

Second-order Propositional Modal Logic: Expressiveness and Completeness Results[☆]

Francesco Belardinelli

Laboratoire IBISC, UEVE & IRIT Toulouse

Wiebe van der Hoek

Department of Computer Science, University of Liverpool

Louwe B. Kuijer

*Department of Computer Science, University of Liverpool
LORIA, CNRS/Université de Lorraine*

Abstract

In this paper we advance the state-of-the-art on the application of second-order propositional modal logic (SOPML) in the representation of individual and group knowledge, as well as temporal and spatial reasoning. The main theoretical contributions of the paper can be summarized as follows. Firstly, we introduce the language of (multi-modal) SOPML and interpret it on a variety of different classes of Kripke frames according to the features of the accessibility relations and of the algebraic structure of the quantification domain of propositions. We provide axiomatisations for some of these classes, and show that SOPML is unaxiomatisable on the remaining classes. Secondly, we introduce novel notions of (bi)simulations and prove that they indeed preserve the interpretation of formulas in (the universal fragment of) SOPML. Then, we apply this formal machinery to study the expressiveness of Second-order Propositional Epistemic Logic (SOPEL) in representing higher-order knowledge, i.e., the knowledge agents have about other agents' knowledge, as well as graph-theoretic notions (e.g., 3-colorability, Hamiltonian paths, etc.) The final outcome is a rich formalism to represent and reason about relevant concepts in artificial intelligence, while still having a model checking problem that is no more computationally expensive than that of the less expressive quantified boolean logic.

Keywords: Modal logic, Knowledge representation, Second-order propositional modal logic, Epistemic logic, Local properties

1. Introduction

Modal logic is nowadays a well-established area in mathematical logic, which has also become one of the most popular formal frameworks in artificial intelligence for knowledge

Email addresses: belardinelli@ibisc.fr (Francesco Belardinelli), wiebe@liverpool.ac.uk (Wiebe van der Hoek), louwe.kuijer@liverpool.ac.uk (Louwe B. Kuijer)

Preprint submitted to Artificial Intelligence

July 16, 2018

representation and reasoning [7, 26]. This success is due to several reasons, including an expressive and flexible formal language, which enjoys nice computational properties. In particular, at the core of the semantics of modal logic lies the notion of *world*, or *state*. Indeed, this concept is very natural when studying computational notions (a system evolving over time from a previous to a successive state), accounts of agency (states that are preferred, desired, or epistemically possible), and of interaction (states that can be winning, losing, terminal, initial, etc.). Indeed, distributed computing [25], reactive systems [43], multi-agent systems [32], and game theory [31] have all benefited from the application of tools and techniques from modal logic, and this list is by no means exhaustive. Most importantly, the worlds in the models for modal logic are connected by means of indexed relations R_a , for some index a , which model (program) transitions, epistemic or desired alternatives, or the effect of possible moves, where index a can assume a number of readings: a specific program, a dimension of time, say, future or past, an agent, a move, etc. Each accessibility relation R_a in the semantics is then paired with a necessity operator \Box_a in the modal language, where a formula $\Box_a\varphi$ is then true in a world w of a model, if φ is true in every world v that is a -accessible from w (see Definition 8 for a formal definition). Informally, this may be read as: after every execution of a , in each future time along dimension a , in every world considered possible or desired by agent a , or in every world that is the result of performing move a , formula φ holds.

The language of modal logic provides a crisp, variable-free way of expressing a variety of properties of interest. It is also important to realise that there is not just one modal logic: although the well-known normal axiomatisation \mathbf{K} characterises the class of validities on all models for modal logic, this does not mean that all logics for, say, agency, are the same and correspond to \mathbf{K} . It only means that they are typically extensions of \mathbf{K} . As a simple example, the property $(i) \Box_a\varphi \rightarrow \varphi$ appears reasonable when \Box_a denotes ‘agent a knows that ...’, but is perhaps less desirable when it is read as ‘agent a believes that ...’, as philosophically knowledge is analysed as truthful belief [28]. One of the reasons for the success of modal logic is that in many relevant cases a syntactic scheme corresponds to an additional constraint on the accessibility relation R_a : in the case of (i) , reflexivity of R_a is, in a precise sense, sufficient and necessary for its validity.

To appreciate this point, we use a little bit more detail (we assume some familiarity with modal logic, precise definitions are given in Section 2). As already mentioned, central in the semantics of modal logic is the notion of (Kripke) frame \mathcal{F} , which comprises of a set W of worlds and accessibility relations R_a , for indices $a \in I$. We can then define a notion of validity \models on frames and formulate the result mentioned above as follows:

$$R_a \text{ is reflexive} \quad \text{iff} \quad \mathcal{F} \models \Box_a\varphi \rightarrow \varphi, \text{ for all formulas } \varphi \quad (1)$$

Characterisations such as (1) are referred to as *correspondence results* [6], because they establish a correspondence between a first-order property on frames (i.e., reflexivity) and a modal validity (i.e., (i)). Another example of correspondence is that between the first-order formula $\forall x\forall y(R_a(x,y) \rightarrow R_b(x,y))$ and modal schema $\Box_b\varphi \rightarrow \Box_a\varphi$, which intuitively says that, e.g., whatever is achieved by program b , is also achieved by a , or that a knows at least as much as b .

Mathematically elegant and powerful as correspondence theory may be, it also has shortcomings. Firstly, note that in the case of (1), correspondence is defined *globally*, i.e., (i) has to be valid throughout the frame. This means that for instance (using a doxastic

reading of (i)), we cannot model situations in which a 's beliefs are true, but b does not know that. Indeed, if the truthfulness of agent a 's beliefs is tantamount to the validity of (i), then (ii) $K_b(\Box_a\varphi \rightarrow \varphi)$ is also a validity, enforcing agent b 's knowledge.

Secondly, in (1) quantification appears at the meta-, and therefore the outermost, level. It is therefore impossible to distinguish (and to express in the language of modal logic) the following two situations: in the first, b knows that a has perfect information and is a perfect reasoner, and therefore, b knows *a priori* that whatever a believes must be correct. Informally, this would be represented as $K_b(\text{for all } \phi (\Box_a\phi \rightarrow \phi))$, which is not a well-formed formula however. In the second situation b has verified, for every φ that a happens to believe, that φ is in fact true. Informally, this would be represented as (for all ϕ , $K_b(\Box_a\phi \rightarrow \phi)$).

As observed in [3], by allowing for quantification over propositions – and thus obtaining the language of second-order propositional modal logic (SOPML) – both issues mentioned above can be addressed. The formal definition of $\forall p\psi$ is given in Definition 8, but informally, given a valuation V which tells us in which worlds $V(p)$ the atom p is true, $\forall p\psi$ holds if for every V' that differs from V in at most the set $V(p)$ (i.e., $V'(q) = V(q)$ for all $q \neq p$), the formula ψ holds. As regards the first example, the SOPML formula $\forall p(\Box_a p \rightarrow p) \wedge \neg K_b \forall p(\Box_a p \rightarrow p)$ intuitively expresses that all beliefs of agent a are correct, but b does not know this fact. Moreover, the two different readings in the second example can be represented by formulas $K_b \forall p(\Box_a p \rightarrow p)$ and $\forall p K_b(\Box_a p \rightarrow p)$, respectively. Readers familiar with the philosophy literature on the topic may recognize the difference between $K_b \forall p(\Box_a p \rightarrow p)$ and $\forall p K_b(\Box_a p \rightarrow p)$ as the distinction between *de dicto* and *de re* quantification.

Importantly, the truth of $\forall p(\Box_a p \rightarrow p)$ at world w enforces the truthfulness of agent a 's beliefs in w only, therefore this is a *local* property of the frame, as opposed to the *global* validity of (i). This fact allows agent b to consider (epistemically) possible a different world w' in which (i) does not hold.

The aim of this paper is to further the applications of propositional quantification and second-order propositional modal logic in knowledge representation and reasoning, through exploring and securing their theoretical foundations. In particular, the original contributions of the paper can be summarised as follows.

Firstly, in Section 2 we introduce the language of multi-agent second-order propositional modal logic, and provide it with a semantics in terms of Kripke frames extended with a domain D of sets of worlds for the interpretation of quantification. The differences between our definition and the existing definitions of SOPML (e.g. [11, 21, 37]) are that (i) in addition to the *full*, *boolean* and *unrestricted* domains of quantification that were studied before, we also consider *modal* domains, and (ii) we use a multi-agent language, which allows us to express higher-order properties of knowledge, i.e., knowledge about other agents' knowledge, including truthfulness of knowledge, inclusion of one agent's knowledge in that of another.

In Section 3 we illustrate the richness of the formal framework, particularly to express local properties in modal logic (LPML) [16, 17]. We compare and contrast our approach with [18], and show that the latter can be subsumed in the account here put forward. This validates our endeavour from the viewpoint of applications. However, we maintain that for SOPML to be adopted as a specification language in artificial intelligence and knowledge representation, appropriate theoretical results and formal tools need to be developed.

95 To this end, in Section 4 we present a number of results about the axiomatisation of several classes of validities. The key findings are that (i) with the exception of single-agent **S5**, SOPML is unaxiomatisable over all of the commonly used classes of frames, when a full domain of quantification is considered; (ii) on frames with a coarser domain of quantification, SOPML without a common knowledge operator is axiomatisable, but
100 SOPML* with common knowledge is unaxiomatisable in general. As a by-product we obtain several undecidability results.

Furthermore, in Section 5 we develop original truth-preserving bisimulations for SOPML. Bisimulations bring to the fore when two models can be considered the same, and they can be used to test the limits of what can be expressed: when two models for
105 a language \mathcal{L} are bisimilar but disagree on some property $\Phi \in \mathcal{L}'$, it shows that Φ is not expressible in \mathcal{L} . We provide several instances of such occurrences. To conclude, our main aim in this paper is to provide formal tools so as to facilitate the use of SOPML as a language for knowledge representation, as well as temporal and spatial reasoning in artificial intelligence.

110 1.1. Related Work.

This contribution is inspired by a series of papers on LPML, an extension of propositional modal logic to express local properties [16, 17, 18]. Here, instead of introducing an *ad hoc* language (with an adjustment for each local property one has in mind), we make use of the general framework of second-order propositional multi-modal logic. In
115 Section 3 we provide a detailed comparison of the two approaches.

Mono-modal SOPML was first considered in [11, 21, 37], mainly in relation with axiomatisability and (un)decidability questions. In particular, [21] provided several axiomatisations for normal modal logics interpreted on a variety of classes of frames. However, it considered only mono-modal languages, whereas here we adopt a multi-modal perspective. Then, [37] proved decidability and independence results pertaining to second-order
120 extensions of the mono-modal logic S5. Notwithstanding these early, significant results, the high computational complexity of SOPML and some undecidability and unaxiomatisability results might partially explain why SOPML has been studied far less than propositional modal logic, and it has been virtually unexplored as a specification language for knowledge representation and reasoning. For instance, only recently SOPML has been
125 proved complete w.r.t. the algebraic semantics in which quantification is interpreted on arbitrary meets and joins [33]. Here we consider a multi-modal version of SOPML, and its epistemic counterpart: second-order propositional epistemic logic (SOPEL).

Among the more recent contributions, [36] shows that there is a validity-preserving
130 translation from second-order logic to SOPML, for modalities weaker than or equal to **S4.2**, implying that for these modalities SOPML is unaxiomatisable. Hereafter we add to the picture and show that multi-modal **S5** is unaxiomatisable as well. Further, [12] provided SOPML with analogues of the van Benthem-Rosen and Goldblatt-Thomason theorems, while in [22] propositional quantification and bisimulations are analysed in the context of modal logic. However, the kind of quantification considered in [22] is preserved
135 by standard bisimulations, and therefore the resulting logic is provably as expressive as epistemic logic, strictly weaker than SOPML. In [39, 40] the author proves that the quantifier alternation hierarchy of SOPML formulas induces an infinite corresponding semantic hierarchy over the class of finite directed graphs. As a by-product, he obtains that, for

140 this class of structures, SOPML with the universal modality and Monadic Second-Order Logic are equally expressive.

Propositional quantification has also been considered in the context of richer modal languages, namely the temporal logics LTL and CTL. A quantified version of LTL, called QLTL, has been introduced and analysed in [49, 48], mainly in relation with the verification of reactive systems. In particular, the model-checking problem for the k -alternation fragment was proved to be k -EXPSPACE-complete. More recently, [41] discusses QCTL, a quantified version of the branching-time temporal logic CTL. The authors prove several complexity and expressivity results for a logic that has more modal operators than SOPML. They also consider two different kinds of semantics for their logic: the former is comparable to SOPML on full frames, while the latter is based on tree-unwindings. To our knowledge, no results are known about the relative expressivity of QCTL and SOPML. It is also outside the scope of this paper to find such results, although it is an interesting question for future research.

More directly related to the present contribution are [3, 4] by some of the authors. In [3] we introduced epistemic quantified boolean logic (EQBL), an epistemic variant of SOPML, and provided axiomatisability and model-checking results. Differently from the reference, here we tackle general SOPML, defined also on modalities strictly weaker than **S5**. Indeed, in this paper we analyse all normal modalities. Moreover, we provide novel unaxiomatisability and undecidability results, as well as give full details on the construction of the canonical models to prove completeness. As regards [4], we define a novel notion of (bi)simulation that generalises the one given therein. Finally, we apply these results to analyse the expressivity of SOPML in capturing relevant properties in temporal and spatial reasoning.

165 These investigations have been extended to public announcement logic (PAL). Specifically, in [5] the authors applied propositional quantification to PAL to analyse arbitrary public announcements and to formalise notions such as preservation, successfulness, and knowability. Hereafter we do not consider such extensions and keep on a purely epistemic setting.

2. Preliminaries

170 In this section we introduce the formal machinery that will be used throughout the rest of the paper, and we prove some preliminary results. First, we present the language of second-order propositional modal logic (SOPML), some of its fragments, and their interpretation on Kripke frames and models.

2.1. The Formal Languages

175 To introduce the language of second-order propositional modal logic, we fix a set AP of atomic propositions and a finite set I of indices. Any language \mathcal{L} built upon AP (using connectives and modal operators) is said to be a language *over* AP .

Definition 1 (SOPML). *The language \mathcal{L}_{sopml} contains formulas ψ as defined by the following BNF:*

$$\psi ::= p \mid \neg\psi \mid (\psi \rightarrow \psi) \mid \Box_a\psi \mid \forall p\psi$$

180 where $p \in AP$ and $a \in I$.

For this language and those to be introduced shortly, we will omit parenthesis if doing so causes no confusion. The language \mathcal{L}_{sopml} contains modal formulas $\Box_a\psi$, for every index $a \in I$. A general reading of this would be ‘according to the aspect or dimension a , formula ψ holds’. The box can have more concrete interpretations, for instance dynamic (after execution of program or action a , ψ holds), temporal or spatial (along dimensions a , ψ), or deontic (in all situations that abide to norm a , ψ is true). Indices may also denote agents, in which case $\Box_a\psi$ can represent attitudes that relate to goals (‘agent a desires ψ ’, or ‘has ψ as a goal’), that are intentional (agent a intends to achieve ψ), or informational (‘agent a believes ψ ’ or ‘ a knows that ψ ’). The latter, epistemic interpretation of \Box_a will obtain some special attention in this paper, and we will write $K_a\psi$ rather than $\Box_a\psi$.

Further, the quantified formula $\forall p\psi$ informally says that ‘for all interpretations of p , ψ obtains’. As usual, the quantifier \exists is dual to \forall : $\exists p\psi ::= \neg\forall p\neg\psi$. Analogously, in \mathcal{L}_{sopml} , $\Diamond_a\phi$ is a shorthand for $\neg\Box_a\neg\phi$, and M_a is dual to K_a .

Hereafter we consider also the extension \mathcal{L}_{sopml}^* of \mathcal{L}_{sopml} obtained by adding the following clause: if ψ is a formula, then $\Box^*\psi$ is also a formula. Instead of $\Box^*\psi$, in the epistemic interpretation we will write $C\psi$ (it is common knowledge that ψ). To give a hint of what this operator means in epistemic logic, define $E\psi$ (everybody knows that ψ) as $\bigwedge_{a \in I} K_a\psi$. Then, formula $C\psi$ intuitively captures the infinite conjunction $\psi \wedge E\psi \wedge EE\psi \wedge EEE\psi \wedge \dots$ (the usual definitions for $\top, \perp, \vee, \wedge$, and \leftrightarrow apply). To sum up, whenever we consider the epistemic interpretation of modal operators, we write K_a and C , and define formulas ψ in the language \mathcal{L}_{sopml}^* of second-order propositional epistemic logic (SOPEL) according to the following BNF:

$$\psi ::= p \mid \neg\psi \mid (\psi \rightarrow \psi) \mid K_a\psi \mid C\psi \mid \forall p\psi$$

for $p \in AP$ and $a \in I$. Standard references for modal logic are [7, 9], while for epistemic logic we refer to [20, 45].

We write SOPML for the family of logics that are based on the languages \mathcal{L}_{sopml} and \mathcal{L}_{sopml}^* . Throughout most of the paper, it is not very important whether we are considering a language with or without operator \Box^* (see Definition 8 for its interpretation). In the places where the difference between \mathcal{L}_{sopml} and \mathcal{L}_{sopml}^* is important, we write SOPML* for the logic based on \mathcal{L}_{sopml}^* .

The name ‘second-order propositional modal (epistemic) logic’ is related to second-order quantification, as will become apparent in Section 3. In particular, this formalism has been studied in relation to monadic second-order logic – MSO, see [12, 36] and also Section 3.

Example 2. To give a flavour of the expressivity of \mathcal{L}_{sopml}^* , we present some specifications written in this language. We use variants of \Box_a in our notation: their meaning will be clear from the context. Using \mathcal{L}_{sopml}^* one can for instance express that agent a believes that agent b always has some unfulfilled desires: $B_a \Box^* \exists p (D_b p \wedge \neg p)$, where operators B_a and D_b are used to represent the doxastic and desire dimensions for agent a and b respectively, whereas \Box^* is interpreted on the reachability relation w.r.t. all agents’ moves.

As a further example, formula (i) $\forall p (\Box_a p \rightarrow \Box_b p)$ expresses, in a dynamic context, that every result guaranteed by program a is also guaranteed by program b , or, provided a doxastic interpretation of the box operator, agent b believes everything that agent a believes. Deontically, the formula $\exists p (Op \wedge \neg p)$ expresses that the current world is not

225 *ideal: there are facts that ought to hold, but they do not. Finally, the doxastic-epistemic*
formula (ii) $K_b \exists p (B_a p \wedge \neg p)$ intuitively expresses that agent b knows that agent a 's beliefs
are incorrect, while (iii) $\forall p (B_a p \rightarrow p) \wedge \Box_a \exists q (B_a q \wedge \neg q)$ denotes that currently, agent
 a 's beliefs are correct, but after executing program α , this ceases to be the case. We
230 *remark that by using propositional quantification we can reason about general properties*
of knowledge, e.g., truthfulness, inclusion, equivalence, of agents' knowledge and beliefs,
as in specifications (i), (ii), and (iii).

In this paper we consider various fragments of \mathcal{L}_{sopml}^* and \mathcal{L}_{sopml} . To begin with, the languages \mathcal{L}_{ml} of (propositional) modal logic and \mathcal{L}_{el} of (propositional) epistemic logic (\mathcal{L}_{ml}^* and \mathcal{L}_{el}^* , respectively) are obtained by removing clause $\forall p\psi$ from the definitions
235 of \mathcal{L}_{sopml} and \mathcal{L}_{sopel} (\mathcal{L}_{sopml}^* and \mathcal{L}_{sopel}^* , respectively). Likewise, the language \mathcal{L}_{qbf} of quantified boolean formulas omits clauses $\Box_a\psi$ from \mathcal{L}_{sopml} , while propositional logic \mathcal{L}_{pl} is defined in a standard way by considering only propositional connectives. Moreover, the *universal* fragment $\mathcal{L}_{a-sopml}^*$ of \mathcal{L}_{sopml}^* is defined by the following BNF:

$$\psi ::= p \mid \neg p \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid \Box_a \psi \mid \Box^* \psi \mid \forall p \psi$$

Notice that in $\mathcal{L}_{a-sopml}^*$ negation applies to atoms only. Hence, $\mathcal{L}_{a-sopml}^*$ contains no
240 formula of the form $\exists p\psi$, $\Diamond_a\psi$, or $\Diamond^*\psi$. For convenience, we will also denote the set of atoms AP by \mathcal{L}_{ap} . A special role in this paper will be played by the languages \mathcal{L}_x of sort x , the set of sort symbols being $\{ap, pl, ml, sopml\}$. We will shortly see that for each sort x , the language \mathcal{L}_x is linked to an interesting class of frames (defined in terms of types y : see the paragraph above Definition 8). This connection is made precise in Lemma 11,
245 item 2. We summarise the main inclusions between languages in Figure 1. We observe that languages \mathcal{L}_x^* are defined only for $x \in \{ml, a-sopml, sopml\}$.

We now introduce some syntactic notions that will be used throughout the paper. Hereafter we use $\#$ as a placeholder for any unary operator \neg , \Box_a , \Box^* , and Q for any quantifier \forall , \exists .

250 **Definition 3 (Subformula and free atoms).** *The sets $Sub(\phi)$ and $fr(\phi)$, for the sub-formulas and free atoms of formula $\phi \in \mathcal{L}_{sopml}^*$, respectively, are recursively defined as follows:*

$$\begin{array}{ll} Sub(p) & = \{p\} & fr(p) & = \{p\} \\ Sub(\# \phi) & = \{\# \phi\} \cup Sub(\phi) & fr(\# \phi) & = fr(\phi) \\ Sub(\phi \rightarrow \phi') & = \{\phi \rightarrow \phi'\} \cup Sub(\phi) \cup Sub(\phi') & fr(\phi \rightarrow \phi') & = fr(\phi) \cup fr(\phi') \\ Sub(\forall p \phi) & = \{\forall p \phi\} \cup Sub(\phi) & fr(\forall p \phi) & = fr(\phi) \setminus \{p\} \end{array}$$

A *sentence* is a formula ϕ with an empty set of free atoms, i.e., $fr(\phi) = \emptyset$. The set $bnd(\phi)$ of bound atoms in ϕ is defined as usual as the set of all atoms q appearing in
255 the scope of any quantifier Qq . We assume that for each formula $\phi \in \mathcal{L}_{sopml}^*$, $fr(\phi)$ and $bnd(\phi)$ are disjoint. Actually, we impose that each quantifier binds a different variable. Both constraints can be enforced without loss of generality by renaming bound variables.

We now define when a formula ψ can substitute an atom p within a formula. In particular, such a substitution should not create any new binding.

260 **Definition 4 (Free for ...).** *Given an atom $p \in fr(\phi)$, a formula ψ is free for p in ϕ iff p does not appear in ψ within the scope of any quantifier Qq for $q \in fr(\psi)$. Alternatively, we can define whether ψ is free for p in ϕ by induction on the structure of ϕ as follows:*

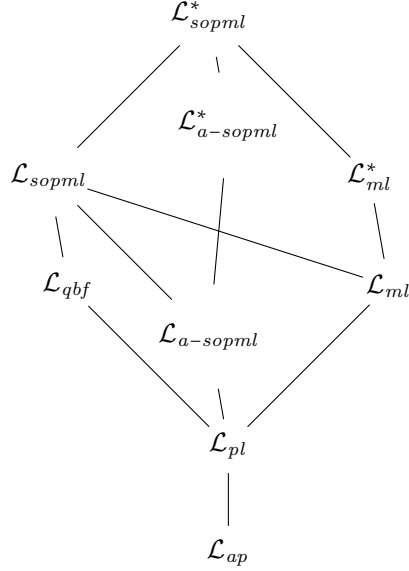


Figure 1: Scheme of inclusions among the different languages.

for ϕ atomic, ψ is free for p in ϕ
 for $\phi = \# \phi'$, ψ is free for p in ϕ iff it is in ϕ'
 for $\phi = \phi' \rightarrow \phi''$, ψ is free for p in ϕ iff it is in ϕ' and ϕ''
 for $\phi = \forall q \phi'$, ψ is free for p in ϕ iff $q \notin fr(\psi)$ and ψ is free for p in ϕ'

We finally introduce a notion of substitution for free formulas.

Definition 5 (Substitution). Whenever ψ is free for $p \in fr(\phi)$, the substitution $\phi[p/\psi]$ is inductively defined as follows:

$$\begin{aligned}
 q[p/\psi] &= \begin{cases} q & \text{for } q \text{ different from } p \\ \psi & \text{otherwise} \end{cases} \\
 (\# \phi')[p/\psi] &= \#(\phi'[p/\psi]) \\
 (\phi' \rightarrow \phi'')[p/\psi] &= (\phi'[p/\psi]) \rightarrow (\phi''[p/\psi]) \\
 (\forall r \phi')[p/\psi] &= \forall r(\phi'[p/\psi]), \text{ where } r \text{ is assumed different from } p \text{ as } p \in fr(\phi)
 \end{aligned}$$

Intuitively, ψ being free for p in ϕ means that a substitution of p by ψ in ϕ does not create any new binding. As an example, $\neg q$ is free for p in $\exists r(r \rightarrow p)$ but not in $\phi = \exists q(p \leftrightarrow q)$. After we have introduced our semantics, it will be clear that, while $\exists q(p \leftrightarrow q)$ is actually a validity, if we were to blindly substitute p with $\neg q$ in ϕ , we would obtain $\exists q(\neg q \leftrightarrow q)$, which is tantamount to a contradiction. But note that, since $\neg q$ is not free for p in ϕ , by Definition 5, $\phi[p/\neg q]$ is not well-defined. Also note that the procedure above does not guarantee that after a substitution a variable r only occurs in the scope of a single quantifier Qr . For instance, $\forall r(r \rightarrow p)[p/\forall r(q \rightarrow r)] = \forall r(r \rightarrow \forall r(q \rightarrow r))$. However, reading the semantics, it will become clear that the latter formula is equivalent

275 to $\forall r(r \rightarrow \forall s(q \rightarrow s))$: bounded variables can always be renamed (so that, in particular, every formula is equivalent to one in which every formula is bound at most once).

Example 6. *As a further example of the expressive power of SOPEL, consider the following specification: agent b knows everything that a knows, and agent c knows this fact, but d does not. This epistemic situation can be recast in $\mathcal{L}_{\text{sopeL}}$ as the following formula:*

$$\forall p(K_{ap} \rightarrow K_{bp}) \wedge K_c \forall p(K_{ap} \rightarrow K_{bp}) \wedge \neg K_d \forall p(K_{ap} \rightarrow K_{bp})$$

In particular, we can reason further about agent d 's knowledge. Indeed, agent d might know that a knows something ignored by b , without being able to explicitly point out the content of a 's extra knowledge. This can be recast in $\mathcal{L}_{\text{sopeL}}$ by the following formula:

$$K_d \exists p(K_{ap} \wedge \neg K_{bp}) \tag{2}$$

280 *However, d could actually know about a specific fact that a knows, but b ignores, as expressed in the following formula:*

$$\exists p K_d(K_{ap} \wedge \neg K_{bp}) \tag{3}$$

285 *We remark that (3) corresponds to the de re reading of our specification, while (2) is its de dicto formalisation. Here we do not discuss in detail the de re/de dicto distinction, as it is beyond the scope of the present contribution, and refer instead to the seminal paper [46]. In particular, according to our semantics (to be introduced next), (3) is strictly stronger than (and entails) (2). Indeed, the implication (3) \Rightarrow (2) is a validity, but the converse implication (2) \Rightarrow (3) does not hold in general. Thus, among other things, SOPEL allows us to distinguish the two readings – de re and de dicto – of individual knowledge.*

290 2.2. Kripke Frames and Models

To provide a meaning to formulas of second-order propositional modal logic, we consider multi-modal Kripke frames and models, extended with a domain for the interpretation of quantifiers.

Definition 7 (Kripke frame). *A Kripke frame is a tuple $\mathcal{F} = \langle W, D, R \rangle$ where*

- 295
- W is a set of possible worlds;
 - D is the domain of propositions, i.e., a subset of 2^W ;
 - $R : I \rightarrow 2^{W \times W}$ assigns a binary relation on W to each index in I .

300 As is common in propositional modal logic (PML), for every index $a \in I$, R_a is an accessibility relation between worlds in W [7]. Differently from standard Kripke frames, Definition 7 includes a set $D \subseteq 2^W$ of “admissible” propositions for the interpretation of atoms and quantifiers. Clearly, the Kripke frames in Definition 7 are related to *general frames* [7, 44]. However, there are some notable differences. Firstly, in general frames the domain D of propositions is a boolean algebra with operators, whereas no such assumption is made in the present case. Secondly, the language interpreted on general frames is

305 usually a plain modal logic, while here we address quantification as well. Indeed, propo-
 sitional quantification makes our language strictly more expressive than propositional
 modal logic interpreted on general frames, as will become apparent later on (see for in-
 stance Example 10 and recall that PML interpreted on general frames is as expressive as
 PML interpreted on Kripke frames).

310 The accessibility relations can satisfy various properties, e.g., seriality, symmetry,
 transitivity, reflexivity, etc. When interpreting the language \mathcal{L}_{sopml}^* we assume that each
 R_a is an equivalence relation (i.e., symmetric, transitive and reflexive), in line with the
 epistemic reading of modal operators [45]. Finally, for each agent index $a \in I$ and $w \in W$,
 we let $R_a(w) = \{w' \mid R_a(w, w')\}$. If R_a is an equivalence relation, then $R_a(w)$ is the
 315 equivalence class of w according to R_a .

To interpret formulas in \mathcal{L}_{sopml}^* on Kripke frames, we introduce *assignments* as func-
 tions $V : AP \rightarrow D$. Also, for $U \in D$, the assignment V_U^p assigns U to p and coincides with
 V on all other atoms. Hence, atoms can only be assigned propositions in $D \subseteq 2^W$. A
Kripke model over \mathcal{F} is then defined as a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$.

320 We now define the notion of satisfaction for formulas in \mathcal{L}_{sopml}^* .

Definition 8 (Semantics). *We define whether Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ satisfies for-
 mula $\varphi \in \mathcal{L}_{sopml}^*$ at world w , or $(\mathcal{M}, w) \models \varphi$, as follows:*

$$\begin{array}{ll}
 (\mathcal{M}, w) \models p & \text{iff } w \in V(p) \\
 (\mathcal{M}, w) \models \neg\psi & \text{iff } (\mathcal{M}, w) \not\models \psi \\
 (\mathcal{M}, w) \models \psi \rightarrow \psi' & \text{iff } (\mathcal{M}, w) \not\models \psi \text{ or } (\mathcal{M}, w) \models \psi' \\
 (\mathcal{M}, w) \models \Box_a \psi & \text{iff for all } w' \in R_a(w), (\mathcal{M}, w') \models \psi \\
 (\mathcal{M}, w) \models \Box^* \psi & \text{iff for all } w' \in R_C(w), (\mathcal{M}, w') \models \psi \\
 (\mathcal{M}, w) \models \forall p \psi & \text{iff for all } U \in D, (\mathcal{M}_U^p, w) \models \psi
 \end{array}$$

where $R_C = (\bigcup_{a \in I} R_a)^*$ is the reflexive transitive closure of $\bigcup_{a \in I} R_a$, and $\mathcal{M}_U^p = \langle \mathcal{F}, V_U^p \rangle$.

Given Definition 8, we say that \Box_a is the necessity operator for R_a and that \Box^* is
 325 the necessity operator for the transitive reflexive closure of $\bigcup_{a \in I} R_a$. By the definition, a
 quantified formula $\forall p \psi$ (respectively, $\exists p \psi$) is true at world w iff for every (respectively,
 some) assignment of propositions in D to atom p , ψ is true. Further, as is the case for
 the common knowledge operator C , $(\mathcal{M}, w) \models \Box^* \psi$ iff $(\mathcal{M}, w') \models \psi$ for every world w'
reachable from w , i.e., for every w' such that for some sequence w_0, \dots, w_k of worlds,
 330 (i) $w_0 = w$, (ii) $w_k = w'$, and (iii) for every $i < k$, $w_i = w_{i+1}$ or $R_a(w_i, w_{i+1})$ for some
 $a \in I$. Hence, in non-epistemic contexts, \Box^* can be interpreted as a reachability operator,
 analogous to the common knowledge operator C .

The satisfaction set $\llbracket \varphi \rrbracket_{\mathcal{M}}$ of formula φ in model \mathcal{M} is defined as $\{w \in W \mid (\mathcal{M}, w) \models \varphi\}$.
 We omit the subscript \mathcal{M} whenever clear by the context. We now introduce vari-
 335 ous notions of truth and validity. First, we write $(\mathcal{F}, V, w) \models \phi$ as a shorthand for
 $(\langle \mathcal{F}, V \rangle, w) \models \phi$. Then, we say that ϕ is *true* at w , or $(\mathcal{F}, w) \models \phi$, iff $(\mathcal{F}, V, w) \models \phi$ for
 every assignment V ; ϕ is *valid* in a frame \mathcal{F} , or $\mathcal{F} \models \phi$, iff $(\mathcal{F}, w) \models \phi$ for every world w
 in \mathcal{F} ; ϕ is *valid* in a class \mathcal{K} of frames, or $\mathcal{K} \models \phi$, iff $\mathcal{F} \models \phi$ for every $\mathcal{F} \in \mathcal{K}$. Also, ϕ is
true in a model \mathcal{M} , or $\mathcal{M} \models \phi$, iff $(\mathcal{M}, w) \models \phi$ for every world w . Finally, ϕ is *satisfiable*
 340 iff for some model \mathcal{M} and world w , $(\mathcal{M}, w) \models \phi$.

In the rest of the paper we consider specific classes of Kripke frames and models,
 which feature pre-eminently in the literature on SOPML [21, 44]. To introduce them,

we first define operators $[a] : 2^W \rightarrow 2^W$, for every $a \in I$, such that $[a](U) = \{w \in W \mid R_a(w) \subseteq U\}$ while operator $[]^* : 2^W \rightarrow 2^W$ is introduced so that $[]^*(U) = \{w \in W \mid$
 345 $\text{for every } n \in \mathbb{N}, \text{ for every sequence } w_0, \dots, w_n, \text{ if } w_0 = w \text{ and for every } i < n, w_i = w_{i+1} \text{ or } R_a(w_i, w_{i+1}) \text{ for some } a \in I, \text{ then } w_n \in U\}$. Notice that for R_C defined as the reflexive transitive closure of $\cup_{a \in I} R_a$, we have $[C](U) = \{w \in W \mid R_C(w) \subseteq U\} = []^*(U)$.

Definition 9. A Kripke frame \mathcal{F} is

- boolean iff D is a boolean algebra, i.e., it is closed under intersection, union and complement
- 350 modal iff D is a boolean algebra closed under operators $[a]$, for every $a \in I$, and $[]^*$
- full iff $D = 2^W$

A Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is *boolean* (*modal*, *full*, respectively) whenever the underlying frame \mathcal{F} is. We distinguish the class \mathcal{K}_{all} of all Kripke frames, the class \mathcal{K}_{bool} of all boolean frames, the class \mathcal{K}_{modal} of all modal frames, and the class \mathcal{K}_{full} of all full frames. Observe that, by using an analogy with monadic second-order logic, the class of
 355 full frames corresponds to the basic interpretation of SOPML, where any frame is uniquely identified by fixing the set W of worlds and accessibility relations, as the domain D is equal to 2^W . On the other hand, the other classes of frames are related to the Henkin interpretation of MSO, where D can be a possibly strict subset of 2^W (cf. [51]). Hereafter, we often refer to \mathcal{K}_{all} -, \mathcal{K}_{bool} -, and \mathcal{K}_{modal} -frames as *non-full* frames, even though they
 360 do contain full frames.

Furthermore, within each of the classes in Definition 9, we will consider further conditions on the accessibility relations R_a : reflexivity r , transitivity t , and symmetry s . Hereafter, given type $y \in Y = \{all, bool, modal, full\}$ and subset $\tau \subseteq \{r, t, s\}$, \mathcal{K}_y^τ denotes the corresponding class of frames satisfying the properties in τ . For simplicity, \mathcal{K}_y^e denotes
 365 class $\mathcal{K}_y^{\{r,t,s\}}$ (which we also write as \mathcal{K}_y^{rts}) of frames in which all accessibility relations are equivalences, that is, the class of epistemic frames for the interpretation of SOPEL. We define a function $\widehat{\cdot} : X \rightarrow Y$ from language sort symbols to type symbols as follows: $\widehat{ap} = all$; $\widehat{pl} = bool$; $\widehat{ml} = modal$; and $\widehat{sopml} = full$. In total, we obtain 32 classes \mathcal{K}_y^τ of frames. However, we only consider 20 of them: the subsets $\tau \subseteq \{r, t, s\}$ corresponding
 370 to the 5 normal modalities **K**, **T**, **S4**, **B**, and **S5**, combined with the 4 types *all*, *bool*, *modal*, and *full*. Further classes of frames could be introduced, for instance the class where every formula in \mathcal{L}_{sopml}^* defines a proposition in D . However, such a class is not directly relevant for the results below and its introduction requires a non-trivial generalisation of Kripke frames [44]. Thus, such extensions are beyond the scope of the present
 375 paper.

Observe that if we define $\text{Th}(\mathcal{K}) = \{\phi \in \mathcal{L}_{sopml}^* \mid \mathcal{K} \models \phi\}$, then clearly

$$\text{Th}(\mathcal{K}_{all}) \subseteq \text{Th}(\mathcal{K}_{bool}) \subseteq \text{Th}(\mathcal{K}_{modal}) \subseteq \text{Th}(\mathcal{K}_{full}) \quad (4)$$

In Section 4.1 we show that these inclusions are strict, but first we illustrate some applications of SOPEL in reasoning about knowledge.

Example 10. To assess the expressivity of SOPEL in knowledge representation, we contrast it with comparative epistemic logic – CEL [18]. CEL extends propositional modal

380 logic with formulas $a \geq b$, the intuitive interpretation of which is: agent b knows at least as much as agent a . Semantically, the clause for satisfaction of such formulas at world w in model \mathcal{M} is given as

$$(\mathcal{M}, w) \models a \geq b \quad \text{iff} \quad R_a(w) \supseteq R_b(w) \quad (5)$$

In this sense $a \geq b$ also expresses a local property of frame \mathcal{F} , namely the inclusion $R_b(w) \subseteq R_a(w)$.

We show that the comparison between agent a 's and agent b 's knowledge can be recast in SOPEL as

$$\forall p(K_a p \rightarrow K_b p) \quad (6)$$

385 In particular, the RHS of (5) is tantamount to the satisfaction of (6) at w , whenever model \mathcal{M} is full. More precisely, for an arbitrary model \mathcal{M} we have

$$(\mathcal{M}, w) \models a \geq b \quad \Rightarrow \quad (\mathcal{M}, w) \models \forall p(K_a p \rightarrow K_b p)$$

The converse also holds for full \mathcal{M} , but has counterexamples in the classes of boolean and modal models. As a result, formulas $a \geq b$ and (6) have the same meaning in the class of full models, and therefore CEL can indeed be mimicked in SOPEL. We discuss this fact in more detail in Section 3.

390 Moreover, in SOPEL we can make distinctions that are not expressible in epistemic logic. Related to Example 2, in $\mathcal{L}_{\text{sopeL}}$ we can state that b knows that a 's beliefs are not truthful by using formula

$$K_b \exists p(B_a p \wedge \neg p) \quad (7)$$

Notice that (7) expresses that b knows that there exists some fact believed by a , which is false, possibly without being able to explicitly point out the actual content of a 's false belief. On the other hand, it could be the case that for some proposition p , agent b knows that a wrongly believes it, as expressed in the following:

$$\exists p K_b(B_a p \wedge \neg p) \quad (8)$$

The formula displayed at (7) is usually referred to as a de dicto reading of the statement above, where quantifier $\exists p$ appears within the scope of modal operator K_a , while (8) corresponds to the de re reading of the same statement, in which $\exists p$ appears outside the scope of K_a (we refer the interested reader in the two different readings to [46]). We remark that (7) and (8) are not equivalent in general, (8) being strictly stronger than (7). Specifically, to account for the difference between (7) and (8), consider frame \mathcal{G} in Fig. 2(a), where the W - and R -components are as depicted, and $D = \{\{w\} \mid w \in W\}$. Clearly, $(\mathcal{G}, V, w_1) \models B_a p \wedge \neg p$ for $V(p) = \{u_1\}$, and similarly $(\mathcal{G}, V', w_2) \models B_a p \wedge \neg p$ for $V'(p) = \{u_2\}$. Hence, $(\mathcal{G}, w) \models (7)$ for $w \in \{w_1, w_2\}$. On the other hand, for no $U \in D$, $(\mathcal{G}, V_U^p, w) \models B_a p \wedge \neg p$. Therefore, $(\mathcal{G}, w) \not\models (8)$ for $w \in \{w_1, w_2\}$. Finally, we observe that $\exists p K_a \phi \rightarrow K_a \exists p \phi$ is a validity in every class of frames. As a result, in SOPEL (8) is strictly stronger than (7), and we can distinguish the de dicto and de re readings of agent b 's higher-level knowledge.

Finally, consider frame \mathcal{G}' in Fig. 2(b) with $D' = \{\{w'\} \mid w' \in W'\}$. Let \mathcal{M} and \mathcal{M}' be models based on \mathcal{G} and \mathcal{G}' respectively, in such a way that assignments V and V' make

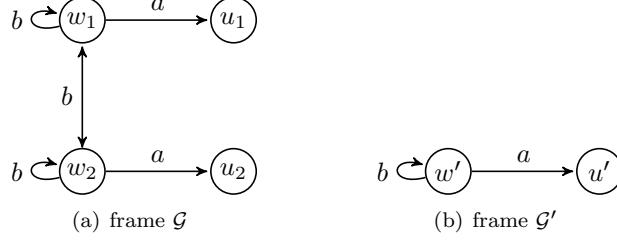


Figure 2: Frames \mathcal{G} and \mathcal{G}' in Example 10.

the same atoms true in w_1 , w_2 , and w' , and similarly for u_1 , u_2 and u' . One can check that $(\mathcal{M}', w') \models (8)$ (and (7) as well). However, (\mathcal{M}, w_2) and (\mathcal{M}', w') , satisfy the same formulas in \mathcal{L}_{ml} (indeed, the two models are bisimilar), implying that the de re formula (8) cannot be expressed in PML. We return to this example in Section 5.

2.3. Preliminary Results

In this section we prove some preliminary results on the model-theory of second-order propositional modal logic, that will be frequently used in the rest of the paper. To start with, in Lemma 11 we extend some basic but useful results in the theory of quantification. In particular, in first-order logic item 1 of Lemma 11 is known as the *coincidence lemma*, and item 2b as the *substitution lemma* (cf. [19]).

Lemma 11.

1. Let ϕ be a formula in \mathcal{L}_{sopml}^* and \mathcal{F} a frame in \mathcal{K}_{all} . If assignments V and V' coincide on $fr(\phi)$, then

$$(\mathcal{F}, V, w) \models \phi \quad \text{iff} \quad (\mathcal{F}, V', w) \models \phi$$

2. Recall that $X = \{ap, pl, ml, sopml\}$ and $\hat{\sim} = \{(ap, all), (pl, bool), (ml, modal), (sopml, full)\}$. Let $x \in X$. Then,

- (a) For every $\psi \in \mathcal{L}_x^*$ and model \mathcal{M} over $\mathcal{F} \in \mathcal{K}_{\hat{x}}$, we have $\llbracket \psi \rrbracket_{\mathcal{M}} \in D$.
- (b) If $\mathcal{F} \in \mathcal{K}_{\hat{x}}$ and $\psi \in \mathcal{L}_x^*$ is free for p in ϕ , then

$$(\mathcal{F}, V_{\llbracket \psi \rrbracket_{(\mathcal{F}, V)}}^p, w) \models \phi \quad \text{iff} \quad (\mathcal{F}, V, w) \models \phi[p/\psi]$$

The proof of this lemma is immediate, so we include it only in the appendix. These results show that quantification in SOPML is “well-behaved”: by item 1 of Lemma 11, models built over the same frame and agreeing on the interpretation of free atoms, satisfy the same formulas. It follows in particular that a sentence ϕ is either satisfied by any assignment or none, that is, $(\mathcal{F}, w) \models \phi$ iff for every model \mathcal{M} over \mathcal{F} , $(\mathcal{M}, w) \models \phi$, iff for some model \mathcal{M} over \mathcal{F} , $(\mathcal{M}, w) \models \phi$. As a consequence of Lemma 11, item 2a, the domain of quantification in a model includes the set of denotations of formulas in that model, according to the various fragments of \mathcal{L}_{sopml}^* . Moreover, by Lemma 11, item 2b, the syntactic operation of substitution $\phi[p/\psi]$ corresponds to the semantic notion of reinterpretation $\mathcal{M}_{\llbracket \psi \rrbracket}^p$.

440 In Section 4 we will make use of *generated submodels*, a concept that is commonly used in modal logic.

Definition 12 (Generated Submodel). *Given model $\mathcal{M} = \langle W, D, R, V \rangle$ and world $w \in W$, the submodel generated by w is the model $\mathcal{M}_w = \langle W_w, D_w, R_w, V_w \rangle$ such that*

- W_w is the set of worlds reachable from w , i.e., $W_w = (\bigcup_{a \in I} R_a)^*(w)$;
- 445 • $D_w = \{U_w \subseteq W_w \mid U_w = U \cap W_w \text{ for some } U \in D\}$;
- for every $a \in I$, $R_{w,a} = R_a \cap W_w^2$;
- for every $p \in AP$, $V_w(p) = V(p) \cap W_w$.

The *subframe generated by w* is then defined as $\mathcal{F}_w = \langle W_w, D_w, R_w \rangle$. The relevant property of a generated submodel \mathcal{M}_w is that $(\mathcal{M}, w) \models \Box^* \phi$ if and only if $(\mathcal{M}_w, w') \models \phi$ 450 for every $w' \in W_w$. It is also important to note that if \mathcal{M} is full, modal or boolean, then so is \mathcal{M}_w .

Proposition 13. *For $y \in \{all, bool, modal, full\}$ and $\tau \subseteq \{r, t, s\}$, if a frame \mathcal{F} belongs to \mathcal{K}_y^τ then also $\mathcal{F}_w \in \mathcal{K}_y^\tau$.*

The proof is immediate, so we omit it.

455 2.3.1. Model Checking

In order to explore the computational properties of SOPML, we consider the complexity of its model checking problem. Then, in the next section we analyse the (lack of) finite model property for SOPML. Before we can determine – or even define – the complexity of model checking, however, we first need to define the size of formulas and models. Our 460 definition of the former is entirely as usual.

Definition 14 (Formula Size). *Let $\phi \in \mathcal{L}_{sopml}^*$ be a formula. The size of ϕ , denoted $|\phi|$, is given recursively by: $|p| = 1$, $|\psi_1 \rightarrow \psi_2| = |\psi_1| + |\psi_2| + 1$, and $|\#\psi| = |\forall p\psi| = |\psi| + 1$.*

Similarly, the size of a model \mathcal{M} can be defined in a straightforward way.

Definition 15 (Model Size). *The size $|\mathcal{M}|$ of model $\mathcal{M} = \langle W, D, R, V \rangle$ is given by 465 $|\mathcal{M}| = |W| + |D| + \sum_{a \in I} |R_a| + \sum_{p \in AP} |V(p)|$. A model \mathcal{M} is finite if $|\mathcal{M}| < \infty$.*

Now that we have defined the sizes of formulas and models, we can define the model checking problem and determine its complexity.

Definition 16 (Model Checking for SOPML). *Given a formula $\phi \in \mathcal{L}_{sopml}^*$, a finite 470 model \mathcal{M} and a world w of \mathcal{M} , the model checking problem for SOPML is to determine whether $(\mathcal{M}, w) \models \phi$.*

Then, we are able to prove the following complexity result.

Theorem 17 (Model Checking Complexity). *The model checking problem for SOPML is PSPACE-complete with respect to $|\phi| + |\mathcal{M}|$.*

475 **Proof.** As regards hardness, we reduce satisfiability of quantified boolean formulas to SOPML model checking. Given a formula $\phi \in \mathcal{L}_{qbf}$, consider the frame $\mathcal{F} = \langle \{w\}, (w, w), \{\{w\}, \emptyset\} \rangle$ and an arbitrary assignment V , and define $\mathcal{M} = \langle \mathcal{F}, V \rangle$. Then, we have that ϕ is satisfiable iff $(\mathcal{M}, w) \models \exists \vec{p} \phi$, where \vec{p} are all the atoms in ϕ . Because the satisfiability problem for quantified boolean formulas is PSPACE-hard, it follows that
 480 model checking SOPML is PSPACE-hard as well.

As regards being in PSPACE, an algorithm in PSPACE for model checking SOPML is shown as Algorithm 1.¹ It is based on standard model checking algorithms for modal logic [8], which run in polynomial time. The difference between these standard algorithms for modal logic and Algorithm 1 is that we need an extra case for the $\forall p$ operator. This
 485 extra case is why Algorithm 1 takes polynomial space as opposed to polynomial time.

Algorithm 1 recursively calls itself. The depth of this recursion is bounded by $|\phi|$. Furthermore, at each stage we need to keep only one of these recursive calls in memory at a time. For example, in the $\forall p\psi$ step, if $U, U' \in D$ we can first compute $X \cap \llbracket \psi \rrbracket_{\mathcal{M}_U^p}$ and then flush the memory dedicated to computing $\llbracket \psi \rrbracket_{\mathcal{M}_U^p}$ before computing $\llbracket \psi \rrbracket_{\mathcal{M}_{U'}^p}$. It
 490 follows that the space requirement of Algorithm 1 is polynomial with respect to $|\phi| + |\mathcal{M}|$. \square

Algorithm 1 Computation of the satisfaction set $\llbracket \phi \rrbracket_{\mathcal{M}}$

```

switch ( $\phi$ ):
  case  $p$ :
    return  $V(p)$ ;
  case  $\neg\psi$ :
    return  $W \setminus \llbracket \psi \rrbracket_{\mathcal{M}}$ ;
  case  $\psi \wedge \psi'$ :
    return  $\llbracket \psi \rrbracket_{\mathcal{M}} \cap \llbracket \psi' \rrbracket_{\mathcal{M}}$ ;
  case  $\Box_a \psi$ :
    return  $\{w \in W \mid R_a(w) \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}\}$ ;
  case  $\Box^* \psi$ :
    return  $\{w \in W \mid (\bigcup_{a \in Ag} R_a)^*(w) \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}\}$ ;
  case  $\forall p\psi$ :
    initialise  $X = W$ 
    for  $U \in D$  do
      set  $X = X \cap \llbracket \psi \rrbracket_{\mathcal{M}_U^p}$ 
    end for
    return  $X$ 

```

As a result, model checking SOPML is no more computationally complex than the corresponding problem for quantified boolean formulas. Thus, the enhanced expressiveness comes at no extra computational cost, when compared with QBF. With respect
 495 to propositional modal logic, the complexity increases from PTIME to PSPACE. However, this is something to be expected given the extra expressive power of propositional quantification.

¹Strictly speaking, the algorithm computes $\llbracket \phi \rrbracket_{\mathcal{M}}$, but we can easily determine whether $\mathcal{M}, w \models \phi$ from that.

We should also note that the complexity of the model checking problem depends more strongly on $|\phi|$ than on $|\mathcal{M}|$. The recursion in Algorithm 1 can be seen as a tree with depth bounded by $|\phi|$ and branching factor bounded by $|\mathcal{M}|$. So the algorithm is called at most $|\mathcal{M}|^{|\phi|}$ times. As a result, while the space complexity of the algorithm is polynomial in both $|\phi|$ and $|\mathcal{M}|$, its time complexity is exponential in $|\phi|$ and polynomial in $|\mathcal{M}|$.

Furthermore, we can obtain the same PSPACE result when using a more concise representation of \mathcal{M} . So far, we have defined the size of \mathcal{M} as $|\mathcal{M}| = |W| + |D| + \sum_{a \in I} |R_a| + \sum_{p \in AP} |V(p)|$. This means that, among other things, we simply count the number of elements in D as one of the components of \mathcal{M} . So, in effect, we are treating D as a list. In some cases, however, there are other natural representations of D that are much more concise. Suppose, for example, that \mathcal{M} has a full domain, so $D = 2^W$. Then instead of representing D as a list of sets we could represent it symbolically as 2^W . More generally, we can assume that D is given by some membership function f , so $D = \{Y \subseteq W \mid f(Y, W)\}$. The only requirement we place upon it is that f should not be too hard to compute; we assume that determining whether $f(Y, W)$ can be done in polynomial space with respect to $|W|$.

Then the model checking problem for SOPML is PSPACE-complete not just in $|\phi| + |\mathcal{M}|$ but also in $|\phi| + |W| + |R| + |V|$, which can be exponentially smaller. With regard to hardness, note that the hardness part of the proof of Theorem 17 uses a fixed model, so in that regard the definition of model size is irrelevant. It follows that the model checking problem is also PSPACE-hard with respect to $|\phi| + |W| + |R| + |V|$. With regard to being in PSPACE, a slight modification to Algorithm 1 suffices. Suppose that we replace the $\forall p\psi$ case of that algorithm with

```

case  $\forall p\psi$ :
  initialise  $X = W$ 
  for  $U \in 2^W$  do
    if  $U \in D$  then
      set  $X = X \cap \llbracket \psi \rrbracket_{\mathcal{M}_U^p}$ 
    end if
  end for
  return  $X$ 

```

The proof that this amended algorithm runs in PSPACE with respect to $|\phi| + |W| + |R| + |V|$ can be done in the same way as the proof of Theorem 17, so we omit it here.

2.3.2. Finite Model Property

We now briefly argue why SOPML does not have the finite model property. Consider the following set of formulas:

$$\Gamma = \{\diamond_a \top, \Box_a \diamond_a \top, \forall p(\Box_a p \rightarrow \Box_a \Box_a p), \Box_a \exists p(p \wedge \Box_a \neg p)\}$$

Now suppose that Γ holds at some pointed model (\mathcal{M}, w) . Then the first two formulas of Γ require R_a to be serial on $\{w\} \cup R_a(w)$, and the third enforces transitivity of R_a (also at w). Finally, if world w satisfies $\Box_a \exists p(p \wedge \Box_a \neg p)$, then, by Example 24 item 1, we know that $\neg R_a(v, v)$ for all $v \in R_a(w)$, which implies that $\neg R_a(w, w)$, so that R_a is irreflexive over $\{w\} \cup R_a(w)$. But it is easy to verify that a transitive, serial, and irreflexive relation on $R_a(w)$ requires $R_a(w)$ to be infinite. In other words, we found a finite set Γ of formulas in SOPML that only has infinite models.

Theorem 18. *The logic SOPML does not have the finite model property.*

540 Theorem 18 is a generalisation of a result presented in [40, Section 3]. As the name of
that section ('SOPMLE = MSO') suggests, it demonstrates that SOPMLE, which is SOPML
with a universal modality, has the same expressive power as MSO. And obviously, MSO
can force a model to be infinite (use the relational properties of our example above), and
therefore SOPMLE can. Note that in our example, we don't assume a universal modality
545 in our language, though.

3. Local Properties in Modal Logic

In the introduction we discussed the difference between a global property as expressed
by the modal schema (i) $\Box_a \varphi \rightarrow \varphi$, whose validity entails that the accessibility relation in
a given frame is reflexive, and a local property such as the one represented by the SOPML
550 formula $\forall p(\Box_a p \rightarrow p)$ that, as we shall see, on full frames holds exactly in reflexive worlds.
Along this line, in [16, 17, 18] a sophisticated account was put forward to express local
properties, by adding dedicated modal operators to a basic propositional modal logic.
To present the language of local properties in modal logic, or LPML, to compare the two
approaches, and more generally to discuss the expressive power of SOPML, we consider a
555 monadic second-order logic and a first-order fragment interpreted on Kripke frames.

Given a frame $\mathcal{F} = \langle W, D, R \rangle$ and a set AP of atoms, we define an MSO alphabet
containing binary predicate constants R_a for every agent index $a \in I$, a unary predicate
variable P for every atom $p \in AP$, and a set \mathcal{X} of individual variables. Then, MSO
formulas Θ in \mathcal{L}_{mso} are defined in BNF as follows:

$$\Theta ::= P(x) \mid x = y \mid R_a(x, y) \mid \neg\Theta \mid \Theta \rightarrow \Theta \mid \forall x\Theta \mid \forall P\Theta$$

560 where $a \in I$ and $x, y \in \mathcal{X}$.

We also consider the first-order fragment \mathcal{L}_{fo} of MSO obtain by removing clause $\forall P\Theta$
from the BNF above. This is indeed the first-order language considered in [18]. Moreover,
we denote as \mathcal{L}_{fo}^1 the fragment of \mathcal{L}_{fo} containing formulas with at most one free individual
variable. This fragment is well-known to be rich enough to express properties of frames
565 such as reflexivity, symmetry, and transitivity (note that more than one variable is needed
for e.g. transitivity, but at most one is free).

As regards the interpretation of MSO and FO (First-Order) formulas, an *assignment*
 ρ now is a function associating a world $w \in W$ to every individual variable x , and a set
 $U \in D$ to every predicate variable P . For $w \in W$ and $U \in D$, the variants ρ_w^x and ρ_U^P are
570 defined similarly to SOPML.

Definition 19 (Semantics of MSO). *We define whether frame $\mathcal{F} = \langle W, D, R \rangle$ satisfies
formula $\Theta \in \mathcal{L}_{mso}$ for an assignment ρ , or $(\mathcal{F}, \rho) \models \Theta$, as follows:*

$$\begin{array}{lll} (\mathcal{F}, \rho) \models P(x) & \text{iff} & \rho(x) \in \rho(P) \\ (\mathcal{F}, \rho) \models x = y & \text{iff} & \rho(x) = \rho(y) \\ (\mathcal{F}, \rho) \models R_a(x, y) & \text{iff} & R_a(\rho(x), \rho(y)) \\ (\mathcal{F}, \rho) \models \neg\Theta & \text{iff} & (\mathcal{F}, \rho) \not\models \Theta \\ (\mathcal{F}, \rho) \models \Theta \rightarrow \Theta' & \text{iff} & (\mathcal{F}, \rho) \not\models \Theta \text{ or } (\mathcal{F}, \rho) \models \Theta' \\ (\mathcal{F}, \rho) \models \forall x\Theta & \text{iff} & \text{for all } w \in W, (\mathcal{F}, \rho_w^x) \models \Theta \\ (\mathcal{F}, \rho) \models \forall P\Theta & \text{iff} & \text{for all } U \in D, (\mathcal{F}, \rho_U^P) \models \Theta \end{array}$$

Obviously Definition 19 induces an interpretation of formulas in \mathcal{L}_{fo} as well. In particular, for a formula $\Theta(x) \in \mathcal{L}_{fo}^1$, we write $(\mathcal{F}, w) \models \Theta$ to denote that $(\mathcal{F}, \rho) \models \Theta$ for $\rho(x) = w$, and $\mathcal{F} \models \Theta$ if $(\mathcal{F}, w) \models \Theta$ for all $w \in W$. The different interpretation of the satisfaction relation \models for SOPML and MSO respectively will be clear from the context. Note that the transitive closure R^* of R can be easily defined in MSO.

We illustrate the relationship between second-order propositional modal logic and monadic second-order logic (MSO) through translation ST that extends the standard translation between modal and first-order logic [7]:

$$\begin{aligned}
ST_x(p) &= P(x) \\
ST_x(\neg\phi) &= \neg ST_x(\phi) \\
ST_x(\phi \rightarrow \phi') &= ST_x(\phi) \rightarrow ST_x(\phi') \\
ST_x(\Box_a \phi) &= \forall y(R_a(x, y) \rightarrow ST_y(\phi)) \\
ST_x(\Box^* \phi) &= \forall y(R^*(x, y) \rightarrow ST_y(\phi)) \\
ST_x(\forall p \phi) &= \forall P(ST_x(\phi))
\end{aligned}$$

Clearly, for every formula $\phi \in \mathcal{L}_{sopml}^*$, $ST_x(\phi) \in \mathcal{L}_{mso}$ is a formula where x is the only free individual variable. If $\psi \in \mathcal{L}_{ml}$ is a purely propositional modal formula, then $ST_x(\psi) \in \mathcal{L}_{fo}$ is a first-order formula, as obtained via the standard translation. In particular, $ST_x(\psi)$ belongs to \mathcal{L}_{fo}^1 .

We now get the following preservation result for the standard translation, that will be used in the completeness proof.

Lemma 20. *For every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, world $w \in W$, and formula $\psi \in \mathcal{L}_{sopml}^*$,*

$$(\mathcal{M}, w) \models \psi \quad \text{iff} \quad (\mathcal{F}, \rho) \models ST_x(\psi)$$

whenever $\rho(x) = w$ and $\rho(P_i) = V(p_i)$.

The proof is mostly standard, and can be found in Appendix A. As a consequence of Lemma 20, there is a one-to-one correspondence between formulas in SOPML and their standard translations in MSO in the following sense: a frame \mathcal{F} validates the universal closure $\forall \vec{p} \psi$ of a formula $\psi \in \mathcal{L}_{sopml}^*$ iff property $\forall \vec{P} ST_x(\psi) \in \mathcal{L}_{mso}$ holds in \mathcal{F} , where \vec{P} are all the unary predicates appearing in $ST_x(\psi)$.

We now briefly recall some basic modal theory on local definability: we refer the interested reader to [7, 9] for further details. We use θ (or $\theta(\vec{a}, \vec{p})$ to emphasise sequences \vec{a} of indices and \vec{p} of atoms) for formulas in \mathcal{L}_{ml} . Likewise, we use $\Theta \in \mathcal{L}_{fo}^1$ for first-order formulas with at most one free variable interpreted over states (or $\Theta(\vec{a}, x)$ to denote that Θ mentions \vec{a} as indices and has x as the free variable).

Definition 21. *Let $\theta \in \mathcal{L}_{ml}$ and $\Theta \in \mathcal{L}_{fo}^1$,*

1. θ defines frame property Θ iff for all frames \mathcal{F} , $\mathcal{F} \models \theta$ iff $\mathcal{F} \models \Theta$.
2. θ locally defines Θ iff for all \mathcal{F} and all $w \in \mathcal{F}$, $(\mathcal{F}, w) \models \theta$ iff $(\mathcal{F}, w) \models \Theta$.

As examples of Definition 21, consider the well-known schemes **T** $\Box_a \varphi \rightarrow \varphi$, **4** $\Box_a \varphi \rightarrow \Box_a \Box_a \varphi$, and **B** $\varphi \rightarrow \Box_a \Diamond_a \varphi$, that (locally) define the properties of reflexivity, transitivity, and symmetry on frames. Furthermore, by Lemma 20 it is clear that every $\theta \in \mathcal{L}_{ml}$ (locally) defines $\forall \vec{P} ST_x(\theta) \in \mathcal{L}_{mso}$, whenever $\forall \vec{P} ST_x(\theta)$ is equivalent to some $\Theta \in \mathcal{L}_{fo}^1$.

In the theory of PML, when formula θ locally defines Θ and some other mild conditions hold, one obtains the following connection between axiomatisation and completeness: if an axiomatisation \mathbf{Ax} is complete for a class \mathcal{K} of frames, then $\mathbf{Ax} + \theta$ is complete for class $\{\mathcal{F} \in \mathcal{K} \mid \mathcal{F} \models \forall x \Theta\}$ of frames satisfying condition Θ . So for instance, taking the basic modal logic \mathbf{K} , which is sound and complete with respect to the class \mathbb{K} of all frames, the logic $\mathbf{K} + \mathbf{T}$ is sound and complete with respect to class $\{\mathcal{F} \in \mathbb{K} \mid \mathcal{F} \models \forall x R_a(x, x) \text{ for all } a \in I\}$, that is, the class of reflexive frames. As further examples, whereas $\mathbf{S5} = \mathbf{K} + \mathbf{T} + \mathbf{4} + \mathbf{B}$ is sound and complete with respect to class $\mathbb{S5} = \{\mathcal{F} \in \mathbb{K} \mid \text{for all } a \in I, R_a \text{ is an equivalence relation}\}$, the logic $\mathbf{S5} + \{\Box_b \varphi \rightarrow \Box_c \varphi \mid b, c \in I\}$ is sound and complete with respect to $\{\mathcal{F} \in \mathbb{S5} \mid \mathcal{F} \models \forall x (R_c(x) \subseteq R_b(x)) \text{ for all } b, c \in I\}$.

This is an appealing modular feature of modal logic. Yet, as also remarked by van Ditmarsch et al. ([16, 17, 18]) this can only be applied if one adds formula θ as a *global* property: assuming θ as an axiom implies that it becomes a validity. For instance, adding formula $B_a \varphi \rightarrow \varphi$ to an axiom system, in order to model that agent a 's beliefs are correct, implies that in the resulting logic, it is common knowledge that a 's beliefs are correct, and this fact will always remain true.

To compare our approach based on SOPML to van Ditmarsch et al.'s LPML, we first provide a brief account of the latter.

3.1. Local properties and LPML

This section on LPML is based on [16, 17, 18]: we refer the reader to these references for a more extensive exposition. The term 'logic' is maybe not appropriate for LPML; rather, it is a specific approach to 'connect', in a modal object language, a modal formula $\theta \in \mathcal{L}_{ml}$ and a first-order property $\Theta \in \mathcal{L}_{fo}^1$ through the introduction of a relational atom \Box (or $\Box(\bar{a})$), in such a way that on Kripke models \Box is interpreted as Θ locally. More precisely, the language of LPML extends \mathcal{L}_{ml} with formulas of type $\Box(\bar{a})$, whose interpretation is provided by an associated formula $\Theta_{\Box}(\bar{a}, x) \in \mathcal{L}_{fo}^1$, according to the following satisfaction clause:

$$(\mathcal{M}, w) \models \Box(\bar{a}) \quad \text{iff} \quad (\mathcal{F}, w) \models \Theta_{\Box}(\bar{a}, x) \quad (9)$$

By clause (9) we say that formula $\Box(\bar{a})$ *expresses locally* first-order property Θ_{\Box} (at w).

Then, LPML investigates how operator \Box can help us, in the object language, to build a bridge between modal formulas θ_{\Box} and first-order properties Θ_{\Box} that θ_{\Box} locally defines. So, for instance, we can have $\Box(a) = \text{Refl}(a)$ for $\Theta_{\Box}(a, x) = R_a(x, x)$, or $\Box(b, c) = \text{Sup}(b, c)$ for $\Theta_{\Box}(b, c, x) = \forall y (R_c(x, y) \rightarrow R_b(x, y))$ (for more examples, see Table 1).

Recalling that operator \Box is part of the object language of LPML, [18] then adds to the basic modal logic \mathbf{K} , for specific formulas $\theta_{\Box} \in \mathcal{L}_{ml}$, an axiom \mathbf{Ax}_{\Box} and an inference rule \mathbf{R}_{\Box} . Further, [18, Theorem 2] provides a sufficient condition on the relationship between θ_{\Box}, \Box and Θ_{\Box} , called *local harmony*, under which $\mathbf{K} + \mathbf{Ax}_{\Box} + \mathbf{R}_{\Box}$ is a sound and complete axiomatisation for the class of models that satisfy Θ_{\Box} .

Definition 22 (Local Harmony). *Formulas $\theta(\bar{a}, \bar{p}) \in \mathcal{L}_{ml}$, $\Theta(\bar{a}, x) \in \mathcal{L}_{fo}^1$, and $\Box(\bar{a})$ in LPML are in local harmony iff (i) θ (locally) defines Θ , and (ii) \Box expresses Θ locally.*

A model \mathcal{M} for LPML is a tuple $\langle W, R, I, V \rangle$ where W, R and V are as for PML and I assigns a first order property to each relational atom \Box . We follow [18] in assuming that

$\theta(\vec{a}, \vec{p})$	$\Theta(\vec{a}, x)$	$\sqsupset(\vec{a})$
$\Box_a p \rightarrow \Box_b p$	$\forall y (R_b(x, y) \rightarrow R_a(x, y))$	$Sup(a, b)$
$\Box_c p \rightarrow \Box_a \Box_b p$	$\forall y, z (R_a(x, y) \wedge R_b(y, z) \rightarrow R_c(x, z))$	$Trans(a, b, c)$
$\neg \Box_a \perp$	$\exists y R_a(x, y)$	$Ser(a)$
$\Box_a p \rightarrow p$	$R_a(x, x)$	$Refl(a)$
$\neg \Box_a p \rightarrow \Box_b \neg \Box_c p$	$\forall y, z (R_a(x, y) \wedge R_b(x, z) \rightarrow R_c(y, z))$	$Eucl(a, b, c)$
$\neg \Box_a p \rightarrow \neg \Box_b \Box_c p$	$\forall z (R_a(x, z) \rightarrow \exists y R_b(x, y) \wedge R_c(y, z))$	$Dens(a, b, c)$
$(\neg \Box_a p \wedge \neg \Box_b q) \rightarrow \neg \Box_c (p \vee q)$	$\forall y, z ((R_a(x, y) \wedge R_b(x, z)) \rightarrow (y = z \wedge R_c(x, y)))$	$Func(a, b, c)$

Table 1: as in [18, Table 1], $\Theta(\vec{a}, x)$ is a property of state x , and $\sqsupset(\vec{a})$ is a name in the object language such that $\sqsupset(\vec{a})$ holds at w iff $\Theta(\vec{a}, x)$ holds in \mathcal{M} for $\rho(x) = w$.

for each symbol $\sqsupset_\theta(\vec{a})$, there is a \mathcal{L}_{ml} -formula $\theta(\vec{a}, \vec{p})$ and a \mathcal{L}_{fo}^1 -formula $\Theta(\vec{a}, x)$ such that the three are in local harmony and $I(\sqsupset_\theta(\vec{a})) = \Theta(\vec{a}, x)$. To be explicit about this, we call such a model \mathcal{M} an *intended model* for LPML.

One could say that LPML as a language can express every formula in \mathcal{L}_{fo}^1 , as there are no restrictions, in the object language, on the relational atoms \sqsupset that can be added to standard PML (i.e., Table 1 can in principle be extended with an atom $\sqsupset(\vec{a})$ for *any* property $\Theta(\vec{a}, x)$). However, the aim of LPML is not to express arbitrary first-order properties Θ , but to reason locally about properties like truthfulness of agent a 's beliefs, or an agent c knowing more than b . In particular, there has to exist a modal formula θ that (locally) defines Θ , that is, Θ has to be equivalent to $\forall \vec{P} ST_x(\theta)$. LPML expresses such first-order properties by adding atoms like $Refl(a)$ and $Sup(b, c)$, respectively. We reckon that SOPML, allowing for quantification over propositions as in $\forall p (B_a p \rightarrow p)$ and $\forall p (K_b p \rightarrow K_a p)$, is an alternative way to study local properties which is at least as natural as LPML and provably as expressive, in a sense we explain below.

3.2. Local properties, LPML and SOPML on full frames

We first compare LPML to SOPML on full frames. On the other classes of frames there are some notable differences that we discuss in Section 3.3. Here we show that if formulas $\Theta_{\sqsupset}(\vec{a}, x)$ and $\theta_{\sqsupset}(\vec{a}, \vec{p})$ are in local harmony with some atom $\sqsupset(\vec{a})$, then formula $\sqsupset(\vec{a})$ is equivalent to $\forall \vec{p} \theta(\vec{a}, \vec{p}) \in \mathcal{L}_{sopml}^*$, within the class of full frames. Hence, SOPML is at least as expressive as LPML. To make this more precise, note that LPML is only able to reason about local properties if all triples $\theta(\vec{a}, \vec{p})$, $\sqsupset(\vec{a})$, and $\Theta(\vec{a})$ are in local harmony. Recall that a LPML model \mathcal{M} that guarantees this is an intended model. We will also interpret such an intended model \mathcal{M} as a model for SOPML: one just discards the LPML information connecting $\sqsupset(\vec{a})$ and $\Theta_{\sqsupset}(\vec{a}, x)$, and then adds the constraint that the model is full. Now, consider the translation t from LPML formulas to SOPML formulas that distributes over all connectives and modal operators, and moreover says

$$t(\sqsupset(\vec{a})) = \forall \vec{p} \theta(\vec{a}, \vec{p})$$

We then obtain the following equivalence result.

Theorem 23. *For every intended LPML model \mathcal{M} , $w \in \mathcal{M}$, and formula φ in LPML, we have*

$$(\mathcal{M}, w) \models \varphi \quad \text{iff} \quad (\mathcal{M}, w) \models t(\varphi)$$

A proof of Theorem 23 is given in Appendix A. This theorem implies, in a sense, that what can be done in LPML, can also be done in SOPML: if $\theta(\vec{a}, \vec{p})$, $\boxplus(\vec{a})$ and $\Theta(\vec{a})$ are in local harmony, then, to reason locally about a scheme θ , one can either use the universal closure $\forall \vec{p}\theta$ in SOPML, or atom $\boxplus(\vec{a})$ in LPML. The result also suggests ways in which SOPML may be more appropriate to reason about local properties, namely cases where Θ is not locally defined by any formula $\theta \in \mathcal{L}_{ml}$ (i.e., there is no θ such that $\forall \vec{P}ST_x(\theta)$ is equivalent to Θ), or, conversely, when θ does not define a first-order property Θ locally (i.e., there is no Θ such that $\forall \vec{P}ST_x(\theta)$ is equivalent to Θ). Hereafter we consider some similar cases.

Example 24. Consider the following first-order formulas:

- $\Theta_1 = \neg R_a(x, x)$ (irreflexivity)
- $\Theta_2 = \exists x_1, \dots, x_n \bigwedge_{i \leq n} (R_a(x, x_i) \wedge \bigwedge_{i \neq j \leq n} x_i \neq x_j)$ (having at least n a -successors)
- $\Theta_3 = \forall y ((R_a(x, y) \wedge R_a(y, x)) \rightarrow x = y)$ (anti-symmetry)
- $\Theta_4 = \forall y (R_b(x, y) \rightarrow \neg R_a(x, y))$ (R_a and R_b have empty intersection)
- $\Theta_5 = \forall y ((R_a(x, y) \wedge R_b(x, y)) \rightarrow R_c(x, y))$ (R_c contains the intersection of R_a and R_b).

It is well-known that these first-order properties are not definable in modal logic [7, 9]. However, consider the following formulas in SOPML:

- $\varphi_1 = \exists p (\Box_a p \wedge \neg p)$
- $\varphi_2 = \exists p_1, \dots, p_n (\bigwedge_{i \leq n} \Diamond_a (p_i \wedge \bigwedge_{j \leq n, j \neq i} \neg p_j))$
- $\varphi_3 = \exists p (p \wedge \forall q (\Diamond_a (q \wedge \Diamond_a p) \rightarrow q))$
- $\varphi_4 = \exists p (\Box_a p \wedge \Box_b \neg p)$
- $\varphi_5 = \forall p (\Box_c p \rightarrow \exists q (\Box_a q \wedge \Box_b (q \rightarrow p)))$

which are such that each φ_i locally defines Θ_i ($1 \leq i \leq 5$). We formalise this result in the following lemma.

Lemma 25. Consider formulas $\varphi_i \in \mathcal{L}_{sopml}$ and $\Theta_i \in \mathcal{L}_{fo}^1$ in Example 24, for $i = 1, \dots, 5$. Let x be the only free variable in Θ_i and assume $\rho(x) = w$. Assume \mathcal{F} is a full frame, then,

$$(\mathcal{F}, w) \models \varphi_i \quad \text{iff} \quad (\mathcal{F}, \rho) \models \Theta_i$$

The proof of some items of the lemma is to be found in Appendix A. In particular, SOPML can express properties that are not definable in standard modal logic.

Example 26. [Distributed Knowledge] To come back to an example from epistemic logic, an interesting notion in collective knowledge is that of distributed knowledge $D\varphi$. The intuition here is that distributed knowledge is the knowledge of a ‘wise man’ (cf. [20]) with whom all agents have shared their knowledge. The typical example is a situation

where, for instance, one agent knows φ , another knows that $\varphi \rightarrow \psi$, implying distributed knowledge of ψ . A more concrete example goes as follows: it is distributed knowledge in every group of agents (provided everybody knows their own birthday) whether two agents share their birthday. The notion of distributed knowledge $D\varphi$ for n agents has an axiomatisation that is sound and complete with respect to models where the corresponding relation R_D is the intersection of all the individual agents' accessibility relations. However, intersection is not locally definable in modal logic (for more on modal properties of distributed knowledge, or implicit knowledge as it is sometimes called, see for instance [30, 47]). However, in SOPML, using Example 24.5 we can express that agent c knows exactly what the distributed knowledge of agents a and b is:

$$\forall p(K_{ap} \rightarrow K_{cp}) \wedge \forall p(K_{bp} \rightarrow K_{cp}) \wedge \forall p(K_{cp} \rightarrow \exists q(K_a q \wedge K_b(q \rightarrow p))) \quad (10)$$

Notice that (10) uses exactly the idea of the typical example of distributed knowledge between two agents discussed above: if agent c knows some fact p , i.e., p is distributed knowledge between a and b , then there exists some fact q such that a knows q and b knows $q \rightarrow p$. So, they are able to derive p by pooling together their knowledge.

Can we generalise this to n agents? Indeed we can, as follows. Define

$$\varphi = \forall p(Dp \rightarrow \exists q_1 \dots \exists q_{n-1}(K_1 q_1 \wedge \dots \wedge K_{n-1} q_{n-1} \wedge K_n(\bigwedge_{i < n} q_i \rightarrow p)))$$

and let

$$\Theta = \forall y((R_1(x, y) \wedge \dots \wedge R_n(x, y)) \rightarrow R_D(x, y))$$

700 Then, we can prove the following result.

Proposition 27. For every full frame \mathcal{F} , $(\mathcal{F}, w) \models \varphi$ iff $(\mathcal{F}, w) \models \Theta(x)$.

The proof is a generalisation of the proof of Lemma 25 for Θ_5 . It follows that operator D locally expresses the distributed knowledge of ψ among agents $1, \dots, n$:

$$\bigwedge_{i \leq n} \forall p(K_i p \rightarrow Dp) \wedge \varphi \wedge D\psi$$

Discussion. From Examples 24 and 26 it follows that SOPML is strictly more expressive than propositional modal logic, and it can also express local properties that cannot be dealt with in LPML. Example 24 also indicates when SOPML can axiomatise frames that cannot be characterised in PML: for instance, formula $\exists p(\Box p \wedge \neg p)$ characterises *irreflexive* frames, in the same way as $\exists p(\Box p \wedge \Diamond \neg p)$ characterises *intransitive* frames. Venema [52] calls such characterisations *negatively definable*. The idea here is the following: suppose that formula $\theta \in \mathcal{L}_{ml}$ locally defines some property Θ ; is there a formula that locally defines $\neg\Theta$? As an example, whereas $R_a(x, x)$ is (locally) defined by $\Box_a p \rightarrow p$, the negation $\neg R(x, x)$ is not (locally) defined by $\neg(\Box p \rightarrow p)$, or equivalently, $\Box_a p \wedge \neg p$, since this would require that on frames for this formula, atom p were false. Gabbay [23] came up with a derivation rule, rather than an axiom, to characterise irreflexivity, while [52] analyses more generally when a negative characterisation of some class of frames also leads to an axiomatisation of such class. For our discussion, it is important to realise that reflexivity is actually characterised by a modal *scheme* $\Box_a \varphi \rightarrow \varphi$, and, in contrast, by 710 formula $\forall p(\Box_a p \rightarrow p)$ in SOPML. But then, irreflexivity is characterised by the negation 715

of that SOPML formula: $\exists p(\Box p \wedge \neg p)$. Moreover, notice that SOPML allows us to interpret such formulas locally, so that we can reason about models that have both reflexive and irreflexive points.

720 From Example 24 we also learn that there are first-order properties Θ that cannot be characterised by any modal formula $\theta \in \mathcal{L}_{ml}$, while we do have a formula in SOPML characterising it. It is also possible to come up with formulas in SOPML that do not correspond to any first-order formula (hence, in SOPML one could reason locally about them, but not in LPML). A first example of such formulas is $\exists p \neg \delta$ for $\delta = (\Diamond p \wedge \Diamond \Box \neg p) \rightarrow \Diamond(\Box^{-1} \Diamond p \wedge \Box \neg p)$ (here \Box^{-1} is interpreted as the converse of relation R for \Box). As argued in [52], although δ as a scheme characterises Dedekind-complete frames among the linear orderings, the frames for $\neg \delta$ are not elementary, that is, not first-order definable. A further example is the Löb formula $\forall p(\Box(\Box p \rightarrow p) \rightarrow \Box p)$: this formula characterises frames with R being transitive and its converse well-founded [52, p. 8].

730 To conclude our comparison between SOPML and LPML, we observe that the \Box operators act in fact as a sort of linguistic black boxes, bringing the metalanguage of the theory of first-order logic into the object language of modal logic. In contrast, SOPML is more transparent, as everything is done in the object language. In addition, for the first-order conditions in [18] there must always be a suitable modal counterpart. Indeed, the axioms $\mathbf{Ax}_{\Box} : \Box(\bar{a}) \rightarrow \theta_{\Box}(\bar{a}, \bar{p})$ in [18] make sense only as long as there is a propositional modal formula θ_{\Box} related to \Box , and this is not always the case as discussed above. We will see later that none of this has to be assumed to axiomatise SOPML.

Revisiting Example 24, it is no surprise that the SOPML formulas in this example all use existential quantification, because we have the following.

740 **Lemma 28.** *For a finite set $\{p_1, \dots, p_n\}$ of atoms, define $\forall \bar{p}$ as $\forall p_1 \dots \forall p_n$ (this is well-defined because $\forall p \forall q \phi$ is equivalent to $\forall q \forall p \phi$). Then for all frames \mathcal{F} , worlds w , assignments V , and formulas ϕ in SOPML we have*

1. $(\mathcal{F}, w) \models \phi$ iff $(\mathcal{F}, w) \models \forall \bar{p} \phi$
2. $(\mathcal{F}, w) \models \forall \vec{r}(\phi)$ iff $(\mathcal{F}, V, w) \models \forall \vec{r}(\phi)$
- 745 3. $(\mathcal{F}, w) \models \phi$ iff $(\mathcal{F}, V, w) \models \phi$, where ϕ is a sentence (i.e., $fr(\phi) = \emptyset$).

3.3. Local properties and SOPML on non-full frames

So far, we have only looked at how SOPML can represent local properties on full frames. Here, we consider local properties on frames with a coarser domain of quantification. Let us return to formula

$$\forall p(K_a p \rightarrow p), \tag{11}$$

which is intended to express that everything a knows is true. Looking at the semantics of SOPML, we can see that (11) holds in (\mathcal{M}, w) if and only if

$$\text{for all } U \in D, R_a(w) \subseteq U \text{ implies } w \in U. \tag{12}$$

If the domain D is equal to the power set 2^W , then (12) is equivalent to $w \in R_a(w)$, so to R_a being locally reflexive. In general, however, there is no guarantee that D is

equal to 2^W . So on non-full frames, (11) does not characterize reflexivity. In fact, on
750 such frames, there is no SOPML formula that characterizes reflexivity.

Whether this is an important downside of SOPML depends on the object of study. If one is after a logic that can reason about graph-theoretic properties like reflexivity, then one should consider SOPML only on full frames, since on other frames SOPML cannot express these properties. If, on the other hand, the goal is to reason about a particular
755 subject (such as knowledge) and only use graphs to represent that subject, then SOPML is useful even on non-full frames. After all, even though (11) does not, in general, correspond to reflexivity, it does still express the fact that everything known by a is true—with one caveat.

The quantifier $\forall p$ quantifies only over those valuations of p that are part of the domain
760 D . So, strictly speaking, (11) means that “for every element $U \in D$, if a knows U then U is true.” There are three main ways to interpret this.

1. We could explicitly retain the reference to D , and interpret (11) as “every atomic (resp. boolean, modal) proposition that a knows is true” if D is any (resp. boolean, modal) domain of quantification.
- 765 2. We could consider D to be the set of properties that are relevant for the problem that we are modeling. In this interpretation, the formula $\forall p(K_a p \rightarrow p)$ might hold even if there is some proposition $T \in 2^W \setminus D$ such that T is false but known by a . However, because $T \notin D$ it is not a relevant property, we don’t care whether a is wrong about it.
- 770 3. We could interpret D as the set of propositions that can be conceptualized. This allows us to interpret (11) as “everything that a knows is true”, where it is understood that being able to conceptualize a proposition is a precondition for knowing that proposition.

As an example of the latter situation, suppose that Alice is looking at a blue object.
775 However, due to a trick of the light, the object seems green to her. She forms the belief that the object is green. This belief is false, so $\forall p(K_a p \rightarrow p)$ does *not* hold. Now, suppose that Bob is looking at the same object, but that Bob is from a culture that does not distinguish between green and blue. Instead, Bob’s culture uses a single concept for these colours that we will translate as “green/blue”. Bob makes the same observation as
780 Alice, but based on that observation he forms the belief that the object is green/blue. This belief is correct, so, assuming that b ’s other beliefs are correct as well, $\forall p(K_b p \rightarrow p)$ holds.² The difference between Alice and Bob does not lie in their accessibility relations. Instead, it is caused by the different ways in which they divide the set of possible worlds into concepts.

785 The conditions on D then place restrictions on the conceptual space that we assume the agents to have. If D is boolean, then the concepts “green” and “blue” need to be accompanied by concepts “not green” and “green or blue”. If D is modal, then the

²We are somewhat over-simplifying here. Bob is probably capable of conceptualizing several different subsets of green/blue, such as dark green/blue and light green/blue. Regardless, it is quite possible for Bob to have no conceptual distinction between the actual and perceived colours of the object while Alice does have such a distinction.

SOPML formula	Model property
$\forall p(\Box_a p \rightarrow \Box_b p)$	$\forall U \in D(R_a(w) \subseteq U \rightarrow R_b(w) \subseteq U)$
$\forall p(\Box_c p \rightarrow \Box_a \Box_b p)$	$\forall U \in D(R_c(w) \subseteq U \rightarrow (R_b \circ R_a)(w) \subseteq U)$
$\forall p(\neg \Box_a \perp)$	$\exists v R_a(w, v)$
$\forall p(\Box_a p \rightarrow p)$	$\forall U \in D(R_a(w) \subseteq U \rightarrow w \in U)$
$\forall p(\neg \Box_a p \rightarrow \Box_b \neg \Box_c p)$	$\forall U \in D(R_b(w) \not\subseteq U \rightarrow \forall v(R_a(w, v) \rightarrow R_c(v) \not\subseteq U))$
$\forall p(\neg \Box_a p \rightarrow \neg \Box_b \Box_c p)$	$\forall U \in D(R_a(w) \not\subseteq U \rightarrow (R_c \circ R_b)(w) \not\subseteq U)$
$\forall p((\neg \Box_a p \wedge \neg \Box_b q) \rightarrow \neg \Box_c (p \vee q))$	$\forall U_1 \in D \forall U_2 \in D((R_a(w) \not\subseteq U_1 \wedge R_b(w) \not\subseteq U_2) \rightarrow (R_c(w) \not\subseteq U_1 \cup U_2))$

Table 2: The model properties expressed by several SOPML formulas.

concept “green” needs to be accompanied by a concept “knowing to be green”, and if D is full then every set of worlds corresponds to some concept.

790 Regardless of the interpretation that we choose, every formula discussed in Table 1 expresses a property of SOPML models. The properties that are expressed by these formulas can be found in Table 2. Note that the properties in Table 2 are generalisations of those in Table 1, and that they are equivalent to their counterparts when D is full. The properties in question require some slightly awkward notation, but conceptually they are
795 not very hard to grasp: they correspond to the properties in Table 1, except we consider only those sets of worlds that are in the domain D . Take, for example, $\forall p(\Box_a p \rightarrow p)$. On full frames, this corresponds to $w \in R_a(w)$, which is equivalent to every superset of $R_a(w)$ containing w , so to $\forall U \in 2^W (R_a(w) \subseteq U \rightarrow w \in U)$. The corresponding property for non-full frames is obtained by replacing 2^W by the domain D .

800 4. (In)completeness

One well-established way to understand a logic is to introduce an axiomatisation for it. After all, since there is an infinite number of valid formulas, we cannot explicitly enumerate all of them. But if we have a complete axiomatisation, then we can at least implicitly know the valid formulas and understand why they are valid.

805 It turns out that not all variants of SOPML are axiomatisable. Still, even if a logic is unaxiomatisable it is worthwhile to prove that it is so, for two reasons. Firstly, of course, if we have a proof that no axiomatisation exists, then we can stop trying to find an axiomatisation. Secondly, even though an unaxiomatisability result arguably provides less insight regarding the theorems of the logic than an axiomatisation, it does still tell
810 us something about the logic, particularly about its computational complexity.

In this paper we discuss many different variants of SOPML, which differ on the domain of quantification (full, modal, boolean, any), restrictions on the accessibility relations (reflexive, transitive, symmetric), availability of common knowledge, and the number of indices. For some of these variants, axiomatisability and unaxiomatisability results are
815 known from [21, 36, 37]. In particular, [21] provides axiomatisations for all single agent normal logics interpreted on boolean and generic frames, as well as an axiomatisation for epistemic full frames. Here we extend several of these results to the multi-modal case for the first time. Tables 3 and 4 give an overview of these results. In summary, the results are that SOPML without common knowledge is unaxiomatisable on full frames (with the exception of the special case of single-agent **S5**), but axiomatisable on modal, boolean
820 and the class of all frames, while SOPML with common knowledge is unaxiomatisable

	S5 ($ I = 1$)	S5 ($ I \geq 2$)	K
full	axiomatisable [21]	unaxiomatisable (Thm. 50)	unaxiomatisable [21]
modal	axiomatisable (Thm. 43)		
boolean	axiomatisable [21]	axiomatisable (Thm. 43)	
all	axiomatisable (Thm. 43)		

Table 3: Axiomatisation results for languages \mathcal{L} without common knowledge.

	S5 ($ I = 1$)	S5 ($ I \geq 2$)	K
full	axiomatisable [21]	unaxiomatisable (Thm. 50)	unaxiomatisable [21]
modal	axiomatisable (Thm. 43)	unaxiomatisable (Thm. 62)	unaxiomatisable (Thm. 57)
boolean	axiomatisable [21]	unaxiomatisable (Cor. 63)	unaxiomatisable (Cor. 60)
all	axiomatisable (Thm. 43)		

Table 4: Axiomatisation results for languages \mathcal{L}^* with common knowledge.

regardless of the domain of quantification³. Note that, if $|I| = 1$ then \Box^* reduces to \Box on **S5** frames, so the entries in the first column of Table 4 follow immediately from the results in the first column of Table 3.

825 So in most cases, adding common knowledge makes the validity problem harder. This is in contrast to the model checking problem, where adding common knowledge is “free”, in the sense that model checking for SOPML is PSPACE-complete, whether or not we have a common knowledge operator (see Theorem 17).

830 We restrict ourselves to the classes of models for logics **S5** and **K** in these tables. This is because these two classes are the most relevant for our analysis. Our axiomatisability results are slightly more general, see Theorem 43 for the exact statement. Our unaxiomatisability results are stated for logics **S5** and **K**, but with a few minor modifications these proofs could easily be adapted to other normal modalities, including **KD45** and **S4.2**. Such results give us useful insights into the computational properties of SOPML, 835 as well as its amenability for knowledge representation and reasoning.

Remark 29. *In most of this paper, we use different notation for SOPML ($\Box_a, \Diamond_a, \Box^*$) and SOPEL (K_a, M_a, C). In this section we discuss both SOPML and SOPEL, so for the sake of readability we only use the SOPML notation.*

840 *Also, we write SOPML to denote generically logics without operator \Box^* , while SOPML* refers to logics with \Box^* . The distinction will be clear from the context.*

4.1. Complete Axiomatisations

845 This section is devoted to axiomatise several classes of validities on Kripke frames built on sets I of agent indices and AP of atomic propositions. We first present a class of logics \mathbf{K}_x , one for each $x \in \{ap, pl, ml\}$. In this section all logics are defined on languages without common knowledge.

³Again, with the exception of the single agent **S5** case.

Definition 30 (Logics \mathbf{K}_x). For each $x \in \{ap, pl, ml\}$, the axioms and inference rules of \mathbf{K}_x are as follows:

Prop	<i>all instances of propositional tautologies</i>
K	$\Box_a(\phi \rightarrow \psi) \rightarrow (\Box_a\phi \rightarrow \Box_a\psi)$
Ex_x	$\forall p\phi \rightarrow \phi[p/\psi]$, where $\psi \in \mathcal{L}_x$
BF	$\forall p\Box_a\phi \rightarrow \Box_a\forall p\phi$
MP	from $\phi \rightarrow \psi$ and ϕ infer ψ
Nec	from ϕ infer $\Box_a\phi$
Gen	from $\phi \rightarrow \psi$ infer $\phi \rightarrow \forall p\psi$, for p not free in ϕ

850 The axioms **Prop** and **K** are standard of any modal logic, as are the rules Modus Ponens (**MP**) and Necessitation (**Nec**). Note how axiom **Ex_x** is parameterised by $x \in \{ap, pl, ml\}$. The axiom specifies the language \mathcal{L}_x which acts as the domain of quantification, or, more precisely, what kind of formulas can be substituted as an instance for the universal quantifier. Axiom **BF** is known as the Barcan formula and it
855 says the following. In our models $\mathcal{M} = \langle W, D, R, V \rangle$ the domain of quantification D is defined globally, and does not depend on the world w of evaluation. To give an example where the dependence of D on world w would cease **BF** to hold, consider a structure $\mathcal{N} = \langle W, \{D_w\}_{w \in W}, R, V \rangle$, with $W = \{x, y, z\}$ and $R = \{(x, y), (y, z)\}$. Also, suppose $D_x = \{\{x\}, \{y, z\}\}$ and $D_y = 2^W \neq D_x$. Then, by restricting the clause for quantification
860 in Definition 8 to each D_w , we have $(\mathcal{N}, x) \models \forall p\Box(p \rightarrow \Box p)$ but not $(\mathcal{N}, x) \models \Box\forall p(p \rightarrow \Box p)$. In Example 31 we prove that the converse of **BF** is derivable in all \mathbf{K}_x .

The scheme of axioms **Ex_x** and the Generalisation rule **Gen** are typical principles of quantification. Axiom **Ex_x** is the elimination axiom for \forall : if something holds for all allowed valuations, then it also holds for each instance from the domain (which can be
865 the set of all atoms, boolean formulas, or modal formulas.) The rule of Generalisation is the introduction rule for \forall : if ψ follows from ϕ for an arbitrary p , we infer that $\forall p\psi$ follows from ϕ .

As customary in PML, by considering a suitable combination of axioms

870 T	$\Box_a\phi \rightarrow \phi$
B	$\phi \rightarrow \Box_a\Diamond_a\phi$
4	$\Box_a\phi \rightarrow \Box_a\Box_a\phi$

we can introduce the following normal extensions of \mathbf{K}_x , also for $x \in \{ap, pl, ml\}$:

875 T_x	$:= \mathbf{K}_x + \mathbf{T}$
S4_x	$:= \mathbf{K}_x + \mathbf{T} + \mathbf{4}$
B_x	$:= \mathbf{K}_x + \mathbf{T} + \mathbf{B}$
S5_x	$:= \mathbf{K}_x + \mathbf{T} + \mathbf{B} + \mathbf{4}$

This gives us 15 logics, 5 for each type $x \in X$.

The notions of *proof* and *theoremhood* are defined as usual. A formula ϕ is *derivable* in logic \mathbf{L} from a set Δ of formulas, or $\Delta \vdash_{\mathbf{L}} \phi$, iff for some $\phi_0, \dots, \phi_m \in \Delta$, formula $\bigwedge_{i \leq m} \phi_i \rightarrow \phi$ is a theorem in \mathbf{L} , or $\vdash_{\mathbf{L}} \bigwedge_{i \leq m} \phi_i \rightarrow \phi$.

880 **Example 31.** As an example, we provide proofs in logic \mathbf{K}_{ap} of the following theorems and derived inference rules, which will be routinely used in the rest of the section, often without explicit mention:

- converse of the Barcan formula **CBF** $\Box\forall p\phi \rightarrow \forall p\Box\phi$:

1. $\forall p\phi \rightarrow \phi$ by axiom **Ex**_{ap}
2. $\Box_a(\forall p\phi \rightarrow \phi) \rightarrow (\Box_a\forall p\phi \rightarrow \Box_a\phi)$ by axiom **K**
3. $\Box_a(\forall p\phi \rightarrow \phi)$ from (1) by rule **Nec**
4. $\Box_a\forall p\phi \rightarrow \Box_a\phi$ from (2), (3) by rule **MP**
5. $\Box_a\forall p\phi \rightarrow \forall p\Box_a\phi$ from (4) by rule **Gen**, as p is not free in $\Box_a\forall p\phi$

- vacuous quantification $\phi \rightarrow \forall p\phi$, whenever p does not appear in ϕ :

1. $\phi \rightarrow \phi$ propositional tautology
2. $\phi \rightarrow \forall p\phi$ by axiom **Gen**

885 By axiom **Ex**_{ap} we then obtain $\phi \leftrightarrow \forall p\phi$, whenever p does not appear in ϕ .

- distribution of quantification $\forall p(\phi \rightarrow \psi) \rightarrow (\forall p\phi \rightarrow \forall p\psi)$:

1. $\forall p(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$ by axiom **Ex**_{ap}
2. $\forall p\phi \rightarrow \phi$ by axiom **Ex**_{ap}
3. $\phi \rightarrow (\forall p(\phi \rightarrow \psi) \rightarrow \psi)$ from (1) by tautology $(A \rightarrow (B \rightarrow C)) \leftrightarrow (B \rightarrow (A \rightarrow C))$
4. $\forall p\phi \rightarrow (\forall p(\phi \rightarrow \psi) \rightarrow \psi)$ from (2), (3) by transitivity of implication
5. $\forall p(\phi \rightarrow \psi) \rightarrow (\forall p\phi \rightarrow \psi)$ from (4) by tautology $(A \rightarrow (B \rightarrow C)) \leftrightarrow (B \rightarrow (A \rightarrow C))$
6. $(\forall p(\phi \rightarrow \psi) \wedge \forall p\phi) \rightarrow \psi$ from (5) by tautology $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \wedge B) \rightarrow C)$
7. $(\forall p(\phi \rightarrow \psi) \wedge \forall p\phi) \rightarrow \forall p\psi$ from (6) by **Gen**
8. $\forall p(\phi \rightarrow \psi) \rightarrow (\forall p\phi \rightarrow \forall p\psi)$ from (7) by tautology $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \wedge B) \rightarrow C)$

890 Since **CBF**, vacuous quantification and distribution of quantification are provable in **K**_{ap}, they are theorems in all the other 14 logics above. We also recall that inferring $\Box_a\phi \rightarrow \Box_a\psi$ from $\phi \rightarrow \psi$ is a derivable rule in modal logic, and whenever p does not appear free in ϕ , formula $\forall p(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall p\psi)$ is a theorem in all our logics.

895 We now prove the soundness and completeness results for logics **L**_x w.r.t. the corresponding class \mathcal{K} of Kripke frames, starting with soundness. In the rest of the paper **L** ranges over $\{\mathbf{K}, \mathbf{T}, \mathbf{S4}, \mathbf{B}, \mathbf{S5}\}$. Given a logic **L**_x, let $\tau(\mathbf{L}_x)$ be a subset of $\{r, t, s\}$, such that **L**_x includes axiom **T** iff τ contains r (for reflexivity), **L**_x includes axiom **4** iff τ contains t (for transitivity), and **L**_x includes axiom **B** iff τ contains s (for symmetry).

Theorem 32 (Soundness). For $x \in \{ap, pl, ml\}$, for every logic **L**_x and formula $\phi \in \mathcal{L}_{sopml}$,

$$\vdash_{\mathbf{L}_x} \phi \text{ implies } \mathcal{K}_{\mathbb{F}}^{\tau(\mathbf{L}_x)} \models \phi$$

Proof. As customary, the axioms of each logic **L**_x are shown to be valid in the corresponding class $\mathcal{K}_{\mathbb{F}}^{\tau(\mathbf{L}_x)}$ of frames, and the inference rules are shown to preserve validity in $\mathcal{K}_{\mathbb{F}}^{\tau(\mathbf{L}_x)}$. Specifically, axioms **Prop**, **K**, **MP**, and **Nec** are valid in any frame. The validity of axioms **T**, **4**, and **B** in specific classes of frames follows as in standard propositional modal logics [7]. The validity of axioms **Ex**_x in each corresponding class of frames follows by Lemma 11.(2), while the validity of **Gen** follows by Lemma 11.(1). We provide a proof for **Ex**_{ap}: suppose that $(\mathcal{M}, w) \models \forall p\phi$, that is, for every $U \in D$,

$(\mathcal{M}_U^p, w) \models \phi$. By Lemma 11.(2a), $[\psi] \in D$, hence in particular $(\mathcal{M}_{[\psi]}^p, w) \models \phi$. Then, by Lemma 11.(2b), $(\mathcal{M}, w) \models \phi[p/\psi]$. As regards **Gen**, suppose that $(\mathcal{M}, w) \models \phi$ and $p \notin \text{fr}(\phi)$. In particular, for every $U \in D$, $V(\text{fr}(\phi)) = V_U^p(\text{fr}(\phi))$, and by Lemma 11.(1), we have $(\mathcal{M}_U^p, w) \models \phi$ as well. By **MP** we obtain that $(\mathcal{M}_U^p, w) \models \psi$, and since U is arbitrary, $(\mathcal{M}, w) \models \forall p\psi$. Moreover, the Barcan formula **BF** is valid as in any frame all worlds have the same domain of quantification, namely $D \subseteq 2^W$. Indeed, $(\mathcal{M}, w) \models \forall p \Box_a \phi$ iff for all $U \in D$, $(\mathcal{M}_U^p, w) \models \Box_a \phi$, iff for every $w' \in R_a(w)$, $(\mathcal{M}_U^p, w') \models \phi$. But this means that for every $w' \in R_a(w)$, $(\mathcal{M}, w') \models \forall p\phi$, that is, $(\mathcal{M}, w) \models \Box_a \forall p\phi$. \square

As a consequence of Theorem 32, all our 15 logics are sound w.r.t. the corresponding classes of frames. Moreover, as a by-product of soundness, we obtain that the inclusions (4) between theories put forward in Section 2.2 are all strict:

$$\text{Th}(\mathcal{K}_{all}) \subset \text{Th}(\mathcal{K}_{bool}) \subset \text{Th}(\mathcal{K}_{modal}) \subset \text{Th}(\mathcal{K}_{full}) \quad (13)$$

To prove this, observe that each axiom \mathbf{Ex}_x holds in \mathcal{K}_x , but in no more general class of frames. Finally, let \mathbf{Ex}_{sopml} be the scheme $\forall p\phi \rightarrow \phi[p/\psi]$, for $\psi \in \mathcal{L}_{sopml}$. It is easy to check that $\mathbf{Ex}_{sopml} \in \text{Th}(\mathcal{K}_{full})$, but $\mathbf{Ex}_{sopml} \notin \text{Th}(\mathcal{K}_{modal})$.

Next we state the completeness result. Here we use the notation of Theorem 32.

Theorem 33 (Completeness). For $x \in \{ap, pl, ml\}$, and every formula $\phi \in \mathcal{L}_{sopml}$,

$$\mathcal{K}_{\widehat{x}}^{\tau(\mathbf{L}_x)} \models \phi \quad \text{implies} \quad \vdash_{\mathbf{L}_x} \phi$$

Theorem 33 guarantees completeness of a logic of sort x , with respect to models of type \widehat{x} , for the sorts of atomic propositions and propositional and modal formulas. Completeness also holds if we add properties such as reflexivity, transitivity, and symmetry to the frames, as long as we add the corresponding axioms from $\{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$ to the logic. To our knowledge this is the first completeness result for SOPML in a multi-agent setting.

To prove Theorem 33 for $x \in \{ap, pl, ml\}$, we show that if a formula ϕ is \mathbf{L}_x -consistent, then we can construct an appropriate model $\mathcal{M}_{\mathbf{L}_x}$ that satisfies ϕ . For logic \mathbf{L}_{ap} (respectively, \mathbf{L}_{pl} , \mathbf{L}_{ml}) this amounts to finding models whose underlying frame is any frame (respectively, a boolean algebra or a boolean algebra with operators). We begin with the cases for \mathbf{L}_{ap} for clarity's sake.

4.1.1. Completeness of \mathbf{L}_{ap}

In this section we show that if a formula ϕ is \mathbf{L}_{ap} -consistent, that is, $\not\vdash_{\mathbf{L}_{ap}} \neg\phi$, then we can construct a (canonical) model $\mathcal{M}_{\mathbf{L}_{ap}} = \langle \mathcal{F}, V \rangle$ that satisfies ϕ . Moreover, \mathcal{F} is shown to belong to the class \mathcal{K}_{all} of all frames. This implies that $\mathcal{K}_{all} \not\models \neg\phi$.

Definition 34. Let $\Lambda \subseteq \mathcal{L}_{sopml}$ be sets of formulas over set AP of atoms, and Y a denumerable set of atoms. We say that Λ is

\mathbf{L}_{ap} -consistent	iff	$\Lambda \not\vdash_{\mathbf{L}_{ap}} \perp$
complete	iff	for every formula $\phi \in \mathcal{L}_{sopml}$, $\phi \in \Lambda$ or $\neg\phi \in \Lambda$
maximal	iff	Λ is consistent and complete
Y -rich	iff	for every $\phi \in \mathcal{L}_{sopml}$, if $\exists p\phi \in \Lambda$ then $\phi[p/q] \in \Lambda$ for some $q \in Y$
saturated	iff	Λ is maximal and Y -rich for some $Y \subseteq AP$

We omit the subscript \mathbf{L}_{ap} whenever clear by the context.

We remark that, by the definition of derivability, a set Λ is inconsistent iff for some $\phi_0, \dots, \phi_m \in \Lambda$, $\vdash \bigwedge_{i \leq m} \phi_i \rightarrow \perp$, that is, $\vdash \bigwedge_{i \leq m} \phi_i \rightarrow \neg\psi$ for every $\psi \in \Lambda$.

935 We now prove that every consistent set can be saturated.

Lemma 35 (Saturation). *Let Δ be a maximal set of formulas over AP . Then there exists a saturated set of formulas Φ over $AP \cup Y$, such that $\Delta \subset \Phi$, where Y is an infinite set of new atoms (i.e., disjoint from AP).*

Proof. Let $\theta_0, \theta_1, \dots$ be an enumeration of the formulas over $AP \cup Y$, and q_0, q_1, \dots an enumeration of atoms in Y . We define by induction a sequence Φ_0, Φ_1, \dots of sets of formulas over $AP \cup Y$ as follows:

$$\begin{aligned} \Phi_0 &= \Delta \\ \Phi_{n+1} &= \begin{cases} \Phi_n \cup \{\theta_n\} & \text{if } \Phi_n \cup \{\theta_n\} \text{ is consistent and } \theta_n \text{ is not of the form } \exists p\chi; \\ \Phi_n \cup \{\theta_n, \chi[p/q]\} & \text{if } \Phi_n \cup \{\theta_n\} \text{ is consistent, } \theta_n \text{ is of the form } \exists p\chi, \\ & \text{and } q \in Y \text{ is the first atom not appearing in } \Phi_n \cup \{\theta_n\}; \\ \Phi_n \cup \{-\theta_n\} & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that, since Y is an infinite set of new atoms and finitely many θ appear in $\Phi_n \setminus \Phi_0$, for each $n \in \mathbb{N}$, we can always find an atom $q \in Y$ that does not appear in $\Phi_n \cup \{\theta_n\}$. Now we prove by induction on n that every Φ_n is consistent. First of all, $\Phi_0 = \Delta$ is consistent by hypothesis. As to the inductive step, suppose that Φ_n is consistent, then consider the various cases. If $\Phi_{n+1} = \Phi_n \cup \{\theta_n\}$ and θ_n is not of the form $\exists p\chi$, then $\Phi_n \cup \{\theta_n\} = \Phi_{n+1}$ has to be consistent, by construction. Further, $\Phi_{n+1} = \Phi_n \cup \{\theta_n, \chi[p/q]\}$ if $\Phi_n \cup \{\theta_n\}$ is consistent, θ_n is of the form $\exists p\chi$, and q is the first atom that does not appear in $\Phi_n \cup \{\theta_n\}$. To obtain a contradiction, suppose that Φ_{n+1} is inconsistent. In particular, for some $\varphi_0, \dots, \varphi_m \in \Phi_n$,

$$\vdash \left(\bigwedge_{i \leq m} \varphi_i \wedge \theta_n \right) \rightarrow \neg\chi[p/q]$$

Since q is assumed not to appear in Φ_n nor in θ_n , by an application of **Gen** we obtain

$$\vdash \left(\bigwedge_{i \leq m} \varphi_i \wedge \theta_n \right) \rightarrow \forall p \neg\chi$$

i.e., $\vdash (\bigwedge_{i \leq m} \varphi_i \wedge \theta_n) \rightarrow \neg\theta_n$, and, since $\vdash (\bigwedge_{i \leq m} \varphi_i \wedge \theta_n) \rightarrow \theta_n$ trivially, we obtain that $\vdash (\bigwedge_{i \leq m} \varphi_i \wedge \theta_n) \rightarrow \perp$, that is, $\Phi_n \cup \{\theta_n\}$ is not consistent, against hypothesis. Hence, $\Phi_{n+1} = \Phi_n \cup \{\theta_n, \chi[p/q]\}$ is indeed consistent.

955 Finally, $\Phi_{n+1} = \Phi_n \cup \{-\theta_n\}$ only if $\Phi_n \cup \{\theta_n\}$ is not consistent. Indeed, if Φ_n is consistent, $\Phi_n \cup \{\theta_n\}$ and $\Phi_n \cup \{-\theta_n\}$ cannot be both inconsistent, since otherwise for some $\varphi_0, \dots, \varphi_m, \varphi'_0, \dots, \varphi'_{m'} \in \Phi_n$,

$$\begin{aligned} \vdash \bigwedge_{i \leq m} \varphi_i \rightarrow \neg\theta_n \\ \vdash \bigwedge_{i \leq m'} \varphi'_i \rightarrow \neg\neg\theta_n \end{aligned}$$

and by propositional reasoning,

$$\vdash \bigwedge_{i \leq m} \varphi_i \wedge \bigwedge_{i \leq m'} \varphi'_i \rightarrow (-\theta_n \wedge \theta_n)$$

and therefore Φ_n itself is inconsistent, a contradiction. Hence, $\Phi_n \cup \{-\theta_n\} = \Phi_{n+1}$ is indeed consistent. 960

Now let $\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n$: Φ is consistent as each Φ_n is. If that were not the case, there would be $\varphi_0, \dots, \varphi_m \in \Phi$ such that $\vdash \bigwedge_{i \leq m} \varphi_i \rightarrow \perp$. Then suppose that k is the smallest index such that all φ_i appear in Φ_k . It follows that Φ_k is inconsistent as well, against hypothesis. Moreover, Φ extends Δ and it is maximal and Y -rich by construction. \square

We now describe informally the construction of the canonical model for a formula ϕ such that $\not\vdash \neg\phi$. First, define W as the set of all saturated sets w of formulas over $AP \cup Y$ as obtained in Lemma 35. Notice that W is non-empty as the set $\{\phi\}$ is consistent by hypothesis, and by Lemma 35 there exists a saturated set $\Phi \supseteq \{\phi\}$ in W . Further, for $w, w' \in W$ and $a \in I$, define $R_a(w, w')$ iff $\{\psi \mid \Box_a \psi \in w\} \subseteq w'$. Finally, for every atom 970 $p \in AP \cup Y$, we consider set $U_p = \{w \in W \mid p \in w\} \subseteq W$ and define the domain D of propositions as $\{U_p \mid p \in AP \cup Y\}$.

Definition 36 (Canonical Model). *The canonical model \mathbf{L} is a tuple $\mathcal{M}_{\mathbf{L}} = \langle W, D, R, V \rangle$ where (i) W, D and R are defined as above; and (ii) V is the assignment such that $V(p) = U_p$.*

Note that every consistent formula ϕ must be contained in some $\Phi \in W$. Next we prove that the canonical model w.r.t. any \mathbf{L}_{ap} is indeed a model based on a frame in $\mathcal{K}_{all}^{\tau(\mathbf{L}_{ap})}$ (recall that $\widehat{ap} = all$ and that \mathbf{L}_{ap} represents 5 different logics: $\mathbf{K}_{ap}, \mathbf{T}_{ap}, \mathbf{S4}_{ap}, \mathbf{B}_{ap}$, and $\mathbf{S5}_{ap}$). 975

Lemma 37. *The canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$ in Definition 36 is a Kripke model based on a frame in $\mathcal{K}_{all}^{\tau(\mathbf{L}_{ap})}$.* 980

Proof. By the remarks above, W is a non-empty set of saturated sets, D is a subset of 2^W , and V is a function from $AP \cup Y$ to D . Moreover, axiom \mathbf{T} (respectively, $\mathbf{4}, \mathbf{B}$) enforces relation R_a on W to be reflexive (respectively, transitive, symmetric), as it is the case for propositional modal logic. As an illustrative example, we consider the case 985 for \mathbf{T} : by maximality, for every $w \in W$, $\Box_a \psi \rightarrow \psi \in w$ for every formula $\psi \in \mathcal{L}_{sopml}$, and by closure under MP we have $\{\psi \in \mathcal{L}_{sopml} \mid \Box \psi \in w\} \subseteq w$, that is, $R_a(w, w)$ by definition. \square

We can finally prove the truth lemma for logics \mathbf{L}_{ap} . Here we adapt the proof in [20] for propositional epistemic languages without common knowledge.

Lemma 38 (Truth lemma). *For every logic \mathbf{L}_{ap} , in the canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$, for every $w \in W$ and every formula ψ over $AP \cup Y$,* 990

$$(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi \quad \text{iff} \quad \psi \in w$$

Proof. The proof is by induction on the length of ψ . As to the base of induction for $\psi = p$, by definition of satisfaction, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models p$ iff $w \in V(p)$ iff $p \in w$.

For $\psi = \neg\chi$, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi$ iff $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \not\models \chi$, iff by induction hypothesis $\chi \notin w$. Since w is maximal, this is the case iff $\psi \in w$.

For $\psi = \chi \rightarrow \chi'$, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi$ iff $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \not\models \chi$ or $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \chi'$. By induction hypothesis this is the case iff $\chi \notin w$ or $\chi' \in w$; in both cases we have that $\psi \in w$, as w is maximal.

Suppose that $\psi = \forall p\chi$. \Leftarrow Let $\psi \in w$. By axiom \mathbf{Ex}_{ap} we have that $\chi[p/q] \in w$ for every $q \in AP \cup Y$. By induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \chi[p/q]$ for every q . Now, take an arbitrary $U_q = \{w \in W \mid q \in w\}$ in the domain D of the canonical model. By Lemma 11.(2b), $((\mathcal{M}_{\mathbf{L}_{ap}})_{U_q}^p, w) \models \chi$, and since variant $V_{U_q}^p$ was chosen arbitrarily, we obtain that $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi$. \Rightarrow Assume that $\psi \notin w$. Since w is maximal, $\exists p\neg\chi \in w$, and w is Y -rich, so $\neg\chi[p/q] \in w$ for some atom $q \in Y$. Then, by induction hypothesis, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \not\models \chi[p/q]$, and by Lemma 11.(2b), $((\mathcal{M}_{\mathbf{L}_{ap}})_{V(q)}^p, w) \not\models \chi$. In particular, for $U_q = V(q) \in D$, $((\mathcal{M}_{\mathbf{L}_{ap}})_{U_q}^p, w) \not\models \chi$, i.e., $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \not\models \psi$.

Suppose that $\psi = \Box_a\chi$. \Leftarrow Assume that $\psi \in w$ and $v \in R_a(w)$. By definition of R_a , $\chi \in v$; therefore by induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \models \chi$. Thus, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi$. \Rightarrow Assume that $\psi \notin w$ and consider set $\{\phi \mid \Box_a\phi \in w\} \cup \{\neg\chi\}$. This set is consistent, for if not, then for some $\phi_1, \dots, \phi_n \in \{\phi \mid \Box_a\phi \in w\}$, $\vdash \bigwedge \phi \rightarrow \chi$. Then, by axiom \mathbf{K} , $\vdash \bigwedge \Box_a\phi \rightarrow \Box_a\chi$ and since $\bigwedge \Box_a\phi \in w$, also $\Box_a\chi \in w$ against hypothesis. Now we want to saturate $\Delta = \{\phi \mid \Box_a\phi \in w\} \cup \{\neg\chi\}$ to obtain $v \in W$ such that $R_a(w, v)$ and $\neg\chi \in v$. However, we cannot directly apply Lemma 35 to Δ , as it is a set of formulas over $AP \cup Y$. We prove that Δ can nonetheless be extended to a saturated set Φ of formulas over $AP \cup Y$. The proof structure is similar to the one for Lemma 35, namely, we define a sequence Φ_0, Φ_1, \dots of sets of formulas over $AP \cup Y$ as follows:

$$\begin{aligned} \Phi_0 &= \Delta \\ \Phi_{n+1} &= \begin{cases} \Phi_n \cup \{\theta_n\} & \text{if } \Phi_n \cup \{\theta_n\} \text{ is consistent and } \theta_n \text{ is not of the form } \exists p\zeta; \\ \Phi_n \cup \{\theta_n, \zeta[p/q]\} & \text{if } \Phi_n \cup \{\theta_n\} \text{ is consistent, } \theta_n \text{ is of the form } \exists p\zeta, \\ & \text{and } q \in AP \cup Y \text{ is the first atom such that} \\ & \Phi_n \cup \{\theta_n, \zeta[p/q]\} \text{ is consistent;} \\ \Phi_n \cup \{-\theta_n\} & \text{otherwise.} \end{cases} \end{aligned}$$

We prove by induction on n that every Φ_n is consistent (and well-defined). First of all, $\Phi_0 = \Delta$ is consistent as shown above. As to the inductive step, suppose that Φ_n is consistent. We only consider the case where $\Phi_n \cup \{\theta_n\}$ is consistent and θ_n is of the form $\exists p\zeta$, and show that Φ_{n+1} is well-defined, that is, there exists $q \in AP \cup Y$ such that $\Phi_n \cup \{\theta_n, \zeta[p/q]\}$ is consistent as well. To obtain a contradiction, suppose that for every $q \in AP \cup Y$, there exist $\Box_a\varphi_0, \dots, \Box_a\varphi_m$ in w such that

$$\vdash \bigwedge_{i \leq m} \varphi_i \rightarrow \left(\bigwedge_{i \leq n} \psi_i \wedge \theta_n \rightarrow \neg\zeta[p/q] \right)$$

where ψ_0, \dots, ψ_n are all the formulas in $\Phi_n \setminus \Delta$.

By axiom \mathbf{K} we obtain

$$\vdash \bigwedge_{i \leq m} \Box_a\varphi_i \rightarrow \Box_a \left(\bigwedge_{i \leq n} \psi_i \wedge \theta_n \rightarrow \neg\zeta[p/q] \right)$$

and since all $\Box_a\varphi_i$ belong to w , by maximality we derive that $\Box_a \left(\bigwedge_{i \leq n} \psi_i \wedge \theta_n \rightarrow \neg\zeta[p/q] \right) \in w$ for all $q \in AP \cup Y$ (*). Take an atom q not occurring in $\bigwedge_{i \leq n} \psi_i \wedge \theta_n$ and consider the

formula $\forall q \Box_a (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \rightarrow \neg \zeta[p/q])$. We claim that this formula is a member of w , because, if not, by maximality, $\exists q \Diamond_a (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \wedge \zeta[p/q]) \in w$, and, by saturation, for some $q' \in AP \cup Y$, we have $\Diamond_a (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \wedge \zeta[p/q']) \in w$, contradicting (*).

1030 Since $\forall q \Box_a (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \rightarrow \neg \zeta[p/q]) \in w$ by axiom **BF** we obtain that $\Box_a \forall q (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \rightarrow \neg \zeta[p/q]) \in w$ as well, and since q is assumed not to appear in $\bigwedge_{i \leq k} \psi_i \wedge \theta_n$ we derive $\Box_a (\bigwedge_{i \leq k} \psi_i \wedge \theta_n \rightarrow \forall q \neg \zeta[p/q]) \in w$, and therefore $\bigwedge_{i \leq k} \psi_i \wedge \theta_n \rightarrow \forall q \neg \zeta[p/q] \in \Delta$. Further, since ψ_0, \dots, ψ_n belong to Φ_n , we obtain that $\Phi_n \cup \{\theta_n\} \vdash \forall q \neg \zeta[p/q]$. But this contradicts the fact that $\Phi_n \cup \{\theta_n\}$ is consistent (recall that $\theta_n = \exists p \zeta$). As a result, $\Phi_n \cup \{\theta_n, \zeta[p/q]\}$ is indeed consistent for some $q \in AP \cup Y$.

The other inductive cases of the construction go as in Lemma 35. Finally, $\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n$ is consistent as each Φ_n is so. In particular, Φ is a saturated set in W such that $\Phi \in R_a(w)$ by construction. By induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, \Phi) \not\models \chi$, that is, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \not\models \psi$. \square

1040 By Lemma 38, if $\not\vdash_{\mathbf{L}_{ap}} \neg \phi$ then there exists a saturated set $w \supseteq \{\phi\}$ such that in the canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$, we have $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \phi$. Moreover, $\mathcal{M}_{\mathbf{L}_{ap}}$ is based on a frame $\mathcal{F} \in \mathcal{K}_{all}$. Thus, $\mathcal{K}_{all} \not\models \neg \phi$. This concludes the completeness proof for \mathbf{L}_{ap} .

4.1.2. Completeness of \mathbf{L}_{pl} and \mathbf{L}_{ml}

1045 In this section we discuss how to adapt the completeness proof for \mathbf{L}_{ap} in the previous section to logics \mathbf{L}_{pl} and \mathbf{L}_{ml} . As regards \mathbf{L}_{pl} , we need to modify the definition of the canonical model and the proof of the truth lemma, starting with the former.

Definition 39 (Canonical Model). *The canonical model for \mathbf{L}_{pl} is a tuple $\mathcal{M}_{\mathbf{L}_{pl}} = \langle W, D, R, V \rangle$ where*

- W, R and V are given as in Definition 36;
 - D is the domain of sets $U_\psi = \{w \in W \mid \psi \in w\} \subseteq W$, for every propositional formula $\psi \in \mathcal{L}_{pl}$ over $AP \cup Y$.
- 1050

Given Definition 39 of canonical model, we can show that it is indeed based on a boolean frame.

Lemma 40. *The canonical model $\mathcal{M}_{\mathbf{L}_{pl}}$ is boolean.*

Proof. We have to prove that domain D is closed under boolean operations. Let U_ϕ and $U_{\phi'}$ be sets in D , we show that $U_\phi \cap U_{\phi'} = U_{\phi \wedge \phi'} \in D$. Clearly, $w \in U_\phi \cap U_{\phi'}$ iff $\phi \in w$ and $\phi' \in w$, and by maximality, this is the case iff $\phi \wedge \phi' \in w$ as well. Closure under disjunction is proved similarly. As to taking complement, we show that $W \setminus U_\phi = U_{\neg \phi} \in D$. Again, $w \notin U_\phi$ iff $\phi \notin w$, and by maximality, this is the case iff $\neg \phi \in w$. \square

1060 As a consequence of Lemma 40, $\mathcal{M}_{\mathbf{L}_{pl}}$ is based on a frame in \mathcal{K}_{bool} . Moreover, we are able to prove the following version of the truth lemma.

Lemma 41 (Truth lemma). *For every logic \mathbf{L}_{pl} , in the canonical model $\mathcal{M}_{\mathbf{L}_{pl}}$, for every $w \in W$ and every formula ψ over $AP \cup Y$,*

$$(\mathcal{M}_{\mathbf{L}_{pl}}, w) \models \psi \quad \text{iff} \quad \psi \in w$$

Proof. To prove the truth lemma for \mathbf{L}_{pl} we have to modify the proof of Lemma 38. Specifically, we can first prove that for every $\psi \in \mathcal{L}_{pl}$, $(\mathcal{M}_{\mathbf{L}_{pl}}, w) \models \psi$ iff $\psi \in w$, as in Lemma 38. In particular, we obtain that $U_\varphi = \llbracket \varphi \rrbracket_{\mathcal{M}_{\mathbf{L}_{pl}}} = \{w \in W \mid (\mathcal{M}_{\mathbf{L}_{pl}}, w) \models \varphi\} \in D$. Then, the proof for any $\psi \in \mathcal{L}_{sopml}$ is given by induction on the length of ψ , similarly as in Lemma 38, the only case of interest being quantified formulas. For $\psi = \forall p\chi$, if $\psi \in w$ then by axiom $\mathbf{E}x_{pl}$ we have that $\chi[p/\varphi] \in w$, for any $\phi \in \mathcal{L}_{pl}$. By induction hypothesis, $(\mathcal{M}_{\mathbf{L}_{pl}}, w) \models \chi[p/\varphi]$. Now consider the set $U_\varphi = \llbracket \varphi \rrbracket_{\mathcal{M}_{\mathbf{L}_{pl}}} \in D$. By Lemma 11.(2b), it is the case that $((\mathcal{M}_{\mathbf{L}_{pl}})_{U_\varphi}^p, w) \models \chi$, and since the choice of φ (and therefore of variant $V_{U_\varphi}^p$) is arbitrary, we obtain that $(\mathcal{M}_{\mathbf{L}_{pl}}, w) \models \forall p\chi$. As to the implication from left to right, the proof is the same as in Lemma 38, as each w is maximal and rich. \square

As a consequence of Lemma 41, the truth lemma also holds for boolean frames and we obtain a completeness proof for \mathbf{L}_{pl} .

We now discuss how to modify the procedure above to obtain completeness results for the logics \mathbf{L}_{ml} . Firstly, the canonical model for \mathbf{L}_{ml} is now defined as a tuple $\mathcal{M}_{\mathbf{L}_{ml}} = \langle W, D, R, V \rangle$ where (i) W , R and V are given as in Definition 36; and (ii) D is the domain of sets $U_\psi = \{w \in W \mid \psi \in w\} \subseteq W$, for every modal formula $\psi \in \mathcal{L}_{ml}$ over $AP \cup Y$. In particular, it is easy to check that the domain D in $\mathcal{M}_{\mathbf{L}_{ml}}$ is a boolean algebra with operators. Secondly, by adapting the proof of Lemma 41, we can prove the truth lemma for $\mathcal{M}_{\mathbf{L}_{ml}}$:

Lemma 42 (Truth lemma). *For every logic \mathbf{L}_{ml} , in the canonical model $\mathcal{M}_{\mathbf{L}_{ml}}$, for every $w \in W$ and every formula ψ over $AP \cup Y$,*

$$(\mathcal{M}_{\mathbf{L}_{ml}}, w) \models \psi \quad \text{iff} \quad \psi \in w$$

This completes the completeness proof for the logics \mathbf{L}_{ml} .

We conclude the section by summarising the soundness and completeness results for our logics w.r.t. the relevant classes of frames.

Theorem 43 (Soundness and Completeness). *For $x \in \{ap, pl, ml\}$, each logic \mathbf{L}_x is sound and complete w.r.t. the class $\mathcal{K}_{\bar{x}}^{\tau(\mathbf{L}_x)}$ of frames that are reflexive (respectively, transitive, symmetric), whenever \mathbf{L}_x includes axiom \mathbf{T} (respectively, $\mathbf{4}$, \mathbf{B}).*

As a result, for types ap , pl , and ml we are able to prove soundness and completeness for all normal modalities (i.e., \mathbf{K} , \mathbf{T} , $\mathbf{S4}$, \mathbf{B} , and $\mathbf{S5}$) in a multi-modal setting.

4.1.3. Generalised Completeness

We now extend the completeness results in the previous section by considering extra axioms expressing properties of frames. Specifically, let \mathbf{L} be any axiomatisation mentioned in Theorem 43. Then, if we extend \mathbf{L} with the universal closure $\forall \vec{p}\psi$ of a formula $\psi \in \mathcal{L}_{sopml}$, the resulting calculus $\mathbf{L} + \forall \vec{p}\psi$ is sound and complete w.r.t. the class of frames satisfying the MSO condition $\forall x \forall \vec{P} ST_x(\psi)$, where \vec{P} are all the unary predicates appearing in $ST_x(\psi)$.

Theorem 44. *Let ψ be a formula in SOPML, then the logic $\mathbf{L} + \forall \vec{p}\psi$ is sound and complete w.r.t. the corresponding class \mathcal{K} of frames satisfying $\forall x \forall \vec{P} ST_x(\psi)$.*

Proof. Soundness follows immediately by Lemma 20, as every frame that satisfies condition $\forall x \forall \vec{P} ST_x(\psi) = \forall x ST_x(\forall \vec{p} \psi)$, also validates $\forall \vec{p} \psi$. As to completeness, if $\forall \vec{p} \psi$ is an axiom, then it appears in every state of the canonical model \mathcal{M} , and by the truth lemma, \mathcal{M} validates $\forall \vec{p} \psi$. Finally, by another application of Lemma 20, \mathcal{F} validates $\forall x \forall \vec{P} ST_x(\psi)$. \square

By the result above we immediately obtain that for every formula $\theta(\vec{a}, \vec{p})$ appearing in Table 1, $\mathbf{L} + \forall p \theta(\vec{a}, \vec{p})$ is sound and complete w.r.t. the class of frames satisfying $\forall x \forall \vec{P} \Theta(\vec{a}, x)$. For instance, $\mathbf{K} + \exists p(\Box_a p \wedge \neg p)$ is a sound and complete axiomatisation of the class of irreflexive frames (notice that, since $\exists p(\Box_a p \wedge \neg p)$ is a sentence, its universal closure is equal to the formula itself.) More generally, there is a one-to-one correspondence between a SOPML axiom $\forall \vec{p} \theta$ and the MSO condition $\forall x \forall \vec{P} ST_x(\theta)$ on the corresponding class of sound and complete frames.

4.2. Incompleteness

In Section 4.1 we provided complete axiomatisations for languages without common knowledge, in the classes of all, boolean, and modal frames. In this section we prove our main unaxiomatisability results for SOPML. Specifically, in Section 4.2.1 we show that the set of validities in SOPEL (without common knowledge) interpreted on full frames is unaxiomatisable whenever we assume at least two agents in our frames. Further, in Section 4.2.2 we prove that, with common knowledge, SOPML is unaxiomatisable on modal frames already when considering a single agent. We then demonstrate (Corollaries 58, 61 and 63 as well as Theorem 62) how this proof can be extended to the other classes of frames. These results complete Tables 3 and 4.

4.2.1. 2-agent SOPEL on Full Frames is Unaxiomatisable

Recall that, as discussed at the beginning of Section 4, an axiomatisation for single-agent SOPEL on full frames was introduced in [21]. Here we show that, this result cannot be generalised to the multi-agent case, i.e., we demonstrate that multi-agent SOPEL is not recursively axiomatisable on full frames.

We prove this unaxiomatisability result by reducing the validity problem of diadic second order logic (DSO) to the validity problem of SOPEL. Since the former is known not to be recursively enumerable, this implies that the latter is also not recursively enumerable, and therefore in particular not axiomatisable. The reduction that we use is somewhat similar to the one used in [36] to prove the unaxiomatisability of single-agent SOPML on full frames with S4.2 or weaker modalities. Specifically, both the proof from [36] and the proof presented here represent a second order domain \mathcal{D} in a Kripke model by taking $\mathcal{D} \subseteq W$ and $(\mathcal{D} \times \mathcal{D}) \subseteq W$. Quantification over diadic relations on Dom then corresponds to propositional (i.e., monadic) quantification over $(\mathcal{D} \times \mathcal{D})$. The difference between the two proofs lies in how they characterize the models where $\mathcal{D} \subseteq W$ and $\mathcal{D} \times \mathcal{D} \subseteq W$, and in how second order formulas are translated once such a characterisation has been established.

Since the reduction is from the validity problem of diadic second order logic, let us first briefly define this logic.

Let a set \mathcal{X} of first-order variables and a set \mathcal{R} of second order variables be given. Then the language of diadic second order logic is given by the following normal form:

$$\theta ::= R(x, y) \mid x = y \mid \neg \theta \mid \theta \rightarrow \theta \mid \forall x \theta \mid \forall R \theta$$

where $x, x \in \mathcal{X}$ and $R \in \mathcal{R}$.

The formulas of DSO can be evaluated on models (Dom, ρ) that consist of a domain Dom and an assignment function ρ that assigns to each first-order variable x an element $\rho(x) \in Dom$ and to each second-order variable R a relation $\rho(R) \subseteq Dom \times Dom$. Given a model (Dom, ρ) an element $d \in Dom$ and a relation $E \subseteq Dom \times Dom$, the assignments $\rho[x \mapsto d]$ and $\rho[R \mapsto E]$ are the modifications of ρ that map $x \in \mathcal{X}$ to $d \in Dom$ and $R \in \mathcal{R}$ to $E \subseteq Dom \times Dom$, respectively.

Given these preliminaries, we can define the semantics of DSO in the usual way.

Definition 45. We define whether a model (Dom, ρ) satisfies a formula θ of DSO recursively as follows:

$$\begin{array}{ll}
(Dom, \rho) \models R(x, y) & \text{iff } (\rho(x), \rho(y)) \in \rho(R) \\
(Dom, \rho) \models x = y & \text{iff } \rho(x) = \rho(y) \\
(Dom, \rho) \models \neg\theta & \text{iff } (Dom, \rho) \not\models \theta \\
(Dom, \rho) \models \theta \rightarrow \theta' & \text{iff } (Dom, \rho) \not\models \theta \text{ or } (Dom, \rho) \models \theta' \\
(Dom, \rho) \models \forall x\theta & \text{iff for every } d \in Dom, (Dom, \rho[x \mapsto d]) \models \theta \\
(Dom, \rho) \models \forall R\theta & \text{iff for every } E \subseteq Dom \times Dom, (Dom, \rho[R \mapsto E]) \models \theta
\end{array}$$

A formula θ of DSO is valid, denoted $\models \theta$, if $(Dom, \rho) \models \theta$ for every model (Dom, ρ) .

In general, a formula of DSO can contain free first- and second-order variables. Our goal is to make a reduction from the validity problem of DSO, however, so we only care about whether a formula is valid. If θ is a DSO formula with a free first-order variable x (a free second-order variable R , respectively), then θ is valid if and only if $\forall x\theta$ ($\forall R\theta$, respectively) is valid. As a result, it suffices for us to consider only the sentences of DSO, i.e., the formulas without free variables. Furthermore, if θ is a sentence then whether $(Dom, \rho) \models \theta$ depends only on Dom . As such, we can consider our models to be given by the domain Dom only, where we say that $Dom \models \theta$ if and only if $(Dom, \rho) \models \theta$ for every assignment ρ .

Now, let us introduce the reduction from the validity problem of DSO to SOPEL's. This reduction has two parts: firstly, we define a formula ψ_{model} of SOPEL and use it to characterize a specific class of pointed models. Then, we define a translation f from the formulas of DSO to the formulas of SOPEL, with the property that

for every formula θ in DSO, there is a model Dom such that $Dom \models \theta$ iff there is a pointed model (\mathcal{M}, w) of SOPML such that $(\mathcal{M}, w) \models \psi_{model} \wedge f(\theta)$.

Before defining ψ_{model} and $f(\phi)$, however, let us present an auxiliary formula $\psi_{unique}(a, \chi)$ that will be useful in several places. This formula is very similar to a uniqueness formula introduced in [12].

Definition 46. Let $a \in I$ be an index and $\chi \in \mathcal{L}_{sopml}$ a formula. Then

$$\psi_{unique}(a, \chi) := \Diamond_a \chi \wedge \forall q (\Diamond_a (q \wedge \chi) \rightarrow \Box_a (\chi \rightarrow q)).$$

Lemma 47. Let a be an index in I , $\chi \in \mathcal{L}_{sopml}$ any formula, \mathcal{M} any full epistemic model, and w any world in \mathcal{M} . Then $(\mathcal{M}, w) \models \psi_{unique}(a, \chi)$ if and only if there is exactly one a -successor w' of w such that $(\mathcal{M}, w') \models \chi$.

Proof. Suppose $(\mathcal{M}, w) \models \psi_{\text{unique}}(a, \chi)$. Then, in particular, $(\mathcal{M}, w) \models \diamond_a \chi$, so there is at least one a -successor w' of w such that $(\mathcal{M}, w') \models \chi$. Suppose now, towards a contradiction, that there are two different a -successors w' and w'' of w such that χ holds on both w' and w'' . Then, since \mathcal{M} is a full model, there is some assignment for q such that q holds in w' but not in w'' . As a result, for this choice of q , we have $(\mathcal{M}, w) \models \diamond_a(q \wedge \chi) \wedge \neg \Box_a(\chi \rightarrow q)$, contradicting the fact that $(\mathcal{M}, w) \models \forall q(\diamond_a(q \wedge \chi) \rightarrow \Box_a(\chi \rightarrow q))$. It follows that the assumption of such w' and w'' existing must be false, so there is at most one a -successor w' of w such that $(\mathcal{M}, w') \models \chi$.

Suppose then that there is exactly one a -successor w' of w such that $(\mathcal{M}, w') \models \chi$. Then there is at least one such successor, so $(\mathcal{M}, w) \models \diamond_a \chi$. Furthermore, for every assignment of q , we have $\diamond_a(q \wedge \chi)$ if and only if $(\mathcal{M}, w') \models q \wedge \chi$, in which case we also have $\Box_a(\chi \rightarrow q)$. It follows that $(\mathcal{M}, w) \models \forall q(\diamond_a(q \wedge \chi) \rightarrow \Box_a(\chi \rightarrow q))$. Together with the previously established $(\mathcal{M}, w) \models \diamond_a \chi$, this implies that $(\mathcal{M}, w) \models \psi_{\text{unique}}(a, \chi)$. \square

Now we can use ψ_{unique} to define ψ_{model} and f .

Definition 48. The formula ψ_{model} is given by

$$\psi_{\text{model}} := \Box_a(\psi_D \wedge \psi_{\text{excl}} \wedge \psi_{\text{connect}} \wedge \psi_{\geq 1} \wedge \psi_{\leq 1})$$

where

$$\begin{aligned} \psi_D &:= p_D \wedge \psi_{\text{unique}}(b, p_D) \\ \psi_{\text{excl}} &:= \Box_b \neg(p_D \wedge (\text{start} \vee \text{end})) \wedge \Box_b \neg(\text{start} \wedge \text{end}) \wedge \Box_b(p_D \vee \text{start} \vee \text{end}) \\ \psi_{\text{connect}} &:= \Box_b(\neg p_D \rightarrow (\Box_a(\text{start} \vee \text{end}) \wedge \psi_{\text{unique}}(a, \text{start}) \wedge \psi_{\text{unique}}(a, \text{end}))) \\ \psi_{\geq 1} &:= \forall p(\diamond_a p \rightarrow \diamond_b(p_{\text{start}} \wedge \diamond_a(p_{\text{end}} \wedge \diamond_b p))) \\ \psi_{\leq 1} &:= \forall p(\psi_{\text{unique}}(a, p) \rightarrow \psi_{\text{unique}}(b, \text{start} \wedge \diamond_a(\text{end} \wedge \diamond_b(p \wedge p_D))) \end{aligned}$$

The meaning of the named subformulas $(\psi_D, \psi_{\text{excl}}, \psi_{\text{connect}}, \psi_{\geq 1}$ and $\psi_{\leq 1})$ is discussed in the proof of Theorem 50.

Definition 49. Let translation function f is recursively defined as follows.

$$\begin{aligned} f(x = y) &= \diamond_a(p_x \wedge p_y) \\ f(R(x, y)) &= \diamond_a(p_x \wedge \diamond_b(p_R \wedge \text{start} \wedge \diamond_a(\text{end} \wedge \diamond_b p_y))) \\ f(\neg \theta) &= \neg f(\theta) \\ f(\theta_1 \rightarrow \theta_2) &= f(\theta_1) \rightarrow f(\theta_2) \\ f(\forall x \theta) &= \forall p_x(\psi_{\text{unique}}(a, p_x) \rightarrow f(\theta)) \\ f(\forall R \theta) &= \forall p_R f(\theta) \end{aligned}$$

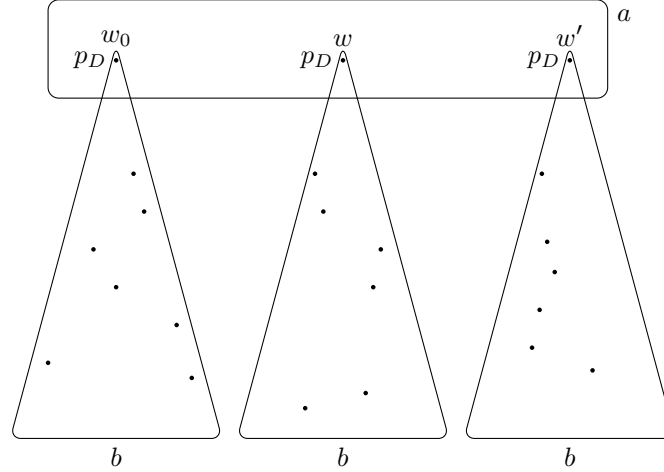
We can now prove the main result of this section.

Theorem 50. For every DSO sentence θ ,

$$\models \theta \quad \text{iff} \quad \mathcal{K}_{\text{full}}^e \models \psi_{\text{model}} \rightarrow f(\theta)$$

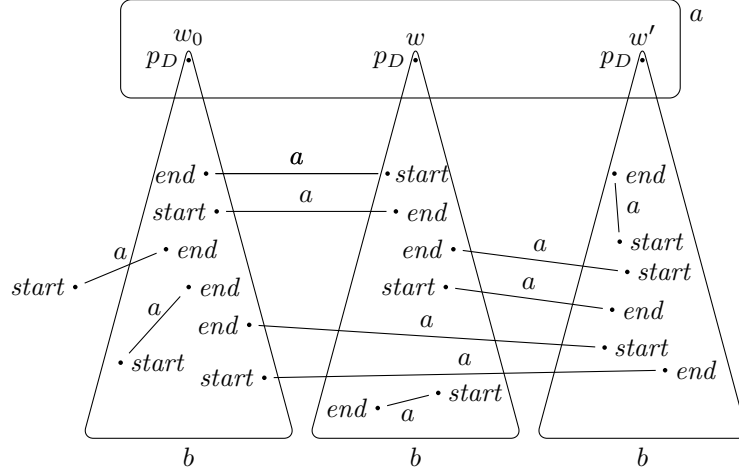
Proof. First, let us consider the models that satisfy ψ_{model} . Let (\mathcal{M}, w_0) be any full epistemic pointed model such that $(\mathcal{M}, w_0) \models \psi_{\text{model}}$, and let $\text{Dom} := R_a(w_0)$. The primary connective of ψ_{model} is \Box_a , so let us consider any $w \in \text{Dom}$.

The conjunct ψ_D holds at w if and only if (i) w satisfies p_D and (ii) w has only one b -successor that satisfies p_D (and by (i) this unique successor has to be w itself). We refer to the b -successors of w as the *cone* on w , as drawn in the following figure. Note that cones of any two different $w, w' \in Dom$ cannot overlap, since by the fact that \mathcal{M} is an epistemic model, this would imply that $R_b(w, w')$, contradicting the uniqueness of the p_D -world.



The conjunct ψ_{excl} holds if and only if every state in a cone satisfies exactly one of p_D , *start* and *end*. Now, consider $\psi_{connect}$. It holds if and only if every $\neg p_D$ state in a cone is connected by the relation a to exactly one state that satisfies *start*, to exactly one state that satisfies *end*, and only to states that satisfy either *start* or *end*. We had already established that no world in a cone can satisfy both *start* and *end*, so it follows that every such $\neg p_D$ state is either a *start* state paired by a with an *end* state, or an *end* state paired by a with a *start* state. Note that if a *start* or *end* state is in the cone of w , then the state that it is paired up with could be in the cone of w , in the cone of some different state w' or it might not be in any cone at all.

Now, consider formula $\psi_{\geq 1}$. It states that for every choice of p , if w has an a -successor w' that satisfies p , then there is a path $w \xrightarrow{b} e_1 \xrightarrow{a} e_2 \xrightarrow{b} w''$ such that e_1 satisfies *start*, e_2 satisfies *end* and w'' satisfies p . This is the case if and only if for every $w, w' \in D$, there is at least one such path $w \xrightarrow{b} e_1 \xrightarrow{a} e_2 \xrightarrow{b} w'$. Note that this also applies for $w = w'$. Our schematic illustration can therefore be extended to the following:



Finally, formula $\psi_{\leq 1}$ states that for any choice of p , if w has a unique a -successor w' that satisfies p , then w also has a unique b -successor e_1 such that (i) e_1 satisfies $start$, (ii) e_1 has an a -successor e_2 that satisfies end , and (iii) e_2 has a b -successor w'' that satisfies $p \wedge p_D$. This is the case if and only if for every $w, w' \in D$ there is at most one path $w \xrightarrow{b} e_1 \xrightarrow{a} e_2 \xrightarrow{b} w'$ where e_1 satisfies $start$ and e_2 satisfies end .

All in all, the conjunction of $\psi_{\geq 1}$ and $\psi_{\leq 1}$ means that there is exactly one path $w \xrightarrow{b} e_1 \xrightarrow{a} e_2 \xrightarrow{b} w'$ with $(\mathcal{M}, e_1) \models start$ and $(\mathcal{M}, e_2) \models end$ between each $w, w' \in D$.

The pointed model (\mathcal{M}, w_0) is then used in the following way. The suggestively named set Dom can be treated as the domain of a DSO model. First-order quantification is then interpreted on selecting exactly one world $w \in Dom$. Note that this is precisely what translation function f does: first-order quantification $\forall x \theta$ is translated as $\forall p_x (\psi_{unique}(a, p_x) \rightarrow f(\theta))$. On the other hand, second-order quantification is interpreted on selecting exactly those $start$ worlds that are on a path $w \xrightarrow{b} e_1 \xrightarrow{a} e_2 \xrightarrow{b} w'$, where pair (w, w') is in the chosen relation. Again, this is exactly what translation function f does: $\forall R \theta$ is translated as $\forall p_R f(\theta)$, and the relational atom $R(x, y)$ is then translated as $\diamond_a(p_x \wedge \diamond_b(p_R \wedge start \wedge \diamond_a(end \wedge \diamond_b p_y)))$, which means exactly that $R(x, y)$ holds iff p_R is true in the $start$ -world in between the unique p_x - and p_y -worlds.

It follows that $Dom \models \theta$ if and only if $(\mathcal{M}, w_0) \models f(\theta)$. Since all we assumed about \mathcal{M} is that it is a full epistemic model and that $(\mathcal{M}, w_0) \models \psi_{model}$, it follows that $\models \theta$ if and only if $\mathcal{K}_{full}^e \models \psi_{model} \rightarrow f(\theta)$. \square

As a consequence of the fact that validity in DSO is not recursively enumerable, we immediately obtain the following corollary.

Corollary 51. *Multi-agent SOPML on full epistemic frames is unaxiomatisable.*

Remark 52. *We used only two indices for modalities in our reduction, so the unaxiomatisability result holds whenever the set of indices contains at least two elements.*

4.2.2. Single-agent SOPML* is Unaxiomatisable on Modal Frames

In Section 4.1.3 we introduced sound and complete axiomatisations for SOPML on the classes of *modal*, *boolean*, and *all* frames. The version of SOPML that we considered

there does not contain common knowledge, however. This is a fundamental restriction; the variant SOPML*, which is obtained by adding common knowledge to SOPML, is not axiomatisable on any class of frames. Here, we show that SOPML* is not axiomatisable with respect to general or epistemic models that have *modal*, *boolean* or *arbitrary* domain of quantification. It can also be shown that SOPML* is unaxiomatisable on intermediate classes of models, such as those for logics **KD45** (i.e., serial, transitive and symmetric models) or **S4.2** (i.e., reflexive, transitive and convergent models). We do not include proofs for these classes, however, since they are very similar to the proofs for the cases we treat.

We prove the unaxiomatisability of SOPML* on *modal* models by a reduction from the non-halting problem to the validity problem of SOPML*. Since the non-halting Turing machines are not recursively enumerable, neither are the validities of SOPML* with respect to *modal* models. The proof that we use here is inspired by a similar proof from [38], in which the non-halting problem is reduced to the validity problem of a logic called *Arbitrary Arrow Update Logic with Common Knowledge*.

Before defining our reduction, let us first define the non-halting problem for Turing machines. A Turing machine, first defined in [50], is an abstract model of computation that consists of (i) an infinite tape, with \mathbb{Z} cells that can each contain a symbol, but that initially contain a “null” symbol α_0 ; (ii) a read/write head that can read the symbol in a cell or write a symbol to it; (iii) a method to keep track of what *state* the machine is currently in; and (iv) a set of instructions that, based on the current state and the current symbol under the read/write head, determines what symbol is to be written, what the next state of the machine should be, and whether the read/write head should move to the left, to the right or remain in place. The Turing machines that we consider here are deterministic so for each combination of state and symbol there is exactly one instruction. Formally, a Turing machine T can be defined as follows.

Definition 53. *A Turing machine T is a triple $T = (\Lambda, S, \Delta)$, where Λ is a finite alphabet containing the symbol α_0 , S is a finite set of states containing the distinct states s_0 and s_{end} , and $\Delta : \Lambda \times S \rightarrow \Lambda \times S \times \{\text{left}, \text{right}, \text{remain}\}$ is a transition function.*

The functions Δ_1, Δ_2 and Δ_3 are the projections of Δ to its first, second and third coordinate, respectively.

We can assume without loss of generality that $\Delta_2(\alpha, s) \neq s_0$ for all $\alpha \in \Lambda$ and $s \in S$, so that the machine never returns to its initial state.⁴ The execution of such a machine T can be represented as a function $E_T : \mathbb{Z} \times \mathbb{N} \rightarrow \Lambda \times S \times \{\text{yes}, \text{no}\}$. The horizontal direction \mathbb{Z} represents positions on the tape. The vertical direction \mathbb{N} represents time. So $E_T(n, m) = (\alpha, s, \text{yes})$ indicates that at time m , the n -th position on the tape contains symbol α , the machine is in state s and the read/write head is in position n . Likewise, $E_T(n, m) = (\alpha, s, \text{no})$ indicates that at time m the read/write head is not at position n . Note that E_T contains some redundant information: the state is a property of the machine, not of any particular position on the tape. Furthermore, the read/write head is in exactly one position at a time. So, for example, if $E_T(n, m) = (\alpha, s, \text{yes})$, then we must have $E_T(n + 1, m) = (\beta, s, \text{no})$ for some symbol β .

⁴Where necessary, an extra copy s_0^{copy} of s_0 can be added; whenever the system would go to state s_0 we can then let it go to s_0^{copy} instead.

A Turing machine T has a designated end state s_{end} . We say that T halts if the machine ever reaches this end state, so if for any (n, m) and any $\alpha \in \Lambda$, we have
1290 $E_T(n, m) = (\alpha, s_{end}, yes)$.

Definition 54 (Halting Problem). *The halting problem is to determine, for a given Turing machine T , whether T halts. The non-halting problem is the complement of the halting problem, i.e., to determine whether a given Turing machine doesn't halt.*

Although neither of them used this terminology, Church [13] and Turing [50] independently showed that the halting problem is undecidable. More precisely, the halting Turing
1295 machines are recursively enumerable, but the non-halting Turing machines are not.

Remark 55. *The intuition behind the end state is that the machine stops when it reaches s_{end} , so no more computation happens after that point. Here, however, in order to avoid special cases it is more convenient to assume that every Turing machine keeps going
1300 forever, whether or not it reaches s_{end} . So, whether or not a machine formally halts (reaches the end state), it never informally halts (stops).*

When reducing the non-halting problem to validity in SOPML*, it is useful to represent time not by the naturals \mathbb{N} but by the integers \mathbb{Z} , since this allows us to avoid special cases at time $t = 0$. For every time $t < 0$ we then say that the machine is in a dummy
1305 state $s_{void} \notin S$, which can be read as “the computation has not started yet.”

Now that the necessary concepts are defined, we can give our reduction of the non-halting problem to the validity problem of SOPML* on modal frames. This reduction has three parts. Firstly, we define a SOPML* formula ψ_{grid} that holds in a pointed model (\mathcal{M}, w) if and only if \mathcal{M} represents a $\mathbb{Z} \times \mathbb{Z}$ grid. Then, we define a formula ψ_{sane}
1310 that enforces a few sanity constraints that are necessary in order for us to interpret the grid as representing the execution of some Turing machine. Finally, we define a formula ψ_T that holds if and only if the Turing machine encoded by the grid is the specific machine T . Before defining these formulas, however, let us explain a “trick” that we will use. Typically, if we wanted to have a $\mathbb{Z} \times \mathbb{Z}$ grid in a modal logic, we would use four
1315 accessibility relations $R_{right}, R_{left}, R_{up}$ and R_{down} . However, we also want to show that even single-agent SOPML* is unaxiomatisable. So we will use only a single relation R .

We still need to represent the four possible directions, however. We do this by “coloring” the worlds of our model with propositional atoms $\{1, \dots, 9\}$, in a repeating pattern

$$\begin{array}{ccc} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}$$

So if a world satisfies 5, then the world above it satisfies 8, and if a world satisfies 6 then the world to its right satisfies 4. For ever direction $x \in dir := \{left, right, up, down\}$, let $f_x : \{1, \dots, 9\} \rightarrow \{1, \dots, 9\}$ be the function that gives the next number in direction x , e.g. $f_{up}(8) = 2$. Further, since we use only one agent a , we write \Box for \Box_a . This operator, together with functions f_x , can then be used to define \Box_x and \Diamond_x for $x \in dir$ as abbreviations:

$$\begin{aligned} \Box_x \phi &:= \bigvee_{1 \leq i \leq 9} (i \wedge \Box(f_x(i) \rightarrow \phi)) \\ \Diamond_x \phi &:= \bigvee_{1 \leq i \leq 9} (i \wedge \Diamond(f_x(i) \wedge \phi)) \end{aligned}$$

$$\begin{aligned}
\psi_{grid} &:= \Box^* (labels \wedge direction \wedge \bigwedge_{x \in dir} unique_x \wedge \\
&\quad \bigwedge_{(x,y) \in inv_dir} inverse_{xy} \wedge \bigwedge_{(x,y) \in perp_dir} commute_{xy}) \\
labels &:= \bigvee_{1 \leq j \leq 9} i \wedge \bigwedge_{i \neq j} \neg(i \wedge j) \\
direction &:= \bigwedge_{x \in dir} \Diamond_x \top \wedge \bigvee_{1 \leq i \leq 9} (i \wedge \Box \bigvee_{x \in dir} f_x(i)) \\
unique_x &:= \forall p (\Diamond_x p \rightarrow \Box_x p) \\
inverse_{xy} &:= \forall p (p \rightarrow \Box_x \Box_y p) \\
commute_{xy} &:= \forall p (\Diamond_x \Diamond_y p \rightarrow \Box_y \Box_x p) \\
dir &:= \{left, right, up, down\} \\
inv_dir &:= \{(left, right), (right, left), (up, down), (down, up)\} \\
perp_dir &:= \{(left, up), (left, down), (right, up), (right, down), \\
&\quad (up, left), (up, right), (down, left), (down, right)\}
\end{aligned}$$

Table 5: The formula ψ_{grid} .

These abbreviations are then used in formulas ψ_{grid} , ψ_{sane} and ψ_T defined as follows.

Definition 56. Let $S, \Lambda, \{pos, rpos, rpos\} \subseteq AP$, and let $T = (\Lambda, S, \Delta)$ be a Turing machine. The formula χ_T is given as $\chi_T := \psi_{grid} \wedge \psi_{sane} \wedge \psi_T \wedge s_0 \wedge pos$, where formulas

1320 ψ_{grid} , ψ_{sane} and ψ_T are as shown in Tables 5–7.

The formulas ψ_{grid} , ψ_{sane} and ψ_T may look complex, but that it slightly misleading. While these formulas are certainly long, every named subformula encodes a property of either a grid or a Turing machine in a rather straightforward way.

We start by considering the formula ψ_{grid} which, as the name implies, encodes a

1325 grid. Specifically, let (\mathcal{M}, w_0) be any pointed model (based on a modal frame) such that $(\mathcal{M}, w_0) \models \psi_{grid}$. We use the worlds of \mathcal{M} to represent the points of $\mathbb{Z} \times \mathbb{Z}$, in the following way: (i) the world w_0 represents $(0, 0)$; (ii) if a world w represents (n, m) , then every *right*-successor of w represents $(n + 1, m)$; and (iii) similarly for the other three directions.

1330 We show that every point (n, m) is represented by at least one world, that every world in the generated submodel represents at least one point, and that all the worlds that represent a single point (n, m) are modally indistinguishable. Let w be any world in the generated submodel of (\mathcal{M}, w_0) . The main connective of ψ_{grid} is \Box^* , so *labels*, *direction*, *unique_x* for every $x \in dir$, *inverse_{xy}* for every $(x, y) \in inv_dir$ and *commute_{xy}*

1335 for all $(x, y) \in perp_dir$ hold at w .

- By *labels*, the world w satisfies exactly one label $i \in \{1, \dots, 9\}$.
- By the first conjunct of *direction*, the world w has at least one x -successor for every $x \in dir$. It follows that every point in $\mathbb{Z} \times \mathbb{Z}$ is represented by at least one world.

1340 Furthermore, by the second conjunct of *direction*, every successor of w satisfies one of the labels that allow us to identify it as an x -successor for some $x \in dir$. It follows that every world in the generated submodel represents at least one point.

- The formula $unique_x$ implies that, for every U in the domain of quantification D , either every x -successor of w is in U , or no x -successor is in U . Since \mathcal{M} is based on a modal frame, this implies that all x -successors of w are modally indistinguishable from one another.
- 1345 • The formula $inverse_{xy}$ implies that, for every $U \in D$, if $w \in U$ then every xy -successor of w is also in U . Again, since \mathcal{M} is based on a modal frame, this implies that w is modally indistinguishable from its xy -successors. Note that this formula holds for all $(x, y) \in inv_dir$.
- 1350 • The formula $commute_{xy}$ implies that, for every $U \in D$, if there is an xy -successor w' of w such that $w' \in U$, then every yx -successor of w is in U . This implies that the xy -successors of w are modally indistinguishable from its yx -successors.
- Taken together, $unique_x$ (for every $x \in dir$), $inverse_{xy}$ (for every $(x, y) \in inv_dir$) and $commute_{xy}$ (for every $(x, y) \in perp_dir$) imply that all worlds representing a single point (n, m) are modally indistinguishable from one another.
- 1355

We say that tape position n at time m contains the symbol α if the worlds representing (n, m) satisfy the propositional atom α . Likewise, if the worlds representing (n, m) satisfy the propositional atom s , then the machine is in state s at time m . Finally, if the worlds representing (n, m) satisfy the atom pos , then the read/write head is in position n at time m , if they satisfy $rpos$ then position n is to the right of the read/write head at time m and if they satisfy $lpos$ then position n is to the left of the head.

Note that, because all worlds representing (n, m) are modally indistinguishable, it does not matter at which world we check whether these atoms are true; if s holds in any world representing (n, m) then s holds on all worlds representing (n, m) . In particular, since we use propositional atoms to represent the state of the Turing machine, the symbols on the tape, as well as the position of the read/write head, the modal indistinguishability of all worlds representing (n, m) means that all those worlds agree on the symbol, state, and on whether the read/write head is in that position. If a modal formula ϕ holds on the worlds that represent (n, m) , we abuse notation by writing $(\mathcal{M}, (n, m)) \models \phi$.⁵

1370 The formula ψ_{sane} imposes a number of sanity constraints: if (\mathcal{M}, w) satisfies both ψ_{grid} and ψ_{sane} , then \mathcal{M} can *almost* be seen as the execution of a Turing machine. We will first discuss the subformulas of ψ_{sane} in detail, and explain which sanity constraints they represent. After that, we will briefly discuss why ψ_{sane} only guarantees that \mathcal{M} can *almost* be seen as the execution of a Turing machine.

1375 As with ψ_{grid} , the main connective of ψ_{sane} is \Box^* . So suppose $(\mathcal{M}, w_0) \models \psi_{grid} \wedge \psi_{sane}$, and let w be any world in the generated submodel of (\mathcal{M}, w_0) .

⁵Note that we have not shown that w represents a *unique* point (n, m) . It is, in fact, true that if $(\mathcal{M}, w_0) \models \chi_T$ then every world in the generated submodel represents exactly one point, but we do not need this fact so we will not prove it here.

$$\begin{aligned}
\psi_{sane} &:= \Box^* (\text{position}_1 \wedge \text{position}_2 \wedge \text{one_state} \wedge \text{same_state} \wedge \text{one_symbol} \wedge \\
&\quad \text{void_state} \wedge \text{initial_symbol} \wedge \text{unchanged}_i) \\
\text{position}_1 &:= \neg(\text{pos} \wedge \text{lpos}) \wedge \neg(\text{pos} \wedge \text{rpos}) \wedge \neg(\text{rpos} \wedge \text{lpos}) \\
\text{position}_2 &:= ((\text{pos} \vee \text{rpos}) \rightarrow \Box_{\text{right}} \text{rpos}) \wedge ((\text{pos} \vee \text{lpos}) \rightarrow \Box_{\text{left}} \text{lpos}) \\
\text{one_state} &:= \bigvee_{s \in \text{states}} (s \wedge \bigwedge_{s' \in \text{states} \setminus \{s\}} \neg s') \\
\text{same_state} &:= \bigwedge_{s \in \text{states}} (s \rightarrow (\Box_{\text{right}} s \wedge \Box_{\text{left}} s)) \\
\text{one_symbol} &:= \bigvee_{\alpha \in \Lambda} (\alpha \wedge \bigwedge_{\beta \in \Lambda \setminus \{\alpha\}} \neg \beta) \\
\text{void_state} &:= (s_0 \vee s_{\text{void}}) \rightarrow \Box_{\text{down}} s_{\text{void}} \\
\text{initial_symbol} &:= s_0 \rightarrow \alpha_0 \\
\text{unchanged} &:= \bigwedge_{\alpha \in \Lambda} ((\neg \text{pos} \wedge \alpha) \rightarrow \Box_{\text{up}} \alpha)
\end{aligned}$$

Table 6: The formula ψ_{sane} .

- The formula position_1 implies that the grid points represented by w being (i) the location of the read/write head, (ii) to the right of the head, and (iii) to the left of the head, are mutually exclusive.
- 1380 • The formula position_2 implies that if the point represented by w is either at the read/write head or to its right, then w 's right-successor is to the right of the read/write head, and similarly for the left side. Together with position_1 , this implies that for every $m \in \mathbb{Z}$, there is at most one n such that $(\mathcal{M}, (n, m)) \models \text{pos}$.
- The formula one_state implies that there is exactly one state s such that $(\mathcal{M}, w) \models s$.
- 1385 • The formula same_state implies that w satisfies the same state as its *left*- and *right*- successors. So for every time m , there is exactly one state s such that $(\mathcal{M}, (n, m)) \models s$ for all n .
- The formula one_symbol implies that there is exactly one symbol α such that $(\mathcal{M}, w) \models \alpha$.
- 1390 • Recall that we use s_{void} as a dummy state which indicates that the execution of T has not started yet. The formula void_state implies that s_{void} holds on all worlds below worlds that satisfy either s_0 or s_{void} , so s_{void} does indeed hold in all worlds that represent points before the execution of T started in s_0 .
- The formula initial_symbol guarantees that, in the initial state, the entire tape contains the symbol α_0 .
- 1395 • The formula symbol_unchanged guarantees that whenever the read/write head is not at a particular position of the tape, then the symbol at that position remains unchanged.

$$\begin{aligned}
\psi_T &:= \Box^* \bigwedge_{1 \leq i \leq 9} (\text{position_change} \wedge \text{state_change} \wedge \text{symbol_change}) \\
\text{position_change} &:= \bigwedge_{\{(s,\alpha) | \Delta_3(s,\alpha)=\text{left}\}} ((pos \wedge s \wedge \alpha) \rightarrow \Box_{up} \Box_{left} pos) \wedge \\
&\quad \bigwedge_{\{(s,\alpha) | \Delta_3(s,\alpha)=\text{right}\}} ((pos \wedge s \wedge \alpha) \rightarrow \Box_{up} \Box_{right} pos) \wedge \\
&\quad \bigwedge_{\{(s,\alpha) | \Delta_3(s,\alpha)=\text{remain}\}} ((pos \wedge s \wedge \alpha) \rightarrow \Box_{up} pos) \\
\text{state_change} &:= \bigwedge_{s' \in \text{states}} \bigwedge_{\{(s,\alpha) | \Delta_2(s,\alpha)=s'\}} ((pos \wedge s \wedge \alpha) \rightarrow \Box_{up} s') \\
\text{symbol_change} &:= \bigwedge_{\beta \in \Lambda} \bigwedge_{\{(s,\alpha) | \Delta_1(s,\alpha)=\beta\}} ((pos \wedge s \wedge \alpha) \rightarrow \Box_{up} \beta)
\end{aligned}$$

Table 7: The formula ψ_T .

The above is almost sufficient to show that \mathcal{M} represents the execution of some Turing machine, except for the following: (i) there is no guarantee that s_0 is satisfied anywhere, so the execution might never start; (ii) while the read/write head is guaranteed to be in at most one position, it is not guaranteed to be in at least position at all times; and (iii) the symbols that are written, the state changes, and the movement of the read/write head could be random, as opposed to fully determined by a set of rules. Whereas ψ_{sane} almost guarantees that \mathcal{M} can be seen as the execution of some Turing machine, the formula ψ_T narrows this down to the specific machine T . In the process, it also solves one of the problems that remained after considering ψ_{sane} .

Suppose that $(\mathcal{M}, w_0) \models \chi_T$, so in addition to $(\mathcal{M}, w_0) \models \psi_{grid} \wedge \psi_{sane}$ we also have $(\mathcal{M}, w_0) \models \psi_T \wedge s_0 \wedge pos$. Then, the subformula *position_change* in ψ_T guarantees that the read/write head moves in accordance with Δ_3 , the subformula *state_change* guarantees that the state changes in accordance with Δ_2 , and the subformula *symbol_change* guarantees that the symbol that is written on the tape by the read/write head is in accordance with Δ_1 . Note that this solves problem (iii) of ψ_{sane} : the movement, state changes, and written symbols are in accordance with the deterministic set of rules that is represented by Δ .

Finally, consider the conjuncts s_0 and pos in χ_T . These solve the other two problems of ψ_{sane} : the execution starts at state s_0 in the point represented by w_0 , which represent point $(0, 0)$. Furthermore, the read/write head starts there as well, so the read/write head is initially in at least one position. From the fact that the head moves deterministically, and that this is encoded in ψ_T , it then follows that the read/write head is also in at least one position at every time after $m = 0$.

To conclude, the above shows that if $(\mathcal{M}, w_0) \models \chi_T$, then the generated submodel of (\mathcal{M}, w_0) represents a grid, and this grid contains the encoding of an execution of T .

Theorem 57. *A Turing machine T is halting if and only if $\mathcal{K}_{modal} \models \chi_T \rightarrow \Diamond^* s_{end}$, and non-halting if and only $\mathcal{K}_{modal} \models \chi_T \rightarrow \Box^* \neg s_{end}$.*

Proof. Let (\mathcal{M}, w_0) be any pointed model such that $(\mathcal{M}, w_0) \models \chi_T$. Then, as shown

above, every world in the generated submodel represents some point (n, m) in a $\mathbb{Z} \times \mathbb{Z}$ grid and, furthermore, the worlds representing a point (n, m) satisfy $s \in S$ if and only if the system is in state s at time m . It follows that the generated submodel contains a s_{end} world if and only if T is halting, from which the theorem follows immediately. \square

As an immediate consequence of Theorem 57, we obtain the following result.

Corollary 58. *SOPML* is not axiomatisable on modal frames for any number of agents.*

This unaxiomatisability result can be extended to the classes of *boolean* and *all* frames, since the modal frames can be characterized inside *boolean* or *any* frame. Let $\phi_{\neg} := \forall p \exists q \square^* (\neg p \leftrightarrow q)$, $\phi_{\vee} := \forall p \forall q \exists r \square^* ((p \vee q) \leftrightarrow r)$ and $\phi_{\square} := \bigwedge_{a \in Ag} \forall p \exists q \square^* (\square_a p \leftrightarrow q)$. The following lemma is entirely straightforward, so we state it without proof.

Lemma 59. *Let (\mathcal{M}, w) be a pointed model based on any frame. Then, the generated submodel of (\mathcal{M}, w) is based on a modal frame if and only if $(\mathcal{M}, w) \models \phi_{\neg} \wedge \phi_{\vee} \wedge \phi_{\square}$.*

Hence, the following corollary follows immediately from Lemma 59.

Corollary 60. *A formula $\phi \in \mathcal{L}_{sopml}$ is valid on modal frames if and only if $(\phi_{\neg} \wedge \phi_{\vee} \wedge \phi_{\square}) \rightarrow \phi$ is valid on all or boolean frames.*

As a result of the unaxiomatisability of SOPML* on modal frames, we finally obtain unaxiomatisability for all classes of frames.

Corollary 61. *SOPML* is not axiomatisable neither on the class of all frames nor on the class of boolean frames, for any number of agents.*

So far, we have shown that SOPML* is unaxiomatisable for modal, boolean, and all frames. This result can be extended to the corresponding classes of epistemic frames as well. The proofs for epistemic frames is very similar to the proofs above however, so we include them in the appendix only.

Theorem 62. *If $|I| \geq 2$, then the validities in SOPML* over modal epistemic frames are not recursively enumerable.*

In particular, SOPML is not axiomatisable on the class of modal epistemic frames.*

The same characterisation of boolean and all frames also applies to epistemic frames. So we also have the following corollary.

Corollary 63. *SOPML* is not axiomatisable on the class of all or boolean epistemic frames, for any number of agents.*

5. Simulations and Bisimulation

In this section we investigate the expressive power of second-order propositional modal logic by introducing truth-preserving (bi)simulation relations for SOPML. Bisimulations are an essential tool for the model theory of propositional modal logic, as they provide non-trivial sufficient conditions under which two models satisfy the same formulas in PML [7, 27]. Moreover, propositional modal logic is characterized by the well-known van Benthem theorem as the bisimulation-invariant fragment of first-order logic [6]. Hereafter

we introduce simulations and bisimulations for SOPML and prove that they are indeed
1465 truth-preserving. Further, in Section 5.2 we present a notion of abstraction for frames
and show that an abstraction of a frame simulates that frame. Finally, in Section 5.3 we
provide examples of the application of (bi)simulations to the analysis of the expressive
power of SOPML in spatial and temporal reasoning.

We should note that SOPML (bi)simulations are somewhat less well behaved than
1470 PML bisimulations. In particular, given two PML models there is a unique greatest
bisimulation between the two. Furthermore, this greatest bisimulation is the bisimilarity
relation, and can also be characterized as the greatest fixed point of a partition refinement
operator. In contrast, between two SOPML models there need not be a unique greatest
bisimulation, see Example 72. While we could define a partition refinement operator like
1475 in the PML case, and any bisimulation would be a fixed point of this operator, it would
not have a unique greatest fixed point. The SOPML bisimilarity relation, meanwhile, is
not guaranteed to be a bisimulation itself.

5.1. Simulations and Bisimulations

We define the notion of (bi)simulations on frames, although it is immediate to extend
1480 this definition to models. In the rest of the section we consider frames $\mathcal{F} = \langle W, D, R \rangle$, $\mathcal{F}' = \langle W', D', R' \rangle$, and models $\mathcal{M} = \langle \mathcal{F}, V \rangle$, $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$ defined on \mathcal{F} and \mathcal{F}' respectively. In the following, Σ denotes a relation $\Sigma \subseteq D \times D$.

Definition 64 (Frame Simulation). *Given frames \mathcal{F} and \mathcal{F}' , a simulation is a pair
1485 (σ, Σ) of relations $\emptyset \neq \sigma \subseteq W \times W'$, $\Sigma \subseteq D \times D'$ such that (i) for every $U \in D$, $\Sigma(U, U')$
for some $U' \in D'$; and (ii) $\sigma(w, w')$ implies*

1. for every $v \in W$, $a \in I$, if $R_a(w, v)$ then $\sigma(v, v')$ for some $v' \in R'_a(w')$;
2. for every $U \in D$, $U' \in D'$, $\Sigma(U, U')$ implies $w \in U$ iff $w' \in U'$.

Notice that condition 1 in Definition 64 expresses the standard notion of simulation
in PML. Hence, simulations for SOPML extend the corresponding definition for PML (we
1490 devote more discussion to this point later on). The definition of simulation above differs
from a similar notion put forward in [4]. Specifically, in [4] only a relation on states is
considered, thus obtaining a strictly weaker notion.

We say that state w' *simulates* w , or $w \leq w'$, iff $\sigma(w, w')$ holds for some simulation
pair (σ, Σ) . Similarly, a set U' *simulates* U , or $U \leq U'$, iff $\Sigma(U, U')$ holds for some
1495 simulation pair (σ, Σ) . Note that it may be that $w < w'$ holds because of (σ_1, Σ_1)
and $U < U'$ holds because of (σ_2, Σ_2) , while (σ_1, Σ_2) is not a simulation pair. To see
this, consider the frames $\mathcal{G}_1 = \langle \{w_1, w_2\}, \{\{w_1\}, \{w_2\}\}, \{(w_1, w_2), (w_2, w_1)\} \rangle$ and $\mathcal{G}_2 = \langle \{x_1, x_2\}, \{\{x_1\}, \{x_2\}\}, \{(x_1, x_2), (x_2, x_1)\} \rangle$. Clearly, $w_1 \leq x_1$ and $\{w_1\} \leq \{x_2\}$. However,
it is not the case that $w_1 \in \{w_1\}$ iff $x_1 \in \{x_2\}$. Nonetheless, each \leq is a preorder, i.e., a
1500 reflexive and transitive relation. Finally, a frame \mathcal{F}' *simulates* \mathcal{F} , or $\mathcal{F} \leq \mathcal{F}'$, iff for every
 $w \in W$, $w \leq w'$ for some $w' \in W'$.

Observe that for PML (that is, whenever we ignore the quantification domain D),
the notion of simulation given on frames is vacuous, as we discard the evaluation of
propositional atoms in the various states. Then, for instance, all serial frames simulate
1505 each other. However, this remark does not apply to SOPML, as we also have to take into
account propositional quantification.

We illustrate the newly introduced notion by an example.

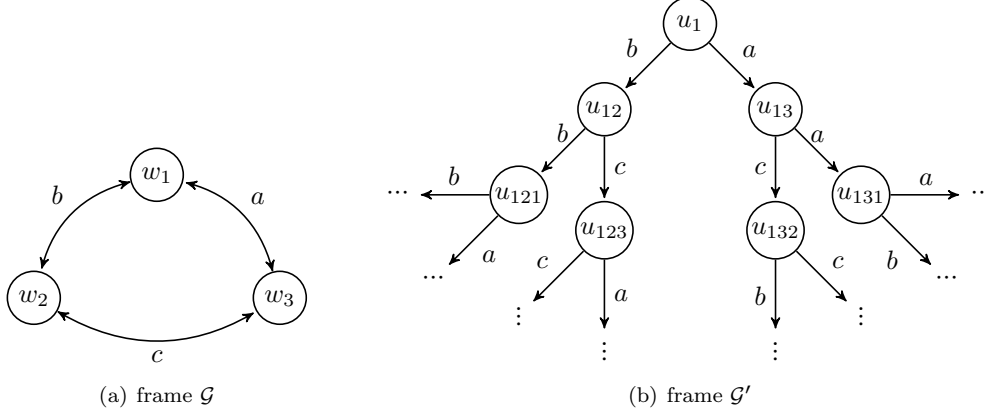


Figure 3: frames \mathcal{G} and \mathcal{G}' in Example 65. D components are omitted for clarity.

Example 65. Consider frames $\mathcal{G} = \langle W, R, D \rangle$ and $\mathcal{G}' = \langle W', R', D' \rangle$ over set $I = \{a, b, c\}$ of indices, depicted in Figure 3, with

- 1510 • $W = \{w_1, w_2, w_3\}$;
- $R_a = \{(w_1, w_3), (w_3, w_1)\}$, $R_b = \{(w_1, w_2), (w_2, w_1)\}$, $R_c = \{(w_2, w_3), (w_3, w_2)\}$;
- $D = \{\{w_1\}, \{w_2\}, \{w_3\}\}$;
- $W' = \{u_s \mid s \text{ is a finite sequence on } \{1, 2, 3\} \text{ starting with } 1, \text{ with no adjacent repetition}\}$;
- 1515 • for every $i \in I$, $R'_i = \{(u_s, u_{s'}) \mid s' = s \cdot m \text{ and } R_i(w_{\text{last}(s)}, w_m)\}$;
- let $U'_n = \{u_s \mid \text{last}(s) = n\}$, then $D' = \{U'_1, U'_2, U'_3\}$.

1520 Intuitively, frame \mathcal{G} can be thought of as a scenario where robots a , b , and c move around locations w_1 , w_2 , w_3 (robot a moves between w_1 and w_3 , etc.) Frame \mathcal{G}' then captures the same scenario but with the additional possibility to reason about some notion of history, or time (one might for instance add an atom p_i which is true exactly at nodes at level i . To do this, one needs to make appropriate assumptions about D' in \mathcal{G}' , like requiring that the frame is full. We do not consider these matters further.)

1525 Now consider the pair (σ, Σ) of relations $\sigma \subseteq W \times W'$ and $\Sigma \subseteq D \times D'$ such that $\sigma(w_n, u_s)$ holds iff $\text{last}(s) = n$ and $\Sigma(\{w_n\}, U'_m)$ holds iff $n = m$. We check that (σ, Σ) is indeed a simulation. Firstly, for every $\{w_n\} \in D$, we have $\Sigma(\{w_n\}, U'_n)$ for $U'_n \in D'$. Secondly, if $\sigma(w_n, u_s)$ and $R_i(w_n, w_m)$, then $s' = s \cdot m$ is such that $R'_i(u_s, u_{s'})$ and $\sigma(w_m, u_{s'})$. Thirdly, if $\sigma(w_n, u_s)$ and $\Sigma(\{w_k\}, U'_m)$, then $\text{last}(s) = n$ and $k = m$. Therefore, $w_n \in \{w_k\}$ iff $n = k$, iff $\text{last}(s) = m$, iff $u_s \in U'_m$.

1530 Finally, we observe that for every $w_n \in W$, $\sigma(w_n, u_s)$ for $\text{last}(s) = n$. Thus, frame \mathcal{G}' simulates \mathcal{G} .

Lemma 66. We have the following regarding the relation between simulations and properties of frames.

1. It can be checked in *NPTIME* whether there exists a simulation relation between two frames.

1535 2. If a frame \mathcal{F}' simulates a boolean (respectively modal, full) frame \mathcal{F} , then \mathcal{F}' need not to be boolean (respectively modal, full). Nor does \mathcal{F}' being boolean (modal, full) imply that \mathcal{F} is also boolean (modal, full).

Proof. As regards the first point, observe that we can guess a pair (σ, Σ) of relations on states and sets respectively, and then check in polynomial time whether it is actually a frame simulation.

As to the second point, consider that a simulating frame \mathcal{F}' may contain sets of states that do not simulate any state in \mathcal{F} , which are not closed under set-theoretic operations. As an example, consider frames $\mathcal{F} = \{\{w_1, w_2\}, \{(w_1, w_2)\}, 2^{\{w_1, w_2\}}\}$ and $\mathcal{F}' = \{\{u_1, u_2, u_3, u_4\}, \{(u_1, u_2), (u_1, u_3), (u_1, u_4)\}, \{\{u_1\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}, \{u_4\}, \emptyset\}\}$. In particular, \mathcal{F}' is a simulation of \mathcal{F} with relations σ and Σ such that $\sigma(w_1, u_1)$, $\sigma(w_2, u_2)$, $\sigma(w_2, u_3)$, and $\Sigma(\{w_1\}, \{u_1\})$, $\Sigma(\{w_2\}, \{u_2, u_3\})$, $\Sigma(\{w_1, w_2\}, \{u_1, u_2, u_3\})$, $\Sigma(\emptyset, \emptyset)$. But \mathcal{F} is a full frame, while \mathcal{F}' is not even boolean.

The other implication can be proved by a similar line of reasoning. Specifically, consider frame $\mathcal{F}'' = \{\{v_1, v_2\}, \{(v_1, v_2)\}, \{\{w_1\}, \{w_2\}\}\}$. Now \mathcal{F}'' is simulated by \mathcal{F} with relations σ and Σ such that $\sigma(v_1, w_1)$, $\sigma(v_2, w_2)$, and $\Sigma(\{v_1\}, \{w_1\})$, $\Sigma(\{v_2\}, \{w_2\})$. But \mathcal{F}'' is not even boolean.

Thus, similar frames do not necessarily belong to the same class. Below we compare these results with those available for bisimulations. \square

We now state the following preservation result for the universal fragment of SOPML.

1555 **Theorem 67.** *If $w \leq w'$, then for every $\varphi \in \mathcal{L}_{a\text{-sopml}}^*$,*

$$(\mathcal{F}', w') \models \varphi \text{ implies } (\mathcal{F}, w) \models \varphi$$

Proof. Since $w \leq w'$, there is a simulation pair (σ, Σ) such that $\sigma(w, w')$. Fix this σ . One can prove by induction on φ that if $(\mathcal{F}, V, w) \not\models \varphi$ for some assignment V , then $(\mathcal{F}', \Sigma(V), w') \not\models \varphi$, where $\Sigma(V)$ is any assignment V' such that for every $p \in AP$, $\Sigma(V(p), V'(p))$. We only show the step for the quantifier. We write $\Sigma(V)(p)$ for $(\Sigma(V))(p)$. By clause (i) of Definition 64, $\Sigma(V)(p) \in D'$.

For $\varphi = \forall p \psi$, $(\mathcal{F}, V, w) \not\models \varphi$ iff for some $U \in D$, $(\mathcal{F}, V_U^p, w) \not\models \psi$. By induction hypothesis, $(\mathcal{F}', \Sigma(V_U^p), w') \not\models \psi$. By condition (i) in Definition 64, for $U \in D$, $\Sigma(U, U')$ for some $U' \in D'$. In particular, we have that $\Sigma(V_U^p) = \Sigma(V)_{U'}^p$ whenever $\Sigma(U, U')$, and therefore $(\mathcal{F}', \Sigma(V)_{U'}^p, w') \not\models \psi$ for $U' \in D'$, that is, $(\mathcal{F}', \Sigma(V), w') \not\models \varphi$. \square

1565 As an immediate consequence of Theorem 67 we obtain the following corollary.

Corollary 68. *If $\mathcal{F} \leq \mathcal{F}'$, then for every $\varphi \in \mathcal{L}_{a\text{-sopml}}^*$,*

$$\mathcal{F}' \models \varphi \text{ implies } \mathcal{F} \models \varphi$$

Thus, the notion of simulation introduced in Definition 64 preserves the universal fragment of SOPML, similarly to the case for standard simulations and PML.

Example 69. Consider again frames \mathcal{G} and \mathcal{G}' in Example 65. We showed that \mathcal{G}' simulates \mathcal{G} . Moreover, we can easily check that \mathcal{G}' validates the following formula in SOPML

$$\forall p \left(p \rightarrow \bigwedge_{i \in I} \Box_i \neg p \right) \quad (14)$$

1570 which intuitively says that the agents can only move to a different position (so they cannot choose to remain in the same state). By Corollary 68 we deduce that (14) is valid in \mathcal{G} as well.

Simulations can naturally be extended to bisimulations. Also in this case, our focus is at the level of frames. In the following the *converse* of a relation R is the relation $R^{-1} = \{(u, v) \mid R(v, u)\}$.

1575 **Definition 70 (Frame Bisimulation).** Given frames \mathcal{F} and \mathcal{F}' , a bisimulation is a pair (ω, Ω) of relations $\emptyset \neq \omega \subseteq W \times W'$, $\Omega \subseteq D \times D'$ such that both (ω, Ω) and $(\omega^{-1}, \Omega^{-1})$ are simulations. That is, (i) for every $U \in D$, $\Omega(U, U')$ for some $U' \in D'$, and for every $U' \in D'$, $\Omega(U, U')$ for some $U \in D$; and (ii) $\omega(w, w')$ implies

1. for every $v \in W$, $a \in I$, if $R_a(w, v)$ then $\omega(v, v')$ for some $v' \in R'_a(w')$;
- 1580 2. for every $v' \in W'$, $a \in I$, if $R'_a(w', v')$ then $\omega(v, v')$ for some $v \in R_a(w)$;
3. for every $U \in D$, $U' \in D'$, $\Omega(U, U')$ implies $w \in U$ iff $w' \in U'$.

1585 States w and w' are *bisimilar*, or $w \approx w'$, iff $\omega(w, w')$ holds for some bisimulation pair (ω, Ω) . Similarly, sets U' and U are *bisimilar*, or $U \approx U'$, iff $\Omega(U, U')$ holds for some bisimulation pair (ω, Ω) . Again, similarly to the case for simulations, the pair (\approx, \approx) is not necessarily a bisimulation (see Example 72), but each \approx is an equivalence relation. This is in contrast to the situation in PML, where the bisimilarity relation is itself a bisimulation. Finally, frames \mathcal{F} and \mathcal{F}' are *bisimilar*, or $\mathcal{F} \approx \mathcal{F}'$, iff (i) for every $w \in W$, $w \approx w'$ for some $w' \in W'$; and (ii) for every $w' \in W'$, $w \approx w'$ for some $w \in W$.

1590 **Example 71.** Notice that frames \mathcal{G} and \mathcal{G}' in Example 65 are actually bisimilar. To prove this fact, we show that the converse relations $\sigma^{-1} \subseteq W' \times W$ and $\Sigma^{-1} \subseteq D' \times D$ form a simulation pair. Firstly, for every $U'_n \in D'$, the set $U = \{w_n\} \in D$ is such that $\Sigma(U, U')$. Secondly, if $\sigma^{-1}(u_s, w_n)$ and $R'_i(u_s, u_{s'})$ then $\text{last}(s) = n$ and $s' = s \cdot m$ for $w_m \in W$ such that $R_i(w_n, w_m)$. Hence, $\sigma^{-1}(u_{s'}, w_m)$. As to (3), the proof is identical as for simulations.

1595 **Example 72.** Let frames \mathcal{H} and \mathcal{H}' be as in Figure 4, where the domain of \mathcal{H} is given by $D = \{\{w_1, w_3\}, \{w_2, w_3\}\}$ and the domain of \mathcal{H}' is given by $D' = \{\{x_1, x_2\}, \{x_2\}\}$. Now, take $\omega_1 = \{(w_1, x_1), (w_3, x_3)\}$, $\Omega_1 = \{(\{w_1, w_3\}, \{x_1, x_2\}), (\{w_2, w_3\}, \{x_2\})\}$, $\omega_2 = \{(w_2, x_1), (w_3, x_2)\}$ and $\Omega_2 = \{(\{w_1, w_3\}, \{x_2\}), (\{w_2, w_3\}, \{x_1, x_2\})\}$.

1600 The pairs (ω_1, Ω_1) and (ω_2, Ω_2) are both bisimulations. Furthermore, it is easy to verify that for every $i \in \{1, 2\}$ and every bisimulation (ω, Ω) , if $\omega_i \subseteq \omega$ and $\Omega_i \subseteq \Omega$, then $(\omega, \Omega) = (\omega_i, \Omega_i)$. In other words, (ω_1, Ω_1) and (ω_2, Ω_2) are maximal bisimulations.

So, there is no unique greatest bisimulation. Furthermore, note that $w_1 \approx x_1$ and $w_2 \approx x_1$, yet there is no bisimulation that relates both w_1 and w_2 with x_1 . In particular, (\approx, \approx) is not a bisimulation.

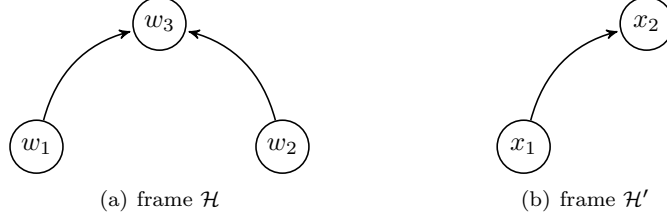


Figure 4: frames \mathcal{H} and \mathcal{H}' in Example 72. D components are omitted for clarity.

1605 We now state the following adaptation of Lemma 66.

Lemma 73.

1. It can be checked in $NPTIME$ whether there exists a bisimulation relation between two frames.
 2. Let \mathcal{F} and \mathcal{F}' be frames and let w and w' be worlds of \mathcal{F} and \mathcal{F}' , respectively, such that $w \approx w'$. Then \mathcal{F}_w is boolean (respectively modal) iff $\mathcal{F}'_{w'}$ is. However, if \mathcal{F}_w is full, then $\mathcal{F}'_{w'}$ need not be full.
- Moreover, if \mathcal{F}_w and $\mathcal{F}'_{w'}$ are full, then they are isomorphic.

1615 With regard to point 1, note that, as with simulations, we can simply guess a pair (ω, Ω) and check in polynomial time whether it is a bisimulation. A proof of point 2 is included in the appendix

Compare the situation for bisimulations with the weaker results available in Lemma 66 for simulations. Specifically, bisimulations preserve the class of boolean and modal frames. Moreover, in the case of full frames, bisimulations collapse into isomorphisms.

1620 We now state the main preservation result of this section. Its proof is similar to that of Theorem 67, and it is in the appendix.

Theorem 74. *If $w \approx w'$, then for every formula $\varphi \in \mathcal{L}_{sopml}^*$,*

$$(\mathcal{F}, w) \models \varphi \quad \text{iff} \quad (\mathcal{F}', w') \models \varphi.$$

As an immediate consequence of Theorem 74 we obtain the following.

Corollary 75. *If $\mathcal{F} \approx \mathcal{F}'$, then for every $\varphi \in \mathcal{L}_{sopml}^*$,*

$$\mathcal{F} \models \varphi \quad \text{iff} \quad \mathcal{F}' \models \varphi$$

1625 We can now infer that bisimulations in SOPML are ‘stronger’ than the corresponding notion for PML: whereas we noticed that the frames of Figure 2 are bisimilar in PML, as a consequence of Theorem 74, and Example 10, which says that the frames do not agree on formula (8), we conclude that they are not bisimilar in the SOPML sense.

Example 76. *We now consider two graph-theoretic properties. First, the notion of 3-colorability, as formalised by the following SOPML formula, where operator \square is interpreted*

as the necessity operator for the edges $E \subseteq W^2$ of a graph $G = \langle W, E \rangle$, while \Box^* is, as usual, the necessity operator for the reflexive and transitive closure of E :

$$\exists p_1, p_2, p_3 \left(\Box^* (p_1 \vee p_2 \vee p_3) \wedge \Box^* \bigwedge_{i \neq j} \neg (p_i \wedge p_j) \wedge \bigwedge_{1,2,3} \Box^* (p_i \rightarrow \neg \Diamond p_i) \right) \quad (15)$$

The truth of (15) in a vertex $v \in G$ implies that (i) all vertices in the subgraph generated by v are coloured by either p_1 , p_2 , or p_3 ; (ii) each vertex has at most one colour; and (iii) no two adjacent vertices have the same colour. Thus, the subgraph generated by v is 3-colorable. Observe that frame \mathcal{G} in Figure 3(a) is indeed 3-colorable, and since states w_1 and u_1 are bisimilar, as an immediate consequence of Theorem 74, also frame \mathcal{G}' is 3-colorable. Notice that the truth of (15) implies that the graph is 3-colorable, while the converse holds only for full frames.

To illustrate further the (in)expressivity of SOPML through simulations, we consider one more graph-theoretic property: the existence of a Hamiltonian path, i.e., a path that visits all vertices in a graph exactly once. Again, frame \mathcal{G} in Figure 3(a) has a Hamiltonian path w_1, w_2, w_3 . Now consider the finite graph \mathcal{G}'' which is obtained from \mathcal{G} and \mathcal{G}' in Figure 3 as follows. In \mathcal{G}' replace the worlds u_{12} and u_{13} by a copy of w_2 and w_3 from \mathcal{G} respectively, and remove all the worlds different from u_1 . It is not difficult to see that \mathcal{G}'' is bisimilar with \mathcal{G} , but does not allow a Hamiltonian path.

Proposition 77. *The property of having a Hamiltonian path is not expressible in SOPML.*

Proposition 77 follows directly from the second part of Example 76 and it implies that such paths are not expressible under the general semantics. Proposition 77 also follows from [42, Corollary 7.24], which says that it is impossible to test in MSO whether a graph is Hamiltonian. Indeed, it is known that such property is expressible in the language MSO_2 , an extension of MSO, which is strictly more expressive than SOPML [15, Prop. 5.13].

Discussion. We now compare our definition of (bi)simulation for SOPML, with the standard notion of (bi)simulation for PML [7]. Observe that if a frame \mathcal{F}' simulates \mathcal{F} in SOPML, with simulation pair (σ, Σ) , then for every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ based on \mathcal{F} , model $\mathcal{M}' = \langle \mathcal{F}', \Sigma(V) \rangle$ on \mathcal{F}' PML-simulates \mathcal{M} . In particular, if $\sigma(w, w')$ then for every $v \in W$, $a \in I$, $R_a(w, v)$ implies that $\sigma(v, v')$ for some $v' \in R'_a(w')$ by condition (ii).1 in Definition 64. Moreover, $w \in V(p) \in D$ iff $w' \in \Sigma(V)(p) \in D'$ by conditions (i) and (ii).2. Therefore, if \mathcal{M}' satisfies any universal formula ϕ in PML, then ϕ also holds in \mathcal{M} . Hence, Definition 64 of simulation for frames in SOPML is indeed a generalisation of the model-theoretic notion in PML. Furthermore, if frames \mathcal{F}' and \mathcal{F} are bisimilar in SOPML, with bisimulation pair (ω, Ω) , then models $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $\mathcal{M}' = \langle \mathcal{F}', \Omega(V) \rangle$ are also bisimilar in PML. Likewise, models $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$ and $\mathcal{M} = \langle \mathcal{F}, \Omega^{-1}(V) \rangle$ are PML-bisimilar as well. So in this case too, SOPML bisimulations on frames generalise PML bisimulations on models.

5.2. Abstraction

This section is devoted to the definition of a notion of abstraction for Kripke frames. Abstractions are deemed useful for system verification, as they allow to ignore some selected features of the system, thus focusing only on the properties relevant for the

verification task [14]. Indeed, a key fact about abstractions is that they simulate the original system. Hereafter we prove such a result for SOPML, starting with a family of equivalence relations on states.

Definition 78 (Equivalence). *Given a frame \mathcal{F} , let \sim be the equivalence relation on W such that for every state $w, w' \in W$, $w \sim w'$ implies that for every $U \in D$, $w \in U$ iff $w' \in U$. Further, denote by $[w] = \{w' \in W \mid w' \sim w\}$ the equivalence class of w in \mathcal{F} , and for a set $U \subseteq W$, let $[U]$ be $\{[w] \mid w \in U\}$.*

Clearly, if we replace ‘implies’ in Definition 78 by ‘iff’, we obtain the coarsest equivalence relation satisfying the conditions in Definition 78.

Definition 79 (Abstraction). *Given a frame \mathcal{F} , the abstraction $\mathcal{F}^A = \langle W^A, D^A, R^A \rangle$ of \mathcal{F} (according to equivalence relation \sim as in Definition 78) is the frame such that*

- $W^A = \{[w] \mid w \in W\}$;
- $D^A = \{[U] \mid U \in D\}$;
- for every $a \in I$, $R_a^A([w], [w'])$ iff $R_a(v, v')$ for some $v \in [w]$, $v' \in [w']$.

Notice that the coarsest abstraction \mathcal{F}^A is finite whenever the interpretation domain D in \mathcal{F} is, and of size $|W^A| = \mathcal{O}(2^D)$ at most.

Example 80. *To illustrate abstractions, we show that the frame \mathcal{G} in Example 65 is (isomorphic to) the coarsest abstraction \mathcal{G}^{A} of \mathcal{G}' . First of all, two worlds u_s and $u_{s'}$ are equivalent according to the coarsest equivalence \sim iff for all $U'_n \in D'$, $u_s \in U'_n$ iff $u_{s'} \in U'_n$, iff $\text{last}(s) = \text{last}(s')$. So, in abstraction \mathcal{G}^{A} we have three equivalence classes $[u_{t.1}]$, $[u_{t.2}]$, and $[u_{t.3}]$, for sequences $t \in \{1, 2, 3\}^+$ beginning with 1. As to the accessibility relations, $R_i^A([u_{t.n}], [u_{t'.m}])$ iff for $u_{t.n}, u_{t'.m}$ in W' , $R'_i(u_{t.n}, u_{t'.m})$, that is, $t' = t \cdot n$ and $R_i(w_n, w_m)$. Hence, for instance, for agent a , we have $R_a^A([u_{t.1}], [u_{t'.3}])$ and $R_a^A([u_{t.3}], [u_{t'.1}])$, as required. Finally, $D^{A} = \{[U'_n] \mid U'_n \in D'\} = \{\{[u_{t.1}]\}, \{[u_{t.2}]\}, \{[u_{t.3}]\}\}$. Clearly, the abstraction \mathcal{G}^{A} of \mathcal{G}' is isomorphic to \mathcal{G} , with mapping $w_i \mapsto [u_{t.i}]$ for $i = 1, 2, 3$.*

We now extend a standard result in modal logic, namely that abstractions are indeed simulations, to SOPML.

Lemma 81. *Given a frame \mathcal{F} with abstraction \mathcal{F}^A , the pair of mappings $w \mapsto [w]$ and $U \mapsto [U]$ is a simulation.*

Proof. We show that the pair (\mapsto, \mapsto) of mappings satisfies Definition 64. As to condition (i), if $U \in D$ then $U \mapsto [U]$ for $[U] \in D^A$. Next, for (i).1 suppose that $R_a(w, v)$. Then, for $[v] \in W^A$ we have that $R_a^A([w], [v])$ and $v \mapsto [v]$. Finally, as to (ii).2, if $w \mapsto [w]$ and $w \in U$, then clearly $[w] \in [U]$. On the other hand, if $[w] \in [U]$ then for some $v \in [w]$, $v \in U$. However, $v \in [w]$ implies that $v \sim w$. In particular, $w \in U$ by the constraint on \sim . \square

We remark that the abstraction \mathcal{F}^A of a full frame \mathcal{F} is isomorphic to \mathcal{F} . In fact, for every $w \in W$, the set $\{w\}$ belongs to D , and since $w \sim w'$ iff for all $U \in D$, $w \in U$ iff $w' \in U$, $w \sim w'$ implies in particular that $w \in \{w'\}$, that is, $w = w'$. As a consequence, $w \mapsto \{w\}$ is

1705 the only simulation on states between \mathcal{F} and \mathcal{F}^A , and it is also an isomorphism. Further, in Example 80 we observed that frame \mathcal{G} is (isomorphic to) the coarsest abstraction of \mathcal{G}' . Hence, Lemma 81 provides an alternative proof of the fact that \mathcal{G} simulates \mathcal{G}' , that we discussed in Example 71.

The following corollary follows immediately from Lemmas 67 and 81.

1710 **Corollary 82.** *Let \mathcal{F} be a frame with abstraction \mathcal{F}^A . For every universal formula $\varphi \in \mathcal{L}_{a\text{-sopml}}^*$,*

$$(\mathcal{F}^A, [w]) \models \varphi \quad \text{implies} \quad (\mathcal{F}, w) \models \varphi$$

The results presented above have an impact that goes beyond their theoretical interest. As an example, we observed that relevant properties P of frames (such as reflexivity, transitivity, symmetry, etc.) are definable in propositional modal logic in the sense that 1715 for some formula ϕ in PML, a frame \mathcal{F} validates ϕ iff \mathcal{F} satisfies property P . In SOPML more properties become frame-definable within the class of full frames. For instance, in Section 3 we showed that a full frame \mathcal{F} is irreflexive iff $\mathcal{F} \models \exists p(\Box p \wedge \neg p)$. On the other hand, whenever we consider the class of all frames, several properties are non-definable (even where they might be definable in PML). For instance, the frame \mathcal{G} in Example 65 1720 is symmetric, while \mathcal{G}' is not. Since, \mathcal{G} and \mathcal{G}' are bisimilar, and therefore satisfy the same formulas in SOPML, we conclude that symmetry is not definable in the class of all frames. Likewise, irreflexivity nor reflexivity are definable on the class of all frames: take \mathcal{F} consisting of only one reflexive world w with $D = \{\{w\}, \emptyset\}$, and \mathcal{F}' consisting of two worlds w'_1, w'_2 with $R' = \{(w'_1, w'_2), (w'_2, w'_1)\}$ and $D' = \{\{w'_1, w'_2\}, \emptyset\}$. The pointed 1725 frames (\mathcal{F}, w) and (\mathcal{F}', w'_1) are bisimilar, but \mathcal{F} is reflexive (hence reflexivity cannot be expressed on all frames) while \mathcal{F}' is irreflexive (hence irreflexivity cannot be expressed). Such results provide us with further knowledge on the expressive power of SOPML.

5.3. Bisimulations and Expressivity

In this section we explore the expressivity of SOPML, also by using the (bi)simulations 1730 introduced in Section 5.1. We focus on some temporal and spatial properties typically used in artificial intelligence. In what follows we say that a property P is *expressible* in a language \mathcal{L} and class \mathcal{K} of frames iff for some formula $\phi \in \mathcal{L}$, we have that for all $\mathcal{F} \in \mathcal{K}$, $\mathcal{F} \models \phi$ iff \mathcal{F} has property P . Sometimes we omit either \mathcal{L} or \mathcal{K} , whenever these are clear from the context.

1735 First of all, consider Dedekind-completeness of a total order \leq , i.e., a total, transitive, and antisymmetric binary relation: a totally ordered set is *Dedekind-complete* if every non-empty subset that has an upper bound, has a least upper bound. We recall that the Dedekind-completeness of the real numbers is not expressible in PML: the proof makes use of a propositional bisimulation between the structure (\mathbb{R}, \leq) of reals and the 1740 rationals (\mathbb{Q}, \leq) [1]. Thus, by using simulations we immediately obtain the following inexpressibility result.

Lemma 83. *Dedekind-completeness is not expressible in the universal fragment $\mathcal{L}_{a\text{-sopml}}$ in the class of full frames.*

Proof. Clearly, the identity relation is a simulation between structures (\mathbb{Q}, \leq) and (\mathbb{R}, \leq) 1745 seen as full frames, i.e., $(\mathbb{Q}, \leq) \leq (\mathbb{R}, \leq)$, and if Dedekind-completeness were expressible as a formula ϕ in A-SOPML, $(\mathbb{R}, \leq) \models \phi$ would imply $(\mathbb{Q}, \leq) \models \phi$, a contradiction. \square

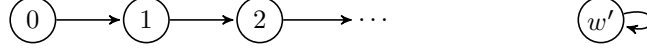


Figure 5: Frames \mathcal{G}_1 and \mathcal{G}_2 . The D -components are omitted for clarity.

Recall that formula $\delta = (\diamond p \wedge \diamond \square \neg p) \rightarrow \diamond(\square^{-1} \diamond p \wedge \square \neg p)$ has been introduced in Section 3.2 to express Dedekind-completeness. Intuitively, δ fails in (\mathbb{Q}, \leq) since, for instance, the set $\{q \in \mathbb{Q} \mid q < \sqrt{2}\}$ is non-empty and upper bounded, and therefore satisfies the antecedent. However, it has no least upper bound to satisfy the consequent.

As a further example, we prove that neither finiteness nor infinity of the state space W are expressible in boolean frames. This is in line with the situation in PML.

Lemma 84. *In language \mathcal{L}_{sopml}^* neither finiteness nor infinity are expressible in the class of boolean frames.*

Proof. Consider frame $\mathcal{G}_1 = \langle \mathbb{N}, succ, \{\mathbb{N}, \emptyset\} \rangle$ of the naturals with the successor relation and the reflexive-point frame $\mathcal{G}_2 = \langle \{w'\}, \{\{w', w'\}\}, \{\{w'\}, \emptyset\} \rangle$ in Figure 5, which are boolean by definition of D_1 and D_2 . In particular, the relations ω mapping every natural $n \in \mathbb{N}$ to w' , and Ω mapping \mathbb{N} to $\{w'\}$ and the empty set \emptyset to itself, form a bisimulation pair. Thus, \mathcal{G}_1 and \mathcal{G}_2 validate the same formulas in SOPML. However, \mathcal{G}_1 is infinite while \mathcal{G}_2 is finite. \square

To conclude our brief review of expressivity results in SOPML, we show that for the sublanguage of \mathcal{L}_{sopml} without the reflexive and transitive closure operator \square^* , finiteness is not even expressible in full frames. For $n \in \mathbb{N}$, let $[n]$ be the set $\{0, \dots, n\}$, \mathcal{G}_n the frame $\langle [n], succ, 2^{[n]} \rangle$, and $\mathcal{G}_{\mathbb{N}} = \langle \mathbb{N}, succ, 2^{\mathbb{N}} \rangle$ the frame isomorphic to the structure of natural numbers endowed with the successor relation. Both $\mathcal{G}_{\mathbb{N}}$ and each \mathcal{G}_n are full. Let G be the class of all frames \mathcal{G}_n , for $n \in \mathbb{N}$, and consider the following result.

Lemma 85. *In language \mathcal{L}_{sopml} the theory $\text{Th}(G)$ is a subset of $\text{Th}(\mathcal{G}_{\mathbb{N}})$.*

Proof. Consider $\phi \notin \text{Th}(\mathcal{G}_{\mathbb{N}})$ with finite modal depth $k \in \mathbb{N}$, where the modal depth is defined as in the propositional case, as the maximum nesting of modal operators [7]. We can assume without loss of generality that $(\mathcal{G}_{\mathbb{N}}, V, 0) \not\models \phi$ for some assignment V . We prove that $(\mathcal{G}_k, V', 0) \not\models \phi$, where assignment V' is such that $V'(p) = V(p) \cap [k]$ for every $p \in AP$ (since our frames are full, $V(p) \cap [k]$ is admissible in \mathcal{G}_k). More precisely we prove, using induction on n , that if ψ is a subformula of ϕ of modal depth $n \leq k$, then for all ℓ with $0 \leq \ell \leq k - n$, it holds that $(\mathcal{G}_{\mathbb{N}}, V, \ell) \models \psi$ iff $(\mathcal{G}_k, V', \ell) \models \psi$.

We start with the case for $n = 0$. If ψ is an atom p , then $(\mathcal{G}_{\mathbb{N}}, V, \ell) \models \psi$ iff $\ell \in V(p)$, iff $\ell \in V'(p)$, iff $(\mathcal{G}_k, V', \ell) \models \psi$, for all $0 \leq \ell \leq k$. The cases for propositional connectives are immediate. Finally, suppose $\psi = \exists p \chi$, and that $(*)$ $(\mathcal{G}_{\mathbb{N}}, V, \ell) \models \chi$ iff $(\mathcal{G}_k, V', \ell) \models \chi$ for all $0 \leq \ell \leq k$. Then $(\mathcal{G}_{\mathbb{N}}, V, \ell) \models \psi$ implies that for some $U \in D$, $(\mathcal{G}_{\mathbb{N}}, V_U^p, \ell) \models \psi$. Consider $U' = U \cap [k] \in D'$. In particular, $(V_U^p)' = V_{U'}^p$. By $(*)$, $(\mathcal{G}_k, V_{U'}^p, \ell) \models \chi$, that is, $(\mathcal{G}_k, V', \ell) \models \psi$.

Now suppose that ψ is a subformula of ϕ of modal depth $n + 1 \leq k$, and let $0 \leq \ell \leq k - (n + 1)$. Assume that $\psi = \diamond \chi$. Then, $(\mathcal{G}_{\mathbb{N}}, V, \ell) \models \psi$ implies that $(\mathcal{G}_{\mathbb{N}}, V, \ell + 1) \models \chi$ for $1 \leq \ell + 1 \leq k - n$. By induction hypothesis $(\mathcal{G}_k, V', \ell + 1) \models \chi$, that is, $(\mathcal{G}_k, V', \ell) \models \psi$. The direction from from $(\mathcal{G}_k, V', \ell)$ to $(\mathcal{G}_{\mathbb{N}}, V, \ell)$ is similar, and the cases for the propositional connectives and the quantifier go as before. \square

Corollary 86. *In language \mathcal{L}_{sopml} finiteness is not expressible on full frames.*

Proof. As a consequence of Lemma 85, $(\mathcal{G}_{\mathbb{N}}, V, 0) \not\models \phi$ implies $(\mathcal{G}_k, V', 0) \not\models \phi$ for some $k \in \mathbb{N}$, i.e., $\phi \notin \text{Th}(G)$. Hence, if ϕ expressed ‘being finite’, then it would be valid in G , and hence also in $\mathcal{G}_{\mathbb{N}}$, a contradiction. Thus, finiteness is not expressible even in the class of full frames. \square

1790

In this section we made use of (bi)simulations to show that SOPML can express notions, such as Dedekind-completeness, that are not expressible in PML; whereas other properties, such as finiteness, cannot even be expressed in SOPML. Together with the remarks in Section 5.1 on 3-colorability and the existence of Hamiltonian paths, these results provide us with some interesting insight into the application of model-theoretic techniques to the analysis of the expressivity of SOPML.

1795

In our opinion bisimulations for SOPML raise a number of interesting questions. We believe that one in particular deserves more attention. The Van Benthem theorem is a well-known result in model theory, stating that modal logic is the bisimulation-invariant fragment of first-order logic [6]. In the light of the notion of bisimulation provided above, it makes sense to ask the same question in the present context: is SOPML the bisimulation-invariant fragment of monadic second-order logic, possibly when interpreted on a particular class of frames? In [35] it is proved that the modal μ -calculus is the bisimulation-invariant fragment of MSO, but according to the standard notion of bisimulation for PML. Presently it is not clear how this result relates to the current setting. We leave this problem open for future work.

1800

1805

6. Conclusion

In this paper we motivated and studied (the use of) second-order propositional modal logic as a specification language for reasoning about knowledge as well as spatial and temporal properties in artificial intelligence. Specifically, we aimed at developing proof- and model-theoretic techniques, notably complete axiomatisations and truth-preserving (bi)simulations, to support the use of SOPML in applications. In Section 2 we introduced 20 different classes of Kripke frames, according to the structure of the domain D of quantification and the features of the accessibility relations. In Section 4 we provided complete axiomatisations for some of these classes, while proving that other classes are unaxiomatisable.

1810

1815

Further, we introduced suitable notions of (bi)simulation and proved that they preserve the satisfaction of (universal) SOPML. Finally, we made use of (bi)simulations to obtain some inexpressibility results. Specifically, we showed that being finite and having a Hamiltonian path are not expressible in SOPML, while other properties, viz. topological completeness and 3-colorability, are indeed expressible. We conclude that SOPML can indeed be used as a modelling language for artificial intelligence, particularly for temporal and spatial reasoning, as well as to describe higher-level knowledge of agents, that is, the knowledge agents have about other agents’ knowledge and beliefs, as shown in Section 3. In this respect, we reckon that the development of model-theoretic techniques is key for applications.

1820

1825

6.1. Future Research

We have presented several results about SOPML, but there are of course also a number of remaining questions for future research.

1830 One such question is the relative expressivity of SOPML and QCTL. As mentioned in Section 1.1, the variant of SOPML with full domain of quantification is similar to one of the variants of QCTL presented in [41], with the main difference being that QCTL is based on CTL, while SOPML is based on basic modal logic. The temporal logic CTL has more operators than modal logic, and it is strictly more expressive. It is not currently clear
1835 whether QCTL is also strictly more expressive than SOPML, however.

Another direction for future research is to find fragments of SOPML that are “well-behaved”. As discussed in Section 4, most of SOPML variants are unaxiomatisable on full frames, and variants that include the reachability operator \Box^* are unaxiomatisable even on frames with coarser domains of quantification. It would be interesting to find
1840 fragments of SOPML that are large enough to be interesting, but that are axiomatisable or even decidable. In this respect the monodic fragment of first-order modal logic, that restrict the number of free variables in the scope of any modal operator, looks promising [2, 29, 53]. We leave this direction for future work.

At the end of Section 5 we mentioned the interest of checking whether SOPML is
1845 the bisimulation-invariant fragment of MSO, according to the newly introduced notion of bisimulation. Such a result would provide us with a precise characterisation of the expressive power of SOPML (with respect to MSO), in the spirit of van Benthem’s result for modal logic [6]. It would also be of interest to define games in the style of Ehrenfeucht-Fraïssé to have a more fine-grained approach to the analysis of the expressivity of SOPML,
1850 similarly to the case for first-order and modal logic [34].

Finally, here we considered a unique domain D of quantification in any of our frames. But more elaborate forms of quantification can be taken into account, for instance, we might consider a different quantification domain for each state in the frame. Then, we might also consider state-dependent assignments. These variants have been studied
1855 extensively in the realm of first-order modal logic [10, 24], but never addressed in SOPML. They might be of interest for specific applications.

Acknowledgements. The authors would like to thank the anonymous reviewers
1860 for valuable comments and input that has significantly improved a previous version of the paper. F. Belardinelli acknowledges the support of the ANR JCJC Project SVEDaS (ANR-16-CE40-0021). L. Kuijter acknowledges support from ERC project EPS 313360.

References

- 1865 [1] A. Baltag, S. Smets (Eds.), *Johan van Benthem on Logic and Information Dynamics*, Springer, 2014.
- [2] F. Belardinelli, A. Lomuscio, Interactions between knowledge and time in a first-order logic for multi-agent systems: Completeness results, *Journal of Artificial Intelligence Research* 45 (2012) 1–45.
- 1870 [3] F. Belardinelli, W. van der Hoek, Epistemic quantified boolean logic: Expressiveness and completeness results, in: Q. Yang, M. Wooldridge (Eds.), *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence*, AAAI Press, 2015, pp. 2748–2754.

- [4] F. Belardinelli, W. van der Hoek, A semantical analysis of second-order propositional modal logic, in: Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI-16), AAAI Press, 2016, pp. 886–892.
- 1875 [5] F. Belardinelli, H. van Ditmarsch, W. van der Hoek, Second-order propositional announcement logic, in: C. Jonker, S. Marsella, J. Thangarajah, K. Tuyls (Eds.), Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems, ACM, 2016, pp. 635–643. URL <http://dl.acm.org/citation.cfm?id=2937019>
- [6] J. van Benthem, Modal correspondence theory, Ph.D. thesis, University of Amsterdam (1976).
- 1880 [7] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Vol. 53 of Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.
- [8] P. Blackburn, J. van Benthem, Modal logic: A semantic perspective, in: Blackburn et al. [9], pp. 1–84.
- [9] P. Blackburn, J. van Benthem, F. Wolter (Eds.), Handbook of Modal Logic, Elsevier, Amsterdam, the Netherlands, 2007.
- 1885 [10] T. Brauner, S. Ghilardi, First-order modal logic, in: P. Blackburn, J. van Benthem, F. Wolter (Eds.), Handbook of Modal Logic, Elsevier, 2007, pp. 549–620.
- [11] R. A. Bull, On modal logic with propositional quantifiers, The journal of symbolic logic 34 (1969) 257–263.
- 1890 [12] B. ten Cate, Expressivity of second order propositional modal logic., Journal of Philosophical Logic 35 (2) (2006) 209–223. URL <http://dblp.uni-trier.de/db/journals/jphil/jphil35.html#Cate06>
- [13] A. Church, A note on the entscheidungsproblem, Journal of Symbolic Logic 1 (1) (1936) 40–41.
- 1895 [14] E. M. Clarke, O. Grumberg, D. E. Long, Model Checking and Abstraction, ACM Transactions on Programming Languages and Systems 16 (5) (1994) 1512–1542.
- [15] P. B. Courcelle, D. J. Engelfriet, Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach, 1st Edition, Cambridge University Press, New York, NY, USA, 2012.
- [16] H. P. van Ditmarsch, W. van der Hoek, B. P. Kooi, Knowing more - from global to local correspondence, in: C. Boutilier (Ed.), IJCAI 2009, Proceedings of the 21st International Joint Conference on Artificial Intelligence, 2009, pp. 955–960. URL <http://ijcai.org/papers09/Papers/IJCAI09-162.pdf>
- 1900 [17] H. P. van Ditmarsch, W. van der Hoek, B. P. Kooi, Reasoning about local properties in modal logic, in: L. Sonenberg, P. Stone, K. Tumer, P. Yolum (Eds.), Proceedings of the 10th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2011), IFAAMAS, 2011, pp. 711–718. URL <http://portal.acm.org/citation.cfm?id=2031719&CFID=54178199&CFTOKEN=61392764>
- 1905 [18] H. P. van Ditmarsch, W. van der Hoek, B. P. Kooi, Local properties in modal logic, Artificial Intelligence 187 (2012) 133–155.
- [19] H. Ebbinghaus, J. Flum, W. Thomas, Mathematical Logic, Springer, 1994.
- 1910 [20] R. Fagin, J. Halpern, Y. Moses, M. Vardi, Reasoning About Knowledge, The MIT Press, 1995.
- [21] K. Fine, Propositional quantifiers in modal logic, Theoria 36 (3) (1970) 336–346.
- [22] T. French, Bisimulation quantifiers for modal logic, Ph.D. thesis, University of Western Australia (2006).
- 1915 [23] D. M. Gabbay, An irreflexivity lemma with applications to axiomatizations of conditions on tense frames, in: U. Mönnich (Ed.), Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy, Springer Netherlands, Dordrecht, 1981, pp. 67–89. doi: 10.1007/978-94-009-8384-7_3. URL http://dx.doi.org/10.1007/978-94-009-8384-7_3
- 1920 [24] J. Garson, Quantification in modal logic, in: D. Gabbay, F. Guenther (Eds.), Handbook of Philosophical Logic, Vol. 3, Reidel, 2001, pp. 267–323.
- [25] J. Halpern, Y. Moses, Knowledge and common knowledge in a distributed environment, Journal of the ACM 37 (3) (1990) 549–587.
- [26] F. van Harmelen, V. Lifschitz, B. Porter, Handbook of Knowledge Representation, Elsevier Science, San Diego, USA, 2007.
- 1925 [27] M. Hennessy, R. Milner, On observing nondeterminism and concurrency, in: Proceedings of the 7th Colloquium on Automata, Languages and Programming, Springer-Verlag, London, UK, UK, 1980, pp. 299–309. URL <http://dl.acm.org/citation.cfm?id=646234.758793>
- 1930 [28] J. Hintikka, Knowledge and belief: an introduction to the logic of the two notions, Contemporary

- philosophy, Cornell University Press, 1962.
 URL <http://books.google.fr/books?id=N280AAAAIAAJ>
- [29] I. Hodkinson, F. Wolter, M. Zakharyashev, Monodic fragments of first-order temporal logics: 2000-2001 a.d., in: R. Nieuwenhuis, A. Voronkov (Eds.), LPAR, Vol. 2250 of Lecture Notes in Computer Science, Springer, 2001, pp. 1–23.
- 1935 [30] W. van der Hoek, J.-J. C. Meyer, Making some issues of implicit knowledge explicit, *International Journal of Foundations of Computer Science* 3 (2) (1992) 193–224.
- [31] W. van der Hoek, M. Pauly, Modal logic for games and information, in: Blackburn et al. [9], pp. 1077–1148.
- 1940 [32] W. van der Hoek, M. Wooldridge, Multi-agent systems, in: F. van Harmelen, V. Lifschitz, B. Porter (Eds.), *Handbook of Knowledge Representation*, Elsevier, 2008, p. 887928.
- [33] W. Holliday, A note on algebraic semantics for s_5 with propositional quantifiers, Tech. rep., UC Berkeley: Group in Logic and the Methodology of Science, to appear in *Notre Dame Journal of Formal Logic* (2017).
 URL <http://escholarship.org/uc/item/303333xr>
- 1945 [34] N. Immerman, *Descriptive Complexity*, Springer-Verlag, 1999.
- [35] D. Janin, I. Walukiewicz, On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic, in: U. Montanari, V. Sassone (Eds.), *Proceedings of the 7th International Conference on Concurrency Theory (CONCUR '96)*, Springer, 1996, pp. 263–277.
 doi:10.1007/3-540-61604-7_60.
 URL http://dx.doi.org/10.1007/3-540-61604-7_60
- 1950 [36] M. Kaminski, M. L. Tiomkin, The expressive power of second-order propositional modal logic., *Notre Dame Journal of Formal Logic* 37 (1) (1996) 35–43.
 URL <http://dblp.uni-trier.de/db/journals/ndjfl/ndjfl137.html#KaminskiT96>
- 1955 [37] D. Kaplan, S_5 with quantifiable propositional variables, *Journal of Symbolic Logic* 35 (1970) 355.
- [38] L. B. Kuijter, Arbitrary arrow update logic with common knowledge is neither re nor co-re, to appear. (2017).
- [39] A. Kuusisto, A modal perspective on monadic second-order alternation hierarchies, in: *Advances in Modal Logic*, 2008, p. 231247.
- 1960 [40] A. Kuusisto, Second-order propositional modal logic and monadic alternation hierarchies, *Annals of Pure and Applied Logic* 166 (1) (2015) 1 – 28. doi:<http://doi.org/10.1016/j.apal.2014.08.003>.
 URL <http://www.sciencedirect.com/science/article/pii/S0168007214000876>
- [41] F. Laroussinie, N. Markey, Quantified CTL: Expressiveness and complexity, *Logical Methods in Computer Science* 10 (4:17). doi:10.2168/LMCS-10(4:17)2014.
 URL <http://www.lsv.ens-cachan.fr/Public/PAPERS/PDF/LM-lmcs14.pdf>
- 1965 [42] L. Libkin, *Elements of Finite Model Theory*, Texts in Theoretical Computer Science. An EATCS Series, Springer Berlin Heidelberg, 2004.
 URL <https://books.google.fr/books?id=zsJ1EK4nK7sC>
- [43] Z. Manna, A. Pnueli, *The Temporal Logic of Reactive and Concurrent Systems*, Vol. 1, Springer-Verlag, 1992.
- 1970 [44] E. Mares, R. Goldblatt, An alternative semantics for quantified relevant logic, *The Journal of Symbolic Logic* 71 (1) (2006) 163–187.
- [45] J.-J. C. Meyer, W. v. d. Hoek, *Epistemic Logic for AI and Computer Science*, Cambridge University Press, New York, NY, USA, 1995.
- 1975 [46] W. Quine, Quantifiers and propositional attitudes, *The Journal of Philosophy* 53 (5) (1956) 177–187.
- [47] F. Roelofsen, Distributed knowledge, *Journal of Applied Non-Classical Logic* 17 (2) (2007) 255–273.
- [48] A. P. Sistla, M. Y. Vardi, P. Wolper, The complementation problem for büchi automata with applications to temporal logic (extended abstract), in: W. Brauer (Ed.), *Automata, Languages and Programming*, 12th Colloquium, Nafplion, Greece, July 15-19, 1985, *Proceedings*, Vol. 194 of Lecture Notes in Computer Science, Springer, 1985, pp. 465–474. doi:10.1007/BFb0015772.
 URL <https://doi.org/10.1007/BFb0015772>
- 1980 [49] A. P. Sistla, *Theoretical issues in the design and verification of distributed systems*, Ph.D. thesis, Carnegie-Mellon University, Cambridge, MA, USA, aAI8403047 (1983).
- 1985 [50] A. M. Turing, On computable numbers, with an application to the entscheidungsproblem, *Proceedings of the London Mathematical Society* s2-42 (1) (1937) 230–265. doi:10.1112/plms/s2-42.1.230.
 URL <http://dx.doi.org/10.1112/plms/s2-42.1.230>
- [51] J. Vaananen, Second-order logic and foundations of mathematics, *The Bulletin of Symbolic Logic*

- 1990 7 (4) (2001) 504–520.
URL <http://www.jstor.org/stable/2687796>
- [52] Y. Venema, Derivation rules as anti-axioms in modal logic, *The Journal of Symbolic Logic* 58 (1993) 1003–1034. doi:10.2307/2275109.
URL http://journals.cambridge.org/article_S0022481200021010
- 1995 [53] F. Wolter, M. Zakharyashev, Axiomatizing the monodic fragment of first-order temporal logic., *Annals of Pure and Applied Logic* 118 (1-2) (2002) 133–145.

Appendix A. Selected Proofs

Lemma 11.

2000 1. Let ϕ be a formula in \mathcal{L}_{sopml}^* and \mathcal{F} a frame in \mathcal{K}_{all} . If assignments V and V' coincide on $fr(\phi)$, then

$$(\mathcal{F}, V, w) \models \phi \quad \text{iff} \quad (\mathcal{F}, V', w) \models \phi$$

2. Recall that $X = \{ap, pl, ml, sopml\}$ and $\sim = \{(ap, all), (pl, bool), (ml, modal), (sopml, full)\}$. Let $x \in X$. Then,

(a) For every $\psi \in \mathcal{L}_x^*$ and model \mathcal{M} over $\mathcal{F} \in \mathcal{K}_{\hat{x}}$, we have $\llbracket \psi \rrbracket_{\mathcal{M}} \in D$.

(b) If $\mathcal{F} \in \mathcal{K}_{\hat{x}}$ and $\psi \in \mathcal{L}_x^*$ is free for p in ϕ , then

$$(\mathcal{F}, V_{\llbracket \psi \rrbracket_{(\mathcal{F}, V)}^p}^p, w) \models \phi \quad \text{iff} \quad (\mathcal{F}, V, w) \models \phi[p/q]$$

2005 **Proof.** The proofs are by induction on the structure of $\phi \in \mathcal{L}_{sopml}^*$.

1. If $\phi = p$, then $fr(\phi) = \{p\}$ and $(\mathcal{M}, w) \models \phi$ iff $w \in V(p) = V'(p)$, iff $(\mathcal{M}', w) \models \phi$. The inductive cases for the propositional connectives are immediate.

2010 If $\phi = \Box_a \psi$, then $(\mathcal{M}, w) \models \phi$ iff for all $w' \in R_a(w)$, $(\mathcal{M}, w') \models \psi$. Since $fr(\phi) = fr(\psi)$, V and V' coincide on $fr(\psi)$ as well, and by induction hypothesis for all $w' \in R_a(w)$, $(\mathcal{M}', w') \models \psi$, i.e., $(\mathcal{M}', w) \models \phi$. The case for $\phi = \Box^* \psi$ is similar.

If $\phi = \forall p \psi$, then $(\mathcal{M}, w) \models \phi$ iff for any $U \in D$, $(\mathcal{M}_U^p, w) \models \psi$. Since $fr(\phi) = fr(\psi) \setminus \{p\}$, V_U^p and $V_U'^p$ coincide on $fr(\psi)$, and by induction hypothesis $(\mathcal{M}'_U, w) \models \psi$. Since U has been chosen arbitrarily, this is the case iff $(\mathcal{M}', w) \models \phi$.

2015 2a The case for $x = ap$ is immediate, as assignments are functions in D . Hence, $V(p) \in D$ for every $p \in AP$.

The case for $x = pl$, follows from equalities $\llbracket \neg \psi \rrbracket = W \setminus \llbracket \psi \rrbracket$, $\llbracket \psi \wedge \psi' \rrbracket = \llbracket \psi \rrbracket \cap \llbracket \psi' \rrbracket$, $\llbracket \psi \vee \psi' \rrbracket = \llbracket \psi \rrbracket \cup \llbracket \psi' \rrbracket$ and the fact that D is a boolean algebra.

As for $x = ml$, notice that $\llbracket \Box_a \psi \rrbracket = [a](\llbracket \psi \rrbracket)$, $\llbracket \Box^* \psi \rrbracket = []^*(\llbracket \psi \rrbracket)$, and D is a boolean algebra closed under operators $[a]$ and $[]^*$.

2020 The case for $x = sopml$, is immediate, as $\llbracket \psi \rrbracket \subseteq W$ for every $\psi \in \mathcal{L}_{sopml}^*$.

2b Let us first consider $x = ap$. If ϕ is an atom r , $(\mathcal{M}_{V(q)}^p, w) \models \phi$ iff $w \in V_{V(q)}^p(r)$, iff $w \in V(r)$ whenever $r \neq p$ or $w \in V(q)$ for $r = p$. In both cases $(\mathcal{M}, w) \models \phi[p/q]$.

The inductive cases for propositional connectives and modal operators are immediate, as these simply commute with substitution.

2025 If $\phi = \forall r \varphi$ for $r \neq p$, then $(\mathcal{M}_{V(q)}^p, w) \models \phi$ iff for every $U \in D$, $((\mathcal{M}_{V(q)}^p)_U^r, w) \models \varphi$. Since $r \neq p$ and q is free for p in ϕ , we have $q \neq r$ and assignment $(V_{V(q)}^p)_U^r$ is equal to $(V_U^r)_{V_U^p(q)}$. As a consequence, we obtain $((\mathcal{M}_U^r)_{V_U^p(q)}^p, w) \models \varphi$, i.e., $(\mathcal{M}_U^r, w) \models \varphi[p/q]$ by induction hypothesis. But this means that $(\mathcal{M}, w) \models \forall r(\varphi[p/q]) = (\forall r \varphi)[p/q]$.

2030 As regards cases $x = pl, ml, sopml$, we make use of item 1. We only prove the inductive step for $\phi = \forall r \varphi$, with $r \neq p$, the other cases being similar to the case for $x = ap$ above. Observe that $(\mathcal{M}_{\llbracket \psi \rrbracket}^p, w) \models \phi$ iff for every $U \in D$, $((\mathcal{M}_{\llbracket \psi \rrbracket}^p)_U^r, w) \models \varphi$.

2035 Since $r \neq p$ and ψ is free for p in ϕ , we have $r \notin fr(\psi)$, and by item 1 above, $[[\psi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}_U^r}$. Therefore assignment $(V_{[[\psi]]}^r)_U^r$ is equal to $(V_U^r)_U^r$. Hence, we obtain $((\mathcal{M}_U^r)_{[[\psi]]_{\mathcal{M}_U^r}}^r, w) \models \varphi$, i.e., $(\mathcal{M}_U^r, w) \models \varphi[p/\psi]$ by induction hypothesis. But this means that $(\mathcal{M}, w) \models \forall r(\varphi[p/\psi]) = (\forall r\varphi)[p/\psi]$.

□

Lemma 20. For every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, world $w \in W$, and formula $\psi \in \mathcal{L}_{sopml}^*$,

$$(\mathcal{M}, w) \models \psi \quad \text{iff} \quad (\mathcal{F}, \rho) \models ST_x(\psi)$$

whenever $\rho(x) = w$ and $\rho(P_i) = V(p_i)$.

2040 **Proof.** The proof is by induction on the structure of ψ . Since the steps for modal logic formulas is common, we only show the case for the quantifier. For $\psi = \forall p\phi$, $(\mathcal{M}, w) \models \psi$ iff for all $U \in D$, $(\mathcal{M}_U^p, w) \models \phi$, that is, $(\mathcal{F}, \rho') \models ST_x(\phi)$ by induction hypothesis, for ρ' that coincides with ρ but $\rho'(P) = U$. However, this means that $(\mathcal{F}, \rho_U^P) \models ST_x(\phi)$, i.e., $(\mathcal{F}, \rho) \models \forall P(ST_x(\phi)) = ST_x(\psi)$. □

2045 **Theorem 23.** For every intended LPML model \mathcal{M} , $w \in \mathcal{M}$, and formula φ in LPML, we have

$$(\mathcal{M}, w) \models \varphi \quad \text{iff} \quad (\mathcal{M}, w) \models t(\varphi)$$

Proof. We will only prove the crucial clause

$$(\mathcal{M}, w) \models \forall \vec{p}\theta(\vec{a}, \vec{p}) \quad \text{iff} \quad (\mathcal{M}, w) \models \boxdot(\vec{a})$$

Before doing that, let us first show what it means for the following specific case: $\theta(a, b, p) = \Box_a p \rightarrow \Box_b p$, $\boxdot(a, b) = Sup(a, b)$, and $\Theta(a, b, x) = \forall y(R_b(x, y) \rightarrow R_a(x, y))$, also written as $R_b(x) \subseteq R_a(x)$.

2050 \Leftarrow Since $(\mathcal{M}, w) \models Sup(a, b)$, we have that $R_b(x) \subseteq R_a(x)$ holds in (\mathcal{M}, w) , and hence in (\mathcal{F}, w) . Since $\Box_a p \rightarrow \Box_b p$ locally defines $R_b(x) \subseteq R_a(x)$, we have $(\mathcal{F}, w) \models (\Box_a p \rightarrow \Box_b p)$, and in particular $(\mathcal{F}, w) \models \forall p(\Box_a p \rightarrow \Box_b p)$. Since $\forall p(\Box_a p \rightarrow \Box_b p)$ is a sentence, we obtain $(\mathcal{M}, w) \models \forall p(\Box_a p \rightarrow \Box_b p)$. \Rightarrow Now suppose that $(\mathcal{M}, w) \not\models Sup(a, b)$. Then $(\mathcal{F}, w) \not\models Sup(a, b)$. Since $\Box_a p \rightarrow \Box_b p$ locally defines $R_b(x) \subseteq R_a(x)$, we know that $(\mathcal{F}, w) \not\models \Box_a p \rightarrow \Box_b p$, and since \mathcal{F} is full, for some assignment V' , we have $(\mathcal{F}, V', w) \not\models \Box_a p \rightarrow \Box_b p$, that is, $(\mathcal{M}, w) \not\models \forall p(\Box_a p \rightarrow \Box_b p)$.

2060 As for the general case: \Leftarrow Since $(\mathcal{M}, w) \models \boxdot(\vec{a})$, we have that $\Theta(\vec{a}, x)$ holds in (\mathcal{M}, w) , and hence in (\mathcal{F}, w) (note that $\Theta \in \mathcal{L}_{fo}^1$ only talks about what is accessible from what). Since $\theta(\vec{a}, \vec{p})$ locally defines $\Theta(\vec{a}, x)$, we have $(\mathcal{F}, w) \models \theta(\vec{a}, \vec{p})$, and in particular $(\mathcal{F}, w) \models \forall \vec{p}\theta(\vec{a}, \vec{p})$. Since $\forall \vec{p}\theta(\vec{a}, \vec{p})$ is a sentence, $(\mathcal{M}, w) \models \forall \vec{p}\theta(\vec{a}, \vec{p})$. \Rightarrow Suppose that $(\mathcal{M}, w) \not\models \boxdot(\vec{a})$. Then, $(\mathcal{F}, w) \not\models \boxdot(\vec{a})$, and therefore $(\mathcal{F}, w) \not\models \Theta(\vec{a}, x)$. Since $\theta(\vec{a}, \vec{p})$ locally defines $\Theta(\vec{a}, x)$, we know that $(\mathcal{F}, w) \not\models \theta(\vec{a}, \vec{p})$, and since \mathcal{F} is full, for some assignment V' , we have $(\mathcal{F}, V', w) \not\models \theta(\vec{a}, \vec{p})$, that is, $(\mathcal{M}, w) \not\models \forall \vec{p}\theta(\vec{a}, \vec{p})$. □

2065 **Lemma 25.** Consider formulas $\varphi_i \in \mathcal{L}_{sopml}$ and $\Theta_i \in \mathcal{L}_{fo}^1$ in Example 24, for $i = 1, \dots, 5$.
 Let x be the only free variable in Θ_i and assume $\rho(x) = w$. Assume \mathcal{F} is a full frame,
 then,

$$(\mathcal{F}, w) \models \varphi_i \quad \text{iff} \quad (\mathcal{F}, \rho) \models \Theta_i$$

Proof. All items are relatively immediate. We only proved the first and the last ones.
 Rather than $(\mathcal{F}, \rho) \models \Theta$, we will also write $\mathcal{F} \models \Theta(w)$, and say that Θ holds for w in \mathcal{F} .

- 2070 1. Suppose \mathcal{F} is full and irreflexive at w , that is, $\neg R_a(w, w)$, then clearly $(\mathcal{F}, w) \models$
 $\exists p(\Box_a p \wedge \neg p)$, by considering the assignment $V(p) = R_a(w)$ for which $w \notin V(p)$. As
 to the converse, suppose that $\mathcal{F}, w \models \exists p(\Box_a p \wedge \neg p)$. Hence, for every model \mathcal{M} on
 \mathcal{F} , $(\mathcal{M}, w) \models \exists p(\Box_a p \wedge \neg p)$, i.e., $(\mathcal{M}, w) \not\models \forall p(\Box_a p \rightarrow p)$. However, by Lemma 20
 below, this is the case iff $R_a(w, w)$ does not hold. Hence, $\Theta_1(w)$ holds in \mathcal{F} .
- 2075 2. Suppose that $\Theta_5(w)$ holds in \mathcal{F} and let V be such that $(\mathcal{F}, V, w) \models \Box_c p$. It is easy
 to check that $(\mathcal{F}, V_{R_a(w)}^q, w) \models \Box_a q \wedge \Box_b(q \rightarrow p)$. In words, if we modify V in such
 a way that q becomes true in exactly w 's a -successors, then for every b -successor
 of w that satisfies q (note that this successor must then also be an a -successor),
 p must be true. Conversely, suppose that $\Theta_5(w)$ does not hold, i.e., for some
 2080 $v \in W$, we have $R_a(w, v)$ and $R_b(w, v)$, but not $R_c(w, v)$. We now show that
 $(\mathcal{F}, w) \models \neg \varphi_5 = \exists p(\Box_c p \wedge \forall q(\Box_a q \rightarrow \Diamond_b(q \wedge \neg p)))$. The assignment V such that
 $V(p) = R_c(w)$ is a witness for this: if p is exactly true in the c -successors of w ,
 then it is false in v , so whenever $\Box_a q$ is true in w , we have that $q \wedge \neg p$ holds in v ,
 and hence $\Diamond_b(q \wedge \neg p)$ holds in w .

2085

□

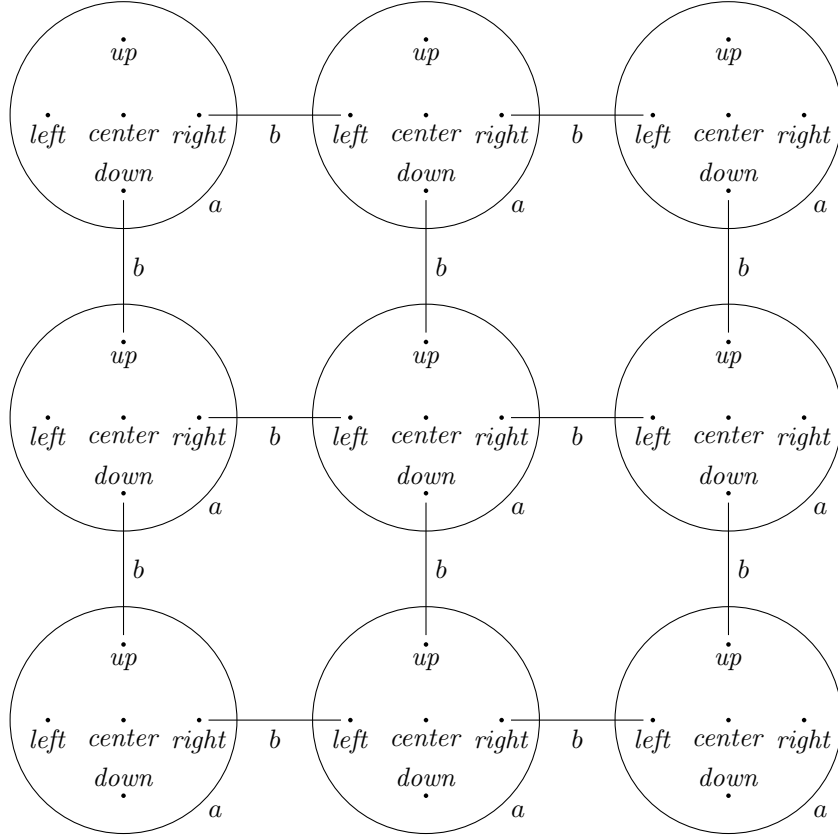
Recall that in Section 4.2, we stated the following theorem without proof:

Theorem 62. If $|I| \geq 2$, then the validities in SOPML* over modal epistemic frames are
 not recursively enumerable.

In particular, SOPML* is not axiomatisable on the class of modal epistemic frames.

2090 Here, we provide the proof. The proof strategy that we use is very similar to the one
 used in the proof of Theorem 57: we define formulas ξ_{grid} , ξ_{sane} and ξ_T that serve the
 same purpose as ψ_{grid} , ψ_{sane} and ψ_T , respectively.

2095 The main difference lies in how we define a grid, now that we use two-agent S5 as
 opposed to single agent K. This time, we use the following pattern: each point $(n, m) \in$
 $\mathbb{Z} \times \mathbb{Z}$ is represented not by a single world, but instead by (at least) five different worlds
 that are related to each other by the relation $R(a)$. One of these five worlds satisfies
 the propositional atom *center*, the other four worlds satisfy the atoms *left*, *right*, *up* and
down, respectively. The *left* world of (n, m) is then related by $R(b)$ to the *right* world
 of $(n - 1, m)$, and similarly for the other directions; see also the following diagram.



2100

As before, this allows us to define \Box_x and \Diamond_x (for $x \in dir$) as abbreviations:

$$\begin{aligned}\Box_x \phi &:= \Box_a(x \rightarrow \Box_b(\neg x \rightarrow \Box_a(center \rightarrow \phi))) \\ \Diamond_x \phi &:= \Diamond_a(x \wedge \Diamond_b(\neg x \wedge \Diamond_a(center \wedge \phi)))\end{aligned}$$

Then we define a grid in almost exactly the same way as the K -case:

$$\begin{aligned}
\xi_{grid} &:= \Box^* (labels \wedge direction \wedge same \wedge remain \wedge \bigwedge_{x \in dir} unique_x \wedge \\
&\quad \bigwedge_{(x,y) \in inv_dir} inverse_{xy} \wedge \bigwedge_{(x,y) \in perp_dir} commute_{xy}) \\
labels &:= \bigvee_{i \in L} (i \wedge \bigwedge_{j \in L \setminus \{i\}} \neg j) \wedge \bigwedge_{l \in L} \Diamond_a l \\
directions &:= ((left \vee right) \rightarrow (\Diamond_b left \wedge \Diamond_b right \wedge \Box_b (left \vee right))) \wedge \\
&\quad ((up \vee down) \rightarrow (\Diamond_b up \wedge \Diamond_b down \wedge \Box_b (up \vee down))) \\
same &:= \bigwedge_{l \in L} \forall p ((l \wedge p) \rightarrow (\Box_a (l \rightarrow p) \wedge \Box_b (l \rightarrow p))) \\
remain &:= \bigwedge_{c \in Ag \setminus \{a,b\}} \forall p (p \leftrightarrow \Box_c p) \\
unique_x &:= \forall p (\Diamond_x p \rightarrow \Box_x p) \\
inverse_{xy} &:= \forall p (p \rightarrow \Box_x \Box_y p) \\
commute_{xy} &:= \forall p (\Diamond_x \Diamond_y p \rightarrow \Box_y \Box_x p)
\end{aligned}$$

where dir , inv_dir and $perp_dir$ are as before, and $L := \{left, right, up, down, center\}$.

Note that $unique_x$, $inverse_{xy}$ and $commute_{xy}$ are identical to their counterparts in the proof of Theorem 57, but the other subformulas are different.

2105 If $\mathcal{M}, w_0 \models \xi_{grid}$, then we associate the worlds of \mathcal{M} with the points of $\mathbb{Z} \times \mathbb{Z}$ in the following way: (i) w_0 and all its a -successors represent $(0, 0)$, (ii) if w represents (n, m) , $\mathcal{M}, w \models left$, w' is a b -successor of w and $\mathcal{M}, w' \models right$, then w' represents $(n - 1, m)$, (iii) similarly for the other directions and (iv) if w represents (n, m) then so does every c -successor of (n, m) for every $c \notin \{a, b\}$.

2110 As before, we show that every point (n, m) is represented by at least one world and that each world represents at least one point. Since every point is represented by at least one $center$ world, one $right$ world, one $left$ world, one up world and one $down$ world, we obviously cannot guarantee that all the worlds representing a point are indistinguishable. We will show, however, that all $center$ worlds representing (n, m) are modally indistinguishable from one another.

- 2115 • $labels$ guarantees that every world (i) satisfies exactly one of the labels from L and (ii) has a successor that satisfies l for every $l \in L$.
- $directions$ guarantees that every $left$ world is paired with a $right$ world through the relation $R(b)$, and similarly for the other directions. Due to how we defined \Diamond_x and \Box_x as abbreviations, this means that every $center$ world has an x -successor that is also a $center$ world, for every $x \in dir$. That, in turn, implies that every point 2120 (n, m) is represented by at least one world.
- $same$ says that for every $U \in D$, if w is in U then so are all of its a - and b -successors that share the same label. Since we are working with a modal domain of quantification, this implies that every label is unique (up to modal indistinguishability) in 2125 its a - and b -equivalence classes. Note that, together with the fact that $left, right, up$

and *down* worlds occur only in pairs, this implies that whenever a world w represents some point (n, m) , then every a - or b -successor of w also represents some point (n', m') . So every world in the generated submodel represents some point (n, m) .

- 2130 • *remain* says that for any agent c other than a and b , w is modally indistinguishable from its c -successors. This implies that if w and w' represent the same point (n, m) due to rule (iv), then w and w' are modally indistinguishable.
- The formulas *unique_x*, *inverse_{xy}* and *commute_{xy}*, for the relevant x and y , guarantee that if w and w' represent the same point (n, m) due to rules (ii) and (iii),
2135 then w and w' are modally indistinguishable.

We have now shown that every point (n, m) is represented, that every world represents a point and that all worlds representing a single point are modally indistinguishable.

Variants ξ_{sane} and ξ_T of ψ_{sane} and ψ_T can then be defined. The only required modification is that in ξ_{sane} and ξ_T we only put requirements on *center* world, e.g.,
2140 the subformula $\bigvee_{s \in states} (s \wedge \bigwedge_{s' \in states \setminus \{s\}} \neg s')$ of ψ_{sane} should be replaced with $center \rightarrow \bigvee_{s \in states} (s \wedge \bigwedge_{s' \in states \setminus \{s\}} \neg s')$ in ξ_{sane} . Since these modifications are rather trivial, we do not list them here in detail.

Overall, if we define $\zeta_T := \xi_{grid} \wedge \xi_{sane} \wedge \xi_T \wedge s_0 \wedge pos$, then $\mathcal{M}, w \models \zeta_T$ implies that \mathcal{M} encodes the execution of T . It follows that T is non-halting if and only if $\mathcal{M}, w \models \zeta_T \rightarrow$
2145 $\Box^*(center \rightarrow \neg s_{end})$. In particular, this implies that the valid formulas of SOPML* over modal S5 frames are not recursively enumerable.

Lemma 87.

1. It can be checked in NPTIME whether there exists a bisimulation relation between two frames.
- 2150 2. Let \mathcal{F} and \mathcal{F}' be frames and let w and w' be worlds of \mathcal{F} and \mathcal{F}' , respectively, such that $w \approx w'$. Then \mathcal{F}_w is boolean (respectively modal) iff $\mathcal{F}'_{w'}$ is. However, if \mathcal{F}_w is full, then $\mathcal{F}'_{w'}$ need not be full.
Moreover, if \mathcal{F}_w and $\mathcal{F}'_{w'}$ are full, then they are isomorphic.

Proof. We provide a proof of point 2. Suppose that $w \approx w'$, so there is some
2155 bisimulation (ω, Ω) such that $\omega(w, w')$. Note that it follows that for every world w_1 of \mathcal{F}_w there is a world w'_1 of $\mathcal{F}'_{w'}$ such that $\omega(w_1, w'_1)$, and vice versa.

Now, suppose that $D'_{w'}$ is closed under complement, and take any $U \in D_w$. There is at least one $U' \in D'_{w'}$ such that $\Omega(U, U')$. Furthermore, since $D'_{w'}$ is closed under complement, we have $W'_{w'} \setminus U' \in D'_{w'}$. There is also at least one $T \in D_w$ such that
2160 $\Omega(T, W'_{w'} \setminus U')$. Now, take any $w_1 \in W_w$. As noted above, there is at least one $w'_1 \in W'_{w'}$ such that $\omega(w_1, w'_1)$. Because (ω, Ω) is a bisimulation, we have $w_1 \in U$ iff $w'_1 \in U'$ and $w_1 \in T$ iff $w'_1 \in W'_{w'} \setminus U'$. Exactly one of $w'_1 \in U'$ and $w'_1 \in W'_{w'} \setminus U'$ are true, so we also have that exactly one of $w_1 \in U$ and $w_1 \in T$ is true. So $W_w \setminus U = T \in D_w$.

We have shown that if $D'_{w'}$ is closed under complement, then so is D_w . It can be
2165 shown in a similar way that if $D'_{w'}$ is closed under intersection or under $[a]$ then so is D_w . So \mathcal{F}_w is boolean (respectively modal) if $\mathcal{F}'_{w'}$ is. Since bisimilarity is symmetrical, the reverse holds as well.

In order to see that \mathcal{F}_w can be full without $\mathcal{F}_{w'}$ being full, consider the frame \mathcal{F} given by $W = \{w\}, R_a = W \times W, D = 2^W$ and the frame \mathcal{F}' given by $W' = \{w', v'\}, R'_a = W' \times W'$ and $D' = \{\emptyset, W'\}$. We have $w \approx w'$, as witnessed by $\omega = \{(w, w'), (w, v')\}$ and $\Omega = \{(\emptyset, \emptyset), (W, W')\}$. Furthermore, both frames are identical to their respective generated submodels. Yet \mathcal{F} is full while \mathcal{F}' is not.

Finally, suppose that \mathcal{F}_w and $\mathcal{F}_{w'}$ are both full. Suppose that for some $w \in W_w$ we have $\omega(w, w'_1)$ and $\omega(w, w'_2)$. Let $U \in D_w$ be such that $\Omega(U, \{w'_1\})$. Then $w'_1 \in \{w'_1\}$ iff $w_1 \in U$ iff $w'_2 \in \{w'_1\}$, which implies that $w'_1 = w'_2$. So ω is injective. Furthermore, as noted above, for every w_1 there is a w'_1 such that $\omega(w_1, w'_1)$ and vice versa, so ω is surjective. It follows that ω is an isomorphism. \square

Theorem 74. *If $w \approx w'$, then for every formula $\varphi \in \mathcal{L}_{sopml}^*$,*

$$(\mathcal{F}, w) \models \varphi \quad \text{iff} \quad (\mathcal{F}', w') \models \varphi.$$

Proof. We prove the implication from right to left, the opposite direction being symmetric. If $w \approx w'$ then $\omega(w, w')$ holds for some bisimulation pair (ω, Ω) . As above, we show by induction on φ that if $(\mathcal{F}, V, w) \not\models \varphi$ for some assignment V , then $(\mathcal{F}', \Omega(V), w') \not\models \varphi$, where $\Omega(V)$ is any assignment such that for every $p \in AP$, $(\Omega(V))(p) = U'$ with $\Omega(V(p), U')$. Since ω is a simulation relation in particular, the base cases for $\varphi = p$ and $\varphi = \neg p$ are as in Theorem 67, as well as the inductive cases for propositional connectives and $\varphi = \Box_a \psi$, $\varphi = \Box^* \psi$, and $\varphi = \forall p \psi$.

For $\varphi = \Diamond_a \psi$, $(\mathcal{F}', \Omega(V), w') \models \varphi$ iff for some $v' \in R'_a(w')$, $(\mathcal{F}', \Omega(V), v') \models \psi$. By bisimulation, for some $v \in R_a(w)$, $\omega(v, v')$. In particular, $(\mathcal{F}, V, v) \models \psi$ by induction hypothesis. That is, $(\mathcal{F}, V, w) \models \varphi$. The case for $\varphi = \Diamond^* \psi$ is similar.

For $\varphi = \exists p \psi$, $(\mathcal{F}', \Omega(V), w') \models \varphi$ iff for some $U' \in D'$, $(\mathcal{F}', (\Omega(V))_{U'}^p, w') \models \psi$. Now consider $U \in D$ such that $\Omega(U, U') \in D$. In particular, assignments $(\Omega(V))_{U'}^p$ and $\Omega(V_U^p)$ coincides. Hence, $(\mathcal{F}', \Omega(V_U^p), w') \models \psi$, and by induction hypothesis, $(\mathcal{F}, V_U^p, w) \models \psi$ for $U \in D$, that is, $(\mathcal{F}, V, w) \models \varphi$. \square