

On Path-Complete Lyapunov Functions: Geometry and Comparison

Matthew Philippe, Nikolaos Athanasopoulos, David Angeli and Raphaël M. Jungers

Abstract—We study optimization-based criteria for the stability of switching systems, known as *Path-Complete Lyapunov Functions*, and ask the question “can we decide algorithmically when a criterion is less conservative than another?”. Our contribution is twofold. First, we show that a Path-Complete Lyapunov Function, which is a multiple Lyapunov function by nature, can always be expressed as a common Lyapunov function taking the form of a combination of minima and maxima of the elementary functions that compose it. Geometrically, our results provide for each Path-Complete criterion an implied invariant set. Second, we provide a linear programming criterion allowing to compare the conservativeness of two arbitrary given Path-Complete Lyapunov functions.

Index Terms—Path-Complete Methods, Lyapunov stability theory, Conservativeness, Automata, Switching Systems.

I. INTRODUCTION

SWITCHING systems [23], [26], [28], [38] present major theoretical challenges [8]. They provide an accurate modeling framework for many processes [13], [20], [21], [29], [37] and can be used as abstractions for more complex hybrid dynamical systems [17]. We focus on discrete-time linear switching systems, with the following dynamics:

$$x(t+1) = A_{\sigma(t)}x(t). \quad (1)$$

There, at any time t , $\sigma(t) \in \{1, \dots, M\}$ is the *mode* of the system and each mode corresponds to a matrix from a set of M matrices $\mathcal{A} = \{A_1, \dots, A_M\}$. We call a sequence of modes $\sigma(0)\sigma(1)\dots$ a *switching sequence*.

The question of the stability of a switching system has been a major challenge in the Control Engineering literature in the past decades [27], [28], [38]. We are

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interested in the study of certificates for stability under arbitrary switching.

Definition I.1. *The System (1) is stable under arbitrary switching if there is $K \in \mathbb{R}$ such that for any switching sequence $\sigma(0)\sigma(1)\dots$, where $\sigma(t) \in \{1, \dots, M\}$, the trajectories satisfy*

$$\forall x(0) \in \mathbb{R}^n, \forall t \in \mathbb{N} : \|x(t)\| \leq K\|x(0)\|.$$

The System is asymptotically stable if it is stable and furthermore, for any switching sequence and initial condition, $\lim_{t \rightarrow \infty} \|x(t)\| = 0$.

The problem of deciding whether or not a switching system is stable under arbitrary switching is in general undecidable (see e.g. [8], [23]). Nevertheless, several tools have been developed, which provide *semi-algorithms* to decide *asymptotic stability* [5]–[7], [14], [25], [30], [31], [33].

A popular approach to assess stability for switching systems is to search for a *common Lyapunov function (CLF)*. The method is attractive because stable arbitrary switching systems always have a CLF (see e.g. [23, Theorem 2.2]). However, in general, whatever the technique used to search for such a function, if it is tractable, it can only provide conservative stability certificates. The search for a *common quadratic Lyapunov function* (see e.g. [28, Section II-A]) illustrates this fact well. There, the goal is to find a *positive definite quadratic function* $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} : x \mapsto x^T Q x$, for a positive definite matrix $Q \succ 0$, such that

$$\forall \sigma \in \{1, \dots, M\}, \forall x \in \mathbb{R}^n : V(A_{\sigma}x) \leq V(x). \quad (2)$$

Checking for the existence of such a function can be done efficiently using convex optimization tools because the *Lyapunov inequalities* (2) are equivalent to a set of *linear matrix inequalities*. Nevertheless, such a Lyapunov function may not exist, even for asymptotically stable systems, see e.g. [27], [28], and Example III.4 below. In order to alleviate this conservativeness, one may rely on more complex parameterizations for the Lyapunov function V at the cost of greater computational efforts (e.g. [31] uses sum-of-squares polynomials, [18] uses max-of-quadratics Lyapunov functions, and [5], [6] uses reachability analysis).

Multiple Lyapunov functions (see [9], [22], [38]) arise as an alternative to the search of common Lyapunov functions. Here again, multiple *quadratic* Lyapunov functions such as those introduced in [7], [12], [14], [25] hold special interest because checking for their existence also amounts to solving a set of linear matrix inequalities. For example, let us consider a switching System (1) on $M = 2$ modes. The multiple Lyapunov function proposed in [11] is composed of two positive definite quadratic functions $V_a, V_b : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ that satisfy the following sets of inequalities $\forall x \in \mathbb{R}^n$:

$$\begin{aligned} V_a(A_1x) &\leq V_a(x), \\ V_b(A_1x) &\leq V_a(x), \\ V_a(A_2x) &\leq V_b(x), \\ V_b(A_2x) &\leq V_b(x). \end{aligned} \quad (3)$$

These tools make use of convex optimization and linear matrix inequalities in order to provide powerful algorithms for the stability analysis of switching systems. In order to further analyze such tools, Ahmadi et al. recently introduced the concept of *Path-Complete Lyapunov functions* [1]. There, multiple Lyapunov functions such as the one of [11] mentioned above are represented by *directed and labeled graphs*, see Figure 1.

More precisely, let $\mathcal{G} = (S, E)$ be a graph where S is the set of nodes, and $E \subset S \times S \times \{1, \dots, M\}$ is the set of directed edges labeled by one of the M modes¹ of the System (1). To each node $s \in S$ of the graph, we assign one positive definite quadratic function $V_s : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$. An edge $(s, d, \sigma) \in E$ then encodes the following inequality:

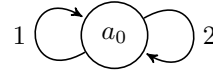
$$\forall x \in \mathbb{R}^n : V_d(A_\sigma x) \leq V_s(x). \quad (4)$$

This formalism provides a framework under which to unify, generalize and study multiple Lyapunov functions such as cited above. Fundamental properties of a multiple Lyapunov function represented by a graph \mathcal{G} can be deduced from the properties of that graph. In particular, in [1], [24] the authors provide a characterization of the set of graphs that *represent stability certificates for switching systems on M modes*. This property, known as *Path-Completeness*, leads to the concept of *Path-Complete Lyapunov functions* (see Definition II.3 below).

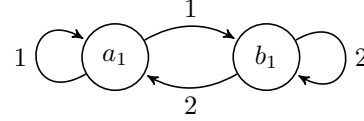
Several challenges exist in the study of Path-Complete Lyapunov functions, in particular for matters related to how these certificates compare with each other with respect to their conservativeness². In this paper we first ask a natural question which aims at revealing the

¹In a more general setting, the labels on the edges can be taken as finite sequences of modes. Our results extend there through the so-called *expanded graph* [1, Definition 2.1].

²Preliminary versions of our results have been presented in [2], [3].



(a) Graph \mathcal{G}_0 , corresponding to the inequalities (2), for a common Lyapunov function.



(b) Graph \mathcal{G}_1 , corresponding to the inequalities (3), leading to the multiple Lyapunov function of [11] for a system on two modes.

Figure 1: Examples of labeled and directed graphs.

connection to classic Lyapunov theory:

Q1: *Can any Path-Complete Lyapunov function be represented as a common Lyapunov function?*

We answer this question affirmatively in Theorem III.8. We show that if a System (1) has a PCLF for a graph $\mathcal{G} = (S, E)$ and with functions $V_s, s \in S$, the system has a common Lyapunov function of the form

$$V(x) = \min_{S_1, \dots, S_k \subseteq S} \left(\max_{s \in S_i} V_s(x) \right), \quad (5)$$

for some finite integer k , where the sets S_i are subsets of the nodes of \mathcal{G} . Our proof is constructive and makes use of a classical tool from automata theory, namely the *observer automaton* [10]. We then discuss in Subsection III-B the conservativeness of these common Lyapunov functions when using quadratic Path-Complete Lyapunov functions and argue that it is ultimately linked with the combinatorial nature of the graph itself.

Motivated by this, we provide in Section IV answers to the following question:

Q2: *When does one graph lead to systematically less conservative stability certificates than another?*

We say that a graph \mathcal{G} is *more conservative* than a graph \mathcal{G}' if for any set of matrices \mathcal{A} , the solvability of the LMIs corresponding to \mathcal{G} implies that of the LMIs for \mathcal{G}' . We provide an algorithmic sufficient condition in Theorem IV.4, inspired by existing ad-hoc proofs for particular cases such as the ones presented in [1, Section 4 and 5], [33, Theorem 3.5], [16, Theorem 20].

Finally, in Section V, we comment on our results and conclude our work.

Remark 1.2. *For the clarity of exposition, and because this is by far the most popular case, we restrict the presentation to linear switching systems under arbitrary switching and consider Path-Complete Lyapunov functions with quadratic functions as pieces. Our results can be generalized in several directions, e.g. to pieces that are continuous, positive definite and*

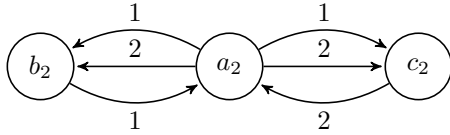
radially increasing functions, or to constrained switching systems [33].

II. PRELIMINARIES

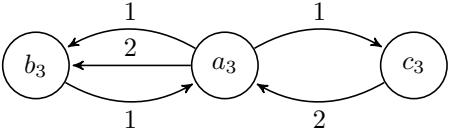
Given an integer M , we let $\langle M \rangle$ denote the set $\{1, \dots, M\}$. Given a discrete set X , we let $|X|$ denote the cardinality of the set.

We now define the central concept of this paper (see Figure 2 for illustrations).

Definition II.1 (Path-Completeness). *A graph $\mathcal{G} = (S, E)$ is Path-Complete if for any $k \geq 1$ and any sequence $\sigma = \sigma_1 \dots, \sigma_k, \sigma_i \in \langle M \rangle$, there is a path in the graph $(s_i, s_{i+1}, \sigma_i)_{i=1,2,\dots,k}$ with $(s_i, s_{i+1}, \sigma_i) \in E$.*



(a) Path-Complete graph \mathcal{G}_2 . It corresponds to a multiple Lyapunov function with 3 functions $V_{a_2}, V_{b_2}, V_{c_2}$ satisfying 6 Lyapunov inequalities.



(b) Graph \mathcal{G}_3 . It is not Path-Complete as it cannot generate the sequence 222.

Figure 2: Illustration for Definition II.1.

As said above, a Path-Complete Lyapunov function is a multiple Lyapunov function where Lyapunov inequalities between a set of *quadratic positive definite functions* are encoded in a labeled and directed graph $\mathcal{G} = (S, E)$, with one function per node in S . We represent a set of such functions in a vector form.

Definition II.2 (VLFC). *A Vector Lyapunov Function Candidate (VLFC) is a vector function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^N$, where each element $V_i : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, $i \in \langle N \rangle$ is a positive definite quadratic function.*

Given a graph $\mathcal{G} = (S, E)$ and a VLFC $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S|}$ we let V_s be the function for the node $s \in S$.

Definition II.3 (Path-Complete Lyapunov function (PCLF)). *Given a switching System (1) on a set of M matrices \mathcal{A} of dimension n and a Path-Complete graph $\mathcal{G} = (S, E)$ on M labels, a Path-Complete Lyapunov function for \mathcal{G} and \mathcal{A} is a VLFC $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S|}$ such that*

$$\forall x \in \mathbb{R}^n, \forall (s, d, \sigma) \in E : V_d(A_\sigma x) \leq V_s(x). \quad (6)$$

We write $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$ to denote the fact that V is a Path-Complete Lyapunov function for \mathcal{G} and \mathcal{A} .

III. EXTRACTING COMMON LYAPUNOV FUNCTIONS

In this section we are given a Path-Complete graph \mathcal{G} , a set of matrices \mathcal{A} , and a Path-Complete Lyapunov function $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$, and we want to construct a common Lyapunov function for the switching system on the set \mathcal{A} . In order to do so, we provide an algorithm relying on concepts from *Automata theory* (see e.g. [10, Chapter 2]). Preliminary versions of our results in this section have been presented in the conference paper [2].

A. Main Result

Our main result exploits the structure of Path-Complete graphs to combine the functions of a PCLF into a common Lyapunov function for the System (1). The following proposition is the first step to achieve this.

Proposition III.1. *Consider a Path-Complete graph $\mathcal{G} = (S, E)$ on M labels, a set of M matrices \mathcal{A} , and a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$. Take two subsets P and Q of S .*

- *If there is a label σ such that*

$$\forall p \in P, \exists q \in Q : (p, q, \sigma) \in E, \quad (7)$$

then

$$\forall x \in \mathbb{R}^n : \min_{q \in Q} V_q(A_\sigma x) \leq \min_{p \in P} V_p(x). \quad (8)$$

- *If there is a label σ such that*

$$\forall q \in Q, \exists p \in P : (p, q, \sigma) \in E, \quad (9)$$

then

$$\forall x \in \mathbb{R}^n : \max_{q \in Q} V_q(A_\sigma x) \leq \max_{p \in P} V_p(x). \quad (10)$$

The geometric proof below provides the intuition for the mechanisms underlying Theorem III.8.

Proof. Take a graph $\mathcal{G} = (S, E)$ with M labels, a set of M matrices \mathcal{A} , and a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$. For any node $s \in S$, define the one-level set

$$X_s = \{x \in \mathbb{R}^n \mid V_s(x) \leq 1\}.$$

For any edge $(s, d, \sigma) \in E$, (2) is equivalent to $A_\sigma X_s \subseteq X_d$, where $A_\sigma X_s = \{A_\sigma x \mid x \in X_s\}$. In light of this, (7) is equivalent to

$$\forall p \in P, \exists q \in Q : A_\sigma X_p \subseteq X_q,$$

and from there it is easy to conclude that

$$A_\sigma \bigcup_{p \in P} X_p = \bigcup_{p \in P} A_\sigma X_p \subseteq \bigcup_{q \in Q} X_q.$$

This is then equivalent to (8) since the union of the level sets of the functions $V_q, q \in Q$ is the level set of the

function $\min_{q \in Q} V_q$.
Similarly, (9) is

$$\forall q \in Q, \exists p \in P : A_\sigma X_p \subseteq X_q$$

which in turn implies

$$A_\sigma \bigcap_{p \in P} X_p = \bigcap_{p \in P} A_\sigma X_p \subseteq \bigcap_{q \in Q} X_q,$$

which is equivalent to (10) since the the intersection level sets of the functions $V_q, q \in Q$ is the level set of the function $\max_{q \in Q} V_q$. \square

The above proposition can already be put to good use to extract common Lyapunov functions from Path-Complete Lyapunov functions where the graph is either *complete* or *co-complete* (see [36] Definition 1.12).

Definition III.2 ((Co-)Complete Graph). A graph $\mathcal{G} = (S, E)$ is complete if for all $s \in S$, for all $\sigma \in \langle M \rangle$ there exists at least one edge $(s, q, \sigma) \in E$.

The graph is co-complete if for all $q \in S$, for all $\sigma \in \langle M \rangle$, there exists at least one edge $(s, q, \sigma) \in E$.

Note that the graph \mathcal{G}_1 in Figure 1b is co-complete. One can check that if a graph is (co-)complete, it is Path-Complete.

Corollary III.3. Consider a graph \mathcal{G} on M modes, a set of M matrices \mathcal{A} , and a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$.

- If \mathcal{G} is complete, then $\bar{V}(x) = \min_{s \in S} V_s(x)$ is a common Lyapunov function for System (1).
- If \mathcal{G} is co-complete, then $\bar{V}(x) = \max_{s \in S} V_s(x)$ is a common Lyapunov function for System (1).

Example III.4. We consider the following switching system consisting of $M = 2$ modes: $x(t+1) = A_{\sigma(t)}x(t)$, $\sigma(t) \in \{1, 2\}$, with

$$A_1 = \alpha \begin{pmatrix} 1.3 & 0 \\ 1 & 0.3 \end{pmatrix}, A_2 = \alpha \begin{pmatrix} -0.3 & 1 \\ 0 & -1.3 \end{pmatrix}, \quad (11)$$

with $\alpha = (1.4)^{-1}$. This system does not have a common quadratic Lyapunov function. However, for the graph \mathcal{G}_1 represented in Figure 1b, we have the following Path-Complete Lyapunov function:

$$\mathcal{V} = \left\{ \begin{array}{l} V_{a_1}(x) = 5x_1^2 + x_2^2, \\ V_{b_1}(x) = x_1^2 + 5x_2^2 \end{array} \right\}. \quad (12)$$

Since \mathcal{G}_1 is co-complete, the function $\bar{V}(x) = \max(V_{a_1}(x), V_{b_1}(x))$ is a common Lyapunov function for the system. This is represented in Figure 3, where we see that the intersection of the level sets of V_{a_1} and V_{b_1} , which is that of \bar{V} , is itself invariant.

In order to tackle Path-Complete graphs that are neither complete nor co-complete (as in Figure 4), we introduce the following concept.

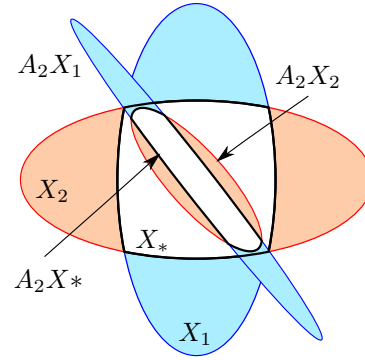


Figure 3: Example III.4: Geometric representation of the Path-Complete stability criterion corresponding to the graph \mathcal{G}_1 in Figure 1b. The ellipsoids X_1 and X_2 are the level sets of the quadratic functions V_{a_1} and V_{b_1} from Example III.4 respectively, and X_* is the level set of $\max(V_{a_1}(x), V_{b_1}(x))$. The set X_* is invariant, as illustrated by the fact that the set A_2X_* is in X_* .

Definition III.5 (Observer Graph, [10, Section 2.3.4]). Consider a graph $\mathcal{G} = (S, E)$. The observer graph $\mathcal{G}^{obs} = (S^{obs}, E^{obs})$ is a graph where each node corresponds to a subset of S , i.e. $S^{obs} \subseteq 2^S$, and is constructed as follows:

1. Initialize $S^{obs} := \{S\}$ and $E^{obs} := \emptyset$.
2. Let $X := \emptyset$. For each pair $(P, \sigma) \in S^{obs} \times \langle M \rangle$:
 - (i) Compute $Q := \bigcup_{p \in P} \{q \mid (p, q, \sigma) \in E\}$.
 - (ii) If $Q \neq \emptyset$, set $E^{obs} := E^{obs} \cup \{(P, Q, \sigma)\}$ then $X := X \cup Q$.
3. If $X \subseteq S^{obs}$, then the observer is given by $\mathcal{G}^{obs} = (S^{obs}, E^{obs})$. Else, let $S^{obs} := S^{obs} \cup X$ and go to step 2.

Example III.6. Consider the graph \mathcal{G}_4 in Figure 4. The

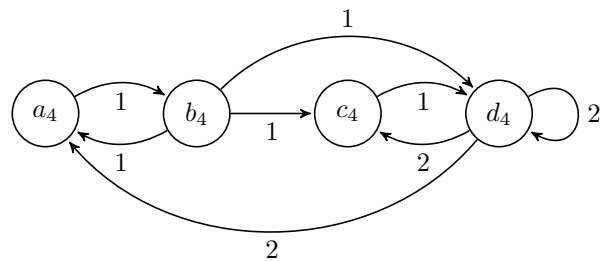


Figure 4: Path-Complete graph \mathcal{G}_4 for Example III.6.

observer graph \mathcal{G}_4^{obs} is given in Figure 5. The first run through step 2 in Definition III.5 is as follows. We have $P = S$. For $\sigma = 1$ the set Q is again S itself: indeed, each node $s \in S$ has at least one inbound edge with the label 1. For $\sigma = 2$, since node b_4 has no inbound edge labeled 2, we get $Q = \{a_4, c_4, d_4\}$. This set is then added to S^{obs} in step 3, and the algorithm repeats step

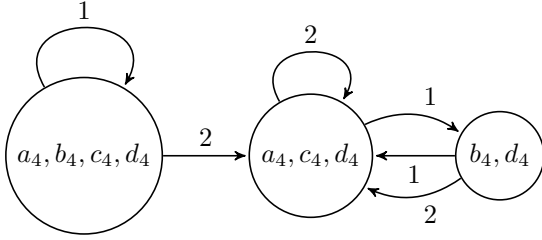


Figure 5: Observer graph \mathcal{G}_4^{obs} of the graph \mathcal{G}_4 in Figure 4. The nodes of \mathcal{G}_4^{obs} are associated to sets of nodes of \mathcal{G}_4 .

2 with the updated S^{obs} .

Remark III.7. The observer automaton is presented in [10, Section 2.3.4]. Our definition of the observer graph is an adaption in the particular case where the automaton considered has all states marked both as starting and accepting states.

We now introduce the main result of this section.

Theorem III.8 (CLF Representation of a PCLF). Consider a set of M matrices \mathcal{A} and a Path-Complete graph \mathcal{G} . If there is a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$, then a common Lyapunov function for System (1) is given by

$$V^*(x) = \min_{Q \in S^{obs}} \left(\max_{s \in Q} V_s(x) \right), \quad (13)$$

where S^{obs} is the set of nodes of the observer graph $\mathcal{G}^{obs} = (S^{obs}, E^{obs})$ of the graph \mathcal{G} .

Proof. Consider a Path-Complete graph $\mathcal{G} = (S, E)$, a set of matrices \mathcal{A} , and a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$. Construct the observer graph $\mathcal{G}^{obs} = (S^{obs}, E^{obs})$. By construction, there is an edge $(P, Q, \sigma) \in E^{obs}$ if and only if $Q = \cup_{p \in P} \{q \mid (p, q, \sigma) \in E\}$. This means that $\forall q \in Q, \exists p \in P$ such that $(p, q, \sigma) \in E$, which is (9). Consequently, from Proposition III.1, we have that

$$(P, Q, \sigma) \in E^{obs} \Rightarrow \forall x \in \mathbb{R}^n : \max_{q \in Q} V_q(A_\sigma x) \leq \max_{p \in P} V_p(x).$$

We now consider a set of functions $\tilde{V} = \{\tilde{V}_Q, Q \in S^{obs}\}$ where $\tilde{V}_Q = \max_{q \in Q} V_q$, and observe that

$$\forall (P, Q, \sigma) \in E^{obs}, \forall x \in \mathbb{R}^n : \tilde{V}_Q(A_\sigma x) \leq \tilde{V}_P(x),$$

which means that these functions satisfy all Lyapunov inequalities encoded in \mathcal{G}^{obs} .

Our next step is now to make use of Proposition III.1 and Corollary III.3 to show that the pointwise-minimum of these functions is a CLF for the system.

To achieve this, first observe that the proof of Proposition III.1 holds verbatim when instead of having a PCLF

where each entry V_s is quadratic, we take each entry to be a *pointwise-maximum of quadratics* as in the above. Second, we claim that the *observer graph of a Path-Complete graph is complete*. We prove this by contraposition: if $\mathcal{G}^{obs} = (S^{obs}, E^{obs})$ is not complete, then it must be so that \mathcal{G} is not Path-Complete. We emphasize that the nodes of an observer graph correspond to sets of nodes in the original graph, and refer to them as thus.

Assume that there is one set of nodes $P \in S^{obs}$, $P \subsetneq S$, and a label $\sigma^* \in \langle M \rangle$ such that there are no edges $(P, Q, \sigma) \in E^{obs}$. By construction of \mathcal{G}^{obs} , there are directed paths from the node $S \in S^{obs}$ (corresponding to the full set of nodes in \mathcal{G}) to the node P above. Take any of such paths, and let $\sigma_1, \dots, \sigma_k$ be the sequence of labels on that path. By the definition of the observer graph, we know that P contains all the nodes of \mathcal{G} that are the destination of some path carrying that sequence of labels. By our choice of P , we know that none of these nodes have an outgoing edge with the label σ^* . Otherwise, there would be an edge $(P, Q, \sigma^*) \in E^{obs}$ for some Q . We conclude that the sequence $\sigma_1 \dots \sigma_k \sigma^*$ can not be found on a path in \mathcal{G} , and \mathcal{G} is therefore not Path-Complete.

Since \mathcal{G}^{obs} is complete, we can use Corollary III.3 to deduce that the function

$$V^* = \min_{Q \in S^{obs}} \tilde{V}_Q$$

is a common Lyapunov function for the system (II.1) on the set \mathcal{A} , which concludes the proof. \square

Example III.9. Consider the following set of matrices taken from [18, Example 11]:

$$\mathcal{A} = \left\{ \alpha \begin{pmatrix} 0.3 & 1 & 0 \\ 0 & 0.6 & 1 \\ 0 & 0 & 0.7 \end{pmatrix}, \alpha \begin{pmatrix} 0.3 & 0 & 0 \\ -0.5 & 0.7 & 0 \\ -0.2 & -0.5 & 0.7 \end{pmatrix} \right\} \quad (14)$$

with the choice of $\alpha = 1.03$. That switching system has a Path-Complete Lyapunov function V for the graph $\mathcal{G}_4 = (S, E)$ in Figure 4. After inspecting the observer graph \mathcal{G}^{obs} , we compute a common Lyapunov function for the system as the minimum of the two functions $\max(V_{a_4}, V_{c_4}, V_{d_4})$ and $\max(V_{b_4}, V_{d_4})$. The last term $\max_{s \in S} (V_s)$ can be omitted here. Figure 6 shows the evolution in time of the four functions, and that of the common Lyapunov function, from the initial condition $x(0) = (0 \ 0 \ -1)^\top$, and for the periodic switching sequence repeating the pattern 21111.

B. Comparison with classical piecewise quadratic Lyapunov Functions.

Our results highlight a link between Path-Complete Lyapunov functions and common Lyapunov functions

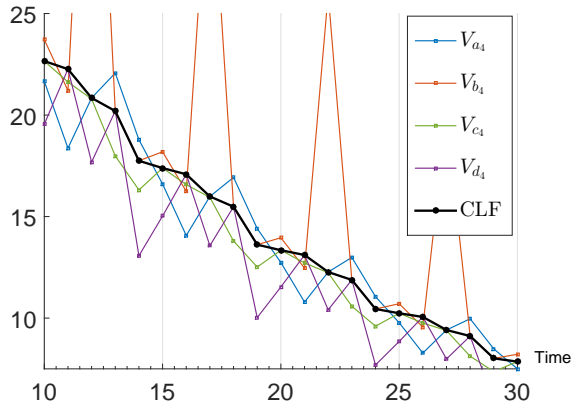


Figure 6: Evolution of a PCLF for the system in Example III.9 for the graph \mathcal{G}_4 in Figure 4. Observe that none of the pieces of the PCLF decreases monotonically. However, the common Lyapunov function (in black) does decrease monotonically.

that are mins or max of functions. It is thus natural to ask whether or not *any* Lyapunov function of the form (5) can be induced from a Path-Complete graph with as many nodes as the number of pieces of the function itself. We give a negative answer to this question, under the form of a counter example. To do so, we use the system in [18, Example 11], where min-of-quadratic and max-of-quadratic Lyapunov functions have been studied.

Example III.10. Consider the switching systems on the two modes with the matrices presented in (14) for $\alpha = 1$. The system has a max-of-quadratics Lyapunov function $x \mapsto \max\{V_1(x), V_2(x)\}$, with $V_i(x) = (x^\top Q_i x)$, where

$$Q_1 = \begin{pmatrix} 36.95 & -36.91 & -5.58 \\ -36.91 & 84.11 & -38.47 \\ -5.58 & -38.47 & 49.32 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 13.80 & -6.69 & 4.80 \\ -6.69 & 21.87 & 10.11 \\ 4.80 & 10.11 & 82.74 \end{pmatrix}.$$

These functions are obtained by solving a set of Bilinear Matrix Inequalities (BMIs) (see [18, Section 5]). However, for the same system, we cannot obtain a Path-Complete Lyapunov function that can then be represented as a max-of-quadratics with two pieces. More precisely, we considered all the graphs $\mathcal{G} = (S, E)$ that are co-complete on two nodes. There are 16 of them in total. For each of them, the convex optimization program corresponding to the search of a Path-Complete Lyapunov function has no solutions.

The example above involves two approaches for trying to compute a max-of-quadratics Lyapunov function. The first approach relies on solving a set of BMIs. As pointed

out in [18], these BMIs can be hard to solve in general (there is no polynomial-time algorithm to solve them). In contrast, searching for a quadratic Path-Complete Lyapunov function can be done efficiently by solving linear matrix inequalities using convex optimization tools. We can therefore efficiently check if a max-of-quadratics common Lyapunov function corresponding to a PCLF exists. Nevertheless, as one can see in Example III.10, the BMI approach can be less conservative than the approach using PCLFs. We leave for further work the question of understanding whether or not the BMI approach is less conservative than the PCLF approach in general.

Additionally, among the 16 Path-Complete graphs tested in Example III.10, some lead to more conservative stability certificates than others. For example, some of these graphs have, at one of their two nodes, two self loops with different labels, as it is the case for the graph \mathcal{G}_0 in Figure 1a. If a Path-Complete Lyapunov Function is found for such a graph, then the function for that node is itself a common quadratic Lyapunov function for the system, and thus that graph is at least as conservative as the common quadratic Lyapunov function technique. However, there are pairs of graphs for which none of them is more conservative than the other. This fact is illustrated in the following example.

Example III.11. Consider the three graphs \mathcal{G}_1 in Figure 1b, \mathcal{G}_5 and \mathcal{G}_6 in Figure 7. These graphs are co-complete.

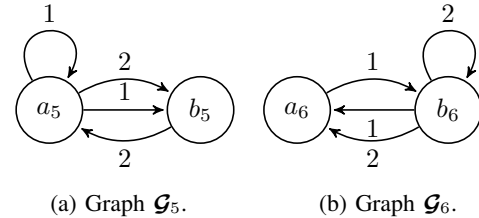


Figure 7: Two co-complete Path-Complete graphs.

The system on two modes with matrices

$$\mathcal{A} = \left\{ \alpha \begin{pmatrix} -0.5 & -1.1 \\ 0.9 & 1.5 \end{pmatrix}, \alpha \begin{pmatrix} 0.2 & 1.0 \\ 0.5 & 0.5 \end{pmatrix} \right\}, \quad (15)$$

with $\alpha = (1.05)^{-1}$, has a quadratic PCLF for \mathcal{G}_1 in Figure 1b but neither for \mathcal{G}_5 nor \mathcal{G}_6 in Figures 7a and 7b. The system on two modes with matrices

$$\mathcal{A} = \left\{ \alpha \begin{pmatrix} 0 & -0.2 \\ 0.8 & 0 \end{pmatrix}, \alpha \begin{pmatrix} 0.25 & 0.4 \\ 0.1 & 0.3 \end{pmatrix} \right\}, \quad (16)$$

with $\alpha = (0.55)^{-1}$ has a quadratic PCLF for the graph \mathcal{G}_6 , but neither for \mathcal{G}_3 or \mathcal{G}_5 . The same set of matrices, when swapping the two modes, will have a PCLF for \mathcal{G}_5 , but not for the two other graphs.

The fact that PCLFs can be represented as common Lyapunov functions closes a gap between multiple and common Lyapunov functions techniques. However, the discussion above illustrates that this does not directly help us to compare the conservativeness of PCLFs. Indeed, we see that several PCLFs lead to CLFs with similar structures, but even among these PCLFs it remains non-trivial to decide which ones are more conservative than others. For this reason, in the next section, we provide novel tools for comparing the conservativeness of Path-Complete Lyapunov functions based on their graphs.

IV. COMPARING GRAPHS

Quadratic Path-Complete Lyapunov functions are particularly attractive in practice since given a graph $\mathcal{G} = (S, E)$ and a set of matrices \mathcal{A} , one can verify their existence by solving a set of LMIs. More precisely, we search for quadratic forms $\{Q_s \succ 0, s \in S\}$ satisfying the matrix inequalities

$$\forall (s, d, \sigma) \in E : A_\sigma^\top Q_d A_\sigma \preceq Q_s. \quad (17)$$

Our goal in this section is to provide systematic tools to decide when, given two graphs \mathcal{G} and \mathcal{G}' , it is true that for any set of matrices \mathcal{A} , the existence of a quadratic Path-Complete Lyapunov function for \mathcal{G} implies that of a quadratic PCLF for \mathcal{G}' .

This relates to the setting of [1], where Path-Complete Lyapunov functions with quadratic pieces are used for the *approximation of the exponential growth rate*, a.k.a. the *joint spectral radius* [23], [35], of switching systems. More precisely, for each graph $\mathcal{G} = (S, E)$ with labels in $\langle M \rangle$, and for any set of M matrices \mathcal{A} we let

$$\begin{aligned} \gamma(\mathcal{G}, \mathcal{A}) &= \inf_{\{Q_s, s \in S\}, \gamma} \gamma : \\ &\forall (s, d, \sigma) \in E : A_\sigma^\top Q_d A_\sigma \preceq \gamma^2 Q_s, \\ &\forall s \in S : Q_s \succ 0. \end{aligned} \quad (18)$$

We can then capture the fact that a graph leads to more conservative stability certificate than another by the following *ordering relation*: given two Path-Complete graphs \mathcal{G} and \mathcal{G}' with labels in $\langle M \rangle$:

$$\begin{aligned} \mathcal{G} \leq \mathcal{G}' &\text{ if } \forall n \in \mathbb{N}, \forall \mathcal{A} \subset \mathbb{R}^{n \times n} : \\ \gamma(\mathcal{G}, \mathcal{A}) &\geq \gamma(\mathcal{G}', \mathcal{A}). \end{aligned} \quad (19)$$

The above relation does not form a total order (see Example III.11, where \mathcal{G}_1 , \mathcal{G}_5 and \mathcal{G}_6 are *incomparable*).

The techniques we develop herein are inspired by existing ad-hoc proofs in the literature applying to particular cases/sets of graphs. Examples of these ad-hoc proofs can be found in [1, Proposition 4.2 and Theorem 5.4], [33, Theorem 3.5]. Typically, given two graphs \mathcal{G} and \mathcal{G}' , these proofs proceed in two steps to show that $\mathcal{G} \leq \mathcal{G}'$. The first step is to propose a construction

to transform any possible quadratic PCLF for \mathcal{G} into a candidate quadratic PCLF for \mathcal{G}' . The second step is to check that the construction does indeed provide a PCLF for \mathcal{G}' .

Example IV.1. Take any set \mathcal{A} of two matrices, and let $V \sim \text{pclf}(\mathcal{G}_2, \mathcal{A})$ be a Path-Complete Lyapunov function for the graph $\mathcal{G}_2 = (S_2, E_2)$ in Figure 2a. Then, one can show that a Path-Complete Lyapunov function for the graph $\mathcal{G}_1 = (S_1, E_1)$ in Figure 1b is given by $U = (U_{a_1}, U_{b_1})^\top$, with

$$U_{a_1} = V_{a_2} + V_{b_2}, U_{b_1} = V_{a_2} + V_{c_2}. \quad (20)$$

To prove this we need to show that each edge $(s, d, \sigma) \in E_2$ represents a valid Lyapunov inequality for the functions in U . For example, consider the edge $(a_1, b_1, 2) \in E_1$. We need to show that

$$\forall x \in \mathbb{R}^n : V_{a_2}(A_2 x) + V_{c_2}(A_2 x) \leq V_{a_2}(x) + V_{b_2}(x)$$

holds true. To do so, it suffices to write down and sum up the inequalities corresponding to the edges $(a_2, b_2, 2)$ and $(c_2, b_2, 2)$ in E_2 , which are assumed to hold true for the choice of functions V . The reader can verify that a similar reasoning can be applied to all edges in E_1 . Therefore, for any set of matrices \mathcal{A} , as long as $V \sim \text{pclf}(\mathcal{G}_2, \mathcal{A})$, the set U defined above satisfies $U \sim \text{pclf}(\mathcal{G}_1, \mathcal{A})$.

In this section, we focus on providing algorithmic tools to decide whenever a construction as in Example IV.1 exists. We formalize this as follows.

Definition IV.2. Consider two Path-Complete graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$ on the same labels $\langle M \rangle$. We write

$$\mathcal{G} \leq_\Sigma \mathcal{G}'$$

if there is a matrix³ $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$, satisfying

$$\forall s' \in S' : \sum_{s \in S} C_{s', s} \geq 1, \quad (21)$$

such that for any set of M matrices \mathcal{A} of dimension n and Path-Complete Lyapunov function $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$, the VLFC $U : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S'|}$ where

$$\forall s' \in S', \forall x \in \mathbb{R}^n : U_{s'}(x) = \sum_{s \in S} C_{s', s} V_s(x)$$

satisfies $U \sim \text{pclf}(\mathcal{G}', \mathcal{A})$.

Given two graphs, the property $\mathcal{G} \leq_\Sigma \mathcal{G}'$ is a *sufficient condition* for the ordering $\mathcal{G} \leq \mathcal{G}'$ (19). This is due to the fact that the set of quadratic functions is closed under positive combinations.

³The element $C_{s', s}$ appearing in (21) is the element for the row of $s' \in S'$ and the column of $s \in S$ of $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$.

The following allows us to express (6) with vector inequalities.

Definition IV.3. Given a graph $\mathcal{G} = (S, E)$ with a set of labels $\langle M \rangle$, and $\sigma \in \langle M \rangle$, we define the two matrices $\mathbf{S}^\sigma(\mathcal{G}) \in \{0, 1\}^{|E_\sigma| \times |S|}$ and $\mathbf{D}^\sigma(\mathcal{G}) \in \{0, 1\}^{|E_\sigma| \times |S|}$ as follows:

$$\begin{aligned} \mathbf{S}_{e,s}^\sigma &= 1 \Leftrightarrow \exists d \in S : e = (s, d, \sigma) \in E, \\ \mathbf{D}_{e,d}^\sigma &= 1 \Leftrightarrow \exists s \in S : e = (s, d, \sigma) \in E, \end{aligned} \quad (22)$$

where $E_\sigma \subset E$ is the set of edges with label σ .

The construction of these matrices is illustrated in Example IV.5. For a graph \mathcal{G} and label σ , the matrix $\mathbf{S}^\sigma(\mathcal{G}) - \mathbf{D}^\sigma(\mathcal{G})$ is the incidence matrix [10] of the subgraph of \mathcal{G} having only the edges with label σ .

Given a set of M matrices \mathcal{A} of dimension n and a Path-Complete graph $\mathcal{G} = (S, E)$ on M labels, a VLFC $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S|}$ satisfies $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$ if and only if

$$\forall \sigma \in \langle M \rangle, \forall x \in \mathbb{R}^n : \mathbf{D}^\sigma(\mathcal{G})V(A_\sigma x) \leq \mathbf{S}^\sigma(\mathcal{G})V(x), \quad (23)$$

where the vector inequality is taken entrywise.

Our main result in this section is the following theorem, whose proof is detailed in Subsection IV-A.

Theorem IV.4. Consider two graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$ on M labels. The following statements are equivalent.

- The graphs satisfy $\mathcal{G} \leq_\Sigma \mathcal{G}'$.
- There exists a matrix $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$ satisfying (21) and one matrix $K_\sigma \in \mathbb{R}_{\geq 0}^{|E'_\sigma| \times |E_\sigma|}$ per label $\sigma \in \langle M \rangle$ such that

$$\mathbf{S}^\sigma(\mathcal{G}')C \geq K_\sigma \mathbf{S}^\sigma(\mathcal{G}), \quad (24)$$

$$\mathbf{D}^\sigma(\mathcal{G}')C \leq K_\sigma \mathbf{D}^\sigma(\mathcal{G}). \quad (25)$$

Example IV.5. In this example, we apply Theorem IV.4 to show that $\mathcal{G}_1 \leq_\Sigma \mathcal{G}_2$, where \mathcal{G}_1 is represented in Figure 1b and \mathcal{G}_2 is represented in Figure 2a.

We first construct the matrices \mathbf{S}^σ and \mathbf{D}^σ for the graphs \mathcal{G}_1 and \mathcal{G}_2 and $\sigma \in \langle 2 \rangle$. In order to do so, we need to set a convention for ordering nodes and edges in a graph (to convene to which edge and node corresponds the entry $\mathbf{S}_{1,1}^\sigma(\mathcal{G}_2)$ for example). In any graph represented in this paper, we let the first node be the node marked with “ a_g ”, the second node be that marked with “ b_g ”, etc..., where the subscript g designates the graph itself. We use a lexicographical ordering for the edges, and sort them according to their source node first, then their destination node, and finally their labels. For example, the first edge in \mathcal{G}_4 in Figure 4 is the edge $(a_4, b_4, 1)$, and its third edge would be the edge $(b_4, c_4, 1)$.

With these conventions, we obtain the following matrices:

$$\mathbf{S}^1(\mathcal{G}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}^1(\mathcal{G}_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{S}^2(\mathcal{G}_1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^2(\mathcal{G}_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{S}^1(\mathcal{G}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{D}^1(\mathcal{G}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{S}^2(\mathcal{G}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D}^2(\mathcal{G}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For these choices, a solution to the inequalities (24), (25) is given by

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad K_1 = K_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

A. From algebraic to linear inequalities

The main challenge for the proof of Theorem IV.4 is to show that $a) \Rightarrow b)$. With our matrix/vector notations, Definition IV.2 can be written as follows: given two graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$, $\mathcal{G} \leq_\Sigma \mathcal{G}'$ if there is a matrix $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$ satisfying (21) such that

$$\forall \mathcal{A}, \forall V \sim \text{pclf}(\mathcal{G}, \mathcal{A}) : U = CV \sim \text{pclf}(\mathcal{G}', \mathcal{A}).$$

In this subsection, we show that these algebraic conditions are equivalent to the inclusion of one polyhedral cone into another. We begin by investigating the range of values that may be taken by a VLFC $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S|}$, independently of the dimension n .

Lemma IV.6. Take any pair of integers $M, N \geq 1$ and any positive vector $\lambda \in \mathbb{R}_{> 0}^{(M+1)N}$. There exists a point $u \in \mathbb{R}^{M+1}$, a set of matrices $\mathcal{T} = \{T_1, \dots, T_M\}$ in $\mathbb{R}^{(M+1) \times (M+1)}$, and a VLFC $U : \mathbb{R}^{M+1} \rightarrow \mathbb{R}_{\geq 0}^N$ such that

$$\lambda = \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ \vdots \\ \lambda^M \end{pmatrix} = \begin{pmatrix} \mathbf{U}(u) \\ \mathbf{U}(T_1 u) \\ \vdots \\ \mathbf{U}(T_M u) \end{pmatrix}, \quad (26)$$

where for $0 \leq i \leq M$, the block λ^i is a positive vector of dimension N .

If furthermore for a graph $\mathcal{G} = (S, E)$ with $|S| = N$, the vector λ satisfies

$$\forall \sigma \in \langle M \rangle : \mathbf{S}^\sigma(\mathcal{G})\lambda^0 \geq \mathbf{D}^\sigma(\mathcal{G})\lambda^\sigma, \quad (27)$$

then we can pick the VLFC U and the matrices \mathcal{T} such that $U \sim \text{pclf}(\mathcal{G}, \mathcal{T})$.

Proof. We begin with the first part of the Lemma. For all $k \in \{1, \dots, N\}$ we define the quadratic function $U_k : \mathbb{R}^{M+1} \rightarrow \mathbb{R}_{\geq 0}$ as follows:

$$U_k(x) = \sum_{i=0}^M \lambda_k^i x_{i+1}^2. \quad (28)$$

We now take the vector $u \in \mathbb{R}^{M+1}$ such that $u_1 = 1$ and $u_i = 0$ for $i \geq 2$. Clearly, $U_k(u) = \lambda_k^0$, so we have $U(u) = \lambda^0$ by definition.

For the matrices \mathcal{T} , we take for each $\sigma \in \langle M \rangle$ the matrix T_σ defined as follows:

$$\forall 1 \leq i, j \leq M+1, T_{\sigma, i, j} = \delta_{i, \sigma+1} \delta_{j, 1}, \quad (29)$$

where for $1 \leq k, \ell \leq M+1$, $\delta_{k, \ell} = 1$ if $k = \ell$ and $\delta_{k, \ell} = 0$ else. We can then verify that for all $\sigma \in \langle M \rangle$, $U(T_\sigma u) = \lambda^\sigma$. This concludes the proof of the first part.

Let us now take a graph $\mathcal{G} = (S, E)$ with N nodes and M labels, and assume (27) holds for a given label σ . This means that $\forall (s, d, \sigma) \in E$, $(\lambda^0)_s \geq (\lambda^\sigma)_d$. Take any edge $(s, d, \sigma) \in E$, and any $x \in \mathbb{R}^{(M+1)}$. Taking again U as in (28), we have

$$\begin{aligned} U_s(x) &= \sum_{i=0}^M \lambda_s^i x_{i+1}^2 \\ &\geq \lambda_s^0 x_1^2 \geq \lambda_d^\sigma x_1^2 = U_d(T_\sigma x). \end{aligned}$$

Since this holds at all edges of the graph with label σ , and all $x \in \mathbb{R}^{(M+1)}$, we can conclude the proof. \square

Proposition IV.7. Consider two graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$ on M labels and a matrix $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$ satisfying (21). The following statements are equivalent.

- The graphs satisfy $\mathcal{G} \leq_\Sigma \mathcal{G}'$ with the matrix C .
- For any set of M matrices $\mathcal{A} = \{A_1, \dots, A_M\}$ in any dimension n , for any VLFC $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}^{|S|}$, for any $\sigma \in \langle M \rangle$ and any $x \in \mathbb{R}^n$:

$$\mathbf{D}^\sigma(\mathcal{G})V(A_\sigma x) \leq \mathbf{S}^\sigma(\mathcal{G})V(x) \Rightarrow \quad (30)$$

$$\mathbf{D}^\sigma(\mathcal{G}')CV(A_\sigma x) \leq \mathbf{S}^\sigma(\mathcal{G}')CV(x). \quad (31)$$

Proof. The difference between the two statements is subtle yet important. Statement a) is equivalent to “if (30) holds for all $\sigma \in \langle M \rangle$ and all $x \in \mathbb{R}^n$, then so does (31)”. In contrast, statement b) merely states that (31) holds at any point and label where (30) holds.

With this in mind, proving b) \Rightarrow a) is immediate as, under b), if (30) holds for all points $x \in \mathbb{R}^n$ and all $\sigma \in \langle M \rangle$, so does (31).

We prove that a) \Rightarrow b) by using the construction in the proof of Lemma IV.6. Assume that a) holds and consider a set of matrices \mathcal{A} , a VLFC V , and one particular point x^* such that (30) holds at x^* . We show that (31) holds at x^* as well. Let

$$\lambda = \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ \vdots \\ \lambda^M \end{pmatrix} = \begin{pmatrix} V(x^*) \\ V(A_1 x^*) \\ \vdots \\ V(A_M x^*) \end{pmatrix}.$$

Without loss of generality, we can assume λ to be strictly positive. For this vector, we know that $\mathbf{S}^\sigma(\mathcal{G})\lambda^0 \geq \mathbf{D}^\sigma(\mathcal{G})\lambda^\sigma$, or more explicitly,

$$\forall (s, d, \sigma) \in E, (\lambda^0)_s \geq (\lambda^\sigma)_d.$$

Using this information, we seek to show that

$$\mathbf{S}^\sigma(\mathcal{G}')C\lambda^0 \geq \mathbf{D}^\sigma(\mathcal{G}')C\lambda^\sigma \quad (32)$$

holds as well.

We now construct a vector $\tilde{\lambda} \in \mathbb{R}^{N(M+1)}$ defined in blocks such that $\tilde{\lambda}^0 = \lambda^0$, $\tilde{\lambda}^\sigma = \lambda^\sigma$, and for all $\sigma' \neq \sigma$, $\tilde{\lambda}^{\sigma'} = (\min_i \lambda_i^0) \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^{|S|}$ is the vector of all ones. By construction, (27) holds for the vector $\tilde{\lambda}$. Using Lemma IV.6, we take a set of matrices $\mathcal{T} = \{T_{\sigma'}, \sigma' \in \langle M \rangle\}$ of dimension $M+1$ and a VLFC $U : \mathbb{R}^{M+1} \rightarrow \mathbb{R}_{\geq 0}^{|S|}$ such that there is a point $u^* \in \mathbb{R}^{M+1}$ where

$$U(u^*) = \tilde{\lambda}^0, \forall \sigma' \in \langle M \rangle : U(T_{\sigma'} u^*) = \tilde{\lambda}^{\sigma'}.$$

Moreover it must be that $U \sim \text{pclf}(\mathcal{G}, \mathcal{T})$ since (27) holds at all $\sigma' \in \langle M \rangle$.

We can now conclude the proof using statement a): since $U \sim \text{pclf}(\mathcal{G}, \mathcal{T})$, (30) holds for U , \mathcal{T} , and all $u \in \mathbb{R}^{M+1}$. Hence, so does (31). In particular, it holds for the mode σ at the point u^* , and we can conclude the proof since $\mathbf{S}^\sigma(\mathcal{G}')C\lambda^0 \geq \mathbf{D}^\sigma(\mathcal{G}')C\lambda^\sigma$ holds true. \square

The next result allows us to characterize the $\mathcal{G} \leq_\Sigma \mathcal{G}'$ property without having to rely on concepts explicitly related to dynamics (e.g. matrix sets and VLFCs).

Proposition IV.8. Consider two graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$ on M labels and a matrix $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$ satisfying (21). The following statements are equivalent.

- The graphs satisfy $\mathcal{G} \leq_\Sigma \mathcal{G}'$ with the matrix C .
- For any vector

$$\lambda = ((\lambda^0)^\top \quad (\lambda^1)^\top \quad \dots \quad (\lambda^M)^\top)^\top \in \mathbb{R}_{\geq 0}^{(M+1)|S|}$$

with $\lambda^i \in \mathbb{R}_{\geq 0}^{|S|}$, for any $\sigma \in \langle M \rangle$:

$$\mathbf{D}^\sigma(\mathcal{G})\lambda^\sigma \leq \mathbf{S}^\sigma(\mathcal{G})\lambda^0 \Rightarrow \quad (33)$$

$$\mathbf{D}^\sigma(\mathcal{G}')C\lambda^\sigma \leq \mathbf{S}^\sigma(\mathcal{G}')C\lambda^0. \quad (34)$$

Proof. By Proposition IV.7, we can as well prove the equivalence between the statement b) above and the statement of Proposition IV.7-b).

With this in mind, b) \Rightarrow a) is direct.

For the other direction, i.e. a) \Rightarrow b), we therefore need to show that Proposition IV.7 - b) implies the current statement b). By Lemma IV.6, we have that any positive vector $\lambda > 0$ satisfying (33) also satisfies (34). Indeed, given any such vector λ , we can construct a point u , a set of matrices \mathcal{T} and a VLFC U such that these satisfy (30). They must therefore satisfy (31), and therefore λ satisfies (34).

Finally, observe that this implies that the open polyhedral cone defined as the set $P_\sigma = \{\lambda > 0 : (33) \text{ holds}\}$ is included into the cone $P'_\sigma = \{\lambda \geq 0 : (34) \text{ holds}\}$. Hence the closure of P_σ is also in P'_σ , which is equivalent to b) and concludes the proof. \square

We now prove Theorem IV.4. The final stage of the proof uses the following formulation of Farkas' Lemma:

Lemma IV.9 (Farkas' Lemma, [19, Lemma II.2]). *Consider two matrices $A \in \mathbb{R}^{p \times n}$ and $B \in \mathbb{R}^{q \times n}$. The following are equivalent:*

$$\begin{aligned} & \{y \in \mathbb{R}^n : Ay \geq \mathbf{0}, y \geq \mathbf{0}\} \subseteq \{y \in \mathbb{R}^n : By \geq \mathbf{0}\} \\ & \Leftrightarrow \exists K \in \mathbb{R}^{m \times p} : KA \leq B, K \geq \mathbf{0}. \end{aligned}$$

Proof of Theorem IV.4. b) \Rightarrow a): Given a set of M matrices \mathcal{A} of dimension n , assume there is a VLFC $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$. By definition, we have $\forall \sigma \in \langle M \rangle, \forall x \in \mathbb{R}^n : \mathbf{D}^\sigma(\mathcal{G})V(A_\sigma x) \leq \mathbf{S}^\sigma(\mathcal{G})V(x)$. This implies, from (24) and (25), that $\forall \sigma \in \langle M \rangle, \forall x \in \mathbb{R}^n : \mathbf{D}^\sigma(\mathcal{G}')CV(A_\sigma x) \leq \mathbf{S}^\sigma(\mathcal{G}')CV(x)$. We conclude that for any set \mathcal{A} such that there is $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$, it is true that $U = CV \sim \text{pclf}(\mathcal{G}', \mathcal{A})$. Therefore, $\mathcal{G} \leq_\Sigma \mathcal{G}'$. a) \Rightarrow b): Take two graphs $\mathcal{G} = (S, E)$ and $\mathcal{G}' = (S', E')$ and assume that $\mathcal{G} \leq_\Sigma \mathcal{G}'$ through some matrix $C \in \mathbb{R}_{\geq 0}^{|S'| \times |S|}$ that satisfies (21). Then, Proposition IV.8 - b) states the following. For all $\sigma \in \langle M \rangle$, the polyhedral sets

$$P_\sigma = \{x, y \in \mathbb{R}_{\geq 0}^{|S|} \mid \mathbf{S}^\sigma(\mathcal{G})x - \mathbf{D}^\sigma(\mathcal{G})y \geq \mathbf{0}\}, \quad (35)$$

$$P'_\sigma = \{x, y \in \mathbb{R}_{\geq 0}^{|S|} \mid \mathbf{S}^\sigma(\mathcal{G}')Cx - \mathbf{D}^\sigma(\mathcal{G}')Cy \geq \mathbf{0}\} \quad (36)$$

satisfy $P_\sigma \subseteq P'_\sigma$.

For the final step of the proof, we apply Farkas' Lemma (Lemma IV.9 above) to the inclusion $P_\sigma \subseteq P'_\sigma$, there must be a matrix $K_\sigma \in \mathbb{R}^{|E'_\sigma| \times |E_\sigma|}$, where E'_σ and E_σ are the sets of edges with label σ in \mathcal{G}' and \mathcal{G} respectively, satisfying

$$K_\sigma (\mathbf{S}^\sigma(\mathcal{G}) \quad -\mathbf{D}^\sigma(\mathcal{G})) \leq (\mathbf{S}^\sigma(\mathcal{G}') \quad -\mathbf{D}^\sigma(\mathcal{G}')) C, \quad (37)$$

which concludes the proof. \square

Remark IV.10. *The characterization of the ordering $\mathcal{G} \leq_\Sigma \mathcal{G}'$ is remarkable in its simplicity. In its essence, the linear program of Theorem IV.4 mimics the proof scheme used in Example IV.1. This is easier to see by inspecting (37). Given a graph \mathcal{G} on an alphabet $\langle M \rangle$, each row of the matrix $(\mathbf{S}^\sigma(\mathcal{G}) \quad -\mathbf{D}^\sigma(\mathcal{G}))$ corresponds to one edge of the graph (with label σ). The inequality (37) is therefore equivalent to checking that, given a matrix C mapping a PCLF for the graph \mathcal{G} into a PCLF for the graph \mathcal{G}' , we can prove that all inequalities of the graph \mathcal{G}' hold true simply by composing those of \mathcal{G} . The matrices K_σ give us the information regarding how to compose these inequalities.*

Note that the tool recovers those from [2, Section 4], which are based on combinatorial criteria, that ultimately rely on constructing a PCLF for one graph from a PCLF for another graph by positive combination of the pieces of the first one.

B. Discussion and extensions

The approach can be naturally extended to encompass more general proofs of ordering.

Example IV.11. *Consider the graph \mathcal{G}_7 in Figure 8.*

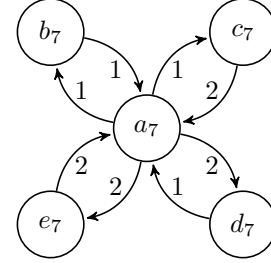


Figure 8: The Path-Complete graph \mathcal{G}_7 .

Given a set of two invertible⁴ matrices $\mathcal{A} = \{A_1, A_2\}$ of dimension n , consider a PCLF $V \sim \text{pclf}(\mathcal{G}, \mathcal{A})$. For any pair of labels $\sigma, \sigma' \in \langle 2 \rangle$, we have

$$\forall x \in \mathbb{R}^n : V_{a_7}(A_\sigma A_{\sigma'} x) \leq V_{a_7}(x). \quad (38)$$

We can show that the existence of a PCLF for this graph implies that of a PCLF for the graph \mathcal{G}_8 in Figure 9 (see e.g. [1, Theorem 5.1], [33, Theorem 3.5]). Indeed,

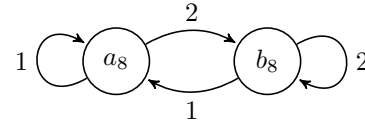


Figure 9: The Path-Complete graph \mathcal{G}_8 .

given the PCLF V above, we can construct a PCLF $U = (U_{a_8}, U_{b_8})^\top$ for \mathcal{G}_8 by taking

$$U_{a_8}(x) = V_{a_7}(x) + V_{a_7}(A_1^{-1}x),$$

$$U_{b_8}(x) = V_{a_7}(x) + V_{a_7}(A_2^{-1}x).$$

This can be proven in a manner similar to that of Example IV.1. For example, take the edge $(a_8, b_8, 2)$ in \mathcal{G}_8 , which corresponds to

$$\forall x \in \mathbb{R}^n : V_{a_7}(A_2 x) + V_{a_7}(x) \leq V_{a_7}(x) + V_{a_7}(A_1^{-1}x).$$

The inequality is satisfied by evaluating the inequality (38) at the point $x' = A_1^{-1}x$, for $\sigma' = 1$ and $\sigma = 2$, which gives $\forall x \in \mathbb{R}^n : V_{a_7}(A_2 x) \leq V_{a_7}(A_1^{-1}x)$.

The main difference here compared to the technique of Example IV.1 is that we need to generate valid inequalities involving a term $A_1^{-1}x$. Such inequalities are easily obtained through a change of variables, as done in

⁴As mentioned in [1, Remark 2.1], we can assume to be dealing with invertible matrices without loss of generality in the current context.

Example IV.11. This leads to a set of linear inequalities on a vector of the form

$$\begin{pmatrix} V(A_2^{-1}x) \\ V(A_1^{-1}x) \\ V(x) \\ V(A_1x) \\ V(A_2x) \end{pmatrix}.$$

By enumerating such inequalities and embedding them in a matrix representation, we can establish sufficient conditions under the form of linear programs similar to that of Theorem IV.4 for when, given a set of matrices \mathcal{A} and two graphs \mathcal{G} and \mathcal{G}' , the existence of a PCLF V on \mathcal{G} for \mathcal{A} implies that of a PCLF U on \mathcal{G}' where each function in U is a combination of functions in V and their compositions with the dynamics and their inverses up to a fixed number of compositions. Further work remains to devise conditions for when this is possible.

V. CONCLUSIONS

Path-Complete criteria are promising tools for the analysis of hybrid and cyber-physical systems. They encapsulate several powerful and popular techniques for the stability analysis of switching systems. Moreover, their range of application seems much wider. First, they can handle switching nonlinear systems as well and are not limited to LMIs and quadratic pieces (see [3], [5]). Second, they have been used to analyze systems where the switching signal is constrained [33]. Third, they can be used beyond stability analysis [15], [32], [34]. However, many questions about these techniques still need to be clarified. In this paper we tackle two fundamental questions. First, we gave a clear interpretation of these criteria in terms of common Lyapunov function: each criterion implies the existence of a common Lyapunov function which can be expressed as the minimum of maxima of sets of functions (Theorem III.8). The combinatorial structure of the graph is used to combine these sets into a CLF. Second, we tackled the problem of deciding when one criterion is more conservative than another, and provide a first systematic approach to the problem (Theorem IV.4). Note that while we focused here on Path-Complete graphs, it is clear that our approach applies to the comparison of any graphs, such as those used in the building of stability certificates for *constrained switching systems* [33]. For further work, the application of the tools used in Section III, such as the observer graph, to refine the analysis of Section IV remains to be investigated. On a more practical side, the common Lyapunov function representation presented in Section III could allow us to better leverage Path-Complete techniques for reachability and invariance analysis [4].

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