Spindown Polyhedra

ANTHONY F. CONSTANTINIDES and GEORGE A. CONSTANTINIDES

ARTICLE HISTORY
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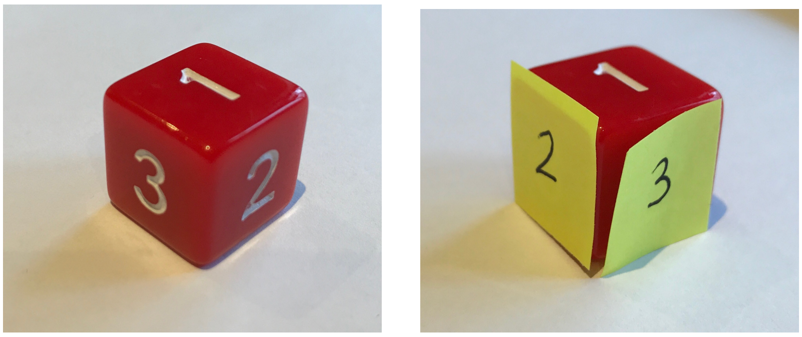
1. Introduction

Magic: The Gathering is a trading card game published by Wizards of the Coast [1]. The aim of most variants of the game is to reduce your opponent's life total from twenty to zero, thus winning the game. As it may take several turns to reduce a player's life total to zero, players need a mechanism to keep track of their current life total. For this purpose, players often use a device called a *spindown life counter*, shown in Figure 1. A spindown life counter is an icosahedron with a special labelling of the faces, such that - starting with 20 life total - a player can reduce their life total in decrements of one by rolling the icosahedron onto an adjacent face each time. A spindown life counter appears similar to a standard icosahedral die, known in gaming as a *d20*, however, the labelling of faces is different, as also shown in Figure 1.

Figure 1. Left: A spindown life counter from Magic: The Gathering. Right: A standard icosahedral die (d20).

The first author posed the question of whether it is possible to construct other polyhedra having a similar 'spindown' property. Some simple experimentation revealed that is it possible to re-label the faces of a standard six-faced die (d6) to produce a spindown cube (see Figure 2); further experimentation revealed this to be the case for all Platonic solids. It can also be seen that it is possible, in all these cases, to chose the labelling such that it is not only spindown but also the face with the lowest label adjacent to the face with the highest label. For example, it is possible to label the faces of a cube such that producing the sequence of faces 6, 5, 4, 3, 2, 1, 6 involves only a single roll to an adjacent face at each step. We refer to such a labelling as 'spinround'.

This observation leads naturally to the more general questions of whether all convex polyhedra can be labelled in a spinround manner and (if not) whether all convex polyhedra can be labelled in a spindown manner. It is the *convex* polyhedra that are



(a) standard labelling - not spindown (b) minimal relabelling - spindown

Figure 2. A standard six-faced die (d6) can be relabelled in a spindown manner.

the main objects of study, because to be meaningful as a spindown life counter, each face must be able to lie on a flat surface (such as a table) such that the entire face is in contact with the table, with all remaining faces above the surface of the table.

To formalise these questions, we provide some definitions.

Definition 1 (Spindown Polyhedron)**.** A *spindown polyhedron* with n faces is a convex polyhedron for which there exists a one-to-one and onto labelling of faces with the numbers $1, 2, \ldots n$ such that the face labelled i is adjacent to the face labelled $i-1$ for all i between 2 and n.

Definition 2 (Spinround Polyhedron)**.** A *spinround polyhedron* with n faces is a spindown polyhedron with the additional property that the face labelled n is adjacent to the face labelled 1.

In this article, we demonstrate that there exist non-spindown polyhedra, for example the cuboctahedron. We also show that there is a non-spinround (but spindown) polyhedron with fewer faces which, to the best of our knowledge, does not have a special name in the literature. We have been able to construct the latter polyhedron - both mathematically and as a 3D printed realisation - which we refer to as a Lich's Nemesis, named after the in-game properties of the card Lich's Mirror in Magic: The Gathering [2].

2. Face Adjacency

It is clear that whether a polyhedron is spindown or spinround is entirely determined by the adjacency relationship between polyhedron faces. We therefore capture this relationship as a graph.

Definition 3. The face adjacency graph of a polyhedron $\mathcal{P}$ is an (undirected) graph with vertices in one-to-one correspondence with the faces of $\mathcal{P}$ and with edges corresponding to adjacency between faces in $\mathcal{P}$.

It is helpful to introduce some standard graph-theoretic definitions:

Definition 4. A *Hamiltonian path* is a path between two vertices of a graph that visits each vertex exactly once. A graph is called *traceable* if it has a Hamiltonian path.

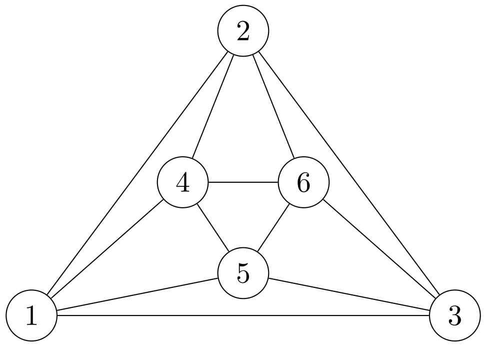


Figure 3. Face adjacency graph of a standard 6-faced die (d6).

Definition 5. A *Hamiltonian cycle* is a cycle (*i.e.* a closed loop) through a graph that visits each vertex exactly once. If there exists a Hamiltonian cycle on a graph, then the graph is called *Hamiltonian.*

Lemma 1 follows directly from the definitions given so far, providing a graph-theoretic characterisation of spindown and spinround polyhedra.

Lemma 1. *A polyhedron is spindown iff its face adjacency graph is traceable. A polyhedron is spinround iff its face adjacency graph is Hamiltonian.*

We may now re-cast our questions in graph-theoretic terms:

(1) Is there a polyhedron whose face adjacency graph is non-Hamiltonian? (This would be a non-spinround polyhedron).
(2) If so, is there also a polyhedron whose face adjacency graph is non-traceable? (This would be a non-spindown polyhedron).

3. Polyhedral Graphs

In order to begin to answer these questions, we need to have a precise understanding of exactly which undirected graphs are face adjacency graphs of polyhedra. We may do this via a result known as Steinitz's Theorem, which we will state after first introducing the standard concepts of the graph of a polyhedron and 3-connectedness of a graph.

Definition 6. The *graph* of a polyhedron $\mathcal{P}$ is the graph whose vertices and edges are in one-to-one correspondence with the vertices and edges of $\mathcal{P}$, respectively.

Definition 7. A graph is said to be *3-connected* if there is no set of two vertices whose removal disconnects the graph into disjoint portions.

An example of the graph of a cube is shown in Figure 4.

Theorem 1 (Steinitz's Theorem, [3])**.** *A graph G is the graph of a convex three-dimensional polyhedron, if and only if G is planar and 3-connected.*

Due to the correspondence in Steinitz's theorem, a graph G is referred to as *polyhedral* if and only if it is both 3-connected and planar. It remains, then, to relate face adjacency graphs to graphs of polyhedra. This relationship is known as duality, and

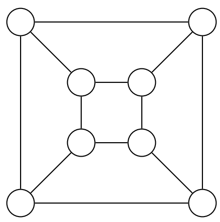


Figure 4. Graph of a cube.

is well-defined for polyhedral graphs, in the sense that every polyhedral graph G has a unique polyhedral dual G' in which the vertices of G' correspond to the faces of G and the edges of G' correspond to the face adjacency relation on the vertices of G, and moreover G is the dual of G' as well [4]. We summarise the implications of this relationship as Lemma 2.

Lemma 2. *A non-traceable polyhedral graph is the dual of the graph of a non-spindown polyhedron. A non-Hamiltonian polyhedral graph is the dual of the graph of a non-spindown polyhedron.*

Rosenthal [5] proved that the graph of a rhombic dodecahedron, shown in Figure 5, is non-traceable. The dual polyhedron of the rhombic dodecahedron is the cuboctahedron [6]. Therefore we may conclude that *the cuboctahedron is not spindown*, answering our two questions.

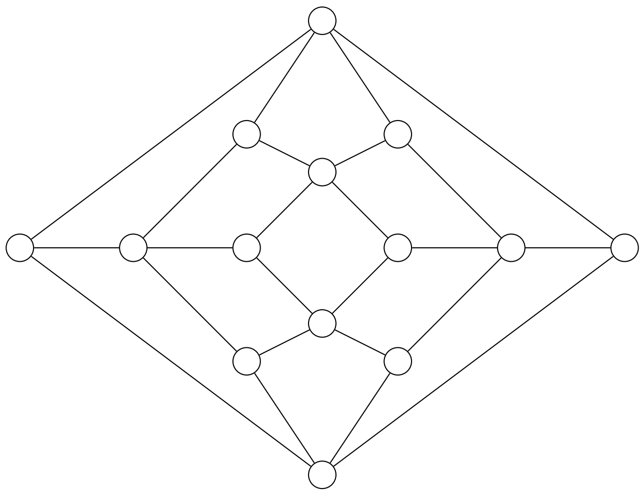
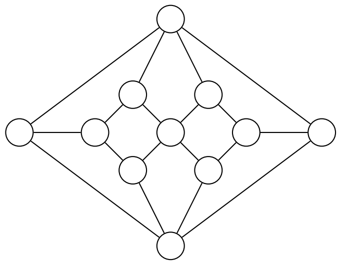
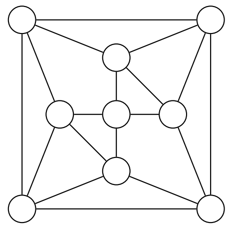


Figure 5. The face adjacency graph of a cuboctahedron (also the graph of a rhombic dodecahedron) is not traceable.

We may also use Lemma 2 to address the question of whether there may be a *smaller* non-spinround polyhedron which is nevertheless spindown. It is known that the Herschel graph, shown in Figure 6(a) is a polyhedral graph with the smallest number of vertices not to admit a Hamiltonian cycle [7]. And so it follows that the dual of the Herschel graph, shown in Figure 6(b) is a graph of a non-spinround polyhedron with the fewest faces. This polyhedron is, however, spindown.



(a) The Herschel Graph



(b) Its dual

Figure 6. The Herschel graph and its dual, the graph of a non-spinround polyhedron with the fewest faces.

4. Realisation as a Polyhedron

The previous section has shown that there exists a non-spindown polyhedron, whose graph is the dual of the Herschel Graph. In this section, we construct one such polyhedron.

Of all the polyhedra with a given graph, there exists a *canonical* one such that all edges are tangent to a unit sphere and where the centre of gravity of the tangent points is the origin [8]. Hart [9] provides an explicit procedure to construct such a canonical polyhedron; the procedure takes an initial set of vertex positions in three-dimensional space, together with a description of the faces as lists of vertices in clockwise or counter-clockwise order, and - if it converges - produces the canonical polyhedron with the same topological structure as the initial polyhedron.

We implemented this approach in Mathematica (a free-to-use computable document format is available at `http://bit.ly/lichnem`), using an arbitrary initialisation of vertex locations derived by imagining the x and y coordinates to be monotonic in the x and y coordinates of the diagram of Figure 6(b) while the z coordinate of each vertex to be rising from left-edge to centre and then falling from centre to right-edge of the same diagram. Hart's algorithm converges, producing the polyhedron shown in Figure 7. To the best of our knowledge, neither the dual of the Herschel graph nor this canonical polyhedron has a specific name in the literature; we refer to it as the *Lich's Nemesis* polyhedron, named after a card in Magic: The Gathering that allows - under certain circumstances - life total to wrap round from one to the starting life total of 20 [2].

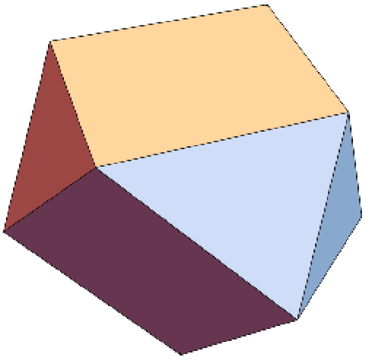
We have constructed a physical realisation of the Lich's Nemesis, using the Prusa i3 Mk2 3D printer. Interested readers wanting to print their own Lich's Nemesis can download the 3D-printer 'stl file' at `http://bit.ly/lichstl`.

5. Conclusion and Future Work

We can summarise the results of this article as a theorem:

Theorem 2. *The cuboctahedron is a non-spindown polyhedron. The Lich's Nemesis polyhedron is a non-spinround polyhedron with the fewest faces.*

Inspired by the game Magic: The Gathering, we initially identified that all Platonic solids are spinround, an experimental result which, in the light of the results in this article, we may now understand follows naturally from the fact that the set of Platonic solids is closed under duality, combined with the fact that the graph of all Platonic



(a) 3D graphic

(b) 3D printed realisation

Figure 7. The Lich's Nemesis Polyhedron. An interactive version can be downloaded from `http://bit.ly/lichnem`, allowing the reader to see the polyhedron from all angles.

solids is Hamiltonian.

We have given examples of polyhedra that are spinround (the cube), non-spinround but spindown (the Lich's Nemesis), and non-spindown (the cuboctahedron).

We have leveraged the knowledge that the Herschel graph has the smallest number of vertices for a non-Hamiltonian graph, when concluding that the Lich's Nemesis is a non-spinround polyhedron with the fewest faces. However, we are not aware of the status of the cuboctahedron. It has 14 faces: is this an example of a non-spindown polyhedron with the fewest faces? Due to the large search space, answering this question requires efficient search methods.

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ANTHONY F. CONSTANTINIDES is a Year 7 secondary school pupil.

GEORGE A. CONSTANTINIDES is Professor of Digital Computation at Imperial College London.

Address: EEE Department, Imperial College London, Exhibition Road, London SW7 2AZ.

Email: *g.constantinides@imperial.ac.uk.*