Majorana Algebras and Subgroups of the Monster

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Declaration of Originality

I, Madeleine L. Whybrow, declare that this thesis and the research to which it refers are the product of my own work except where acknowledged in accordance with the standard referencing practices of the discipline.

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What we know is a drop, what we don't know is an ocean.

Isaac Newton

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Abstract

Majorana theory was introduced by A. A. Ivanov in [Iva09] as an axiomatisation of certain properties of the $2A$ axes of the Griess algebra. This work was inspired by that of S. Sakuma [Sak07] who reproved certain important properties of the Monster simple group and the Griess algebra using the framework of vertex operator algebras.

The objects at the centre of Majorana theory are known as *Majorana algebras* and are real, commutative, non-associative algebras that are generated by idempotents known as *Majorana axes*. To each Majorana axis, we associate a unique involution in the automorphism group of the algebra, known as a *Majorana involution*.

These involutions form an important link between Majorana theory and group theory. In particular, Majorana algebras can be studied either in their own right or as *Majorana representations* of finite groups.

The main aim of this work is to classify and construct Majorana algebras generated by three axes a_0, a_1 and a_2 such that the subalgebra generated by a_0 and a_1 is isomorphic to a $2A$ dihedral subalgebra of the Griess algebra.

We first show that such an algebra must occur as a Majorana representation of one of 26 subgroups of the Monster. These groups coincide with the list of *triangle-point subgroups* of the Monster given by S. P. Norton in [Nor85]. In particular, our result reproves the completeness of Norton's list. This work builds on that of S. Decelle in [Dec13].

Next, inspired by work of Á. Seress, we design and implement an algorithm to construct the Majorana representations of a given group. We use this to construct a number of important Majorana representations which are independent of the main aim of this work.

Finally, we use this algorithm along with our first result to construct all possible Majorana algebras generated by three axes, two of which generate a $2A$-dihedral algebra. We use these constructions to show that each of these algebras must be isomorphic to a subalgebra of the Griess algebra.

This is our main result and can equivalently be thought of as the construction of the subalgebras of the Griess algebra which correspond to the groups in Norton's list of triangle-point groups.

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Notation

$\mathbb{M}$	the Monster simple group
$V_{\mathbb{M}}$	the Griess or Monster algebra
m^n	the elementary abelian group of order m^n
D_{2n}	the dihedral group of order $2n$
S_n	the symmetric group of degree n
A_n	the alternating group of degree n
$L_n(p^k)$	the projective special linear group of dimension n over the field of order p^k
p_+^{1+2n}	the extraspecial group of order p^{1+2n} and type +
$GL(V)$	the general linear group of a vector space V.

For a Majorana algebra V:

$\langle X \rangle$	the smallest subspace of V containing the set $X \subseteq V$
$\langle\langle X \rangle\rangle$	the smallest subalgebra of V containing the set $X \subseteq V$
$\tau(a)$	the Majorana involution corresponding to the Majorana axis a
$V_\mu^{(a)}$	the μ-eigenspace of (the adjoint action of) the Majorana axis a

For two groups G and H:

$\langle X \rangle$	the smallest subgroup of G containing the set $X \subseteq G$
$G \times H$	the direct product of G and H
$G.H$	an extension of G by H
$G : H$	a semidirect product of G and H
G wr H	the wreath product of G and H

Chapter 1

Background

In the classification of finite simple groups, it was shown that there are exactly 26 *sporadic simple groups*, groups that do not lie in any of the infinite families that make up the rest of the classification. The *Monster simple group*, $\mathbb{M}$, is the largest of these sporadic groups and contains 20 of the others as subgroups or quotients of subgroups. It has order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}.$$

The existence of the Monster was independently conjectured by B. Fischer and R. Griess in 1973. It was first constructed by Griess in [Gri82] as the automorphism group of $V_{\mathbb{M}}$, a 196,884-dimensional commutative, non-associative, real algebra known as the *Griess* or *Monster algebra*. The algebra $V_{\mathbb{M}}$ is the direct sum (as a vector space) of an irreducible module of $\mathbb{M}$ and the one-dimensional trivial module. This construction was later simplified by J. H. Conway [Con84] and J. Tits [Tit84].

It is well known (see [CCN$^+$85]) that $\mathbb{M}$ contains two conjugacy classes of involutions which we will refer to as $2A$ and $2B$, where $2A$ is the smaller of the two. These two classes play an important role in the study of the Monster. The $2A$ involutions are *6-transpositions* in that the product of any two has order at most 6. Moreover, $\mathbb{M}$ is generated by the $2A$ involutions, making it a *6-transposition group*.

Conway [Con84] showed that for each $x \in 2A$, we may define an idempotent vector $\psi(x) \in V_{\mathbb{M}}$ called the $2A$-*axis* corresponding to x. S. P. Norton and Conway [Con84, Nor96] have described all the subalgebras generated by two $2A$-axes $\psi(x)$ and $\psi(y)$. They are known as *dihedral* subalgebras and are completely determined by the conjugacy class in $\mathbb{M}$ of the product xy, for which there are nine distinct possibilities:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$

The *type* of the dihedral subalgebra generated by $\psi(x)$ and $\psi(y)$ is defined to be the conjugacy class of the product xy.

Simply by its position as the largest of the sporadic groups, the Monster is an object of great significance in group theory. However, as mathematicians began to work on the so-called "Friendly Giant", they began to see some surprising connections between the Monster and another area of mathematics, modular functions.

In 1978, J. McKay noticed that the first coefficient in the q-expansion of the elliptic modular function (commonly denoted j) is equal to 196,884, the dimension of the Griess algebra. He informed J. Thompson of this remarkable observation who then went on to show that in fact all of the first few coefficients in the q-expansion of the function j can be expressed as linear combinations of the dimensions of the smallest irreducible representations of $\mathbb{M}$ [Tho79].

This numerical link between the Monster and the function j appeared to suggest a deeper connection between the two objects. In [CN79], Conway and Norton proposed a number of conjectures formalising this idea, which they christened "Monstrous Moonshine".

In [FLM88], I. Frenkel, J. Lepowsky and A. Meurman constructed an infinite graded algebra $V^\natural$ known as the *moonshine module* such that $\mathrm{Aut}(V^\natural) = \mathbb{M}$ and such that the graded dimension of $V^\natural$ is given by the function j. In 1992, R. Borcherds [Bor92] used the moonshine module to prove Conway and Norton's Monstrous Moonshine conjectures in work that contributed to his receipt of a Fields medal in 1998.

The module $V^\natural$ lies in a class of mathematical objects known as *vertex operator algebras*, or *VOAs*. Both VOAs and their close relatives, *vertex algebras*, were first introduced as purely mathematical tools but have since been shown to have applications in certain areas of physics. They play a key role in the motivation behind Majorana theory.

Suppose that $V = \bigoplus_{n=0}^{\infty} V_n$ is a real VOA such that $V_0 = \mathbb{R}\mathbf{1}$ and $V_1 = 0$. Then the space V_2 has the structure of a commutative non-associative algebra and is referred to as the *generalised Griess algebra* of V. Crucially, when $V = V^\natural$, $V_2 \cong V_{\mathbb{M}}$, which means that the Griess algebra is an example of a generalised Griess algebra.

M. Miyamoto [Miy96] showed that, for a VOA V as above, there exist involutions $\tau_a \in \mathrm{Aut}(V)$ that correspond to special generators $a \in V_2$ known as *Ising vectors*. Moreover, when $V = V^\natural$, the vectors $\frac{1}{2}a$ are $2A$-axes of $V_{\mathbb{M}}$ and the τ_a are $2A$-involutions of $\mathbb{M}$. S. Sakuma, a student of Miyamoto, then proved the following result, which we refer to later in the text as *Sakuma's theorem*.

Theorem 1.0.1. *If V_2 is a generalised Griess algebra and $a_1, a_2 \in V_2$ are Ising vectors then the subalgebra $\langle\langle a_1, a_2 \rangle\rangle$ is isomorphic to one of the nine dihedral subalgebras of the Monster algebra.*

This was a remarkable result that reproved the classification of the dihedral algebras of the Griess algebra, but in the more general setting of VOAs. It offered hope that this approach might lead to a new way of studying the Monster. However, VOAs are very complex objects and it is difficult to fully exploit their potential in studying the Monster.

Majorana theory offers a solution to this problem. In 2009, A. A. Ivanov [Iva09] introduced Majorana theory as an axiomatisation of certain properties of generalised Griess algebras which reframed the approach of VOAs in a way which could easily be applied to group theoretic problems.

In particular, the Majorana axioms allow the construction of non-associative real algebras known as *Majorana algebras*, which can be considered as analogues of generalised Griess algebras. Majorana algebras are generated by certain idempotents, known as *Majorana axes*, that correspond to involutions in the automorphism group of the algebra, known as *Majorana involutions*. Majorana axes and involutions correspond to Ising vectors and Miyamoto involutions respectively.

In 2010, [IPSS10] Ivanov *et al.* proved that a Majorana algebra generated by two Majorana axes is isomorphic to one of the dihedral subalgebras of $V_{\mathbb{M}}$. This reproved Sakuma's theorem using the Majorana axioms, demonstrating the potential of the new theory.

Since then, a number of publications have further developed the theory (cf. [CRI14], [Dec14], [IPSS10], [Iva11b], [Iva11a], [IS12a], [IS12b], [Ser12]). In particular, Majorana theory has been used to construct a number of important subalgebras of the Griess algebra including two algebras of dimension 20 and 26 corresponding to the 2A-generated A_5-subgroups in the Monster [IS12a].

Majorana algebras show a remarkable tendency to embed into the Griess algebra, with only a few of the known Majorana algebras existing independently. In fact A. A. Ivanov has posed the following *Straight Flush Conjecture*.

Conjecture 1.0.2. *Every indecomposable Majorana algebra in which 2,3,4,5 and 6 appear as the order of the product of two Majorana involutions, always embeds into the Griess algebra.*

If the conjecture was true, it would place the Griess algebra as the universal object in the class of Majorana algebras, adding weight to the belief that Majorana theory will prove to be a crucial tool in the study of the Monster and the Griess algebra.

We will later show (Lemma 2.3.1) that the eigenvectors of the adjoint action of a Majorana axis must obey what are known as the *Majorana fusion rules*. This is a phenomenon which has long been known to hold in the Griess algebra, as well as certain other important examples of non-associative algebras.

In 2015, J. I. Hall, F. Rehren and S. Shpectorov [HRS15b] defined an *axial algebra* to be a commutative, non-associative algebra that is generated by idempotents whose eigenvectors obey fusion rules. Majorana algebras are one of the earliest, and most important, examples of axial algebras and play a crucial role in this new theory.

Since their inception, a number of papers ([CRMR17], [CRM18], [HRS15a], [DR17], [DV17], [HRS15a], [HSS17], [Reh15], [Yab17]) have further developed the theory of axial algebras. In particular, many of the definitions and results in Chapter 2, have analogues in the language of axial algebras.

Virasoro algebras are objects occuring in mathematical physics, notably relating to the Ising and n-state Potts models in statistical mechanics. The discrete series $\mathrm{Vir}(p, q)$ of Virasoro algebras gives rise to many examples of fusion rules. In particular, the Virasoro operator algebra $\mathrm{Vir}(4, 3)$ is the operator algebra of the Ising model of free Majorana fermions and its associated fusion rules are those of Majorana algebras, giving the theory its name.

This thesis primarily addresses the problem of classifying and constructing certain 3-generated Majorana algebras as a natural part of the classification of low-dimensional Majorana algebras. In particular, these algebras are closely related to *triangle-point groups*, a class of finitely generated groups which play an important role in the study of the Monster group.

In Chapter 3, we use the work of S. Decelle [Dec13] on triangle-point groups to prove the following classification result.

Theorem 1.0.3. *Suppose that V is a Majorana algebra which obeys axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$. Then V must occur as a Majorana representation of one of 27 groups, each of which occurs as a subgroup of the Monster.*

In Chapter 4, we give the details of the implementation of an algorithm to construct generic finite dimensional Majorana representations. This is work is inspired by that of Á. Seress [Ser12]. We also give some details of the algebras constructed using this algorithm, some of which are examples which have not before been constructed. This work is independent of Chapter 3.

In Chapter 5, we combine the work in the preceding two chapters to give a full classification and construction of Majorana algebras as in the hypothesis of Theorem 1.0.3. The following is the main theorem of this thesis.

Theorem 1.0.4. *Suppose that V is a Majorana algebra which obeys axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$. Then V is isomorphic to a subalgebra of the Griess algebra $V_{\mathbb{M}}$.*

In [Nor85], Norton gives a list (up to conjugacy) of all triangle-point subgroups $G = \langle a, b, c\rangle$ of the Monster such that $a, b, c, ab \in 2A$. The work in this thesis proves that this list is complete and provides constructions of each of the corresponding subalgebras of the Griess algebra, details of which are given in Table 5.1.

In most cases, the number of times that a triangle-point group $G = \langle a, b, c\rangle$ appears in Norton's list is equal to the number, up to isomorphism, of Majorana representations of the form (G, T, V) where $a, b, c, ab \in 2A$. However, this is not the case for the groups $L_2(11)$, $2^4.D_{10}$ and S_5. This implies the following result.

Corollary 1.0.5. *Let G be one of the groups $L_2(11)$, $2^4 : D_{10}$ and S_5. Then, from [Nor85], there are two non-conjugate embeddings ι_0 and ι_1 of G as a triangle-point group into $\mathbb{M}$. The subalgebras $\langle\langle \psi(\iota_0(G) \cap 2A)\rangle\rangle$ and $\langle\langle \psi(\iota_1(G) \cap 2A)\rangle\rangle$ of $V_{\mathbb{M}}$ are isomorphic.*

Chapter 2

An introduction to Majorana theory

In this section we go over some basic definitions and results in the Majorana theory. Our main reference for this section is [Iva09].

2.1 Majorana algebras

Majorana algebras are the objects at the heart of Majorana theory. They are defined using a series of axioms which have their roots in the theory of VOAs. These axioms are all known to hold in the Griess algebra $V_{\mathbb{M}}$.

Let V be a real vector space equipped with a positive-definite, symmetric, bilinear form $(\ ,\ )$ and a bilinear, commutative, non-associative algebra product $\cdot$ such that

M1 $(\ ,\ )$ associates with $\cdot$ in the sense that

$$(u, v \cdot w) = (u \cdot v, w)$$

for all $u, v, w \in V$;

M2 the *Norton inequality* holds so that

$$(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$$

for all $u, v \in V$.

Let A be a subset of $V \backslash \{0\}$ and suppose that for every $a \in A$ the following conditions M3 to M7 hold:

M3 $(a, a) = 1$ and $a \cdot a = a$, so that the elements of A are idempotents of length 1;

M4 $V = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)} \oplus V_{\frac{1}{2^5}}^{(a)}$ where $V_\mu^{(a)} = \{v \mid v \in V,\, a \cdot v = \mu v\}$ is the set of μ-eigenvectors of the adjoint action of a on V;

M5 $V_1^{(a)} = \{\lambda a \mid \lambda \in \mathbb{R}\}$;

M6 the linear transformation $\tau(a)$ of V defined via

$$\tau(a) : u \mapsto (-1)^{2^5 \mu} u$$

for $u \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}, \frac{1}{2^5}$, preserves the algebra product (i.e. $u^{\tau(a)} \cdot v^{\tau(a)} = (u \cdot v)^{\tau(a)}$ for all $u, v \in V$);

M7 if $V_+^{(a)}$ is the centraliser of $\tau(a)$ in V, so that $V_+^{(a)} = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$, then the linear transformation $\sigma(a)$ of $V_+^{(a)}$ defined via

$$\sigma(a) : u \mapsto (-1)^{2^2 \mu} u$$

for $u \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}$, preserves the restriction of the algebra product to $V_+^{(a)}$ (i.e. $u^{\sigma(a)} \cdot v^{\sigma(a)} = (u \cdot v)^{\sigma(a)}$ for all $u, v \in V_+^{(a)}$.

Definition 2.1.1. *The elements of A are called* Majorana axes *while the automorphisms $\tau(a)$ are called* Majorana involutions*. A real, commutative, non-associative algebra $(V, \cdot, (\ ,\ ))$ equipped with an inner product $(\ ,\ )$ is called a* Majorana algebra *if it satisfies axioms M1 and M2 and is generated by a set of Majorana axes.*

Definition 2.1.2. *If V is a Majorana algebra then its* dimension *is its dimension as a vector space over $\mathbb{R}$.*

Definition 2.1.3. *Two Majorana algebras V_1 and V_2 are* isomorphic *if there exists a linear map ϕ from V_1 to V_2 which preserves the inner and algebra products and which induces a bijection from the Majorana axes of V_1 to the Majorana axes of V_2. If $V_1 = V_2$ then ϕ is an* automorphism *of V_1.*

The Majorana axes and Majorana involutions correspond to the $2A$-axes in the Griess algebra and the $2A$-involutions in the Monster respectively. The fact that the $2A$-axes obey the axioms M3 - M7 was implicitly stated in [Nor96] and was explicitly shown in Proposition 8.6.2 of [Iva09].

The following is a natural axiomatisation of the relationship between the Monster and the Griess algebra.

Definition 2.1.4. *A* Majorana representation *is a tuple*

$$\boldsymbol{R} = (G, T, V, \cdot, (,), \varphi, \psi)$$

where

- *G is a finite group;*
- *T is a G-invariant set of generating involutions of G;*

- *V is a real vector space equipped with an inner product $(\ ,\ )$ and bilinear, commutative, non-associative algebra product $\cdot$ satisfying M1 and M2 such that V is generated by a set A of Majorana axes;*
- *$\varphi : G \to GL(V)$ is a linear representation that preserves both products,*
- *$\psi : T \to A$ is a bijective mapping such that for all $t \in T$ and $g \in G$*

$$\psi(t^g) = \psi(t)^{\varphi(g)}.$$

We also require that if $\tau(\psi(t))$ is the involution defined as in axiom M6 then $\tau(\psi(t)) = \varphi(t)$ for all $t \in T$.

Majorana representations have proved to be a very powerful tool in Majorana theory, allowing us to exploit the natural relationships between Majorana algebras and groups in order to study and construct Majorana algebras. Almost all known examples of Majorana algebras have been constructed as Majorana representations of finite groups.

A key example of a Majorana representation is when G is isomorphic to $\mathbb{M}$, T is the conjugacy class of $2A$-involutions of $\mathbb{M}$ and V is $V_{\mathbb{M}}$. In this case, the map ψ is Conway's bijection from the $2A$-involutions of $\mathbb{M}$ to the $2A$ axes of $V_{\mathbb{M}}$ [Con84].

We say that a Majorana representation is *based on an embedding* into $\mathbb{M}$ if G is isomorphic to a subgroup of $\mathbb{M}$ generated by $2A$-involutions and V is isomorphic to the subalgebra of $V_{\mathbb{M}}$ generated by the corresponding $2A$ axes.

We note that it is possible for a Majorana algebra to be infinite dimensional. However, there are no known non-trivial examples of such an algebra. In fact, almost all known examples of Majorana algebras are equal to the linear span of the set of their elements which are the product of most two Majorana axes or, equivalently, are 2-*closed*, as defined below.

Definition 2.1.5. *Let V be a commutative algebra and let $X \subseteq V$. Then we say that the subalgebra $\langle\langle X \rangle\rangle$ is k-*closed *(with respect to X) if it is equal to the linear span of the set*

$$\bigcup_{n=1}^{k} \{x_1 \cdot x_2 \cdots x_n \,:\, x_i \in X\}.$$

In this definition, as the algebra product is not necessarily associative, we must specify that the notation $x_1 \cdot x_2 \cdots x_n$ refers to the algebra products obtained from *all* possible combinations of brackets on the elements. For example, if $n = 3$, we include $x_1 \cdot (x_2 \cdot x_3)$ as well as $(x_1 \cdot x_2) \cdot x_3$.

2.2 Sakuma's theorem

The seminal paper in Majorana theory was that of Ivanov et al. [IPSS10]. In the first part of the paper, they reproved Sakuma's theorem (Theorem 1.0.1) as stated below. This result truly forms the foundation of Majorana theory. It can equivalently be thought of as the classification of 2-generated Majorana algebras, i.e. those generated by at most two Majorana axes.

Theorem 2.2.1 ([IPSS10]). *Let $\boldsymbol{R} = (G, T, V, \cdot, (\,,\,), \varphi, \psi)$ be a Majorana representation of G, as defined above. For $t_0, t_1 \in T$ let $a_0 = \psi(t_0)$, $a_1 = \psi(t_1)$, $\tau_0 = \varphi(t_0)$, $\tau_1 = \varphi(t_1)$ and $\rho = t_0t_1$. Finally, let $D \leq GL(V)$ be the dihedral group $\langle \tau_0, \tau_1 \rangle$. Then*

(i) $|D| = 2N$ for some N with $1 \leq N \leq 6$;

(ii) the subalgebra $U = \langle\langle a_0, a_1 \rangle\rangle$ is isomorphic to a dihedral algebra of type NX for $X \in \{A, B, C\}$, the structure of which is given in Table 2.1;

(iii) for $i \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$, the image of a_ϵ under the i-th power of ρ, which we denote $a_{2i+\epsilon}$, is a Majorana axis and $\tau(a_{2i+\epsilon}) = \rho^{-i}\tau_\epsilon\rho^i$.

Table 2.1 does not show all pairwise algebra and inner products of the basis vectors. Those that are missing can be recovered from the action of the group $\langle \tau_0, \tau_1 \rangle$ together with the symmetry between a_0 and a_1. We also note that the dihedral algebra of type $1A$ is a 1-dimensional algebra generated by one Majorana axis and so is omitted from Table 2.1.

We can use the values in Table 2.1 to calculate the eigenvectors of these algebras with respect to the axis a_0.

Proposition 2.2.2. *The eigenspace decompositions with respect to the axis a_0 for each dihedral Majorana algebra are given in Table 2.2. In each case, the 1-eigenspace is the 1-dimensional space spanned by a_0 and so is omitted from this table.*

The following result is also a direct consequence of the values in Table 2.1.

Proposition 2.2.3. *Let V be a Majorana algebra and let $a_0, a_1 \in A$. Let $U := \langle\langle a_0, a_1 \rangle\rangle$ and let $t_0 := \tau(a_0)$ and $t_1 := \tau(a_1)$.*

If U is of type $3A$ or $4A$ then U contains the additional basis vector $u_{\rho(t_0,t_1)}$ or $v_{\rho(t_0,t_1)}$ respectively where $u_{\rho(t_0,t_1)}$ and $v_{\rho(t_0,t_1)}$ depend only on the cyclic subgroup $\langle t_0t_1 \rangle$. That is to say,

$$u_{\rho(t_0,t_1)} = u_{\rho(t_0,t_0t_1t_0)} \text{ and } v_{\rho(t_0,t_1)} = v_{\rho(t_0,t_0t_1t_0)}.$$

Similarly, if U is of type $5A$ then U contains an additional basis vector $w_{\rho(t_0,t_1)}$ which, up to a possible change of sign, depends only on the cyclic subgroup $\langle t_0t_1 \rangle$, as below

$$w_{\rho(t_0,t_1)} = -w_{\rho(t_0,t_1t_0t_1)} = -w_{\rho(t_0,t_0t_1t_0t_1t_0)} = w_{\rho(t_0,t_0t_1t_0)}. \tag{2.1}$$

Moreover, if $U := \langle\langle a_0, a_1 \rangle\rangle$ is a dihedral algebra of type $3A$, $4A$ or $5A$ then $u_{\rho(t_0,t_1)} = u_{\rho(t_1,t_0)}$, $v_{\rho(t_0,t_1)} = v_{\rho(t_1,t_0)}$ or $w_{\rho(t_0,t_1)} = w_{\rho(t_1,t_0)}$ respectively.

Sakuma's theorem provides a crucial tool in Majorana theory. If the dihedral subalgebras of a Majorana algebra are known, they provide initial values for the inner and algebra products which can be used to explore the structure of the whole algebra. As such, it is important to classify the possibilities for the dihedral algebras which are contained in a given Majorana algebra. This idea is formalised in the following definition.

Type	Basis	Products and angles
2A	$a_0, a_1, a_{\rho(t_0,t_1)}$	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_{\rho(t_0,t_1)}),\ a_0 \cdot a_{\rho(t_0,t_1)} = \frac{1}{2^3}(a_0 + a_{\rho(t_0,t_1)} - a_1)$ $a_{\rho(t_0,t_1)} \cdot a_{\rho(t_0,t_1)} = a_{\rho(t_0,t_1)}$ $(a_0, a_1) = (a_0, a_{\rho(t_0,t_1)}) = (a_{\rho(t_0,t_1)}, a_{\rho(t_0,t_1)}) = \frac{1}{2^3}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0,\ (a_0, a_1) = 0$
3A	$a_{-1}, a_0, a_1,$ $u_{\rho(t_0,t_1)}$	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho(t_0,t_1)}$ $a_0 \cdot u_{\rho(t_0,t_1)} = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_{\rho(t_0,t_1)}$ $u_{\rho(t_0,t_1)} \cdot u_{\rho(t_0,t_1)} = u_{\rho(t_0,t_1)}$ $(a_0, a_1) = \frac{13}{2^8},\ (a_0, u_{\rho(t_0,t_1)}) = \frac{1}{2^2},\ (u_{\rho(t_0,t_1)}, u_{\rho(t_0,t_1)}) = \frac{2^3}{5}$
3C	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1}),\ (a_0, a_1) = \frac{1}{2^6}$
4A	$a_{-1}, a_0, a_1,$ $a_2, v_{\rho(t_0,t_1)}$	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v_{\rho(t_0,t_1)})$ $a_0 \cdot v_{\rho(t_0,t_1)} = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_{\rho(t_0,t_1)})$ $v_{\rho(t_0,t_1)} \cdot v_{\rho(t_0,t_1)} = v_{\rho(t_0,t_1)},\ \ a_0 \cdot a_2 = 0$ $(a_0, a_1) = \frac{1}{2^5},\ (a_0, a_2) = 0,\ (a_0, v_{\rho(t_0,t_1)}) = \frac{3}{2^3},\ (v_{\rho(t_0,t_1)}, v_{\rho(t_0,t_2)}) = 2$
4B	$a_{-1}, a_0, a_1,$ $a_2, a_{\rho(t_0,t_2)}$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho(t_0,t_2)})$ $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho(t_0,t_2)})$ $(a_0, a_1) = \frac{1}{2^6},\ (a_0, a_2) = (a_0, a_{\rho(t_0,t_1)}) = \frac{1}{2^3}$
5A	$a_{-2}, a_{-1}, a_0,$ $a_1, a_2, w_{\rho(t_0,t_1)}$	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_{\rho(t_0,t_1)}$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_{\rho(t_0,t_1)}$ $a_0 \cdot w_{\rho(t_0,t_1)} = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_{\rho(t_0,t_1)}$ $w_{\rho(t_0,t_1)} \cdot w_{\rho(t_0,t_1)} = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $(a_0, a_1) = \frac{3}{2^7},\ (a_0, w_{\rho(t_0,t_1)}) = 0,\ (w_{\rho(t_0,t_1)}, w_{\rho(t_0,t_1)}) = \frac{5^3 \cdot 7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, a_3 $a_{\rho(t_0,t_3)}, u_{\rho(t_0,t_2)}$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho(t_0,t_3)}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho(t_0,t_2)}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho(t_0,t_2)}$ $a_0 \cdot u_{\rho(t_0,t_2)} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho(t_0,t_2)}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho(t_0,t_3)}),\ a_{\rho(t_0,t_3)} \cdot u_{\rho(t_0,t_2)} = 0$ $(a_{\rho(t_0,t_3)}, u_{\rho(t_0,t_2)}) = 0,\ (a_0, a_1) = \frac{5}{2^8},\ (a_0, a_2) = \frac{13}{2^8},\ (a_0, a_3) = \frac{1}{2^3}$

Table 2.1: The dihedral Majorana algebras

Type	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$2A$	$a_1 + a_{\rho(t_0,t_1)} - \frac{1}{2^2}$	$a_1 - a_{\rho(t_0,t_1)}$	
$2B$	a_1		
$3A$	$u_{\rho(t_0,t_1)} + \frac{2\cdot 5}{3^3}a_0 + \frac{2^5}{3^3}(a_1 + a_{-1})$	$u_{\rho(t_0,t_1)} - \frac{2^3}{3^2\cdot 5}a_0$ $-\frac{2^5}{3^2\cdot 5}(a_1 + a_{-1})$	$a_1 - a_{-1}$
$3C$	$a_1 + a_{-1} - \frac{1}{2^5}a_0$		$a_1 - a_{-1}$
$4A$	$v_{\rho(t_0,t_1)} - \frac{1}{2}a_0 + 2(a_1 + a_{-1}), a_2$	$v_{\rho(t_0,t_1)} - \frac{1}{3}a_0$ $-\frac{2}{3}(a_1 + a_{-1}) - \frac{1}{3}a_2$	$a_1 - a_{-1}$
$4B$	$a_1 + a_{-1} - \frac{1}{2^5}a_0 - \frac{1}{2^3}(a_{\rho(t_0,t_2)} - a_2),$ $a_2 + a_{\rho(t_0,t_2)} - \frac{1}{2^2}a_0$	$a_2 - a_{\rho(t_0,t_2)}$	$a_1 - a_{-1}$
$5A$	$w_{\rho(t_0,t_1)} + \frac{3}{2^9}a_0 - \frac{3\cdot 5}{2^7}(a_1 + a_{-1}) - \frac{1}{2^7}(a_2 - a_{-2}),$ $w_{\rho(t_0,t_1)} - \frac{3}{2^9}a_0 + \frac{1}{2^7}(a_1 + a_{-1}) + \frac{3\cdot 5}{2^7}(a_2 + a_{-2})$	$w_{\rho(t_0,t_1)} + \frac{1}{2^7}(a_1 + a_{-1})$ $-\frac{1}{2^7}(a_2 + a_{-2})$	$a_1 - a_{-1},$ $a_2 - a_{-2}$
$6A$	$u_{\rho(t_0,t_2)} + \frac{2}{3^2\cdot 5}a_0 - \frac{2^8}{3^2\cdot 5}(a_1 - a_{-1})$ $-\frac{2^5}{3^2\cdot 5}(a_2 + a_{-2} + a_3 - a_{\rho(t_0,t_3)}),$ $a_3 + a_{\rho(t_0,t_3)} - \frac{1}{2^2}a_0,$ $u_{\rho(t_0,t_2)} - \frac{2\cdot 5}{3^3}a_0 + \frac{2^5}{3^3}(a_2 + a_{-2})$	$u_{\rho(t_0,t_2)} - \frac{2^3}{3^2\cdot 5}a_0$ $-\frac{2^5}{3^2\cdot 5}(a_2 + a_{-2} + a_3)$ $+\frac{2^5}{3^2\cdot 5}a_{\rho(t_0,t_3)},$ $a_3 - a_{\rho(t_0,t_3)}$	$a_1 - a_{-1},$ $a_2 - a_{-2}$

Table 2.2: The eigenspace decomposition of the dihedral Majorana algebras

Definition 2.2.4. *If $\boldsymbol{R} = (G, T, V)$ is a Majorana representation then we define a map Ψ which sends $(t, s) \in T^2$ to the type of the dihedral Majorana algebra $\langle\langle a_t, a_s \rangle\rangle$. Then the* shape *of $\boldsymbol{R}$ is the multiset $[\Psi((t_{i_1}, t_{j_1})), \Psi((t_{i_2}, t_{j_2})) \dots, \Psi((t_{i_n}, t_{j_n}))]$ where the (t_{i_k}, t_{j_k}) are representatives of the orbitals of G on T.*

It is worth noting that, whilst it is not necessarily the case that two non-isomorphic Majorana representations must have different shapes, no such examples have been found where this is not the case. It is an open question as to whether this is true in general.

When determining the shape of a Majorana representation, if $t, s \in T$ and $\langle t, s \rangle \cong D_{2N}$ then $\Psi((t, s)^G) = NX$ for $X \in \{A, B, C\}$. If $N = 1, 5$ or 6, there is only one option for the value of X. If N takes any other value, we must use other results to restrict the possible value of X. In particular, the following lemma can be deduced from the structure of the dihedral algebras (see Lemma 2.20, [IPSS10]).

Lemma 2.2.5. *Let U be an algebra of type NX (as in Table 2.1) that is generated by Majorana axes a_0 and a_1. Then U contains no proper, non-trivial subalgebras, with the exception of the following cases.*

(i) If U is of type $4A$ or $4B$ then the subalgebras $\langle\langle a_0, a_2 \rangle\rangle$ and $\langle\langle a_1, a_{-1} \rangle\rangle$ are of type $2B$ or $2A$ respectively.

(ii) If U is of type $6A$ then the subalgebras $\langle\langle a_0, a_3 \rangle\rangle$ and $\langle\langle a_1, a_{-2} \rangle\rangle$ are of type $3A$ and the subalgebras $\langle\langle a_0, a_2 \rangle\rangle$, $\langle\langle a_1, a_{-1} \rangle\rangle$, $\langle\langle a_0, a_{-2} \rangle\rangle$ and $\langle\langle a_1, a_3 \rangle\rangle$ are of type $2A$.

Informally, this means that we have the following *inclusions* of algebras:

$$2A \hookrightarrow 4B, \qquad 2B \hookrightarrow 4A, \qquad 2A \hookrightarrow 6A, \qquad \text{and} \qquad 3A \hookrightarrow 6A$$

and that these are the only possible inclusions of non-trivial algebras.

Given a group G and a set of G-invariant involutions T, we let Γ be the directed graph whose vertex set is the set of orbitals of G on T and where $(t_0, t_1)^G \to (t_2, t_3)^G$ if and only if $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle \hookrightarrow \langle\langle a_{t_2}, a_{t_3} \rangle\rangle$ and $t_0 \neq t_1$. If we fix the type of the dihedral algebra corresponding to one vertex $v \in V(\Gamma)$ then this determines the types of the algebras corresponding to all vertices in its connected component. In particular, there are at most 2^c possible shapes for a representation of the form (G, T, V), where c is the number of connected components of Γ.

2.3 Basic results

We now survey a few basic results which follow as a consequence of the Majorana axioms. We will frequently refer to this section in the remainder of the text.

Perhaps the most important consequence of the Majorana axioms is that the eigenspace decomposition of (the adjoint action of) a given Majorana axis must obey the *Majorana fusion rules*.

	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	0	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

Table 2.3: The Majorana fusion rules

Lemma 2.3.1. *For a fixed Majorana axis a, if $u \in V_\mu^{(a)}$, $v \in V_\nu^{(a)}$ then the product $u \cdot v$ lies in the sum of eigenspaces with corresponding eigenvalues given by the (μ, ν)-th entry of Table 2.3.*

Proof. As in the hypothesis, let $u \in V_\mu^{(a)}$, $v \in V_\nu^{(a)}$. If $1 \in \{\mu, \nu\}$ then the result follows from axiom M4. If either zero or two of $\{\mu, \nu\}$ are equal to $\frac{1}{2^5}$ then $\tau(a)$ preserves $u \cdot v$ which must then lie in $V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$. However, if exactly one of μ and ν is equal to $\frac{1}{2^5}$, then $\tau(a)$ inverts $u \cdot v$ and so $u \cdot v \in V_{\frac{1}{2^5}}^{(a)}$.

A similar analysis using the action of $\sigma(a)$ shows that if $\{\mu, \nu\} = \{0, \frac{1}{2^2}\}$ then $u \cdot v \in V_{\frac{1}{2^2}}^{(a)}$ and if $\mu = \nu$ is equal to 0 or $\frac{1}{2^2}$ then $u \cdot v \in V_1^{(a)} \oplus V_0^{(a)}$. Finally, axioms M1 and M4 show that if $\mu = \nu = 0$ then the projection of $u \cdot v$ on to $V_1^{(a)}$ is zero. □

The fusion rules give Majorana algebras much of their structure and play a crucial role in the construction of these algebras. In particular, although Majorana algebras are, in general, non-associative, the fusion rules allow us to consider products of the form $a \cdot (u \cdot v)$ for $a \in A$ and $u, v \in V$, even if the value of the product $u \cdot v$ is not known. As mentioned in Chapter 1, the fusion rules mean that Majorana algebras are examples of a larger class of algebras, known as axial algebras.

A further important consequence of the fusion rules is the *resurrection principle.* This result first appeared in its current form in [IPSS10] but the basic ideas were used in [Sak07]. It takes its name from that fact that the vector v disappears at the start of the proof, only to reappear at the end.

Proposition 2.3.2 (The Resurrection Principle)**.** *Let V be a Majorana algebra and let $a \in V$ be a fixed Majorana axis. Let W be an a-stable subspace of V (i.e. a subspace such that $a \cdot w \in W$ for all $w \in W$). For $v \in V$ suppose that*

$$\alpha_v = v + w_\alpha \in V_0^{(a)} \text{ and } \beta_v = v + w_\beta \in V_{\frac{1}{2^2}}^{(a)}$$

for some $w_\alpha, w_\beta \in W$. Then

$$v = -[4a \cdot (w_\alpha - w_\beta) + w_\beta]$$

in particular, $v \in W$.

Proof. From the fusion rules,

$$a \cdot (w_\alpha - w_\beta) = a \cdot (\alpha_v - \beta_v) = -\frac{1}{2^2}\beta_v = -\frac{1}{2^2}v - \frac{1}{2^2}w_\beta.$$

□

The following results are further consequences of the Majorana axioms.

Lemma 2.3.3. *Let V be a Majorana algebra and let $a \in V$ be a Majorana axis. If $v \in V$ then*

$$v - v^{\tau(a)} \in V^{(a)}_{\frac{1}{2^5}}.$$

Moreover, if $v \in V^{(a)}_{+}$ then

$$v - v^{\sigma(a)} \in V^{(a)}_{\frac{1}{2^2}}.$$

Proof. Let $v \in V$ then as $\tau(a)$ is an involution,

$$(v - v^{\tau(a)})^{\tau(a)} = -(v - v^{\tau(a)})$$

and so we must have $v - v^{\tau(a)} \in V^{(a)}_{\frac{1}{2^5}}$. As similar argument follows for the second statement. □

Lemma 2.3.4. *Let V be a real vector space satisfying the axioms M1 to M5, and let a be a Majorana axis of V. Then the eigenspace decomposition in M4 is orthogonal with respect to $(\ ,\ )$ (i.e. $(u, v) = 0$ for all $u \in V^{(a)}_{\mu}$, $v \in V^{(a)}_{\nu}$ with $\mu, \nu \in \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$ and $\mu \neq \nu$).*

Proof. Let $u \in V^{(a)}_{\mu}$, $v \in V^{(a)}_{\nu}$ with $\mu \neq \nu$ as in the hypothesis. Suppose first that $\mu = 0$ and $\nu \neq 0$. Then, from axiom M1,

$$(u, v) = \frac{1}{\nu}(u, a \cdot v) = \frac{1}{\nu}(a \cdot u, v) = 0$$

If both μ and ν are non-zero then

$$\frac{1}{\mu}(a \cdot u, v) = (u, v) = \frac{1}{\nu}(u, a \cdot v) = \frac{1}{\nu}(a \cdot u, v)$$

and so, as $\mu \neq \nu$, $(a \cdot u, v) = (u, v) = 0$. □

Lemma 2.3.5. *Let V be a real vector space satisfying the axioms M1 to M5, and let a be a Majorana axis of V. Then for all $v \in V$, the projection of v onto the eigenspace $V^{(a)}_1$ is $(a, v)a$.*

Proof. Let $v \in V$ and write

$$v = \lambda a + v_0 + v_{\frac{1}{2^2}} + v_{\frac{1}{2^5}}$$

where $\lambda \in \mathbb{R}$ and $v_\mu \in V^{(a)}_{\mu}$ for $\mu \in \{0, \frac{1}{2^2}, \frac{1}{2^5}\}$. Then, from Lemma 2.3.4,

$$(a, v) = \lambda(a, a) + (a, v_0) + (a, v_{\frac{1}{2^2}}) + (a, v_{\frac{1}{2^5}}) = \lambda$$

and so the projection of v onto $V^{(a)}_1$ is $\lambda a = (a, v)a$. □

Lemma 2.3.6. *Let V be a Majorana algebra and let a be a Majorana axis of V. Then the linear transformation $\tau(a)$ defined in axiom M6 preserves the inner product $(\ ,\ )$ on V, i.e. $(u^{\tau(a)}, v^{\tau(a)}) = (u, v)$ for all $u, v \in V$. Similarly, the linear transformation $\sigma(a)$ defined in axiom M7 preserves the inner product $(\ ,\ )$ on $V^{(a)}_{+}$.*

Proof. We start by writing u and v as

$$u = u_1 + u_0 + u_{\frac{1}{2^2}} + u_{\frac{1}{2^5}}$$
$$v = v_1 + v_0 + v_{\frac{1}{2^2}} + v_{\frac{1}{2^5}}$$

where $u_\mu, v_\mu \in V_\mu^{(a)}$ for $\mu \in \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$. From Lemma 2.3.4, the eigenspaces of a are mutually orthogonal with respect to the inner product and so

$$(u, v) = (u_1, v_1) + (u_0, v_0) + (u_{\frac{1}{2^2}}, v_{\frac{1}{2^2}}) + (u_{\frac{1}{2^5}}, v_{\frac{1}{2^5}}).$$

Now as

$$u^{\tau(a)} = u_1 + u_0 + u_{\frac{1}{2^2}} - u_{\frac{1}{2^5}}$$
$$v^{\tau(a)} = v_1 + v_0 + v_{\frac{1}{2^2}} - v_{\frac{1}{2^5}}$$

we clearly have

$$(u^{\tau(a)}, v^{\tau(a)}) = (u_1, v_1) + (u_0, v_0) + (u_{\frac{1}{2^2}}, v_{\frac{1}{2^2}}) + (u_{\frac{1}{2^5}}, v_{\frac{1}{2^5}}) = (u, v).$$

A similar argument follows for the transformation $\sigma(a)$. □

Lemma 2.3.7. *Let V be a Majorana algebra and let a and b be two Majorana axes of V. Then $\tau(a^{\tau(b)}) = \tau(b)\tau(a)\tau(b)$ on V and $\sigma(a^{\tau(b)}) = \tau(b)\sigma(a)\tau(b)$ on $V_+^{(a^{\tau(b)})}$.*

Proof. We will show that for all $v \in V$, $v^{\tau(a^{\tau(b)})} = v^{\tau(b)\tau(a)\tau(b)}$. Recall that

$$v^{\tau(a^{\tau(b)})} = \begin{cases} v & \text{if } v \in V_+^{(a^{\tau(b)})} \\ -v & \text{if } v \in V_{\frac{1}{2^5}}^{(a^{\tau(b)})}. \end{cases}$$

Now, by definition, $\tau(b)$ is an involution which preserves the algebra product on V and so $v \in V_\mu^{(a^{\tau(b)})}$ if and only if $v^{\tau(b)} \in V_\mu^{(a)}$ for $\mu \in \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$. Thus if $v \in V_+^{(a^{\tau(b)})}$ then $v^{\tau(b)} \in V_+^{(a)}$ and so

$$v^{\tau(b)\tau(a)\tau(b)} = (v^{\tau(b)})^{\tau(a)\tau(b)} = v^{\tau(b)\tau(b)} = v.$$

Similarly, if $v \in V_{\frac{1}{2^5}}^{(a^{\tau(b)})}$ then

$$v^{\tau(b)\tau(a)\tau(b)} = (v^{\tau(b)})^{\tau(a)\tau(b)} = -v^{\tau(b)\tau(b)} = -v.$$

and so the action of $\tau(b)\tau(a)\tau(b)$ on V coincides with that of $\tau(a^{\tau(b)})$.

A similar argument follows to show that $\sigma(a^{\tau(b)}) = \tau(b)\sigma(a)\tau(b)$. We only note that, again, if $v \in V_+^{(a^{\tau(b)})}$ then $v^{\tau(b)} \in V_+^{(a)}$ and so the action of $\tau(b)\sigma(a)\tau(b)$ is well-defined on $V_+^{(a^{\tau(b)})}$. □

Proposition 2.3.8. *Let V be a Majorana algebra and let a and b be two Majorana axes of V. Then $a^{\tau(b)}$ obeys the axioms M3 - M7.*

Proof. By axiom M6 and Lemma 2.3.6, the action of $\tau(b)$ preserves the inner and algebra products on V and so axioms M3 - M5 hold for $a^{\tau(b)}$. By Lemma 2.3.7, $\tau(a^{\tau(b)})$ and $\sigma(a^{\tau(b)})$ are both products of linear transformations which preserve the algebra product on V and $V_+^{(a^{\tau(b)})}$ respectively. Thus $\tau(a^{\tau(b)})$ and $\sigma(a^{\tau(b)})$ must also be linear transformations preserving the algebra product on their respective spaces, and so $a^{\tau(b)}$ also obeys axioms M6 and M7. □

We note that in the definition of a Majorana representation (Definition 2.1.4), as is customary, we require that the set T be closed under the action of G. This proposition shows that such an assumption is not in any way restrictive as any additional axes of the form a^g for $a \in A$ and $g \in G$ are already required to obey the Majorana axioms.

Chapter 3

Classifying Majorana representations of triangle-point groups

With Sakuma's classification of the Majorana algebras generated by two Majorana axes complete, it is natural to consider the classification of algebras generated by three Majorana axes. Whilst the question of classifying all 3-generated Majorana algebras is clearly beyond the scope of this work (the Griess algebra itself is one such example), we consider a natural first step towards this question.

In particular, we are interested in the classification of Majorana algebras generated by three axes, two of which generate a $2A$ dihedral algebra. In this chapter, we will prove the following result.

Theorem 3.0.1. *Suppose that V is a Majorana algebra that satisfies axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$. Then V must occur as a Majorana representation of one of 26 groups, each of which occurs as a subgroup of the Monster.*

In order to prove this result, we will consider a class of groups known as triangle-point groups, which are finitely generated groups of importance in the study of the Monster group. Before defining these groups, we first discuss the role of the axiom M8, which we state below.

M8 Suppose that V is a Majorana algebra and suppose that $a_0, a_1 \in V$ are Majorana axes such that the dihedral algebra $U := \langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$. Then the basis vector

$$a_\rho := a_0 + a_1 - 8a_0 \cdot a_1$$

is a Majorana axis of V and $\tau(a_\rho) = \tau(a_0)\tau(a_1)$. Conversely, we require that the map $\tau : A \to \mathrm{Aut}(V)$ is injective and that if $a_0, a_1, a_2 \in V$ are Majorana axes such that $\tau(a_0)\tau(a_1) = \tau(a_2)$ then the algebra $\langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$ and $a_\rho = a_2$.

This axiom is a very natural additional condition which is known to hold in the Griess algebra. A similar (but slightly weaker) version for Majorana representations is assumed in nearly all published works in Majorana theory. Moreover, this additional assumption means that the Majorana axioms model the behaviour of the Griess algebra extremely closely.

In particular, of all known Majorana algebras which have been constructed, there is only one (a representation of A_6 constructed in [Iva11a]) which obeys axiom M8 and which does not embed as a subalgebra of the Griess algebra. As one of the main aims of Majorana theory is to study the Griess algebra as closely as possible, it is reasonable to include this axiom in our assumptions.

Axiom M8 and the inclusions of dihedral algebras imply the following useful lemma.

Lemma 3.0.2. *Suppose that (G, T, V) is a Majorana representation which obeys axiom M8 and suppose that $t_0, t_1 \in T$. Then*

(i) if $o(t_0t_1) = 2$ then the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $2A$ if and only if $t_0t_1 \in T$;

(ii) if $o(t_0t_1) = 4$ then the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $4B$ if and only if $(t_0t_1)^2 \in T$;

(iii) if $o(t_0t_1) = 6$ then we must have $(t_0t_1)^3 \in T$.

Proof. The claim (i) follows from axiom M8. In particular, if $t_0, t_1 \in T$ such that $t_0t_1 \in T$ then $\psi(t_0t_1)$ is a Majorana axis such that $\tau(\psi(t_0t_1)) = t_0t_1 = \tau(\psi(t_0))\tau(\psi(t_1))$. Axiom M8 then implies that $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $2A$.

Conversely, if $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $2A$ then from axiom M8, the basis vector $a_\rho := \psi(t_0) + \psi(t_1) - 8\psi(t_0) \cdot \psi(t_1)$ is a Majorana axis and $\tau(a_\rho) = \tau(\psi(t_0))\tau(\psi(t_1)) = t_0t_1 \in T$ as required.

If $t_0, t_1 \in T$ such that $o(t_0t_1) = 4$ then the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $4B$ if and only if the subalgebra $\langle\langle \psi(t_0), \psi(t_0^{t_1})\rangle\rangle$ is of type $2A$. Part (i) implies that this occurs if and only if $(t_0t_1)^2 \in T$.

Similarly, if $t_0, t_1 \in T$ such that $o(t_0t_1) = 6$ then the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1)\rangle\rangle$ is of type $6A$ and contains the $2A$ subalgebra $\langle\langle \psi(t_0), \psi(t_1^{t_0t_1})\rangle\rangle$. Part (i) then implies that $(t_0t_1)^3 \in T$. □

The work in this chapter has been published by the author in [Why18].

3.1 Triangle-point groups

We begin by considering the structure of the groups G that may admit a Majorana representation (G, T, V) where V is of the desired form.

Definition 3.1.1. *Let G be a group such that*

(i) G is generated by three elements a, b, c of order 2 such that ab is also of order 2;

(ii) for any two elements $t, s \in X := a^G \cup b^G \cup c^G \cup (ab)^G$, the product ts has order at most 6

then G is a triangle-point *group.*

Proposition 3.1.2. *Suppose that V is a Majorana algebra that satisfies axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$. Then V must exist as a Majorana representation (G, T, V) where $G = \langle a, b, c\rangle$ is a triangle-point group and $T \subset G$ is such that $a, b, c, ab \in T$.*

Proof. Let $G := \langle \tau(a_0), \tau(a_1), \tau(a_2)\rangle \leq \mathrm{Aut}(V)$. From Proposition 2.3.8, if $a \in V$ is a Majorana axis and $g \in G$ then a^g is also a Majorana axis. Thus the elements of $B := a_0^G \cup a_1^G \cup a_2^G$ are all Majorana axes.

As V obeys axiom M8, if $a, b \in B$ such that $\langle\langle a, b\rangle\rangle$ is a $2A$ dihedral algebra then the element $a + b - 8a \cdot b$ is also a Majorana axis with Majorana involution $\tau(a)\tau(b)$. Thus if we let

$$A := B \cup \{a + b - 8a \cdot b \mid a, b \in B \text{ s.t. } \langle\langle a, b\rangle\rangle \text{ is of type } 2A\}$$

then we can take A to be the Majorana axes of V. Note that this set is closed under the action of G.

We will now choose a set of involutions T and maps φ and ψ such that (G, T, V, φ, ψ) is a Majorana representation. We let

$$T := \{\tau(a) \mid a \in A\} = \{\tau(a) \mid a \in B\} \cup \{\tau(a)\tau(b) \mid a, b \in B \text{ s.t. } \langle\langle a, b\rangle\rangle \text{ is of type } 2A\} \leq G.$$

As V obeys axiom M8, the map $\tau : A \to T$ is a bijection and so we can take $\psi := \tau^{-1}$ and, as we already have $G \leq \mathrm{Aut}(V)$, we can choose φ to be the identity map. It is then easy to check that (G, T, V, φ, ψ) is a Majorana representation as required.

We will now show that $G := \langle \tau(a_0), \tau(a_1), \tau(a_2)\rangle$ is a triangle-point group. As the subalgebra $\langle\langle a_0, a_1\rangle\rangle$ is of type $2A$ and the algebra V obeys axiom M8, the element $\tau(a_0)\tau(a_1)$ is a Majorana involution and so is of order 2, as required.

Now, if $X := \tau(a_0)^G \cup \tau(a_1)^G \cup \tau(a_2)^G \cup (\tau(a_0)\tau(a_1))^G$ then $X \subseteq T$ and so ψ is well defined on X. Then if $t, s \in X$, then the algebra $\langle\langle \psi(t), \psi(s)\rangle\rangle$ is a dihedral algebra and so we must have $o(ts) \leq 6$. Thus G is a triangle-point as required. □

Triangle-point groups are of particular interest in the context of the Monster group. The *Monster graph* is defined to be the graph whose vertices are the $2A$-involutions of $\mathbb{M}$ such that two vertices are joined by an edge if and only their product is also a $2A$-involution of $\mathbb{M}$. Norton [Nor85] studied the subgraphs of the Monster graph induced by sets of vertices the form a, b, ab, c. He named these subgraphs *triangle-point configurations* of the Monster graph and studied the possibility of constructing the Griess algebra as a permutation representation on these configurations.

In particular, his work includes a list of subgroups of the form $G = \langle a, b, c\rangle$ such that $a, b, c, ab \in 2A$ which we partially reproduce in Table 3.1. In each case he gives explicit generators of these subgroups (usually as elements of A_{12}) and so it is easy to check that these are indeed triangle-point subgroups of the Monster. However, Norton does not prove that this list is complete. A consequence of this work is to give a proof, independent of the Monster, that this list is indeed complete (see the discussion following Theorem 3.2.1).

In general, we say that a triangle-point group $G = \langle a, b, c\rangle$, $2A$-*embeds* into the Monster if there exists an injective homomorphism $\iota : G \to \mathbb{M}$ such that $\iota(a), \iota(b), \iota(c), \iota(ab) \in 2A$.

In [Nor85], for each triple (a, b, c) of $2A$ involutions, Norton defines the corresponding *ancestor subgroups* of $\langle a, b, c\rangle$ to be

$$A_1 := \langle a, b, a^c\rangle,\ A_2 := \langle a, b, b^c\rangle \text{ and } A_3 := \langle a, b, (ab)^c\rangle.$$

In Table 5 of [Nor85], as in our reproduction in Table 3.1, for each triple (a, b, c), the column labelled "ancestors" gives the indices of the rows corresponding to the groups A_1, A_2 and A_3.

No.	Order	Iso. Type	Classes of ac, bc, abc	Ancestors	No.	Order	Iso. Type	Classes of ac, bc, abc	Ancestors
1	4	2^2	$(1A\ 2A\ 2A)$	$(1,1,1)$	20	32	$2^4.2$	$(4A\ 4B\ 4B)$	$(8,2,2)$
2	8	2^3	$(2A\ 2A\ 2B)$	$(1,1,1)$	21	160	$2^4 : D_{10}$	$(4A\ 5A\ 5A)$	$(8,27,27)$
3	12	D_{12}	$(2A\ 3A\ 6A)$	$(1,3,3)$	22	240	$2 \times S_5$	$(4A\ 5A\ 6A)$	$(9,22,11)$
4	16	$2 \times D_8$	$(2A\ 4A\ 4A)$	$(1,2,2)$	23	96	$2^2 \times S_4$	$(4A\ 6A\ 6A)$	$(7,11,11)$
5	8	D_8	$(2A\ 4B\ 4B)$	$(1,1,1)$	24	192	$2^4 : D_{12}$	$(4A\ 6A\ 6A)$	$(8,10,10)$
6	8	2^3	$(2A\ 2B\ 2B)$	$(1,1,1)$	25	64	$2^4.2^2$	$(4B\ 4B\ 4B)$	$(4,4,4)$
7	16	$2 \times D_8$	$(2B\ 4A\ 4A)$	$(1,2,2)$	26	72	$3^2.D_8$	$(4B\ 4B\ 6A)$	$(3,3,13)$
8	16	$2 \times D_8$	$(2B\ 4B\ 4B)$	$(1,2,2)$	27	160	$2^4 : D_{10}$	$(4B\ 5A\ 5A)$	$(4,21,21)$
9	24	$2^2 \times S_3$	$(2B\ 6A\ 6A)$	$(1,3,3)$	28	120	S_5	$(4B\ 5A\ 5A)$	$(3,28,10)$
10	24	S_4	$(3A\ 3A\ 4B)$	$(10,10,5)$	29	48	$2 \times S_4$	$(4B\ 5A\ 6A)$	$(5,10,10)$
11	48	$2 \times S_4$	$(3A\ 4A\ 6A)$	$(11,7,11)$	30	384	$2^5.D_{12}$	$(4B\ 6A\ 6A)$	$(4,11,11)$
12	60	A_5	$(3A\ 5A\ 5A)$	$(12,12,12)$	31	660	$L_2(11)$	$(5A\ 5A\ 5A)$	$(33,33,33)$
13	36	$S_3 \times S_3$	$(3A\ 6A\ 6A)$	$(13,3,3)$	32	960	$2^4.A_5$	$(5A\ 5A\ 6A)$	$(32,32,12)$
14	24	S_4	$(3C\ 3C\ 4B)$	$(14,14,5)$	33	660	$L_2(11)$	$(5A\ 6A\ 6A)$	$(31,12,12)$
15	60	A_5	$(3C\ 5A\ 5A)$	$(15,15,15)$	34	1440	$2 \times S_6$	$(5A\ 6A\ 6A)$	$(34,11,11)$
16	108	$3^{1+2}.2^2$	$(3C\ 6A\ 6A)$	$(16,3,3)$	35	120	S_5	$(6A\ 6A\ 6A)$	$(12,12,12)$
17	32	2^2 wr 2	$(4A\ 4A\ 4A)$	$(6,6,6)$	36	576	$\frac{1}{2}.S_4$ wr 2	$(6A\ 6A\ 6A)$	$(12,12,12)$
18	64	$2^3.2^3$	$(4A\ 4A\ 4B)$	$(7,7,4)$	37	3840	$2^5.S_5$	$(6A\ 6A\ 6A)$	$(12,12,12)$
19	144	$2 \times 3^2.D_8$	$(4A\ 4A\ 6A)$	$(9,9,13)$					

Table 3.1: Norton's list of triangle-point subgroups of the Monster

Name	Isomorphism type	$G^{(m,n,p)}$ (m,n,p)	Added relations $R_i^{r_i}$ (r_1,r_2,r_3,r_4,r_5)	$2A$-embeds in $\mathbb{M}$
G_1	$2^3.2^3$	$(4,4,4)$		Y
G_2	$(S_3 \times S_3) : 2^2$	$(4,4,6)$		Y
G_3	$2^4 : D_{10}$	$(4,5,5)$		Y
G_4	$2 \times S_5$	$(4,5,6)$		Y
G_5	$L_2(11)$	$(5,5,5)$		Y
G_6	$(2^4 : D_{12}) \times 2$	$(4,6,6)$	$(4,-,-,-,-)$	Y
G_7	$2^4 : A_5$	$(6,5,5)$	$(5,-,-,-,-)$	Y
G_8	$2 \times S_6$	$(6,6,5)$	$(4,-,-,-,-)$	Y
G_9	$(2^4 : (S_3 \times S_3)) \times 2$	$(6,6,6)$	$(4,\ 6,\ 6,-,-)$ $(6,\ 4,\ 6,-,-)$ $(6,\ 6,\ 4,-,-)$	N
G_{10}	$2^5 : S_5$	$(6,6,6)$	$(5,\ 5,\ 5,\ 4,-)$	Y
G_{11}	$(3^4 : 2) : (3_+^{1+2} : 2^2)$	$(6,6,6)$	$(6,\ 6,\ 6,-,\ 3)$	N

Table 3.2: The groups $G_1, \ldots, G_{11}$

A crucial first step in the proof of our main result was completed by Decelle in Theorem 3.3 of [Dec13].

Theorem 3.1.3 ([Dec13])**.** *Each triangle-point group must occur as a quotient of at least one of the* 11 *groups given in Table 3.2. Each of these groups occurs as a quotient of a group of the form*

$$G^{(m,n,p)} := \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^m, (bc)^n, (abc)^p \rangle$$

for $m, n, p \in [1..6]$*, potentially with additional relations of the form* $R_i^{r_i}$ *for* $i \in [1..5]$ *and*

$$R_1 := a \cdot b^c,\ R_2 := ab \cdot b^c,\ R_3 := ab \cdot a^c,\ R_4 := c \cdot b^{ca},\ R_5 := c^a \cdot c^{bc}.$$

Our first aim is to use Theorem 3.1.3 to construct a list of all possible triangle-point groups. We do this by classifying the normal subgroups, and the corresponding quotients, of the groups $G_1, \ldots, G_{11}$. However, some smaller examples will appear as quotients of many of the above groups. Thus, to significantly reduce the number of normal subgroups that we must classify, we first consider small examples of triangle-point groups.

Proposition 3.1.4. *Suppose that G is a triangle-point group of order at most 12. Then G is either a dihedral group or an elementary abelian group of order 8.*

Proof. Suppose that $G = \langle a, b, c \rangle$ is a triangle-point group. Then G contains the subgroup $\langle a, b \rangle \cong 2^2$ and so the order of G must be a multiple of 4 and so must be equal to 4, 8 or 12. Up to isomorphism, the only groups of these orders that are generated by their involutions are D_4, D_8, 2^3 and D_{12} and it is easy to check that each of these is indeed a triangle-point group. □

In Tables 3.3 and 3.4 below, for each $i \in [1..11]$ we give a complete list of the non-trivial normal subgroups $N \triangleleft G_i$ such that $[G_i : N] > 12$. Note that the groups G_3 and G_5 have no such normal subgroups and are thus omitted from these tables.

The lists of normal subgroups in Table 3.3 have been calculated in GAP [GAP16] using explicit generators of each of the groups $G_1, \ldots, G_{10}$ and by using the group presentation in the case of G_{11}. In most cases, the generators used below are given in [Nor85]. In particular, where possible, we choose these generators to be elements of A_{12}.

Proposition 3.1.5. *For* $1 \leq i \leq 10$*, Table 3.3 gives*

- *elements* $a, b, c \in G_i$ *such that* $G_i = \langle a, b, c \rangle$ *as a triangle-point group;*
- *generators in terms of* a, b, c *of all normal subgroups* $N \trianglelefteq G_i$ *such that* $[G_i : N] > 12$*;*
- *the isomorphism types for the corresponding quotients* G_i/N*.*

Table 3.4 gives generators of all normal subgroups $N \trianglelefteq G_{11}$ *such that* $[G_{11} : N] > 12$ *and the isomorphism types for the corresponding quotients* G_{11}/N*.*

Using Tables 3.3 and 3.4 and Proposition 3.1.4, we have compiled a complete list (up to isomorphism) of triangle-point groups which we give in Table 3.5. With the exception of the groups of order less than or equal to 12, for each triangle-point group G, we give the indices i of the groups $G_1, \ldots, G_{11}$ such that G occurs as a quotient of the group G_i.

3.2 The main theorem

In this section we prove the main result of this chapter.

Theorem 3.2.1. *Suppose that* V *is a Majorana algebra that satisfies axiom M8 and which is generated by three Majorana axes* a_0, a_1, a_2 *such that the dihedral algebra* $\langle\langle a_0, a_1 \rangle\rangle$ *is of type* $2A$*. Then* V *must occur as a Majorana representation of one of* 26 *groups, each of which occurs as a subgroup of the Monster.*

By comparing the list of triangle-point groups in Table 3.5 with Norton's list of triangle-point subgroups of the Monster (Table 3 of [Nor85]), we see that of the 36 triangle-point groups, there are 10 that do not appear on Norton's list. We reproduce the rows of Table 3.5 which contain these 10 groups in Table 3.6.

In the remainder of this section, we consider these ten groups and show that none of them can admit a Majorana representation of the form (G, T, V) where $G = \langle a, b, c \rangle$ and $a, b, c, ab \in T$. In doing so, we also show that these groups cannot exist as triangle-point subgroups of the Monster (as otherwise they would have to admit such a Majorana representation) and so prove that Norton's list of triangle-point groups is complete.

Throughout this section, we make use of the following result, which is Lemma 8.6.3 in [Iva09].

G	Generators a, b, c	N	$X \subset G$ s.t. $N = \langle X \rangle^G$	G/N
G_1	$(1,2)(3,4)$ $(1,3)(2,4)(5,6)(7,8)$ $(1,5)(2,7)$	2^2 2^2 2^2 2	$(ac)^2$ $(bc)^2$ $(abc)^2$ $(a \cdot b^c)^2$	$2 \times D_8$ $2 \times D_8$ $2 \times D_8$ $2^4 : 2$
G_2	$(1,2)(3,4)$ $(5,6)(7,8)$ $(1,2)(3,9)(4,5)(6,10)$	3^2 2	$(a \cdot b^c)^2$ $(a \cdot b^c)^3$	$2 \times D_8$ $(S_3 \times S_3) : 2$
G_4	$(1,2)(3,4)$ $(1,2)(3,4)(5,6)(7,8)$ $(1,9)(2,5)(3,4)(7,8)$	2	$(ac)^2$	S_5
G_6	$(1,2)(3,4)$ $(1,3)(2,4)(5,6)(7,8)(9,10)(11,12)$ $(1,2)(3,5)(4,7)(6,9)(8,11)(10,12)$	2^4 2^4 2^3 2^3 2^3 2^2 2	$(bc)^3, (abc)^3$ $(ac)^2$ $(ab \cdot b^c)^3, (a^c \cdot c^b)^2$ $(bc)^3$ $(abc)^3$ $(a^c \cdot c^b)^2$ $(ab \cdot b^c)^3$	S_4 $2^2 \times S_3$ $2 \times S_4$ $2 \times S_4$ $2 \times S_4$ $2^2 \times S_4$ $2^4 : D_{12}$
G_7	$(1,3)(2,15)(4,13)(6,12)(7,11)(14,16)$ $(1,11)(2,12)(3,9)(4,10)(5,6)(13,14)$ $(1,3)(2,15)(4,13)(6,12)(7,11)(14,16)$	2^4	$(ac)^3$	A_5
G_8	$(1,2)(7,8)$ $(1,2)(3,4)(5,6)(9,10)$ $(1,3)(4,5)(7,8)(9,10)$	2	$((bc)^3 \cdot b^{ca})^3$	S_6
G_9	$(1,2)(3,4)(5,6)(7,8)$ $(1,8)(2,7)(3,4)(5,6)$ $(2,5)(3,6)(9,10)(11,12)$	$2^4 : 3$ $2^4 : 3$ 2^5 2^4 2	$(ac)^2, (a \cdot b^c)^2$ $(bc)^2, (a \cdot b^c)^2$ $(abc)^2, (a \cdot b^c)^2$ $(a \cdot b^c)^2$ $a \cdot (b \cdot c^{ac})^3$	$2^2 \times S_3$ $2^2 \times S_3$ $S_3 \times S_3$ $2 \times S_3 \times S_3$ $2^4 : (S_3 \times S_3)$
G_{10}	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ $(1,3)(2,4)(5,8)(6,7)(9,12)(10,11)$ $(1,7)(2,6)(3,9)(4,11)(5,10)(8,12)$	2^5 2	$((ac)^2(bc)^2)^2$ $(c^a \cdot (bc)^3)^3$	S_5 $2^4 : S_5$

Table 3.3: Some normal subgroups of $G_1, \ldots, G_{10}$

N	$X \subset G_{11}$ s.t. $N = \langle X \rangle^{G_{11}}$	G_{11}/N
$(3^4 : 3) : 3$	$(ac)^2$	$2^2 \times S_3$
$(3^4 : 3) : 3$	$(bc)^2, (a \cdot b^c)^2$	$2^2 \times S_3$
$(3^4 : 3) : 3$	$(abc)^2$	$2^2 \times S_3$
$(3^4 : 3) : 2$	$(ac)^3, (ab \cdot b^c)^2$	$S_3 \times S_3$
$(3^4 : 3) : 2$	$(bc)^3, (ab \cdot a^c)^2$	$S_3 \times S_3$
$(3^4 : 3) : 2$	$(abc)^3, (a \cdot a^c)^2$	$S_3 \times S_3$
$3^4 : 3$	$(a \cdot b^c)^2$	$2 \times S_3 \times S_3$
$3^4 : 3$	$(ab \cdot b^c)^2$	$2 \times S_3 \times S_3$
$3^4 : 3$	$(ab \cdot a^c)^2$	$2 \times S_3 \times S_3$
$3^4 : 2$	$(ac)^3$	$3^{1+2}_+ : 2^2$
$3^4 : 2$	$(bc)^3$	$3^{1+2}_+ : 2^2$
$3^4 : 2$	$(abc)^3$	$3^{1+2}_+ : 2^2$
3^4	$(acbcacb)^2$	$S_3 \times S_3 \times S_3$
3^4	$(a \cdot c^{bc})^2$	$2 \times (3^{1+2}_+ : 2^2)$
3^4	$(b \cdot c^{ac})^2$	$2 \times (3^{1+2}_+ : 2^2)$
3^4	$(ab \cdot c^{ac})^2$	$2 \times (3^{1+2}_+ : 2^2)$
3^3	$(ab \cdot a^{cbc})^2, (a \cdot b^{cabc})^2$	$S_3 : (3^{1+2}_+ : 2^2)$
3^3	$(ab \cdot b^{cac})^2, (a \cdot b^{cabc})^2$	$S_3 : (3^{1+2}_+ : 2^2)$
3^3	$(ab \cdot b^{cac})^2, (ab \cdot a^{cbc})^2$	$S_3 : (3^{1+2}_+ : 2^2)$
3^2	$(a \cdot b^{cabc})^2,$	$(3^2 : 2) : (3^{1+2}_+ : 2^2)$
3^2	$(ab \cdot b^{cac})^2,$	$(3^2 : 2) : (3^{1+2}_+ : 2^2)$
3^2	$(ab \cdot a^{cbc})^2$	$(3^2 : 2) : (3^{1+2}_+ : 2^2)$
3	$c^{acbcacb} \cdot c^{bcacbca}$	$(3^3 : 2) : (3^{1+2}_+ : 2^2)$

Table 3.4: Some normal subgroups of G_{11}

i	G	$\|G\|$	Quotient of G_i for i in	i	G	$\|G\|$	Quotient of G_i for i in
1	2^2	4	–	19	$2^4 : D_{10}$	160	3
2	2^3	8	–	20	$2^4 : D_{12}$	192	6
3	D_8	8	–	21	$S_3 \times S_3 \times S_3$	216	11
4	D_{12}	12	–	22	$2 \times (3_+^{1+2} : 2^2)$	216	11
5	$2 \times D_8$	16	$1, 2$	23	$2 \times S_5$	240	4
6	$2^2 \times S_3$	24	$6, 9, 11$	24	$2^5 : D_{12}$	384	6
7	S_4	24	6	25	$2^4 : (S_3 \times S_3)$	576	9
8	$2^4.2$	32	1	26	$(3 : 2) : (3_+^{1+2} : 2^2)$	648	11
9	$S_3 \times S_3$	36	$9, 11$	27	$L_2(11)$	660	6
10	$2 \times S_4$	48	6	28	S_6	720	8
11	A_5	60	7	29	$2^4.A_5$	960	7
12	$2^3.2^3$	64	1	30	$(2^4 : (S_3 \times S_3)) \times 2$	1152	9
13	$2 \times S_3 \times S_3$	72	$9, 11$	31	$2 \times S_6$	1440	8
14	$(S_3 \times S_3) : 2$	72	2	32	$2^4 : S_5$	1920	10
15	$2^2 \times S_4$	96	6	33	$(3^2 : 2) : (3_+^{1+2} : 2^2)$	1944	11
16	$3_+^{1+2}.2^2$	108	11	34	$2^5.S_5$	3840	10
17	S_5	120	10	35	$(3^3 : 2) : (3_+^{1+2} : 2^2)$	5832	11
18	$(S_3 \times S_3) : 2^2$	144	2	36	$(3^4 : 2) : (3_+^{1+2} : 2^2)$	17496	11

Table 3.5: A complete list of triangle-point groups

i	G	$\lvert G \rvert$	Quotient of G_i for i in
13	$2 \times S_3 \times S_3$	72	$9, 11$
21	$S_3 \times S_3 \times S_3$	216	11
22	$2 \times (3_+^{1+2} : 2^2)$	216	11
26	$(3 : 2) : (3_+^{1+2} : 2^2)$	648	11
28	S_6	720	8
30	$(2^4 : (S_3 \times S_3)) \times 2$	1152	9
32	$2^4 : S_5$	1920	10
33	$(3^2 : 2) : (3_+^{1+2} : 2^2)$	1944	11
35	$(3^3 : 2) : (3_+^{1+2} : 2^2)$	5832	11
36	$(3^4 : 2) : (3_+^{1+2} : 2^2)$	17496	11

Table 3.6: The triangle-point groups that do not $2A$-embed into the Monster

Lemma 3.2.2. *Suppose that there exists a group G that admits a Majorana representation (G, T, V). Suppose also that G contains a subgroup K isomorphic to the elementary abelian group of order 8. Then there must exist at least one non-identity element of K that does not lie in T.*

Proof. Suppose that $K := \langle t_0, t_1, t_2 \rangle$ and suppose for contradiction that all non-identity elements of K lie in T. Since any two axes $\psi(t_i)$ and $\psi(t_j)$ generate a $2A$ algebra,

$$\psi(t_1) - \psi(t_0 t_1),\ \psi(t_2) - \psi(t_0 t_2) \text{ and } \psi(t_1 t_2) - \psi(t_0 t_1 t_2)$$

are all $\frac{1}{2^2}$-eigenvectors of $\psi(t_0)$. However,

$$(\psi(t_1) - \psi(t_0 t_1)) \cdot (\psi(t_2) - \psi(t_0 t_2)) = -\frac{1}{2^2}(\psi(t_1 t_2) - \psi(t_0 t_1 t_2)).$$

is also a $\frac{1}{2^2}$-eigenvector of $\psi(t_0)$. This contradicts the fusion rules and so such a representation cannot exist. ☐

In most of the cases below, we have explicit generators for the groups in question. However, as Tables 3.3 and 3.4 provide an exhaustive list of all triangle-point groups of order greater than 12, we can also use these to determine the exact presentations of these groups. Whether we use explicit generators or the group presentation for our calculations is simply a question of clarity.

3.2.1 The group $2 \times S_3 \times S_3$

Proposition 3.2.3. *Suppose that $G = \langle a, b, c \rangle \cong 2 \times S_3 \times S_3$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. From Tables 3.3 and 3.4, we see that G occurs either as a quotient of G_9, or as a quotient of G_{11}. In either case, we must have

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^6, (a \cdot b^c)^{r_1}, (ab \cdot b^c)^{r_2}, (ab \cdot a^c)^{r_3} \rangle$$

where $(r_1, r_2, r_3) \in \{(2, 6, 6), (6, 2, 6), (6, 6, 2)\}$. We first suppose that $(r_1, r_2, r_3) = (2, 6, 6)$ and show that the group

$$K := \langle a, b, (abc)^3 \rangle \leq G$$

is isomorphic to 2^3 and that all of its non-identity elements are contained in T. Using the presentation of G in GAP, we have checked that $o(ac) = o(bc) = o(abc) = 6$. By assumption, $a, b, ab \in T$ and, from Lemma 3.0.2, as $o(abc) = 6$, $(abc)^3 \in T$. Using the presentation of G, and in particular the relation $(a \cdot b^c)^2 = 1$, we can show that

$$a \cdot (abc)^3 = ((bc)^3)^{ac} \in T$$
$$b \cdot (abc)^3 = ((ac)^3)^{bc} \in T$$
$$ab \cdot (abc)^3 = c^{abc} \in T$$

and so $2^3 \cong K \subset T \cup \{e\}$. This is a contradiction with Lemma 3.2.2 and so no such representation can exist. In the case that $(r_1, r_2, r_3) = (6, 2, 6)$ or $(6, 6, 2)$, we take K to be $\langle a, b, (ac)^3 \rangle$ or $\langle a, b, (bc)^3 \rangle$ respectively. In either case, we find that $2^3 \cong K \subset T \cup \{e\}$, again giving a contradiction. □

3.2.2 The group S_6

Proposition 3.2.4. *Suppose that $G = \langle a, b, c \rangle \cong S_6$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as the group

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^5, (a \cdot b^c)^4, x^3 \rangle$$

where $x = (bc)^3 \cdot b^{ca}$, so that x^3 is the central element of G_8. In this case, we use explicit generators. If we pick

$$a := (1,2)(3,4)(5,6)$$
$$b := (5,6)$$
$$c := (2,3)(4,5)$$

then a, b, c satisfy the relations above and generate the group S_6. Thus we may take $G = \langle a, b, c \rangle$. By definition, T must contain the conjugacy classes

$$a^G = (1,2)(3,4)^G,\ b^G = (5,6)^G,\ (ab)^G = (1,2)(3,4)(5,6)^G.$$

In particular, as the conjugacy classes in S_6 are indexed by the cycle types of their elements, this must mean that *all* involutions of G are contained in T. Finally, G contains the subgroup $\langle (1,2), (3,4), (5,6) \rangle \cong 2^3$, all of whose non-identity elements must be contained in T. This is in contradiction with Lemma 3.2.2 and the result follows. □

3.2.3 The group $(2^4 : (S_3 \times S_3)) \times 2$

Proposition 3.2.5. *Suppose that $G = \langle a, b, c \rangle \cong (2^4 : (S_3 \times S_3)) \times 2$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. In this case, we must have $G = G_9$. Although we have explicit generators of the group, it is easier to consider G as a finitely presented group with generators a, b, c. We now let $x := ab \cdot c^{ac}$ and claim that

$$K := \langle a, b, x^3 \rangle$$

is isomorphic to 2^3 and that all the non-identity elements of K lie in T. By definition, $a, b, ab \in T$. We calculate that

$$\begin{aligned} a \cdot x^3 &= (b \cdot c^{ac})^3 \\ b \cdot x^3 &= ((ac)^3)^{bcacac} \\ ab \cdot x^3 &= c^{acabcacac}. \end{aligned}$$

Either by using explicit generators, or by calculating with the group presentation in GAP, we see that x, $b \cdot c^{ac}$ and ac are all of order 6. Moreover, as they are each the product of two elements of T, by Lemma 3.0.2, their cubes must all also lie in T. This shows that K must be isomorphic to 2^3 and that all non-identity elements of K are contained in T. This is in contradiction with Lemma 3.2.2 and the result follows. □

3.2.4 The group $2^4 : S_5$

We deal with this group using slightly different techniques to the other cases. We begin by noting that from Tables 3.3 and 3.4, we see that G must occur as the group

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^6, (a \cdot b^c)^5, (ab \cdot b^c)^5, (ab \cdot a^c)^5, (c \cdot b^{ca})^4, x^3 \rangle$$

where $x = c^a \cdot (bc)^3$, so that x^3 is the central element of G_{10}. If we take

$$\begin{aligned} a &:= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) \\ b &:= (1,3)(2,4)(5,6)(7,8)(13,14)(15,16) \\ c &:= (1,12)(3,14)(4,6)(5,16)(7,11)(9,13). \end{aligned}$$

then a, b, c generate a group of order 1920 and satisfy the presentation of G and so we may take $G = \langle a, b, c \rangle$.

We will show that G contains a subgroup $K \cong 2 \times D_8$ and that there exist no representations of the form $(K, K \cap T, U)$. This will in turn show that there exist no representations of the form (G, T, V).

Lemma 3.2.6. *Let $K := \langle (1,2), (1,3)(2,4), (5,6) \rangle \cong 2 \times D_8$ then K contains eleven involutions, which we label t_i for $1 \leq i \leq 11$ as below.*

i	t_i	i	t_i	i	t_i
1	$(1,2)$	5	$(1,3)(2,4)(5,6)$	9	$(1,2)(3,4)$
2	$(3,4)$	6	$(1,4)(2,3)(5,6)$	10	$(1,2)(3,4)(5,6)$
3	$(1,2)(5,6)$	7	$(1,3)(2,4)$	11	$(5,6)$
4	$(3,4)(5,6)$	8	$(1,4)(2,3)$		

If we let $S := \{t_1, \ldots, t_{10}\}$ then there exist no Majorana representations of the form (K, S, U).

Proof. We suppose for contradiction that such a representation exists and show that it cannot obey axiom M1. In the following, we let $a_i := \psi(t_i)$ for $1 \leq i \leq 10$. Note that $t_1 t_4 =$

$(1,2)(3,4)(5,6) = t_{10}$ and so the algebra $\langle\langle a_1, a_4\rangle\rangle$ is of type $2A$ and

$$a_1 \cdot a_4 = \frac{1}{2^3}(a_1 + a_4 - a_{10}).$$

Now $o(t_1t_5) = o(t_4t_5) = 4$ and $(t_1t_5)^2, (t_4t_5)^2 \in S$ and so, by Lemma 3.0.2, the algebras $\langle\langle a_1, a_5\rangle\rangle$ and $\langle\langle a_4, a_5\rangle\rangle$ are of type $4B$ and

$$(a_1, a_5) = (a_4, a_5) = \frac{1}{2^6}.$$

Finally, $t_{10}t_5 = (1,4)(2,3) = t_8$ and so $\langle\langle a_{10}, a_5\rangle\rangle$ is of type $2A$, $(a_{10}, a_5) = \frac{1}{2^3}$ and

$$(a_1 \cdot a_4, a_5) = -\frac{3}{2^8}.$$

Similarly, we calculate that

$$a_4 \cdot a_5 = \frac{1}{2^6}(a_4 + a_5 - a_3 - a_6 + a_9)$$

and that

$$(a_1, a_4 \cdot a_5) = \frac{1}{2^6} \neq (a_1 \cdot a_4, a_5)$$

which is in contradiction with axiom M1, showing that such an algebra cannot exist. □

Proposition 3.2.7. *Suppose that $G = \langle a, b, c\rangle \cong 2^4 : S_5$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. If we let

$$x := a,\ y := b^{cacac} \text{ and } z := ((ac)^3)^b$$

then it is easy to check that the map f that sends

$$x \mapsto (1,2)(3,4)(5,6)$$
$$y \mapsto (3,4)$$
$$z \mapsto (1,4)(2,3)$$

is an isomorphism from $K := \langle x, y, z\rangle$ to $2 \times D_8$.

By definition and by Lemma 3.0.2, we have $x, y, z \in T$. Moreover,

$$xy = (ab)^{cacac} \in T,$$
$$xz = c^{acb} \in T,$$
$$(yz)^2 = (b \cdot b^{cacac})^3 \in T.$$

By considering the conjugacy classes of $2 \times D_8$, we see that

$$|x^K \cup y^K \cup z^K \cup (xy)^K \cup (xz)^K \cup ((yz)^2)^K| = 10$$

and so, as K contains 11 involutions in total, $|K \cap T|$ is equal to 10 or 11.

If $|K \cap T| = 11$ then K contains an elementary abelian subgroup of order 8 all of whose involutions are contained in T and so the representation $(K, K \cap T, U)$ cannot exist. If $|K \cap T| = 10$ then $(K, K \cap T, U)$ is the representation in Lemma 3.2.6 and so equally cannot exist. Thus we may conclude that there exist no representations of the form (G, T, V), as required. □

3.2.5 The group $S_3 \times S_3 \times S_3$

Proposition 3.2.8. *Suppose that $G = \langle a, b, c \rangle \cong S_3 \times S_3 \times S_3$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as the group

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^6, (a \cdot b^c)^6, (ab \cdot b^c)^6, (ab \cdot a^c)^6, (c^a \cdot c^{bc})^3, x^2 \rangle$$

where $x = acbcacb$. In this case, we use explicit generators. If we pick

$$\begin{aligned} a &:= (1,2)(4,5) \\ b &:= (4,5)(7,8) \\ c &:= (1,3)(4,6)(7,9) \end{aligned}$$

then a, b, c satisfy the relations above and generate the group $S_3 \times S_3 \times S_3$. Thus we may take $G = \langle a, b, c \rangle$. By assumption, T must contain the conjugacy classes $a^G, b^G, c^G, (ab)^G$. By Lemma 3.0.2, it must also contain

$$\begin{aligned} (ac)^3 &= (7,9) \\ (bc)^3 &= (1,3) \\ (abc)^3 &= (4,6). \end{aligned}$$

We now let

$$K := \langle (1,3), (4,6), (7,9) \rangle \leq G.$$

then K is clearly elementary abelian of order 8 and, from the above discussion, all the non-identity elements of K are contained in T. This is a contradiction with Lemma 3.2.2 and so such a representation cannot exist. □

3.2.6 The group $2 \times (3_+^{1+2} : 2^2)$

Proposition 3.2.9. *Suppose that $G = \langle a, b, c \rangle \cong 2 \times (3_+^{1+2} : 2^2)$ is a triangle-point group and suppose that $T \subseteq G$ such that $a, b, c, ab \in T$. Then there exist no Majorana representations of the form (G, T, V) which obey axiom M8.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as the group

$$\langle a, b, c \mid, a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^6, (a \cdot b^c)^6, (ab \cdot b^c)^6, (ab \cdot a^c)^6, (c^a \cdot c^{bc})^3, y^2 \rangle$$

where $y \in \{a \cdot c^{bc}, b \cdot c^{ac}, ab \cdot c^{ac}\}$.

Note that any of the possible values for y can be sent to any other by a suitable permutation of the generators a, b, ab. Moreover, such a permutation preserves all other relations in the presentation of these groups (to show this for the relation $(c^a \cdot c^{bc})^3 = 1$ requires some calculation, in all other cases it is clear). Thus, permutating a, b and ab induces pairwise isomorphisms between the three groups arising from the different choices of y.

Without loss of generality, we can now pick $y = a \cdot c^{bc}$ and let

$$a = (1,4)(2,6)(3,5)(8,9), b = (1,4)(2,8)(6,9)(10,11), c = (2,7)(3,4)(5,9).$$

Then it is easy to check that a, b, c satisfy the presentation of G and generate a group of order 216. Thus we may take $G = \langle a, b, c \rangle$. We then calculate that

$$\begin{aligned} ac &= (1,3,9,8,5,4)(2,6,7) \\ b \cdot c^{ac} &= (1,8,6)(2,4,9)(10,11) \\ ab \cdot c^{ac} &= (1,9,6,4,8,2)(3,5)(10,11) \end{aligned}$$

are all of order 6. We now let

$$K := \langle a, b, y \rangle.$$

where $y := (b \cdot c^{ac})^3$. From Lemma 3.0.2, $(b \cdot c^{ac})^3, (ab \cdot c^{ac})^3, (ac)^3 \in T$. Thus

$$\begin{aligned} a \cdot y &= (ab \cdot c^{ac})^3 \in T \\ b \cdot y &= ((ac)^3)^{b(ac)^3} \in T \\ ab \cdot y &= c^{acabc(ac)^2} \in T. \end{aligned}$$

We now have $K \cong 2^3 \subseteq T \cup \{e\}$, which is a contradiction with Lemma 3.2.2 and so such a representation cannot exist. □

3.2.7 The remaining quotients of the group G_{11}

Here we consider the case where $G = \langle a, b, c \rangle \cong (3^i : 2) : (3_+^{1+2} : 2^2)$ for $i = 1, 2, 3, 4$. The following is Proposition 3.52 in [Dec13].

Lemma 3.2.10. *Let K be the quotient of $G^{(6,6,6)}$ with the additional relations*

$$(a \cdot b^c)^6 = (ab \cdot b^c)^6 = (ab \cdot a^c)^6 = (c \cdot b^{ca})^{r_4} = 1$$

then

- *if $r_4 \in \{1, 5\}$ then $K \cong D_{12}$;*
- *if $r_4 \in \{3\}$ then $K \cong G^{(3,6,6)} \cong 3_+^{1+2} : 2^2$;*
- *if $r_4 \in \{2, 4\}$ then $K \cong 2 \times G^{(3,6,6)} \cong 2 \times (3_+^{1+2} : 2^2)$.*

We will also require the following result.

Lemma 3.2.11. *Let $G := G^{(m,6,6)}$ then*

- *if $m = 1$, $G \cong 2^2$;*

- *if* $m = 2$, $G \cong 2 \times D_{12}$;
- *if* $m = 3$, $G \cong 3_+^{1+2} : 2^2$.

In particular, if $m \in \{1, 2, 3\}$ *then* $|G^{(m,6,6)}| \leq 108$.

Proof. We let $G := G^{(m,6,6)}$ and deal with the cases $m = 1, 2, 3$ in turn.

- $m = 1$: Here $a = c$ and so $G = \langle a, b \mid a^2, b^2, (ab)^2 \rangle \cong 2^2$.
- $m = 2$: Here a commutes with b and c and is of order 2 and so $G = 2 \times \langle b, c \rangle \cong 2 \times D_{12}$.
- $m = 3$: This case is given in Lemma 3.2.10 above. □

Proposition 3.2.12. *Suppose that* $G = \langle a, b, c \rangle \cong (3^i : 2) : (3_+^{1+2} : 2^2)$ *for* $i = \{1, 2, 3, 4\}$ *is a triangle-point group and suppose that* $T \subseteq G$ *such that* $a, b, c, ab \in T$. *Then there exist no Majorana representations of the form* (G, T, V) *which obey axiom M8.*

Proof. Let $m := o(ac)$, $n := o(bc)$ and $p := o(abc)$. We will show that we must have $(m, n, p) = (6, 6, 6)$. Suppose for contradiction that this is not true. Then G must be isomorphic to a quotient of $G^{(m,6,6)}$ for $m \in \{1, 2, 3\}$. However, $|G| \geq 648$, in contradiction with with Lemma 3.2.11 above, and so we must have $(m, n, p) = (6, 6, 6)$. With Lemma 3.0.2, this implies that $(ac)^3, (bc)^3, (abc)^3 \in T$.

We now consider the element $x := b \cdot (ac)^3$. As G is a triangle-point group and $(ac)^3 \in T$, we must have $o(x) \leq 6$. If we were to have $o(x) < 6$ then $R_4 = x^{cac}$ would also be of order strictly less than 6 and so G would have to exist as the quotient of one of the groups in Lemma 3.2.10. Comparison of orders again shows that this cannot be the case, and so we get $o(R_4) = o(x) = 6$.

We claim that
$$K := \langle a, b, x^3 \rangle$$
is elementary abelian of order 8 and that all its non-identity elements lie in T. By assumption, $a, b, ab \in T$ and, by Lemma 3.0.2, $x^3, (abc)^3, (ac)^3 \in T$.

$$a \cdot x^3 = ((abc)^3)^{bcabcabca}$$
$$b \cdot x^3 = c^{acbcacac}$$
$$ab \cdot x^3 = (ac)^3.$$

We now have $K \cong 2^3 \subseteq T \cup \{e\}$, which is a contradiction with Lemma 3.2.2 and so such a representation cannot exist. □

Chapter 4

An algorithm for constructing n-closed Majorana representations

4.1 Background

Since its inception, Majorana theory has proved to be an impressive tool with which to study the Griess algebra and related objects. Over the past few years, a number of important subalgebras of the Griess algebra have been constructed using Majorana theory. Most of the early work on Majorana theory was completed by hand. However, it soon became clear that, in order to construct larger and more complex algebras, a more computational approach was necessary.

In 2012, Seress published a celebrated paper [Ser12] in which he announced the existence of an algorithm to construct 2-closed Majorana algebras. This was hailed "a groundbreaking work" that "marks a turning point in Majorana Theory." Sadly, shortly after the publication of this paper, Seress passed away and recovering the full details of his algorithm and results has been an important aim of the theory ever since.

We have sucessfully used GAP to implement and run an algorithm based largely on Seress' method and have completely recovered his results, given in Section 4.6. Moreover, we have also extended Seress' methods and are also able to construct generic Majorana algebras, including those which are not 2-closed, as explained in Section 4.5.

This work is joint with Markus Pfeiffer of the University of St Andrews who has contributed invaluable help in the implementation of the algorithm. All of our work can be found at `https://github.com/mwhybrow92/MajoranaAlgebras`, where it is also possible to see the exact contributions of the two authors.

4.2 Additional axioms

Recall that if (G, T, V, φ, ψ) is a Majorana representation and $t_0, t_1 \in T$ then, from Theorem 2.2.1, the algebra $U := \langle\langle \psi(t_0), \psi(t_1) \rangle\rangle$ must be isomorphic to one of the nine dihedral algebras of the Griess algebra. In particular, if U is of type $3A$, $4A$ or $5A$ respectively then U is spanned as a vector space by its Majorana axes and one further axis, denoted $u_{\rho(t_0,t_1)}$, $v_{\rho(t_0,t_1)}$ or $w_{\rho(t_0,t_1)}$ and referred to as a $3A$, $4A$ or $5A$ *axis* respectively.

As in Seress' original work, in the implementation of our algorithm we assume five axioms 2Aa, 2Ab, 3A, 4A and 5A as described below in addition to the Majorana axioms M1 - M7. These enforce certain equalities on the $2A$, $3A$, $4A$ and $5A$ axes.

Definition 4.2.1. *Let (G, T, V) be a Majorana representation and let $t_0, t_1, t_2, t_3 \in T$, with corresponding Majorana axes $a_{t_i} = \psi(t_i)$ for $0 \leq i \leq 3$. We define the axioms 2Aa, 2Ab, 3A, 4A and 5A as below.*

2Aa *If $t_0t_1t_2 = 1$ and $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ is of type $2A$ then $a_{t_2} \in \langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ and $a_{t_2} = a_\rho$ for the basis element $a_\rho = a_{\rho(t_0,t_1)}$ of $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$.*

2Ab *If $o(t_0t_1) = o(t_2t_3) = 2$ and $\langle t_0t_1 \rangle = \langle t_2t_3 \rangle$, and both $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ and $\langle\langle a_{t_2}, a_{t_3} \rangle\rangle$ are of type $2A$ then the basis elements $a_{\rho(t_0,t_1)}$ and $a_{\rho(t_2,t_3)}$ of the two subalgebras are equal.*

3A *If $o(t_0t_1) = o(t_2t_3) = 3$ and $\langle t_0t_1 \rangle = \langle t_2t_3 \rangle$, and both $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ and $\langle\langle a_{t_2}, a_{t_3} \rangle\rangle$ are of type $3A$ then the basis elements $u_{\rho(t_0,t_1)}$ and $u_{\rho(t_2,t_3)}$ are equal.*

4A *If $o(t_0t_1) = o(t_2t_3) = 4$ and $\langle t_0t_1 \rangle = \langle t_2t_3 \rangle$, and both $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ and $\langle\langle a_{t_2}, a_{t_3} \rangle\rangle$ are of type $4A$ then the basis elements $v_{\rho(t_0,t_1)}$ and $v_{\rho(t_2,t_3)}$ are equal.*

5A *If $o(t_0t_1) = o(t_2t_3) = 5$ and $\langle t_0t_1 \rangle = \langle t_2t_3 \rangle$, and both $\langle\langle a_{t_0}, a_{t_1} \rangle\rangle$ and $\langle\langle a_{t_2}, a_{t_3} \rangle\rangle$ are of type $5A$ then, up to a possible change of sign, the basis elements $w_{\rho(t_0,t_1)}$ and $w_{\rho(t_2,t_3)}$ are equal as in (2.1).*

We note that axiom 2Aa above is a weaker version of axiom M8.

In general, the axioms above say that if two axes are indexed by the same group element then they are equal, even if they arise from different dihedral algebras. Thus, where we assume these axioms, if $t, s \in T$ and $h := ts$ then we write

$$u_h := u_{\rho(t,s)},\ v_h := v_{\rho(t,s)},\ w_h := w_{\rho(t,s)}.$$

In particular, we can represent a spanning set of the algebra by a duplicate-free list of group elements corresponding to the Majorana axes and the $3A$, $4A$ and $5A$ axes.

The axioms 2Aa - 5A are known to hold in the Griess algebra and, crucially, these axioms restrict the cardinality of the spanning set used in the algorithm. That is to say, to assume axioms 2Aa - 5A gives a more restrictive but more efficient algorithm.

As the main goal of our computational work is to use the Majorana axioms to construct large subalgebras of the Griess algebra, it is advantageous to assume these additional axioms. However, we have in fact implemented two different versions of the algorithm; one which obeys the axioms 2Aa - 5A above, and another which requires no axioms, other than the Majorana axioms M1 - M7.

Whilst this more general version of the algorithm is less efficient, it is necessary to have such a version if we wish to use our algorithm(s) to prove full classification results in Majorana theory. In particular, we use the more general version of the algorithm to prove the main theorem in Chapter 5. We give full details of this version of the algorithm in Section 5.1.

For the remainder of this chapter, we assume that the axioms 2Aa - 5A hold and consider the algorithm in which these hold to be the default method.

Note that if we assume axiom 2Aa then to additionally assume axiom M8 leads only to restrictions on the possible shapes of the Majorana representation, not in the construction itself. As such, in the main version of the algorithm, the user can choose to consider only shapes which obey axiom M8, by using the function `ShapesOfMajoranaRepresentationAxiomM8` in place of the function `ShapesOfMajoranaRepresentation` in Step 1 of Section 4.3 below.

Finally, we discuss the role of axiom M2, known as Norton's inequality, in this work. This axiom is used neither in Seress' algorithm nor in any other published papers which construct Majorana algebras and it is an open problem as to whether it is a consequence of the other Majorana axioms. Like Seress, we do not use axiom M2 in our algorithm.

Using the result below, we are able to check if an algebra obeys axiom M2. However, it is computationally very expensive to do so and so we do not routinely perform this check.

Lemma 4.2.2 ([IS12a, Lemma 7.8]). *Let V be an n-dimensional algebra with commutative algebra product $\cdot$ and scalar product $(\ ,\ )$. Let $\{v_i : 1 \leq i \leq n\}$ be a basis of V, and define a $(n^2 \times n^2)$-dimensional matrix $B = (b_{ij,kl})$ in the following way. The rows and columns are indexed by the ordered pairs (i, j) for $1 \leq i, j \leq n$ and*

$$b_{ij,kl} = (v_i \cdot v_k, v_j \cdot v_l) - (v_j \cdot v_k, v_i \cdot v_l).$$

If B is positive semidefinite then V satisfies Norton's inequality M2.

Proof. For $u, v \in V$, write u and v as linear combinations

$$u = \sum_{i=1}^{n} \lambda_i v_i \text{ and } v = \sum_{j=1}^{n} \mu_j v_j$$

and form the n^2-long vector z with entries $\lambda_i \mu_j$. In this vector, the coordinate $\lambda_i \mu_j$ is in the position indexed by (i, j) in the matrix B. Then the inequality $(u \cdot u, v \cdot v) - (u \cdot v, u \cdot v) \geq 0$ is equivalent to $zBz^T \geq 0$. Hence, if B is positive semidefinite then M2 must hold in V. □

4.3 Notes on the implementation

The construction of these algebras is expensive both in terms of time and memory and a large part of the implementation of the algorithm involves mitigating these factors. In particular, we exploit the fact that the algebra and inner products on the algebra are preserved by the action of the group on the algebra. That is to say, for all $u, v \in V$ and $g \in G$, $(u^g, v^g) = (u, v)$ and $u^g \cdot v^g = (u \cdot v)^g$.

Thus, given a spanning set C of V, we store the products corresponding to a proper subset of all possible pairs of elements of C. From this subset we can use the action of the G to recover the full product values. Similarly, we store eigenvectors only for representatives of the orbits of G on the axes of V.

In the following, we take C to be the union of the Majorana axes and the $3A$, $4A$ and $5A$ axes of V. Note that the value of C can be immediately determined from the shape of the representation. Moreover, if V is 2-closed then C will be a spanning set of V.

Recall from Proposition 2.2.3 that we have the following equalities on $3A$, $4A$ and $5A$ axes:

$$u_h = u_{h^2},\ v_h = v_{h^3},\ \text{and } w_h = -w_{h^2} = -w_{h^3} = w_{h^4}.$$

In the implementation of the algorithm, we exploit these equalities, as well as those from the axioms above, whilst keeping track of any sign changes from the $5A$ axes.

Note that the spanning set C is invariant under the action of G, up to possible changes of sign. In a departure from Seress' methods, if the set C is of size n, we express the elements of G as signed permutations on the n points. This is more efficient both in terms of time and memory and also makes it easier to implement an n-closed version of the algorithm (see Section 4.5). In particular, we express an element g of G as

$$[\pm i_1, \pm i_2, \dots, \pm i_n]$$

where $C[j]^g = \pm C[i_j]$.

For each algebra, we store the following data structures which enable the calculation of the signed permutation corresponding to a given element. As we assume the axioms 2Aa - 5A, the elements of C can be indexed by elements of the group and we store C in terms of these elements, rather then vectors themselves.

- `coordinates`: This is the set C, a sorted, duplicate-free list consisting of all elements of T, as well as, or including, exactly one generator of each cyclic group of the form $\langle t_0 t_1 \rangle$ for $t_0, t_1 \in T$ such that the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1) \rangle\rangle$ is of type $2A$, $3A$, $4A$ or $5A$.
- `longcoordinates`: This is a set consisting of all elements of T as well as, or including, *all* generators of each cyclic group $\langle t_0 t_1 \rangle$ for $t_0, t_1 \in T$ such that the dihedral algebra $\langle\langle \psi(t_0), \psi(t_1) \rangle\rangle$ is of type $2A$, $3A$, $4A$ or $5A$.
- `positionlist`: This is a list whose order is equal to the cardinality of `longcoordinates`. The absolute value of `positionlist[i]` is the index of the element of `coordinates` which corresponds to `longcoordinates[i]`. The entry `positionlist[i]` is negative if and only if `longcoordinates[i]` is of order 5 and is equal to h^2 or h^3, where h is the corresponding element of `coordinates`.

We note that in the case of the n-closed algorithm, we store further elements in the lists `coordinates` and `longcoordinates` as described in detail in Section 4.5.

We now describe how we use these signed permutations to recover generic product values. The set `longcoordinates` is invariant (up to signs) under the action of G and so this action partitions the set `longcoordinates` $\times$ `longcoordinates`. We consider the classes formed by the closure of this partition with respect to the following conditions, where P is a class and $u, v, w, z \in V$:

- if $(u, v) \in P$ then $(v, u) \in P$;
- if $(u, v) \in P$, $\langle u \rangle = \langle w \rangle$ and $\langle v \rangle = \langle z \rangle$ then $(v, w) \in P$.

We label these equivalence classes $P_1, \ldots, P_k$. We do not need to explicitly store these classes. Instead, we have implemented a bespoke orbits algorithm which outputs the following structures.

- `pairrepresentatives:` From each equivalence class P_i, we pick a representative p_i. The entry `pairrepresentatives[i]` is a list of length two consisting of the positions in `coordinates` of the elements of p_i.
- `pairorbitlist:` This is a matrix of size $|C| \times |C|$ whose entries lie in $\{1, 2, \ldots, k\}$. If [`coordinates[i]`, `coordinates[j]`] $\in P_k$ then `pairorbitlist[i][j]` is equal to k.
- `pairconjelements:` This is a list of length $|G|$ consisting of the signed permutations corresponding to each element of G.
- `pairconj:` This is a matrix of size $|C| \times |C|$ whose entries lie in $\{1, 2, \ldots, |G|\}$. If [`coordinates[i]`, `coordinates[j]`] $\in P_k$ then the value of `pairconj[i][j]` gives the index in `pairconjelements` of a signed permutation which sends the representative p_k of P_k to [`coordinates[i]`, `coordinates[j]`].

In order to recover the product of the two basis vectors corresponding to the elements $C[i]$ and $C[j]$, we execute the following steps:

1. Let `k := pairorbit[i][j]` and let `l := pairconj[i][j]`.
2. Then `u := algebraproducts[k]` will be a row vector and `g := pairconj[l]` will be a signed permutation representing a group element which sends the representative of the orbital P_k to `[coordinates[i], coordinates[j]]`.
3. The desired product will be equal to the row vector `v` where

$$\texttt{v[i]} := \begin{cases} \texttt{u[g[i]]} & \text{if } \texttt{g[i]} > 0 \\ \texttt{-u[-g[i]]} & \text{if } \texttt{g[i]} < 0. \end{cases}$$

Finally, we store eigenvectors only for representatives of the orbits of G on T. Again, instead of storing the full orbits, we use a bespoke orbits algorithm which outputs the following two structures.

- `orbitrepresentatives:` This is a list whose entries give the indices in `coordinates` of representatives of orbits of G on T.

- `conjelements:` This is a duplicate free list of signed permutations corresponding to the group elements which send an element of `coordinates` to one of the representatives in `orbitrepresentatives`.

Finally, we note that the methods used in this algorithm require the solving of potentially large systems of linear equations over the rational numbers. Storing the matrices involved and reducing them to row echelon form takes a large amount of the memory and time required by the program. In what we believe to be an improvement on Seress' methods, we use the sparse matrix format provided by the GAP package `Gauss` [GBG] as an efficient way to store and compute with the matrices in question.

4.4 The algorithm

Input: A finite group G and a G-invariant set of involutions such that $G = \langle T \rangle$. The user must also choose which one of the possible shapes found by the function `ShapesOfMajoranaRepresentation` is to be considered by the algorithm.

Output: The algorithm returns a record with the following components.

- `group` and `involutions:` The group G and generating set of involutions T, as inputed by the user.

- `shape:` The shape of the representation, as chosen by the user.

- `setup:` A record whose components are the 9 structures given in Section 4.3.

- `algebraproducts:` A list of row vectors (in sparse matrix format) where `algebraproducts[i]` gives the algebra product of the two basis vectors whose indices are given by `setup.pairreps[i]`.

- `innerproducts:` A list where `innerproducts[i]` gives the inner product of the two basis vectors whose indices are given by `setup.pairreps[i]`.

- `evecs:` If i is in `setup.orbitrepresentatives` then for $j = 1, 2, 3$, `evecs[i][j]` gives respectively a basis (in sparse matrix format) of the 0-, $\frac{1}{2^2}$- or $\frac{1}{2^5}$-eigenspace of the ith axis.

- `nullspace:` A matrix (in sparse matrix format) which forms a basis of the nullspace of the algebra with respect to the spanning set C (as defined below).

Suppose that V is an algebra with a spanning set C. Then the space $\langle C \rangle$ will also be an algebra but, as C is not necessarily a basis of V, there might be some linear combinations of vectors of C which are equal to zero. As the bilinear form $(\, , \,)$ is positive definite, we can use this form to determine which vectors of $\langle C \rangle$ are equal to 0 in V. In particular, we define the *nullspace* of V with respect to C to be

$$N(C) := \{v \in \langle C \rangle \mid (v, v) = 0\}.$$

Then, $V = \langle C \rangle / N(C)$ and the ideal $N(C)$ is equal to the row space of the nullspace of the Gram matrix of $(\, , \,)$ on C.

Step 1 - Shapes

The first step is to find all possible shapes of a Majorana representation of the form (G, T, V). That is to say, we find representatives of the orbitals of G on $T \times T$ and determine the possibilities for the types of the dihedral algebras generated by the Majorana axes corresponding to each of these representatives. Note that the possible shapes must respect the inclusions of dihedral algebras, as described in Lemma 2.2.5.

The function `ShapesOfMajoranaRepresentation` takes as its input a group G and a set of involutions T and returns a record, of which one of the components is labelled `shapes` and gives a list of possible shapes for a representation of the form (G, T, V). The user may then choose which of these possible shapes they want to use for the constructive part of the algorithm.

This output is then used as the first input variable in the function `MajoranaRepresentation`. The second variable is an integer i which signifies that the user has chosen the shape at the position i.

Step 2 - Set Up

The first step is to build the nine objects which form the record `setup`, as described in Section 4.3. We then record all product values and eigenvectors which are given from the known values on dihedral algebras, i.e. those in Tables 2.1 and 2.2.

Step 3 - Inner Products

The first step in the main part of the algorithm is to find inner product values on the spanning set C of V given by `setup.coordinates`. Let u, v and w be elements of C. Then, from axiom M1

$$(u, v \cdot w) = (u \cdot v, w).$$

We consider all cases where the algebra products $v \cdot w$ and $u \cdot v$ are known, but at least one of the inner products above are not. Using equations of this form, we build systems of linear equations of the unknown products (u, v) where u and v are elements of C. We then solve this system and record any new inner products.

Example 4.4.1. *Suppose that $t_0, t_1, t_2 \in T$ such that the algebras $\langle\langle \psi(t_0), \psi(t_1) \rangle\rangle$ and $\langle\langle \psi(t_1), \psi(t_2) \rangle\rangle$*

are of types $3A$ *and* $2B$ *respectively. Then*

$$(\psi(t_0), \psi(t_1) \cdot \psi(t_2)) = 0$$

and so, by axiom M1,

$$(\psi(t_0) \cdot \psi(t_1), \psi(t_2)) = \frac{1}{2^5}(2\psi(t_0) + 2\psi(t_1) + \psi(t_1^{t_0}) - \frac{3^3 \cdot 5}{2^{11}} u_{t_0 t_1}, \psi(t_2)) = 0.$$

As the inner product value of any two Majorana axes is known, we can use this expression in order to determine the value of $(u_{t_0t_1}, \psi(t_2))$*.*

If at this stage, all inner product values are known then we construct the Gram matrix of the inner product and store basis vectors of its nullspace as the component `nullspace`. In particular, as the inner product is positive definite, these vectors will form a basis of the nullspace of the algebra.

We note that Seress instead uses the orthogonality of eigenvectors (Lemma 2.3.4) to determine new inner products. We have tested both methods and have found that our approach tends to find more products and is more efficient. However, in either case, finding inner products tends to take only a small amount of the total running time.

Step 4 - Fusion of Eigenvectors

For each of the orbit representatives of G on T given by `setup.orbitrepresentatives` we take the corresponding Majorana axis a and consider all pairs of known eigenvectors $\alpha \in V_\mu^{(a)}$ and $\beta \in V_\nu^{(a)}$. If the product $\alpha \cdot \beta$ is known then we can use this along with the fusion rules to find further eigenvectors.

If $\mu \neq \nu$, or if $\mu = \nu = 0$ then $\alpha \cdot \beta$ is itself an eigenvector, and is added to the relevant eigenspace. If $\mu = \nu = \frac{1}{2^2}$ then, using the fusion rules and Lemma 2.3.5.

$$\alpha \cdot \beta = v_0 + (a, \alpha \cdot \beta)a = v_0 + \frac{1}{2^2}(\alpha, \beta)a$$

where $v_0 \in V_0^{(a)}$. Thus, if the value of (α, β) is known, then we can recover the eigenvector v_0 and add it to the 0-eigenspace of a.

Similarly, if $\mu = \nu = \frac{1}{2^5}$ then

$$\alpha \cdot \beta = v_0 + v_{\frac{1}{2^2}} + \frac{1}{2^5}(\alpha, \beta)a$$

where $v_0 \in V_0^{(a)}$ and $v_{\frac{1}{2^2}} \in V_{\frac{1}{2^2}}^{(a)}$. If the value of (α, β) is known, then we further calculate

$$a \cdot (\alpha \cdot \beta) = \frac{1}{2^2} v_{\frac{1}{2^2}} + \frac{1}{2^5}(\alpha, \beta)a$$

and so we can recover both v_0 and $v_{\frac{1}{2^2}}$ and add them to their respective eigenspaces.

Step 5 - Algebra Products

We seek products of the form $u \cdot v$ for $u, v \in C$. We write a system of linear equations of the unknown products from the following sources:

- if $v \in V_\mu^{(a)}$ for some $a \in A$ and $\mu \in \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$ then $a \cdot v = \mu v$;
- if $u \in C$ and $v \in V$ such that $(v, v) = 0$ then $u \cdot v = 0$.

If there remains unknown algebra products then we again construct a system of linear equations, this time making use of the resurrection principle.

Proposition 4.4.2 (The Generalised Resurrection Principle)**.** *Fix $a \in A$ and let $\alpha, \gamma \in V_0^{(a)}$ and $\beta \in V_\mu^{(a)}$ for $\mu \in \{\frac{1}{2^2}, \frac{1}{2^5}\}$. Then*

$$a \cdot ((\alpha - \beta) \cdot \gamma) = -\mu(\gamma \cdot \beta).$$

Similarly, if $\alpha, \gamma \in V_{\frac{1}{2^2}}^{(a)}$ and $\beta \in V_\mu^{(a)}$ for $\mu \in \{0, \frac{1}{2^5}\}$. Then

$$a \cdot ((\alpha - \beta) \cdot \gamma) = \frac{1}{2^2}(\alpha, \gamma)a - \nu(\gamma \cdot \beta)$$

where $\nu = \frac{1}{2^2}$ if $\mu = 0$ and $\nu = \frac{1}{2^5}$ if $\mu = \frac{1}{2^5}$.

Proof. Firstly,

$$a \cdot ((\alpha - \beta) \cdot \gamma) = a \cdot (\alpha \cdot \gamma) - a \cdot (\beta \cdot \gamma).$$

The result then follows from the fusion rules. □

In particular, where possible, we choose α, β and γ such that the product $(\alpha - \beta) \cdot \gamma$ is known, but $\beta \cdot \gamma$ is not known. In this way, we obtain a linear combination of terms $a \cdot x$ for the $x \in C$ occuring in $(\alpha - \beta) \cdot \gamma$, and $y \cdot z$ for $y \in \beta$ and $z \in \gamma$. Using these, we construct and solve a system of linear equations.

Example 4.4.3. *Suppose again that $t_0, t_1, t_2 \in T$ such that the algebras $\langle\langle \psi(t_0), \psi(t_1) \rangle\rangle$ and $\langle\langle \psi(t_1), \psi(t_2) \rangle\rangle$ are of types $3A$ and $2B$ respectively. Then the following are eigenvectors of $\psi(t_1)$ given by the known values of the dihedral algebras of type $2B$ and $3A$:*

$$\begin{aligned}
\alpha &:= u_{t_0 t_1} - \frac{2 \cdot 5}{3^3}\psi(t_1) + \frac{2^5}{3^3}(\psi(t_0) + \psi(t_1^{t_0})) \in V_0^{(\psi(t_1))} \\
\beta &:= u_{t_0 t_1} - \frac{2^3}{3^2 \cdot 5}\psi(t_1) - \frac{2^5}{3^2 \cdot 5}(\psi(t_0) + \psi(t_1^{t_0})) \in V_{\frac{1}{2^2}}^{(\psi(t_1))} \\
\gamma &:= \psi(t_2) \in V_0^{(\psi(t_1))}.
\end{aligned}$$

We consider the equality

$$\psi(t_1) \cdot ((\alpha - \beta) \cdot \gamma) = -\frac{1}{2^2}\beta \cdot \gamma.$$

All product values required to calculate $\beta \cdot \gamma$ are known, with the possible exception of the product $u_{t_0 t_1} \cdot \psi(t_2)$. Similarly, the product $(\alpha - \beta) \cdot \gamma$ can be calculated using the known values of dihedral *algebras. If the product $\psi(t_1) \cdot ((\alpha - \beta) \cdot \gamma)$ is also known then we can immediately record the value $u_{t_0 t_1} \cdot \psi(t_2)$. Otherwise, this equality gives rise to a linear equation where the unknown values are $u_{t_0 t_1} \cdot \psi(t_2)$ as well as some products of the form $\psi(t_1) \cdot v$ for $v \in C$.*

In some cases, there still remains unknown algebra and inner product values and so we run steps 3 - 5 repeatedly until all values have been found. In some rare cases, it is not possible to find all products and the program exits with the output `fail`.

Step 6 - Check the algebra

Finally, we check that the algebra obtained in these calculations is indeed a Majorana algebra. We first check that all triples of basis vectors obey axiom M1. This then implies that axiom M1 is satisfied by all triples of vectors in V.

We also check that, for each $a \in A$, the eigenspaces of the adjoint action of a on V obey the fusion rules. From the following result, this is sufficient to show that the algebra V is indeed a Majorana algebra.

Lemma 4.4.4. *Suppose that V is a real vector space equipped with a commutative algebra product $\cdot$. Suppose also that V is generated by a set A of idempotents such that for some $a \in A$, a obeys the axiom M4 (Definition 2.1.1) as well as the fusion rules of Majorana algebras (Table 2.3). Then a also satisfies axioms M6 and M7. That is to say, the linear transformation $\tau(a)$ of V defined via*

$$\tau(a) : u \mapsto (-1)^{2^5 \mu} u$$

for $u \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}, \frac{1}{2^5}$ and the linear transformation $\sigma(a)$ of $V_+^{(a)} := V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$ defined via

$$\sigma(a) : v \mapsto (-1)^{2^2 \mu} v$$

for $v \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}$ preserve the algebra product. That is to say, for all $u_1, u_2 \in V$ and for all $v_1, v_2 \in V_+^{(a)}$

$$u_1^{\tau(a)} \cdot u_2^{\tau(a)} = (u_1 \cdot u_2)^{\tau(a)} \text{ and } v_1^{\sigma(a)} \cdot v_2^{\sigma(a)} = (v_1 \cdot v_2)^{\sigma(a)}.$$

Proof. Let $u, v \in V$, let $a \in A$, and suppose that V obeys axiom M4 and the fusion rules. Then u and v possess unique presentations of the form

$$u = x_1 + \gamma_1$$
$$v = x_2 + \gamma_2$$

where $x_1, x_2 \in V_0^{(a)} \oplus V_1^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$ and $\gamma_1, \gamma_2 \in V_{\frac{1}{2^5}}^{(a)}$. Then

$$\begin{aligned} u \cdot v &= (x_1 + \gamma_1) \cdot (x_2 + \gamma_2) \\ &= x_1 \cdot x_2 + \gamma_1 \cdot x_2 + x_1 \cdot \gamma + \gamma_1 \cdot \gamma_2. \end{aligned}$$

By the fusion rules,

$$u \cdot v + \gamma_1 \cdot \gamma_2 \in V_0^{(a)} \oplus V_1^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)} \text{ and } \gamma_1 \cdot x_2 + x_1 \cdot \gamma \in V_{\frac{1}{2^5}}^{(a)}.$$

As τ acts as the identity on $V_0^{(a)} \oplus V_1^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$ and as multiplication by the scalar -1 on $V_{\frac{1}{2^5}}^{(a)}$. So

$$\begin{aligned}(u \cdot v)^{\tau(a)} &= x_1 \cdot x_2 - \gamma_1 \cdot x_2 - x_1 \cdot \gamma + \gamma_1 \cdot \gamma_2 \\ &= (x_1 - \gamma_1) \cdot (x_2 - \gamma_2) = u^{\tau(a)} \cdot v^{\tau(a)}\end{aligned}$$

as required. A similar argument applies for the transformation $\sigma(a)$. □

4.5 The n-closed algorithm

In a significant improvement on Seress' work, we have been able to implement a version of the algorithm which allows the construction of n-closed representations, theoretically for any given n. Of course, technology limits the value of n in practice. We have implemented this part of the algorithm so that the first step is to attempt construction of the 2-closed part of the algebra as, in the vast majority of cases, this is sufficient.

If the algorithm is unable to determine all algebra products of the 2-closed algorithm then the user may pass the incomplete algebra outputted by the function `MajoranaRepresentation` to the function `NClosedMajoranaRepresentation` in order to attempt construction of the 3-closed algorithm. In order to attempt construction of the n-closed part of the algebra, the user must pass the incomplete algebra to the function `NClosedMajoranaRepresentation` $n-2$ times for $n > 2$.

We describe the implementation of the function `NClosedMajoranaRepresentation`. We note that this method crucially relies on our use of signed permutations to encode the action of the group on the algebra, as described in Section 4.3.

The main steps of the n-closed algorithm are the same as those for the 2-closed algorithm. The function `NClosedMajoranaRepresentation` extends the spanning set of the algebra and adjusts the record encoding the algebra as described below. We then perform Steps 2 - 5 of the main algorithm until no more algebra products can be found. We describe this process in detail below.

Input: A record which has been outputted by the function `MajoranaRepresentation` where at least one of the entries in the component `algebraproducts` has the value `false`, indicating a product which has not yet been determined.

Output: The function has no output. We record additional values in the components of the input record.

We first record a list of indices `k` such that `algebraproducts[k]` has the value `false`. For each entry k of this list, if the value of algebraproducts[k] is still false, then we perform the following steps.

Step 1 - Extend the spanning set of the algebra

For all $i, j \in \{1, \ldots, |C|\}$, if the absolute value of pairorbitlist[i][j] is equal to k, then we add the ordered pair [i,j] to the list coordinates. This corresponds to adding the vector $C[i] \cdot C[j]$ to the spanning set of the algorithm.

Step 2 - Extend the known values

For all known products in algebraproducts and all vectors in evecs and nullspace, extend the length of the vectors to the new cardinality of coordinates by adding zeroes to the end. Set the value of the entry algebraproducts[k] to be the vector with a one at position i and zeroes elsewhere, where i is such that coordinates[i] is equal to pairrepresentatives[k].

Step 3 - Extend the signed permutations

Let p be an element of pairconjelements or conjelements corresponding to an element $g \in G$. Let l be such that coordinates[l] is equal to [i,j] where [i,j] is one of the additional elements recorded in Step 1.

Let x be the ordered list consisting of the abolute values of p[i] and p[j]. Then

```
p[l] = ± Position(coordinates, x)
```

where p[l] > 0 if and only if p[i]*p[j] > 0.

The generality of the method of signed permutations means that this method works regardless of whether the vectors corresponding to coordinates[i] and coordinates[j] are in the 2-closed part of the algebra and are thus stored as group elements, or whether they are in the n-closed part for $n > 2$ and are stored as ordered pairs of integers.

Step 4 - New orbitals

We can now use the orbits function developed for the main algorithm along with the signed permutations found in the previous step to find any additional orbitals of G on coordinates $\times$ coordinates. We add any new entries to pairrepresentatives and add new values to the matrices pairorbitlist and pairconj.

Step 5 - New eigenvectors

Finally, if $t \in T$ and $u, v \in V$ then, from Lemma 2.3.3, the vector $u \cdot v - (u \cdot v)^t$ is a $\frac{1}{2^5}$-eigenvector of the $2A$ axis $\psi(t)$. We use the signed permutations found in Step 3 to find and record any vectors of this form.

Step 6 - The main steps

The extended algebra can now be passed through Steps 3 - 5 of Section 4.4. If at any point all products have been found then the algebra is complete and the function exits. If no more products can be found, but there are still missing values then we repeat the above Steps 1 - 6 with the next index `k` from our original list of unknown algebra product values.

If we have performed these steps for all values in this list and there still remains unknown products then the function exits and the algebra is still incomplete. The user may then decide to again run the function `NClosedMajoranaRepresentation` on the incomplete representation in order to attempt the construction of the n-closed algebra for the next value of n.

4.6 Results

We now present the basic details for some of the representations which we have constructed using our program. For each representation (G, T, V), we give the following information

- the isomorphism type of G;
- the cardinality of T, where $c_1 + c_2 + \ldots + c_k$ indicates that T is the union of k conjugacy classes of size $c_1, c_2, \ldots, c_k$;
- a subset of the shape of V, showing only the values of Ψ for the orbitals where a choice has been made on the type of the corresponding dihedral algebra;
- the cardinality of the spanning set C of the 2-closed part of the algebra, which consists of the $2A$, $3A$, $4A$ and $5A$ axes;
- the dimension of the algebra V as a vector space over $\mathbb{R}$;
- whether the algebra is 2-closed or not, all those which are not 2-closed are 3-closed.

We note that, for a given group G and set of involutions for a T, and for a given shape, if the algorithm returns a completed algebra then this must be the unique representation of the form (G, T, V) with this shape. In particular, our work reproves the constructive results found in [IPSS10], [IS12a], [IS12b], [Iva11b], [Iva11a] and [Dec14]. Additionally, many of the above algebras had not been constructed before Seress' work [Ser12] and the algebras in the above list which are not 3-closed are completely new examples.

i	G	$\|T\|$	Shape	$\|C\|$	dim.	2-closed
1	S_4	$6+3$	$(2B,3C)$	12	12	Y
2	S_4	$6+3$	$(2A,3C)$	9	9	Y
3	S_4	$6+3$	$(2B,3A)$	10	25	N
4	S_4	$6+3$	$(2A,3A)$	13	13	Y
5	S_4	6	$(2B,3C)$	6	6	Y
6	S_4	6	$(2A,3C)$	9	9	Y
7	S_4	6	$(2B,3A)$	28	13	N
8	A_5	15	$(2B,3C)$	21	21	Y
9	A_5	15	$(2A,3C)$	21	20	Y
10	A_5	15	$(2B,3A)$	31	46	N
11	A_5	15	$(2A,3A)$	31	26	Y
12	S_5	$15+10$	$(2B,2A)$	41	36	Y
13	S_5	$15+10$	$(2A,2A)$	41	36	Y
14	$L_3(2)$	21	$(2A,3C)$	21	21	Y
15	$L_3(2)$	21	$(2A,3A)$	49	49	Y
16	A_6	45	$(2A,3C)$	81	70	Y
17	A_6	45	$(2A,3A)$	121	76	Y
18	S_6	$45+15$	$(2A,2B,3A)$	136	91	Y
19	$3.A_6$	45	$(2A,3C,3C)$	81	70	Y
20	$3.A_6$	45	$(2A,3A,3C)$	141	105	Y
21	$3.A_6$	45	$(2A,3A,3C)$	201	76	Y
22	$3.S_6$	$45+45$	$(2A,3A,3C)$	187	136	Y
23	$(S_4\times S_3)\cap A_7$	$18+3$	$(2A,3A)$	34	30	Y
24	$(S_4\times S_3)\cap A_7$	18	$(2A,3A)$	34	30	Y
25	A_7	105	$(2A,3A)$	406	196	Y
26	$3.A_7$	105	$(2A,3A,3C)$	336	211	Y
27	$3.A_7$	105	$(2A,3A,3A)$	756	196	Y
28	$3.S_7$	$105+63$	$(2A,3A,3C)$	400	254	Y
29	$L_2(11)$	55	$(2A,3A)$	176	101	Y
30	$L_3(3)$	117	$(2A,3C,3A)$	169	144	Y
31	M_{11}	165	$(2A,3A)$	781	286	Y
32	$(S_5\times S_3)\cap A_8$	$30+15$	$(3A)$	62	67	N

Table 4.1: Certain constructed Majorana representations

The groups which have been considered above are all know to exist as subgroups of the Monster. However, it is of course possible to use this algorithm to construct Majorana representations of groups which are not known to embed into the Monster.

Chapter 5

Constructing Majorana representations of triangle-point groups

In this section, we contribute to the classification of Majorana representations of triangle-point groups. We will prove the following result.

Theorem 5.0.1. *Suppose that V is a Majorana algebra which obeys axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$. Then V is must be isomorphic to one of the 34 Majorana algebras whose dimensions are given in Table 5.1. In particular, V must be isomorphic to a subalgebra of the Griess algebra.*

Recall that axiom M8 is defined as below. For a full discussion about its use, see Chapter 3. For the remainder of this chapter, we assume that axiom M8 holds.

M8 Suppose that V is a Majorana algebra and suppose that $a_0, a_1 \in V$ are Majorana axes such that the dihedral algebra $U := \langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$. Then the basis vector

$$a_\rho := a_0 + a_1 - 8a_0 \cdot a_1$$

is a Majorana axis of V and $\tau(a_\rho) = \tau(a_0)\tau(a_1)$. Conversely, we require that the map $\tau : A \to \mathrm{Aut}(V)$ is injective and that if $a_0, a_1, a_2 \in V$ are Majorana axes such that $\tau(a_0)\tau(a_1) = \tau(a_2)$ then the algebra $\langle\langle a_0, a_1 \rangle\rangle$ is of type $2A$ and $a_\rho = a_2$.

In Theorem 3.2.1, we show that an algebra as in the hypothesis of Theorem 5.0.1 must occur as a Majorana representation of the form (G, T, V) where $G = \langle a, b, c \rangle$ is isomorphic to one of the 26 triangle-point subgroups of the Monster and T is such that $a, b, c, ab \in T$.

The orders and isomorphism types of these 26 groups are given in Table 5.1. This table also gives the cardinality of the set T and the dimension of the algebra V for all possible Majorana representations (G, T, V) such that $a, b, c, ab \in T$.

In order to prove the main theorem, we take each of these triangle-point groups in turn. We classify all possible values for the set T such that $a, b, c, ab \in T$ and such that (G, T, V) is a Majorana representation of G.

Finally, for each choice of G and T, we use a slightly generalised version of the algorithm described in Chapter 4 to classify and construct all possible Majorana representations of the form (G, T, V). This is equivalent to constructing all Majorana algebras as in the hypothesis of the main theorem above. We then check that each of these Majorana representation is based on an embedding on G into the Monster.

No.	G	$\lvert G\rvert$	$\lvert T\rvert$	dim V	No.	G	$\lvert G\rvert$	$\lvert T\rvert$	dim V
1	2^2	4	3	3	14	$2^2 \times S_4$	96	30	36
2	2^3	8	4 6	4 6	15	$3^{1+2}_{+} : 2^2$	108	27	32
3	D_8	8	5	5	16	S_5	120	25	36
4	D_{12}	12	7	8	17	$(S_3 \times S_3) : 2^2$	144	34	45
5	$2 \times D_8$	16	8 8 10	8 10 11	18	$2^4 : D_{10}$	160	30	46
6	$2^2 \times S_3$	24	12	13	19	$2^4 : D_{12}$	192	28	54
7	S_4	24	9 9	9 13	20	$2 \times S_5$	240	36	61
8	$2^4.2$	32	10 14	18 15	21	$2^5 : D_{12}$	384	60	76
9	$S_3 \times S_3$	36	15	18	22	$2^4 : (S_3 \times S_3)$	576	66	93
10	$2 \times S_4$	48	16 16	20 23	23	$L_2(11)$	660	55	101
11	A_5	60	15 15	20 26	24	$2^4.A_5$	960	70	125
12	$2^3.2^3$	64	18 22	28 24	25	$2 \times S_6$	1440	106	151
13	$(S_3 \times S_3) : 2$	72	21	25	26	$2^5 : S_5$	3840	156	231

Table 5.1: The triangle-point subgroups of the Monster and their Majorana representations

5.1 A generalised version of the constructive algorithm

Before we commence with the main body of this chapter, we describe the implementation of the modified algorithm which we have used in the proof of our main theorem above.

Recall that in the main implementation of the algorithm we assume five axioms 2Aa, 2Ab, 3A, 4A and 5A in addition to the Majorana axioms M1 - M7. In order to prove Theorem 5.4.1, we have implemented and used a version of the algorithm which assumes none of these further axioms, but does assume axiom M8.

For completeness, we have also implemented a version of the algorithm which assumes no axioms other than the main Majorana axioms M1 - M7. However, we have not had need to use this third version in this work.

It is important to note that, although we are no longer assuming that two axes indexed by the same group element are equal, certain equalities between axes still hold as a consequence of Theorem 2.2.1.

Recall from Proposition 2.2.3 that if (G, T, V) is a Majorana representation and if $t_0, t_1 \in T$ such that the dihedral algebra $\langle\langle\psi(t_0), \psi(t_1)\rangle\rangle$ is of type $3A$, $4A$ or $5A$ then

$$u_{\rho(t_0,t_1)} = u_{\rho(t_0,t_0t_1t_0)},\ v_{\rho(t_0,t_1)} = v_{\rho(t_0,t_0t_1t_0)}$$

or

$$w_{\rho(t_0,t_1)} = -w_{\rho(t_0,t_1t_0t_1)} = -w_{\rho(t_0,t_0t_1t_0t_1t_0)} = w_{\rho(t_0,t_0t_1t_0)}.$$

respectively.

Furthermore, the structure of $6A$ dihedral algebras gives further equalities on the $3A$-axes.

Proposition 5.1.1. *Suppose that (G, T, V) is a Majorana representation. If $t_0, t_1 \in T$ such that the dihedral algebra $\langle\langle\psi(t_0), \psi(t_1)\rangle\rangle$ is of type $6A$ then*

$$u_{\rho(t_0,t_1t_0t_1)} = u_{\rho(t_1,t_0t_1t_0)}.$$

Proof. From the structure of the dihedral algebras given in Table 2.1, we see that the algebra $\langle\langle\psi(t_0), \psi(t_1)\rangle\rangle$ contains two distinct dihedral algebras of type $3A$. By considering the action of the group $\langle t_0, t_1\rangle$ on the algebra, we see that the $3A$ axes contained in these two algebras must be equal. $\square$

In particular, if the intersection of two distinct $6A$ dihedral algebras contains a $3A$ dihedral algebra then the $3A$ axes from both algebras must be equal. In this version of the algorithm, we exploit the equalities given by Proposition 2.2.3 and the intersection of $6A$ algebras whilst assuming no further equalities.

We now describe the implementation of this algorithm. We note that this version differs from the main algorithm only in the set up of the algebra. The main steps described in Section 4.4 are common to both.

Moreover, the data structures used in the main algorithm play a similar role in this algorithm, as described below.

- `coordinates`: This is a set consisting of all elements of T and exactly one ordered pair `[i,j]` for each $3A$, $4A$ and $5A$ axis such that the dihedral algebra generated by the axes corresponding to `T[i]` and `T[j]` contains the $3A$, $4A$ or $5A$ axis in question.
- `longcoordinates`: This is a set consisting of all elements of T and all ordered pairs `[i,j]` such that the dihedral algebra generated by the axes corresponding to `T[i]` and `T[j]` is of type $3A$, $4A$ or $5A$.
- `positionlist`: This is a list whose order is equal to the cardinality of `longcoordinates`. The absolute value of `positionlist[i]` is the index of the element of `coordinates` which represents the $3A$, $4A$ or $5A$ axis contained in the dihedral algebra corresponding to the involutions whose indices are given by `longcoordinates[i]`. The entry `positionlist[i]` is negative if and only if the involutions in `longcoordinates[i]` correspond to a $5A$ axis which is equal to the negative of its representative axis.

If we want to implement the algorithm in full generality (i.e. without the assumption of axiom M8) then we also record indices corresponding to the $2A$ axes of the algebra in `coordinates` and `longcoordinates`.

In order to find the signed permutations which encode the group's action on the set `coordinates`, we first use the group structure to find the signed permutations of the action on T using the method described in Section 4.3. We then use the method used in the n-closed version of the algorithm, as described in Step 3 of Section 4.5, to extend the signed permutation to the full spanning set of V.

We can then use the orbits function from the main algorithm to construct the components `pairrepresentatives`, `pairorbitlist`, `pairconjelements` and `pairconj`. These components play exactly the same role here as in the main version of the algorithm.

As in the case of the n-closed algorithm, once this setup has been achieved, all components are compatible with the main part of the algorithm, and we can run Steps 2 - 5 in Section 4.4 until all possible products have been found.

5.2 Constructions

In the following, if (G, T, V, φ, ψ) is a Majorana representation then we write

$$a_i := a_{t_i} := \psi(t_i)$$

for $t_i \in T$.

The following result shows that we may be able to exploit automorphisms of the group G in order to restrict the number of possibilities for the set T which we must consider.

Lemma 5.2.1. *Suppose that G is a finite group and that (G, T_1, V_1) and (G, T_2, V_2) are two Majorana representations of G. Suppose further that there exists an automorphism α of G such that the image of T_1 under α is T_2. Then the possible shapes for the Majorana representation (G, T_1, V_1) will be the same as those for (G, T_2, V_2) .*

Proof. The possible shapes of a Majorana representation (G, T_1, V_1) are determined by the isomorphism types and inclusions of the dihedral groups of the form $\langle t, s\rangle$ for $s, t \in T_1$. These are preserved under the action of the automorphism α. □

Lemma 5.2.2. *Suppose that G is a finitely presented group with generators $X := \{x_1, \ldots, x_n\}$ and relations $R_X := \{R_1(x_1, \ldots, x_n) = 1, \ldots, R_k(x_1, \ldots x_n) = 1\}$. Suppose further that there exists a second set of generators $Y := \{y_1, \ldots, y_n\}$ of G such that for all $1 \leq i \leq k$, $R_i(y_1, \ldots, y_n) = 1$. Then there exists $\alpha \in \mathrm{Aut}(G)$ such that*

$$T_Y := \bigcup_{i=1}^{n} y_i^G = \bigcup_{i=1}^{n} \alpha(x_i^G).$$

Proof. As X and Y both generate G and satisfy the same relations, we may choose α to be the automorphism which maps $x_i \mapsto y_i$ for $1 \leq i \leq n$. Then α will also map x_i^G to y_i^G, and the result follows. □

Suppose that G is a triangle-point group and suppose that we know all possible presentations for G as a triangle-point group. Then these two results show that for each possible presentation of G it is sufficient to classify representations of the form (G, T, V) where $a, b, c, ab \in T$ for *any* set of generators $\{a, b, c\}$ which generate G and which satisfy the relations in question.

In most cases, this is the first time that these groups have been considered in the Majorana framework. We now briefly discuss the groups where this is not the case. Each of the dihedral groups have at most two possibilities for the set T, and their Majorana representations are classified in [IPSS10].

The group S_4 has exactly two conjugacy classes of involutions, $C_1 := (1,2)^{S_4}$ and $C_2 := (1,2)(3,4)^{S_4}$. As the conjugacy class C_2 does not generate the whole group, we must have $C_1 \subseteq T$. Moreover, if $a, b \in C_1$ such that $o(ab) = 2$ then we must have $ab \in C_2$ and so we must take $T = C_1 \cup C_2$. Consider a Majorana representation of the form (G, T, V). Then the algebra generated by $a_{(1,2)}$ and $a_{(1,3)}$ may be of type $3A$ or $3C$. Each of these possibilities gives rise to a Majorana representation, as shown in Section 4 of [IPSS10].

The group A_5 has exactly one conjugacy class of involutions so there is only one choice for the value of the set T. There are exactly two Majorana representations of A_5, as shown in [IS12a]. Finally, in [Dec14], Decelle shows that there is exactly one Majorana representation of $L_2(11)$.

We now consider each of the remaining groups in turn. Recall that, as the inner product (,) is assumed to be positive definite, the nullspace of an algebra V with respect to a spanning set C is defined to be

$$N(C) := \{v \in \langle C\rangle \mid (v, v) = 0\}.$$

In particular, the dimension of the algebra is given by the rank of the Gram matrix of $(\, , \,)$ on C.

In the following, all representations have been shown to be 2-closed unless otherwise stated.

5.2.1 The Group 2^3

Proposition 5.2.3. *Let $G = \langle a, b, c \rangle \cong 2^3$ be a triangle-point group. Then G admits exactly two Majorana representations of the form (G, T, V) where $a, b, c, ab \in T$.*

With the exception of 2^2, which is a dihedral group, 2^3 is the only abelian group to be considered and thus is dealt with slightly differently to the other groups. In particular, every conjugacy class of G is of size 1 and there are seven conjugacy classes of involutions (one for each non-identity element).

From Lemma 5.2.1, if $T_1, T_2 \subseteq G$ and there exists an automorphism of G sending T_1 to T_2, then the Majorana representations (G, T_1, V_1) and (G, T_2, V_2) have the same shape. Our first task is thus to determine the orbits of $\mathrm{Aut}(G)$ on candidates for the set T in a Majorana representation of G.

Lemma 5.2.4. *If $G \cong 2^3$ then the automorphism group $Aut(G)$ is transitive on the following sets for x, y and z pairwise distinct non-identity elements*

i) $X_1 := \{\{a, b, c, ab\} \mid G = \langle a, b, c \rangle\}$;

ii) $X_2 := \{\{a, b, c, ab, x\} \mid G = \langle a, b, c \rangle, \; x \in G, x \notin \{a, b, c, ab\}\}$;

iii) $X_3 := \{\{a, b, c, ab, x, y\} \mid G = \langle a, b, c \rangle, \; x, y \in G, x, y \notin \{a, b, c, ab\}\}$;

iv) $X_4 := \{\{a, b, c, ab, x, y, z\} \mid G = \langle a, b, c \rangle, \; x, y, z \in G, x, y, z \notin \{a, b, c, ab\}\}$.

Proof. Considering 2^3 as the vector space of dimension 3 over $GF(2)$, we see that $\mathrm{Aut}(G) \cong GL(3, 2)$. Moreover, if three elements $a, b, c \in G$ generate G then their corresponding vectors must be linearly independent. It follows from basic linear algebra that $GL(3, 2)$ is transitive on 3-tuples of linearly independent vectors and so $\mathrm{Aut}(G)$ is transitive on minimal generating sets of G.

Now let $\{a, b, c, ab\}, \{d, e, f, de\} \in X_1$. Then, from the preceding discussion, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = d$, $\alpha(b) = e$ and $\alpha(c) = f$. Then we also have $\alpha(ab) = \alpha(de)$ and α maps the first set to the second which implies that $\mathrm{Aut}(G)$ acts transitively on X_1.

Now, if $\{a, b, c, ab, x\}, \{d, e, f, de, y\} \in X_2$ then, by permuting the elements $\{a, b, ab\}$, we may assume without loss of generality that $x = ac$. Similarly, we may assume that $y = df$. Then the element $\alpha \in \mathrm{Aut}(G)$ which maps $\{a, b, c, ab\}$ to $\{d, e, f, de\}$ also sends x to y and we are done.

Similar arguments follow for the final two cases. $\square$

Proof of Proposition 5.2.3. The above lemma shows that when classifying the Majorana representations of G as a triangle-point group we need only classify the Majorana representations of the form (G, T_i, V) where T_i is a representative of X_i for $1 \leq i \leq 4$.

Propositions 5.2.6 and 5.2.8 below show that there are no representations of the form (G, T_2, V) or (G, T_4, V). We have checked in GAP that T_1 and T_3 each give exactly one representation, details of which are given in Propositions 5.2.5 and 5.2.7 below. □

We now label the elements of the T_i as below.

i	t_i	i	t_i
1	a	5	ac
2	b	6	bc
3	c	7	abc
4	ab		

Proposition 5.2.5. *There is exactly one Majorana representation of the form* (G, T_1, V) *where*

$$T_1 := \{t_i \mid 1 \leq i \leq 4\}.$$

The algebra V *contains no further axes and its nullspace is zero dimensional and so the dimension of* V *is* 4.

Note that this algebra is in fact the orthogonal annihilating sum of the $2A$ algebra $\langle\langle a_{(1,2)}, a_{(3,4)}\rangle\rangle$ and the one-dimensional algebra spanned by $a_{(5,6)}$.

Proposition 5.2.6. *There are no Majorana representations of the form* (G, T_2, V) *where*

$$T_2 := \{t_i \mid 1 \leq i \leq 5\}.$$

Proof. Suppose for contradiction that there exists a Majorana representation of the form (G, T_2, V). As $t_1 t_2 = t_4 \in T_2$ and $t_2 t_3 = t_6 \notin T_2$, the algebras $\langle\langle a_1, a_2\rangle\rangle$ and $\langle\langle a_2, a_3\rangle\rangle$ are of types $2A$ and $2B$ respectively and so from the known values of dihedral algebras

$$a_1 \cdot a_2 = \frac{1}{8}(a_1 + a_2 + a_4)$$
$$a_2 \cdot a_3 = 0.$$

Similarly, we calculate that

$$(a_1 \cdot a_2, a_3) = \frac{1}{16}$$
$$(a_1, a_2 \cdot a_3) = 0.$$

In particular, $(a_1 \cdot a_2, a_3) \neq (a_1, a_2 \cdot a_3)$ which contradicts axiom M1. Thus such a representation cannot exist. □

Proposition 5.2.7. *There is exactly one Majorana representation of the form* (G, T_3, V) *where*

$$T_3 := \{t_i \mid 1 \leq i \leq 6\}.$$

The algebra V *contains no further axes and its nullspace is zero dimensional and so the dimension of* V *is* 6.

Proposition 5.2.8. *There are no Majorana representations of the form* (G, T_4, V) *where*

$$T_4 := \{t_i \mid 1 \leq i \leq 7\}.$$

Proof. In this case, the set T_4 consists of all non-identity elements of G. We have already shown in Lemma 3.2.2 that there can be no Majorana representations of this form. □

5.2.2 The Group $2 \times D_8$

Proposition 5.2.9. *Let* $G = \langle a, b, c \rangle \cong 2 \times D_8$ *be a triangle-point group. Then* G *admits exactly three Majorana representations of the form* (G, T, V) *where* $a, b, c, ab \in T$.

Proof. From Tables 3.3 and 3.4, we can see that we must have $G \cong G^{(4,4,2)}$. If we let

$$a := (1,2),\ b := (1,2)(5,6),\ \text{and}\ c := (1,3)(2,4)(5,6)$$

then a, b, c generate G and satisfy the presentation of $G^{(4,4,2)}$. We label the conjugacy classes of involutions of G $C_1, \ldots C_7$, as below.

i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$
1	a^G	2	5	$(abc)^G$	2
2	b^G	2	6	$((ac)^2)^G$	1
3	c^G	2	7	$(a \cdot b^c)^G$	1
4	$(ab)^G$	1			

By assumption, T must contain $X := C_1 \cup \ldots \cup C_4$.

Furthermore, G contains a subgroup

$$K := \langle (1,2), (3,4), (5,6) \rangle \cong 2^3.$$

If $T \subseteq G$ is a set of involutions such that $|T \cap K| \in \{5, 7\}$ then section 5.2.1 shows that there are no representations of the form (G, T, V). As

$$|K \cap X| = 5 \text{ and } |K \cap C_6| = |K \cap C_7| = 1,$$

T must contain exactly one of C_6 or C_7. Thus there are four possibilities for the value of T, as shown below.

i	T_i	$\lvert T_i\rvert$	i	T_i	$\lvert T_i\rvert$
1	$X \cup C_6$	8	3	$X \cup C_5 \cup C_6$	10
2	$X \cup C_7$	8	4	$X \cup C_6 \cup C_7$	10

Proposition 5.2.12 below shows that there is no representation of the form (G, T_3, V). We have checked in GAP that T_1, T_2 and T_4 each give exactly one representation, details of which are given in Propositions 5.2.10, 5.2.11 and 5.2.13 below. □

In the following, we label the elements of the T_i as below.

i	t_i	i	t_i	i	t_i
1	a	5	c	9	$(abc)^a$
2	a^c	6	c^a	10	$(ac)^2$
3	b	7	ab	11	$a \cdot b^c$
4	b^c	8	abc		

Proposition 5.2.10. *There is exactly one Majorana representation of the form* (G, T_1, V) *where*

$$T_1 := X \cup C_6 = \{t_i \mid 1 \leq i \leq 7\} \cup \{t_{10}\}.$$

The algebra contains no further axes and its nullspace is zero dimensional and so the dimension of the algebra is 8.

Proposition 5.2.11. *There is exactly one Majorana representation of the form* (G, T_2, V) *where*

$$T_2 := X \cup C_7 = \{t_i \mid 1 \leq i \leq 7\} \cup \{t_{11}\}.$$

The algebra V *contains two* $4A$*-axes,* $v_{\rho(t_1,t_5)}$ *and* $v_{\rho(t_3,t_5)}$*. The nullspace of* V *is zero dimensional and so the dimension of* V *is 10.*

Proposition 5.2.12. *There are no Majorana representations of the form* (G, T_3, V) *where*

$$T_3 := X \cup C_5 \cup C_6 = \{t_i \mid 1 \leq i \leq 10\}.$$

Proof. Suppose for contradiction that there exists a Majorana representation of the form (G, T_3, V). As $t_1t_3 = (5,6) \in T_3$, $t_3t_5 = (1,4,2,3)$ and $(t_3t_5)^2 = (1,2)(3,4) \in T_3$, the algebras $\langle\langle a_1, a_3 \rangle\rangle$ and $\langle\langle a_3, a_5 \rangle\rangle$ are of type $2A$ and $4B$ respectively. Thus

$$\begin{aligned} a_1 \cdot a_3 &= \frac{1}{2^3}(a_1 + a_3 - a_7) \\ a_3 \cdot a_5 &= \frac{1}{2^6}(a_3 - a_4 + a_5 - a_6 + a_{10}). \end{aligned}$$

Similarly, we calculate that

$$\begin{aligned} (a_1 \cdot a_3, a_5) &= \frac{1}{2^8} \\ (a_1, a_3 \cdot a_5) &= -\frac{3}{2^8}. \end{aligned}$$

In particular, $(a_1 \cdot a_3, a_5) \neq (a_1, a_3 \cdot a_5)$ which contradicts axiom M1. Thus such a representation cannot exist. □

Proposition 5.2.13. *There is exactly one Majorana representation of the form* (G, T_4, V) *where*

$$T_4 := X \cup C_6 \cup C_7 = \{t_i \mid 1 \leq i \leq 9\} \cup \{t_{11}\}.$$

The algebra V *contains four* $4A$*-axes,* $v_{\rho(t_1,t_5)}$*,* $v_{\rho(t_1,t_8)}$*,* $v_{\rho(t_3,t_5)}$ *and* $v_{\rho(t_3,t_8)}$*. The nullspace of* V *is spanned by the vectors*

$$\begin{aligned} n_1 &:= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - a_7 + a_8 + a_9 - a_{10} - \frac{3}{2}\left(v_{\rho(t_1,t_5)} + v_{\rho(t_1,t_8)}\right) \\ n_2 &:= v_{\rho(t_1,t_5)} - v_{\rho(t_3,t_8)} \\ n_3 &:= v_{\rho(t_1,t_8)} - v_{\rho(t_3,t_5)} \end{aligned}$$

and so the dimension of V *is 11.*

5.2.3 The Group $2^2 \times S_3$

Proposition 5.2.14. *Let $G = \langle a, b, c \rangle \cong 2^2 \times S_3$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$*

Proof. We first show that we must have $G \cong G^{(2,6,6)}$. From Tables 3.3 and 3.4, we can see that G must occur as a quotient of G_i for $i \in \{6, 9, 11\}$. If G occurs as a quotient of G_6, then we must have

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^2, (bc)^6, (abc)^6, (a \cdot b^c)^4 \rangle.$$

However, if $(ac)^2 = 1$ then

$$o(a \cdot b^c) = o((ab)^c) = 2$$

and so we have $G = G^{(2,6,6)}$.

If G occurs as a quotient of G_9 then we must have either

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^2, (bc)^6, (abc)^6, (a \cdot b^c)^2, (ab \cdot b^c)^6, (ab \cdot a^c)^6 \rangle$$

or

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^2, (abc)^6, (a \cdot b^c)^2, (ab \cdot b^c)^6, (ab \cdot a^c)^6 \rangle.$$

In this first case, as $a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^6 = (abc)^6 = 1$,

$$o(a \cdot b^c) = o(cacb) = o(ab) = 2,$$
$$o(ab \cdot a^c) = o(bacac) = o(b) = 2$$

and $o(a) = 2$, $o(bcbc) = 3$ and $[a, bcbc] = 1$ and so

$$o(ab \cdot b^c) = 6$$

as required. Similarly, in the second case, we can show that $G = G^{(6,2,6)} \cong G^{(2,6,6)}$.

If G occurs as a quotient of G_{11} then we can again show that each of the three possibilities from Table 3.4 gives $G = G^{(2,6,6)}$.

If we let

$$a := (1, 2),\ b := (3, 4),\ c := (4, 5)(6, 7)$$

then a, b and c generate G and satisfy the presentation of $G^{(2,6,6)}$. We label the conjugacy classes of involutions of G $C_1, \ldots C_7$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a^G	1	5	$((bc)^3)^G$	1
2	b^G	3	6	$((abc)^3)^G$	1
3	c^G	3	7	$(ac)^G$	3
4	$(ab)^G$	3			

Note that, as $o(bc) = o(abc) = 6$, the elements $(bc)^3 = (6, 7)$ and $(abc)^3 = (1, 2)(6, 7)$ must be contained in T and so $X := C_1 \cup \ldots \cup C_6 \subseteq T$. Moreover, G clearly contains an elementary abelian subgroup of order 8 and so T cannot contain all involutions of G. Thus we must have $T = X$. We have checked in GAP that T gives exactly one Majorana representation of G, details of which are given in 5.2.15 below. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	a	5	c	9	$(ab)^c$
2	b	6	c^b	10	$(ab)^{cb}$
3	b^c	7	c^{bc}	11	$(bc)^3$
4	b^{cb}	8	ab	12	$(abc)^3$

Proposition 5.2.15. *There is exactly one Majorana representation of the form (G, T, V) where*

$$T := \{t_1, \ldots, t_{12}\}.$$

The algebra V contains one $3A$-axis, $u_{\rho(t_2,t_3)}$. The nullspace of the algebra V is zero dimensional and so the dimension of the representation is 13.

5.2.4 The Group $2^4.2$

Proposition 5.2.16. *Let $G = \langle a, b, c\rangle \cong 2^4.2$ be a triangle-point group. Then G admits two Majorana representations of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as the quotient of G_1 and that

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^4, (bc)^4, (abc)^4, (a \cdot b^c)^2\rangle.$$

If we let

$$a := (1, 2)(3, 4),\ b := (5, 6)(7, 8),\ c := (1, 3)(5, 7)$$

then a, b, c satisfy the relations of G and generate a group of order 32 and so we can take $G = \langle a, b, c\rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_{10}$, as below.

i	C_i	$\|C_i\|$	i	C_i	$\|C_i\|$	i	C_i	$\|C_i\|$
1	a^G	2	5	$((ac)^2)^G$	1	9	$(ab \cdot b^c)^G$	2
2	b^G	2	6	$((bc)^2)^G$	1	10	$(ab \cdot a^c)^G$	2
3	c^G	4	7	$((abc)^2)^G$	1			
4	$(ab)^G$	2	8	$(a \cdot b^c)^G$	2			

Now note that the group G contains the following subgroups:

$$K_1 := \langle a, b, b^c \rangle \cong 2^3$$
$$K_2 := \langle a, b, a^c \rangle \cong 2^3$$
$$K_3 := \langle ab, a^c, b^c \rangle \cong 2^3$$
$$K_4 := \langle c^b, c^{ab}, (bc)^2 \rangle \cong 2^3.$$

If we let $X := a^G \cup b^G \cup c^G \cup (ab)^G$ then by assumption we have $X \subseteq T$. Moreover, we see that $|K_i \cap X| = 4$ for $1 \leq i \leq 4$ and

$$|K_1 \cap C_6| = |K_1 \cap C_8| = |K_1 \cap C_9| = 1$$
$$|K_2 \cap C_5| = |K_2 \cap C_8| = |K_2 \cap C_{10}| = 1$$
$$|K_3 \cap C_7| = |K_3 \cap C_9| = |K_3 \cap C_{10}| = 1$$
$$|K_4 \cap C_5| = |K_4 \cap C_6| = |K_4 \cap C_7| = 1.$$

Thus T must contain exactly 0 or 2 of the conjugacy classes C_i, C_j and C_k for

$$\{i, j, k\} \in \{\{6, 8, 9\}, \{5, 8, 10\}, \{7, 9, 10\}, \{5, 6, 7\}\}.$$

This means that there are no possibilities for T which contain exactly 1, 5 or 6 of $\{C_5, \dots, C_{10}\}$. We can also put strong restrictions on the remaining possibilities for T. At this stage, all possibilities are shown below. Here, as before, we denote $X := a^G \cup b^G \cup c^G \cup (ab)^G$.

i	T_i	$\lvert T_i \rvert$	i	T_i	$\lvert T_i \rvert$
1	X	10	5	$X \cup C_8 \cup C_9 \cup C_{10}$	16
2	$X \cup C_5 \cup C_6 \cup C_8$	14	6	$X \cup C_5 \cup C_6 \cup C_9 \cup C_{10}$	16
3	$X \cup C_5 \cup C_7 \cup C_{10}$	14	7	$X \cup C_5 \cup C_7 \cup C_8 \cup C_9$	16
4	$X \cup C_6 \cup C_7 \cup C_9$	14	8	$X \cup C_6 \cup C_7 \cup C_8 \cup C_{10}$	16

We further restrict the possibilities for T by considering automorphisms of G. As the relations of G are preserved by permuting $\{a, b, ab\}$, any such permutation induces an automorphism of G. In particular, there exist permutations of $\{a, b, ab\}$ sending T_2 to T_3 and T_4 and permutations sending T_6 to T_7 and T_8. Thus we need only consider representations of the form (G, T_i, V) for $i \in \{1, 2, 5, 6\}$.

Propositions 5.2.19 and 5.2.20 below show that there are no representations of the form (G, T_5, V) or (G, T_6, V). We have checked in GAP that T_1 and T_2 give exactly one representation each, details of which are given in Propositions 5.2.17 and 5.2.18 below. □

In the following, we label the elements of the T_i as below.

i	t_i	i	t_i	i	t_i
1	a	8	c^{ab}	15	$(a \cdot b^c)^c$
2	a^c	9	ab	16	$ab \cdot b^c$
3	b	10	$(ab)^c$	17	$(ab \cdot b^c)^c$
4	b^c	11	$(ac)^2$	18	$ab \cdot a^c$
5	c	12	$(bc)^2$	19	$(ab \cdot a^c)^c$
6	c^a	13	$(abc)^2$		
7	c^b	14	$a \cdot b^c$		

Proposition 5.2.17. *There exists exactly one Majorana representations of the form* (G, T_1, V) *where*

$$T_1 := \{t_i \mid 1 \leq i \leq 10\}.$$

The algebra is spanned by the ten $2A$ *axes, six* $4A$ *axes and also by the products* $a_1 \cdot v_{\rho(t_3,t_5)}$ *and* $a_5 \cdot v_{\rho(t_1,t_7)}$. *The nullspace on the algebra generated by this spanning set is zero dimensional and so the dimension of the algebra is* 18.

Unlike all other algebras that we have constructed in this chapter, this is an example of an algebra which is 3-closed not 2-closed.

Proposition 5.2.18. *There is exactly one Majorana representation of the form* (G, T_2, V) *where*

$$T_2 := \{t_i \mid 1 \leq i \leq 12\} \cup \{t_{14}, t_{15}\}.$$

The algebra contains four $4A$ *axes,* $v_{\rho(t_5,t_9)}$, $v_{\rho(t_5,t_{14})}$, $v_{\rho(t_6,t_9)}$ *and* $v_{\rho(t_6,t_{14})}$. *The nullspace of* V *is spanned by the vector*

$$n_1 := a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} - a_{11} - a_{12} + a_{13} + a_{14} - \frac{2}{3}(v_{\rho(t_5,t_9)} + v_{\rho(t_5,t_{14})})$$

along with two vectors of the form

$$v_{\rho(s_0,s_1)} - v_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T_2$ *and* $\langle s_0 s_1 \rangle = \langle s_2 s_3 \rangle$. *Thus the dimension of* V *is* 15.

Proposition 5.2.19. *There exist no Majorana representations of the form* (G, T_5, V) *where*

$$T_5 := \{t_i \mid 1 \leq i \leq 10\} \cup \{t_i \mid 14 \leq i \leq 19\}.$$

It is not possible to show that this algebra does not exist by considering only the 2-closed part of the algebra, one must attempt a construction of the 3-closed part before a contradiction is found. This proof is given in full in Section 5.3.1.

Proposition 5.2.20. *There exist no Majorana representations of the form* (G, T_6, V) *where*

$$T_6 := \{t_i \mid 1 \leq i \leq 12\} \cup \{t_i \mid 16 \leq i \leq 19\}.$$

Proof. Suppose for contradiction that there exists a Majorana representation of the form (G, T_6, V). As $t_1 t_3 = ab = t_9$, the dihedral algebra $\langle\langle a_1, a_3 \rangle\rangle$ is of type $2A$ and so

$$a_1 \cdot a_3 = \frac{1}{2^3}(a_1 + a_3 - a_9).$$

Using the known values of the inner product on dihedral algebras, we can then calculate that

$$(a_1 \cdot a_3, a_5) = 0.$$

Now, as $t_3 t_5 = bc$ is of order 4 and $(t_3 t_5)^2 = t_{12}$, the dihedral algebra $\langle\langle a_3, a_5 \rangle\rangle$ is of type $4B$ and so

$$a_3 \cdot a_5 = \frac{1}{2^6}(a_3 - a_4 + a_5 - a_7 + a_{12}).$$

Similarly, we calculate that

$$(a_1, a_3 \cdot a_5) = \frac{1}{2^8}.$$

In particular, $(a_1, a_3 \cdot a_5) \neq (a_1 \cdot a_3, a_5)$, in contradiction with axiom M1, and so such a representation cannot exist. □

5.2.5 The Group $S_3 \times S_3$

Proposition 5.2.21. *Let $G = \langle a, b, c \rangle \cong S_3 \times S_3$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. We use the natural embedding of $S_3 \times S_3$ into S_6, that is to say

$$\langle (1,2), (1,3), (4,5), (5,6) \rangle \cong S_3 \times S_3.$$

We can see that G has only three conjugacy classes: $(1,2)^G$, $(4,5)^G$ and $(1,2)(4,5)^G$. Moreover, considering the values of a, b and ab, we see that they must each belong to different conjugacy classes. Thus the only possibility for T is

$$T := (1,2)^G \cup (4,5)^G \cup (1,2)(4,5)^G.$$

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	$(1,2)$	6	$(5,6)$	11	$(1,3)(4,6)$
2	$(1,3)$	7	$(1,2)(4,5)$	12	$(1,3)(5,6)$
3	$(2,3)$	8	$(1,2)(4,6)$	13	$(2,3)(4,5)$
4	$(4,5)$	9	$(1,2)(5,6)$	14	$(2,3)(4,6)$
5	$(4,6)$	10	$(1,3)(4,5)$	15	$(2,3)(5,6)$

Note that $t_7 t_{11} = (1,2,3)(4,5,6)$ and there exists no elements $t_i, t_j \in T$ such that $o(t_i t_j) = 6$ and $(t_i t_j)^2 = t_7 t_{11}$. Thus the axes a_7 and a_{11} generate an algebra which may be of type $3A$ or $3C$.

In Proposition 5.2.23 below, we show that this algebra cannot be of type $3C$. We have checked in GAP that when this algebra is of type $3A$, T gives exactly one representation, details of which are given in Proposition 5.2.22 below. □

Proposition 5.2.22. *There is exactly one Majorana representations of the form (G, T, V) where the algebra $\langle\langle a_7, a_{11}\rangle\rangle$ is of type $3A$. The algebra contains eight $3A$ axes and the nullspace of V is spanned by the vector*

$$n_1 := -\frac{2^5}{3^2 \cdot 5}\sum_{i=7}^{15} a_i + u_{\rho(t_1,t_2)} + u_{\rho(t_4,t_5)} + u_{\rho(t_7,t_{11})} + u_{\rho(t_7,t_{12})}$$

along with four vectors of the form

$$u_{\rho(s_0,s_1)} - u_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T$ and $\langle s_0 s_1\rangle = \langle s_2 s_3\rangle$. Thus the dimension of V is 18.

Proposition 5.2.23. *There exist no representations of the form (G, T, V) where the algebra $\langle\langle a_7, a_{11}\rangle\rangle$ is of type $3C$.*

Proof. We will show that such a representation cannot obey axiom M1. Suppose that such a representation exists. Firstly, the axes a_1 and a_4 generate an algebra of type $2A$ and so

$$a_1 \cdot a_4 = \frac{1}{8}(a_1 + a_4 - a_7)$$

and

$$(a_1 \cdot a_4, a_{11}) = \frac{3}{2^{10}}.$$

However, the axes a_4 and a_{11} generate an algebra of type $6A$ and so

$$a_4 \cdot a_{11} = \frac{1}{2^6}(a_2 + a_4 - a_5 - a_6 - a_{10} + a_{11} - a_{12}) + \frac{3^2 \cdot 5}{2^{20}} u_{\rho(t_4,t_5)}.$$

The axes a_1 and $u_{\rho(t_4,t_5)}$ are contained in the $6A$ algebra generated by a_5 and a_7 and so $(a_1, u_{\rho(t_4,t_5)}) = 0$. We can now calculate that

$$(a_1, a_4 \cdot a_{11}) = -\frac{3}{2^{11}} \neq (a_1 \cdot a_4, a_{11})$$

which is in contradiction with axiom M1. Thus such a representation cannot exist. □

5.2.6 The Group $2 \times S_4$

Proposition 5.2.24. *Let $G = \langle a, b, c\rangle \cong 2 \times S_4$ be a triangle-point group. Then G admits exactly two Majorana representations of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_6. If we let $m := o(ac)$, $n := o(bc)$, $p := o(abc)$ then we must have $(m, n, p) \in \{(4, 3, 6), (4, 6, 3), (4, 4, 6)\}$. In the first two cases, we in fact have $G = G^{(m,n,p)}$.

In order to procede, we need to determine the conjugacy classes of G. To do so, we use the following embedding of G into S_6:

$$G \cong \langle (1, 2), (1, 3), (1, 4), (5, 6)\rangle.$$

We label the conjugacy classes of involutions of G $C_1, \ldots, C_5$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	$(1,2)^G$	6	4	$(1,2)(3,4)(5,6)^G$	3
2	$(1,2)(3,4)^G$	3	5	$(5,6)^G$	1
3	$(1,2)(5,6)^G$	6			

We first suppose that $(m, n, p) \in \{(4, 3, 6), (4, 6, 3)\}$ in which case we have $G = G^{(m,n,p)}$. Moreover, as $G^{(4,3,6)} \cong G^{(4,6,3)}$, without loss of generality, we can take $(m, n, p) = (4, 3, 6)$. From Lemma 5.2.1, we need only consider one set of generators a, b, c of G. If we pick

$$a := (1,2)(3,4)(5,6),\ b := (1,2),\ c := (1,3)$$

then a, b and c generate G and obey the presentation of $G^{(m,n,p)}$. By assumption, and from axiom M8, we must have $X_1 \subseteq T$ where

$$X_1 := C_1 \cup C_3 \cup C_4 \cup C_5.$$

Now note that G contains a subgroup $K := \langle (1,2), (3,4), (5,6) \rangle \cong 2^3$. If T were to contain all involutions of G, then there would exist a subrepresentation $(K, T \cap K, U)$, such that $|T \cap K| = 7$. This is a contradiction from Proposition 5.2.8. We conclude that if $(m, n, p) = (4, 3, 6)$ then we must have $T = C_1 \cup C_3 \cup C_4 \cup C_5$.

We now turn to the case $(m, n, p) = (4, 6, 6)$ and put restrictions on the value of T by considering the conjugacy classes of G. We let $C_i := a^G$, $C_j := b^G$, $C_k := c^G$ and $C_l := (ab)^G$ where $1 \leq i, j, k, l \leq 5$. Either by inspection, or by considering the structure constants of S_4, we see that, as $o(bc) = o(abc) = 6$, we must have

$$(j, k), (l, k) \in \{(1, 3), (3, 1)\}.$$

We suppose first that $k = 1$ then $j = l = 3$ and so we must have $i = 2$. Similarly, if $k = 3$ then $j = l = 1$ and, again, $i = 2$. In each of these cases, $(bc)^3 = (abc)^3 = (5, 6)$ and so, by axiom M8, we must have $(5, 6) \in T$. Thus, if $(m, n, p) = (4, 6, 6)$, we must have

$$X_2 := C_1 \cup C_2 \cup C_3 \cup C_5 \leq T.$$

As before, T cannot consist of all involutions of G, else we would have a contradiction with the representations of 2^3 and so in this case we have $T = C_1 \cup C_2 \cup C_3 \cup C_5$. This is indeed a possibility, as can be shown by choosing, for example, the generators

$$a := (1,2)(3,4),\ b := (1,2)(5,6),\ c := (1,3).$$

Thus the possibilities for T are given below.

i	T_i	$\lvert T_i \rvert$
1	$C_1 \cup C_3 \cup C_4 \cup C_5$	16
2	$C_1 \cup C_2 \cup C_3 \cup C_5$	16

We have checked in GAP that T_1 and T_2 each give exactly one Majorana representation, details of which are given in Propositions 5.2.25 and 5.2.26 below. □

In the following, we label the elements of T_1 and T_2 as below.

i	t_i	i	t_i	i	t_i
1	$(1,2)$	8	$(1,3)(2,4)$	15	$(3,4)(5,6)$
2	$(1,3)$	9	$(1,4)(2,3)$	16	$(1,2)(3,4)(5,6)$
3	$(1,4)$	10	$(1,2)(5,6)$	17	$(1,3)(2,4)(5,6)$
4	$(2,3)$	11	$(1,3)(5,6)$	18	$(1,4)(2,3)(5,6)$
5	$(2,4)$	12	$(1,4)(5,6)$	19	$(5,6)$
6	$(3,4)$	13	$(2,3)(5,6)$		
7	$(1,2)(3,4)$	14	$(2,4)(5,6)$		

Proposition 5.2.25. *There is exactly one Majorana representation of the form (G, T_1, V) where*

$$T_1 := \{t_i \mid 1 \leq i \leq 15\} \cup \{t_{19}\}.$$

The algebra V contains four $3A$-axes and the nullspace of V is zero dimensional and so the dimension of the representation is 20.

Proposition 5.2.26. *There is exactly one Majorana representation of the form (G, T_2, V) where*

$$T_2 := \{t_i \mid 1 \leq i \leq 6\} \cup \{t_i \mid 10 \leq i \leq 19\}.$$

The algebra V contains four $3A$-axes and six $4A$-axes. The nullspace of V is spanned by the vectors

$$\begin{aligned}
n_1 :=& \frac{2}{3^2}(3a_1 + 3a_6 - a_{10} + 2a_{11} + 2a_{12} + 2a_{13} + 2a_{14} - a_{15} - 3a_{16} - 3a_{17} - 3a_{18} - 3a_{19}) \\
&- \frac{5}{2^3}(u_{\rho(t_1,t_2)} + u_{\rho(t_1,t_3)} + u_{\rho(t_2,t_3)} + u_{\rho(t_4,t_5)}) + v_{\rho(t_{10},t_{17})} \\
&- \frac{1}{3}(v_{\rho(t_1,t_{17})} - 2v_{\rho(t_2,t_{16})} - 2v_{\rho(t_3,t_{16})}) \\
n_2 :=& \frac{2}{3^2}(3a_2 + 3a_5 + 2a_{10} - a_{11} + 2a_{12} + 2a_{13} - a_{14} + 2a_{15} - 3a_{16} - 3a_{17} - 3a_{18} - 3a_{19}) \\
&- \frac{5}{2^3}(u_{\rho(t_1,t_2)} + u_{\rho(t_1,t_3)} + u_{\rho(t_2,t_3)} + u_{\rho(t_4,t_5)}) + v_{\rho(t_{11},t_{16})} \\
&- \frac{1}{3}(-2v_{\rho(t_1,t_{17})} + v_{\rho(t_2,t_{16})} - 2v_{\rho(t_3,t_{16})}) \\
n_3 :=& \frac{2}{3^2}(3a_3 + 3a_4 + 2a_{10} + 2a_{11} - a_{12} - a_{13} + 2a_{14} + 2a_{15} - 3a_{16} - 3a_{17} - 3a_{18} - 3a_{19}) \\
&- \frac{5}{2^3}(u_{\rho(t_1,t_2)} + u_{\rho(t_1,t_3)} + u_{\rho(t_2,t_3)} + u_{\rho(t_4,t_5)}) + v_{\rho(t_{12},t_{16})} \\
&- \frac{1}{3}(-2v_{\rho(t_1,t_{17})} - 2v_{\rho(t_2,t_{16})} + v_{\rho(t_3,t_{16})})
\end{aligned}$$

and so the dimension of the representation is 23.

Remark 5.2.27. *We note that $2 \times S_4$ contains two subgroups isomorphic to S_4, that there are four Majorana representations of S_4 and that these have shapes $(2B, 3A)$, $(2A, 3A)$, $(2B, 3C)$ and $(2A, 3C)$ [IPSS10]. The representation given in Proposition 5.2.26 contains subrepresentations $U_1, U_2 \leq V$ of S_4 of type $(2B, 3A)$.*

As representations of S_4, U_1 and U_2 are not 2-closed, each containing 3 basis vectors in addition to the $2A$ and $3A$ axes. The above result shows that in V these vectors are equal to $\{v_{\rho(t_{10},t_{17})}, v_{\rho(t_{11},t_{16})}, v_{\rho(t_{12},t_{16})}\}$ and $\{v_{\rho(t_1,t_{17})}, v_{\rho(t_2,t_{16})}, v_{\rho(t_3,t_{16})}\}$ in the case of U_1 and U_2 respectively.

5.2.7 The Group $2^3.2^3$

Proposition 5.2.28. *Let $G = \langle a, b, c\rangle \cong 2^3.2^3$ be a triangle-point group. Then G admits exactly two Majorana representations of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_1, in which case we in fact have $G = G_1$. Thus, the presentation of G is known and, by Lemma 5.2.1, we may consider some fixed generators a, b, c of G.

If we let

$$a := (1,2)(3,4),\ b := (1,3)(2,4)(5,6)(7,8)\,,c := (1,5)(2,7)$$

then a, b, c satisfy the relations of G_1 and generate a group of order 64 and so we can take $G = \langle a, b, c\rangle$.

We label the conjugacy classes of involutions of G $C_1 \dots C_9$ as below.

i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$
1	a^G	4	4	$(ab)^G$	4	7	$((abc)^2)^G$	2
2	b^G	4	5	$((ac)^2)^G$	2	8	$(acabcbc)^G$	4
3	c^G	4	6	$((bc)^2)^G$	2	9	$((a\cdot b^c)^2)^G$	1

The group G contains the following subgroups:

$$K_1 = \langle a, a^{cb}, a^{cbc}\rangle \cong 2^3$$
$$K_2 = \langle c, c^b, (ac)^2\rangle \cong 2^3.$$

If we were to have $T \subseteq G$ such that $\lvert T\cap K_1\rvert \in \{5,7\}$ or $\lvert T\cap K_2\rvert \in \{5,7\}$ then, from the classification of Majorana representations of the group 2^3, there would be no Majorana representations of the form (G, T, V). We can thus use these groups to restrict the possibilities for the set T.

Note first that

$$\lvert K_1 \cap C_1\rvert = 4,\ \lvert K_1 \cap C_3\rvert = 2,\ \lvert K_1 \cap C_9\rvert = 1.$$

Thus, as we must have $C_1 \subseteq T$, we cannot have $C_9 \subseteq T$. Furthermore,

$$\lvert K_2 \cap C_3\rvert = 3,\ \lvert K_2 \cap C_5\rvert = \lvert K_2 \cap C_6\rvert = \lvert K_2 \cap C_7\rvert = \lvert K_2 \cap C_8\rvert = 1.$$

As we must have $C_3 \in T$, we must have precisely 0, 1 or 3 of $\{C_5, C_6, C_7, C_8\}$ contained in T.

At this stage, we have the following possibilities for the values of T. In the table below we denote $X := a^G \cup b^G \cup c^G \cup (ab)^G$.

i	T_i	$\|T_i\|$	i	T_i	$\|T_i\|$
1	X	16	6	$X \cup C_5 \cup C_6 \cup C_7$	22
2	$X \cup C_5$	18	7	$X \cup C_5 \cup C_6 \cup C_8$	24
3	$X \cup C_6$	18	8	$X \cup C_5 \cup C_7 \cup C_8$	24
4	$X \cup C_7$	18	9	$X \cup C_6 \cup C_7 \cup C_8$	24
5	$X \cup C_8$	20			

We can restrict these choices further by considering the automorphisms of G. The follow map is an automorphism of G (as can be verified by checking that it preserves the presentation of G).

$$\alpha : a \mapsto b, \; b \mapsto ab, \; c \mapsto c^a.$$

Moreover, it is an outer automorphism inducing the following action on the conjugacy classes of G,

$$a^G \mapsto b^G \mapsto (ab)^G \mapsto a^G,$$
$$((ac)^2)^G \mapsto ((bc)^2)^G \mapsto ((abc)^2)^G \mapsto ((ac)^2)^G.$$

In particular, either α or its inverse maps T_3 and T_4 to T_2 and T_8 and T_9 to T_7. Thus, by Lemma 5.2.1, we need only consider representations of the form (G, T_i, V) for $i \in \{1, 2, 5, 6, 7\}$.

In Propositions 5.2.29 and 5.2.31, we show that there exist no representations of the form (G, T_1, V) or (G, T_5, V). We have checked in GAP that T_2 and T_6 each give exactly one Majorana representation, details of which are given in Propositions 5.2.30 and 5.2.32 below. □

In the following, we label the elements of the T_i as below.

i	t_i	i	t_i	i	t_i
1	a	10	c^a	19	$(bc)^2$
2	a^c	11	c^b	20	$((bc)^2)^a$
3	a^{cb}	12	c^{ab}	21	$(abc)^2$
4	a^{cbc}	13	ab	22	$((abc)^2)^a$
5	b	14	$(ab)^c$	23	$acabcbc$
6	b^c	15	$(ab)^{ca}$	24	$(acabcbc)^a$
7	b^{ca}	16	$(ab)^{cac}$	25	$(acabcbc)^b$
8	b^{cac}	17	$(ac)^2$	26	$(acabcbc)^{ab}$
9	c	18	$((ac)^2)^b$	27	$(a \cdot b^c)^2$

Proposition 5.2.29. *There are no representations of the form (G, T_1, V) where*

$$T_1 := \{t_i \mid 1 \leq i \leq 16\}.$$

It is not possible to show that this algebra does not exist by considering only the 2-closed part of the algebra, one must attempt a construction of the 3-closed part before a contradiction is found. This proof is much more involved than the others in this chapter and is given in full in Section 5.3.2.

Proposition 5.2.30. *There is exactly one representation of the form* (G, T_2, V) *where*

$$T_2 := \{t_i \mid 1 \leq i \leq 18\}.$$

The algebra V *contains eighteen* $4A$ *axes. The nullspace of* V *is spanned by the vectors*

$$\begin{aligned}
n_1 :=& \frac{2}{3}(-a_1 + a_2 + a_3 - a_4 - a_6 - a_7 - a_{14} - a_{15} - a_{17} - a_{18}) + v_{\rho(t_1,t_6)} + v_{\rho(t_1,t_{14})} \\
n_2 :=& \frac{2}{3}(a_1 - a_2 - a_3 + a_4 - a_5 - a_8 - a_{13} - a_{16} - a_{17} - a_{18}) + v_{\rho(t_2,t_5)} + v_{\rho(t_2,t_{13})} \\
n_3 :=& \frac{2}{3}(a_5 + a_8 + a_{14} + a_{15} + a_{17} + a_{18}) - v_{\rho(t_1,t_6)} - v_{\rho(t_2,t_{13})} - v_{\rho(t_5,t_{14})} + v_{\rho(t_6,t_{13})} \\
n_4 :=& \frac{2}{3}(2a_5 + 2a_6 + 2a_7 + 2a_8 - a_{13} - a_{14} - a_{15} - a_{16} + a_{17} + a_{18}) - v_{\rho(t_1,t_6)} - v_{\rho(t_2,t_5)} \\
& - v_{\rho(t_5,t_9)} - v_{\rho(t_5,t_{10})} - v_{\rho(t_6,t_{10})} - v_{\rho(t_7,t_9)} + v_{\rho(t_9,t_{13})} + v_{\rho(t_9,t_{15})} + v_{\rho(t_{10},t_{13})} + v_{\rho(t_{10},t_{14})}
\end{aligned}$$

along with four vectors of the form

$$v_{\rho(s_0,s_1)} - v_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T_2$ *and* $\langle s_0 s_1 \rangle = \langle s_2 s_3 \rangle$*. Thus the dimension of* V *is* 28.

Proposition 5.2.31. *There are no representations of the form* (G, T_5, V) *where*

$$T_5 := \{t_i \mid 1 \leq i \leq 16\} \cup \{t_i \mid 23 \leq i \leq 26\}.$$

Proof. We will show that if such a representation were to exist then it would contain a subalgebra U which is a Majorana representation of the form (G, T_1, V). As we have shown in Proposition 5.2.29 that such a representation cannot exist, this implies that the representation (G, T_5, V) cannot exist either.

As $T_1 \subseteq T_5$, we can take $U = \langle\langle \psi(t) \mid t \in T_1 \rangle\rangle$, where ψ is the bijective mapping associated with the representation (G, T_5, V). Then this algebra is clearly a Majorana representation of the form (G, T_1, U) which is contained in V.

In particular, we have checked that for all $t, s \in T_1$, $ts \notin \{t_{23}, \ldots, t_{26}\}$ this means that the representation (G, T_1, U) obeys axiom M8 and so neither U nor V can exist. □

Proposition 5.2.32. *There is exactly one Majorana representation of the form* (G, T_6, V) *where*

$$T_6 := \{t_i \mid 1 \leq i \leq 22\}.$$

The algebra contains eighteen $4A$ *axes. The nullspace of* V *is spanned by the vectors*

$$\begin{aligned}
n_1 :=& \frac{2}{3}(-a_1 + a_2 + a_3 - a_4 - a_6 - a_7 - a_{14} - a_{15} - a_{17} - a_{18}) + v_{\rho(t_1,t_6)} + v_{\rho(t_1,t_{14})} \\
n_2 :=& \frac{2}{3}(a_1 - a_2 - a_3 + a_4 - a_5 - a_8 - a_{13} - a_{16} - a_{17} - a_{18}) + v_{\rho(t_2,t_5)} + v_{\rho(t_2,t_{13})} \\
n_3 :=& \frac{2}{3}(-a_1 - a_4 + a_5 - a_6 - a_7 + a_8 - a_{13} - a_{16} - a_{19} - a_{20}) + +v_{\rho(t_1,t_6)} + v_{\rho(t_1,t_{19})} \\
n_4 :=& \frac{2}{3}(-a_2 - a_3 - a_5 + a_6 + a_7 - a_8 + a_{13} + a_{16} + a_{17} + a_{18} - a_{21} - a_{22}) - v_{\rho(t_1,t_6)} + v_{\rho(t_1,t_{21})} \\
n_5 :=& \frac{2}{3}(a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20} - a_{21} - a_{22}) - v_{\rho(t_1,t_6)} - v_{\rho(t_2,t_5)}
\end{aligned}$$

along with twelve vectors of the form

$$v_{\rho(s_0,s_1)} - v_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T_6$ *and* $\langle s_0 s_1 \rangle = \langle s_2 s_3 \rangle$. *Thus the dimension of* V *is* 23.

Proposition 5.2.33. *There are no representations of the form* (G, T_7, V) *where*

$$T_7 := \{t_i \mid 1 \leq i \leq 20\} \cup \{t_i \mid 23 \leq i \leq 26\}.$$

Proof. We will show that such an algebra cannot obey axiom M1. Suppose that such a representation exists. Firstly, the axes a_1 and a_2 generate an algebra of type $2A$ and so

$$a_1 \cdot a_2 = \frac{1}{2^3}(a_1 + a_2 - a_{17})$$

and

$$(a_1 \cdot a_2, a_{11}) = -\frac{3}{2^8}.$$

Conversely, the axes a_2 and a_{11} generate an algebra of type $4B$ and so

$$a_2 \cdot a_{11} = \frac{1}{2^6}(a_2 - a_4 + a_{11} - a_{12} + a_{18})$$

and

$$(a_1, a_2 \cdot a_{11}) = \frac{1}{2^6}.$$

In particular,

$$(a_1 \cdot a_2, a_{11}) \neq (a_1, a_2 \cdot a_{11})$$

which contradicts axiom M1 and so such an algebra cannot exist. □

5.2.8 The Group $(S_3 \times S_3) : 2$

Proposition 5.2.34. *Let* $G = \langle a, b, c \rangle \cong (S_3 \times S_3) : 2$ *be a triangle-point group. Then* G *admits exactly one representation of the form* (G, T, V) *where* $a, b, c, ab \in T$.

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_2 and must have presentation

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^4, (bc)^4, (abc)^6, (a \cdot b^c)^3 \rangle.$$

If we let

$$a := (1, 2),\ b := (4, 5),\ c := (1, 6)(2, 5)(4, 3)$$

then a, b, c satisfy the relations of G and generate a group of order 72 and so we can take $G = \langle a, b, c \rangle$. Then G has just three conjugacy classes

$$C_1 := a^G,\ C_2 := c^G,\ C_3 := (ab)^G$$

and so T must contain all involutions of the group G. We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.35 below. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	a	8	c^a	15	$(ab)^{ca}$
2	a^c	9	c^b	16	$(ab)^{cb}$
3	a^{cb}	10	c^{ab}	17	$(ab)^{cab}$
4	b	11	c^{abc}	18	$(ab)^{cac}$
5	b^c	12	c^{abca}	19	$(ab)^{cbc}$
6	b^{ca}	13	ab	20	$(ab)^{cacb}$
7	c	14	$(ab)^c$	21	$(ab)^{cbca}$

Proposition 5.2.35. *There exists exactly one Majorana representation of the form (G, T_1, V) where*

$$T := \{t_i \mid 1 \leq i \leq 21\}.$$

The algebra V contains four $3A$ axes and the nullspace of V is spanned by the vector

$$n_1 := -\frac{2^5}{3^2 \cdot 5} \sum_{i=13}^{21} a_i + u_{\rho(t_1,t_6)} + u_{\rho(t_2,t_3)} + u_{\rho(t_7,t_{10})} + u_{\rho(t_8,t_9)}$$

and so the dimension of V is 24.

5.2.9 The Group $2^2 \times S_4$

Proposition 5.2.36. *Let $G = \langle a, b, c \rangle \cong 2^2 \times S_4$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V), where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we can see that G must occur as a quotient of G_6 and must have presentation

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^4, (bc)^6, (abc)^6, (a \cdot b^c)^4, (a^c \cdot c^b)^2 \rangle.$$

If we take

$$a := (1,2)(3,4)(5,6),\ b := (1,2)(7,8),\ c := (2,3)(5,6)$$

then a, b, c satisfy the relations of G and generate a group of order 96 and so we can take $G = \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots, C_{11}$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a^G	3	5	$((ac)^2)^G$	3	9	$(a^c \cdot c^b)^G$	6
2	b^G	6	6	$((bc)^3)^G$	1	10	$(ab \cdot c^{bc})^G$	3
3	c^G	6	7	$((abc)^3)^G$	1	11	$(b \cdot c^{abc})^G$	3
4	$(ab)^G$	6	8	$((ab \cdot b^c)^3)^G$	1			

As $o(bc) = o(abc) = (ab \cdot b^c) = 6$, T must contain $X := C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_6 \cup C_7 \cup C_8$ The group G contains the subgroups

$$K_1 := \langle a^c \cdot c^b, (bc)^3, (abc)^3 \rangle = \langle (2,4), (5,6)(7,8), (7,8) \rangle \cong 2^3$$
$$K_2 := \langle a^c \cdot c^b, (a^c \cdot c^b)^a, (ab \cdot b^c)^3 \rangle = \langle (2,4), (1,3), (5,6) \rangle \cong 2^3$$
$$K_3 := \langle a^c \cdot c^b, (a^c \cdot c^b)^a, (bc)^3 \rangle = \langle (2,4), (1,3)(5,6), (7,8) \rangle \cong 2^3$$
$$K_4 := \langle (ac)^2, (bc)^3, (ab \cdot b^c)^3 \rangle = \langle (1,2)(3,4), (5,6), (7,8) \rangle \cong 2^3.$$

Note that

$$|K_1 \cap X| = 6,\ |K_1 \cap C_9| = 1$$
$$|K_2 \cap X| = 4,\ |K_2 \cap C_5| = 1,\ |K_2 \cap C_9| = 2$$
$$|K_3 \cap X| = 5,\ |K_3 \cap C_9| = 1,\ |K_3 \cap C_{11}| = 1$$
$$|K_4 \cap X| = 4,\ |K_4 \cap C_5| = 1,\ |K_4 \cap C_{10}| = 1,\ |K_4 \cap C_{11}| = 1.$$

Thus T must contain C_{10} and C_{11} but cannot contain C_5 or C_9, leaving just one possibility for T:

$$T_1 := X \cup C_{10} \cup C_{11}.$$

We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.37 below. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	a	11	c^a	21	$(ab)^{cac}$
2	a^c	12	c^b	22	$(bc)^3$
3	a^{cb}	13	c^{ab}	23	$(abc)^3$
4	b	14	c^{bc}	24	$(ab \cdot b^c)^3$
5	b^c	15	c^{abc}	25	$ab \cdot c^{bc}$
6	b^{ca}	16	ab	26	$(ab \cdot c^{bc})^c$
7	b^{cb}	17	$(ab)^c$	27	$(ab \cdot c^{bc})^{cb}$
8	b^{cab}	18	$(ab)^{ca}$	28	$b \cdot c^{abc}$
9	b^{cac}	19	$(ab)^{cb}$	29	$(b \cdot c^{abc})^c$
10	c	20	$(ab)^{cab}$	30	$(b \cdot c^{abc})^{cb}$

Proposition 5.2.37. *There exists exactly one Majorana representation of the form (G, T, V) where*

$$T := \{t_i \mid 1 \leq i \leq 30\}.$$

The algebra V contains four $3A$-axes and twenty-seven $4A$-axes and the nullspace of V is spanned by the vectors $n_1, \ldots, n_{10}$, as below, along with fifteen vectors of the form

$$v_{\rho(s_0,s_1)} - v_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T$ and $\langle s_0 s_1 \rangle = \langle s_2 s_3 \rangle$. Thus the dimension of V is 36.

i	n_i
1	$-\frac{2}{3}(a_1+a_3+a_5+a_6+a_{12}+a_{13}-a_{22}+a_{25}+a_{27}-a_{29})+v_{\rho(t_1,t_5)}+v_{\rho(t_1,t_{12})}$
2	$-\frac{2}{3}(a_1+a_2+a_7+a_8+a_{10}+a_{11}-a_{22}+a_{25}+a_{26}-a_{30})+v_{\rho(t_1,t_7)}+v_{\rho(t_1,t_{10})}$
3	$-\frac{2}{3}(a_1+a_2+a_{10}+a_{11}+a_{19}+a_{20}-a_{23}-a_{27}+a_{28}+a_{29})+v_{\rho(t_1,t_{10})}+v_{\rho(t_1,t_{19})}$
4	$-\frac{2}{3}(a_1+a_3+a_{12}+a_{13}+a_{17}+a_{18}-a_{23}-a_{26}+a_{28}+a_{30})+v_{\rho(t_1,t_{12})}+v_{\rho(t_1,t_{17})}$
5	$-\frac{2}{3}(a_2+a_3+a_4+a_9+a_{14}+a_{15}-a_{22}+a_{26}+a_{27}-a_{28})+v_{\rho(t_2,t_4)}+v_{\rho(t_2,t_{14})}$
6	$-\frac{2}{3}(a_2+a_3+a_{14}+a_{15}+a_{16}+a_{21}-a_{23}-a_{25}+a_{29}+a_{30})+v_{\rho(t_2,t_{14})}+v_{\rho(t_2,t_{16})}$
7	$-\frac{2}{3}(-a_1+a_4+a_9+a_{16}+a_{21}-a_{24}+a_{26}+a_{27}+a_{29}+a_{30})+v_{\rho(t_4,t_{26})}+v_{\rho(t_4,t_{29})}$
8	$-\frac{2}{3}(-a_2+a_5+a_6+a_{17}+a_{18}-a_{24}+a_{25}+a_{27}+a_{28}+a_{30})+v_{\rho(t_5,t_{25})}+v_{\rho(t_5,t_{28})}$
9	$-\frac{2}{3}(-a_3+a_7+a_8+a_{19}+a_{20}-a_{24}+a_{25}+a_{26}+a_{28}+a_{29})+v_{\rho(t_7,t_{25})}+v_{\rho(t_7,t_{28})}$
10	$-\frac{1}{3}\sum_{i=1}^{3}a_i-\frac{2}{3}\sum_{i=4}^{21}a_i+\sum_{i=22}^{24}a_i-\frac{1}{3}\sum_{i=25}^{30}a_i+\frac{15}{2^4}\sum_{u\in V^{(3)}}u+v_{\rho(t_1,t_{10})}$ $+v_{\rho(t_1,t_{12})}+v_{\rho(t_2,t_{14})}$

5.2.10 The Group $3^{1+2}_{+}:2^2$

Proposition 5.2.38. *Let $G=\langle a,b,c\rangle\cong 3^{1+2}_{+}:2^2$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G,T,V) where $a,b,c,ab\in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of $G^{(m,n,p)}$ for $(m,n,p)\in\{(6,6,3),(6,3,6),(3,6,6)\}$. In fact, by Lemma 3.2.11, we have $G=G^{(3,6,6)}$. Furthermore, any permutation of a, b and ab induces an automorphism of G which permutes the values m, n and p and so, without loss of generality, we may take $G=G^{(3,6,6)}$.

Now that we have the presentation of G, Lemma 5.2.1 allows us to consider fixed generators a,b,c of G. If we let

$$a:=(1,2)(3,4)(5,6),\ b:=(1,3)(2,4)(7,8),\ c:=(1,9)(3,8)(5,7)$$

then a,b,c satisfies the relations of $G^{(3,6,6)}$ and generate a group of order 108 and so we can take $G=\langle a,b,c\rangle$.

Using these generators, we calculate there are just three conjugacy classes of involutions, $C_1=a^G=c^G$, $C_2=b^G$ and $C_3=(ab)^G$. Thus there is only one choice for T:

$$T:=C_1\cup C_2\cup C_3.$$

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	a	10	b	19	ab
2	a^c	11	b^c	20	$(ab)^c$
3	a^{ca}	12	b^{ca}	21	$(ab)^{ca}$
4	a^{cb}	13	b^{cb}	22	$(ab)^{cb}$
5	a^{cab}	14	b^{cab}	23	$(ab)^{cab}$
6	a^{cbc}	15	b^{cabc}	24	$(ab)^{cbc}$
7	a^{cbca}	16	b^{cabcb}	25	$(ab)^{cbcb}$
8	a^{cbcac}	17	b^{cabcbc}	26	$(ab)^{cbcbc}$
9	a^{cbcaca}	18	$b^{cabcbcb}$	27	$(ab)^{cbcbca}$

Note that $t_1t_2 = (ac)^2$ is of order 3 and there exists no elements $t_i, t_j \in T$ such that $o(t_it_j) = 6$ and $(t_it_j)^2 = t_1t_2$. Thus the axes a_1 and a_2 generate an algebra which may be of type $3A$ or $3C$.

Proposition 5.2.39 below shows that if this algebra is of type $3A$ then the representation does not exist. We have checked in GAP that if this algebra is of type $3C$ then there is exactly one representation of the form (G, T, V), details of which are given in Proposition 5.2.40. □

Proposition 5.2.39. *There are no Majorana representations of the form* (G, T, V) *where the algebra* $\langle\langle a_1, a_2\rangle\rangle$ *is of type* $3A$.

Proof. Suppose for contradiction that G does admit such a representation. As $t_1t_{10} = ab \in T$, the dihedral algebra $\langle\langle a_1, a_{10}\rangle\rangle$ is of type $2A$ and

$$a_1 \cdot a_{10} = \frac{1}{2^3}(a_1 + a_{10} - a_{19}).$$

We now let $w := a_3 + 3a_{11}$ then

$$(a_1 \cdot a_{10}, w) = \frac{13}{2^9}.$$

Now, the algebra $\langle\langle a_3, a_{10}\rangle\rangle$ is of type $6A$ and contains the algebra $\langle\langle a_{10}, a_{11}\rangle\rangle$. Therefore the algebra $\langle\langle a_{10}, a_{11}\rangle\rangle$ must be of type $3A$ and we have the products

$$a_{10} \cdot a_{11} = \frac{1}{2^5}(2a_{10} + 2a_{11} + a_{13}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho(t_{10},t_{11})}.$$

and

$$a_3 \cdot a_{10} = \frac{1}{2^6}(a_3 - a_5 - a_9 + a_{10} - a_{11} - a_{13} + a_{27}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho(t_{10},t_{11})}.$$

Thus

$$a_{10} \cdot w = \frac{1}{2^6}(3a_3 - 3a_5 - 3a_9 + 7a_{10} + a_{11} - a_{13} + 3a_{27})$$

and so we can use the known values of the inner product on dihedral algebras to calculate that

$$(a_1, a_{10} \cdot w) = \frac{25}{2^{11}}.$$

In particular, $(a_1 \cdot a_{10}, w) \neq (a_1, a_{10} \cdot w)$, which contradicts axiom M1. Thus such a representation cannot exist. □

Proposition 5.2.40. *There is exactly one Majorana representation of the for (G, T, V) where the algebra $\langle\langle a_{19}, a_{20}\rangle\rangle$ is of type $3C$. The algebra contains seven $3A$ axes and the nullspace of V is spanned by the vectors*

$$n_1 := -\frac{2^5}{3^3 \cdot 5} \sum_{i=10}^{1} 8a_i + u_{\rho(t_1,t_6)} + u_{\rho(t_2,t_4)} + u_{\rho(t_3,t_5)} + u_{\rho(t_{10},t_{15})}$$

$$n_2 := -\frac{2^5}{3^3 \cdot 5} \sum_{i=19}^{26} a_i + u_{\rho(t_1,t_9)} + u_{\rho(t_2,t_5)} + u_{\rho(t_3,t_4)} + u_{\rho(t_{10},t_{15})}$$

and so the dimension of the representation is 32.

5.2.11 The Group S_5

Proposition 5.2.41. *Let $G = \langle a, b, c\rangle \cong S_5$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. It is well known that G contains just two conjugacy classes of involutions: $C_1 := (1,2)^G$ and $C_2 := (1,2)(3,4)^G$ and $G = \langle C_1\rangle$. However, if $a, b \in C_1$ and $o(ab) = 2$ then we must have $ab \in C_2$. Thus we must take

$$T := (1,2)^G \cup (1,2)(3,4)^G.$$

We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.42 below. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	$(1,2)$	10	$(4,5)$	19	$(1,5)(2,4)$
2	$(1,3)$	11	$(1,2)(3,4)$	20	$(1,3)(4,5)$
3	$(1,4)$	12	$(1,3)(2,4)$	21	$(1,4)(3,5)$
4	$(1,5)$	13	$(1,4)(2,3)$	22	$(1,5)(3,4)$
5	$(2,3)$	14	$(1,2)(3,5)$	23	$(2,3)(4,5)$
6	$(2,4)$	15	$(1,3)(2,5)$	24	$(2,4)(3,5)$
7	$(2,5)$	16	$(1,5)(2,3)$	25	$(2,5)(3,4)$
8	$(3,4)$	17	$(1,2)(4,5)$		
9	$(3,5)$	18	$(1,4)(2,5)$		

Proposition 5.2.42. *There is exactly one Majorana representation of the form (G, T, V) where*

$$T := \{t_1, \ldots, t_{25}\}.$$

The algebra contains ten $3A$ axes and six $5A$ axes. The nullspace of V is 5-dimensional and so the dimension of V is 36.

5.2.12 The Group $(S_3 \times S_3) : 2^2$

Proposition 5.2.43. *Let $G = \langle a, b, c\rangle \cong (S_3 \times S_3) : 2^2$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_2, and in fact we have $G = G_2$.

If we let

$$a := (1, 2)(3, 4),\ b := (1, 2)(5, 6),\ c := (1, 10)(2, 9)(3, 8)(4, 5)(6, 7)$$

then a, b, c satisfy the relations of G_2 and generate a group of order 144 and so we can take $G = \langle a, b, c\rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots, C_7$, as below.

i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$
1	a^G	6	5	$((ac)^2)^G$	9
2	b^G	6	6	$((abc)^3)^G$	6
3	c^G	6	7	$((a \cdot b^c)^3)^G$	1
4	$(ab)^G$	9			

The group G contains the subgroup

$$K := \langle a, b, (a \cdot b^c)^3\rangle \cong 2^3.$$

If we were to have $T \subseteq G$ such that $\lvert K \cap T\rvert \in \{5, 7\}$ then, from the classification of Majorana representations of the group 2^3, there would be no Majorana representations of the form (G, T, V). We can thus use this group to restrict the possibilities for the set T.

We let $X := a^G \cup b^G \cup c^G \cup (ab)^G$ and note that, by assumption, we must have $X \subseteq T$. Note further that

$$\lvert K \cap X\rvert = 5,\ \lvert K \cap C_5\rvert = \lvert K \cap C_7\rvert = 1.$$

Therefore T must contain *exactly* one of C_5 and C_7. Finally, as $o(abc) = o(a \cdot b^c) = 6$, from axiom M8, T must also contain C_6 and C_7.

Thus we have just one possibility for T:

$$T := X \cup C_6 \cup C_7.$$

We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.44. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	a	13	c	25	$(ab)^{cbc}$
2	a^c	14	c^a	26	$(ab)^{cacb}$
3	a^{cb}	15	c^b	27	$(ab)^{cbca}$
4	a^{cbc}	16	c^{ab}	28	$(abc)^3$
5	a^{cbca}	17	c^{abc}	29	$((abc)^3)^a$
6	a^{cbcac}	18	c^{abca}	30	$((abc)^3)^{ac}$
7	b	19	ab	31	$((abc)^3)^{aca}$
8	b^c	20	$(ab)^c$	32	$((abc)^3)^{acb}$
9	b^{ca}	21	$(ab)^{ca}$	33	$((abc)^3)^{acab}$
10	b^{cac}	22	$(ab)^{cb}$	34	$(a \cdot b^c)^3$
11	b^{cacb}	23	$(ab)^{cab}$		
12	b^{cacbc}	24	$(ab)^{cac}$		

Proposition 5.2.44. *There exists exactly one Majorana representation of the form (G, T_1, V) where*

$$T := \{t_i \mid 1 \leq i \leq 34\}.$$

The algebra V contains four $3A$ axes and thirty-six $4A$ axes. The nullspace of V is spanned by the vectors $n_1, \ldots, n_{11}$, as below along with eighteen vectors of the form

$$v_{\rho(s_0,s_1)} - v_{\rho(s_2,s_3)}$$

where $s_0, s_1, s_2, s_3 \in T$ and $\langle s_0 s_1 \rangle = \langle s_2 s_3 \rangle$. Thus the dimension of V is 45.

i	n_i
1	$-\frac{2}{3}(a_1 + a_2 + a_{11} + a_{12} + a_{13} + a_{14} - a_{26} + a_{32} + a_{33} - a_{34}) + v_{\rho(t_1,t_{13})} + v_{\rho(t_1,t_{32})}$
2	$-\frac{2}{3}(a_1 + a_3 + a_{10} + a_{12} + a_{15} + a_{16} - a_{24} + a_{30} + a_{31} - a_{34}) + v_{\rho(t_1,t_{15})} + v_{\rho(t_1,t_{30})}$
3	$-\frac{2}{3}(a_1 + a_6 + a_7 + a_{12} + a_{17} + a_{18} - a_{19} + a_{28} + a_{29} - a_{34}) + v_{\rho(t_1,t_{17})} + v_{\rho(t_1,t_{28})}$
4	$-\frac{2}{3}(a_2 + a_4 + a_9 + a_{11} + a_{15} + a_{17} - a_{21} + a_{29} + a_{30} - a_{34}) + v_{\rho(t_2,t_{15})} + v_{\rho(t_2,t_{29})}$
5	$-\frac{2}{3}(a_2 + a_5 + a_8 + a_{11} + a_{16} + a_{18} - a_{20} + a_{28} + a_{30} - a_{34}) + v_{\rho(t_2,t_{16})} + v_{\rho(t_2,t_{28})}$
6	$-\frac{2}{3}(a_3 + a_4 + a_9 + a_{10} + a_{13} + a_{18} - a_{23} + a_{28} + a_{33} - a_{34}) + v_{\rho(t_3,t_{13})} + v_{\rho(t_3,t_{28})}$
7	$-\frac{2}{3}(a_3 + a_5 + a_8 + a_{10} + a_{14} + a_{17} - a_{22} + a_{29} + a_{32} - a_{34}) + v_{\rho(t_3,t_{14})} + v_{\rho(t_3,t_{29})}$
8	$-\frac{2}{3}(a_4 + a_6 + a_7 + a_9 + a_{14} + a_{16} - a_{25} + a_{30} + a_{32} - a_{34}) + v_{\rho(t_4,t_{14})} + v_{\rho(t_4,t_{30})}$
9	$-\frac{2}{3}(a_5 + a_6 + a_7 + a_8 + a_{13} + a_{15} - a_{27} + a_{31} + a_{33} - a_{34}) + v_{\rho(t_7,t_{13})} + v_{\rho(t_7,t_{32})}$
10	$v_{\rho(t_1,t_{13})} + v_{\rho(t_1,t_{15})} + v_{\rho(t_1,t_{17})} + v_{\rho(t_2,t_{15})} + v_{\rho(t_2,t_{16})} + v_{\rho(t_3,t_{13})} + v_{\rho(t_3,t_{14})} + v_{\rho(t_4,t_{14})}$ $+ v_{\rho(t_4,t_{15})} - v_{\rho(t_1,t_{30})} - v_{\rho(t_1,t_{32})} - v_{\rho(t_2,t_{28})} - v_{\rho(t_2,t_{29})} - v_{\rho(t_3,t_{28})} - v_{\rho(t_3,t_{29})}$ $- v_{\rho(t_4,t_{30})} - v_{\rho(t_5,t_{31})} - v_{\rho(t_7,t_{17})}$
11	$-\frac{2^5}{3^2 \cdot 5} \sum_{i=19}^{27} a_i + u_{\rho(t_1,t_4)} + u_{\rho(t_2,t_3)} + u_{\rho(t_{13},t_{16})} + u_{\rho(t_{14},t_{15})}$

5.2.13 The Group $2^4 : D_{10}$

Proposition 5.2.45. *Let $G = \langle a, b, c\rangle \cong 2^4 : D_{10}$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_3, and in fact we have $G = G_3$.

If we let

$$a := (1, 2)(3, 4),\ b := (1, 3)(2, 4)(5, 6)(7, 8)(9, 10),\ c := (1, 2)(3, 5)(4, 7)(6, 9)(8, 10)$$

then a, b, c satisfy the relations of G_3 and generate a group of order 160 and so we can take $G = \langle a, b, c\rangle$. We label the conjugacy classes of involutions of G $C_1, \dots C_4$, as below.

i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$
1	a^G	5	3	$((ac)^2)^G$	5
2	b^G	20	4	$((a \cdot b^c)^2)^G$	5

By assumption, T must contain the classes C_1 and C_2. Note that G contains the group

$$K := \langle a, b, ((a \cdot b^c)^2)^c\rangle \cong 2^3.$$

Moreover,

$$|K \cap C_2|= 4,\ |K \cap C_1|= |K \cap C_3|= |K \cap C_4|= 1$$

and so T must contain exactly one of C_3 and C_4. At this stage, we have two possibilities for the value of T, as shown below. Here, we denote $X := a^G \cup b^G$.

i	T_i	$\lvert T_i\rvert$	i	T_i	$\lvert T_i\rvert$
1	$X \cup C_3$	30	2	$X \cup C_4$	30

We can restrict these choices further by considering the automorphisms of G. The follow map is an automorphism of G (as can be verified by checking that it preserves the presentation of G).

$$\alpha :\ a \mapsto ((a \cdot b^c)^2)^c,\ b \mapsto b,\ c \mapsto c.$$

Moreover, it is an outer automorphism inducing the following action on the conjugacy classes of G,

$$\begin{aligned} a^G &\mapsto ((a \cdot b^c)^2)^G \\ b^G &\mapsto b^G \\ ((ac)^2)^G &\mapsto a^G \\ ((a \cdot b^c)^2)^G &\mapsto ((ac)^2)^G. \end{aligned}$$

In particular, α maps T_1 to T_2. Thus by Lemma 5.2.1, we need only consider representations of the form (G, T_1, V).

We have checked in GAP that T_1 gives exactly one Majorana representation, details of which are given in Proposition 5.2.46. □

In the following, we label the elements of T_1 as below.

i	t_i	i	t_i	i	t_i
1	a	11	b^{cbc}	21	$(ab)^{cbc}$
2	a^c	12	c	22	$(ab)^{cbcb}$
3	a^{cb}	13	c^a	23	$(ab)^{cbca}$
4	a^{cbc}	14	c^{ab}	24	$(ab)^{cbcab}$
5	a^{cbcb}	15	c^{abc}	25	$(ab)^{cbcac}$
6	b	16	ab	26	$(ac)^2$
7	b^c	17	$(ab)^c$	27	$((ac)^2)^b$
8	b^{ca}	18	$(ab)^{ca}$	28	$((ac)^2)^{bc}$
9	b^{cb}	19	$(ab)^{cb}$	29	$((ac)^2)^{bcb}$
10	b^{cac}	20	$(ab)^{cac}$	30	$((ac)^2)^{bcbc}$

Proposition 5.2.46. *There exists exactly one Majorana representation of the form (G, T_1, V) where*

$$T_1 := \{t_i \mid 1 \leq i \leq 30\}.$$

The algebra V contains twenty $4A$ axes and sixteen $5A$ axes. The nullspace of V is 20-dimensional and so the dimension of V is 46.

5.2.14 The Group $2^4 : D_{12}$

Proposition 5.2.47. *Let $G = \langle a, b, c \rangle \cong 2^4 : D_{12}$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must occur as a quotient of G_6 and must have presentation

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^4, (bc)^6, (abc)^6, (a \cdot b^c)^4, (ab \cdot b^c)^3 \rangle.$$

If we let

$$a := (1,2)(3,4),\ b := (1,2)(5,6),\ c := (1,5)(2,7)(3,6)(4,8)$$

then a, b, c satisfy the relations of G and generate a group of order 192 and so we can take $G = \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_6$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a^G	6	4	$((ac)^2)^G$	6
2	b^G	12	5	$((bc)^3)^G$	4
3	c^G	12	6	$(ab \cdot b^{cac})^G$	3

By assumption, T must contain the classes C_1, C_2 and C_3 and, by axiom M8, T must contain C_5. In the following we let $X = C_1 \cup C_2 \cup C_3 \cup C_5$. Note that G contains the group

$$K := \langle a, b, b^{cac} \rangle \cong 2^3.$$

Moreover,

$$|K \cap C_1| = 2,\ |K \cap C_2| = 4, |K \cap C_6| = 1$$

and so T cannot contain C_6. At this stage, we have the following possibilities for the value of T.

i	T_i	$\|T_i\|$	i	T_i	$\|T_i\|$
1	X	34	2	$X \cup C_4$	40

Proposition 5.2.49 below shows that there are no Majorana representations of the form (G, T_2, V). We have checked in GAP that T_1 gives exactly one Majorana representation, details of which are given in Proposition 5.2.48. □

In the following, we label the elements of the T_i as below.

i	t_i	i	t_i	i	t_i	i	t_i
1	a	11	b^{cab}	21	c^b	31	$(bc)^3$
2	a^c	12	b^{cac}	22	c^{ab}	32	$((bc)^3)^a$
3	a^{cb}	13	b^{cabc}	23	c^{bc}	33	$((bc)^3)^{ac}$
4	a^{cbc}	14	ab	24	c^{abc}	34	$((bc)^3)^{aca}$
5	a^{cbcb}	15	$(ab)^c$	25	c^{bca}	35	$(ac)^2$
6	a^{cbcbc}	16	$(ab)^{ca}$	26	c^{abca}	36	$((ac)^2)^b$
7	b	17	$(ab)^{cab}$	27	c^{bcac}	37	$((ac)^2)^{bc}$
8	b^c	18	$(ab)^{cac}$	28	c^{abcac}	38	$((ac)^2)^{bcb}$
9	b^{ca}	19	c	29	c^{bcacb}	39	$((ac)^2)^{bcbc}$
10	b^{cb}	20	c^a	30	c^{abcacb}	40	$((ac)^2)^{bcbcb}$

Proposition 5.2.48. *There exists exactly one Majorana representation of the form (G, T_1, V) where*

$$T_1 := \{t_i \mid 1 \leq i \leq 34\}.$$

The algebra V contains sixteen 3A axes and forty-two 4A axes. The nullspace of V is 38-dimensional and so the algebra is 54-dimensional.

Proposition 5.2.49. *There exist no Majorana representations of the form (G, T_2, V) where*

$$T_2 := \{t_i \mid 1 \leq i \leq 40\}.$$

Proof. Suppose for contradiction that such an algebra exists. Then $t_1 t_2 = (ac)^2 = t_{35}$ and so the algebra $\langle\langle a_1, a_2 \rangle\rangle$ is of type $2A$ and

$$a_1 \cdot a_2 = \frac{1}{2^3}(a_1 + a_2 - a_{35}).$$

We then calculate that

$$(a_1 \cdot a_2, a_7) = \frac{1}{2^6}.$$

However, $t_2 t_7 = a^c \cdot b = (1,2)(5,7,6,8)$ and $(t_2 t_7)^2 = t_6 \in T_2$ and so the algebra $\langle\langle a_2, a_7 \rangle\rangle$ is of type $4B$ and

$$a_2 \cdot a_7 = \frac{1}{2^6}(a_2 - a_3 + a_6 + a_7 - a_{12}).$$

We then calculate that

$$(a_1, a_2 \cdot a_7) = 0.$$

Thus we have

$$(a_1 \cdot a_2, a_7) \neq (a_1, a_2 \cdot a_7)$$

and so the algebra does not obey axiom M1. □

5.2.15 The Group $2 \times S_5$

Proposition 5.2.50. *Let $G = \langle a, b, c \rangle \cong 2 \times S_5$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must be equal to $G_4 = G^{(4,5,6)}$.

If we let

$$a := (1,2),\ b := (1,2)(3,4)(6,7),\ c := (1,5)(2,3)(6,7)$$

then a, b, c satisfy the relations of G_4 and generate a group of order 240 and so we can take $G = \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_5$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a^G	10	4	$(ac)^2$	15
2	b^G	15	5	$(ab \cdot a^c)^3$	1
3	$(ab)^G$	10			

The group G clearly contains a subgroup K which is elementary abelian of order 8 and so T cannot contain all involutions of G. By assumption, T must contain the classes a^G, b^G, c^G and $(ab)^G$. By axiom M8, T must also contain $(ab \cdot a^c)^3$. This leaves us with just one possibility for T:

$$T := C_1 \cup C_2 \cup C_3 \cup C_5.$$

We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.51. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i
1	$(1,2)$	13	$(1,4)(2,3)(6,7)$	25	$(2,5)(3,4)(6,7)$
2	$(1,3)$	14	$(1,2)(3,5)(6,7)$	26	$(1,2)(6,7)$
3	$(1,4)$	15	$(1,3)(2,5)(6,7)$	27	$(1,3)(6,7)$
4	$(1,5)$	16	$(1,5)(2,3)(6,7)$	28	$(1,4)(6,7)$
5	$(2,3)$	17	$(1,2)(4,5)(6,7)$	29	$(1,5)(6,7)$
6	$(2,4)$	18	$(1,4)(2,5)(6,7)$	30	$(2,3)(6,7)$
7	$(2,5)$	19	$(1,5)(2,4)(6,7)$	31	$(2,4)(6,7)$
8	$(3,4)$	20	$(1,3)(4,5)(6,7)$	32	$(2,5)(6,7)$
9	$(3,5)$	21	$(1,4)(3,5)(6,7)$	33	$(3,4)(6,7)$
10	$(4,5)$	22	$(1,5)(3,4)(6,7)$	34	$(3,5)(6,7)$
11	$(1,2)(3,4)(6,7)$	23	$(2,3)(4,5)(6,7)$	35	$(4,5)(6,7)$
12	$(1,3)(2,4)(6,7)$	24	$(2,4)(3,5)(6,7)$	36	$(6,7)$

Proposition 5.2.51. *There exists exactly one Majorana representation of the form (G,T,V) where*

$$T := \{t_i \mid 1 \leq i \leq 36\}.$$

The algebra V also contains ten $3A$ axes, thirty $4A$ axes and six $5A$ axes. Its nullspace is 21 dimensional and so the algebra V is 61 dimensional.

5.2.16 The Group $2^5 : D_{12}$

Proposition 5.2.52. *Let $G = \langle a,b,c \rangle \cong 2^5 : D_{12}$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G,T,V) where $a,b,c,ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must be equal to G_6. If we let

$$a := (1,2)(3,4)(9,10),\ b := (1,2)(5,6),\ c := (1,5)(2,7)(3,8)(4,6)(9,10)$$

then a,b,c satisfy the relations of G and generate a group of order 384 and so we can take $G := \langle a,b,c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_{13}$, as below.

i	C_i	$\lvert C_i\rvert$	i	C_i	$\lvert C_i\rvert$
1	a^G	6	8	$((ac)^2)^G$	6
2	b^G	12	9	$((a \cdot b^c)^2)^G$	6
3	c^G	12	10	$(ab \cdot b^{cac})^G$	3
4	$(ab)^G$	12	11	$((a \cdot c^{bc})^2)^G$	3
5	$((bc)^3)^G$	4	12	$(bc \cdot (ab)^{cbca})^G$	12
6	$((abc)^3)^G$	4	13	$(ac \cdot c^{bcacb})^G$	6
7	$((ab \cdot b^c)^3)^G$	1			

By assumption and by axiom M8, T must contain $X := C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7$.

The group G contains the following subgroups

$$K_1 := \langle a, b, (ab \cdot b^c)^3 \rangle \cong 2^3$$
$$K_2 := \langle a, b, b^{cac} \rangle \cong 2^3$$
$$K_3 := \langle a, b, a^{cbcbc} \rangle \cong 2^3$$
$$K_4 := \langle b, (bc)^3, ab^{cac} \rangle \cong 2^3$$
$$K_5 := \langle (a \cdot b^c)^2, ((a \cdot b^c)^2)^c, (ab \cdot b^c)^3 \rangle \cong 2^3.$$

We use these groups, along with the fact that if, for some i, $|K_i \cap T| \in \{5, 7\}$ then the representation (G, T, V) cannot exist. Note that

$$|K_1 \cap X| = 6,\ |K_1 \cap C_9| = 1$$
$$|K_2 \cap X| = 5,\ |K_2 \cap C_9| = 1,\ |K_2 \cap C_{10}| = 1$$
$$|K_3 \cap X| = 6,\ |K_3 \cap C_{11}| = 1$$
$$|K_4 \cap X| = 5,\ |K_4 \cap C_{10}| = 1,\ |K_4 \cap C_{12}| = 1$$
$$|K_5 \cap X| = 3,\ |K_5 \cap C_8| = 1,\ |K_5 \cap C_9| = 2,\ |K_5 \cap C_{13}| = 2$$

and so T must contain C_{10}, cannot contain any of C_9, C_{11}, C_{12} and cannot contain both C_8 and C_{13}. Thus there are just three possibilities for the value of T.

i	T_i	$\|T_i\|$	i	T_i	$\|T_i\|$
1	$X \cup C_{10}$	54	3	$X \cup C_{10} \cup C_{13}$	60
2	$X \cup C_{10} \cup C_8$	60			

We show in Propositions 5.2.53 and 5.2.55 that there are no representations of the form (G, T_1, V) or (G, T_3, V). We have checked in GAP that T_2 admits exactly one Majorana representation, details of which are given in Proposition 5.2.54. □

In the following, we label the elements of the T_i as below.

i	t_i	i	t_i	i	t_i	i	t_i
1	a	18	$b^{cabcabc}$	35	$(ab)^{cab}$	52	$ab \cdot b^{cac}$
2	a^c	19	c	36	$(ab)^{cac}$	53	$(ab \cdot b^{cac})^c$
3	a^{cb}	20	c^a	37	$(ab)^{cbc}$	54	$(ab \cdot b^{cac})^{cb}$
4	a^{cbc}	21	c^b	38	$(ab)^{cbca}$	55	$(ac)^2$
5	a^{cbcb}	22	c^{ab}	39	$(ab)^{cbcb}$	56	$((ac)^2)^b$
6	a^{cbcbc}	23	c^{bc}	40	$(ab)^{cbcab}$	57	$((ac)^2)^{bc}$
7	b	24	c^{abc}	41	$(ab)^{cbcbc}$	58	$((ac)^2)^{bcb}$
8	b^c	25	c^{bca}	42	$(ab)^{cbcabc}$	59	$((ac)^2)^{bcbc}$
9	b^{ca}	26	c^{abca}	43	$(bc)^3$	60	$((ac)^2)^{bcbcb}$
10	b^{cb}	27	c^{bcac}	44	$((bc)^3)^a$	61	$ac \cdot c^{bcacb}$
11	b^{cab}	28	c^{abcac}	45	$((bc)^3)^{ac}$	62	$(ac \cdot c^{bcacb})^b$
12	b^{cac}	29	c^{bcacb}	46	$((bc)^3)^{aca}$	63	$(ac \cdot c^{bcacb})^c$
13	b^{cabc}	30	c^{abcacb}	47	$(abc)^3$	64	$(ac \cdot c^{bcacb})^{bc}$
14	b^{cabca}	31	ab	48	$((abc)^3)^a$	65	$(ac \cdot c^{bcacb})^{cb}$
15	b^{cabcb}	32	$(ab)^c$	49	$((abc)^3)^{ac}$	66	$(ac \cdot c^{bcacb})^{bcb}$
16	b^{cabcab}	33	$(ab)^{ca}$	50	$((abc)^3)^{aca}$		
17	b^{cabcbc}	34	$(ab)^{cb}$	51	$(ab \cdot b^c)^3$		

Proposition 5.2.53. *There are no Majorana representations of the form (G, T_1, V) where*

$$T_1 := X \cup C_{10} = \{t_i \mid 1 \leq i \leq 54\}.$$

Proof. Suppose for contradiction that such an algebra exists. From the known values of the algebra and inner products on dihedral algebras, we calculate that

$$(a_1 \cdot a_7, a_{19}) = \frac{1}{2^3}(a_1 + a_7 - a_{31}, a_{19}) = \frac{1}{2^8}.$$

Now, the dihedral algebra $\langle\langle a_7, a_{19}\rangle\rangle$ is of type $6A$ and contains the $3A$ algebra $\langle\langle a_7, a_{11}\rangle\rangle$ and so

$$a_7 \cdot a_{19} = \frac{3}{2^6}(a_7 - a_{11} - a_{16} + a_{19} - a_{25} - a_{28} + a_{31}) + \frac{45}{2^{11}} u_{\rho(t_7,t_{11})}.$$

We now determine the value of $(a_1, u_{\rho(t_7,t_{11})})$. Firstly, using the known values of dihedral algebras as before, we calculate that

$$(a_1 \cdot a_7, a_{11}) = \frac{1}{2^3}(a_1 + a_7 - a_{31}, a_{11}) = \frac{1}{2^7}.$$

Conversely,

$$(a_1, a_7 \cdot a_{11}) = \frac{1}{2^5}(a_1, 2a_7 + 2a_{11} + a_{16}) - \frac{135}{2^{11}}(a_1, u_{\rho(t_7,t_{11})}) = \frac{11}{2^{10}} - \frac{135}{2^{11}}(a_1, u_{\rho(t_7,t_{11})})$$

therefore, by axiom M1,

$$(a_1, u_{\rho(t_7,t_{11})}) = \frac{2}{3^2 \cdot 5}.$$

We can now calculate that

$$(a_1, a_7 \cdot a_{19}) = \frac{1}{2^9}.$$

This contradicts axiom M1 and so such an algebra cannot exist. ◻

Proposition 5.2.54. *There is exactly one Majorana representation of the form* (G, T_2, V) *where*

$$T_2 := X \cup C_8 \cup C_{10} = \{t_i \mid 1 \le i \le 60\}.$$

The algebra V *contains sixteen* $3A$ *axes and one hundred and fifty-six* $4A$ *axes. Its nullspace is* 132*-dimensional and so the algebra* V *is* 100*-dimensional.*

Proposition 5.2.55. *There are no Majorana representations of the form* (G, T_3, V) *where*

$$T_3 := X \cup C_{10} \cup C_{13} = \{t_i \mid 1 \le i \le 54\} \cup \{t_i \mid 61 \le i \le 66\}.$$

Proof. We will show that if such a representation were to exist then it would contain a subalgebra U which is a Majorana representation of the form (G, T_1, V). As we have shown in Proposition 5.2.53 that such a representation which obeys axiom M8 cannot exist, this implies that the representation (G, T_3, V) cannot exist either.

As $T_1 \subseteq T_3$, we can take $U = \langle\langle \psi(t) \mid t \in T_1 \rangle\rangle$, where ψ is the bijective mapping associated with the representation (G, T_3, V). Then this algebra is clearly a Majorana representation of the form (G, T_1, U) which is contained in V.

In particular, we have checked that for all $t, s \in T_1$, $ts \notin \{t_{61}, \ldots, t_{66}\}$ this means that the representation (G, T_1, U) obeys axiom M8 and so neither U nor V can exist. □

5.2.17 The Group $2^4 : (S_3 \times S_3)$

Proposition 5.2.56. *Let* $G = \langle a, b, c \rangle \cong 2^4 : (S_3 \times S_3)$ *be a triangle-point group. Then* G *admits exactly one Majorana representation of the form* (G, T, V).

Proof. From Tables 3.3 and 3.4, we see that we must have

$$G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^6, (bc)^6, (abc)^6, (a \cdot b^c)^4, a \cdot (b \cdot c^{ac})^3 \rangle.$$

If we let

$$\begin{aligned} a &:= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) \\ b &:= (1,3)(2,4)(5,6)(7,8)(9,10)(11,12) \\ c &:= (2,5)(6,7). \end{aligned}$$

Then a, b and c satisfy the presentation of G and generate a group of order 576, so we may take $G = \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_5$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a	12	4	$(abc)^3$	6
2	b	12	5	$(a \cdot b^c)^2$	9
3	c	36			

By assumption, T must contain C_1, C_2, C_3 and C_4. However, as G clearly contains an elementary abelian subgroup of order 8 so T cannot contain all involutions of the group G. Thus we have just one choice for the value of T:

$$T := C_1 \cup C_2 \cup C_3 \cup C_4.$$

We have checked in GAP that T gives exactly one Majorana representation, details of which are given in Proposition 5.2.57. □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i	i	t_i
1	a	18	b^{cac}	35	c^{acbc}	52	$c^{bcacabc}$
2	a^c	19	b^{cabc}	36	c^{bcac}	53	$c^{abcacac}$
3	a^{ca}	20	b^{caca}	37	c^{abcac}	54	$c^{abcacbc}$
4	a^{cb}	21	b^{cabcb}	38	c^{acbca}	55	$c^{acbcaca}$
5	a^{cab}	22	b^{cacac}	39	c^{acbcb}	56	$c^{acbcacb}$
6	a^{cbc}	23	b^{cabcbc}	40	c^{acbcab}	57	$c^{acbcabca}$
7	a^{cabc}	24	b^{cacacb}	41	c^{bcaca}	58	$c^{acbcabcb}$
8	a^{cbcb}	25	c	42	c^{bcacb}	59	$c^{acbcabcac}$
9	a^{cabca}	26	c^a	43	c^{bcacab}	60	ab
10	a^{cbcbc}	27	c^b	44	c^{abcaca}	61	$(abc)^3$
11	a^{cabcac}	28	c^{ab}	45	c^{abcacb}	62	$((abc)^3)^a$
12	a^{cbcbca}	29	c^{ac}	46	$c^{abcacab}$	63	$((abc)^3)^{ac}$
13	b	30	c^{bc}	47	c^{acbcac}	64	$((abc)^3)^{aca}$
14	b^c	31	c^{abc}	48	c^{acbcbc}	65	$((abc)^3)^{acb}$
15	b^{ca}	32	c^{acb}	49	$c^{acbcabc}$	66	$((abc)^3)^{acab}$
16	b^{cb}	33	c^{bca}	50	c^{bcacac}		
17	b^{cab}	34	c^{abca}	51	c^{bcacbc}		

Proposition 5.2.57. *There is exactly one Majorana representation of the form* (G, T, V) *where*

$$T := \{t_i \mid 1 \leq i \leq 66\}.$$

The algebra V *contains fifty-six* $3A$ *axes and ninety* $4A$ *axes. Its nullspace is* 119*-dimensional and so the algebra is* 93*-dimensional.*

5.2.18 The Group $2^4 : A_5$

Proposition 5.2.58. *Let* $G = \langle a, b, c \rangle \cong 2^4 : A_5$ *be a triangle-point group. Then* G *admits exactly one Majorana representation of the form* (G, T, V) *where* $a, b, c, ab \in T$*.*

Proof. From Tables 3.3 and 3.4, we see that G must be equal to G_7. If we let

$$a := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$
$$b := (1,11)(2,12)(3,9)(4,10)(5,6)(13,14)$$
$$c := (1,3)(2,15)(4,13)(6,12)(7,11)(14,16)$$

then a, b, c satisfy the relations of G_7 and generate a group of order 960 and so we can take $G := \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_3$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a^G	60	3	$((b \cdot c^{ac})^2)^G$	5
2	$((ac)^3)^G$	10			

By assumption, $a \in T$ and by axiom M8, $(ac)^3 \in T$. Moreover, G contains an elementary abelian subgroup of order 8 and so T cannot contain all involutions of G. Thus we have one choice for the value of T:

$$T_1 := C_1 \cup C_2.$$

We have checked in GAP that T_1 gives exactly one Majorana representation, details of which are given in Proposition 5.2.59 □

In the following, we label the elements of T as below.

i	t_i	i	t_i	i	t_i	i	t_i
1	a	19	$a^{cabcbcac}$	37	b^{cabcac}	55	$(ab)^{cacbcab}$
2	a^c	20	$a^{cabcbcaca}$	38	b^{cabcbc}	56	$(ab)^{cacbcac}$
3	a^{ca}	21	$a^{cabcbcacb}$	39	$b^{cabcaca}$	57	$(ab)^{cacbcbc}$
4	a^{cb}	22	b	40	$b^{cabcacb}$	58	$(ab)^{cacbcbcb}$
5	a^{cab}	23	b^c	41	$b^{cabcacab}$	59	$(ab)^{cacbcbcbc}$
6	a^{cbc}	24	b^{ca}	42	$b^{cabcacbc}$	60	$(ab)^{cacbcacabc}$
7	a^{cabc}	25	b^{cb}	43	ab	61	$(ac)^3$
8	a^{cbca}	26	b^{cab}	44	$(ab)^c$	62	$((ac)^3)^b$
9	a^{cbcb}	27	b^{cac}	45	$(ab)^{ca}$	63	$((ac)^3)^{bc}$
10	a^{cbcab}	28	b^{cbc}	46	$(ab)^{cb}$	64	$((ac)^3)^{bca}$
11	a^{cabca}	29	b^{cabc}	47	$(ab)^{cac}$	65	$((ac)^3)^{bcb}$
12	a^{cabcb}	30	b^{caca}	48	$(ab)^{cbc}$	66	$((ac)^3)^{bcab}$
13	a^{cabcab}	31	b^{cbca}	49	$(ab)^{caca}$	67	$((ac)^3)^{bcbc}$
14	a^{cbcac}	32	b^{cbcb}	50	$(ab)^{cacb}$	68	$((ac)^3)^{bcabc}$
15	a^{cbcabc}	33	b^{cabca}	51	$(ab)^{cacac}$	69	$((ac)^3)^{bcbca}$
16	a^{cabcac}	34	b^{cabcb}	52	$(ab)^{cacbc}$	70	$((ac)^3)^{bcabca}$
17	a^{cabcbc}	35	b^{cabcab}	53	$(ab)^{cacbca}$		
18	$a^{cabcbca}$	36	b^{cbcac}	54	$(ab)^{cacbcb}$		

Proposition 5.2.59. *There exists exactly one Majorana representation of the form* (G, T_1, V) *where*

$$T_1 := \{t_i \mid 1 \leq i \leq 70\}.$$

The algebra V contains forty $3A$ axes, ninety $4A$ axes and ninety-six $5A$ axes. Its nullspace is 171-dimensional and so the algebra is 125 dimensional.

5.2.19 The Group $2 \times S_6$

Proposition 5.2.60. *Let $G = \langle a, b, c \rangle \cong 2 \times S_6$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must be equal to G_8. If we let

$$a := (1,2),\ b := (1,2)(3,4)(5,6)(7,8),\ c := (2,3)(4,5)(7,8)$$

then a, b and c generate a group of order 1440 and satisfy the presentation of G_8 and so we may take $G := \langle a, b, c \rangle$. We label the conjugacy classes of involutions of G $C_1, \dots C_7$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a	15	5	$(bc)^3$	15
2	b	15	6	$(a \cdot ((ac)^3)^{bc})^3$	1
3	c	45	7	$(a \cdot b^c)^2$	45
4	$(ac)^3$	15			

By assumption and by axiom M8, T must contain $C_1 \cup \ldots \cup C_6$. The group G clearly contains an elementary abelian subgroup of order 8 and so T cannot contain all involutions in G. Thus we have only one choice for the value of T:

$$T := C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6.$$

We have checked in GAP that T admits exactly one Majorana representation, details of which are given in Proposition 5.2.61. □

Proposition 5.2.61. *There exists exactly one Majorana representation of the form (G, T, V) where*

$$T := \bigcup_{i=1}^{6} C_i = \{t_i \mid 1 \leq i \leq 106\}.$$

The algebra contains forty $3A$ axes, four hundred and five $4A$ axes and thirty-six $5A$ axes. The nullspace is 436-dimensional and so the algebra is 151-dimensional.

5.2.20 The Group $2^5 : S_5$

Proposition 5.2.62. *Let $G = \langle a, b, c \rangle \cong 2^5 : S_5$ be a triangle-point group. Then G admits exactly one Majorana representation of the form (G, T, V) where $a, b, c, ab \in T$.*

Proof. From Tables 3.3 and 3.4, we see that G must be equal to G_{10}. If we let

$$a := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$
$$b := (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$$
$$c := (1,8)(2,6)(3,9)(4,12)(5,10)(7,11)$$

then a, b and c generate a group of order 3840 and satisfy the presentation of G_{10} and so we may take $G := \langle a,b,c \rangle$. We label the conjugacy classes of involutions of G $C_1, \ldots C_7$, as below.

i	C_i	$\lvert C_i \rvert$	i	C_i	$\lvert C_i \rvert$
1	a	60	5	$a \cdot (b \cdot c^{ac})^2$	15
2	c	40	6	$((ac)^2 \cdot (bc)^2)^2$	15
3	$(ac)^3$	40	7	$(a \cdot c^{bc})^2$	60
4	$((ac)^3 \cdot c^{abc})^3$	1			

By assumption and by axiom M8, T must contain $X := C_1 \cup C_2 \cup C_3 \cup C_4$. The group G contains the subgroups

$$K_1 := \langle c, (ac)^3, (c \cdot c^{acb})^2 \rangle \cong 2^3$$
$$K_2 := \langle a, c^{acb}, ((abc)^3)^{a \cdot b^c \cdot abc} \rangle \cong 2^3$$
$$K_3 := \langle a, c^{ac}, (b \cdot c^{ac})^2 \rangle \cong 2^3.$$

Note that

$$|K_1 \cap X| = 6, |K_1 \cap C_6| = 1$$
$$|K_2 \cap X| = 6, |K_2 \cap C_7| = 1$$
$$|K_3 \cap X| = 5, |K_3 \cap C_5| = 1, |K_3 \cap C_7| = 1$$

and so T must contain C_5 but cannot contain either C_6 or C_7. Thus we must have $T := X \cup C_5$. ☐

Proposition 5.2.63. *There is exactly one Majorana representation of the form (G,T,V) where $T = X \cup C_5$. The algebra contains one hundred and twenty-nine $3A$ axes, one thousand and twenty $4A$ axes and ninety-six $5A$ axes. Its nullspace is 1201-dimensional and so the algebra is 231-dimensional.*

5.3 The 3-closed examples

We now deal with the two non-existence results which involve algebras which are not 2-closed. These proofs are rather involved and are presented here in order to show the logical steps of the argument rather than the full details of the calculations. All calculations have been performed in GAP and are reproducible using the tools available as part of the package `MajoranaAlgebras` [WP].

5.3.1 The Group $2^4.2$

Let

$$a := (1,2)(3,4),\ b := (5,6)(7,8),\ c := (1,3)(5,7)$$

and let $G = \langle a, b, c \rangle \cong 2^4.2$.

In the following, we label certain involutions of G as below.

i	t_i	i	t_i	i	t_i
1	a	8	c^{ab}	15	$(a \cdot b^c)^c$
2	a^c	9	ab	16	$ab \cdot b^c$
3	b	10	$(ab)^c$	17	$(ab \cdot b^c)^c$
4	b^c	11	$(ac)^2$	18	$ab \cdot a^c$
5	c	12	$(bc)^2$	19	$(ab \cdot a^c)^c$
6	c^a	13	$(abc)^2$		
7	c^b	14	$a \cdot b^c$		

Proposition 5.3.1. *There exist no Majorana representations of the form* (G, T_5, V)

$$T_5 := \{t_i \mid 1 \leq i \leq 10\} \cup \{t_i \mid 14 \leq i \leq 19\}.$$

In the following, we suppose that there exists such an algebra V and let $a_i := \psi(t_i)$ for $1 \leq i \leq 10$ and $14 \leq i \leq 19$.

Lemma 5.3.2. *Suppose that V is a Majorana algebra as in Proposition 5.3.1. Then Table 5.2 gives the value of the inner product on certain vectors of V.*

Proof. We use axiom M1 to find the required products.

Row 1 First,

$$(a_1, a_6 \cdot a_9) = \frac{1}{2^6}(a_1, 3a_6 + a_7 + 3a_9 + a_{10} - 3v_{\rho(t_6,t_9)}) = \frac{5}{2^9} - \frac{3}{2^6}(a_1, v_{\rho(t_6,t_9)}).$$

Conversely,

$$(a_1 \cdot a_9, a_6) = \frac{1}{2^3}(a_1 - a_3 + a_9, a_6) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_1, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}$ as required.

i	u	v	(u,v)
1	a_1	$v_{\rho(t_6,t_9)}$	$\frac{1}{2^3}$
2	a_3	$v_{\rho(t_1,t_5)}$	$\frac{1}{2^3}$
3	a_3	$v_{\rho(t_6,t_9)}$	$\frac{1}{2^3}$
4	a_3	$v_{\rho(t_5,t_{16})}$	$\frac{1}{2^3}$
5	a_{14}	$v_{\rho(t_6,t_9)}$	$\frac{1}{2^3\cdot 3}$
6	a_{16}	$v_{\rho(t_6,t_9)}$	$\frac{1}{2^3}$
7	a_{18}	$v_{\rho(t_6,t_9)}$	$\frac{1}{2^3}$
8	a_3	$a_1 \cdot v_{\rho(t_6,t_9)}$	$-\frac{1}{2^6}$

Table 5.2: Some inner product values on V for $G \cong 2^4.2$

Row 2 First,

$$(a_3, a_1 \cdot a_5) = \frac{1}{2^6}(a_3, 3a_1 + a_2 + 3a_5 + a_6 - 3v_{\rho(t_1,t_5)}) = \frac{5}{2^9} - \frac{3}{2^6}(a_3, v_{\rho(t_1,t_5)}).$$

Conversely,

$$(a_1 \cdot a_3, a_5) = \frac{1}{2^3}(a_1 + a_3 - a_9, a_5) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_3, v_{\rho(t_1,t_5)}) = \frac{1}{2^3}$ as required.

Row 3 First,

$$(a_3, a_6 \cdot a_9) = \frac{1}{2^6}(a_3, 3a_6 + a_7 + 3a_9 + a_{10} - 3v_{\rho(t_6,t_9)}) = \frac{5}{2^9} - \frac{3}{2^6}(a_3, v_{\rho(t_6,t_9)}).$$

Conversely,

$$(a_3 \cdot a_9, a_6) = -\frac{1}{2^3}(a_1 - a_3 - a_9, a_6) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_3, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}$ as required.

Row 4 First,

$$(a_3, a_5 \cdot a_{16}) = \frac{1}{2^6}(a_3, 3a_5 + a_6 + 3a_{16} + a_{17} - 3v_{\rho(t_5,t_{16})}) = \frac{5}{2^9} - \frac{3}{2^6}(a_3, v_{\rho(t_5,t_{16})}).$$

Conversely,

$$(a_3 \cdot a_{16}, a_5) = \frac{1}{2^3}(a_3 - a_{14} + a_{16}, a_5) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_3, v_{\rho(t_5,t_{16})}) = \frac{1}{2^3}$ as required.

Row 5 First,

$$(a_{14}, a_6 \cdot a_9) = \frac{1}{2^6}(a_{14}, 3a_6 + a_7 + 3a_9 + a_{10} - v_{\rho(t_6,t_9)}) = \frac{1}{2^9} - \frac{3}{2^6}(a_{14}, v_{\rho(t_6,t_9)}).$$

Conversely, $a_9 \cdot a_{14} = 0$ and so $(a_9 \cdot a_{14}, a_{16}) = 0$. Axiom M1 then implies that $(a_{14}, v_{\rho(t_6,t_9)}) = \frac{1}{2^3\cdot 3}$ as required.

Row 6 First,

$$(a_{16}, a_6 \cdot a_9) = \frac{1}{2^6}(a_{16}, 3a_6 + a_7 + 3a_9 + a_{10} - 3v_{\rho(t_6,t_9)}) = \frac{5}{2^9} - \frac{3}{2^6}(a_{16}, v_{\rho(t_6,t_9)}).$$

Conversely,

$$(a_9 \cdot a_{16}, a_6) = -\frac{1}{2^3}(a_4 - a_9 - a_{16}, a_6) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_{16}, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}$ as required.

Row 7 First,

$$(a_{18}, a_6 \cdot a_9) = \frac{1}{2^6}(a_{18}, 3a_6 + a_7 + 3a_9 + a_{10} - 3v_{\rho(t_6,t_9)}) = \frac{5}{2^9} - \frac{3}{2^6}(a_{18}, v_{\rho(t_6,t_9)})).$$

Conversely,

$$(a_9 \cdot a_{18}, a_6) = -\frac{1}{2^3}(a_2 - a_9 - a_{18}, a_6) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_{18}, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}$ as required.

Row 8 Using the inner product values given by rows 1 and 3 of this table, we calculate that

$$(a_3, a_1 \cdot v_{\rho(t_6,t_9)}) = (a_1 \cdot a_3, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}(a_1 + a_3 - a_9, v_{\rho(t_6,t_9)}) = -\frac{1}{2^6}.$$

□

Lemma 5.3.3. *Suppose that V is a Majorana algebra as in 5.3.1. Then*

$$\begin{aligned} a_{10} \cdot (v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) = &-\frac{1}{2^4 \cdot 3}(7a_1 - a_2 - 2a_4 + a_9 - 5a_{10} + 2a_{16} - 2a_{19}) \\ &- \frac{1}{2^4}(v_{\rho(t_1,t_5)} + v_{\rho(t_1,t_7)}) - \frac{1}{2^5}(v_{\rho(t_5,t_9)} + 3v_{\rho(t_6,t_9)}) \\ &+ \frac{1}{2^3}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) + 2a_1 \cdot v_{\rho(t_6,t_9)}. \end{aligned}$$

Proof. The dihedral algebras $\langle\langle a_1, a_{10}\rangle\rangle$ and $\langle\langle a_6, a_{10}\rangle\rangle$ are of types $2A$ and $4A$ respectively and so

$$\begin{aligned} \alpha_0 &:= 2a_6 + 2a_7 - \frac{1}{2}a_{10} + v_{\rho(t_6,t_9)} \in V_0^{(a_{10})} \\ \beta_0 &:= -\frac{1}{3}(2a_6 + 2a_7 + a_9 + a_{10}) + v_{\rho(t_6,t_9)} \in V_{\frac{1}{2^2}}^{(a_{10})} \\ \alpha_1 &:= a_1 - \frac{1}{2^2}a_{10} + a_{19} \in V_0^{(a_{10})} \\ \beta_1 &:= a_1 - a_{19} \in V_{\frac{1}{2^2}}^{(a_{10})}. \end{aligned}$$

Using the inner product values given by rows 1 and 7 of Table 5.2, we calculate that $(\beta_0, \beta_1) = 0$. Thus, using the fusion rules,

$$a_{10} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1)) = -\frac{1}{2^2}(\alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0). \tag{5.1}$$

We calculate that

$$\begin{aligned} (\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1) = &\frac{1}{2^3 \cdot 3}(a_1 + 2a_5 + 5a_6 + 2a_7 + 2a_8) + \frac{1}{2^4}(a_9 + a_{10}) - \frac{1}{2^2 \cdot 3}(a_{17} - 6a_{18}) \\ &+ \frac{13}{2^3 \cdot 3}a_{19} + \frac{1}{2^4}v_{\rho(t_6,t_9)} - \frac{1}{2^2}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}). \end{aligned}$$

Thus

$$\begin{aligned} a_{10} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1)) = &-\frac{1}{2^4}a_1 + \frac{1}{2^6 \cdot 3}(2a_3 + a_5 + a_6 + a_7 + a_8 + a_9 + 13a_{10}) \\ &- \frac{1}{2^5 \cdot 3}(2a_{16} + a_{17} - a_{18} - 6a_{19}) - \frac{1}{2^7}(v_{\rho(t_5,t_9)} + v_{\rho(t_6,t_9)}) \\ &- \frac{1}{2^2}a_{10} \cdot (v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}). \end{aligned}$$

It remains only to calculate that

$$\begin{aligned}\alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0 = & -\frac{1}{2^3 \cdot 3}(a_1 - a_2 - a_3) - \frac{1}{2^4 \cdot 3}(a_5 + a_6 + a_7 + a_8) - \frac{1}{2^4}(a_9 + a_{10}) \\ & + \frac{1}{2^3 \cdot 3}(a_{17} - 2a_{18} - 4a_{19}) - \frac{1}{2^4}(v_{\rho(t_1,t_5)} + v_{\rho(t_1,t_7)} + v_{\rho(t_6,t_9)}) \\ & + \frac{1}{2^3}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) + 2a_1 \cdot v_{\rho(t_6,t_9)}.\end{aligned}$$

Putting these values into (5.1) give the value of $a_{10} \cdot (v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})})$ as required. ☐

Lemma 5.3.4. *Suppose that V is a Majorana algebra as in Proposition 5.3.1. Then*

$$\begin{aligned}a_{16} \cdot v_{\rho(t_6,t_9)} = & -\frac{1}{2^6 \cdot 3}(5a_1 - a_2) + \frac{1}{2^3 \cdot 3}(a_3 - a_4) + \frac{1}{2^6 \cdot 3}(5a_{16} - a_{17}) - \frac{1}{2^3 \cdot 3}(a_{18} - a_{19}) \\ & - \frac{1}{2^5}(v_{\rho(t_1,t_5)} + v_{\rho(t_1,t_7)}) + \frac{1}{2^5}(v_{\rho(t_5,t_{16})} - v_{\rho(t_7,t_{16})}) + a_1 \cdot v_{\rho(t_6,t_9)}.\end{aligned}$$

Proof. The dihedral algebras $\langle\langle a_6, a_{10}\rangle\rangle$ and $\langle\langle a_{10}, a_{16}\rangle\rangle$ are of types $4A$ and $2A$ respectively and so

$$\begin{aligned}\alpha_0 &:= 2a_6 + 2a_7 - \frac{1}{2}a_{10} + v_{\rho(t_6,t_9)} \in V_0^{(a_{10})} \\ \beta_0 &:= -\frac{1}{3}(2a_6 + 2a_7 + a_9 + a_{10}) + v_{\rho(t_6,t_9)} \in V_{\frac{1}{2^2}}^{(a_{10})} \\ \alpha_1 &:= -\frac{1}{2^2}a_{10} + a_{16} + a_{18} \in V_0^{(a_{10})} \\ \beta_1 &:= a_{16} - a_{18} \in V_{\frac{1}{2^2}}^{(a_{10})}.\end{aligned}$$

Using the inner product values given by rows 6 and 7 of Table 5.2, we calculate that $(\beta_0, \beta_1) = 0$. Thus, using the fusion rules,

$$a_{10} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1)) = -\frac{1}{2^2}(\alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0). \tag{5.2}$$

We calculate that

$$\begin{aligned}(\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1) = & -\frac{1}{2^3 \cdot 3}(2a_2 - 2a_5 - 5a_6 - 5a_7 - 2a_8) + \frac{1}{2^4}(a_9 - a_{10}) \\ & + \frac{1}{2^3 \cdot 3}(a_{16} + 13a_{18} - 4a_{19}) - \frac{1}{2^2}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) + \frac{1}{2^4}v_{\rho(t_6,t_9)}.\end{aligned}$$

Then, using the value of $a_{10} \cdot (v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})})$ given by Lemma 5.3.3, we calculate that

$$\begin{aligned}a_{10} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1)) = & \frac{1}{2^5 \cdot 3}(5a_1 - 2a_2 - 2a_3 + a_4) + \frac{1}{2^6 \cdot 3}(a_5 + a_6 + a_7 + a_8) \\ & + \frac{1}{2^6}(a_9 + a_{10}) - \frac{1}{2^4 \cdot 3}(2a_{16} - 3a_{18}) + \frac{1}{2^6}(v_{\rho(t_1,t_5)} + v_{\rho(t_1,t_7)}) \\ & + \frac{1}{2^6}v_{\rho(t_6,t_9)} - \frac{1}{2^5}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) - \frac{1}{2}a_1 \cdot v_{\rho(t_6,t_9)}.\end{aligned}$$

It remains only to calculate that

$$\begin{aligned}\alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0 = & \frac{1}{2^3 \cdot 3}(a_2 + a_4) - \frac{1}{2^4 \cdot 3}(a_5 + a_6 + a_7 + a_8) - \frac{1}{2^4}(a_9 + a_{10}) \\ & - \frac{1}{2^3 \cdot 3}(a_{16} - a_{17} + 4a_{18} + 2a_{19}) - \frac{1}{2^4}(v_{\rho(t_5,t_{16})} + v_{\rho(t_6,t_{16})} + v_{\rho(t_6,t_9)}) \\ & + \frac{1}{2^3}(v_{\rho(t_5,t_{18})} + v_{\rho(t_6,t_{18})}) + 2a_{16} \cdot v_{\rho(t_6,t_9)}.\end{aligned}$$

Putting these values into (5.2) gives the value of $a_{16} \cdot v_{\rho(t_6,t_9)}$ as required. ☐

Proof of Proposition 5.3.1. We will show that such a Majorana representation cannot obey axiom M1. Using the inner product values in rows 3, 5 and 6 of Table 5.2 we calculate that

$$(a_3 \cdot a_{16}, v_{\rho(t_6,t_9)}) = \frac{1}{2^3}(a_3 - a_{14} + a_{16}, v_{\rho(t_6,t_9)}) = \frac{5}{2^6 \cdot 3}.$$

Now, the value of $a_{16} \cdot v_{\rho(t_6,t_9)}$ is given by Lemma 5.3.4. Using the inner product values in rows 2, 4 and 8 of Table 5.2 we calculate that

$$(a_3, a_{16} \cdot v_{\rho(t_6,t_9)}) = -\frac{5}{2^8 \cdot 3}.$$

In particular, $(a_3 \cdot a_{16}, v_{\rho(t_6,t_9)}) \neq (a_3, a_{16} \cdot v_{\rho(t_6,t_9)})$, in contradiction with axiom M1. □

5.3.2 The Group $2^3.2^3$

Let

$$a := (1,2)(3,4),\ b := (1,3)(2,4)(5,6)(7,8)\,, c := (1,5)(2,7)$$

and let $G = \langle a, b, c \rangle \cong 2^3.2^3$.

In the following, we label certain involutions of G as below.

i	t_i	i	t_i	i	t_i
1	a	10	c^a	19	$(bc)^2$
2	a^c	11	c^b	20	$((bc)^2)^a$
3	a^{cb}	12	c^{ab}	21	$(abc)^2$
4	a^{cbc}	13	ab	22	$((abc)^2)^a$
5	b	14	$(ab)^c$	23	$acabcbc$
6	b^c	15	$(ab)^{ca}$	24	$(acabcbc)^a$
7	b^{ca}	16	$(ab)^{cac}$	25	$(acabcbc)^b$
8	b^{cac}	17	$(ac)^2$	26	$(acabcbc)^{ab}$
9	c	18	$((ac)^2)^b$	27	$(a \cdot b^c)^2$

Proposition 5.3.5. *There are no representations of the form (G, T_1, V) where*

$$T_1 := \{t_i \mid 1 \leq i \leq 16\}.$$

We first give a brief summary of the proof of this result. In Lemma 5.3.6 we give some of the inner products on the 2-closed part of the algebra. In Lemma 5.3.10 we give some of the inner products on the 3-closed part of the algebra, using the algebra products given in Lemmas 5.3.7, 5.3.8 and 5.3.9. Finally, Lemma 5.3.11 gives one further algebra product which then allow us to prove our main result.

In the following, we suppose that there exists such an algebra V and let $a_i := \psi(t_i)$ for $1 \leq i \leq 16$.

Lemma 5.3.6. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then Table 5.3 gives the value of the inner product on certain vectors in the 2-closed part of V.*

i	u	v	(u,v)
1	a_1	$v_{\rho(t_5,t_9)}$	$\frac{3}{2^3}$
2	a_3	$v_{\rho(t_1,t_9)}$	$\frac{1}{2^3\cdot 3}$
3	a_5	$v_{\rho(t_1,t_9)}$	$\frac{3}{2^5}$
4	a_9	$v_{\rho(t_1,t_6)}$	$\frac{1}{2^4}$
5	a_9	$v_{\rho(t_1,t_{11})}$	$\frac{1}{2^3\cdot 3}$
6	a_9	$v_{\rho(t_5,t_{10})}$	$\frac{1}{2^3\cdot 3}$
7	a_9	$v_{\rho(t_6,t_{13})}$	$\frac{1}{2^4}$
8	a_{13}	$v_{\rho(t_1,t_{11})}$	$\frac{3}{2^5}$
9	a_{13}	$v_{\rho(t_5,t_9)}$	$\frac{3}{2^5}$
10	$v_{\rho(t_1,t_9)}$	$v_{\rho(t_1,t_{11})}$	$\frac{1}{3^2}$
11	$v_{\rho(t_1,t_9)}$	$v_{\rho(t_3,t_9)}$	$\frac{2}{3^2}$
12	$v_{\rho(t_1,t_9)}$	$v_{\rho(t_5,t_{10})}$	$\frac{7}{2\cdot 3^2}$
13	$v_{\rho(t_1,t_{14})}$	$v_{\rho(t_2,t_{11})}$	$\frac{19}{2^3\cdot 3^2}$

Table 5.3: Some inner product values on the 2-closed part of V for $G \cong 2^3.2^3$

Proof. In most cases, we use the orthogonality of eigenvectors (Lemma 2.3.4) in order to calculate these inner product values. As such, we begin by listing some eigenvectors of axes of V. These eigenvectors can all be deduced from the shape of the algebra and the known eigenvectors of dihedral algebras.

The following are eigenvectors of a_1:

$$\begin{aligned}
\alpha_0^{(a_1)} &:= a_3 \in V_0^{(a_1)} \\
\alpha_1^{(a_1)} &:= -\frac{1}{2}a_1 + 2a_{11} + 2a_{12} + v_{\rho(t_1,t_{11})} \in V_0^{(a_1)} \\
\beta_0^{(a_1)} &:= -\frac{1}{3}(a_1 + a_2 + a_9 + a_{12}) + v_{\rho(t_1,t_9)} \in V_{\frac{1}{2^2}}^{(a_1)}.
\end{aligned}$$

The following are eigenvectors of a_{10}:

$$\begin{aligned}
\alpha_0^{(a_{10})} &:= a_9 \in V_0^{(a_{10})} \\
\alpha_1^{(a_{10})} &:= 2a_3 + 2a_4 - \frac{1}{2}a_{10} + v_{\rho(t_3,t_9)} \in V_0^{(a_{10})} \\
\alpha_2^{(a_{10})} &:= 2a_1 + 2a_2 - \frac{1}{2}a_{10} + v_{\rho(t_1,t_9)} \in V_0^{(a_{10})} \\
\beta_0^{(a_{10})} &:= -\frac{1}{3}(2a_1 + 2a_2 + a_9 + a_{10}) + v_{\rho(t_1,t_9)} \in V_{\frac{1}{2^2}}^{(a_{10})} \\
\beta_1^{(a_{10})} &:= -\frac{1}{3}(2a_5 + 2a_7 + a_{10} + a_{12}) + v_{\rho(t_5,t_{10})} \in V_{\frac{1}{2^2}}^{(a_{10})}.
\end{aligned}$$

The following are eigenvectors of a_{11}:

$$\alpha_0^{(a_{11})} := a_9 \in V_0^{(a_{11})}$$
$$\beta_0^{(a_{11})} := -\frac{1}{3}(2a_1 + 2a_3 + a_{11} + a_{12}) + v_{\rho(t_1,t_{11})} \in V_{\frac{1}{2^2}}^{(a_{11})}.$$

Row 1 First,

$$(a_1, a_5 \cdot a_9) = \frac{1}{2^6}(a_1, 3a_5 + a_6 + 3a_9 + a_{11} - 3v_{\rho(t_5,t_9)}) = \frac{17}{2^{11}} - \frac{3}{2^6}(a_1, v_{\rho(t_5,t_9)}).$$

Conversely,

$$(a_1 \cdot a_5, a_9) = \frac{1}{2^3}(a_1 + a_5 - a_{13}, a_9) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_1, v_{\rho(t_5,t_9)}) = \frac{3}{2^3}$, as required.

Row 2 This is given by the equality $(\alpha_0^{(a_1)}, \beta_0^{(a_1)}) = 0$.

Row 3 First,

$$(a_5, a_1 \cdot a_9) = \frac{1}{2^6}(a_5, 3a_1 + a_2 + 3a_9 + a_{10} - 3v_{\rho(t_1,t_9)}) = \frac{17}{2^{11}} - \frac{3}{2^6}(a_5, v_{\rho(t_1,t_9)})$$

Conversely,

$$(a_5 \cdot a_1, a_9) = \frac{1}{2^3}(a_1 + a_5 - a_{13}, a_9) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_5, v_{\rho(t_1,t_9)}) = \frac{3}{2^5}$ as required.

Row 4 Using row 1 for the value of $(a_1, v_{\rho(t_5,t_9)})$, we calculate that

$$(a_1, a_6 \cdot a_9) = \frac{1}{2^6}(a_1, a_5 + 3a_6 + 3a_9 + a_{11} - 3v_{\rho(t_5,t_9)}) = \frac{1}{2^{10}}.$$

Conversely,

$$(a_1 \cdot a_6, a_9) = \frac{1}{2^6}(3a_1 + a_4 + 3a_6 + a_7 - 3v_{\rho(t_1,t_6)}, a_9) = \frac{1}{2^8} - \frac{3}{2^6}(a_9, v_{\rho(t_1,t_6)}).$$

Axiom M1 then implies that $(a_9, v_{\rho(t_1,t_6)}) = \frac{1}{2^4}$, as required.

Row 6 This is given by the equality $(\alpha_0^{(a_{11})}, \beta_0^{(a_{11})}) = 0$.

Row 7 This is given by the equality $(\alpha_0^{(a_{10})}, \beta_1^{(a_{10})}) = 0$.

Row 8 Using row 9 for the value of $(a_{13}, v_{\rho(t_5,t_9)})$, we calculate that

$$(a_{13}, a_6 \cdot a_9) = \frac{1}{2^6}(a_{13}, a_5 + 3a_6 + 3a_9 + a_{11} - 3v_{\rho(t_5,t_9)}) = \frac{1}{2^{10}}.$$

Conversely,

$$(a_{13} \cdot a_6, a_9) = \frac{1}{2^6}(3a_6 + a_7 + 3a_{13} + a_{16} - 3v_{\rho(t_6,t_{13})}, a_9) = \frac{1}{2^8} - \frac{3}{2^6}(a_9, v_{\rho(t_6,t_{13})}).$$

Axiom M1 then implies that $(a_9, v_{\rho(t_6,t_{13})}) = \frac{1}{2^4}$, as required.

Row 9 First,

$$(a_{11}, a_1 \cdot a_{13}) = \frac{1}{2^3}(a_{11}, a_1 + a_{13} - a_5) = \frac{1}{2^8}.$$

Conversely,

$$(a_{11} \cdot a_1, a_{13}) = \frac{1}{2^6}(3a_1 + a_3 + 3a_{11} + a_{12} - 3v_{\rho(t_1,t_{11})}, a_{13}) = \frac{17}{2^{11}} - \frac{3}{2^6}(a_{13}, v_{\rho(t_1,t_{11})}).$$

Axiom M1 then implies that $(a_{13}, v_{\rho(t_1,t_{11})}) = \frac{3}{2^5}$, as required.

Row 10 First,

$$(a_{13}, a_5 \cdot a_9) = \frac{1}{2^6}(a_{13}, 3a_5 + a_6 + 3a_9 + a_{11} - 3v_{\rho(t_5,t_9)}) = \frac{17}{2^{11}} - \frac{3}{2^6}(a_1, v_{\rho(t_5,t_9)}).$$

Conversely,

$$(a_{13} \cdot a_5, a_9) = \frac{1}{2^3}(a_{13} + a_5 - a_1, a_9) = \frac{1}{2^8}.$$

Axiom M1 then implies that $(a_{13}, v_{\rho(t_5,t_9)}) = \frac{3}{2^5}$, as required.

Row 11 This is given by the equality $(\alpha_1^{(a_1)}, \beta_0^{(a_1)}) = 0$ along with the values of $(a_2, v_{\rho(t_1,t_{11})})$ and $(a_{11}, v_{\rho(t_1,t_9)}) = (a_{12}, v_{\rho(t_1,t_9)}) = (a_9, v_{\rho(t_1,t_{11})})$ as given by rows 2 and 5 respectively.

Row 12 This is given by the equality $(\alpha_1^{(a_{10})}, \beta_0^{(a_{10})}) = 0$ along with the value of $(a_3, v_{\rho(t_1,t_9)}) = (a_4, v_{\rho(t_1,t_9)}) = (a_1, v_{\rho(t_3,t_9)}) = (a_2, v_{\rho(t_3,t_9)})$ as given by row 2.

Row 13 This is given by the equality $(\alpha_2^{(a_{10})}, \beta_1^{(a_{10})}) = 0$ along with the values of $(a_1, v_{\rho(t_5,t_{10})}) = (a_2, v_{\rho(t_5,t_{10})})$, $(a_5, v_{\rho(t_1,t_9)}) = (a_7, v_{\rho(t_1,t_9)})$ and $(a_{12}, v_{\rho(t_1,t_9)})$ as given by rows 1, 3 and 5 respectively.

Row 14 First, using the value of $(a_{15}, v_{\rho(t_2,t_{11})})$ as given by row 8, we calculate that

$$(a_{15}, a_4 \cdot v_{\rho(t_2,t_{11})}) = -\frac{1}{2^4}(a_{15}, a_2 - 5a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) = \frac{3}{2^8}.$$

Conversely, using the values of $(a_1, v_{\rho(t_2,t_{11})})$ and $(a_{14}, v_{\rho(t_2,t_{11})}) = (a_{15}, v_{\rho(t_2,t_{11})})$ as given by rows 2 and 8 respectively, we calculate that

$$\begin{aligned}(a_{15} \cdot a_4, v_{\rho(t_2,t_{11})}) &= \frac{1}{2^6}(a_1 + 3a_4 + a_{14} + 3a_{15} - 3v_{\rho(t_1,t_{14})}, v_{\rho(t_2,t_{11})}) \\ &= \frac{37}{2^9 \cdot 3} - \frac{3}{2^6}(v_{\rho(t_2,t_{11})}, v_{\rho(t_1,t_{14})}).\end{aligned}$$

Axiom M1 then implies that $(v_{\rho(t_1,t_{14})}, v_{\rho(t_2,t_{11})}) = \frac{19}{2^3 \cdot 3^2}$, as required.

□

Lemma 5.3.7. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then*

$$a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{1}{2^6 \cdot 3}(a_1 - 3a_2 + a_3 - a_4 - 3v_{\rho(t_1,t_{11})} + 3v_{\rho(t_2,t_{11})}) + \frac{1}{2^2}(a_2 \cdot v_{\rho(t_1,t_{11})}).$$

Proof. The dihedral algebras $\langle\langle a_2, a_{11}\rangle\rangle$, $\langle\langle a_2, a_1\rangle\rangle$ and $\langle\langle a_2, a_3\rangle\rangle$ are of types $4A$, $2B$ and $2B$ respectively and so

$$\begin{aligned}\alpha_0 &:= -\frac{1}{2}a_2 + 2a_{11} + 2a_{12} + v_{\rho(t_2,t_{11})} \in V_0^{(a_2)} \\ \alpha_1 &:= a_1 \in V_0^{(a_2)} \\ \alpha_2 &:= a_3 \in V_0^{(a_2)} \\ \beta_0 &:= -\frac{1}{3}(a_2 + a_4 + 2a_{11} + 2a_{12}) + v_{\rho(t_2,t_{11})} \in V_{\frac{1}{2^2}}^{(a_2)}.\end{aligned}$$

We now use the fusion rules to determine further eigenvectors. We calculate that

$$\alpha_3 := \alpha_0 \cdot \alpha_1 - \frac{1}{2^4}(3\alpha_1 + \alpha_2) = \frac{1}{2^4}(2a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) + a_1 \cdot v_{\rho(t_2,t_{11})} \in V_0^{(a_2)}$$

and

$$\beta_1 := \alpha_0 \cdot \beta_0 = -\frac{1}{2^4 \cdot 3}(3a_1 + a_3 + 2a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) + a_1 \cdot v_{\rho(t_2,t_{11})} \in V^{(a_2)}_{\frac{1}{2^2}}.$$

We now use the resurrection principle to find the value of $a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})$. Firstly,

$$\begin{aligned} a_2 \cdot ((\alpha_3 - \beta_1) \cdot \alpha_2) &= \frac{1}{2^4 \cdot 3} a_2 \cdot (a_1 - 2a_3 + a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) \\ &= \frac{1}{2^8 \cdot 3}(3a_2 + a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) - \frac{1}{2^4} a_2 \cdot v_{\rho(t_1,t_{11})}. \end{aligned}$$

However, using the fusion rules,

$$\begin{aligned} a_2 \cdot ((\alpha_3 - \beta_1) \cdot \alpha_2) &= -\frac{1}{2^2}\beta_1 \cdot \alpha_2 \\ &= \frac{1}{2^8 \cdot 3}(a_1 + a_3 + 2a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) - \frac{1}{2^2} a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}). \end{aligned}$$

Equating these two expressions gives the value of $a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})$ as required. □

Lemma 5.3.8. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then*

$$a_1 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{1}{2^5} a_1 + \frac{1}{2^2} a_1 \cdot v_{\rho(t_2,t_{11})}.$$

Proof. The dihedral algebras $\langle\langle a_1, a_4 \rangle\rangle$ and $\langle\langle a_1, a_{11} \rangle\rangle$ are of type $2B$ and $4A$ respectively. Thus from the known eigenvectors of dihedral algebras,

$$\begin{aligned} \alpha_0 &= a_4 \in V^{(a_1)}_0 \\ \alpha_1 &= -\frac{1}{2} a_1 + 2a_{11} + 2a_{12} + v_{\rho(t_1,t_{11})} \in V^{(a_1)}_0 \\ \beta_0 &= -\frac{1}{3}(a_1 + a_3 + 2a_{11} + 2a_{12}) + v_{\rho(t_1,t_{11})} \in V^{(a_1)}_{\frac{1}{2^2}}. \end{aligned}$$

We now use the fusion rules to determine further eigenvectors. We calculate that

$$\beta_1 := \alpha_0 \cdot \beta_0 = -\frac{1}{2^4 \cdot 3}(a_2 + 3a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) + a_4 \cdot v_{\rho(t_1,t_{11})} \in V^{(a_1)}_{\frac{1}{2^2}}$$

and

$$\alpha_2 := \alpha_0 \cdot \alpha_1 = \frac{1}{2^4}(a_2 + 2a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) + a_4 \cdot v_{\rho(t_1,t_{11})} \in V^{(a_1)}_0.$$

We now determine the product $a_4 \cdot (a_4 \cdot v_{\rho(t_1,t_{11})})$. We do this using the resurrection principle with α_2 and β_1. Firstly,

$$\begin{aligned} a_1 \cdot ((\alpha_2 - \beta_1) \cdot \alpha_0) &= a_1 \cdot \left(\frac{1}{2^2 \cdot 3}(a_2 + 3a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) \cdot \alpha_0 \right) \\ &= a_1 \cdot \left(\frac{1}{2^4 \cdot 3}(a_2 + 9a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) \right) \\ &= \frac{1}{2^8 \cdot 3}(3a_1 + a_3 + 2a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) - \frac{1}{2^4} a_1 \cdot v_{\rho(t_2,t_{11})}. \end{aligned}$$

However, using the fusion rules, we have

$$\begin{aligned} a_1 \cdot ((\alpha_2 - \beta_1) \cdot \alpha_0) &= a_1 \cdot (\alpha_2 \cdot \alpha_0 - \beta_1 \cdot \alpha_0) \\ &= -\frac{1}{2^2}\beta_1 \cdot \alpha_0 \\ &= \frac{1}{2^8 \cdot 3}(a_2 + 9a_4 + 2a_{11} + 2a_{12} - 3v_{\rho(t_2,t_{11})}) - \frac{1}{2^2} a_4 \cdot (a_4 \cdot v_{\rho(t_1,t_{11})}). \end{aligned}$$

Equating these two expressions gives

$$a_4 \cdot (a_4 \cdot v_{\rho(t_1,t_{11})}) = \frac{1}{2^6 \cdot 3}(3a_1 - a_2 + a_3 - 9a_4 - 3v_{\rho(t_1,t_{11})} + 3v_{\rho(t_2,t_{11})}) + \frac{1}{2^2} a_1 \cdot v_{\rho(t_2,t_{11})}.$$

Using this, we can now calculate that

$$\beta_2 := 4\alpha_0 \cdot \beta_1 = -\frac{1}{2^4 \cdot 3}(3a_1 + a_3 + 2a_{11} + 2a_{12} - 3v_{\rho(t_1,t_{11})}) + a_1 \cdot v_{\rho(t_2,t_{11})} \in V^{(a_1)}_{\frac{1}{2^2}}.$$

Moreover,

$$\beta_3 := \beta_2 - 4\beta_0 = \beta_3 = -\frac{1}{2^3 \cdot 3} a_1 + a_1 \cdot v_{\rho(t_2,t_{11})} \in V^{(a_1)}_{\frac{1}{2^2}}.$$

We can then use β_3 to calculate the value of $a_1 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})$ as required. □

Lemma 5.3.9. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then*

$$\begin{aligned} a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) =& \frac{1}{2^7 \cdot 3}(a_1 + a_2 + a_3 + a_4 + a_{11} + 7a_{12}) - \frac{1}{2^8}(v_{\rho(t_1,t_{11})} + v_{\rho(t_2,t_{11})}) \\ &+ \frac{1}{2^6}\left((5a_1 + 3a_3) \cdot v_{\rho(t_2,t_{11})} - (4a_2 + 4a_4) \cdot v_{\rho(t_1,t_{11})}\right). \end{aligned}$$

Proof. The dihedral algebras $\langle\langle a_1, a_{12} \rangle\rangle$ and $\langle\langle a_2, a_{12} \rangle\rangle$ are both of type $4A$ and so

$$\begin{aligned} \alpha_0 &:= 2a_1 + 2a_3 - \frac{1}{2^2} a_{12} + v_{\rho(t_1,t_{11})} \in V^{(a_{12})}_0 \\ \alpha_1 &:= 2a_2 + 2a_4 - \frac{1}{2^2} a_{12} + v_{\rho(t_2,t_{11})} \in V^{(a_{12})}_0 \\ \beta_0 &:= -\frac{1}{3}(2a_1 + 2a_3 + a_{11} + a_{12}) + v_{\rho(t_1,t_{11})} \in V^{(a_{12})}_{\frac{1}{2^2}} \\ \beta_1 &:= -\frac{1}{3}(2a_2 + 2a_4 + a_{11} + a_{13}) + v_{\rho(t_2,t_{11})} \in V^{(a_{12})}_{\frac{1}{2^2}}. \end{aligned}$$

From the fusion rules,

$$a_{12} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 + \beta_1)) = \frac{1}{2^2}(\alpha_0 \cdot \beta_1 - \alpha_1 \cdot \beta_0 - (\beta_0, \beta_1) a_{12}). \tag{5.3}$$

We calculate that

$$\begin{aligned} (\alpha_0 - \beta_0) \cdot (\alpha_1 + \beta_1) =& -\frac{1}{2^2 \cdot 3^2}(7a_1 + a_3 + 7a_2 + a_4) - \frac{1}{2^2} a_{12} + \frac{1}{2^3 \cdot 3}(7v_{\rho(t_1,t_{11})} + v_{\rho(t_2,t_{11})}) \\ &+ \frac{16}{3}(a_1 \cdot v_{\rho(t_2,t_{11})} + a_3 \cdot v_{\rho(t_2,t_{11})}). \end{aligned}$$

Thus

$$\begin{aligned} a_{12} \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 + \beta_1)) =& -\frac{1}{2^4 \cdot 3^2}(7a_1 + a_3 + 7a_2 + a_4) - \frac{1}{2^2 \cdot 3^2} a_{11} - \frac{1}{2 \cdot 3} a_{12} \\ &+ \frac{1}{2^5 \cdot 3}(7v_{\rho(t_1,t_{11})} + v_{\rho(t_2,t_{11})}) + \frac{16}{3}(a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) + a_{12} \cdot (a_3 \cdot v_{\rho(t_2,t_{11})})). \end{aligned}$$

We now consider the right hand side of (5.3). We calculate that $(\beta_0, \beta_1) = -\frac{1}{3^2}$ using the values of $(a_1, v_{\rho(t_2,t_{11})}) = (a_3, v_{\rho(t_2,t_{11})}) = (a_2, v_{\rho(t_1,t_{11})}) = (a_2, v_{\rho(t_3,t_9)})$ and $(a_{13}, v_{\rho(t_1,t_{11})})$ as given by rows 2 and 8 of Table 5.3 respectively.

Moreover,

$$\begin{aligned} \alpha_0 \cdot \beta_1 - \alpha_1 \cdot \beta_0 =& -\frac{1}{2^2 \cdot 3}(a_1 - a_2 + a_3 - a_4) + \frac{1}{2^3}(v_{\rho(t_1,t_{11})} - v_{\rho(t_2,t_{11})}) \\ &+ \frac{8}{3}((a_1 + a_3) \cdot v_{\rho(t_2,t_{11})} - (a_2 + a_4) \cdot v_{\rho(t_1,t_{11})}) \end{aligned}$$

i	u	v	(u,v)
1	a_1	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^3 \cdot 3}$
2	a_2	$a_1 \cdot v_{\rho(t_2,t_{11})}$	0
3	a_3	$a_1 \cdot v_{\rho(t_2,t_{11})}$	0
4	a_4	$a_1 \cdot v_{\rho(t_2,t_{11})}$	0
5	a_9	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^6 \cdot 3}$
6	a_9	$a_1 \cdot v_{\rho(t_5,t_9)}$	$\frac{3}{2^9}$
7	a_9	$a_1 \cdot v_{\rho(t_5,t_{10})}$	$-\frac{7}{2^9 \cdot 3}$
8	a_{11}	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^6}$
9	a_{13}	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^6 \cdot 3}$
10	a_{14}	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^8 \cdot 3}$
11	$v_{\rho(t_1,t_9)}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	0
12	$v_{\rho(t_1,t_{11})}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$-\frac{1}{2^3 \cdot 3}$
13	$v_{\rho(t_2,t_{11})}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{1}{2^3 \cdot 3}$
14	$v_{\rho(t_3,t_9)}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$-\frac{1}{2^3 \cdot 3^2}$
15	$v_{\rho(t_9,t_{13})}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{7}{2^7 \cdot 3^2}$
16	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$\frac{3}{2^8}$
17	$a_1 \cdot v_{\rho(t_2,t_{11})}$	$a_1 \cdot v_{\rho(t_3,t_9)}$	$-\frac{5}{2^8 \cdot 3^2}$

Table 5.4: Some inner product values on the 3-closed part of V for $G \cong 2^3.2^3$

where all values are given by the known values of the algebra product on dihedral algebras, except for the products $v_{\rho(t_1,t_{11})} \cdot v_{\rho(t_2,t_{11})}$ which cancel out.

Putting these values into (5.3) gives

$$\begin{aligned} a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) + a_{12} \cdot (a_3 \cdot v_{\rho(t_1,t_{11})}) &= \frac{1}{2^6 \cdot 3}(a_1 + a_2 + a_3 + a_4 + a_{11} + 7a_{12}) \\ - \frac{1}{2^7}(v_{\rho(t_1,t_{11})} + v_{\rho(t_2,t_{11})}) &+ \frac{1}{2^3}((a_1 + a_3) \cdot v_{\rho(t_2,t_{11})} - (a_2 + a_4) \cdot v_{\rho(t_1,t_{11})}). \end{aligned}$$

Finally, we note that $(a_1 \cdot v_{\rho(t_2,t_{11})})^{t_{12}} = a_3 \cdot v_{\rho(t_2,t_{11})}$ so that

$$a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})} - a_3 \cdot v_{\rho(t_2,t_{11})}) = \frac{1}{2^5}(a_1 \cdot v_{\rho(t_2,t_{11})} - a_3 \cdot v_{\rho(t_2,t_{11})})$$

which allows us to calculate $a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})$ as required. □

Lemma 5.3.10. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then Table 5.4 gives the value of the inner product on certain vectors in the 3-closed part of V.*

Proof. We use axiom M1 to recover the inner product values in all cases below.

Row 1 From row 2 of Table 5.3, $(a_1, v_{\rho(t_2,t_{11})}) = \frac{1}{2^3 \cdot 3}$ and so

$$(a_1, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot a_1, v_{\rho(t_2,t_{11})}) = (a_1, v_{\rho(t_2,t_{11})}) = \frac{1}{2^3 \cdot 3}.$$

Rows 2 - 4 For $v \in \{a_2, a_3, a_4\}$, $v \cdot a_1 = 0$ and so $(v, a_1 \cdot v_{\rho(t_2,t_{11})}) = (v \cdot a_1, v_{\rho(t_2,t_{11})}) = 0$.

Rows 5 - 7 The value of $a_1 \cdot a_9$ is given by the known values of dihedral algebras as

$$a_1 \cdot a_9 = \frac{1}{2^6}(3a_1 + a_2 + 3a_9 + a_{10} - 3v_{\rho(t_1,t_9)}).$$

Then, using the values of $(a_1, v_{\rho(t_5,t_9)}) = (a_1, v_{\rho(t_5,t_{10})})$, $(a_1, v_{\rho(t_2,t_{11})})$, $(a_9, v_{\rho(t_2,t_{11})})$, $(a_9, v_{\rho(t_5,t_9)}) = (a_9, v_{\rho(t_5,t_{10})})$ and $(v_{\rho(t_1,t_9)}, v_{\rho(t_2,t_{11})})$ as given by rows 1, 2, 5, 6 and 10 of Table 5.3 respectively, we calculate that

$$\begin{aligned}
(a_9, a_1 \cdot v_{\rho(t_2,t_{11})}) &= (a_1 \cdot a_9, v_{\rho(t_2,t_{11})}) = \frac{1}{2^6 \cdot 3} \\
(a_9, a_1 \cdot v_{\rho(t_5,t_9)}) &= (a_1 \cdot a_9, v_{\rho(t_5,t_9)}) = \frac{3}{2^9} \\
(a_9, a_1 \cdot v_{\rho(t_5,t_{10})}) &= (a_1 \cdot a_9, v_{\rho(t_5,t_{10})}) = -\frac{7}{2^9 \cdot 3}.
\end{aligned}$$

Row 8 The value of $a_1 \cdot a_{11}$ is given by the known values of dihedral algebras as

$$a_1 \cdot a_{11} = \frac{1}{2^6}(3a_1 + a_3 + 3a_{11} + a_{12} - 3v_{\rho(t_1,t_{11})}).$$

Then, using the values of $(a_1, v_{\rho(t_2,t_{11})}) = (a_3, v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_1,t_{11})}, v_{\rho(t_2,t_{11})})$ as given by rows 2 and 11 of Table 5.3 respectively, we calculate that

$$(a_{11}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot a_{11}, v_{\rho(t_2,t_{11})}) = \frac{1}{2^6}.$$

Row 9 The value of $a_1 \cdot a_{13}$ is given by the known values of dihedral algebras as

$$a_1 \cdot a_{13} = \frac{1}{2^3}(a_1 - a_5 + a_{13}).$$

Then, using the values of $(a_1, v_{\rho(t_2,t_{11})})$, $(a_5, v_{\rho(t_2,t_{11})})$ and $(a_{13}, v_{\rho(t_2,t_{11})})$ as given by rows 2, 3 and 8 of Table 5.3 respectively, we calculate that

$$(a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot a_{13}, v_{\rho(t_2,t_{11})}) = \frac{1}{2^6 \cdot 3}.$$

Row 10 The value of $a_1 \cdot a_{14}$ is given by the known values of dihedral algebras as

$$a_1 \cdot a_{14} = \frac{1}{2^6}(3a_1 + a_4 + 3a_{14} + a_{15} - 3v_{\rho(t_1,t_{14})}).$$

Then, using the values of $(a_1, v_{\rho(t_2,t_{11})})$, $(a_{14}, v_{\rho(t_2,t_{11})}) = (a_{15}, v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_2,t_{11})}, v_{\rho(t_1,t_{14})})$ as given by rows 2, 8 and 13 of Table 5.3 respectively, we calculate that

$$(a_{14}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot a_{14}, v_{\rho(t_2,t_{11})}) = \frac{1}{2^8 \cdot 3}.$$

Row 11 The value of $a_1 \cdot v_{\rho(t_1,t_9)}$ is given by the known values of dihedral algebras as

$$a_1 \cdot v_{\rho(t_1,t_9)} = \frac{1}{2^4}(5a_1 - a_2 - 2a_9 - 2a_{10} + 3v_{\rho(t_1,t_9)}).$$

Then, using the values of $(a_1, v_{\rho(t_2,t_{11})})$, $(a_9, v_{\rho(t_2,t_{11})}) = (a_{10}, v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_1,t_9)}, v_{\rho(t_2,t_{11})})$ as given by rows 2, 5 and 11 of Table 5.3 respectively, we calculate that

$$(v_{\rho(t_1,t_9)}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot v_{\rho(t_1,t_9)}, v_{\rho(t_2,t_{11})}) = 0.$$

Row 12 The value of $a_1 \cdot v_{\rho(t_1,t_{11})}$ is given by the known values of dihedral algebras as

$$a_1 \cdot v_{\rho(t_1,t_{11})} = \frac{1}{2^4}(5a_1 - a_3 - 2a_{11} - 2a_{12} + 3v_{\rho(t_1,t_{11})}).$$

Then, using the values of $(a_1, v_{\rho(t_2,t_{11})}) = (a_3, v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_1,t_{11})}, v_{\rho(t_2,t_{11})})$ as given by rows 2 and 11 of Table 5.3 respectively, we calculate that

$$(v_{\rho(t_1,t_{11})}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_1 \cdot v_{\rho(t_1,t_{11})}, v_{\rho(t_2,t_{11})}) = -\frac{1}{2^3 \cdot 3}.$$

Row 13 From row 2 of Table 5.3, $(a_1, v_{\rho(t_2,t_{11})}) = \frac{1}{2^3 \cdot 3}$ and so

$$(v_{\rho(t_2,t_{11})}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (v_{\rho(t_2,t_{11})} \cdot v_{\rho(t_2,t_{11})}, a_1) = (a_1, v_{\rho(t_2,t_{11})}) = \frac{1}{2^3 \cdot 3}.$$

Rows 14 From Lemma 5.3.7,

$$a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{1}{2^6 \cdot 3}(a_1 - 3a_2 + a_3 - a_4 - 3v_{\rho(t_1,t_{11})} + 3v_{\rho(t_2,t_{11})}) + \frac{1}{2^2}(a_2 \cdot v_{\rho(t_1,t_{11})}).$$

Using the value of $(a_9, v_{\rho(t_1,t_{11})}) = (a_9, v_{\rho(t_2,t_{11})})$ as given by row 5 of Table 5.3, as well as the value of $(a_9, a_2 \cdot v_{\rho(t_3,t_9)})$ as given by row 5 of this table, we calculate that

$$(a_9, a_3 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) = \frac{1}{2^{10}}$$

Conversely, using the value of $(a_5, v_{\rho(t_1,t_9)})$ as given by row 3 of Table 5.3 as well as the values of $(a_3, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_4, a_1 \cdot v_{\rho(t_2,t_{11})})$ and $(a_9, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_{10}, a_1 \cdot v_{\rho(t_2,t_{11})})$ as given by rows 3, 4 and 5 respectively of this table, we calculate that

$$\begin{aligned}(a_9 \cdot a_3, a_1 \cdot v_{\rho(t_2,t_{11})}) &= \frac{1}{2^6}(3a_3 + a_4 + 3a_9 + a_{10} - 3v_{\rho(t_3,t_9)}, a_1 \cdot v_{\rho(t_2,t_{11})}) \\ &= \frac{1}{2^{10} \cdot 3} - \frac{3}{2^6}(v_{\rho(t_3,t_9)}, a_1 \cdot v_{\rho(t_2,t_{11})}).\end{aligned}$$

Thus $(v_{\rho(t_3,t_9)}, a_1 \cdot v_{\rho(t_2,t_{11})}) = -\frac{1}{2^3 \cdot 3^2}$ as required.

Row 15 From Lemma 5.3.9,

$$\begin{aligned}a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) =& \frac{1}{2^7 \cdot 3}(a_1 + a_2 + a_3 + a_4 + a_{11} + 7a_{12}) - \frac{1}{2^8}(v_{\rho(t_1,t_{11})} + v_{\rho(t_2,t_{11})}) \\ &+ \frac{1}{2^6}((5a_1 + 3a_3) \cdot v_{\rho(t_2,t_{11})} - 4(a_2 + a_4) \cdot v_{\rho(t_1,t_{11})}).\end{aligned}$$

Using the value of $(a_{13}, v_{\rho(t_1,t_{11})}) = (a_{13}, v_{\rho(t_2,t_{11})})$ as given by row 8 of Table 5.3, as well as the values of $(a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (a_{13}, a_4 \cdot v_{\rho(t_1,t_{11})})$ and $(a_{13}, a_3 \cdot v_{\rho(t_2,t_{11})}) = (a_{13}, a_2 \cdot v_{\rho(t_1,t_{11})})$ as given by rows 9 and 10 respectively of this table, we calculate that

$$(a_{13}, a_{12} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) = \frac{13}{2^{14}}.$$

Conversely,

$$a_{12} \cdot a_{13} = \frac{1}{2^6}(a_9 + 3a_{12} + 3a_{13} + a_{14} - 3v_{\rho(t_9,t_{13})}).$$

Using the values of $(a_9, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_{12}, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})})$ and $(a_{14}, a_1 \cdot v_{\rho(t_2,t_{11})})$, given by rows 5, 8, 9, and 10 respectively of this table, we calculate that

$$(a_{12} \cdot a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{53}{2^{14} \cdot 3} - \frac{3}{2^6}(v_{\rho(t_9,t_{13})}, a_1 \cdot v_{\rho(t_2,t_{11})}).$$

Thus $(v_{\rho(t_9,t_{13})}, a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{7}{2^7 \cdot 3^2}$ as required.

Rows 16 and 17 From Lemma 5.3.8,

$$a_1 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{1}{2^5} a_1 + \frac{1}{2^2} a_1 \cdot v_{\rho(t_2,t_{11})}.$$

Thus, using the value of $(a_1, v_{\rho(t_2,t_{11})})$ as given by row 2 of Table 5.3 as well as the values of $(v_{\rho(t_2,t_{11})}, a_1 \cdot v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_3,t_9)}, a_1 \cdot v_{\rho(t_2,t_{11})})$ from rows 13 and 14 respectively of this table, we calculate that

$$(a_1 \cdot v_{\rho(t_2,t_{11})}, a_1 \cdot v_{\rho(t_2,t_{11})}) = (v_{\rho(t_2,t_{11})}, a_1 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) = \frac{3}{2^8}$$

and

$$(a_1 \cdot v_{\rho(t_2,t_{11})}, a_1 \cdot v_{\rho(t_3,t_9)}) = (v_{\rho(t_3,t_9)}, a_1 \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) = -\frac{5}{2^8 \cdot 3^2}.$$

□

Lemma 5.3.11. *Suppose that V is a Majorana algebra as in Proposition 5.3.5. Then*

$$\begin{aligned} a_{13} \cdot v_{\rho(t_1,t_9)} = &- \frac{1}{2^6} a_1 - \frac{1}{2^6 \cdot 3}(a_4 + 8a_5 - 2a_6 - 2a_7 - 23a_{13} + 2a_{16} + 2a_{15} - a_{16}) \\ &+ \frac{1}{2^6}(v_{\rho(t_1,t_6)} + 3v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})} - v_{\rho(t_2,t_{13})} - 4v_{\rho(t_5,t_9)} - 4v_{\rho(t_5,t_{10})}) \\ &+ \frac{1}{2^5}(v_{\rho(t_9,t_{13})} + v_{\rho(t_{10},t_{13})}) + \frac{1}{2}(a_1 \cdot v_{\rho(t_5,t_9)} + a_1 \cdot v_{\rho(t_5,t_{10})}). \end{aligned}$$

Proof. The dihedral algebras $\langle\langle a_1, a_9 \rangle\rangle$ and $\langle\langle a_1, a_5 \rangle\rangle$ are of type $4A$ and $2A$ respectively and so

$$\begin{aligned} \alpha_0 &:= -\frac{1}{2} a_1 + 2(a_9 + a_{10}) + v_{\rho(t_1,t_9)} \in V_0^{(a_1)} \\ \alpha_1 &:= -\frac{1}{2^2} a_1 + a_5 + a_{13} \in V_0^{(a_1)}. \\ \beta_0 &:= -\frac{1}{3}(a_1 + a_2 + 2a_9 + 2a_{10}) + v_{\rho(t_1,t_9)} \in V_{\frac{1}{2^2}}^{(a_1)} \\ \beta_1 &:= a_5 - a_{13} \in V_{\frac{1}{2^2}}^{(a_1)}. \end{aligned}$$

From the fusion rules,

$$a_1 \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 - \beta_1)) = -\frac{1}{2^2}(\alpha_0 \cdot \beta_1 + \alpha_1 \cdot \beta_0 - (\beta_0, \beta_1)a_1). \tag{5.4}$$

We calculate that

$$\begin{aligned} (\alpha_0 - \beta_0) \cdot (\alpha_1 + \beta_1) = &\frac{1}{2^5 \cdot 3}(-6a_1 + a_2 + a_3 + 4a_5 + 8a_6 + 8a_7 + a_8 + 20a_9 + 20a_{10} \\ &+ 8a_{11} + 8a_{12} + 47a_{13} + 6v_{\rho(t_1,t_9)} - 3v_{\rho(t_2,t_5)}) - \frac{1}{2^2}(v_{\rho(t_5,t_9)} + v_{\rho(t_5,t_{10})}). \end{aligned}$$

This then gives

$$\begin{aligned} a_1 \cdot ((\alpha_0 - \beta_0) \cdot (\alpha_1 + \beta_1)) = &\frac{7}{2^7} a_1 + \frac{1}{2^8 \cdot 3}(2a_2 + 2a_3 + 2a_4 + 43a_5 + 4a_6 + 4a_7 + a_8 + 4a_9 \\ &+ 4a_{10} + 4a_{11} + 4a_{12} - 43a_{13} - a_{14}) - \frac{1}{2^7}(v_{\rho(t_1,t_6)} + v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})}) \\ &- \frac{1}{2^8}(v_{\rho(t_2,t_5)} - v_{\rho(t_2,t_{13})}) - \frac{1}{2^2}(a_1 \cdot v_{\rho(t_5,t_9)} + a_1 \cdot v_{\rho(t_5,t_{10})}). \end{aligned}$$

We now consider the right hand side of (5.4). Using the values of $(a_5, v_{\rho(t_1,t_9)})$ and $(a_{13}, v_{\rho(t_1,t_9)})$ as given by rows 3 and 8 of Table 5.3 respectively, we calculate that $(\beta_0, \beta_1) = 0$.

It remains to calculate that

$$\begin{aligned}\alpha_0 \cdot \beta_1 - \alpha_1 \cdot \beta_0 = \frac{1}{2^4}a_1 + \frac{1}{2^6 \cdot 3}(2a_2 + 2a_3 + 27a_5 + 8a_6 + 8a_7 + a_8 + 4a_9 + 4a_{10} + 4a_{11} + 4a_{12} \\ + 3a_{13} - 4a_{14} - 4a_{15} + a_{16}) + \frac{1}{2^4}(v_{\rho(t_1,t_9)} - 2v_{\rho(t_5,t_9)} - 2v_{\rho(t_5,t_{10})} + v_{\rho(t_9,t_{13})} + v_{\rho(t_{10},t_{13})}) \\ - \frac{1}{2^6}(v_{\rho(t_2,t_5)} + v_{\rho(t_2,t_{13})}) - 2a_{13} \cdot v_{\rho(t_1,t_9)}.\end{aligned}$$

We can then use these values in (5.4) to give the value of $a_{13} \cdot v_{\rho(t_1,t_9)}$ as required. □

Proof of Proposition 5.3.5. Suppose for contradiction that G does admit such a representation. We will show that

$$(a_9 \cdot a_{13}, (a_1 \cdot v_{\rho(t_2,t_{11})})) \neq (a_9, a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) \tag{5.5}$$

which is in contradiction with axiom M1.

We start with the left hand side of (5.5). The algebra $\langle\langle a_9, a_{13}\rangle\rangle$ is of type $4A$ and contains the $4A$ axis $v_{\rho(t_9,t_{13})}$ and so

$$a_9 \cdot a_{13} = \frac{1}{2^6}(3a_9 + a_{12} + 3a_{13} + a_{14} - 3v_{\rho(t_9,t_{13})}).$$

Using the values of $(a_9, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_{12}, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})})$, $(a_{14}, a_1 \cdot v_{\rho(t_2,t_{11})})$ and $(v_{\rho(t_9,t_{13})}, a_1 \cdot v_{\rho(t_2,t_{11})})$ as given by rows 5, 8, 9, 10 and 15 of Table 5.4 respectively, we calculate that

$$(a_9 \cdot a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})}) = \frac{23}{2^{14} \cdot 3}.$$

We now consider the right hand side of (5.5) and begin by calculating the value of $a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})$. As $(a_1 \cdot v_{\rho(t_2,t_{11})})^{t_{13}} = a_1 \cdot v_{\rho(t_3,t_9)}$, from Lemma 2.3.3 we have

$$a_1 \cdot v_{\rho(t_2,t_{11})} - a_1 \cdot v_{\rho(t_3,t_9)} \in V^{(a_{13})}_{\frac{1}{2^5}}$$

and so

$$a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) - a_{13} \cdot (a_1 \cdot v_{\rho(t_3,t_9)}) = \frac{1}{2^5}(a_1 \cdot v_{\rho(t_2,t_{11})} - a_1 \cdot v_{\rho(t_3,t_9)}). \tag{5.6}$$

We can check using the inner product values in rows 2, 5 and 10 of Table 5.3 and rows 1, 2, 3, 5, 8, 11, 12, 16 and 17 of Table 5.4 that the following is an element of the nullspace of V

$$\begin{aligned}n := -\frac{1}{2^4 \cdot 3}(6a_1 + a_2 + a_3 + 2a_9 + 2a_{10} + 2a_{11} + 2a_{12}) + \frac{1}{2^4}(v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})}) \\ + a_1 \cdot (v_{\rho(t_2,t_{11})} + v_{\rho(t_3,t_9)})\end{aligned}$$

and so $a_{13} \cdot n = 0$. Note that $v_{\rho(t_1,t_9)} - v_{\rho(t_1,t_{11})} \in V^{(a_{13})}_{\frac{1}{2^5}}$ and so

$$\begin{aligned}a_{13} \cdot (v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})}) &= a_{13} \cdot (2v_{\rho(t_1,t_9)} + (v_{\rho(t_1,t_9)} - v_{\rho(t_1,t_{11})})) \\ &= 2a_{13} \cdot v_{\rho(t_1,t_9)} + \frac{1}{2^5}(v_{\rho(t_1,t_9)} - v_{\rho(t_1,t_{11})})\end{aligned}$$

where the value of $a_{13} \cdot v_{\rho(t_1,t_9)}$ is given in Lemma 5.3.11.

We can now calculate that, for n as above,

$$\begin{aligned} a_{13} \cdot n = &- \frac{13}{2^9} a_1 - \frac{1}{2^9 \cdot 3}(2a_2 + 2a_3 + a_4 + 16a_5 - 2a_6 - 2a_7) \\ &- \frac{1}{2^7 \cdot 3}(a_9 + a_{10} + a_{11} + a_{12} + 4a_{13} + a_{14} + a_{15}) + \frac{1}{2^9} v_{\rho(t_1,t_6)} \\ &+ \frac{1}{2^8}(v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})}) - \frac{1}{2^7}(v_{\rho(t_5,t_9)} + v_{\rho(t_5,t_{10})} - v_{\rho(t_9,t_{13})} - v_{\rho(t_{10},t_{13})}) \\ &+ a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})} + a_1 \cdot v_{\rho(t_3,t_9)}) \end{aligned}$$

Using this equation with the fact that $a_{13} \cdot n = 0$ gives the value of $a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})} + a_1 \cdot v_{\rho(t_3,t_9)})$. From (5.6), we have the value of $a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})} - a_1 \cdot v_{\rho(t_3,t_9)})$ and so we can calculate that

$$\begin{aligned} a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})}) = &\frac{13}{2^{10}} a_1 + \frac{1}{2^{10} \cdot 3}(2a_2 + 2a_3 + a_4 - 16a_5 - 2a_6 - 2a_7) \\ &+ \frac{1}{2^8 \cdot 3}(a_9 + a_{10} + a_{11} + a_{12} + 4a_{13} + a_{14} + a_{15}) - \frac{1}{2^{10}} v_{\rho(t_1,t_6)} \\ &- \frac{1}{2^9}(v_{\rho(t_1,t_9)} + v_{\rho(t_1,t_{11})}) + \frac{1}{2^8}(v_{\rho(t_5,t_9)} + v_{\rho(t_5,t_{10})} - v_{\rho(t_9,t_{13})} - v_{\rho(t_{10},t_{13})}) \\ &+ \frac{1}{2^6}(a_1 \cdot v_{\rho(t_2,t_{11})} - a_1 \cdot v_{\rho(t_3,t_9)}) - \frac{1}{2^5}(a_1 \cdot v_{\rho(t_5,t_9)} + a_1 \cdot v_{\rho(t_5,t_{10})}). \end{aligned}$$

Finally, we use rows 4 - 7 of Table 5.3 and rows 5 - 8 of Table 5.4 to calculate that

$$(a_9, a_{13} \cdot (a_1 \cdot v_{\rho(t_2,t_{11})})) = \frac{35}{2^{14} \cdot 3} \neq (a_9 \cdot a_{13}, a_1 \cdot v_{\rho(t_2,t_{11})}).$$

□

5.4 The main theorem

We can now prove the main results of this work.

Theorem 5.4.1. *Suppose that V is a Majorana algebra which obeys axiom M8 and which is generated by three Majorana axes a_0, a_1, a_2 such that the dihedral algebra $\langle\langle a_0, a_1 \rangle\rangle$ is of type 2A. Then V is must be isomorphic to one of the 34 Majorana algebras whose dimensions are given in Table 5.1.*

Proof. In Theorem 3.2.1, we show that such an algebra must occur as a Majorana representation of the form (G, T, V) where $G = \langle a, b, c \rangle$ is isomorphic to one of the 26 triangle-point groups in Table 5.1 and T is such that $a, b, c, ab \in T$.

In Section 5.2, for each of these groups we have classified the possible values for the set T. Then, for each choice of G and T, we have used the algorithm described in Chapter 4 to classify and construct all representations of the form (G, T, V). Their dimensions are given in Table 5.1. □

Theorem 5.4.2. *Each of the 34 Majorana algebras whose dimensions are given in Table 5.1 is isomorphic to a subalgebra of the Griess algebra.*

Before proving this result, we first recall some important results concerning Majorana representations and the Griess algebra.

Definition 5.4.3. *A Majorana representation* (G, T, V) *is* based on an embedding *into the Monster if there exists an injective homomorphism* $\iota : G \hookrightarrow \mathbb{M}$ *such that* $\iota(T) \subseteq 2A$*.*

Proposition 5.4.4. *Suppose that* G *is a finite group generated by a* G*-closed set of involutions* T*. Suppose further that there exists an injective homomorphism* $\iota : G \hookrightarrow \mathbb{M}$ *such that* $\iota(T) = \iota(G) \cap 2A$*. Then the subalgebra* V *of the Griess algebra that is generated by the* $2A$ *axes corresponding to* $\iota(T)$ *is a Majorana representation of* G *of the form* (G, T, V) *that obeys axiom M8.*

Proof. As the Majorana axioms M1 - M8 are known to hold in the Griess algebra [Iva09, Proposition 8.6.2], V is certainly a Majorana algebra. If ψ is Conway's bijection [Con84] between the $2A$ involutions and $2A$ axes then $(G, T, V, \iota, \iota \circ \psi)$ is the required Majorana representation. □

In the following, as in [Nor85], we will consider the subgroups in question as subgroups of the group A_{12}, or in one case as a subgroup of the group ${}^2D_5(2) \cong O_{10}^-(2)$. Both these groups are known to $2A$-embed into the Monster, as described below.

Lemma 5.4.5 ([Nor85], Lemma 6)**.**

1. *Let* ι *be an embedding of* A_{12} *into* $\mathbb{M}$ *and suppose that* $t \in A_{12}$*. Then* $\iota(t)$ *is a* $2A$ *involution of* $\mathbb{M}$ *if and only if* t *is an involution of* A_{12} *of cycle type* 2^2 *or* 2^6*.*
2. *Let* ι *be an embedding of* ${}^2D_5(2)$ *into* $\mathbb{M}$ *and suppose that* $t \in {}^2D_5(2)$*. Then* $\iota(t)$ *is a* $2A$ *involution of* $\mathbb{M}$ *if and only if* t *is an involution of* ${}^2D_5(2)$ *and is a product of two commuting* 3*-transpositions in the automorphism group of* ${}^2D_5(2)$*.*

Proof of Theorem 5.4.2. For each of the 24 triangle-point groups $G = \langle a, b, c \rangle$ given in Table 5.1, Norton [Nor85, Table 3] gives embeddings $\iota : G \hookrightarrow \mathbb{M}$ such that $\iota(a), \iota(b), \iota(c), \iota(ab) \in 2A$. We do not assume that this list is exhaustive.

From Proposition 5.4.4, if ι is such an embedding, then the subalgebra of $V_{\mathbb{M}}$ of the form $\langle\langle \psi(\iota(G) \cap 2A) \rangle\rangle$ must be isomorphic to one of the Majorana algebras whose dimensions are given in Table 5.1. We therefore need to show that, for each group $G = \langle a, b, c \rangle$ given in Table 5.1, the number (up to isomorphism) of Majorana representations of the form (G, T, V) with $a, b, c, ab \in T$ is equal to the number (up to isomorphism) of subalgebras of $V_{\mathbb{M}}$ of the form $\langle\langle \psi(\iota(G) \cap 2A) \rangle\rangle$ for the embeddings ι given by [Nor85, Table 3].

For each group in Table 5.1, the corresponding Majorana representations are all of different dimensions and so must be pairwise non-isomorphic. Moreover, for each group, the number of embeddings given in [Nor85, Table 3] is equal to the number of Majorana representations given in Table 5.1. Thus it suffices to check that for each group, the embeddings given in [Nor85, Table 3] give rise to pairwise non-isomorphic subalgebras of $V_{\mathbb{M}}$.

In the cases where a group admits only one Majorana representation (and equivalently has only one embedding into $\mathbb{M}$), there is nothing to check and we are done. For the cases where $G \cong S_4$ or G is dihedral and the case where $G \cong A_5$ then the fact that the corresponding Majorana representations are isomorphic to subalgebras of $V_{\mathbb{M}}$ is given by [IPSS10] and [IS12a] respectively. The remaining cases are treated below. In each case, for each embedding we state to which Majorana representation it corresponds.

The group $2 \times D_8$. Let $G \cong 2 \times D_8$. Then, up to conjugacy, there are three embeddings of G into $\mathbb{M}$ and these are given by rows 4, 7 and 8 of Table 3 in [Nor85]. We label these embeddings ι_1, ι_2 and ι_3 respectively. Using the generators given in the seventh column of Table 3 of [Nor85], we calculate that

$$|\iota_1(G) \cap 2A| = 10,\ |\iota_2(G) \cap 2A| = 8 \text{ and } |\iota_3(G) \cap 2A| = 8.$$

Thus the representation constructed in Proposition 5.2.13 is based on the embedding ι_1.

As $|\iota_2(G) \cap 2A| = |\iota_3(G) \cap 2A|$, we must check that these two embeddings give non-isomorphic algebras. Firstly, for all $t, s \in \iota_2(G) \cap 2A$ such that $o(ts) = 4$, we find that $(ts)^2 \notin 2A$ and so the corresponding subalgebra of $V_{\mathbb{M}}$ contains dihedral algebras of type $4A$ but none of type $4B$.

Conversely, for all $t, s \in \iota_3(G) \cap 2A$ such that $o(ts) = 4$, we find that $(ts)^2 \in 2A$ and so the corresponding subalgebra of $V_{\mathbb{M}}$ contains dihedral algebras of type $4B$ but none of type $4A$. Thus these two algebras are indeed non-isomorphic and the Majorana representations constructed in 5.2.10 and 5.2.11 are based on the embeddings ι_3 and ι_2 respectively.

The group $2^4.2$. Let $G \cong 2^4.2$. Then, up to conjugacy, there are two embeddings of G into $\mathbb{M}$ and these are given by rows 17 and 20 of Table 3 in [Nor85]. We label these embeddings ι_1 and ι_2 respectively. Using the generators given in the seventh column of Table 3 of [Nor85], we calculate that

$$|\iota_1(G) \cap 2A| = 10 \text{ and } |\iota_2(G) \cap 2A| = 14.$$

Thus the Majorana representations constructed in Propositions 5.2.17 and 5.2.18 are based on the embeddings ι_1 and ι_2 respectively.

The group $2 \times S_4$. Let $G \cong 2 \times S_4$. Then, up to conjugacy, there are two embeddings of G into $\mathbb{M}$ and these are given by rows 11 and 29 of Table 3 in [Nor85]. We label these embeddings ι_1 and ι_2 respectively. Using the generators given in the seventh column of Table 3 of [Nor85], we calculate that

$$|\iota_1(G) \cap 2A| = 16 \text{ and } |\iota_2(G) \cap 2A| = 16.$$

As $|\iota_1(G) \cap 2A| = |\iota_2(G) \cap 2A|$, we must check that these two embeddings truly give different algebras. Firstly, for all $t, s \in \iota_1(G) \cap 2A$ such that $o(ts) = 4$, we find that $(ts)^2 \notin 2A$ and so the corresponding subalgebra of $V_{\mathbb{M}}$ contains dihedral algebras of type $4A$ but none of type $4B$.

Conversely, for all $t, s \in \iota_2(G) \cap 2A$ such that $o(ts) = 4$, we find that $(ts)^2 \in 2A$ and so the corresponding subalgebra of $V_{\mathbb{M}}$ contains dihedral algebras of type $4B$ but none of type $4A$. Thus these two algebras are indeed non-isomorphic and the Majorana representations constructed in 5.2.25 and 5.2.26 are non-isomorphic and are based on the embeddings ι_2 and ι_1 respectively.

The group $2^3.2^3$. Let $G \cong 2^3.2^3$. Then, up to conjugacy, there are two embeddings of G into $\mathbb{M}$ and these are given by rows 18 and 25 of Table 3 in [Nor85]. We label these embeddings ι_1 and ι_2 respectively.

We calculate in Proposition 5.2.28 that a triangle-point group $G = \langle a, b, c \rangle \cong 2^3.2^3$ has three conjugacy classes of size two. We label these classes

$$C_5 := ((ac)^2)^G, \, C_6 := (bc)^2 \text{ and } C_7 := (abc)^2.$$

Moreover, there are two possiblilities for the value of a set T such that (G, T, V) is a Majorana representation and $a, b, c, ab \in T$. The smaller of the two, which is of size 18, contains exactly one of the conjugacy classes C_5, C_6 and C_7. The larger of the two, which is of size 22, contains all three classes.

The generators of $\iota_1(G)$ are given in [Nor85] as elements of A_{12} and so it is straightforward to calculate that

$$|\iota_1(G) \cap 2A| = 18.$$

This implies that the Majorana representation constructed in Proposition 5.2.30 is based on the embedding ι_1.

The generators of $\iota_2(G)$ are given in [Nor85] as elements of ${}^2D_5(2)$ as follows

$$a := (1\ 2).(3\ 4), \; b := (1\ 3).(2\ 4) \text{ and } c := (4\ 5).(1\ 2\ 3\ 6\ 7\ 8 \mid 4\ 5\ 9\ 10\ 11\ 12).$$

We can then calculate that

$$\begin{aligned} (ac)^2 &= (1\ 2).(1\ 2\ 5\ 6\ 7\ 8 \mid 3\ 4\ 9\ 10\ 11\ 12) \\ (bc)^2 &= (2\ 4).(1\ 3\ 5\ 6\ 7\ 8 \mid 2\ 4\ 9\ 10\ 11\ 12) \\ (abc)^2 &= (2\ 3).(1\ 4\ 5\ 6\ 7\ 8 \mid 2\ 3\ 9\ 10\ 11\ 12). \end{aligned}$$

Thus, by Lemma 5.4.5, $\iota_2((ac)^2), \iota_2((bc)^2), \iota_2((abc)^2) \in 2A$ and so $|\iota_2(G) \cap 2A| = 22$. This implies that the Majorana representation constructed in Proposition 5.2.32 is based on the embedding ι_2.

An explanation of the notation of elements of ${}^2D_5(2)$ and full details of the calculations in the group ${}^2D_5(2) \cong O_{10}^-(2)$ can be found in Appendix A. □

Finally, we recall that the Majorana axioms, including M8, are known to hold in the Griess algebra ([Iva09, Proposition 8.6.2]) and so any subalgebra of the Griess algebra which is generated by $2A$ axes is a Majorana algebra which obeys axiom M8. In particular, Theorem 5.4.1 directly implies the following corollaries.

Corollary 5.4.6. *Let V be a subalgebra of the Griess algebra generated by three $2A$ axes a_0, a_1 and a_2 such that a_0 and a_1 generate a $2A$ dihedral algebra. Then V must be isomorphic to one of the 34 Majorana algebras whose dimensions are given in Table 5.1.*

Corollary 5.4.7. *Let G be one of the groups $L_2(11)$, $2^4 : D_{10}$ and S_5. Then, from [Nor85, Table 3], there are two non-conjugate embeddings ι_0 and ι_1 of G as a triangle-point group into $\mathbb{M}$. The subalgebras $\langle\langle \iota_0(G) \cap 2A \rangle\rangle$ and $\langle\langle \iota_1(G) \cap 2A \rangle\rangle$ of $V_{\mathbb{M}}$ are isomorphic.*

Appendix A

Calculations in ${}^2D_5(2)$

A.1 The group ${}^2D_5(2)$

Our main references for this appendix are [Nor85] and [CCN$^+$85]. We consider the group ${}^2D_5(2) \cong O_{10}^-(2)$ and describe in detail various calculations which appear in the proof of Theorem 5.4.1.

These calculations are best performed in the automorphism group of $O_{10}^-(2)$ which is isomorphic to $GO_{10}^-(2)$, the group consisting of all 10×10 matrices over $GF(2)$ preserving a quadratic form of Witt defect 1.

We first construct the quadratic space upon which this group will act. Take the vector space V to be the set of all even weight length 12 vectors over $GF(2)$ modulo complementation. We define an orthogonal form $(\ ,\ )$ on V where for all $u, v \in V$ such that $u \neq v$,

$$(u,v) = \begin{cases} 0 & \text{if } u+v \text{ is of even weight} \\ 1 & \text{if } u+v \text{ is of odd weight} \end{cases}$$

and

$$(u,u) = \begin{cases} 0 & \text{if } u \text{ is of weight 0 mod 4} \\ 1 & \text{if } u \text{ is of weight 2 mod 4.} \end{cases}$$

The set of linear transformations preserving this form will be $GO_{10}^-(2) = O_{10}^-(2).2$. In particular, this group contains transvections of the form

$$T(x) : v \mapsto v + (v,x)x$$

where x is any vector of norm 1. These transvections form a set of 3-transpositions.

In a slight abuse of notation, we identify the group element $T(x)$ with the vector x. In particular, if x is of weight 2 with ones in positions i and j then we denote the transvection $T(x)$ by the pair $(i\ j)$. In fact, there is a natural embedding of S_{12} into $GO_{10}^-(2)$ such that the permutation

(i, j) is mapped to the transvection $(i\ j)$ for all $1 \le i < j \le 12$. Otherwise, if x is of weight 6 then we used bifid notation $(a\ b\ c\ d\ e\ f \mid g\ h\ i\ j\ k\ l)$ to denote the transvection $T(x)$.

The following lemma is easy to check using the definitions above.

Lemma A.1.1. *Suppose that $u, v \in V$. If $(u, v) = 0$ then $T(u)^{T(v)} = T(u)$. Otherwise, if $(u, v) = 1$, then $T(u)^{T(v)} = T(u + v)$.*

A.2 Calculations for the proof of Theorem 5.4.1

Now, recall from the proof of Theorem 5.4.1 that if

$$a := (1\ 2).(3\ 4),\ b := (1\ 3).(2\ 4) \text{ and } c := (4\ 5).(1\ 2\ 3\ 6\ 7\ 8 \mid 4\ 5\ 9\ 10\ 11\ 12).$$

then a, b, c generate $2^3.2^3$ as a triangle-point group.

We now denote

$$a_1 := (1\ 2),\ a_2 := (3\ 4),\ b_1 := (1\ 3),\ b_2 := (2\ 4),\ c_1 := (4\ 5),\ c_2 := (1\ 2\ 3\ 6\ 7\ 8 \mid 4\ 5\ 9\ 10\ 11\ 12)$$

so that $a = a_1a_2$, $b = b_1b_2$ and $c = c_1c_2$.

We pick the following set to be a basis of V:

$$\{(1\ i) \mid 2 \le i \le 11\}.$$

With respect to this basis, we express the transvections $a_1, a_2, b_1, b_2, c_1, c_2$ as 10×10 matrices over $GF(2)$, as below.

$$a_1 := \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1\\ .&1&.&.&.&.&.&.&.&.\\ .&.&1&.&.&.&.&.&.&.\\ .&.&.&1&.&.&.&.&.&.\\ .&.&.&.&1&.&.&.&.&.\\ .&.&.&.&.&1&.&.&.&.\\ .&.&.&.&.&.&1&.&.&.\\ .&.&.&.&.&.&.&1&.&.\\ .&.&.&.&.&.&.&.&1&.\\ .&.&.&.&.&.&.&.&.&1 \end{bmatrix} \qquad a_2 := \begin{bmatrix} 1&.&.&.&.&.&.&.&.&.\\ .&.&1&.&.&.&.&.&.&.\\ .&1&.&.&.&.&.&.&.&.\\ .&.&.&1&.&.&.&.&.&.\\ .&.&.&.&1&.&.&.&.&.\\ .&.&.&.&.&1&.&.&.&.\\ .&.&.&.&.&.&1&.&.&.\\ .&.&.&.&.&.&.&1&.&.\\ .&.&.&.&.&.&.&.&1&.\\ .&.&.&.&.&.&.&.&.&1 \end{bmatrix}$$

$$b_1 := \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{bmatrix} \quad b_2 := \begin{bmatrix} . & . & 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{bmatrix}$$

$$c_1 := \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{bmatrix} \quad c_2 := \begin{bmatrix} 1 & . & 1 & 1 & . & . & . & 1 & 1 & 1 \\ . & 1 & 1 & 1 & . & . & . & 1 & 1 & 1 \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & 1 & 1 & 1 & . & . & 1 & 1 & 1 \\ . & . & 1 & 1 & . & 1 & . & 1 & 1 & 1 \\ . & . & 1 & 1 & . & . & 1 & 1 & 1 & 1 \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & 1 \end{bmatrix}$$

Using these matrix representations, we can now check that

$$\begin{aligned} (ac)^2 &= a_2 \cdot c_2^{a_2 c_1} \\ (bc)^2 &= b_2 \cdot c_2^{b_2 c_1} \\ (abc)^2 &= b_2^{a_1} \cdot c_2^{b_2 a_1 c_1}. \end{aligned}$$

Finally, using Lemma A.1.1, it is straightforward to calculate that

$$\begin{aligned} (ac)^2 &= (1\ 2).(1\ 2\ 5\ 6\ 7\ 8 \mid 3\ 4\ 9\ 10\ 11\ 12) \\ (bc)^2 &= (2\ 4).(1\ 3\ 5\ 6\ 7\ 8 \mid 2\ 4\ 9\ 10\ 11\ 12) \\ (abc)^2 &= (2\ 3).(1\ 4\ 5\ 6\ 7\ 8 \mid 2\ 3\ 9\ 10\ 11\ 12). \end{aligned}$$

Thus the elements $(ac)^2$, $(bc)^2$ and $(abc)^2$ are each the product of commuting transvections, as required in the proof of Theorem 5.4.1.

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