Adaptive IDA-PBC for Underactuated Mechanical Systems with Constant Disturbances

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Abstract: This work investigates the control of nonlinear underactuated mechanical systems with matched and unmatched constant disturbances. To this end, a new control strategy is proposed which builds upon the interconnection-and-damping-assignment passivity-based-control (IDA-PBC), augmenting it with an additional term for the purpose of disturbance compensation. In particular, the disturbances are estimated adaptively and then accounted for in the control law employing a new matching condition of algebraic nature. Stability conditions are discussed and, for comparison purposes, an alternative controller based on partial feedback-linearization is presented. The effectiveness of the proposed approach is demonstrated with numerical simulations for three motivating examples: the inertia-wheel pendulum; the disk-on-disk system; the pendulum-on-cart.

Keywords: Adaptive Control; Underactuated Mechanical Systems; Nonlinear Control; Matched and Unmatched Disturbances.

1. Introduction

Control of underactuated mechanical systems represents a challenging engineering problem which has motivated a vast body of research in recent years. The most notable results include the method of Controlled Lagrangians (CL) based on the Euler-Lagrange formulation [1], [2], and the interconnection-and-damping-assignment passivity-based control (IDA-PBC) for Port-controlled Hamiltonian (PCH) systems [3]. Although these approaches have proved effective for a wide range of applications [4] including discrete-time systems [5], most studies have been focussing on ideal systems free from uncertainties and disturbances [6]. However, the practical importance of disturbances in real systems has recently been attracting increasing interest within the research community. Important results in this sense include the investigation of linear viscous friction within CL [7], [8]. In parallel, the effect of continuous and smooth physical dissipation within IDA-PBC was studied in [9], [10], while Dahl friction on actuated joints was considered in [11]. Recently, an IDA-PBC design with adaptive friction compensation was proposed in [12]. Besides IDA-PBC, a sliding-mode-control formulation was proposed in [13] for underactuated mechanical systems with Coulomb friction of known amplitude. Furthermore, an H-infinity control was proposed for the inertia-wheel pendulum with bounded disturbances in [14] and an adaptive neural network control was employed to compensate friction forces for the Furuta pendulum in [15].

In summary, most research on nonlinear underactuated mechanical systems with disturbances has been focusing on friction, while the constant disturbance rejection problem has remained relatively unexplored. Initial results in this respect were presented in [16], [17], where integral control was overlaid to IDA-PBC in order to compensate constant matched disturbances (i.e. only affecting the actuated joints). More sophisticated integral IDA-PBC designs for underactuated mechanical systems with matched disturbances, constant or bounded, were proposed in [18], [19]. The extension of integral IDA-PBC to the case of unmatched disturbances for mechanical systems

with constant inertia matrix was investigated in [20]. In this respect, it must be highlighted that unmatched disturbances represent a particularly challenging condition even for linear systems, since control input and disturbances are not linearly dependent [21], [22]. Additionally, differently from friction, constant disturbances do not vanish at equilibrium. As a result, the control of underactuated mechanical systems with constant unmatched disturbances remains an active area of research.

In this work the control of nonlinear underactuated mechanical systems with constant matched and unmatched disturbances is investigated. The main contribution is a new IDA-PBC design which includes a disturbance-compensation term resulting from a new matching condition of algebraic nature. Additionally, the disturbances are estimated adaptively hence no prior knowledge of their value is required. Stability conditions are discussed and, for comparison purposes, an alternative solution based on partial feedback-linearization is also presented. Differently from other approaches, the proposed method is also applicable to systems with non-constant inertia matrix and non-vanishing unmatched disturbances. The effectiveness of the new control is demonstrated with simulations for three motivating examples: the inertia-wheel pendulum, the disk-on-disk system, and the pendulum-on-cart system.

The rest of the paper is organised as follows: Section 2 contains a brief review of IDA-PBC, while Section 3 outlines the problem formulation. Section 4 presents the new IDA-PBC design, discusses stability conditions, and briefly outlines an alternative approach based on partial feedback-linearization. Section 5 presents simulation results for the inertia-wheel pendulum, the disk-on-disk system, and the pendulum-on-cart. Section 6 contains concluding remarks and suggestions for future work.

2. Overview of IDA-PBC

Consider an underactuated mechanical system with position $q \in \mathbb{R}^n$, momenta $p = M\dot{q} \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and mapping $G(q) \in \mathbb{R}^{n\times m}$, with $\mathrm{rank}(G) = m < n$. Define the open-loop Hamiltonian $H = T(q,p) + V(q)$, where $T(q,p) = \frac{1}{2}p^T M^{-1} p$ is the kinetic energy, $M(q)$ is the positive definite and invertible inertia matrix, and $V(q)$ is the open-loop potential energy. The open-loop system dynamics is defined as follows [23]:

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I^n\\ -I^n & 0\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix}u \tag{1}$$

The $n \times n$ identity matrix is indicated with I^n, the symbol $\nabla_q(\cdot)$ represents the vector of partial derivatives in q, the symbol $\nabla_p(\cdot)$ represents the vector of partial derivatives in p, and the dependency on q, p is omitted for brevity. The control aim typically corresponds to stabilising the equilibrium $(q,p) = (q^*, 0)$, which is unstable in open-loop and satisfies the condition $\nabla_q V(q^*) = 0$. To this end, the IDA-PBC control is constructed to achieve the following closed-loop dynamics [3], where $H_d = \frac{1}{2}p^T M_d^{-1} p + V_d$ is the closed-loop Hamiltonian and $q^* = \mathrm{argmin}(V_d)$ corresponds to a strict minimum of the closed-loop potential energy V_d.

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d\\ -M_d M^{-1} & J_2 - G k_v G^T\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix} \tag{2}$$

The parameter $M_d = M_d^T > 0$ is the closed-loop inertia matrix, $J_2 = -J_2^T$ is a free-parameter matrix typically defined as a linear function of the momenta, and $k_v = k_v^T > 0$ is a constant gain matrix.

Introducing $G^{\dagger} = (G^T G)^{-1} G^T$, the IDA-PBC control law that achieves the closed-loop dynamics (2) is the sum of an energy-shaping component u_{es}, which assigns the closed-loop equilibrium q^*, and of a damping-assignment component u_{di}, which injects damping in the system:

$$\begin{aligned} u &= u_{es} + u_{di} \\ u_{es} &= G^{\dagger}\left(\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p\right) \\ u_{di} &= -k_v G^T \nabla_p H_d \end{aligned} \tag{3}$$

The matrix M_d and the potential energy V_d, which are defined as part of the IDA-PBC design, should satisfy the following partial-differential-equations (PDE), termed kinetic-energy PDE and potential-energy PDE, where $G^{\perp}$ is a full-rank left annihilator of G so that $G^{\perp} G = 0$:

$$G^{\perp}\left(\nabla_q\left(\frac{1}{2} p^T M^{-1} p\right) - M_d M^{-1} \nabla_q\left(\frac{1}{2} p^T M_d^{-1} p\right) + J_2 M_d^{-1} p\right) = 0 \tag{4}$$

$$G^{\perp}\left(\nabla_q V - M_d M^{-1}\left(\nabla_q V_d\right)\right) = 0 \tag{5}$$

If (4),(5) are satisfied $\forall (q,p) \in \mathbb{R}^{2n}$, control (3) in closed-loop with (1) achieves stability of the equilibrium $(q,p) = (q^*, 0)$, and if $\nabla_q V_d(q^*) = 0$ and $\nabla_q^2 V_d(q^*) > 0$ the equilibrium is a strict-minimum of V_d. Finally, asymptotic stability is concluded if the output $y = G^T \nabla_p H_d$ is detectable.

3. Problem Formulation

In this section an underactuated mechanical systems with disturbances $\delta \in \mathbb{R}^n$ is considered:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I^n \\ -I^n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u - \begin{bmatrix} 0 \\ \delta \end{bmatrix} \tag{6}$$

The following assumptions are introduced and briefly discussed.

Assumption 1: The matching conditions (4),(5) are satisfied $\forall (q,p) \in \mathbb{R}^{2n}$ resulting in a stabilising controller (3) for system (1), and the output $y = G^T \nabla_p H_d$ is detectable, as defined in [3].

Assumption 2: The disturbance $\delta \in \mathbb{R}^n$ is constant but unknown.

Assumption 3: There exists an assignable equilibrium q^* which is unstable in open-loop and satisfies the following conditions:

$$G^{\perp}\left(\nabla_q V(q^*) + \delta\right) = 0 \tag{7}$$

$$\nabla_q^2 V(q^*) \neq 0 \tag{8}$$

While *Assumption 1* is typically verified by classical examples, the solution of the PDE (4),(5) can be a limiting factor in practical applications and has been the subject of extensive research [24], [25]. *Assumption 2* restricts the study to constant disturbance, which are the focus of this work. This is a specific class of disturbances that has practical relevance for underactuated systems [18]. The case of varying matched disturbances within the context of discrete-time systems was studied in [26], while the extension to varying unmatched disturbances is part of future work. *Assumption 3* requires the existence of an open-loop unstable equilibrium, which is a common prerequisite for IDA-PBC. In particular, computing (6) at the equilibrium $(q,p) = (q^*, 0)$ and pre-multiplying by $G^{\perp}$ recovers (7), while open-loop instability implies (8). Notably, q^* depends on the unmatched

disturbance $G^{\perp}\delta$ and on the mechanical structure of the system according to (7). If $G^{\perp}\nabla_q V$ is bounded, (7) imposes a bound on the maximum value of the unmatched disturbance that can be compensated. This limitation also applies to the baseline IDA-PBC (3) and to the integral IDA-PBC designs [17], [18] (ref. Section 5).

Finally, the control aim for system (6) under *Assumption 1-3* corresponds to stabilising the equilibrium $(q, p) = (q^*, 0)$.

4. Main Result

The proposed control strategy entails two main steps: I) the adaptive estimation of the disturbances; II) the inclusion of a disturbance-compensation term within the IDA-PBC (3). The first step is addressed in the following proposition which employs the Immersion & Invariance method [27], [28].

Proposition 1: Consider system (6) under *Assumption 1-2* and define the vector of estimation errors $z \in \mathbb{R}^n$ as:

$$z = \tilde{\delta} - \delta = \hat{\delta} + \beta(p) - \delta \tag{9}$$

where the disturbance estimate is $\tilde{\delta} = \hat{\delta} + \beta(p)$ and $\tilde{\delta}, \hat{\delta}, \beta(p) \in \mathbb{R}^n$. In particular, the functions $\hat{\delta}$ and $\beta(p)$ are the state-independent part and the state-dependent part of the disturbance estimate $\tilde{\delta}$. The estimation errors z are bounded and converge to zero employing the following adaptation law with $\alpha > 0$:

$$\begin{gathered} \dot{\hat{\delta}} = -\nabla_p \beta^T \left(-\nabla_q H + Gu - \hat{\delta} - \beta\right) \\ \beta = -\alpha I^n p \end{gathered} \tag{10}$$

Proof: Computing the time-derivative of (9) and substituting (6),(9) we obtain:

$$\dot{z} = \dot{\hat{\delta}} + \nabla_p \beta^T (-\nabla_q H + Gu - \hat{\delta} - \beta + z) \tag{11}$$

Substituting (10) into (11) gives:

$$\dot{z} = -\alpha I^n z \tag{12}$$

Choosing the Lyapunov function candidate $W = \frac{1}{2} z^T I^n z$, computing its time-derivative and substituting (12) gives:

$$\dot{W} = z^T \dot{z} = -\alpha\left(z^T I^n z\right) \leq 0 \tag{13}$$

As a result, z is bounded and converges to zero exponentially concluding the proof ■

While δ is unknown, computing the estimate $\tilde{\delta}$ allows verifying that *Assumption 3* holds for the assignable equilibrium q^* (ref. Section 5). To construct the new control law for system (6) we introduce the disturbance estimate $\tilde{\delta}$ in the potential-energy PDE (5) as follows, where V_d' is the new closed-loop potential energy:

$$G^{\perp}\left(\nabla_q V + \tilde{\delta} - M_d M^{-1}\left(\nabla_q V_d'\right)\right) = 0 \tag{14}$$

Considering that under *Assumption 1* there exist a solution V_d' of (14), without loss of generality we introduce the term $\Lambda(q) = \nabla_q V_d' - \nabla_q V_d$. Subtracting (5) from (14) and substituting $\Lambda(q)$ gives:

$$G^{\perp}\left(\tilde{\delta} - M_d M^{-1}\Lambda(q)\right) = 0 \tag{15}$$

which shall be verified for all $q \in \mathbb{R}^n$. Additionally, the equilibrium q^* is a strict-minimum of V_d' if $\nabla_q V_d'(q^*) = 0$ and $\nabla_q^2 V_d'(q^*) > 0$, which correspond to the following conditions on $\Lambda(q)$:

$$\nabla_q V_d(q^*) + \Lambda(q^*) = 0 \tag{16}$$

$$\nabla_q^2 V_d(q^*) + \nabla_q \Lambda(q^*) > 0 \tag{17}$$

Notably, combining (16) and (7) while replacing the disturbance by its estimate verifies (14) at the equilibrium q^*. Finally, the new control law which represents the main contribution of this work is obtained adding to (3) the disturbance-compensation term u^*:

$$\begin{aligned} u &= u_{es} + u_{di} + u^* \\ u^* &= G^{\dagger}(\hat{\delta} - \alpha I^n p - M_d M^{-1}\Lambda(q)) \\ \dot{\hat{\delta}} &= \alpha I^n\left(-\nabla_q H + Gu - \hat{\delta} + \alpha I^n p\right) \end{aligned} \tag{18}$$

In particular, introducing (15) the matching conditions (4),(5) and their respective solutions M_d, V_d remain unchanged. In this sense, (15) can be interpreted as an additional matching condition of algebraic nature, which is therefore always solvable, and relates the disturbance estimate $\tilde{\delta}$ to $\Lambda(q)$ by means of the coupling term $M_d M^{-1}$.

Theorem 1: Consider system (6) under *Assumption 1-3* in closed-loop with control (18), where $\Lambda(q)$ is defined according to (15),(16),(17) and $\tilde{\delta}$ is estimated according to (10). Then q^* is a strict minimum of V_d', all trajectories $q(t), p(t)$ are bounded and the equilibrium $(q,p) = (q^*, 0)$ is asymptotically stable for some parameters $\alpha, k_v > 0$.

Proof: To prove the first claim we observe that q^* is a strict-minimum of V_d' due to (16),(17). To prove the stability claim we substitute (18) into (6) obtaining the following cascaded system:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - G k_v G^T \end{bmatrix} \begin{bmatrix} \nabla_q H_d + \Lambda(q) \\ \nabla_p H_d \end{bmatrix} + \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{19}$$

$$\dot{z} = -\alpha I^n z \tag{20}$$

Since the kinetic-energy PDE (4) is unchanged, $\nabla_p H_d' = \nabla_p H_d$, and $\nabla_q H_d' = \nabla_q H_d + \Lambda(q)$. We initially consider system (19) with $z = 0$. Choosing the Lyapunov function candidate $H_d' = \frac{1}{2} p^T M_d^{-1} p + V_d'$, and computing its time-derivative gives:

$$\dot{H}_d' = \nabla_q {H_d'}^T \dot{q} + \nabla_p H_d^T \dot{p} = -\nabla_p H_d^T G k_v G^T \nabla_p H_d \leq 0 \tag{21}$$

Consequently, the equilibrium $(q,p) = (q^*, 0)$ is asymptotically stable if $z = 0$. According to *Proposition 1*, z is vanishing exponentially hence $(q^*, 0)$ is also a locally stable equilibrium for the complete system (19)-(20). In order to prove global asymptotic stability it is necessary to establish boundedness of the trajectories $q(t), p(t)$ [29], [30]. To this end we consider the Lyapunov function candidate $W' = H_d + \frac{1}{2} c_2 z^T z$ and observe that $W' \geq H_d \geq \frac{1}{2} p^T M_d^{-1} p \geq c_1 |p|^2$, where c_1, c_2 are arbitrary positive constants. Computing the time derivative of W' along the trajectories of (19)-(20) gives:

$$\dot{W}' = -\nabla_p H_d^T G k_v G^T \nabla_p H_d + \nabla_p H_d^T z - \alpha c_2 (z^T I^n z) \leq 0 \tag{22}$$

Employing Young's inequality in (22) and introducing the positive constants c_3, c_4 gives:

$$\dot{W}' \leq c_3 |p||z| - \alpha c_2 |z|^2 \leq c_3 c_4 |p|^2 + \left(\frac{c_3}{4c_4} - \alpha c_2\right)|z|^2 \leq c_3 c_4 |p|^2 \leq \frac{c_3 c_4}{c_1} W' \tag{23}$$

which holds true for some $\alpha c_2 > \frac{c_3}{4c_4} > 0$. Invoking the Comparison Lemma [31] in (23) confirms that $W' \geq H_d$ is bounded. Consequently $H_d \geq \frac{1}{2} p^T M_d^{-1} p$ is bounded and therefore $q(t), p(t)$ are bounded [32]. Finally, the equilibrium $(q, p) = (q^*, 0)$ is asymptotically stable ■

Corollary 1: If in addition to the conditions stated in *Theorem 1* the disturbances are only matched, the equilibrium q^* is asymptotically stable if $\lambda_{min}\{G k_v G^T\}\alpha > 1/4$, where $\lambda_{min}\{\cdot\}$ represents the minimum eigenvalue of its argument.

Proof

In case of matched disturbances $z \in \mathbb{R}^m$, (22) can be expressed as follows with $c_2 = 1$:

$$\dot{W}' = -\begin{bmatrix} \nabla_p H_d^T & z^T \end{bmatrix} \begin{bmatrix} G k_v G^T & -I^m/2 \\ -I^m/2 & \alpha I^m \end{bmatrix} \begin{bmatrix} \nabla_p H_d \\ z \end{bmatrix} \leq 0 \tag{24}$$

Recalling that $\text{rank}(G k_v G^T) = m$ and employing a Schur complement argument, (24) holds true if $\lambda_{min}\{G k_v G^T\}\alpha > 1/4$ concluding the proof ■

Remark 1: If system (6) has a constant inertia matrix M, the kinetic-energy PDE (4) is trivially satisfied with $J_2 = 0$ and a constant M_d. If in addition $G^\perp$ is constant, then (15),(16),(17) admit a constant solution $\Lambda(q) = \Lambda(q^*) = -\nabla_q V_d(q^*)$ and consequently (17) reduces to the strict-minimum condition for the disturbance-free system: $\nabla_q^2 V_d(q^*) > 0$. This is the case for the inertia-wheel pendulum and the disk-on-disk systems. Alternatively, if the disturbance is only matched (i.e. $G^\perp \delta = 0$), then (15) can be solved with $\Lambda(q) = 0$ resulting in $u^* = G^\dagger \tilde{\delta}$ and (17) reduces again to $\nabla_q^2 V_d(q^*) > 0$. In general, $\Lambda(q)$ is not constant and not null hence it contributes to the control input. If for comparison purposes, the unmatched disturbance $G^\perp \delta$ is ignored in the control law, the Lyapunov derivative (21) becomes $\dot{H}_d = -\nabla_p H_d^T G k_v G^T \nabla_p H_d + \nabla_p H_d^T G^\perp \delta$ which is in general not negative semi-definite hence convergence to the equilibrium q^* cannot be concluded.

Remark 2: A physical interpretation of $\Lambda(q)$ as non-conservative forces is possible drawing a parallel to the Euler-Lagrange formulation as outlined below. Defining the open-loop Lagrangian $L = \frac{1}{2}\dot{q}^T M \dot{q} - V$ and the closed-loop Lagrangian $L_c = \frac{1}{2}\dot{q}^T M_c \dot{q} - V_c$, the open-loop dynamics (6) with the disturbance estimate $\tilde{\delta}$ and the corresponding closed-loop dynamics are respectively:

$$\frac{d}{dt}\nabla_{\dot{q}} L - \nabla_q L = Gu - \tilde{\delta} \tag{25}$$

$$\frac{d}{dt}\nabla_{\dot{q}} L_c - \nabla_q L_c = -\Lambda(q) \tag{26}$$

Since M, M_c are invertible and positive definite, (25) and (26) can be rewritten as:

$$\ddot{q} = M^{-1}\left(Gu - \tilde{\delta} + \nabla_q\left(\frac{1}{2}\dot{q}^T M \dot{q}\right) - \nabla_q (M\dot{q})\dot{q} - \nabla_q V\right) \tag{27}$$

$$\ddot{q} = M_c{}^{-1}\left(\nabla_q\left(\frac{1}{2}\dot{q}^T M_c \dot{q}\right) - \nabla_q(M_c\dot{q})\dot{q} - \nabla_q V_c - \Lambda(q)\right) \tag{28}$$

Equating (27) and (28) gives the following matching conditions that correspond to (4),(5),(15) if $V_c = V_d; M_c = MM_d^{-1}M; J_2 = M_d M^{-1}\left(\nabla_q\left(MM_d^{-1}p\right)^T - \nabla_q\left(MM_d^{-1}p\right)\right)M^{-1}M_d$ [23]:

$$\begin{gathered} G^{\perp}\left(\nabla_q\left(\frac{1}{2}\dot{q}^T M\dot{q}\right) - \nabla_q(M\dot{q})\dot{q} - MM_c^{-1}\left(\nabla_q\left(\frac{1}{2}\dot{q}^T M_c\dot{q}\right) - \nabla_q(M_c\dot{q})\dot{q}\right)\right) = 0 \\ G^{\perp}\left(\nabla_q V - MM_c^{-1}\left(\nabla_q V_c\right)\right) = 0 \\ G^{\perp}\left(\tilde{\delta} - MM_c^{-1}\Lambda(q)\right) = 0 \end{gathered} \tag{29}$$

Remark 3: For comparison purposes an alternative control approach based on feedback-linearization [33] is briefly outlined although the control aim is different from that of IDA-PBC. Consider system (6) in Euler-Lagrange form (27) for the case $n = 2$ with $q = (q_1, q_2)$ and define the position error $\tilde{q}_1 = q_1^* - q_1$, where q_1^* is the desired equilibrium with the following target dynamics:

$$\begin{gathered} \dot{\tilde{q}}_1 = \dot{q}_1^* - \dot{q}_1 \\ \ddot{\tilde{q}}_1 = \ddot{q}_1^* - \ddot{q}_1 = -k_p\tilde{q}_1 - k_d\dot{\tilde{q}}_1 \end{gathered} \tag{30}$$

where $k_p, k_d > 0$ are tuning parameters. Substituting $\ddot{q}_1$ from (27) and considering that $\dot{q}_1^*, \ddot{q}_1^* = 0$ for setpoint regulation, the partial feedback-linearization control law is:

$$u = \frac{-[1 \quad 0]M^{-1}\left(-\tilde{\delta} + \nabla_q\left(\frac{1}{2}\dot{q}^T M\dot{q}\right) - \nabla_q(M\dot{q})\dot{q} - \nabla_q V\right) + k_p\tilde{q}_1 + k_d\dot{\tilde{q}}_1}{[1 \quad 0]M^{-1}G} \tag{31}$$

It follows from *Proposition 1* and from (30) (see details in [33]) that $\tilde{q}_1$ converges to zero for $k_p, k_d > 0$. Similarly, the control law for the position error $\tilde{q}_2 = q_2^* - q_2$ is:

$$u = \frac{-[0 \quad 1]M^{-1}\left(-\tilde{\delta} + \nabla_q\left(\frac{1}{2}\dot{q}^T M\dot{q}\right) - \nabla_q(M\dot{q})\dot{q} - \nabla_q V\right) + k_p\tilde{q}_2 + k_d\dot{\tilde{q}}_2}{[0 \quad 1]M^{-1}G} \tag{32}$$

While IDA-PBC (18) aims to stabilise $q^* = (q_1^*, q_2^*)$, the control aim of (31) is only the dynamics of $\tilde{q}_1$ and no conclusion can be drawn about q_2 (conversely for (32)). Nevertheless, (31) can in theory stabilise any position $q_1^* \in \mathbb{R}$, even if *Assumption 3* is not satisfied.

5. Simulation Results

5.1. Inertia-wheel pendulum

This system consists of an unactuated pendulum with a balanced actuated rotor at the tip and the equations of motion are defined as follows, where a_1, a_2, a_3 are constant positive parameters:

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -a_3\sin(q_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}u - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{33}$$

The pendulum angle $q_1 \in (-\pi/2; \pi/2)$ is measured from the vertical, while the rotor angle q_2 is measured relative to the pendulum. The open-loop potential energy is $V = a_3\cos(q_1)$ and M is

constant, therefore (4),(5) are solvable with $J_2 = 0$ and a constant $M_d = (a_1 a_2 - a_2^2)\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$ hence a stabilising controller (3) can be constructed (ref. Appendix).

Evaluating (7) and replacing the disturbances with their estimates, the open-loop unstable equilibrium of (33) is $q^* = (q_1^*, q_2^*)$ with $q_1^* = \sin^{-1}(\tilde{\delta}_1/a_3)$. Notably, *Assumption 3* is only satisfied if $|\tilde{\delta}_1| \leq a_3$, which limits the magnitude of the unmatched disturbance that the system can withstand depending on the parameter a_3. It must be highlighted that this is not dependent on the specific IDA-PBC design but is common to (3) and to the integral IDA-PBC designs [17], [18]. Conversely, no condition is imposed on the matched disturbance $\tilde{\delta}_2$ and q_2^* can be freely chosen. Since G, M, M_d are constant, $\Lambda(q)$ is also constant and is defined as follows (see *Remark 1*), where $\gamma = -(m_2 a_1 - m_1 a_2)/((m_1 - m_2)a_2)$ and $k_p > 0$ is a tuning parameter (ref Appendix):

$$\Lambda(q) = \Lambda(q^*) = -\nabla_q V_d(q^*) = \begin{bmatrix} \dfrac{\tilde{\delta}_1}{(m_1 - m_2)a_2} - k_p \gamma^2 q_1^* \\ -k_p \gamma q_1^* \end{bmatrix} \tag{34}$$

Verifying the strict-minimum conditions confirms that (16) is trivially satisfied, while from (17) we have $\nabla_q^2 V_d(q_1^*, q_2^*) > 0, \forall q_2^*, \forall q_1^* \in (-\pi/2; \pi/2)$ if $m_1 < m_2$. In conclusion q^* is a strict-minimum of V_d' and the control law (18) is defined as follows, where u_{es}, u_{di} are defined as in [17] and $\gamma_3$is a constant parameter (ref. Appendix):

$$\begin{gathered} u = u_{es} + u_{di} + u^* \\ u^* = \tilde{\delta}_2 - \tilde{\delta}_1 \frac{(m_2 - m_3)}{(m_1 - m_2)} + k_p \gamma_3 \gamma q_1^* \end{gathered} \tag{35}$$

The disturbance estimates $\tilde{\delta}_1 = \hat{\delta}_1 + \beta_1$ and $\tilde{\delta}_2 = \hat{\delta}_2 + \beta_2$ are computed according to (10):

$$\begin{gathered} \dot{\hat{\delta}}_1 = \alpha\left(a_2 a_3 \sin(q_1) - a_2\, u - a_2 \tilde{\delta}_1 + a_2 \tilde{\delta}_2\right) \\ \dot{\hat{\delta}}_2 = \alpha\left(-a_2 a_3 \sin(q_1) + a_1 u + a_2 \tilde{\delta}_1 - a_1 \tilde{\delta}_2\right) \\ \beta_1 = -\alpha(a_1 a_2 - a_2^2)\dot{q}_1 \\ \beta_2 = -\alpha(a_1 a_2 - a_2^2)\dot{q}_2 \end{gathered} \tag{36}$$

Simulations were conducted for system (33) with initial position $q_1 = 0.1$; $q_2 = 0.2$ and $p = 0$ employing the parameters $a_1 = 0.0124$; $a_2 = 0.0025$; $a_3 = 0.4446$; $m_1 = 0.40$; $m_2 = 1.08$; $m_3 = 5$. Control (35) was implemented with the parameters $k_p = 1$; $k_v = 0.001$; $\alpha = 50$. The Integral IDA-PBC design [17] was employed for comparison purposes with the parameters $k_p = 1$; $k_v = 0.001$; $K_i = 0.0015$.

$$\begin{gathered} u = u_{es} + u_{di} + \frac{k_v K_i}{(a_1 a_2 - a_2^2)(m_1 m_3 - m_2^2)}(m_2 a_1 - m_1 a_2)x_v - K_i k_2 k_p (q_2 + \gamma q_1) \\ \dot{x}_v = k_p (q_2 + \gamma q_1) \end{gathered} \tag{37}$$

Figure 1 represents q_1, q_2 in the presence of the disturbances $\delta_1 = 0$; $\delta_2 = -0.1$ only affecting the actuated rotor. The results confirm that both control (35) and the Integral IDA-PBC (37) effectively compensate matched disturbances stabilising the assignable equilibrium $q^* = (0,0)$. Conversely, the baseline IDA-PBC (3) would result in noticeable steady-state errors on both positions q_1, q_2 [17]. Figure 2 refers to the disturbances $\delta_1 = -0.2$; $\delta_2 = -0.1$ affecting both joints and in this case the assignable equilibrium is $q^* = (q_1^*, 0)$, where $q_1^* = \sin^{-1}(\tilde{\delta}_1/a_3) = -0.467$.

With the Integral IDA-PBC (37) the position settles at $q = (-0.467, 3.50)$ corresponding to a large error on q_2. Increasing K_i, k_p in (37) slightly improves performance up to a certain point (e.g. the position settles at $q = (-0.467, 3.40)$ with $k_p = 10, K_i = 0.015$), however much larger values (e.g. $k_p = 100, K_i = 0.15$) can lead to instability. Conversely, control (35) correctly stabilises the equilibrium q^*, while the disturbances are accurately estimated by the adaptation law (36) resulting in a smooth control input (Figure 3). In conclusion, while both controllers (35) and (37) stabilise the equilibrium position $q_1^* = -0.467$, only control (35) can regulate q_2 to its desired value $q_2^* = 0$ in the presence of unmatched disturbances. Comparable results are achieved for larger disturbances, as long as *Assumption 3* is satisfied.

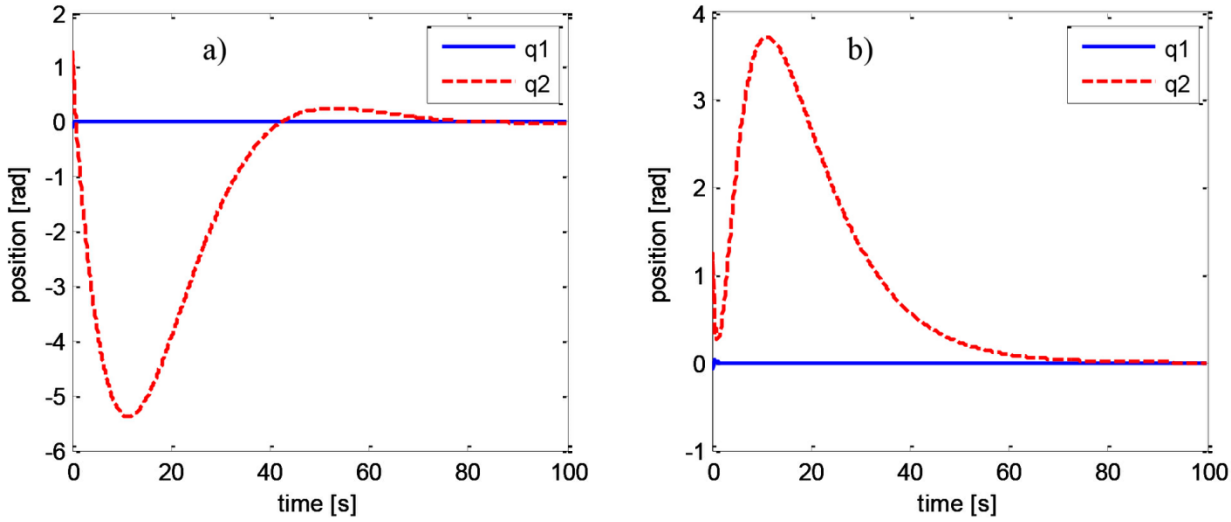

Figure 1. Position of pendulum and rotor with $\delta_1 = 0$; $\delta_2 = -0.1$: (a) Integral IDA-PBC (37); (b) control (35).

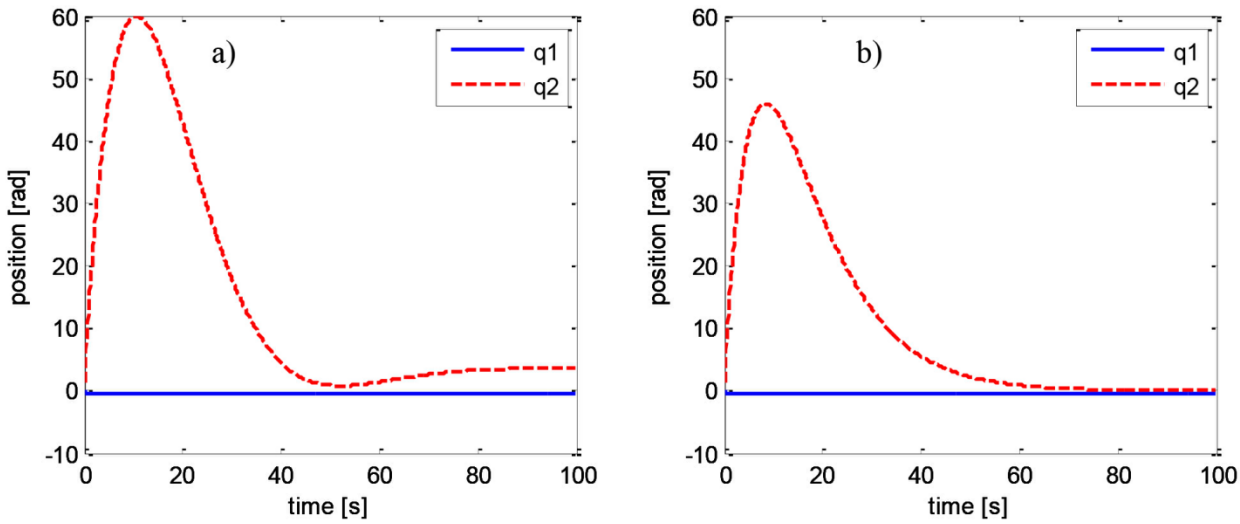

Figure 2. Position of pendulum and rotor with $\delta_1 = -0.2$; $\delta_2 = -0.1$: (a) Integral IDA-PBC (37); (b) control (35).

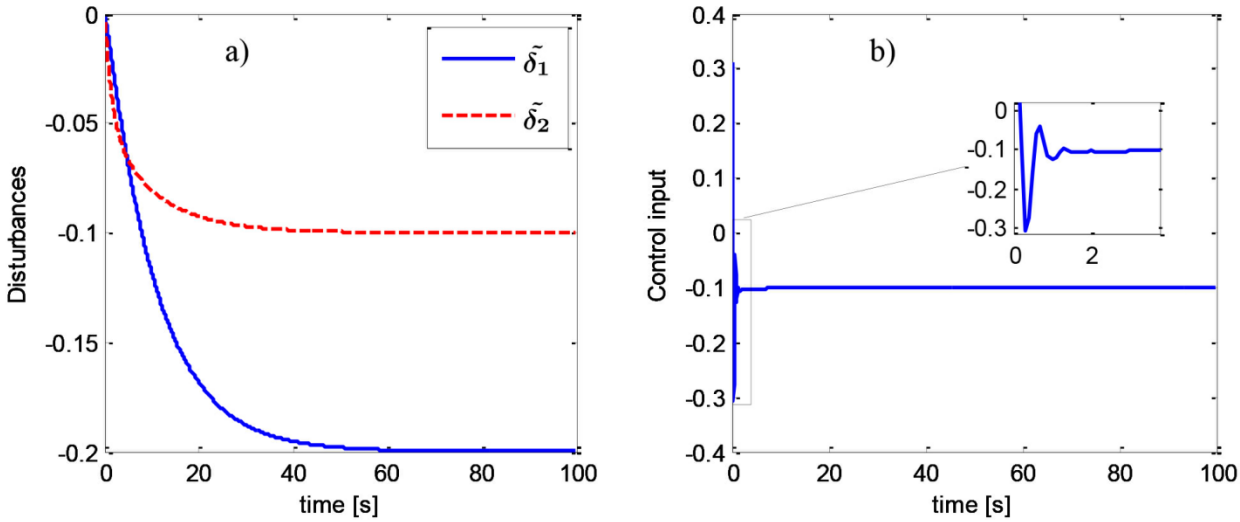

Figure 3. Control (35) with $\delta_1 = -0.2$; $\delta_2 = -0.1$: (a) disturbance estimates; (b) control input.

5.2. Disk-on-disk system

The disk-on-disk system consists of one unactuated disks that rolls without slipping on an actuated disk. The equations of motion for this system are defined as follows, where $m_{11}, m_{12}, m_{22}, m_3$ are constant parameters:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -m_3 \sin(q_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{38}$$

The angle q_1 of the actuated disk and the angle $q_2 \in (-\pi/2; \pi/2)$ of the unactuated disk are measured from the vertical. The open-loop potential energy is $V = m_3 \cos(q_2)$ and M is constant, therefore (4),(5) are solvable with $J_2 = 0$ and a constant $M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$, and a stabilising controller (3) can be constructed (ref. Appendix).

Evaluating (7) and replacing the disturbance with their estimates, the open-loop unstable equilibrium of (38) is $q^* = (q_1^*, q_2^*)$ with $q_2^* = \sin^{-1}(\tilde{\delta}_2/m_3)$. In this case, *Assumption 3* is only satisfied if $|\tilde{\delta}_2| < m_3$. Since G, M, M_d are constant, $\Lambda(q)$ is also constant and is defined as follows (see *Remark 1*), where $\gamma = -(m_{22}k_2 - m_{12}k_3)/(m_{11}k_3 - m_{12}k_2)$ and $k_p > 0$.

$$\Lambda(q) = \Lambda(q^*) = -\nabla_q V_d(q^*) = \begin{bmatrix} -k_p \gamma q_2^* \\ \dfrac{\tilde{\delta}_2 (m_{11}m_{22} - m_{12}^2)}{(m_{11}k_3 - m_{12}k_2)} - k_p \gamma^2 q_2^* \end{bmatrix} \tag{39}$$

Consequently, (16) is trivially satisfied, while computing (17) gives $\nabla_q^2 V_d(0, q_2^*) > 0, \forall q_1^*, \forall q_2^* \in (-\pi/2; \pi/2)$ if $(m_{12}k_2 - m_{11}k_3) > 0$. In conclusion q^* is a strict-minimum of V_d' and the control law (18) is defined as follows, where u_{es}, u_{di} are defined as in [18] (ref. Appendix):

$$\begin{gathered} u = u_{es} + u_{di} + u^* \\ u^* = \tilde{\delta}_1 - \frac{(m_{11}k_2 - m_{12}k_1)}{(m_{11}k_3 - m_{12}k_2)} \tilde{\delta}_2 + k_p \gamma \frac{(k_1 k_3 - k_2^2)}{(m_{11}k_3 - m_{12}k_2)} q_2^* \end{gathered} \tag{40}$$

From (10), the disturbance estimates are $\tilde{\delta}_1 = \hat{\delta}_1 + \beta_1$ and $\tilde{\delta}_2 = \hat{\delta}_2 + \beta_2$ where:

$$
\begin{aligned}
\dot{\tilde{\delta}}_1 &= \alpha\left(u - \tilde{\delta}_1\right) \\
\dot{\tilde{\delta}}_2 &= \alpha\left(m_3 \sin(q_2) - \tilde{\delta}_2\right) \\
\beta_1 &= -\alpha(m_{11}\dot{q}_1 + m_{12}\dot{q}_2) \\
\beta_2 &= -\alpha(m_{12}\dot{q}_1 + m_{22}\dot{q}_2)
\end{aligned} \tag{41}
$$

For comparison purposes, the adaptive feedback-linearization (32) for system (38) becomes:

$$
u = \frac{m_{11}}{m_{12}} m_3 \sin(q_2) + \tilde{\delta}_1 - \frac{m_{11}}{m_{12}} \tilde{\delta}_2 + \frac{(m_{11}m_{22} - m_{12}^2)}{m_{12}} \left(k_p q_2 + k_d \dot{q}_2\right) \tag{42}
$$

where $k_p, k_d > 0$ are tuning parameters, while the adaptation law (41) remains unchanged. Differently from (40), the control law (42) only depends on the position q_2 therefore no attempt is made to regulate q_1. Additionally, only the elements of M appear in (42), while more design freedom is provided by (40). Nevertheless, there is a clear analogy in how the disturbances are accounted for in (40) and (42): in both designs the matched component $\tilde{\delta}_1$ is simply added to the control input, while $\tilde{\delta}_2$ is multiplied by the same coefficient as the nonlinear term $m_3 \sin(q_2)$.

Simulations were conducted for system (38) with initial position $q_1 = 0$; $q_2 = 0.12$ and $p = 0$ employing the parameters [18]: $m_{11} = 5.77 \cdot 10^{-3}$; $m_{12} = -7.28 \cdot 10^{-4}$; $m_{22} = 2.18 \cdot 10^{-3}$; $m_3 = 47.6 \cdot 10^{-3}$ with $k_1 = 0.4$; $k_2 = -0.03$; $k_3 = 0.003$. Control (40) was implemented with the following tuning parameters: $k_p = 0.00005$; $k_v = 0.8$; $\alpha = 5$. The PID IDA-PBC design [18] was employed for comparison purposes with the following parameters: $k_p = 0.00025$; $k_v = 0.8$; $K_1 = 0.012$; $K_3 = 0.06$; $K_i = 0.2$; $K_p = 0.8$, and $K_2 = (G^T M_d^{-1} G)^{-1}$:

$$
\begin{aligned}
u = u_{es} + u_{di} &- \left[K_p G^T M_d^{-1} G K_1 G^T M^{-1} + K_2 K_i (K_2^T + K_3^T G^T M_d^{-1} G K_1) G^T M^{-1}\right] \nabla_q V_d \\
&- \left[K_i G^T M^{-1} \nabla_q^2 V_d M^{-1} + K_2 K_i K_3^T G^T M_d^{-1}\right] p - \left(K_p G^T M_d^{-1} G K_2 + K_3\right) K_i \zeta \\
\dot{\zeta} &= (K_2^T G^T M^{-1} + K_3^T G^T M_d^{-1} G K_1 G^T M^{-1}) \nabla_q V_d + K_3^T G^T M_d^{-1} p
\end{aligned} \tag{43}
$$

Figure 4 represents q_1, q_2 in the presence of the matched disturbance $\delta_1 = -0.01$; $\delta_2 = 0$ in which case both control approaches correctly stabilise the assignable equilibrium $q^* = (0,0)$. Conversely, the baseline IDA-PBC (3) with the same parameters would produce a large steady-state error [18]. Figure 5 depicts the disks' position in the presence of the disturbances $\delta_1 = -0.01$; $\delta_2 = -0.001$, in which case the assignable equilibrium is $q^* = (0, q_2^*)$, with $q_2^* = \sin^{-1}\left(\tilde{\delta}_2 / m_3\right) = -0.021$. In spite of the small value of δ_2, with the PID IDA-PBC (43) the position settles at $q = (-1.265, -0.021)$ corresponding to a large steady-state error on q_1. Conversely, control (40) stabilises the equilibrium position q^*, while both components of the disturbance are estimated accurately by (41) resulting in a smooth control input (Figure 6). In conclusion, while both controllers (40) and (43) stabilise the equilibrium position $q_2^* = -0.021$, only control (40) can regulate q_1 to its desired value $q_1^* = 0$ in the presence of unmatched disturbances. Figure 7 shows the time history of q_2 and the disturbance estimates with the adaptive feedback-linearization control (42) in the presence of the disturbances $\delta_1 = -0.01$; $\delta_2 = -0.05$, where $|\delta_2| > m_3$. Notably, the position of the unactuated disk is regulated at $q_2^* = 0$ in spite of the fact that *Assumption 3* is not satisfied (ref. *Remark 3*). Nevertheless, the position q_1 keeps increasing in absolute value without settling around an equilibrium point, which represents the main difference from IDA-PBC.

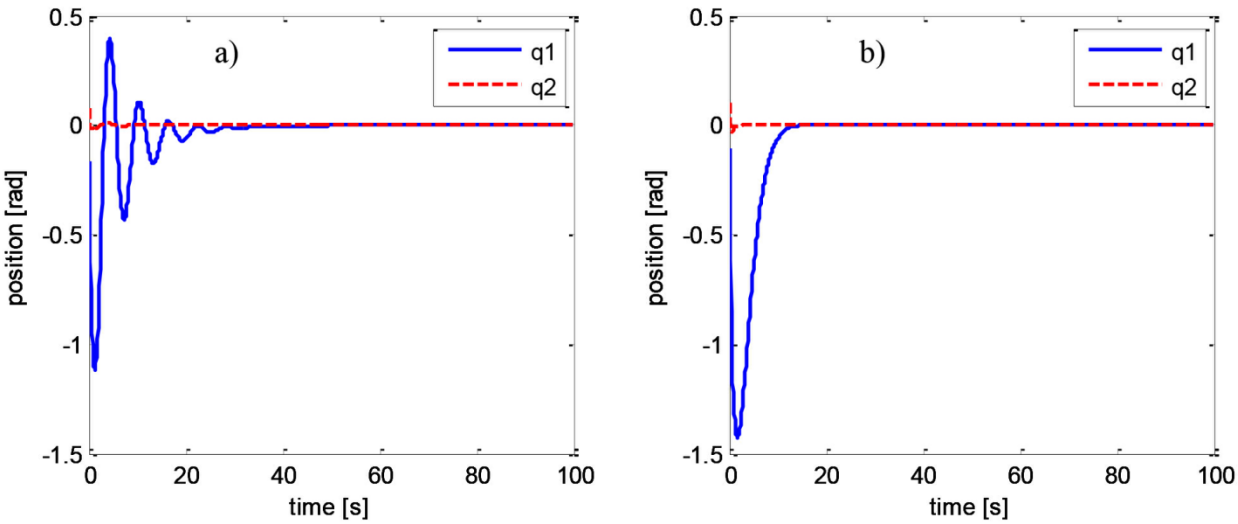

Figure 4. Disks position with $\delta_1 = -0.01;\ \delta_2 = 0$: (a) PID IDA-PBC (43); (b) control (40).

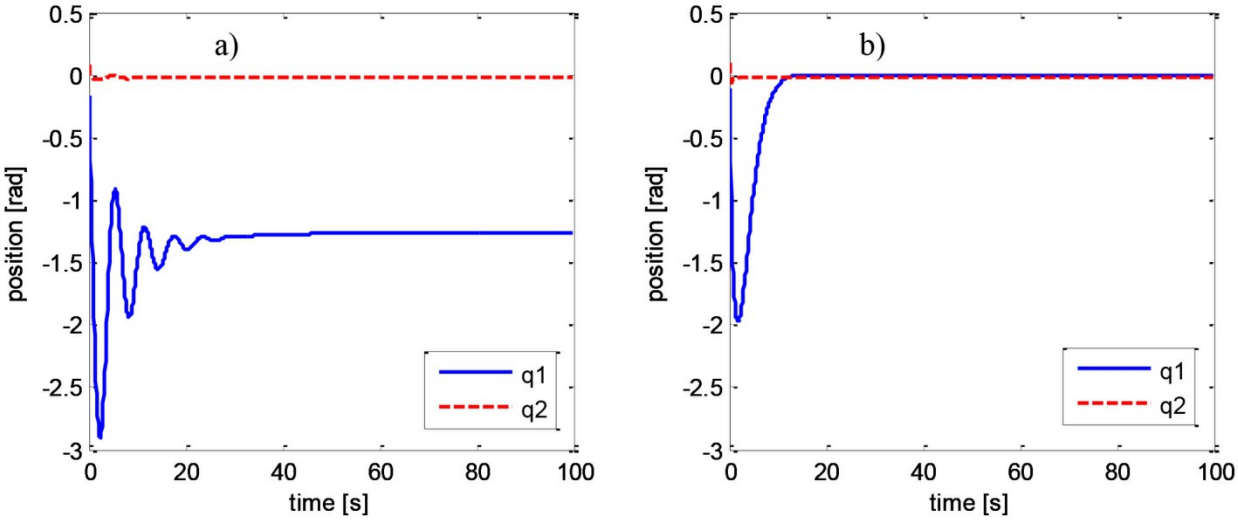

Figure 5. Disks position with $\delta_1 = -0.01;\ \delta_2 = -0.001$: (a) PID IDA-PBC (43); (b) control (40).

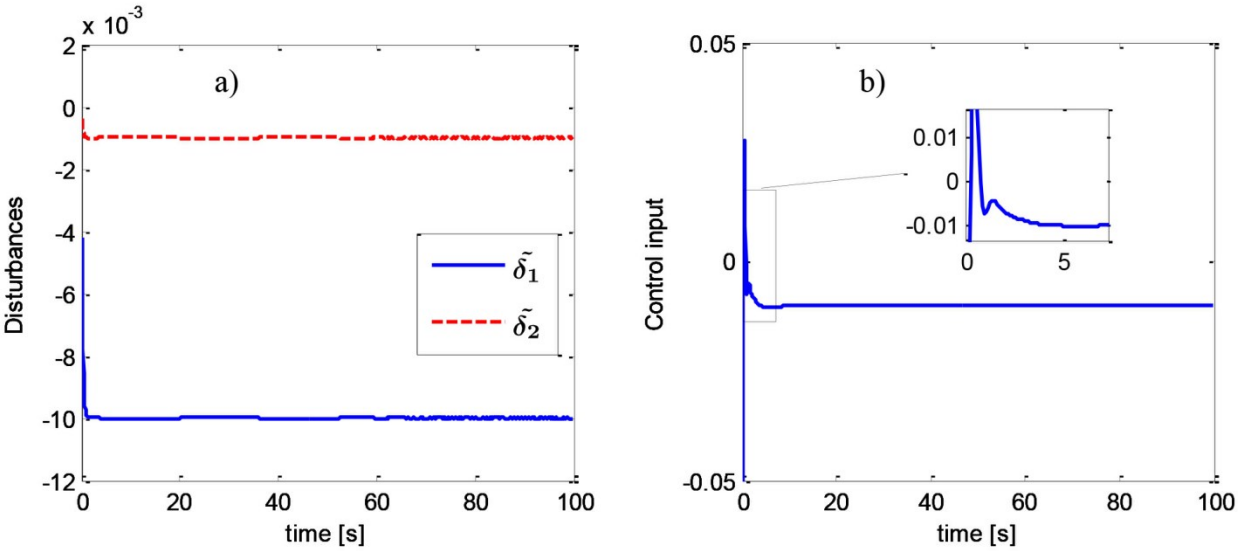

Figure 6. Control (40) for $\delta_1 = -0.01; \delta_2 = -0.001$: (a) disturbance estimates; (b) control input.

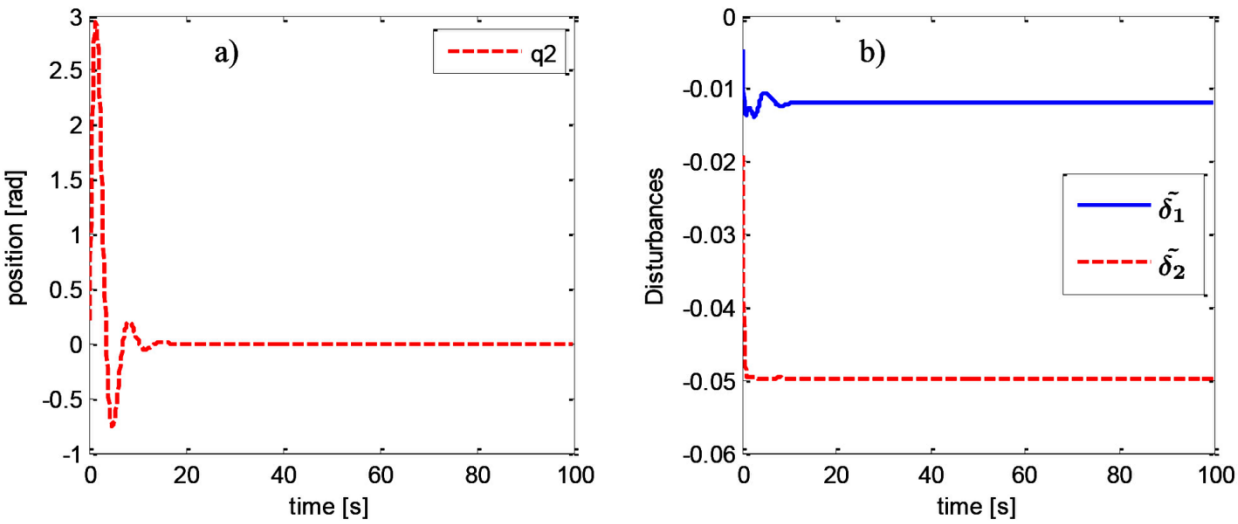

Figure 7. Adaptive feedback-linearization (42) with $\delta_1 = -0.01$; $\delta_2 = -0.05$: (a) position of the unactuated disk; (b) disturbance estimates.

5.3. Pendulum-on-cart

The pendulum-on-cart system consists of an unactuated pendulum of length l with a point mass at the tip mounted on an actuated cart. The equations of motion for this system after partial feedback-linearization are defined as follows, where a, b are positive constants:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -a\sin(q_1) \\ 0 \end{bmatrix} = \begin{bmatrix} -b\cos(q_1) \\ 1 \end{bmatrix} u - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{44}$$

The angle $q_1 \in (-\pi/2; \pi/2)$ of the unactuated pendulum is measured from the vertical, while the position q_2 of the actuated cart is measured from an arbitrary origin. The open-loop potential energy is $V = a\cos(q_1)$, the open-loop inertia matrix is $M = I^2$, while the closed-loop inertia matrix $M_d = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$ is computed according to Proposition 3 in [34]. Also in this case (4),(5) are solvable and control (3) can be constructed (ref. Appendix).

Evaluating (7) for system (44), for which $G^{\perp} = [1 \quad b\cos(q_1)]$, and replacing the disturbances with their estimates gives:

$$a\sin(q_1^*) = \tilde{\delta}_1 + \tilde{\delta}_2 b\cos(q_1^*) \tag{45}$$

For sufficiently small angles q_1^*, (45) can be approximated with Taylor series and admits the solutions $q_1^* = \left(-a \pm \sqrt{a^2 + 2b\tilde{\delta}_1\tilde{\delta}_2 + 2b^2(\tilde{\delta}_2)^2}\right) / (b\tilde{\delta}_2)$. Since G, M_d depend on q_1, the term $\Lambda(q)$ is not constant and is defined as follows, with $k_p > 0$:

$$\Lambda = \begin{bmatrix} \dfrac{-6\tilde{\delta}_1 - 6\tilde{\delta}_2 b\cos(q_1)}{kb^2\cos^3(q_1)} - 9k_p \dfrac{\ln\left(\dfrac{1+\sin(q_1^*)}{\cos(q_1^*)}\right)\left(2+\cos(q_1^*)\right)}{b^2\cos^2(q_1^*)} - 9k_p \dfrac{4\sin(q_1^*) + \sin(2q_1^*)}{b^2\cos^3(q_1^*)} \\ -3\dfrac{k_p}{b}\left(\ln\left(\dfrac{1+\sin(q_1^*)}{\cos(q_1^*)}\right) + 2\tan(q_1^*)\right) \end{bmatrix} \tag{46}$$

Proceeding to evaluate the strict-minimum conditions confirms that (16) follows from (46) within the limits of the approximation on q_1^*. Instead, (17) results in the following sufficient condition:

$$(3\tilde{\delta}_1 \sin(q_1^*) + \tilde{\delta}_2 b \sin(2q_1^*)) < a(1 + 2\sin^2(q_1^*))$$

In particular, if the disturbances are only unmatched, (45) can be solved analytically for $q_1^* = \sin^{-1}(\tilde{\delta}_1/a)$, and $\nabla_q^2 V_d'(q_1^*, 0) > 0 \; \forall \; q_1^* \in (-\pi/2; \pi/2)$ if $|\tilde{\delta}_1| < a$.
In summary, the control law (18) is defined as follows, with u_{es}, u_{di} from [34] (ref. Appendix):

$$\begin{aligned} u &= u_{es} + u_{di} + u^* \\ u^* &= G^{\dagger}\left(\begin{bmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{bmatrix} - M_d M^{-1} \Lambda(q)\right) \end{aligned} \tag{47}$$

From (10), the disturbance estimates $\tilde{\delta}_1 = \hat{\delta}_1 + \beta_1$ and $\tilde{\delta}_2 = \hat{\delta}_2 + \beta_2$ are in this case:

$$\begin{aligned} \dot{\hat{\delta}}_1 &= \alpha\left(a\sin(q_1) - b\cos(q_1)u - \tilde{\delta}_1\right) \\ \dot{\hat{\delta}}_2 &= \alpha\left(u - \tilde{\delta}_2\right) \\ \beta_1 &= -\alpha \dot{q}_1 \\ \beta_2 &= -\alpha \dot{q}_2 \end{aligned} \tag{48}$$

Simulations were conducted for system (44) from the initial position $q_1 = 1.37$; $q_2 = -0.1$ and $p = 0$ employing the parameters $a = 1$; $b = 1$. Control (47) was implemented with the following tuning parameters: $k = 0.01, k_p = 1$; $k_v = 0.01$; $\alpha = 1$. Figure 8 represents q_1, q_2 in the presence of the disturbances $\delta_1 = -0.1$; $\delta_2 = -0.05$. In this case the assignable equilibrium is $q^* = (q_1^*, 0)$, with $q_1^* = -0.15$ and is correctly stabilised by (47). Additionally, both components of the disturbance are estimated correctly by (48). In comparison, with the baseline IDA-PBC [34] the position settles at $q = (-0.15, 17.53)$ corresponding to a large steady-state error on q_2. Finally, the integral IDA-PBC designs [17], [18] are not applicable in this case since G, M, M_d are not constant.

Although beyond the scope of the current work, similar results to Figure 8 are obtained introducing small parameter uncertainties (e.g. $b = 1.01$ in (44); $b = 1$ in (47)). In particular, these effects are treated by the control as additional disturbances and accounted for by the estimates (48). Large uncertainties result in large varying disturbances hence *Assumption 2* is no longer satisfied.

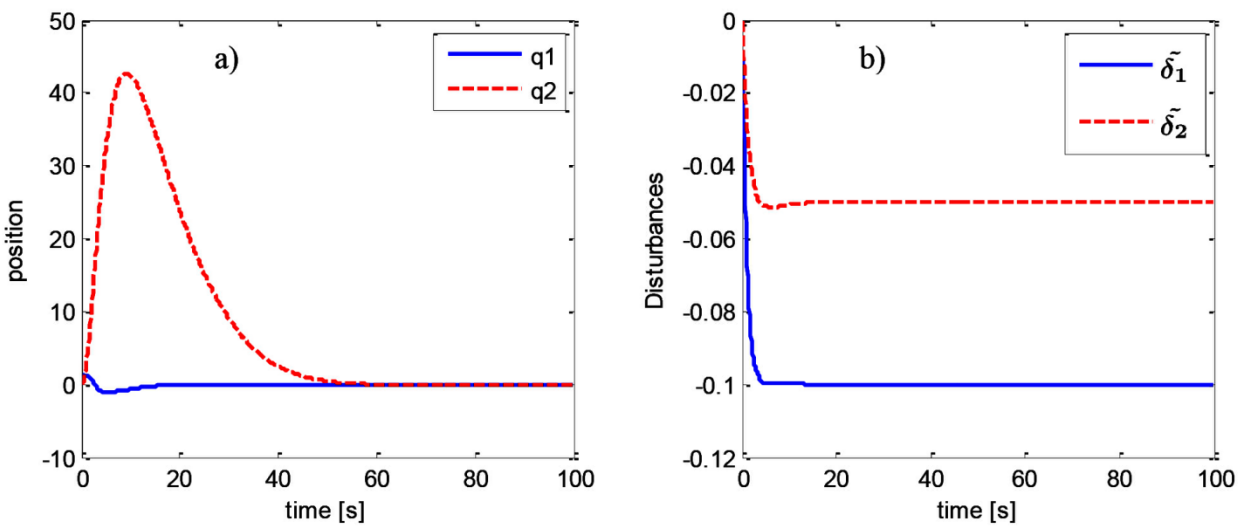

Figure 8. Pendulum-on-cart system with disturbances $\delta_1 = -0.1$; $\delta_2 = -0.05$ and control (47): (a) position of pendulum and cart; (b) disturbance estimates.

6. Conclusions

This paper presented a new IDA-PBC design for nonlinear underactuated mechanical systems subject to matched and unmatched constant disturbances. The proposed control was constructed from the baseline IDA-PBC introducing a disturbance-compensation term computed with a new algebraic matching condition and estimating the disturbances adaptively. For comparison purposes, an alternative solution based on adaptive feedback-linearization was also presented highlighting the differences from IDA-PBC. Simulations on three motivating examples demonstrated that the proposed control is comparably effective to integral IDA-PBC designs in the presence of matched disturbances, and results in better performance if the disturbances are unmatched. Additionally, the control can be used for systems with non-constant inertia matrix. Aims for future work include the investigation of variable and state-dependent disturbances in nonlinear underactuated systems, the study of uncertain model parameters in the open-loop Hamiltonian, and the experimental validation of the results.

7. Acknowledgement

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8. Appendix

Baseline IDA-PBC for the Inertia-wheel pendulum

The solutions of the matching conditions (4),(5) are defined as follows, with $m_1, m_2, m_3, \varepsilon$ constant parameters [17]:

$$\begin{gathered} M_d = (a_1 a_2 - a_2^2)\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \\ m_2 = m_1 a_2 / a_1 + \varepsilon \\ V_d = a_3 \cos(q_1)\frac{1}{(m_1 - m_2)a_2} + \frac{k_p}{2}(q_2 + \gamma q_1)^2 \\ \gamma = -(m_2 a_1 - m_1 a_2)/((m_1 - m_2)a_2) \end{gathered} \tag{A.1}$$

The IDA-PBC (3) is defined as follows:

$$\begin{gathered} u = u_{es} + u_{di} \\ u_{es} = \gamma_2 \sin(q_1) - k_p \gamma_3 (q_2 + \gamma q_1) \\ u_{di} = -\frac{k_v}{(a_1 a_2 - a_2^2)(m_1 m_3 - m_2^2)}(m_1 p_2 - m_2 p_1) \end{gathered} \tag{A.2}$$

The terms $k_p, k_v > 0$ are constant tuning parameters. The constant terms γ_2, γ_3 are:

$$\begin{gathered} \gamma_2 = a_3(m_2 - m_3)/(m_1 - m_2) \\ \gamma_3 = (\varepsilon a_1 (m_2 - m_3)/(m_1 - m_2) - (m_3 a_1 - m_2 a_2)) \end{gathered} \tag{A.3}$$

Baseline IDA-PBC for the Disk-on-disk system

The solutions of the matching conditions (4),(5) are defined as follows, with k_1, k_2, k_3 constant parameters [18]:

$$M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$$
$$V_d = m_3 \cos(q_2) \frac{m_{11}m_{22} - m_{12}^2}{(m_{11}k_3 - m_{12}k_2)} + \frac{k_p}{2}(q_1 + \gamma q_2)^2 \tag{A.4}$$
$$\gamma = -(m_{22}k_2 - m_{12}k_3)/(m_{11}k_3 - m_{12}k_2)$$

The IDA-PBC (3) is defined as follows, with $k_p, k_v > 0$ constant tuning parameters:

$$u = u_{es} + u_{di}$$
$$u_{es} = m_3 \sin(q_2) \frac{(m_{11}k_2 - m_{12}k_1)}{(m_{11}k_3 - m_{12}k_2)} - k_p \frac{(k_1k_3 - k_2^2)}{(m_{11}k_3 - m_{12}k_2)}(q_1 + \gamma q_2) \tag{A.5}$$
$$u_{di} = -\frac{k_v}{(k_1k_3 - k_2^2)}(k_3p_1 - k_2p_2)$$

Baseline IDA-PBC for the Pendulum-on-cart system

The solutions of the matching conditions (4),(5) are defined as follows, with k constant [34]:

$$M_d = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} = \begin{bmatrix} \frac{kb^2}{3}\cos^3(q_1) & -\frac{kb}{2}\cos^2(q_1) \\ -\frac{kb}{2}\cos^2(q_1) & k\cos(q_1) + k \end{bmatrix} \tag{A.6}$$
$$V_d = \frac{3a}{kb^2\cos^2(q_1)} + \frac{k_p}{2}\left(q_2 + \frac{3}{b}\ln\left(\frac{1+\sin(q_1)}{\cos(q_1)}\right) + \frac{6}{b}\tan(q_1)\right)^2$$

The IDA-PBC (3) is defined as follows:

$$u = u_{es} + u_{di}$$
$$u_{es} = A_1k_pq_2 + p^TA_2p + A_3 \tag{A.7}$$
$$u_{di} = -k_vA_4p$$

The terms $k_p, k_v > 0$ are constant tuning parameters, while A_1, A_2, A_3, A_4 are defined as follows:

$$A_1 = -\left(m_{12}\left(\frac{3}{b\cos(q_1)} + \frac{6}{b\cos^2(q_1)}\right) + m_{22}\right)$$
$$A_2 = -\frac{1}{2}m_{12}M_d^{-1}\begin{bmatrix} \frac{dm_{11}}{dq_1} + \frac{k^2b^3}{12}\sin(q_1)\cos^4(q_1) & \frac{dm_{12}}{dq_1} \\ \frac{dm_{12}}{dq_1} - \frac{k^2b^2}{12}\sin(q_1)\cos^3(q_1) & \frac{dm_{22}}{dq_1} \end{bmatrix}M_d^{-1} \tag{A.8}$$
$$A_3 = -m_{12}\left(\frac{6a\sin(q_1)}{kb^2\cos^3(q_1)}\right) + k_p\left(\frac{3}{b}\ln\left(\frac{1+\sin(q_1)}{\cos(q_1)}\right) + \frac{6}{b}\tan(q_1)\right)A_1$$
$$A_4 = [-b\cos(q_1) \quad 1]M_d^{-1}$$

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