^{1/3}\, \sin\left(\frac{\theta - \varepsilon^{1/3}}{2}\right)^2 + \frac{1}{\varepsilon}\left[\sin^2\left(\frac{\theta}{2}\right) - \varepsilon^3\right]^3 \\
&\simeq \frac{\varepsilon^{6+\frac{1}{3}}}{8}\left(\theta^2 + \varepsilon^{\frac{2}{3}} - 2\varepsilon^{\frac{1}{3}}\theta\right) + \frac{1}{\varepsilon}\left(\frac{\theta^2}{4} - \varepsilon^3\right)^3 \\
&\geq \frac{\varepsilon^7}{8} + \frac{1}{\varepsilon}\left(-\frac{3\varepsilon^3}{16}\theta^4 + \frac{3\varepsilon^6}{4}\theta^2 - \varepsilon^9\right).
\end{aligned} \tag{67}$$

Expression (67) is positive if

$$3\theta^4 - 12\varepsilon^3\theta^2 - 2\varepsilon^5 + 16\varepsilon^6 < 0. \tag{68}$$

The two roots of the polynomial in (68) are given by:

$$\theta_{1,2}^2 = 2\varepsilon^3 \pm \frac{1}{6}\sqrt{(12\varepsilon^3)^2 - 12(16\varepsilon^6 - 2\varepsilon^5)}. \tag{69}$$

Since $\varepsilon^{5/2}$ dominates over $\varepsilon^3$ for all $\varepsilon$s small enough, one of the roots in (69) becomes negative, thus there exists $\varepsilon_4^\star \in (0, \varepsilon_3^\star]$ such that (68) is satisfied for all $\varepsilon \in (0, \varepsilon_4^\star]$ and

$$\forall\, \theta \in \left[-\sqrt{\frac{2}{\sqrt{6}}}\, \varepsilon^{\frac{5}{4}},\ 0\right]. \tag{70}$$

Since $\varepsilon^{5/4}$ dominates over $\varepsilon^{3/2}$ for all $\varepsilon$s small enough, there exists $\varepsilon_5^\star \in (0, \varepsilon_4^\star]$ such that, for all $\varepsilon \in (0, \varepsilon_5^\star]$, conditions (66) and (70) overlap. In addition, (65) holds, and we can thus conclude that (61) is satisfied for all $\varepsilon \in (0, \varepsilon_5^\star]$.

Since $R_\varepsilon$ does not contain a fixed point, we conclude by Poincaré-Bendixson theorem that system (56)-(57) admits the existence of a limit cycle for all $\varepsilon \in (0, \varepsilon_5^\star]$.

## VI. Conclusions

In [19], [20] the authors have shown that parameterized families of nonlinear systems whose solutions converge to those of an ISS system inherit ISS in a semi-global practical fashion. Similarly, in [15], [4], the authors have shown that singular perturbation models inherit ISS from the reduced subsystem, whenever the boundary-layer subsystem satisfies uniform asymptotic stability.

In this work, we first have considered a notion of multistability based on the existence of a compact, invariant, globally attractive set whose decomposition in compact and invariant subsets satisfies specific acyclicity conditions. Second, we have recalled the notion of Input-to-State Multistability introduced in [1]. Finally, we have extended the aforementioned results on perturbation theory and singular perturbations to the case of Input-to-State Multistable systems, respectively in Theorems 1 and 2. Throughout this process, we obtained a convergence result in singular perturbations models where the reduced subsystem globally converges to hyperbolic fixed points only (Theorem 3).

A central role in the proof of our main results is played by acylicity of the invariant set, as highlighted in [9, Section V.C] and Section V-C of the present work. The proposed perturbation theory finds immediate applications in systems where averaging techniques would typically be employed, and in highly oscillatory control systems as those studied in [29]. Singular perturbations of multistable systems are typically ubiquitous in gene regulatory circuits when non-neglecting RNA dynamics.

## Appendix A
## Proof of Theorem 1

Let $\hat{\eta}_u, \breve{\eta}_u, \eta_u$ be given by Corollary 6, Lemma 4, and Definition 7 respectively. Let $\tilde{\eta}_u$ be the class-$\mathcal{K}_\infty$ defined as $\tilde{\eta}_u(s) := \max\{\hat{\eta}_u(s), \breve{\eta}_u(s), \eta_u(s)\}$ for all $s \geq 0$. Let $D := \min_{i,j \in \{1,\ldots,N\}} \mathfrak{d}[\mathcal{W}_i, \mathcal{W}_j]$ be the minimum distance among the atoms of the decomposition of $\mathcal{W}$. Let $d_u^\star, d_u^{\star\star} > 0$ be such that $\tilde{\eta}_u(d_u^\star) = 7D/16$ and $\tilde{\eta}_u(d_u^{\star\star}) = D/2$ respectively.

Following [1] and subsequent publications [8], Input-to-State Multistability of (2) wrt set $\mathcal{W}$ and input $u$ entails the practical global stability (pGS) property, namely the existence of a class-$\mathcal{K}_\infty$ function $\beta$ and a constant $Q \geq 0$ such that

$$\begin{aligned}
&|\psi(t, q; w)|_{\mathcal{W}} \leq Q + \beta\left(\max\left\{|q|_{\mathcal{W}}, \|w\|\right\}\right), \\
&\quad \forall t \geq 0\ \ \forall q \in \mathcal{M}\ \ \forall w \in \mathcal{U}.
\end{aligned} \tag{71}$$

Let $\eta_u^\star$ be the class-$\mathcal{K}_\infty$ function satisfying $\eta_u^\star(s) = \tilde{\eta}_u(s)$ for all $s \in [0, d_u^\star)$, $\eta_u^\star(s) > Q + \beta(\max\{\tilde{\eta}_u(s), s\})$ for all $s \geq d_u^{\star\star}$, and $\eta_u^\star(s) \geq \tilde{\eta}_u(s)$ for all $s \in [d_u^\star, d_u^{\star\star})$.

Semi-global practical Input-to-State Multistability reads as:

$$\begin{aligned}
&\forall d_1, d_2, d_u > 0,\ \ \exists \varepsilon^\star > 0\ :\ \ \forall \varepsilon \in (0, \varepsilon^\star), \\
&\forall x \text{ with } |x| \leq d_1,\ \ \forall u \in \mathcal{L}(d_u),\ \ \forall c > 0,\ \ \exists T > 0 : \\
&\forall t \geq T,\ \ |\phi_\varepsilon(t, x; u)|_{\mathcal{W}} \leq \eta_u^\star(\|u\|) + d_2 + c.
\end{aligned} \tag{72}$$

Assume by contradiction that:

$$\begin{aligned}
&\exists d_1, d_2, d_u > 0,\ \ \forall \varepsilon^\star > 0\ \ \exists \varepsilon \in (0, \varepsilon^\star), \\
&\exists x \text{ with } |x| \leq d_1,\ \ \exists u \in \mathcal{L}(d_u),\ \ \exists c > 0 : \\
&\forall T > 0 :\ \ \exists \bar{t} \geq T,\ \ |\phi_\varepsilon(\bar{t}, x; u)|_{\mathcal{W}} > \eta_u^\star(\|u\|) + d_2 + c.
\end{aligned} \tag{73}$$

Contradiction hypothesis (73) reads as:

$$\begin{aligned}
&\exists\, d_1, d_2, d_u > 0,\ \ \exists\{\varepsilon_n\}_{n\in\mathbb{N}} > 0,\ \ \exists\{x_n\}_{n\in\mathbb{N}} \in \mathcal{M}, \\
&\exists\{u_n\}_{n\in\mathbb{N}} \in \mathcal{L}(d_u),\ \exists\{c_n\}_{n\in\mathbb{N}} > 0,\ \ \exists\{t_{n,m}\}_{n,m\in\mathbb{N}} > 0 : \\
&\varepsilon_n \in \left[0, \frac{1}{n}\right],\ \ |x_n| \leq d_1,\ \ t_{n,m} \geq m,\ \text{and} \\
&|\phi_{\varepsilon_n}(t_{n,m}, x_n; u_n)|_{\mathcal{W}} > \eta_u^\star(\|u_n\|) + d_2 + c_n.\ \ \forall n, m \in \mathbb{N}
\end{aligned} \tag{74}$$

By virtue of Lemma 3, the flow $\phi_\varepsilon$ is semiglobally practically Input-to-State Stable wrt compact set $A_N$, and thus there exists $\tilde{d}_1 > 0$ such that, for all $n$ large enough and all $t \geq 0$, $|\phi_{\varepsilon_n}(t, x_n; u_n)| \leq \tilde{d}_1$. Without loss of generality, we are now going to consider the following sequences defined for all $n \in \mathbb{N}$: $\tilde{x}_n := \phi_{\varepsilon_n}(t_{n,1}, x_n; u_n)$, $\tilde{u}_n(\cdot) := u_n(\cdot + t_{n,1})$, and $\tilde{t}_{n,m} := t_{n,m+1}$, for all $n, m \in \mathbb{N}$. For such sequences, property (74) reads as:

$$\begin{aligned} &\exists\, \tilde{d}_1, d_2, d_u > 0,\ \ \exists\{\varepsilon_n\}_{n\in\mathbb{N}} > 0,\ \ \exists\{\tilde{x}_n\}_{n\in\mathbb{N}} \in \mathcal{M}, \\ &\exists\{\tilde{u}_n\}_{n\in\mathbb{N}} \in \mathcal{L}(d_u),\ \exists\{c_n\}_{n\in\mathbb{N}} > 0,\ \ \exists\{\tilde{t}_{n,m}\}_{n,m\in\mathbb{N}} > 0: \\ &\varepsilon_n \in \left[0, \frac{1}{n}\right],\ \ |\tilde{x}_n| \leq \tilde{d}_1,\ \ \tilde{t}_{n,m} \geq m,\ \text{and} \\ &|\phi_{\varepsilon_n}(\tilde{t}_{n,m}, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} > \eta_u^\star(\|\tilde{u}_n\|) + d_2 + c_n.\ \ \forall n, m \in \mathbb{N} \end{aligned} \tag{75}$$

Furthermore, we have that:

$$|\tilde{x}_n|_{\mathcal{W}} > \eta_u^\star(\|\tilde{u}_n\|) + d_2 + c_n \text{ for all } n \in \mathbb{N}. \tag{76}$$

However, due to $|\tilde{x}_n| \leq \tilde{d}_1$ and $\tilde{u}_n \in \mathcal{L}(d_u)$, we may select a subsequence of the $\tilde{x}_n$s and $\tilde{u}_n$s such that $\lim_{n\to+\infty}\|\tilde{u}_n\| = \underline{d}_u$ for some $\underline{d}_u \leq d_u$ and $\lim_{n\to+\infty}\tilde{x}_n =: \bar{x}$ for some $\bar{x} \in \mathfrak{B}(\mathcal{W}, d_1)$. For such subsequences of the $\tilde{x}_n$s and $\tilde{u}_n$s, property (76) implies that:

$$\bar{x} \in \operatorname{clos}\left[\mathfrak{B}(\mathcal{W}, d_1) \setminus \mathfrak{B}(\mathcal{W}, \eta^\star(\underline{d}_u) + d_2)\right]. \tag{77}$$

Two cases arise: $\underline{d}_u \geq d_u^{\star\star}$ and $\underline{d}_u < d_u^{\star\star}$.

*(Case $\underline{d}_u \geq d_u^{\star\star}$).*

We may further pass to two subsequences of the $\tilde{x}_n$s and $\tilde{u}_n$s in order to have property (75) holding true together with:

$$\begin{aligned} &|\phi_{\varepsilon_n}(\tilde{t}_{n,m}, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}_h} \geq |\phi_{\varepsilon_n}(\tilde{t}_{n,m}, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \\ &\quad > \eta_u^\star(\underline{d}_u) + d_2/2 + c_n,\quad \forall n, m \in \mathbb{N}\ \ \forall h \in \{1, \ldots, N\}, \end{aligned} \tag{78}$$

$$\text{and } \|\tilde{u}_n\| \in (\underline{d}_u - d_2/2, \underline{d}_u + d_2/2)\,,\quad \forall n \in \mathbb{N}. \tag{79}$$

By virtue of Corollary 4, we can select a strictly decreasing sequence of positive constants $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v \in \mathcal{L}(\underline{d}_u + d_2/2)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence of the $\varepsilon_n$s, $\tilde{x}_n$s, and $\tilde{u}_n$s such that:

$$\mathfrak{d}\left[\psi(t, \bar{x}; v), \phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)\right] \leq d_{\kappa,i}\ \ \forall t \in [i, i+1]\ \ \forall n \geq n_i. \tag{80}$$

By making use of [14, Lemma 10.4.4], Input-to-State Multistability of (2) entails the asymptotic gain property:

$$\limsup_{t\to+\infty}|\psi(t, \bar{x}; v)|_{\mathcal{W}} \leq \tilde{\eta}_u\big(\limsup_{t\to+\infty}|v(t)|\big) \leq \eta_u^\star(\underline{d}_u). \tag{81}$$

Then, by combining (80) and (81), we have that:

$$\begin{aligned} &\forall\theta > 0\ \ \exists T_\theta > 0\ \ \forall i \in \mathbb{N}_{\geq T_\theta}\ \ \exists n_i \in \mathbb{N}:\ \ \forall n \geq n_i \\ &\quad [\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)]_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + \theta\ \ \ \forall t \in [i, i+1]. \end{aligned} \tag{82}$$

Select $\theta = d_2/4$ and $i = \lceil T_{d_2/4}\rceil$ in (82) to obtain $|\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/4$ for all $t \in [\lceil T_{d_2/4}\rceil, \lceil T_{d_2/4}\rceil + 1]$ and all $n \geq n_{\lceil T_{d_2/4}\rceil}$. However, recall that (78) holds with $t_{n,m} \geq m$ for all $m \in \mathbb{N}$. Therefore, for all $n \geq n_{\lceil T_{d_2/4}\rceil}$, there exists a maximal time

$$\begin{aligned} \tau_n := \inf\big\{&t \geq \lceil T_{d_2/4}\rceil + 1 \text{ such that} \\ &|\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} > \eta_u^\star(\underline{d}_u) + d_2/2\big\}. \end{aligned}$$

before which the $\phi_{\varepsilon_n}$ are contained in $\mathfrak{B}(\mathcal{W}, \eta^\star(\underline{d}_u) + d_2/2)$, namely:

$$\begin{aligned} &|\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/2, \\ &\qquad \forall t \in [\lceil T_{d_2/4}\rceil, \tau_n]\ \ \ \forall n \geq n_{\lceil T_{d_2/4}\rceil}. \end{aligned} \tag{83}$$

*Claim 1:* $\lim_{n\to+\infty}\tau_n = +\infty$.

*Proof:* By virtue of (82) and without loss of generality, we can select a non-decreasing subsequence of $n_i$s such that $|\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/4$ for all $t \in [i, i+1]$, all $n \geq n_i$, and all $i \in \mathbb{N}$ with $i \geq T_{d_2/4}$. Since the $n_i$s are non-decreasing, we have that $|\phi_{\varepsilon_n}(t, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/4$ for all $t \in [\lceil T_{d_2/4}\rceil, \lceil T_{d_2/4} + j + 1\rceil]$, all $n \geq n_j$, and all $j \in \mathbb{N}$. It then follows that $\tau_n > \lceil T_{d_2/4}\rceil + j + 1$ for all $n \geq n_j$, and all $j \in \mathbb{N}$. It thus follows: $\lim_{n\to+\infty}\tau_n = \lim_{j\to+\infty}\tau_{n_j} \geq \lim_{j\to+\infty} j = +\infty$. ■

Let $y_n := \phi_{\varepsilon_n}(\tau_n, \tilde{x}_n; \tilde{u}_n)$ for all $n \in \mathbb{N}$ large enough. By definition of $\tau_n$, it holds that $y_n \in \mathfrak{d}\mathfrak{B}(\mathcal{W}, \eta_u^\star(\underline{d}_u) + d_2/2)$, and thus there exists a subsequence of the $y_n$s such that

$$\lim_{n\to+\infty} y_n =: \bar{y}_1 \in \mathfrak{d}\mathfrak{B}(\mathcal{W}, \eta_u^\star(\underline{d}_u) + d_2/2). \tag{84}$$

Property (83) reads as:

$$\begin{aligned} &|\phi_{\varepsilon_n}(t, y_n; \Delta^{\tau_n} u_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/2, \\ &\quad \forall t \in [-\tau_n + \lceil T_{d_2/4}\rceil, 0]\ \ \ \forall n \text{ large enough.} \end{aligned} \tag{85}$$

By virtue of Corollary 5, we can select a strictly decreasing sequence of positive constants $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v_{1,b} \in \mathcal{L}(\underline{d}_u + d_2/2)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v_{1,b}(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence of the $\varepsilon_n$s, $y_n$s, and $u_n$s such that

$$\begin{aligned} &\mathfrak{d}\left[\psi(t, \bar{y}_1; v_{1,b}), \phi_{\varepsilon_n}(t, y_n; \Delta^{\tau_n} u_n)\right] \leq d_{\kappa,i}, \\ &\quad \forall t \in [-i-1, -i]\ \ \forall n \geq n_i. \end{aligned} \tag{86}$$

*Claim 2:* There exists $L > 0$ such that $|\psi(t, \bar{y}_1; v_{1,b})|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + L$ for all $t \leq 0$.

*Proof:* Fix $t \leq 0$. Select $\bar{n} \in \mathbb{N}$ such that $t \in [-\tau_n + \lceil T_{d_2/4}\rceil, 0]$ for all $n \geq \bar{n}$. Then, from (85), we have that $|\phi_{\varepsilon_n}(t, y_n; \Delta^{\tau_n} u_n)|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/2$ holds for all $n$ large enough. The combination of the latter inequality and property (86) with $i \in \mathbb{N}$ such that $t \in [i, i+1]$ yields $|\psi(t, \bar{y}_1; v_{1,b})|_{\mathcal{W}} \leq \eta_u^\star(\underline{d}_u) + d_2/2 + d_{\kappa,0}$. ■

By virtue of Claim 2 and Corollary 6, it holds that: $\limsup_{t\to-\infty}|\psi(t, \bar{y}_1; v_{1,b})|_{\mathcal{W}} \leq \tilde{\eta}(\limsup_{t\to-\infty}|v_{1,b}(t)|) = \tilde{\eta}(\underline{d}_u)$. The latter property implies that

$$|\psi(\bar{T}, \bar{y}_1; v_{1,b})|_{\mathcal{W}} \leq \tilde{\eta}(\underline{d}_u) + d_2/2, \tag{87}$$

for some $\bar{T} < 0$. It follows from (71) and (87) that:

$$\begin{aligned} |\bar{y}_1|_{\mathcal{W}} &\leq Q + \beta\left(\max\{|\psi(\bar{T}, \bar{y}_1; v_{1,b})|_{\mathcal{W}}, \|v_{1,b}\|\}\right) \\ &\leq Q + \beta\left(\max\{\tilde{\eta}(\underline{d}_u) + d_2/2, \underline{d}_u + d_2/2\}\right). \end{aligned} \tag{88}$$

However, due to (84), our definition of $\eta_u^\star$, and the fact that $\underline{d}_u \geq d_u^{\star\star}$, it holds that:

$$|\bar{y}_1|_{\mathcal{W}} = \eta_u^\star(\underline{d}_u) + d_2/2 > d_2/2 + Q + \beta(\max\{\tilde{\eta}_u(\underline{d}_u), \underline{d}_u\}) \tag{89}$$

which contradicts (88).

*(Case $\underline{d}_u < d_u^{\star\star}$).* We may consider the sequences of the $\tilde{x}_n$s, $\tilde{u}_n$s, $c_n$s, and $\tilde{t}_{n,m}$s satisfying (75), (76), and (77). From our definition of $d_u^{\star\star}$, it holds that $\tilde{\eta}_u(\underline{d}_u) < D/2$, and we can thus select $\bar{\theta} > 0$ such that $\tilde{\eta}_u(\underline{d}_u + \bar{\theta}) < D/2$ and $\tilde{\eta}_u(\underline{d}_u + \bar{\theta}) < \eta_u^\star(\underline{d}_u) + d_2/2$.

*(Step 1: there exist an input $v_{0,f} \in \mathcal{L}(\underline{d}_u + \bar{\theta}/2)$ and an atom index $h_0 \in \{1, \ldots, N\}$ such that $\limsup_{t\to+\infty}|\psi(t, \bar{x}; v_{0,f})|_{\mathcal{W}_{h_0}} \leq \tilde{\eta}_u(\underline{d}_u)$).* We may further pass to two subsequences of the $\tilde{x}_n$s and $\tilde{u}_n$s in order to have property (75) holding true together with:

$$\begin{aligned} &|\phi_{\varepsilon_n}(\tilde{t}_{n,m}, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}_h} \geq |\phi_{\varepsilon_n}(\tilde{t}_{n,m}, \tilde{x}_n; \tilde{u}_n)|_{\mathcal{W}} \\ &\quad > \eta_u^\star(\underline{d}_u) + d_2/2 + c_n,\quad \forall n, m \in \mathbb{N}\ \ \forall h \in \{1, \ldots, N\}, \end{aligned} \tag{90}$$

$$\text{and } \|\tilde{u}_n\| \in \left(\underline{d}_u - \bar{\theta}/2, \underline{d}_u + \bar{\theta}/2\right),\quad \forall n \in \mathbb{N}. \tag{91}$$

By virtue of Corollary 4, we can select a strictly decreasing sequence of positive constants $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v_{0,f} \in \mathcal{L}(\underline{d}_u + \bar{\theta}/2)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v_{0,f}(t)| = \underline{d}_u$ and, for any

$i \in \mathbb{N} \cup \{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence of the $\varepsilon_n$s, $x_n$s, and $u_n$s such that

$$\mathfrak{d}\left[\psi(t,\bar{x};v_{0,f}),\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)\right] \le d_{\kappa,i} \quad \forall t \in [i,i+1] \quad \forall n \ge n_i. \quad (92)$$

By making use of [14, Lemma 10.4.4], Input-to-State Multistability of (2) entails the asymptotic gain property:

$$\limsup_{t\to+\infty} |\psi(t,\bar{x};v_{0,f})|_{\mathcal{W}} \le \tilde{\eta}_u\big(\limsup_{t\to+\infty}|v_{0,f}(t)|\big) \le \eta_u^\star(\underline{d}_u). \quad (93)$$

Since $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_N$ and $\tilde{\eta}_u(\underline{d}_u) < D/2$, there exists an index $h_0 \in \{1,\ldots,N\}$ such that the AG property reads as:

$$\limsup_{t\to+\infty} |\psi(t,\bar{x};v_{0,f})|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u). \quad (94)$$

*(Step 2: solutions $\phi_{\varepsilon_n}$ enter and escape the ball $\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$ for $n$ large enough).*

By combining (92) and (94), we have that:

$$\begin{aligned}&\exists T_{\bar\theta} > 0 \ \ \forall i \in \mathbb{N}_{\ge T_{\bar\theta}} \ \ \exists n_i \in \mathbb{N}: \ \ \forall n \ge n_i \\ &|\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta}) < D/2 \quad \forall t \in [i,i+1]. \quad (95)\end{aligned}$$

Select $i = \lceil T_{\bar\theta} \rceil$ in (95) to obtain $|\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta})$ for all $t \in [\lceil T_{\bar\theta}\rceil, \lceil T_{\bar\theta}\rceil+1]$ and all $n \ge n_{\lceil T_{\bar\theta}\rceil}$. However, recall that (90) holds with $t_{n,m} \ge m$ for all $m \in \mathbb{N}$. Therefore, since $\eta_u^\star(\underline{d}_u) + d_2/2 > \tilde{\eta}_u(\underline{d}_u+\bar{\theta})$, it follows for all $n \ge n_{\lceil T_{\bar\theta}\rceil}$ that there exists a maximal time

$$\begin{aligned}\tau_n := \inf\big\{t \ge \lceil T_{\bar\theta}\rceil + 1 \text{ such that} \\ |\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}_{h_0}} > \tilde{\eta}_u(\underline{d}_u+\bar{\theta})\big\}.\end{aligned}$$

before which the $\phi_{\varepsilon_n}$ are contained in $\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$, namely:

$$\begin{aligned}&|\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta}), \\ &\forall t \in [\lceil T_{\bar\theta}\rceil, \tau_n] \quad \forall n \ge n_{\lceil T_{\bar\theta}\rceil}. \quad (96)\end{aligned}$$

*Claim 3:* $\lim_{n\to+\infty} \tau_n = +\infty$.

*Proof:* By virtue of (95) and without loss of generality, we can select a non-decreasing subsequence of $n_i$s such that $|\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta})$ for all $t \in [i,i+1]$, all $n \ge n_i$, and all $i \in \mathbb{N}$ with $i \ge T_{\bar\theta}$. Since the $n_i$s are non-decreasing, we have that $|\phi_{\varepsilon_n}(t,\tilde{x}_n;\tilde{u}_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta})$ for all $t \in [\lceil T_{\bar\theta}\rceil, \lceil T_{\bar\theta}+j+1\rceil]$, all $n \ge n_j$, and all $j \in \mathbb{N}$. It then follows that $\tau_n > \lceil T_{\bar\theta}\rceil + j + 1$ for all $n \ge n_j$, and all $j \in \mathbb{N}$. It thus follows: $\lim_{n\to+\infty}\tau_n = \lim_{j\to+\infty}\tau_{n_j} \ge \lim_{j\to+\infty} j = +\infty$. ■

*(Step 3: there exist an input $v_{1,b} \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ and a point $\bar{y}_1 \in \eth\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$ such that $\limsup_{t\to-\infty}|\psi(t,\bar{y}_1;v_{1,b})|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u)$).*

Let $y_n := \phi_{\varepsilon_n}(\tau_n,\tilde{x}_n;\tilde{u}_n)$ for all $n \in \mathbb{N}$ large enough. By definition of $\tau_n$, it holds that $y_n \in \eth\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$, and thus there exists a subsequence of the $y_n$s such that

$$\lim_{n\to+\infty} =: \bar{y}_1 \in \eth\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u+\bar{\theta})). \quad (97)$$

Property (96) reads as:

$$\begin{aligned}&|\phi_{\varepsilon_n}(t,y_n;\Delta^{\tau_n}u_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta}), \\ &\forall t \in [-\tau_n+\lceil T_{\bar\theta}\rceil, 0] \quad \forall n \text{ large enough.} \quad (98)\end{aligned}$$

By virtue of Corollary 5, we can select a strictly decreasing sequence of positive constants $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v_{1,b} \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v_{1,b}(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence of the $\varepsilon_n$s, $x_n$s, and $u_n$s such that (86) holds.

*Claim 4:* There exists $\bar{i} \in \mathbb{N}$ such that $|\psi(t,\bar{y}_1;v_{1,b})|_{\mathcal{W}_{h_0}} < \tilde{\eta}_u(\underline{d}_u+\bar{\theta}) + d_{\kappa,\bar{i}} < D/2$ for all $t \le -\bar{i}$.

*Proof:* Since $\tilde{\eta}_u(\underline{d}_u+\bar{\theta}) < D/2$ and $\lim_{i\to+\infty} d_{\kappa,i} = 0$, we can select $\bar{i} \in \mathbb{N}$ such that $\tilde{\eta}_u(\underline{d}_u+\bar{\theta}) + d_{\kappa,\bar{i}} < D/2$. Fix $t \le -\bar{i}$. Select $\bar{n} \in \mathbb{N}$ such that $t \in [-\tau_n+\lceil T_{\bar\theta}\rceil, 0]$ for all $n \ge \bar{n}$. Then, from (98), we have that $|\phi_{\varepsilon_n}(t,y_n;\Delta^{\tau_n}u_n)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u+\bar{\theta})$ holds for all $n$ large enough. The combination of the latter inequality and property (86) with $i \in \mathbb{N}$ such that $i \ge \bar{i}$ and $t \in [-i-1,-i]$ yields $|\psi(t,\bar{y}_1;v_{1,b})|_{\mathcal{W}_{h_0}} \le \eta_u^\star(\underline{d}_u+\bar{\theta}) + d_{\kappa,\bar{i}} < D/2$. ■

By virtue of Claim 4 and Corollary 6, and due to the fact that $\tilde{\eta}_u(\underline{d}_u+\bar{\theta}) < D/2$, we can conclude that:

$$\limsup_{t\to-\infty}|\psi(t,\bar{y}_1;v_{1,b})|_{\mathcal{W}_{h_0}} \le \tilde{\eta}(\limsup_{t\to-\infty}|v_{1,b}(t)|) = \tilde{\eta}(\underline{d}_u). \quad (99)$$

*(Step 4: there exist an input $v_{1,f} \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ and an atom index $h_1 \in \{1,\ldots,N\}$ such that $\limsup_{t\to+\infty}|\psi(t,\bar{y}_1;v_{1,f})|_{\mathcal{W}_{h_1}} \le \tilde{\eta}_u(\underline{d}_u)$).*

By virtue of Corollary 4, we can select a strictly decreasing sequence of positive constants $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v_{1,f} \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v_{1,f}(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence of the $\varepsilon_n$s, $y_n$s, and $u_n$s such that:

$$\mathfrak{d}\left[\psi(t,\bar{y}_1;v_{1,f}),\phi_{\varepsilon_n}(t,y_n;\Delta^{\tau_n}u_n)\right] \le d_{\kappa,i} \quad \forall t \in [i,i+1] \quad \forall n \ge n_i. \quad (100)$$

Since $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_N$ and $\tilde{\eta}_u(\underline{d}_u) < D/2$, there exists an atom index $h_1 \in \{1,\ldots,N\}$ such that the AG property reads as:

$$\limsup_{t\to+\infty}|\psi(t,\bar{y}_1;v_{1,f})|_{\mathcal{W}_{h_1}} \le \tilde{\eta}_u\big(\limsup_{t\to+\infty}|v_{0,f}(t)|\big) = \tilde{\eta}_u(\underline{d}_u). \quad (101)$$

*(Step 5: there exists an input $v_1 \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ such that $\psi(t,\bar{y}_1;v_1)$ asymptotically approaches $\mathfrak{B}(\mathcal{W}_{h_0},\tilde{\eta}_u(\underline{d}_u))$ in backward time and $\mathfrak{B}(\mathcal{W}_{h_1},\tilde{\eta}_u(\underline{d}_u))$ in forward time).*

Define $v_1 \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ as the concatenation: $v_1(s) := v_{1,b}(s)$ for all $s < 0$ and $v_1(s) := v_{1,f}(s)$ for all $s \ge 0$. By virtue of (99) and (101), we have that:

$$\begin{aligned}&\limsup_{t\to-\infty}|\psi(t,\bar{y}_1;v_1)|_{\mathcal{W}_{h_0}} \le \tilde{\eta}_u(\underline{d}_u) \\ &\limsup_{t\to+\infty}|\psi(t,\bar{y}_1;v_1)|_{\mathcal{W}_{h_1}} \le \tilde{\eta}_u(\underline{d}_u).\end{aligned}$$

*(Iteration step).* By relabeling $x_n := y_n$, and due to our contradiction assumption (78) and (79), it is possible to show along the lines of Step 2 that solutions $\phi_{\varepsilon_n}$ enter and escape the ball $\mathfrak{B}(\mathcal{W}_{h_1},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$ for $n$ large enough. It is then possible to show along the lines of Step 3 that there exist an input $v_{2,b} \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ and a point $\bar{y}_2 \in \eth\mathfrak{B}(\mathcal{W}_{h_1},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}))$ such that $\limsup_{t\to-\infty}|\psi(t,\bar{y}_2;v_{2,b})|_{\mathcal{W}_{h_1}} \le \tilde{\eta}_u(\underline{d}_u))$. It is then possible to show along the lines of Steps 4 and 5 that there exist an input $v_2 \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$ and an atom index $h_2 \in \{1,\ldots,N\}$ such that $\psi(t,\bar{y}_2;v_2)$ asymptotically approaches $\mathfrak{B}(\mathcal{W}_{h_1},\tilde{\eta}_u(\underline{d}_u))$ in backward time and $\mathfrak{B}(\mathcal{W}_{h_2},\tilde{\eta}_u(\underline{d}_u))$ in forward time.

*(Conclusions).* We can iterate the previous step in order to construct a sequence of points $\{\bar{y}_j\}_{j\in\mathbb{N}}$, inputs $\{v_j\}_{j\in\mathbb{N}}$, and atom indices $\{h_j\}_{j\in\mathbb{N}}$ which satisfy $\bar{y}_j \notin \mathfrak{B}(\mathcal{W},\tilde{\eta}_u(\underline{d}_u+\bar{\theta}/2))$, $v_j \in \mathcal{L}(\underline{d}_u+\bar{\theta}/2)$, and

$$\begin{aligned}&\limsup_{t\to-\infty}|\psi(t,\bar{y}_j;v_j)|_{\mathcal{W}_{h_{j-1}}} \le \tilde{\eta}_u(\underline{d}_u) \\ &\limsup_{t\to+\infty}|\psi(t,\bar{y}_j;v_j)|_{\mathcal{W}_{h_j}} \le \tilde{\eta}_u(\underline{d}_u),\end{aligned}$$

for all $j \in \mathbb{N}$. However, the existence of such sequences together with the fact that the number of atoms in the decomposition of $\mathcal{W}$ is finite contradicts Lemma 4. Q.E.D.

## Appendix B
## Proof of Theorem 2

Recall that compactness of $A_N$ and inclusion $\mathcal{W} \subseteq A_N$ imply the existence of $Q > 0$ such that:

$$|q|_{A_N} \le |q|_{\mathcal{W}} \le |q|_{A_N} + Q, \quad \forall q \in \mathcal{X}. \quad (102)$$

Let $\beta_x, \beta_y, \gamma^\star$ be given by Lemma 5. Let $B : \mathcal{X} \to \mathbb{R}_{m\times m}$ and $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be given by Lemma 7. Pick any $d_1, d_2 > 0$. Define:

$$\tilde{b} := b\left(\beta_x(d_1,0) + \gamma^\star(d_1) + Q + d_2\right), \tag{103}$$

$$\tilde{d}_2 := \frac{d_2}{4}\,\tilde{b}, \tag{104}$$

$$\tilde{d}_1 := \max\left\{d_1,\ \frac{\beta_y\left(d_1,0\right) + \tilde{d}_2}{\tilde{b}}\right\}. \tag{105}$$

Let $\varepsilon_0^\star$ be the $\varepsilon^\star(d_1, \tilde{d}_2)$ given by Lemma 5. It then follows from (102) and Lemma 5 that the solutions of (26) are bounded for all forward times, i.e.

$$\begin{aligned} |X_\varepsilon(t,w;\theta)|_\mathcal{W} &\leq \beta_x(|x|_\mathcal{W}, t) + \gamma^\star(\|\theta\|) + Q + \tilde{d}_2 \\ &\leq \beta_x(d_1,0) + \gamma^\star(d_1) + Q + \tilde{d}_2, \end{aligned} \tag{106}$$

$$\begin{aligned} |Y_\varepsilon(t,w;\theta)| &\leq \beta_y\left(|y|, \frac{t}{\varepsilon}\right) + \tilde{d}_2 \\ &\leq \beta_y\left(d_1,0\right) + \tilde{d}_2 \end{aligned} \tag{107}$$

hold for all $t \geq 0$, $\varepsilon \in (0, \varepsilon_0^\star]$, and any $w = (x,y) \in \mathcal{X} \times \mathcal{Z}$ and $\theta \in \mathcal{L}(\Theta)$ such that $\max\left\{|x|_\mathcal{W}, |y|, \|\theta\|, \left\|\dot{\theta}\right\|\right\} \leq d_1$. Observe that (107) immediately proves (30).

By virtue of Lemma 7, system (115) is Input-to-State Multistable wrt set $\mathcal{W}$ and inputs $\theta$, $\tilde{y}$. Due to continuity of $f$ wrt to $\varepsilon$, we observe that the solutions of the perturbed system

$$\dot{x} = f(x, h(x,\theta) + B(x)\tilde{y}, \theta, \varepsilon) \tag{108}$$

converge uniformly to those of system (115) on compact time intervals, i.e. the flows of (115) and (108) satisfy Assumption 2. It then follows from Theorem 1 that (108) is semi-globally practically Input-to-State Multistable wrt set $\mathcal{W}$ and inputs $\theta$, $\tilde{y}$. Denote with $X_\varepsilon^B(t,x;\theta,\tilde{y})$ the solution of (108) with initial condition $x$ and inputs $\theta, \tilde{y}$. We can select $\varepsilon_1^\star = \varepsilon(d_1, d_2/2, \tilde{d}_1)$ in Corollary 2 so as to have:

$$\begin{aligned} \limsup_{t\to+\infty} |X_\varepsilon^B(t,x;\theta,\tilde{y})|_\mathcal{W} &\leq \eta_\theta^\star(\limsup_{t\to+\infty}|\theta(t)|) \\ &+ \eta_{\tilde{y}}^\star(\limsup_{t\to+\infty}|\tilde{y}(t)|) + \frac{d_2}{2}. \end{aligned} \tag{109}$$

for any $\varepsilon \in (0, \varepsilon_1^\star]$, any $|x| \leq d_1$, and any $\tilde{y}, \theta \in \mathcal{L}(\tilde{d}_1)$.

We are now going to prove Theorem 2 by selecting $\varepsilon^\star := \min\{\varepsilon_0^\star, \varepsilon_1^\star\}$ and by picking $\varepsilon \in (0, \varepsilon^\star]$, $w = (x,y) \in \mathcal{X} \times \mathcal{Z}$, and $\theta \in \mathcal{L}(\Theta)$ satisfying $\max\left\{|x|_\mathcal{W}, |y|, \|\theta\|\ \left\|\dot{\theta}\right\|\right\} \leq d_1$. Define:

$$\tilde{y}(t) := B^{-1}(X_\varepsilon(t,w;\theta))Y_\varepsilon(t,w;\theta), \ \ \forall t \geq 0. \tag{110}$$

In particular, due to $b(\cdot)$ being non-increasing and due to definitions (110), (104), (103), (105) and inequalities (106)-(107), we have that:

$$\begin{aligned} |\tilde{y}(t)| &\leq \frac{|Y_\varepsilon(t,w;\theta)|}{b(|X_\varepsilon(t,w;\theta)|_\mathcal{W})} \\ &\leq \frac{\beta_y\left(d_1, \frac{t}{\varepsilon}\right) + \tilde{d}_2}{\tilde{b}} \leq \tilde{d}_1, \ \ \forall t \geq 0. \end{aligned} \tag{111}$$

By definition of perturbed system (108) and signal $\tilde{y}$ in (110), it follows that:

$$X_\varepsilon^B(t,x;\theta,\tilde{y}) \equiv X_\varepsilon(t,w;\theta), \ \ \forall t \geq 0. \tag{112}$$

It then follows from (109), (111), and (112) that:

$$\begin{aligned} \limsup_{t\to+\infty} |X_\varepsilon(t,w;\theta)|_\mathcal{W} &\leq \eta_\theta^\star(\limsup_{t\to+\infty}|\theta(t)|) \\ &+ \eta_{\tilde{y}}^\star(\limsup_{t\to+\infty}|\tilde{y}(t)|) + \frac{d_2}{2}. \end{aligned} \tag{113}$$

Let $T \geq 0$ be such that $\beta_y\left(d_1, t/\varepsilon\right) \leq \tilde{d}_2$ for all $t \geq T$. It then follows from definitions (103)-(104) and property (111) that

$$|\tilde{y}(t)| \leq \frac{\beta_y\left(d_1, \frac{t}{\varepsilon}\right) + \tilde{d}_2}{\tilde{b}} \leq \frac{2\tilde{d}_2}{\tilde{b}} \leq \frac{d_2}{2}, \ \ \forall t \geq T.$$

By making use of the latter inequality, (113) implies (29). Q.E.D.

*Lemma 7:* Assume that

$$\frac{dx}{dt} = f(x, h(x,\theta) + y, \theta, 0). \tag{114}$$

with $y(t) \equiv 0$, $\forall t \geq 0$ is Input-to-State Multistable wrt set $\mathcal{W}$ and input $\theta$. Then, there exists an $m \times m$ matrix $B(x)$ of smooth functions, invertible for all $x \in \mathcal{M}$ and satisfying $B(x) \equiv I_{m\times m}$ in a neighborhood of $\mathcal{W}$, such that

$$\dot{x} = f(x, h(x,\theta) + B(x)\tilde{y}, \theta, 0) \tag{115}$$

is Input-to-State Multistable wrt set $\mathcal{W}$ and inputs $\theta, \tilde{y}$. In particular, there exist two class-$\mathcal{K}_\infty$ functions $\eta_\theta$ and $\eta_{\tilde{y}}$ such that:

$$\limsup_{t\to+\infty} |X(t,x;\theta(\cdot),\tilde{y}(\cdot))|_\mathcal{W} \leq \eta_\theta(\|\theta\|) + \eta_{\tilde{y}}(\|\tilde{y}\|), \tag{116}$$

for all $x \in \mathcal{M}$, $\theta \in \mathcal{L}(\Theta)$, and $\tilde{y} \in \mathcal{L}(\mathcal{Z})$.

*Proof:* Following [1], there exist a smooth ISS-Lyapunov function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, flat on $\mathcal{W}$, and two class-$\mathcal{K}_\infty$ functions $\alpha, \gamma$ such that:

$$\alpha(|x|_\mathcal{W}) \geq \gamma(|\theta|) \ \Rightarrow\ DV(x)\cdot f(x, h(x,\theta), \theta, 0) \leq -\alpha(|x|_\mathcal{W}), \tag{117}$$

for all $x \in \mathcal{X}$ and all $\theta \in \Theta$. It then follows from (117) that, whenever $\alpha(|x|_\mathcal{W}) \geq \gamma(|\theta|)$, it holds:

$$\begin{aligned} DV(x)\cdot f(x, h(x,\theta)+y, \theta, 0) &\leq -\alpha(|x|_\mathcal{W}) \qquad (118)\\ &+ DV(x)\cdot\left(f(x,h(x,\theta)+y,\theta,0) - f(x,h(x,\theta),\theta,0)\right) \\ &\leq -\alpha(|x|_\mathcal{W}) + |y|\left(L + \Psi_1(\max\{|x|_\mathcal{W}, |\theta|, |y|\})\right), \qquad (119) \end{aligned}$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Z}$, and $\theta \in \Theta$. where the existence of $L$ and $\Psi_1$ follow from smoothness of $V$ and local Lipschitz continuity of $f$ wrt its arguments. Let $\gamma_{\tilde{y}}, \gamma_\theta$ be the class-$\mathcal{K}_\infty$ functions defined as $\gamma_{\tilde{y}}(s) := \max\left\{s,\ \alpha^{-1}\left(2(1+L)s\right)\right\}$ and $\gamma_\theta(s) := \max\left\{s,\ (\alpha^{-1}\circ\gamma)(|\theta|)\right\}$ for all $s \geq 0$. Let $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-increasing function satisfying

$$0 < b(s) \leq \min\left\{\frac{1}{0.5 + \Psi_1(s)},\ 1\right\}.$$

Then, $b(s) \equiv 1$ on a neighborhood of the origin, and

$$\max\{b(|q|_\mathcal{W}),\ b(|q|_\mathcal{W})\Psi_1(|q|_\mathcal{W})\} \leq 1, \ \ \forall q \in \mathcal{M}. \tag{120}$$

Define $B(q) := b(|q|_\mathcal{W})I_{m\times m}$ for all $q \in \mathcal{M}$. Then, given any $x \in \mathcal{M}$, $\theta \in \Theta$, and $\tilde{y} \in \mathcal{Z}$, it follows from (119) and (120) and our definitions of $\gamma_\theta$ and $\gamma_{\tilde{y}}$ that:

$$\begin{aligned} &|x|_\mathcal{W} \geq \max\{\gamma_\theta(|\theta|), \gamma_{\tilde{y}}(|\tilde{y}|)\} \\ &\Rightarrow DV(x)\cdot f(x, h(x,\theta) + B(x)\tilde{y}, \theta, 0) \leq -\alpha(|x|_\mathcal{W}) \qquad (121)\\ &\quad + b(|x|_\mathcal{W})|\tilde{y}|\left(L + \Psi_1(\max\{|x|_\mathcal{W}, |\theta|, b(|x|_\mathcal{W})|\tilde{y}|\})\right) \\ &\quad \leq -\alpha(|x|_\mathcal{W}) + b(|x|_\mathcal{W})|\tilde{y}|\left(L + \Psi_1(|x|_\mathcal{W})\right) \\ &\quad \leq -\alpha(|x|_\mathcal{W}) + (L+1)|\tilde{y}| \leq -\frac{1}{2}\alpha(|x|_\mathcal{W}). \qquad (122) \end{aligned}$$

By virtue of [27, Remark 2.4] and $dV(\mathcal{W}) = 0$, there exist two class-$\mathcal{K}_\infty$ functions $\tilde{\gamma}_\theta$ and $\tilde{\gamma}_{\tilde{y}}$ such that estimate (119) reads as:

$$\begin{aligned} DV(x)\cdot f(x, h(x,\theta) + B(x)\tilde{y}, \theta, 0) &\leq -\alpha(|x|_\mathcal{W}) \\ &+ \tilde{\gamma}_\theta(|\theta|) + \tilde{\gamma}_{\tilde{y}}(|\tilde{y}|),\ \forall x \in \mathcal{X},\ \forall \theta \in \Theta,\ \forall \tilde{y} \in \mathcal{Z}. \end{aligned} \tag{123}$$

Denote with $X(t,x;\theta(\cdot),\tilde{y}(\cdot))$ the solution of (114) with initial condition at $x \in \mathcal{M}$ and inputs $\theta(\cdot) \in \mathcal{L}(\Theta), \tilde{y}(\cdot) \in \mathcal{L}(\mathcal{Z})$. We can then show along the lines of [1, Claims 1,2, and 3] that there exist two class-$\mathcal{K}_\infty$ functions $\eta_\theta$ and $\eta_{\tilde{y}}$ such that (116) holds for all $x \in \mathcal{M}$, $\theta \in \mathcal{L}(\Theta)$, and $\tilde{y} \in \mathcal{L}(\mathcal{Z})$. ∎

## Appendix C
## Convergence of perturbed solutions under forcing

*Lemma 8:* Let Assumptions 1 and 2 hold. Assume that (2) is Input-to-State Multistable with respect to set $\mathcal{W}$ and input $u$. Let $\{x_j\}_{j\in\mathbb{N}}$, $\{u_j\}_{j\in\mathbb{N}}$, and $\{\varepsilon_j\}_{j\in\mathbb{N}}$ respectively be a sequence of states, input signals, and positive reals satisfying $|x_j| \le d_1$, $u_j \in \mathcal{L}(d_u)$ for all $j \in \mathbb{N}$, $\lim_{j\to+\infty} \|u_j\| = \underline{d}_u$, and $\lim_{j\to+\infty} \varepsilon_j = 0$, for some $d_1, d_u, \underline{d}_u \ge 0$. Then, there exists a continuous function $\kappa(t)$, $t \ge 0$ and, for any $i \in \mathbb{N}$, there exists a subsequence $\{\sigma_i(j)\}_{j\in\mathbb{N}} \subset \{j\}_{j\in\mathbb{N}}$ such that solutions $\phi_{\varepsilon_{\sigma_i(j)}}(t, x_{\sigma_i(j)}; u_{\sigma_i(j)})$ uniformly converge to $\kappa(t)$ on the compact time interval $[i-1, i]$. Furthermore, for any $d_{\kappa,0} > 0$, there exist a strictly decreasing sequence $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v \in \mathcal{L}(d_u)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty} |v(t)| = \underline{d}_u$ and

$$\mathfrak{d}\left[\psi(t, q; v), \kappa(t)\right] \le d_{\kappa,i} \quad \forall t \in [i, i+1] \;\; \forall q \in \mathfrak{B}(\kappa(0), d_{\kappa,0}) \;\; \forall i \in \mathbb{N} \cup \{0\}. \tag{124}$$

*Proof:* The proof of this Lemma makes use of the approximation technique introduced in [26, Lemma III.2]. Since $|x_j| \le d_1$, we can pass to a subsequence satisfying $\lim_{j\to+\infty} x_j =: q$. Define $\theta_j(t) := \phi_{\varepsilon_j}(t, x_j; u_j)$ for all $t \ge 0$ and all $j \in \mathbb{N}$. Since semiglobal practical ISS holds in the classical sense for solutions $\phi_{\varepsilon_j}(\cdot)$, functions $\theta_j(t)$ and points $p_j$ belong to some compact set $S_1 \subset \mathcal{M}$ for all $t \ge 0$, $j \in \mathbb{N}$, and all $v \in \mathcal{L}(d_u)$, and thus sequence $\{\theta_j(t)\}_{j\ge 1}$ is uniformly bounded and equicontinuous on any compact time interval $[0, T]$ with $T \ge 0$. Therefore, following the proof in [26, Lemma III.2], there exist a continuous function $\kappa(t)$, $t \ge 0$ and subsequences

$$\{j\}_{j\ge1} \supset \{\sigma_1(j)\}_{j\ge1} \supset \{\sigma_2(j)\}_{j\ge1} \supset \dots \tag{125}$$

such that, for any $k \in \mathbb{N}$, $\kappa(t)$ is the uniform limit of $\{\theta_{\sigma_k(j)}(t)\}_{j\in\mathbb{N}}$ on the time interval $[k, k+1]$.

Define $d_{\kappa,0} := d_\kappa$ and $p_i := \kappa(i)$ for all $i \in \mathbb{N}$. We are going to prove by induction the following Claim.

*Claim 5:* For each $i \ge 1$, there exist $j_i \in \mathbb{N}$, $0 < d_{\kappa,i} < d_{\kappa,i-1}$, and $w_i \in \mathcal{L}(d_u)$ of the form $w_i = \Delta^{i-1} u_{\sigma_i(j_i)}$ so that, for all $p \in \mathfrak{B}(p_i, d_{\kappa,i})$, it holds:

$$\mathfrak{d}\left[\psi(-t, p; \Delta w_i), \kappa(i-t)\right] \le d_{\kappa,i-1} \quad \forall t \in [0, 1] \tag{126}$$

Furthermore, $\{j_i\}_{i\in\mathbb{N}}$ is non-decreasing.

*Proof:* (*Base of induction:* $i = 1$). We wish to study the trajectory $\psi(-t, p_1; \Delta u_{\sigma_1(j)})$ for $t \in [0, 1]$. This may be a priori undefined for all such $t$. However, since $S_1$ is compact, we may pick another compact set $\tilde{S}_1$ containing $\mathcal{B}(S_1, d_{\kappa,0})$ in its interior, and we may also pick a vector field $\tilde{g} \in \mathfrak{X}(\mathcal{M}, \mathcal{U})$ which is equal to $g$ for all $(x, u) \in \tilde{S}_1 \times \mathcal{U}$ and has compact support; now the system $\dot{x} = \tilde{g}(x, u)$ is complete, namely its solutions $\tilde{\psi}(t, \cdot; \cdot)$ exists for all $t \in (-\infty, +\infty)$. Observe that, for any trajectory $\tilde{\psi}(t, \xi; u)$ which remains in $\tilde{S}_1$, $\psi(t, \xi; u)$ is also defined and coincides with $\tilde{\psi}(t, \xi; u)$. Since $\dot{x} = \tilde{g}(x, u)$ is complete, Gronwall's estimate entails the the existence of some $L > 1$ such that:

$$\mathfrak{d}\left[\tilde{\psi}(-t, p_1; \Delta u_{\sigma_1(j)}), \tilde{\psi}(-t, p; \Delta u_{\sigma_1(j)})\right] \le L\, \mathfrak{d}[p_1, p] \quad \forall j \ge 1,\ \forall t \in [0, 1],\ \forall p \in \mathfrak{B}(p_1, d_{\kappa,0}/2). \tag{127}$$

In particular,

$$\mathfrak{d}\left[\tilde{\psi}(-t, p_1; \Delta u_{\sigma_1(j)}), \tilde{\psi}(-t, p; \Delta u_{\sigma_1(j)})\right] \le \frac{d_{\kappa,0}}{4} \quad \forall j \ge 1,\ \forall t \in [0, 1],\ \forall p \in \mathfrak{B}(p_1, d_{\kappa,0}/(4L)). \tag{128}$$

Since $\mathfrak{d}[\theta_{\sigma_1(j)}(1), p_1] \to 0$ as $j \to +\infty$, there exists $j_{1,a} \ge 1$ such that

$$\theta_{\sigma_1(j)}(1) \in \mathfrak{B}(p_1, d_{\kappa,0}/(8L)) \quad \forall j \ge j_{1,a}. \tag{129}$$

Since $|x_{\sigma_1(j)}| \le d_1$, by virtue of Assumption 2 there exist some $j_1 > j_{1,a}$ and a sequence of $y_j$ such that:

$$\mathfrak{d}\left[\phi_{\varepsilon_{\sigma_1(j)}}(t, x_{\sigma_1(j)}; u_{\sigma_1(j)}),\ \psi(t, y_j; u_{\sigma_1(j)})\right] \le \frac{d_{\kappa,0}}{8L} \quad \forall j \ge j_1,\ \forall t \in [0, 1]. \tag{130}$$

Denote $Y_j := \psi(1, y_j; u_{\sigma_1(j)})$ for all $j \ge j_1$. Since $\phi_{\varepsilon_{\sigma_1(j)}}(1, x_{\sigma_1(j)}; u_{\sigma_1(j)}) = \theta_{\sigma_1(j)}(1)$, inequality (130) reads as:

$$\mathfrak{d}\left[\phi_{\varepsilon_{\sigma_1(j)}}(-t, \theta_{\sigma_1(j)}(1); \Delta u_{\sigma_1(j)}),\ \psi(-t, Y_j; \Delta u_{\sigma_1(j)})\right] \le \frac{d_{\kappa,0}}{8L} \quad \forall j \ge j_1,\ \forall t \in [0, 1]. \tag{131}$$

In particular, since $\phi_{\varepsilon_{\sigma_1(j)}}(-t, \theta_{\sigma_1(j)}(1); \Delta u_{\sigma_1(j)}) \in S_1$ for all $j \ge j_1$ and all $t \in [0, 1]$, inequality (131) implies that $\psi(-t, Y_j; \Delta u_{\sigma_1(j)}) \in \tilde{S}_1$ for all such $j$s and $t$s, and thus we can replace $\psi$ with $\tilde{\psi}$ in (131). Now, combining (129) with inequality (131) at $t = 0$ implies that $\mathfrak{d}[Y_j, p_1] \le d_{\kappa,0}/(4L)$ for all $j \ge j_1$, and thus we can use (128) with $p = Y_j$. We can then combine (128) with (131) to obtain:

$$\mathfrak{d}\left[\phi_{\varepsilon_{\sigma_1(j)}}(-t, \theta_{\sigma_1(j)}(1); \Delta u_{\sigma_1(j)}),\ \psi(-t, p_1; \Delta u_{\sigma_1(j)})\right] \le \frac{d_{\kappa,0}}{2} \quad \forall j \ge j_1,\ \forall t \in [0, 1]. \tag{132}$$

where we have replaced $\tilde{\psi}$ with $\psi$ due to the fact that $\phi_{\varepsilon_{\sigma_1(j)}} \in S_1$. Let $d_{\kappa,1} := d_{\kappa,0}/(4L)$. By combining (128) with (132) and by dropping again the $\tilde{}$ sign, we have:

$$\mathfrak{d}\left[\phi_{\varepsilon_{\sigma_1(j)}}(-t, \theta_{\sigma_1(j)}(1); \Delta u_{\sigma_1(j)}),\ \psi(-t, p; \Delta u_{\sigma_1(j)})\right] \le d_{\kappa,0}, \quad \forall p \in \mathfrak{B}(p_1, d_{\kappa,1}) \;\; \forall j \ge j_1,\ \forall t \in [0, 1]. \tag{133}$$

Let $w_1(t) := u_{\sigma_1(j_1)}(t)$ for all $t \in [0, 1]$. Then, property (133) implies (126) by uniform convergence of the $\phi_{\varepsilon_{\sigma_1(j)}}(\cdot)$ to $\kappa(\cdot)$ on $[0, 1]$.

(*Induction step*). Assume that for some $i \ge 1$, there exist $j_i \ge j_{i-1}$, $0 < d_{\kappa,i} < d_{\kappa,i-1}$, and $w_i \in \mathcal{L}(d_u)$ of the form $w_i = \Delta^{i-1} u_{\sigma_i(j_i)}$ so that, for all $p \in \mathfrak{B}(p_i, d_{\kappa,i})$, it holds:

$$\mathfrak{d}\left[\psi(-t, p; \Delta w_i), \kappa(i-t)\right] \le d_{\kappa,i-1} \quad \forall t \in [i-1, i]. \tag{134}$$

We are then going to prove that there exist $j_{i+1} \ge j_i$, $0 < d_{\kappa,i+1} < d_{\kappa,i}$, and $w_{i+1} \in \mathcal{L}(d_u)$ of the form $w_{i+1} = \Delta^i u_{\sigma_{i+1}(j_{i+1})}$ so that, for all $p \in \mathfrak{B}(p_{i+1}, d_{\kappa,i+1})$, it holds:

$$\mathfrak{d}\left[\psi(-t, p; \Delta w_{i+1}), \kappa(i+1-t)\right] \le d_{\kappa,i} \quad \forall t \in [i, i+1]. \tag{135}$$

To this end, we study the trajectory $\psi(-t, p_{i+1}, \Delta^{i+1} u_{\sigma_{i+1}(j)})$ for $t \in [0, 1]$. This may be a priori undefined for all such $t$. However, since $p_{i+1} \in S_1$ and $\dot{x} = \tilde{g}(x, u)$ is complete and coincides with $\dot{x} = g(x, u)$ for all $x \in \tilde{S}_1$, Gronwall's estimate entails the the existence of some $L > 1$ such that:

$$\mathfrak{d}\left[\tilde{\psi}(-t, p_{i+1}; \Delta^{i+1} u_{\sigma_{i+1}(j)}), \tilde{\psi}(-t, p; \Delta^{i+1} u_{\sigma_{i+1}(j)})\right] \le L\, \mathfrak{d}[p_{i+1}, p], \quad \forall j \ge 1,\ \forall t \in [0, 1],\ \forall p \in \mathfrak{B}(p_{i+1}, d_{\kappa,i}/2). \tag{136}$$

In particular,

$$\mathfrak{d}\left[\tilde{\psi}(-t, p_{i+1}; \Delta^{i+1} u_{\sigma_{i+1}(j)}), \tilde{\psi}(-t, p; \Delta^{i+1} u_{\sigma_{i+1}(j)})\right] \le \frac{d_{\kappa,i}}{4}, \quad \forall j \ge 1,\ \forall t \in [0, 1],\ \forall p \in \mathfrak{B}(p_{i+1}, d_{\kappa,i}/(4L)). \tag{137}$$

Since $\mathfrak{d}[\theta_{\sigma_{i+1}(j)}(i+1), p_{i+1}] \to 0$ as $j \to +\infty$, there exists $j_{i+1,a} \ge j_i$ such that

$$\theta_{\sigma_{i+1}(j)}(i+1) \in \mathfrak{B}(p_{i+1}, d_{\kappa,i}/(8L)) \quad \forall j \ge j_{i+1,a}. \tag{138}$$

Since $|x_{\sigma_{i+1}(j)}| \leq d_1$, by virtue of Assumption 2 there exist some $j_{i+1} > j_{i+1,a}$ and a sequence of $y_j$ such that:

$$\mathfrak{d}\left[\phi_{\varepsilon_{\sigma_{i+1}(j)}}(t, x_{\sigma_{i+1}(j)}; u_{\sigma_{i+1}(j)}),\ \psi(t, y_j; u_{\sigma_{i+1}(j)})\right] \leq \frac{d_{\kappa,0}}{8L} \quad \forall j \geq j_{i+1},\ \forall t \in [0, i+1]. \tag{139}$$

Denote $Y_j := \psi(i+1, y_j; u_{\sigma_{i+1}(j)})$ for all $j \geq j_{i+1}$. Since $\phi_{\varepsilon_{\sigma_{i+1}(j)}}(i+1, x_{\sigma_{i+1}(j)}; u_{\sigma_{i+1}(j)}) = \theta_{\sigma_{i+}(j)}(1)$, inequality (139) reads as:

$$\mathfrak{d}[\phi_{\varepsilon_{\sigma_{i+1}(j)}}(-t, \theta_{\sigma_{i+1}(j)}(1); \Delta^{i+1}u_{\sigma_{i+1}(j)}), \psi(-t, Y_j; \Delta^{i+1}u_{\sigma_{i+1}(j)})] \leq \frac{d_{\kappa,i}}{8L} \quad \forall j \geq j_{i+1},\ \forall t \in [0,1]. \tag{140}$$

In particular, since $\phi_{\varepsilon_{\sigma_{i+1}(j)}}(-t, \theta_{\sigma_{i+1}(j)}(1); \Delta^{i+1}u_{\sigma_{i+1}(j)}) \in S_1$ for all $j \geq j_{i+1}$ and all $t \in [0,1]$, inequality (140) implies that $\psi(-t, Y_j; \Delta^{i+1}u_{\sigma_{i+1}(j)}) \in \tilde{S}_1$ for all such $j$s and $t$s, and thus we can replace $\psi$ with $\tilde{\psi}$ in (140). Now, combining (129) with inequality (140) at $t=0$ implies that $\mathfrak{d}[Y_j, p_{i+1}] \leq d_{\kappa,i}/(4L)$ for all $j \geq j_{i+1}$, and thus we can use (137) with $p = Y_j$. We can then combine (137) with (140) to obtain:

$$\mathfrak{d}[\phi_{\varepsilon_{\sigma_{i+1}(j)}}(-t, \theta_{\sigma_{i+1}(j)}(1); \Delta^{i+1}u_{\sigma_{i+1}(j)}), \psi(-t, p_{i+1}; \Delta^{i+1}u_{\sigma_{i+1}(j)})] \leq \frac{d_{\kappa,i}}{2} \quad \forall j \geq j_{i+1},\ \forall t \in [0,1]. \tag{141}$$

where we have replaced $\tilde{\psi}$ with $\psi$ due to the fact that $\phi_{\varepsilon_{\sigma_{i+1}(j)}} \in S_1$. Let $d_{\kappa,i+1} := d_{\kappa,i}/(4L)$. By combining (137) with (141) and by dropping again the $\tilde{\cdot}$ sign, we have:

$$\mathfrak{d}[\phi_{\varepsilon_{\sigma_{i+1}(j)}}(-t, \theta_{\sigma_{i+1}(j)}(1); \Delta^{i+1}u_{\sigma_{i+1}(j)}), \psi(-t, p; \Delta^{i+1}u_{\sigma_{i+1}(j)})] \leq d_{\kappa,i}, \quad \forall p \in \mathfrak{B}(p_{i+1}, d_{\kappa,i+1})\ \forall j \geq j_{i+1},\ \forall t \in [0,1]. \tag{142}$$

Let $w_{i+1}(t) := u_{\sigma_{i+1}(j_{i+1})}(t)$ for all $t \in [i, i+1]$. Then, property (142) implies (135) by uniform convergence of the $\phi_{\varepsilon_{\sigma_{i+1}(j)}}(\cdot)$ to $\kappa(\cdot)$ on $[i, i+1]$. ∎

Finally, we define a control $v \in \mathcal{L}(d_u)$ as follows:

$$v(t) := w_i(t - i + 1) \text{ if } t \in [i-1, i)$$

for each integer $i \in \mathbb{N}$. By definition of $v$ and $w_i$, and due to $\lim_{i\to+\infty}\|w_i\| \leq \lim_{i\to+\infty}\|u_i\| = \underline{d}_u$, it immediately follows that $\limsup_{t\to+\infty}|v(t)| = \underline{d}_u$.

Property (124) is then proved for all $q \in \mathfrak{B}(\kappa(0), d_\kappa)$ by inductively applying Claim 5. ∎

*Corollary 4:* Under the same assumptions of Lemma 8, for any $d_{\kappa,0} > 0$, there exist a strictly decreasing sequence $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v \in \mathcal{L}(d_u)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to+\infty}|v(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence $\{\sigma_i(n)\}_{n\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ such that:

$$\mathfrak{d}\left[\psi(t, q; v), \phi_{\varepsilon_{\sigma_i(n)}}(t, x_{\sigma_i(n)}; u_{\sigma_i(n)})\right] \leq d_{\kappa,i} \quad \forall t \in [i, i+1]\ \forall q \in \mathfrak{B}(\bar{x}, d_{\kappa,0})\ \forall n \geq n_i, \tag{143}$$

where $\bar{x} := \lim_{n\to+\infty} x_n$.

*Proof:* It follows from Lemma 8. ∎

*Corollary 5:* Let Assumptions 1 and 2 hold. Assume that (2) is Input-to-State Multistable with respect to set $\mathcal{W}$ and input $u$. Let $\{x_j\}_{j\in\mathbb{N}}$, $\{u_j\}_{j\in\mathbb{N}}$, and $\{\varepsilon_j\}_{j\in\mathbb{N}}$ respectively be a sequence of states, input signals, and positive reals satisfying $u_j \in \mathcal{L}(d_u)$ for all $j \in \mathbb{N}$, $\lim_{j\to+\infty}\|u_j\| = \underline{d}_u$, and $\lim_{j\to+\infty}\varepsilon_j = 0$, for some $d_1, d_u, \underline{d}_u \geq 0$. Furthermore, assume that solutions $\phi_{\varepsilon_j}(t, x_j; u_j)$ exist and are bounded for all $t \in [\tau_j, 0]$ and all $j \in \mathbb{N}$, where $\lim_{j\to+\infty}\tau_j = -\infty$. for any $d_{\kappa,0} > 0$, there exist a strictly decreasing sequence $\{d_{\kappa,i}\}_{i\in\mathbb{N}}$ and an input $v \in \mathcal{L}(d_u)$ such that $\lim_{i\to+\infty} d_{\kappa,i} = 0$, $\limsup_{t\to-\infty}|v(t)| = \underline{d}_u$ and, for any $i \in \mathbb{N}\cup\{0\}$, there exist $n_i \in \mathbb{N}$ and a subsequence $\{\sigma_i(n)\}_{n\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ such that:

$$\mathfrak{d}\left[\psi(t, q; v), \phi_{\varepsilon_{\sigma_i(n)}}(t, x_{\sigma_i(n)}; u_{\sigma_i(n)})\right] \leq d_{\kappa,i} \quad \forall t \in [-i-1, -i]\ \forall q \in \mathfrak{B}(\kappa(0), d_{\kappa,0})\ \forall n \geq n_i, \tag{144}$$

where $\bar{x} := \lim_{n\to+\infty} x_n$.

*Proof:* It follows along the lines of Lemma 8 and Corollary 4. ∎

## Appendix D
## Acyclicity under Forcing

Throughout this Appendix, we let Assumption 1 hold true and we further assume that (2) is Input-to-State Multistable with respect to set $\mathcal{W}$ and input $u$. First, we prove a backward-time property for Input-to-State Multistable systems, i.e. Corollary 5, which refers to Lemmas 9, 10, and 11. Second, we provide Lemmas 12 and 13 which are instrumental in establishing that $\mathcal{W}$ is acyclic under forcing, according to Definition 13. The proof of both properties hinges upon the existence of an ISS-Lyapunov function for (2) which in turn hinges upon acyclicity of $\mathcal{W}$ along the solutions of the autonomous system (4). The existence of ISS-Lyapunov functions for systems with an acyclic $\mathcal{W}$-limit set has been proved in [1] and detailed in [8].

Let $D := \min_{i\neq j}\mathfrak{d}[\mathcal{W}_i, \mathcal{W}_j]$ be the minimum distance among the atoms of the decomposition of $\mathcal{W}$.

*Lemma 9:* Let $d_u \geq 0$. Then, for all $x \in \mathcal{M}$ and all $u \in \mathcal{L}(d_u)$, either $\lim_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} = +\infty$ or $\limsup_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} < +\infty$.

*Proof:* Assume by contradiction that, for some $x \in \mathcal{M}$ and some $u \in \mathcal{L}(d_u)$, it holds: $\limsup_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} = +\infty$ and $\liminf_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} < +\infty$. Then, there exist sequences $\{\underline{t}_n\}_{n\in\mathbb{N}}$, $\{\bar{t}_n\}_{n\in\mathbb{N}}$, $\{M_n\}_{n\in\mathbb{N}}$, and a constant $L > 0$ such that $\bar{t}_{n+1} < \underline{t}_n < \bar{t}_n < -n$ for all $n \in \mathbb{N}$, $\lim_{n\to+\infty} M_n \to +\infty$, and

$$|\psi(\underline{t}_n, x; u)|_{\mathcal{W}} \leq L, |\psi(\bar{t}_n, x; u)|_{\mathcal{W}} > M_n, \quad \forall n \in \mathbb{N}. \tag{145}$$

Input-to-State Multistability of (2) entails the pGS property [1], namely the existence of a class-$\mathcal{K}^\infty$ function $\beta$ and a constant $Q \geq 0$ such that $|\psi(t,q;w)|_{\mathcal{W}} \leq Q + \beta(\max\{|q|_{\mathcal{W}}, \|w\|\})$ for all $q \in \mathcal{M}$ and all $w \in \mathcal{U}$. By making use of the latter property with $q = \psi(\underline{t}_n, x; u)$ as the initial condition and $w(\cdot) = u(\cdot + \underline{t}_n)$ as the input, we immediately have that:

$$|\psi(\bar{t}_n, x; u)|_{\mathcal{W}} \leq Q + \beta(L, d_u) \ \ \forall n \in \mathbb{N},$$

which contradicts (145). ∎

*Lemma 10:* For all $\theta > 0$ there exists $d_u > 0$ such that, for all $x \in \mathcal{M}$ and all $u \in \mathcal{L}(d_u)$,

$$\text{if } \limsup_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} < +\infty, \tag{146}$$

$$\text{then } \limsup_{t\to-\infty}|\psi(t,x;u)|_{\mathcal{W}} \leq \theta. \tag{147}$$

*Proof:* As studied in [1], Input-to-State Multistability of (2) implies the existence of a smooth Lyapunov function $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$ which is flat on each atom $\mathcal{W}_i$, i.e. $V(\mathcal{W}_i)$ is a singleton, and whose time derivative along the solutions of (2) satisfies the following inequality for some class-$\mathcal{K}_\infty$ functions $\alpha, \gamma$:

$$\dot{V} \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|u|), \quad \forall x \in \mathcal{M}\ \ \forall u \in \mathcal{U}. \tag{148}$$

Define $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as the following function:

$$\xi(s) := \max_{i\in\{1,\ldots,N\}} \Big\{ \max_{|x|_{\mathcal{W}_i}\leq s} V(x) - \min_{|x|_{\mathcal{W}_i}\leq s} V(x)\Big\}. \tag{149}$$

It follows by smoothness of $V$ and its flatness on each atom $\mathcal{W}_i$ that $\xi$ is a non-decreasing and continuous function of $s \geq 0$ with $\xi(0) = 0$. Assume by contradiction that there exists $\theta > 0$ such that, for all $d_u > 0$, there exist $x \in \mathcal{M}$ and $u \in \mathcal{L}(d_u)$ such that (146) holds but (147) does not. Let $\bar{s} \in (0, \theta/4)$ be such that $\xi(\bar{s}) \leq \frac{\theta}{4G}\alpha(\frac{\theta}{2})$, where:

$$G := \max_{|q|_{\mathcal{W}}\leq 2\theta, |w|\leq \bar{d}_u} \{|g(q,w)|_{\mathfrak{g}}\}. \tag{150}$$

for some fixed $\bar{d}_u > 0$. In particular, select $d_u := \min\{(\gamma^{-1}\circ\alpha/2)(\bar{s}/2), \bar{d}_u\}$ and the corresponding $x \in \mathcal{M}$ and $u \in \mathcal{L}(d_u)$ such that hypothesis (146) holds together with:

$$\limsup_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}} > \theta. \tag{151}$$

*Claim 6:* There exists $i \in \{1,\ldots,N\}$ such that $\liminf_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}_i} \leq \bar{s}/2$.

*Proof:* Assume by contradiction that $\liminf_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}_i} > \bar{s}/2$ for all $i \in \{1,\ldots,N\}$. It follows that, for any $i \in \{1,\ldots,N\}$, there exists $T_i < 0$ such that $|\psi(t,x;u)|_{\mathcal{W}_i} > \bar{s}/2$ for all $t \leq T_i$. In particular, there exists $T = \min_{i\in\{1,\ldots,N\}}\{T_i\}$ such that $|\psi(t,x;u)|_{\mathcal{W}} > \bar{s}/2$ for all $t \leq T$. Observe that, due to our particular selection of $d_u$, inequality (148) reads as $\dot{V} \leq -\alpha(\bar{s}/2)/2$ whenever $x \notin \mathcal{B}(\mathcal{W}, \bar{s}/2)$ and $u \in \mathcal{U}$ satisfies $|u| \leq d_u$. We can then integrate the latter inequality along the solution $\psi(t,x;u)$ for any $t \leq T$ so as to obtain:

$$V(\psi(T,x;u)) - V(\psi(t,x;u)) \leq -\frac{1}{2}\alpha\left(\frac{\bar{s}}{2}\right)(T-t)\,. \tag{152}$$

By taking the limit of both sides of (152) for $t \to -\infty$ and rearranging terms, we have that $\lim_{t\to-\infty} V(\psi(t,x;u)) = +\infty$ which is a contradiction since $V$ is smooth and, by (146), solution $\psi(t,x;u)$ is eventually bounded backward in time. ■

Claim (6) and property (151) entail the existence of an index $i \in \{1,\ldots,N\}$ and two sequences $\{\underline{t}_n\}_{n\in\mathbb{N}}$, $\{\bar{t}_n\}_{n\in\mathbb{N}}$ such that $\bar{t}_{n+1} < \underline{t}_n < \bar{t}_n < -n$ for all $n \in \mathbb{N}$ and

$$|\psi(\underline{t}_n,x;u)|_{\mathcal{W}_i} \leq \bar{s}, |\psi(\bar{t}_n,x;u)|_{\mathcal{W}_i} > \theta, \quad \forall n \in \mathbb{N}. \tag{153}$$

Select any $\bar{n} \in \mathbb{N}$. Let:

$$\begin{aligned}
&\bar{t} := \bar{t}_{\bar{n}}, \ \ \underline{t}_0 := \underline{t}_{\bar{n}+1}, \ \ \underline{t}_1 := \underline{t}_{\bar{n}},\\
&t_a := \sup\{t < \bar{t}: \ |\psi(t,x;u)|_{\mathcal{W}_i} < \bar{s}\},\\
&t_b := \sup\{t < \bar{t}: \ |\psi(t,x;u)|_{\mathcal{W}_i} < \theta/2\},\\
&t_c := \inf\{t > \bar{t}: \ |\psi(t,x;u)|_{\mathcal{W}_i} < \theta/2\},\\
&t_d := \inf\{t > \bar{t}: \ |\psi(t,x;u)|_{\mathcal{W}_i} < \bar{s}\}.
\end{aligned} \tag{154}$$

Maximality of $t_a, t_b, t_c, t_d$ implies $|\psi(t,x;u)|_{\mathcal{W}_i} \geq \theta/2$ for all $t \in [t_b, t_c]$ and $|\psi(t,x;u)|_{\mathcal{W}_i} \geq \bar{s}$ for all $t \in [t_a, t_d]$. Observe that the path taken by solution $\psi$ both in forward and backward time to reach any point $\psi_0 \in \mathfrak{B}(\mathcal{W}_i, \theta/2)$ from $\psi(\bar{t},x;u)$ has length greater or equal to $\theta/2$. It then follows from definition (150) that $t_c - \bar{t} \geq \theta/(2G)$ and $\bar{t} - t_b \geq \theta/(2G)$, and thus $t_c - t_b \geq \theta/G$ Observe that, due to our particular selection of $d_u$, inequality (148) reads as $\dot{V} \leq -\alpha(\bar{s})/2$ whenever $x \notin \mathcal{B}(\mathcal{W}, \bar{s})$ and $u \in \mathcal{U}$ satisfies $|u| \leq d_u$. Furthermore, due to our particular selection of $d_u$, inequality (148) reads as $\dot{V} \leq -\alpha(\theta/2)/2$ whenever $x \notin \mathcal{B}(\mathcal{W}, \theta/2)$ and $u \in \mathcal{U}$ satisfies $|u| \leq d_u$.

We can then integrate the latter inequality along the solution $\psi(t,x;u)$ on the time interval $[t_a, t_d]$ so as to obtain:

$$\begin{aligned}
&V(\psi(t_d,x;u)) - V(\psi(t_a,x;u))\\
&\leq V(\psi(t_c,x;u)) - V(\psi(t_b,x;u)) \leq -\frac{1}{2}\alpha\left(\frac{\theta}{2}\right)\frac{\theta}{G}.
\end{aligned} \tag{155}$$

Since $\psi(t_d,x;u), \psi(t_a,x;u) \in \mathfrak{B}(\mathcal{W}_i, \bar{s})$, it follows from our definition of $\bar{s}$ that $|V(\psi(t_d,x;u)) - V(\psi(t_a,x;u))| \leq \frac{\theta}{4G}\alpha(\frac{\theta}{2})$ which contradicts (155). ■

*Lemma 11:* For all $d_u > 0$ there exists $\theta > 0$ such that, for all $x \in \mathcal{M}$ and all $u \in \mathcal{L}(d_u)$,

$$\text{if } \limsup_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}} < +\infty, \tag{156}$$

$$\text{then } \limsup_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}} \leq \theta. \tag{157}$$

*Proof:* Assume by contradiction that for some $d_u > 0$ there exist sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, and $\{M_n\}_{n\in\mathbb{N}}$ such that, for any $n \in \mathbb{N}$, it holds that $M_n > 0$, $x_n \in \mathcal{M}$, $u_n \in \mathcal{L}(d_u)$, and:

$$n < \limsup_{t\to-\infty} |\psi(t,x_n;u_n)|_{\mathcal{W}} < M_n. \tag{158}$$

Following [1] and subsequent publications [8], Input-to-State Multi-stability of (2) wrt set $\mathcal{W}$ and input $u$ entails Input-to-State Stability of (2) wrt set $A_N$ and input $u$, with $A_N$ as in Lemma 1. Namely, there exist a class-$\mathcal{KL}$ function $\beta$ and a class-$\mathcal{K}_\infty$ function $\gamma_{A_N}$ such that

$$\begin{aligned}
&|\psi(t,q;w)|_{A_N} \leq \beta(|q|_{A_N}, t) + \gamma_{A_N}(\|w\|),\\
&\quad \forall t \geq 0 \ \ \forall q \in \mathcal{M} \ \ \forall w \in \mathcal{U}.
\end{aligned} \tag{159}$$

Furthermore, due to compactness of $A_N$ and $\mathcal{W}$ and due to inclusion $\mathcal{W} \subseteq A_N$, there exists a constant $Q \geq 0$ such that

$$|q|_{A_N} \leq |q|_{\mathcal{W}} \leq |q|_{A_N} + Q, \quad \forall q \in \mathcal{M}. \tag{160}$$

Fix $n \in \mathbb{N}$ so that $n > 2(Q + \gamma_{A_N}(d_u))$. From (158) it follows that there exists a diverging sequence of negative times $\{t_m\}_{m\in\mathbb{N}}$ such that

$$2(Q + \gamma_{A_N}(d_u)) < n < |\psi(t_m,x_n;u_n)|_{\mathcal{W}} < M_n, \quad \forall m \in \mathbb{N}. \tag{161}$$

However, from (159) and (160), we have that:

$$\begin{aligned}
&|\psi(t,q;w)|_{\mathcal{W}} \leq \beta(M_n, t) + \gamma_{A_N}(d_u) + Q,\\
&\quad \forall t \geq 0 \ \ \forall q \in \mathfrak{B}(\mathcal{W}, M_n) \ \ \forall w \in \mathcal{L}(d_u).
\end{aligned} \tag{162}$$

Let $T > 0$ be such that $\beta(M_n, t) < Q$ for all $t \geq T$. Since $t_m \to -\infty$, we can select $\bar{m}, \underline{m} \in \mathbb{N}$ such that $t_{\bar{m}} - t_{\underline{m}} > T$. It then follows from (161) and (162) that

$$\begin{aligned}
&2(Q + \gamma_{A_N}(d_u)) < |\psi(t_{\bar{m}}, x_n; u_n)|_{\mathcal{W}}\\
&\quad < \beta(M_n, t_{\bar{m}} - t_{\underline{m}}) + Q + \gamma_{A_N}(d_u) < 2Q + \gamma_{A_N}(d_u),
\end{aligned}$$

which is a contradiction. ■

*Corollary 6:* There exists a class-$\mathcal{K}_\infty$ function $\hat{\eta}_u$ such that, for all $x \in \mathcal{M}$ and all $u \in \mathcal{U}$, either $\lim_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}} = +\infty$ or

$$\limsup_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}} \leq \hat{\eta}(\|u\|).$$

*Proof:* It follows from Lemmas 9, 10, and 11. ■

*Lemma 12:* For any $\theta \in (0, D/2)$, there exists $d_u > 0$ such that, for all $i \in \{1,\ldots,N\}$, all $x \notin \mathfrak{B}(\mathcal{W}, \theta)$ and all $u \in \mathcal{L}(d_u)$, the following implication holds:

$$\text{if } \limsup_{t\to-\infty} |\psi(t,x;u)|_{\mathcal{W}_i} \leq \theta, \tag{163}$$

$$\text{then } \limsup_{t\to+\infty} |\psi(t,x;u)|_{\mathcal{W}_i} > \theta. \tag{164}$$

*Proof:* Assume by contradiction that there exists $\theta \in (0, D/2)$ such that, for all $d_u > 0$, there exist an index $i \in \{1, \ldots, N\}$, an initial condition $x \notin \mathfrak{B}(\mathcal{W}, \theta)$, and an input $u \in \mathcal{L}(d_u)$ such that (163) holds but (164) does not. Select $V : \mathcal{M} \to \mathbb{R}_{\geq 0}$, $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $G > 0$, $\bar{d}_u > 0$, and $\bar{s} \in (0, \theta/4)$ as in the proof of Lemma 10. Let $\hat{\eta}_u$ be the class-$\mathcal{K}_\infty$ function given by Corollary 6. Let $\eta_u$ be the class-$\mathcal{K}_\infty$ satisfying the AG property (9). In particular, select

$$d_u := \min\{(\gamma^{-1} \circ \alpha/2)(\bar{s}/2), \bar{d}_u, \hat{\eta}_u^{-1}(\bar{s}/2), \eta_u^{-1}(\bar{s}/2)\},$$

and the corresponding $i \in \{1, \ldots, N\}$, $x \notin \mathfrak{B}(\mathcal{W}, \theta)$, and $u \in \mathcal{L}(d_u)$ such that hypothesis (163) holds together with:

$$\limsup_{t \to +\infty} |\psi(t, x; u)|_{\mathcal{W}_i} \leq \theta. \qquad (165)$$

Corollary 6 implies that $\limsup_{t \to -\infty} |\psi(t, x; u)|_{\mathcal{W}} \leq \bar{s}/2$. It follows from (163) and the fact that $\theta < D/2$ that $\limsup_{t \to -\infty} |\psi(t, x; u)|_{\mathcal{W}_i} \leq \bar{s}/2$. Similarly, due to AG, (165) and $\theta < D/2$, it holds that $\limsup_{t \to +\infty} |\psi(t, x; u)|_{\mathcal{W}_i} \leq \bar{s}/2$. Therefore, since $\bar{s}/2 \leq \theta/4$, we can define $\bar{t} := 0$ and $t_a, t_b, t_c, t_d$ as in (154). We can then prove along the lines of Lemma 10 that (155) holds. Since $\psi(t_d, x; u), \psi(t_a, x; u) \in \mathfrak{B}(\mathcal{W}_i, \bar{s})$, it follows from our definition of $\bar{s}$ that $|V(\psi(t_d, x; u)) - V(\psi(t_a, x; u))| \leq \frac{\theta}{4G}\alpha(\frac{\theta}{2})$ which contradicts (155). ■

*Lemma 13:* For any $d_u > 0$ there exists a $\Theta > 0$ such that, for all $i \in \{1, \ldots, N\}$, all $x \notin \mathfrak{B}(\mathcal{W}, \Theta)$ and all $u \in \mathcal{L}(d_u)$, the following implication holds:

$$\text{if } \limsup_{t \to -\infty} |\psi(t, x; u)|_{\mathcal{W}_i} \leq \Theta, \qquad (166)$$

$$\text{then } \limsup_{t \to +\infty} |\psi(t, x; u)|_{\mathcal{W}_i} > \Theta. \qquad (167)$$

*Proof:* Assume by contradiction that there exist $d_u > 0$, an index $i \in \{1, \ldots, N\}$, two sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{u_n\}_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, it holds that $x_n \notin \mathfrak{B}(\mathcal{W}, n)$, $u_n \in \mathcal{L}(d_u)$, and $\limsup_{t \to \pm\infty} |\psi(t, x_n; u_n)|_{\mathcal{W}_i} \leq n$. By virtue of Corollary 6, there exists a class-$\mathcal{K}_\infty$ function $\hat{\eta}_u$ such that

$$\limsup_{t \to -\infty} |\psi(t, x_n; u_n)|_{\mathcal{W}} \leq \hat{\eta}_u(d_u) \ \ \forall n \in \mathbb{N}. \qquad (168)$$

By virtue of the pGS property, there exist $Q > 0$ and a class-$\mathcal{K}_\infty$ function $\beta$ such that:

$$|\psi(t, q; w)|_{\mathcal{W}} \leq Q + \beta(|q|_{\mathcal{W}} + d_u) \quad \forall q \in \mathcal{M} \ \ \forall w \in \mathcal{L}(d_u). \qquad (169)$$

Fix $n \in \mathbb{N}$ such that $n > Q + \beta(d_u + \hat{\eta}(d_u) + 1)$. For such $n$, it follows from (168) that there exists $T < 0$ such that $|\psi(t, x_n; u_n)|_{\mathcal{W}} \leq \hat{\eta}_u(d_u) + 1$ for all $t \leq T$. From the latter property and from (169) with $q = \psi(T, x_n; u_n)$ and $w(\cdot) = u_n(\cdot - T)$, it follows that:

$$\begin{aligned} |x_n|_{\mathcal{W}} &= |\psi(-T, \psi(T, x_n; u_n); u_n(\cdot - T))|_{\mathcal{W}} \\ &\leq Q + \beta(\hat{\eta}_u(d_u) + 1 + d_u) < n. \end{aligned} \qquad (170)$$

Property (170) contradicts our assumption that $|x_n|_{\mathcal{W}} > n$. ■

## References


[1] D. Angeli and D. Efimov, "Characterizations of Input-to-State Stability for Systems With Multiple Invariant Sets," *IEEE Transactions on Automatic Control*, vol. 60, no. 12, pp. 3242–3256, Dec 2015.

[2] D. Angeli and E. Sontag, "Monotone control systems," *Automatic Control, IEEE Transactions on*, vol. 48, no. 10, pp. 1684–1698, Oct 2003.

[3] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov Functions for Hybrid Systems. Part I: Existence Is Equivalent to Robustness," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1264–1277, July 2007.

[4] P. D. Christofides and A. R. Teel, "Singular perturbations and input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 11, pp. 1645–1650, Nov 1996.

[5] F. Clarke, Y. Ledyaev, and R. Stern, "Asymptotic Stability and Smooth Lyapunov Functions," *Journal of Differential Equations*, vol. 149, no. 1, pp. 69 – 114, 1998.

[6] M. P. do Carmo, *Riemannian Geometry*. Birkhuser, 1992.

[7] D. Efimov, "Global Lyapunov Analysis of Multistable Nonlinear Systems," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 3132–3154, 2012.

[8] P. Forni and D. Angeli, "A converse Lyapunov theorem with prescribed dissipation rate for systems with multiple hyperbolic nonresonant equilibria," *ESAIM: Control, Optimisation and Calculus of Variations*, submitted.

[9] ——, "Smooth Lyapunov functions for multistable hybrid systems on manifolds," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, Dec 2017, pp. 5481–5486.

[10] ——, "The ISS approach to the stability and robustness properties of nonautonomous systems with decomposable invariant sets: An overview," *European Journal of Control*, vol. 30, pp. 50 – 60, 2016, 15th European Control Conference, {ECC16}.

[11] T. S. Gardner, C. R. Cantor, and J. J. Collins, "Construction of a genetic toggle switch in escherichia coli," *Nature*, vol. 403, no. 20, pp. 339–342, Apr 2000.

[12] W. Hahn, *Stability of Motion*, ser. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[13] P. Hartman, "On local homeomorphisms of euclidean spaces," *Boletn de la Sociedad Matemtica Mexicana*, vol. 5, pp. 220–241, 1960.

[14] A. Isidori, *Nonlinear Control Systems II*. London, UK, UK: Springer-Verlag, 2000.

[15] H. Khalil, *Nonlinear Systems*, ser. Pearson Education. Prentice Hall, 2002.

[16] P. Kokotović, H. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, ser. Mathematics in science and engineering. Academic Press, 1986.

[17] J. Kurzweil, "On the inversion of ljapunov's second theorem on stability of motion," *Amer. Math. Soc. Trans. Ser.*, vol. 24, pp. 19–77, 1956.

[18] W. Liu, "An approximation algorithm for nonholonomic systems," *SIAM Journal on Control and Optimization*, vol. 35, no. 4, pp. 1328–1365, 1997.

[19] L. Moreau and D. Aeyels, "Practical stability and stabilization," *IEEE Transactions on Automatic Control*, vol. 45, no. 8, pp. 1554–1558, Aug 2000.

[20] L. Moreau, D. Nešić, and A. R. Teel, "A trajectory based approach for robustness of input-to-state stability," in *American Control Conference, 2001. Proceedings of the 2001*, vol. 5, 2001, pp. 3570–3575 vol.5.

[21] P. Morin., J.-B. Pomet, and C. Samson, "Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of lie brackets in closed loop," *SIAM Journal on Control and Optimization*, vol. 38, no. 1, pp. 22–49, 1999.

[22] Z. Nitecki and M. Shub, "Filtrations, decompositions, and explosions," *American Journal of Mathematics*, vol. 97, no. 4, pp. pp. 1029–1047, 1975.

[23] L. Perko, *Differential Equations and Dynamical Systems*, ser. Texts in Applied Mathematics. Springer New York, 2008.

[24] J. Sanders, F. Verhulst, and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, ser. Applied Mathematical Sciences. Springer New York, 2007.

[25] E. Sontag, "Smooth stabilization implies coprime factorization," *Automatic Control, IEEE Transactions on*, vol. 34, no. 4, pp. 435–443, Apr 1989.

[26] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *Automatic Control, IEEE Transactions on*, vol. 41, no. 9, pp. 1283–1294, Sep 1996.

[27] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351 – 359, 1995.

[28] S. Sternberg, "On local $\mathcal{C}^n$ contractions of the real line," *Duke Mathematical Journal*, vol. 24, pp. 97–102, 1967.

[29] H. J. Sussmann and W. Liu, "Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories," in *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*, Dec 1991, pp. 437–442 vol.1.

[30] A. R. Teel, L. Moreau, and D. Nešić, "A unified framework for input-to-state stability in systems with two time scales," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1526–1544, Sept 2003.

[31] A. R. Teel, J. Peuteman, and D. Aeyels, "Semi-global practical asymptotic stability and averaging," *Systems & Control Letters*, vol. 37, no. 5, pp. 329 – 334, 1999.

[32] A. Teel and D. Nešić, “Averaging with disturbances and closeness of solutions,” *Systems & Control Letters*, vol. 40, no. 5, pp. 317 – 323, 2000.

[33] D. Vecchio and R. Murray, *Biomolecular Feedback Systems*. Princeton University Press, 2014.

[34] T. Yoshizawa, *Stability Theory by Liapunov’s Second Methods*, ser. Publications of the Mathematical society of Japan. Mathematical society of Japan, 1966.

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